AN UP-TO-THE-BOUNDARY VERSION OF FRIEDRICHS’ LEMMA
AND APPLICATIONS TO THE LINEAR KOITER SHELL MODEL

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Abstract. In this work, we introduce a variant of the standard mollifier technique that is valid up to the boundary of a Lipschitz domain in $\mathbb{R}^n$. A version of Friedrichs’ lemma is derived that gives an estimate up to the boundary for the commutator of the multiplication by a Lipschitz function and the modified mollification. We use this version of Friedrichs’ lemma to prove the density of smooth functions in the new function space introduced in our earlier work concerning the linear Koiter shell model for shells with little regularity. The density of smooth functions is in turn used to prove continuous dependence of the solution of Koiter’s model on the midsurface. This provides a complete justification of our new formulation of the Koiter model.

Résumé. Nous introduisons dans ce travail une variante de la technique usuelle de régularisation par convolution. Cette variante permet de travailler jusqu’à la frontière d’un domaine lipschitzien de $\mathbb{R}^n$. Nous montrons une version du lemme de Friedrichs pour estimer le commutateur de cette régularisation et de la multiplication par une fonction lipschitzienne, version qui est également valable jusqu’à la frontière. Nous utilisons ce lemme pour montrer la densité des fonctions régulières dans le nouvel espace fonctionnel introduit dans un travail antérieur pour formuler et analyser le modèle linéaire de coques de Koiter pour des coques dont la surface moyenne est peu régulière. Nous en déduisons la dépendance continue de la solution du modèle de Koiter par rapport à la surface moyenne. Ce dernier résultat fournit une justification complète de notre nouvelle formulation du modèle de Koiter.

1. INTRODUCTION

Mollification is a basic technique in analysis. It is classically performed by convolution with a compactly supported mollifier. In order for the convolution to be defined, it is necessary either to work on the whole of $\mathbb{R}^n$ or in a compactly contained subdomain $\omega$ of the domain of interest $\Omega$. For many classical function spaces on a domain $\Omega$, approximation by smooth functions is quite straightforward, if the boundary of the domain $\Omega$ is regular enough. In effect, in the latter case, there usually is a continuous extension operator that reduces the case of the domain to that of $\mathbb{R}^n$. It suffices to perform the mollification on the extended function and then restrict the mollified function to the domain. See [1], [13], [14], [22] and [24] among others.

Our main field of application here is the linear Koiter shell model in elasticity. In [5], [6], we introduced a new formulation of this model that makes sense and is well-posed for mid-surfaces of class $W^{2,\infty}$ instead of $C^3$, as was customarily assumed earlier, see also [9] and [11] for shell models in the same context of regularity. The simplest and most natural examples of $W^{2,\infty}$-shells

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are given by globally $C^1$- and piecewise $C^3$-midsurfaces. Consider for instance a shell consisting of a planar part that is connected to a circular cylinder part or an egg-shaped shell made of a quarter of a sphere and a quarter of an ellipsoid glued together along a circle. Our new formulation entails the introduction of a new functional setting. The new function space involves multiplication of distributional partial derivatives of the functions by given Lipschitz functions related to the geometry of the midsurface. It is required that such quantities be square integrable. To the best of our knowledge, this specific kind of function space was not studied before as regards such fundamental issues as the density or non density of smooth functions. There are relatively close ideas in transport equation theory, see [2], [12], [10], although the techniques used therein do not apply in our case. For the function space introduced in [5], there is no obvious extension operator and it is not natural to work on the whole of $\mathbb{R}^2$. Thus different ideas were needed to address the density question.

One such idea was put forth in [23] and was rediscovered later on simultaneously by the authors and by [15] in a slightly different form. The idea consists in defining a new mollification technique in which the mollifier is simultaneously scaled and translated inside the domain. For the technique to work, it is necessary that the domain satisfy a uniform cone condition, which is practically equivalent to being Lipschitz, see [1], [16]. This simple but powerful idea yields mollified functions defined on the whole domain without any need for an extension operator to provide values for the function outside of its domain of definition.

It is a straightforward matter to reproduce the proof of Friedrichs’ lemma using the above convolution-translation in place of the convolution itself. This gives an $L^p$ estimate on the whole domain for the commutator of the convolution-translation and the multiplication by a Lipschitz function applied to partial derivatives of an $L^p$ function.

This version of Friedrichs’ lemma is the main tool in proving the density of smooth functions in the new function space for Koiter’s model. This density has important consequences. For example, it shows that standard finite element methods actually approximate the solution of the newly formulated variational problem for Koiter’s model, see [19] and [20]. Another consequence that we develop here is that, if we consider a sequence of $W^{2,\infty}$-midsurfaces that converge in a natural sense toward a given midsurface, and a sequence of loads that also converge, then the corresponding sequence of solutions to Koiter’s model converge in a natural sense too. Since the new model coincides with the classical model for $C^3$-midsurfaces, taking a sequence of such $C^3$-midsurfaces converging to a midsurface that is only $W^{2,\infty}$ shows that our new formulation is an appropriate extension of the classical formulation to less regular midsurfaces from the mathematical point of view.

2. A MODIFIED MOLLIFICATION TECHNIQUE

In this section, we develop the convolution-translation technique introduced in [23], see also [15],
that allows for up-to-the-boundary mollification without requiring an extension operator. It is well
known that the density of smooth functions in, for example, Sobolev spaces, may fail if the domain
under consideration is not regular. It is thus natural that the regularity of the boundary should come
into play.

First of all, let us recall the uniform cone property for a domain $\Omega$ in $\mathbb{R}^n$. We refer the reader to
[1], [13], [16] and [24] for details.

**Definition 2.1.** An open set $\Omega \subset \mathbb{R}^n$ is said to satisfy the cone property if there exists an open cone

$$C = \{ x = (x', x_n) \in \mathbb{R}^n; 0 < x_n < h, |x'| < x_n \tan(\theta/2) \},$$

with $h > 0$ and $0 < \theta < \pi$, such that for every point $x$ in $\bar{\Omega}$, there is a rotation $R_x$ such that
$C_x = x + R_x C \subset \Omega$. In other words, any point $x$ is the vertex of a cone congruent to $C$ and
included in $\bar{\Omega}$ (or $\Omega$). Here, $|\cdot|$ denotes the standard Euclidean norm either on $\mathbb{R}^{n-1}$ or on $\mathbb{R}^n$.

The set $\Omega$ is said to satisfy the uniform cone property if there exists a locally finite open covering
$\{U_i\}_{i \geq 1}$ of $\partial \Omega$, and a corresponding sequence $\{C_i\}$ of cones, each congruent to some fixed cone $C$, such that:

i) There exists $M$ such that every $U_i$ has diameter less than $M$.

ii) For some $\delta > 0$, $\Omega_\delta = \{ x \in \Omega; \text{dist}(x; \partial \Omega) < \delta \} \subset \bigcup_{i=1}^{\infty} U_i$.

iii) For every $i \in \mathbb{N}$, $Q_i = \bigcup_{x \in \Omega \cap U_i} (x + C_i) \subset \Omega$.

iv) There exists $N \in \mathbb{N}$ such that every collection of $N+1$ of the sets $Q_i$ has an empty intersection.

**Remarks 2.2.** 1) Obviously, a set which satisfies the uniform cone property also satisfies the cone property.

2) If $\Omega$ has a bounded boundary of class $C^1$, then it can be shown that $\Omega$ satisfies the uniform cone property, cf. [14], p. 409.

3) Let us note that if a given open set satisfies the uniform cone property, then its boundary has no interior or exterior cusp. For example, the set $\{(x, y) \in \mathbb{R}^2; -1 < x < 1, |y| < x^2 \text{ for } x \geq 0\}$ has an interior cusp at the origin and does not satisfy the uniform cone property. It does however satisfy the cone property.

4) The set $\{(x, y) \in \mathbb{R}^2; 0 < x < 1, |y| < x^3\}$ has an exterior cusp at the origin and so does not satisfy the cone property.

5) Finally, we remark that a bounded domain of $\mathbb{R}^n$ satisfies the uniform cone property if and only if its boundary is Lipschitz, see [7] or [16], Theorem 1.2.2.2.

Let us now introduce some notation for the convolution-translation operator which is the basic
tool for subsequent developments.

**Definition 2.3.** Let $e$ be a unit vector in $\mathbb{R}^n$ and $\tau > 0$. For all $u, v \in L^1(\mathbb{R}^n)$, we define their
convolution-translation (of amount $\tau$ in the direction $e$) $u *_{\tau,e} v$ by
\[ u *_{\tau,e} v(x) = u * v(x - \tau e), \tag{2.1} \]
that is to say,
\[ u *_{\tau,e} v(x) = \int_{\mathbb{R}^n} u(x - \tau e - y)v(y) \, dy. \tag{2.2} \]

Obviously, $u *_{\tau,e} v \in L^1(\mathbb{R}^n)$, $u *_{\tau,e} v = v *_{\tau,e} u$ and if $v$ is $C^\infty$, so is $u *_{\tau,e} v$ with $\partial^\alpha (u *_{\tau,e} v) = u *_{\tau,e} \partial^\alpha v$ for any multi-index $\alpha \in \mathbb{N}^n$.

The interesting feature of this slightly modified convolution is that it can be used to define a mollification technique for Sobolev spaces that is valid up to the boundary of any domain in $\mathbb{R}^n$ that satisfies the uniform cone condition, without using any extension operator. Let us now describe how this can be achieved.

Let $\rho$ be a standard mollifier, i.e., a positive $C^\infty$ function on $\mathbb{R}^n$ supported in the unit ball and such that $\int_{\mathbb{R}^n} \rho(x) \, dx = 1$. Let $\Omega$ be a domain in $\mathbb{R}^n$ that satisfies the uniform cone condition, and denote by $U_i$ the locally finite covering of the boundary from Definition 2.1. To entirely cover $\Omega$, we let $U_0 = \{ x \in \Omega; \text{dist}(x; \partial \Omega) > \delta/2 \}$. We denote by $(\varphi_i)_{i \in \mathbb{N}}$ an associated $C^\infty$ partition of unity.

**Theorem 2.4.** For all $u \in W^{m,p}(\Omega)$, there exists a sequence $u_\varepsilon$ in $C^\infty(\overline{\Omega})$ such that
\[ u_\varepsilon \to u \quad \text{in} \quad W^{m,p}(\Omega) \quad \text{when} \quad \varepsilon \to 0. \tag{2.3} \]

**Proof.** This is of course a very classical result. We only include it here to show how it can be proved using the convolution-translation instead of standard mollification together with an extension operator.

We begin by localizing $u = \sum_{i \in \mathbb{N}} u_i$, with $u_i = \varphi_i u$. Each $u_i$ has compact support in $U_i \cap \overline{\Omega}$. The “interior” part $u_0$ does not pose any problem and can be approximated by standard mollification. Let us concentrate on what happens near the boundary.

From now on, we may thus assume that $u$ has support in, say, $U_1 \cap \overline{\Omega} = U$ without loss of generality. As far as cones are concerned, we may as well assume that $C = C_1$.

We consider an open subset $\Omega'$ of $\mathbb{R}^n$ such that $U_1 \cap \overline{\Omega}' = U$ and that satisfies conditions i), ii) and iii) of Definition 2.1 with just one cone equal to $C$. Such a set clearly exists. Indeed, in view of [16], Theorem 1.2.2.2., $\partial \Omega \cap U$ is the graph of a Lipschitz function $\Phi$ from a compact subset $K$ of a hyperplane in $\mathbb{R}^n$ into $\mathbb{R}$ (using an appropriate coordinate system $(x', x_n)$). If $M$ denotes the Lipschitz constant of $\Phi$, the standard McShane, or Whitney, extension of $\Phi$ to the whole hyperplane defined by
\[ \tilde{\Phi}(x') = \min_{y \in K} (\Phi(y) + M|x' - y|) \]
provides a globally defined Lipschitz extension of $\Phi$ with the same Lipschitz constant $M$. It is thus sufficient to set $\Omega' = \{ x \in \mathbb{R}^n; x_n < \tilde{\Phi}(x') \}$. Now, extension of $u$ by zero to $\Omega' \setminus \Omega$ clearly yields a function in $W^{m,p}(\Omega')$.

It follows from the previous considerations that we may assume that $\Omega$ satisfies the uniform cone condition with just one cone, equal to $C$. Let $\theta_C$, $\epsilon_C$ and $h_C$ be respectively the cone’s angle, outward unit axis vector and height.

For all $0 < \epsilon$, we now define

$$\eta(\epsilon) = \epsilon \sin(\theta_C/2),$$

and

$$\rho_\epsilon(y) = \eta(\epsilon)^{-n} \rho(y/\eta(\epsilon)).$$

Let $\epsilon_C = \frac{h_C}{1+\sin(\theta_C/2)}$. For all $0 < \epsilon < \epsilon_C$ and all $x \in \bar{\Omega}$, we then let

$$u_\epsilon(x) = \int_{B(0,\eta(\epsilon))} u(x - \epsilon e - y) \rho_\epsilon(y) \, dy.$$  \hfill (2.6)

By construction, we have $x - \epsilon e - B(0, \eta(\epsilon)) \subset x + C \subset \Omega$, therefore $u_\epsilon(x)$ is well defined up to the boundary, see Figure 1. Moreover, since $\rho_\epsilon$ has support in $\bar{B}(0, \eta(\epsilon))$, we have that

$$u_\epsilon(x) = \int_{\mathbb{R}^n} \tilde{u}(x - \epsilon e - y) \rho_\epsilon(y) \, dy = \tilde{u} *_{\epsilon} \rho_\epsilon(x),$$

i.e., $u_\epsilon = (\tilde{u} *_{\epsilon} \rho_\epsilon)_{|\bar{\Omega}}$, where $\tilde{u}$ is any extension of $u$ to $\mathbb{R}^n$, for instance the extension by 0. It follows that $u_\epsilon \in C^\infty(\bar{\Omega})$. Moreover, it is easy to see that, for any multi-index $\alpha$ and all $x \in \bar{\Omega}$, we also have

$$\partial^\alpha u_\epsilon(x) = \int_{\mathbb{R}^n} \tilde{\partial^\alpha u}(x - \epsilon e - y) \rho_\epsilon(y) \, dy = \tilde{\partial^\alpha u} *_{\epsilon} \rho_\epsilon(x).$$

To conclude, it is thus sufficient to show that $u_\epsilon \to u$ in $L^p(\Omega)$ when $\epsilon \to 0$. This follows from the same argument as used in standard mollification. For all $\eta > 0$, first choose $v \in C^0_c(\Omega)$ such that $\|u - v\|_{L^p(\Omega)} \leq \eta/3$. Clearly, $\|u_\epsilon - v_\epsilon\|_{L^p(\Omega)} \leq \eta/3$. Indeed,

$$\|u_\epsilon - v_\epsilon\|_{L^p(\Omega)} \leq \|u_\epsilon - v_\epsilon\|_{L^p(\mathbb{R}^n)}$$

$$= \|\tilde{u} *_{\epsilon} \rho_\epsilon\|_{L^p(\mathbb{R}^n)}$$

$$= \|\tilde{u} *_{\epsilon} \rho_\epsilon\|_{L^p(\mathbb{R}^n)}$$

$$\leq \|u - v\|_{L^p(\mathbb{R}^n)}$$

$$= \|u - v\|_{L^p(\Omega)},$$

since $\|\rho_\epsilon\|_{L^1(\mathbb{R}^n)} = 1$. Consequently, it suffices to estimate $\|v - v_\epsilon\|_{L^p(\Omega)}$, which is a simple matter since $v$ is continuous with compact support in $\Omega$. Let $K$ be a compact set that contains the supports of $(v - v_\epsilon)$ for all $\epsilon \leq \epsilon_C$. Clearly,

$$\|v - v_\epsilon\|_{L^p(\Omega)} \leq \omega(\epsilon(1 + \sin \theta_C/2)) \text{meas } K^{1/p},$$
where \( \omega \) is the modulus of continuity of \( v \). Therefore, we can choose \( \varepsilon_0 \) such that

\[
\|v - v_{\varepsilon}\|_{L^p(\Omega)} \leq \eta/3
\]

for all \( 0 < \varepsilon < \varepsilon_0 \).

3. A GENERALIZED VERSION OF FRIEDRICH'S LEMMA

Friedrichs' lemma was introduced to deal with partial differential equations with varying coefficients. There are many different versions of the lemma. We are concerned here with the version that estimates the commutator of the multiplication by a Lipschitz function and the convolution by a mollifier on \( \mathbb{R}^n \) or on a compactly contained subset of the domain \( \Omega \), see [17] and [18]. We replace here the usual convolution by the previously introduced up-to-the-boundary convolution-translation.

In what follows, \( \Omega \) is a domain in \( \mathbb{R}^n \) that satisfies the uniform cone condition with just one cone, as before. For all \( v \in L^p(\Omega) \), we denote by \( \partial_\alpha v \) its distributional partial derivative with respect to \( x_\alpha \) (we thus do not use the multi-index notation). Therefore, \( \partial_\alpha v \in W^{-1,p}(\Omega) \). We need to define the convolution-translation for such distributions:

\[
\partial_\alpha v *_{\varepsilon,e} \rho_\varepsilon(x) = \int_{B(0,\eta(\varepsilon))} v(x - \varepsilon e - y) \partial_\alpha \rho_\varepsilon(y) \, dy,
\]

where \( \eta(\varepsilon) \) and \( \rho_\varepsilon \) are defined as in formulas (2.4) and (2.5). Clearly, this definition agrees with the former one (2.6) when \( v \) is \( C^\infty \), the resulting function is in \( C^\infty(\overline{\Omega}) \) and if we take a sequence
$v_k \in C^\infty(\bar{\Omega})$ such that $v_k \to v$ in $L^p(\Omega)$ when $k \to +\infty$, then $\partial_\alpha v_k \ast_{\varepsilon,e} \rho_\varepsilon \to \partial_\alpha v \ast_{\varepsilon,e} \rho_\varepsilon$ in $L^p(\Omega)$.

Our version of Friedrichs’ lemma is as follows.

**Lemma 3.1.** Let $v \in L^p(\Omega)$ and $a \in W^{1,\infty}(\Omega)$, then there exists a constant $M$ which depends only on $\rho$ and on the cone angle, such that

$$
\|(a\partial_\alpha v) \ast_{\varepsilon,e} \rho_\varepsilon - a(\partial_\alpha v \ast_{\varepsilon,e} \rho_\varepsilon)\|_{L^p(\Omega)} \leq M\|a\|_{W^{1,\infty}(\Omega)}\|v\|_{L^p(\Omega)}.
$$

**Proof.** Note first that if $v \in L^p(\Omega)$ and $a \in W^{1,\infty}(\Omega)$, then $a\partial_\alpha v \in W^{-1,p}(\Omega)$ so that all terms are well defined. Moreover, in view of the above remarks, it is sufficient to establish estimate (3.2) for functions $v$ in $\mathcal{D}(\Omega)$, which is a dense subset of $L^p(\Omega)$. In this case, $a\partial_\alpha v$ belongs to $W^{1,\infty}(\Omega)$ with compact support.

Let us estimate the commutator. For all $x \in \bar{\Omega}$, we have

$$
[(a\partial_\alpha v) \ast_{\varepsilon,e} \rho_\varepsilon - a(\partial_\alpha v \ast_{\varepsilon,e} \rho_\varepsilon)](x) = \int_{B(0,\eta(\varepsilon))} [a(x - \varepsilon e - y) - a(x)]\partial_\alpha v(x - \varepsilon e - y)\rho_\varepsilon(y) \, dy.
$$

(3.3)

Integrating the right-hand side of equation (3.3) by parts in the ball, we obtain

$$
[(a\partial_\alpha v) \ast_{\varepsilon,e} \rho_\varepsilon - a(\partial_\alpha v \ast_{\varepsilon,e} \rho_\varepsilon)](x) = \int_{B(0,\eta(\varepsilon))} \partial_\alpha a(x - \varepsilon e - y)v(x - \varepsilon e - y)\rho_\varepsilon(y) \, dy
$$

$$
+ \int_{B(0,\eta(\varepsilon))} [a(x - \varepsilon e - y) - a(x)]v(x - \varepsilon e - y)\partial_\alpha \rho_\varepsilon(y) \, dy.
$$

(3.4)

Note that both integral quantities vanish for $x$ outside of a compact neighborhood $K$ of the support of $v$ that is independent of $\varepsilon$ for $\varepsilon \leq \varepsilon_C$. We denote by $\tilde{a}$ the McShane extension of the restriction of $a$ to $K$, and by $\tilde{\varepsilon}$ the extension of $v$ by zero. It is then clear that for all $x \in \Omega$,

$$
[(a\partial_\alpha v) \ast_{\varepsilon,e} \rho_\varepsilon - a(\partial_\alpha v \ast_{\varepsilon,e} \rho_\varepsilon)](x) = f_1(x) + f_2(x),
$$

(3.5)

where

$$
f_1(x) = \int_{\mathbb{R}^n} \partial_\alpha \tilde{a}(x - \varepsilon e - y)\tilde{\varepsilon}(x - \varepsilon e - y)\rho_\varepsilon(y) \, dy
$$

and

$$
f_2(x) = \int_{\mathbb{R}^n} [\tilde{a}(x - \varepsilon e - y) - \tilde{a}(x)]\tilde{\varepsilon}(x - \varepsilon e - y)\partial_\alpha \rho_\varepsilon(y) \, dy
$$

for all $x \in \mathbb{R}^n$. It is thus enough to estimate $f_1$ and $f_2$ in $L^p$ separately.

For the first term, we have

$$
|f_1(x)| \leq \int_{\mathbb{R}^n} |\partial_\alpha \tilde{a}(x - \varepsilon e - y)||\tilde{\varepsilon}(x - \varepsilon e - y)|\rho_\varepsilon(y) \, dy
$$

$$
\leq \|\partial_\alpha \tilde{a}\|_{L^\infty(\mathbb{R}^n)} \int_{\mathbb{R}^n} |\tilde{\varepsilon}(x - \varepsilon e - y)|\rho_\varepsilon(y) \, dy
$$

$$
\leq \|a\|_{W^{1,\infty}(\Omega)}(|\tilde{\varepsilon}| \ast_{\varepsilon,e} \rho_\varepsilon)(x - \varepsilon e),
$$
for all \( x \in \mathbb{R}^n \). Therefore,
\[
\| f_1 \|_{L^p(\Omega)} \leq \| f_1 \|_{L^p(\mathbb{R}^n)} \\
\leq \| a \|_{W^{1,\infty}(\Omega)} \| (| \vec{v}| * \rho_\varepsilon) \|_{L^p(\mathbb{R}^n)} \\
\leq \| a \|_{W^{1,\infty}(\Omega)} \| \vec{v} \|_{L^p(\mathbb{R}^n)} \\
= \| a \|_{W^{1,\infty}(\Omega)} \| v \|_{L^p(\Omega)}.
\]

Similarly, for the second term,
\[
|f_2(x)| \leq \int_{\mathbb{R}^n} |\tilde{a}(x-\varepsilon e-y) - \tilde{a}(x)| |\tilde{v}(x-\varepsilon e-y)| |\partial_\alpha \rho_\varepsilon(y)|\ dy \\
\leq \| \nabla \tilde{a} \|_{L^1(\mathbb{R}^n)} \int_{\mathbb{R}^n} |\tilde{v}(x-\varepsilon e-y)| |\varepsilon e - y| |\partial_\alpha \rho_\varepsilon(y)|\ dy \\
\leq \| a \|_{W^{1,\infty}(\Omega)} (| \tilde{v}| * g_\varepsilon)(x-\varepsilon e),
\]
for all \( x \in \mathbb{R}^n \), where
\[ g_\varepsilon(y) = |\varepsilon e - y| |\partial_\alpha \rho_\varepsilon(y)|. \]

In view of definitions (2.4) and (2.5), it is easy to see that
\[
\| g_\varepsilon \|_{L^1(\mathbb{R}^n)} \leq \left( \frac{1}{\sin(\theta_C/2)} + 1 \right) \| \partial_\alpha \rho \|_{L^1(\mathbb{R}^n)}.
\]

Therefore,
\[
\| f_2 \|_{L^p(\Omega)} \leq \left( \frac{1}{\sin(\theta_C/2)} + 1 \right) \| \partial_\alpha \rho \|_{L^1(\mathbb{R}^n)} \| a \|_{W^{1,\infty}(\Omega)} \| v \|_{L^p(\Omega)},
\]
hence the result with \( M = 1 + \left( \frac{1}{\sin(\theta_C/2)} + 1 \right) \| \partial_\alpha \rho \|_{L^1(\mathbb{R}^n)}. \)

The following corollary will be the basic tool for our density results in the context of the Koiter shell model.

**Corollary 3.2.** For all \( v \in L^p(\Omega) \) and \( a \in W^{1,\infty}(\Omega) \),
\[
\| (a \partial_\alpha v) *_{\varepsilon,e} \rho_\varepsilon - a(\partial_\alpha v *_{\varepsilon,e} \rho_\varepsilon) \|_{L^p(\Omega)} \to 0 \quad \text{when} \quad \varepsilon \to 0. \quad (3.6)
\]

**Proof.** For all \( n \in \mathbb{N}^* \), let us first choose a function \( \varphi_n \in D(\Omega) \) such that
\[
\| v - \varphi_n \|_{L^p(\Omega)} \leq \frac{1}{2nM \| a \|_{W^{1,\infty}(\Omega)}}.
\]

Let us set
\[
T_\varepsilon(v) = (a \partial_\alpha v) *_{\varepsilon,e} \rho_\varepsilon - a(\partial_\alpha v *_{\varepsilon,e} \rho_\varepsilon).
\]
By estimate (3.2), we have
\[ \|T_\varepsilon (v - \varphi_n)\|_{L^p(\Omega)} \leq \frac{1}{2n}. \]

Note that \( a\partial_\alpha \varphi_n \) now belongs to \( W^{1,\infty}(\omega) \), hence to \( L^p(\Omega) \). Therefore,
\[ (a\partial_\alpha \varphi_n) \star_{\varepsilon,e} \rho_\varepsilon \rightarrow a\partial_\alpha \varphi_n \text{ strongly in } L^p(\Omega) \text{ when } \varepsilon \rightarrow 0, \]
as can be seen from the proof of Theorem 2.4. Similarly,
\[ \partial_\alpha \varphi_n \star_{\varepsilon,e} \rho_\varepsilon \rightarrow \partial_\alpha \varphi_n \text{ strongly in } L^p(\Omega) \text{ when } \varepsilon \rightarrow 0, \]
and since \( a \in W^{1,\infty}(\Omega) \), it follows clearly that
\[ a(\partial_\alpha \varphi_n) \star_{\varepsilon,e} \rho_\varepsilon \rightarrow a\partial_\alpha \varphi_n \text{ strongly in } L^p(\Omega) \text{ when } \varepsilon \rightarrow 0 \]
as well. Consequently, we can choose \( \varepsilon_n \) such that for all \( 0 < \varepsilon \leq \varepsilon_n \),
\[ \|T_\varepsilon (\varphi_n)\|_{L^p(\Omega)} \leq \frac{1}{2n}, \]
which concludes the proof.

\[\square\]

4. APPLICATION TO THE KOITER SHELL MODEL

4.1. FORMULATION OF THE PROBLEM

In this section, we briefly recall the formulation of the linear Koiter shell model introduced in [5] and [6]. This formulation is much simpler than the classical formulation and is furthermore valid for midsurfaces that can have discontinuous curvatures. We refer to [3] and [8] for general elastic shell theory.

In the sequel, Greek indices and exponents always belong to the set \( \{1, 2\} \), while Latin indices and exponents belong to the set \( \{1, 2, 3\} \). We use the Einstein summation convention, unless otherwise specified.

Let \( (e_1, e_2, e_3) \) be the orthonormal canonical basis of the Euclidean space \( \mathbb{R}^3 \). We note \( u \cdot v \) the inner product of \( \mathbb{R}^3 \), \( |u| = \sqrt{u \cdot u} \) the associated Euclidean norm and \( u \wedge v \) the vector product of \( u \) and \( v \).

Let \( \omega \) denote a Lipschitz domain of \( \mathbb{R}^2 \). We consider a shell with midsurface \( S = \varphi(\bar{\omega}) \), where \( \varphi \in W^{2,\infty}(\omega; \mathbb{R}^3) \) is a one-to-one mapping such that the two vectors
\[ a_\alpha = \partial_\alpha \varphi \]
are linearly independent at each point \( x \in \bar{\omega} \). We let
\[ a_3 = \frac{a_1 \wedge a_2}{|a_1 \wedge a_2|} \]
be the unit normal vector on the midsurface at point \( \varphi(x) \). The vectors \( a_i \) define the covariant basis at point \( \varphi(x) \). The regularity of the midsurface chart and the hypothesis of linear independence on \( \bar{\omega} \) imply that the vectors \( a_i \) belong to \( W^{1,\infty}(\omega; \mathbb{R}^3) \). The contravariant basis \( a^i \) is defined by the relations

\[
a^i(x) \cdot a_j(x) = \delta^i_j
\]

where \( \delta^i_j \) is the Kronecker symbol. In particular \( a^3(x) = a_3(x) \). As before, \( a^i \in W^{1,\infty}(\omega; \mathbb{R}^3) \). We let \( a(x) = |a_1(x) \wedge a_2(x)|^2 \), so that \( \sqrt{a} \) is the area element of the midsurface in the chart \( \varphi \).

The first and second fundamental forms of the surface are given in covariant components by

\[
a_{\alpha\beta} = a_\alpha \cdot a_\beta \quad \text{and} \quad b_{\alpha\beta} = a_3 \cdot \partial_\beta a_\alpha = -a_\alpha \cdot \partial_\beta a_3.
\]

Since \( W^{1,\infty}(\omega; \mathbb{R}^3) \) is a Banach algebra, it follows that \( a_{\alpha\beta} \in W^{1,\infty}(\omega) \) and \( b_{\alpha\beta} \in L^\infty(\omega) \). We further introduce the contravariant components of the first fundamental form

\[
a^{\alpha\beta} = a^\alpha \cdot a^\beta
\]

and the mixed components of the second fundamental form

\[
b^\beta_\alpha = a^{\beta\rho} b_{\rho\alpha}.
\]

Again, \( a^{\alpha\beta} \in W^{1,\infty}(\omega) \) and \( b^\beta_\alpha \in L^\infty(\omega) \). Finally, the Christoffel symbols of the midsurface are given by

\[
\Gamma^\rho_{\alpha\beta} = \Gamma^\rho_{\beta\alpha} = a^\rho \cdot \partial_\beta a_\alpha
\]

and we have \( \Gamma^\rho_{\alpha\beta} \in L^\infty(\omega) \).

Let us recall the new expressions for the various strain tensors that were introduced in [5] and [6]. Let \( u \in H^1(\omega; \mathbb{R}^3) \) be a displacement of the midsurface, i.e., a regular mapping from \( \bar{\omega} \) into \( \mathbb{R}^3 \). Its linearized strain tensor is given by

\[
\gamma(u) = \gamma_{\alpha\beta}(u) a^\alpha \otimes a^\beta \quad \text{with}
\]

\[
\gamma_{\alpha\beta}(u) = \frac{1}{2} (\partial_\alpha u \cdot a_\beta + \partial_\beta u \cdot a_\alpha) \in L^2(\omega)
\]

and its linearized change of curvature tensor by

\[
\Upsilon(u) = \Upsilon_{\alpha\beta}(u) a^\alpha \otimes a^\beta \quad \text{with}
\]

\[
\Upsilon_{\alpha\beta}(u) = (\partial_{\alpha\beta} u - \Gamma^\rho_{\alpha\beta} \partial_\rho u) \cdot a_3 \in H^{-1}(\omega).
\]

Note that in the classical approach, the displacement is identified with the triple of its covariant components. In particular, the linearized change of curvature tensor is usually given by:

\[
\Upsilon_{\alpha\beta}(u) = u_{3|\alpha\beta} - b_{\alpha\rho} b_{\beta\rho} u_3 + b_{\beta\rho} u_{\rho|\alpha} + b_{\alpha\rho} u_{\rho|\beta} + b_{\beta\rho} u_{\rho|\alpha} u_\rho,
\]
where \( u_{3\alpha\beta} = \partial_{\alpha\beta}u_3 - \Gamma^\rho_{\alpha\beta}\partial_\rho u_3 \) and \( b^\rho_{\beta\alpha} = \partial_\alpha b_\beta^\rho + \Gamma^\rho_{\alpha\sigma}b^\sigma_\beta - \Gamma^\sigma_{\beta\alpha}b^\rho_\sigma \), see [21]. It is this last expression that restricts the regularity of chart in the classical approach. Indeed, in [4] and all subsequent works on existence and uniqueness for the Koiter model, the tangential components of the displacement \( u_\rho \) belong to \( H^1(\omega) \). For the term \( \partial_\alpha b^\rho_\beta u_\beta \) to make sense, the above mentioned authors are led to assume that \( \varphi \) is of class at least \( C^3 \), which precludes many interesting cases as we mentioned earlier.

In [5] and [6], we considered instead the displacement \( u \) as a mapping from \( \omega \) into \( \mathbb{R}^3 \), and not as the triple of its covariant components. We introduced the function space

\[
W = \left\{ v \in H^1(\omega; \mathbb{R}^3), \partial_{\alpha\beta}v \cdot a_3 \in L^2(\omega) \right\}.
\]

(4.3)

Note that if \( v \in H^1(\omega; \mathbb{R}^3) \), then \( \partial_{\alpha\beta}v \cdot a_3 \) is a priori in \( H^{-1}(\omega) \). In view of formulas (4.1) and (4.2), it is apparent that displacements in \( W \) are such that their linearized strain and change of curvature tensors are square-integrable. When equipped with its natural norm

\[
\|v\|_W = \left( \|v\|^2_{H^1(\omega; \mathbb{R}^3)} + \sum_{\alpha, \beta} \|\partial_{\alpha\beta}v \cdot a_3\|^2_{L^2(\omega)} \right)^{1/2},
\]

(4.4)

the space \( W \) is a Hilbert space. To formulate an equilibrium problem for the shell, we consider \( \varepsilon > 0 \) to be the thickness of the shell and an elasticity tensor \( a^{\alpha\beta\rho\sigma} \in L^\infty(\omega) \), which we assume to satisfy the usual symmetries and to be uniformly strictly positive. These hypotheses are for example satisfied by a homogeneous, isotropic material with Lamé moduli \( \mu > 0 \) and \( \lambda \geq 0 \), in which case

\[
a^{\alpha\beta\rho\sigma} = 2\mu(a^{\alpha\beta}a^{\rho\sigma} + a^{\alpha\sigma}a^{\beta\rho}) + \frac{4\lambda\mu}{\lambda + 2\mu} a^{\alpha\beta}a^{\rho\sigma}.
\]

(4.5)

In terms of boundary conditions, for a simply supported shell, we introduced the space

\[
V_0 = \{ v \in W; v = 0 \text{ on } \partial\omega, \text{ in the sense of traces} \}.
\]

(4.6)

In [5] and [6], we proved the following existence and uniqueness result for Koiter’s model in the case of shells with possibly discontinuous curvatures.

THEOREM 4.1. Let \( f \in L^2(\omega; \mathbb{R}^3) \) be a given force resultant density. Then there exists a unique solution to the variational problem: Find \( u \in V_0 \) such that

\[
\forall v \in V_0, \quad \int_\omega \varepsilon a^{\alpha\beta\rho\sigma} (\gamma_{\alpha\beta}(u)\gamma_{\rho\sigma}(v) + \frac{\varepsilon^2}{12} \Upsilon_{\alpha\beta}(u)\Upsilon_{\rho\sigma}(v)) \sqrt{a} \, dx = \int_\omega f \cdot v \sqrt{a} \, dx.
\]

(4.7)

Remarks 4.2. 1) The proof of this theorem relies on the new version of the rigid displacement lemma and the Korn inequality for surfaces with \( W^{2, \infty} \) regularity, see [6] for details.
2) Also in [6], it is proved that the space $W$ defines an extension of the classical framework of [4] to our case. Indeed, when $\varphi \in C^3$, the function space of [4] (corresponding to boundary conditions of simple support) is canonically isomorphic to the space $V_0$. Moreover, the new and classical expressions for the linearized strain and change of curvature tensors coincide under this isomorphism. Consequently, the solution given by Theorem 4.1 is in this case equal, modulo the isomorphism, to the solution found in [4].

3) The case of a shell clamped on a part of its boundary $\gamma_0 \subset \partial \omega$ (hard boundary conditions) and submitted to tractions and moments on the remaining part is also treated in [6]. The relevant function space is

$$V_{1,\gamma_0} = \{ v \in W; v = \partial_\alpha v \cdot a_3 = 0 \text{ on } \gamma_0 \},$$

(4.8)

which can be shown to be a closed subspace of $W$. We thus obtain another existence and uniqueness result in this case.

4.2. DENSITY RESULTS

One fundamental issue that was not addressed in [6] is the density of smooth functions in the various function spaces introduced in our new formulation of the Koiter model. The density of smooth functions is for instance required in order to make sure that standard finite element methods will actually approximate the solution of the continuous problem. Another use of this density will be to show the continuous dependence of the solution of the model on the midsurface, in an appropriate sense. In [6], the consistency of our formulation with the classical formulation is an a priori consistency: we know that our formulation is more general than and coincides with the classical formulation when both are applicable. Continuous dependence is a way to prove a posteriori consistency, via a convergence result.

It should be noted that such a density result cannot be taken for granted since these spaces are not of a standard kind. Similar questions arise in transport theory, see [2], [12] and [15] for example. In the case of the transport equation, the definition of the relevant function spaces involves a directional derivative of the form $a \nabla u$, with $a$ a vector field and $u$ a scalar unknown, that is required to satisfy some integrability condition. Although formally slightly reminiscent of this situation, our function space setting is different, since the quantities of interest in shell theory, $\partial_{\alpha \beta} u \cdot a_3$, are not directional derivatives—$a_3$ does not “live” in the same space as $u$—and we cannot adapt techniques based on this special structure to our case.

It should be noted that if $\omega = \mathbb{R}^2$, then the density of smooth functions in $W$ follows more or less readily from the classical version of Friedrichs’ lemma, see [18], as will be made clear in the ensuing proofs. However, from the point of view of the applications, a shell whose midsurface was described by a chart over $\mathbb{R}^2$ would be of little interest, since the chart could not represent the shell’s boundary. Besides, it would then be diffeomorphic to an open disk, which is restrictive in terms of the topology of the shell since multiply connected shells would not be allowed. Thus, there is no
escaping the difficulties that arise at the boundary.

As we mentioned earlier, the standard way of performing mollification up to the boundary consists in using an extension operator and then mollifying over \( \mathbb{R}^2 \). This does not seem to be of much help here. Indeed, if \( E_1(u) \) and \( E_2(\varphi) \) denote respectively an \( H^1 \)-extension operator for the displacement \( u \) and a \( W^{2,\infty} \)-extension operator for the chart \( \varphi \) to \( \mathbb{R}^2 \), there does not seem to be an easy way of devising \( E_1 \) and \( E_2 \) in a way that ensures that \( \partial_{\alpha\beta}(E_1(u)) \cdot \bar{a}_3 \) will belong to \( L^2(\mathbb{R}^2) \), where \( \bar{a}_3 \) denotes the corresponding extended normal vector (assuming it is defined). In particular, the classical techniques using reflections or integral operators do not seem to work very well. The same remark applies if we try to extend \( a_3 \) itself without any reference to the geometrical underpinnings of the situation.

It is this failure that prompted us to look for an alternative and eventually rediscover a mollification technique essentially already put forth in [23].

Let us start with the larger function space, without boundary conditions.

**Theorem 4.3.** Assume that \( \omega \) satisfies the uniform cone condition. Then \( C^\infty(\bar{\omega}; \mathbb{R}^3) \) is dense in \( \tilde{\omega} \).

**Proof.** First of all, it clear that \( C^\infty(\bar{\omega}; \mathbb{R}^3) \) is dense in \( \tilde{\omega} \).

Let \( u \in \tilde{\omega} \). We want to construct a sequence \( u_\varepsilon \) of \( C^\infty(\bar{\omega}; \mathbb{R}^3) \) functions that converges to \( u \) in the norm of \( \tilde{\omega} \), i.e., such that \( u_\varepsilon \to u \) in \( H^1(\omega; \mathbb{R}^3) \) and \( \partial_{\alpha\beta}u_\varepsilon \cdot a_3 \to \partial_{\alpha\beta}u \cdot a_3 \) in \( L^2(\omega) \) for all indices \( \alpha, \beta \).

It is not difficult to check that the space \( \tilde{\omega} \) can be localized using a partition of unity that is adapted to the uniform cone condition satisfied by \( \omega \). We can thus assume that \( u \) is compactly supported in one of the sets \( U_i \cap \bar{\omega} \) introduced in the proof of Theorem 2.4. We leave the case of the “interior” part in \( U_0 \cap \bar{\omega} \) aside for the time being.

Let \( U = U_1 \cap \bar{\omega} \). As in the proof of Theorem 2.4, we can assume that \( \omega \) satisfies the uniform cone condition with just one cone and that \( u \) is compactly supported in \( U \). Introducing then

\[
    u_\varepsilon = \tilde{u} *_{\varepsilon,\rho} \rho_{\varepsilon},
\]

Theorem 2.4 shows that

\[
    u_\varepsilon \to u \quad \text{in} \quad H^1(\omega; \mathbb{R}^3).
\]

Let \( u_i, u_\varepsilon,i \) and \( a_{3,i} \) denote the Cartesian components of \( u, u_\varepsilon \) and \( a_3 \) respectively, so that \( \partial_{\alpha\beta}u \cdot a_3 = (\partial_{\alpha\beta}u_i) a_{3,i} \). Applying Corollary 3.2 to \( \partial_{\beta}u_i \in L^2(\omega) \), we obtain

\[
    \|(a_{3,i}\partial_{\alpha\beta}u_i)^*_{\varepsilon,\rho} - a_{3,i}(\partial_{\alpha\beta}u_i)^*_{\varepsilon,\rho}\|_{L^2(\omega)} \to 0 \quad \text{when} \quad \varepsilon \to 0.
\]

Now, since \( u \in \tilde{\omega} \), we also have that \( (a_{3,i}\partial_{\alpha\beta}u_i) \in L^2(\omega) \). Therefore, by Theorem 2.4, it follows that

\[
    \|a_{3,i}\partial_{\alpha\beta}u_i - (a_{3,i}\partial_{\alpha\beta}u_i)^*_{\varepsilon,\rho}\|_{L^2(\omega)} \to 0 \quad \text{when} \quad \varepsilon \to 0.
\]
as well. Since
\[ \partial_{\alpha\beta} u_i = (\partial_{\alpha\beta} u_i) \ast_{\varepsilon, e} \rho_\varepsilon, \]
we have thus shown that
\[ \| a_{3,i} \partial_{\alpha\beta} u_i - a_{3,i} \partial_{\alpha\beta} u_i, \varepsilon \|_{L^2(\omega)} \rightarrow 0 \text{ when } \varepsilon \rightarrow 0 \]
which concludes the proof near the boundary.

Concerning the interior part of \( u_i \), it is apparent that the same proof works using standard mollification and the classical Friedrichs lemma.

Let us now consider the case of various boundary conditions. The simplest case is that of a shell clamped on all its boundary. This corresponds to the space
\[ V_1 = \{ v \in W; v = \partial_\alpha v \cdot a_3 = 0 \text{ on } \partial \omega \}, \quad (4.9) \]
which is a closed subspace of \( W \), endowed with the norm of \( W \). As before, we assume that \( \omega \) satisfies the uniform cone condition.

**Theorem 4.4.** For a totally clamped shell, \( \mathcal{D}(\omega; \mathbb{R}^3) \) is dense in \( V_1 \).

**Proof.** Localize as before near the boundary. We claim that the extension \( \tilde{u} \) of \( u \) by zero to the whole of \( \mathbb{R}^2 \) is such that \( \tilde{u} \in H^1(\mathbb{R}^2; \mathbb{R}^3) \) and \( \partial_{\alpha\beta} \tilde{u} \cdot \tilde{a}_3 \in L^2(\mathbb{R}^2) \). Indeed, both conditions are equivalent to having \( \tilde{u} \in H^1(\mathbb{R}^2; \mathbb{R}^3) \) and \( \partial_\alpha \tilde{u} \cdot \tilde{a}_3 \in H^1(\mathbb{R}^2) \). Since both functions are piecewise \( H^1 \) and have no jump on \( \partial \omega \), the claim is true.

Instead of translating inside the domain as earlier, we translate here outside and let \( u_\varepsilon = \tilde{u} \ast_{\varepsilon, -\varepsilon} \rho_\varepsilon \). The same proof as in Theorem 4.3 shows that \( (u_\varepsilon)|_\omega \rightarrow u \) in \( W \) and that \( u_\varepsilon \in C^\infty(\mathbb{R}^2; \mathbb{R}^3) \). Moreover, since \( \tilde{u} \) is identically zero outside of \( \omega \), it is clear that \( u_\varepsilon \) has compact support in \( \omega \). It may be necessary to change the cone in such a way that the exterior cone condition also be satisfied to achieve this, which is possible since \( \omega \) is locally Lipschitz. Hence the result.

Theorem 4.4 above can be seen as an intermediary step for the following density result.

**Theorem 4.5.** Assume that \( \gamma_0 \) consists of a finite union of open arcs in \( \partial \omega \) and let \( C^\infty_{c, \gamma_0}(\bar{\omega}; \mathbb{R}^3) \) denote the set of functions in \( C^\infty(\bar{\omega}; \mathbb{R}^3) \) that are equal to 0 in a neighborhood of \( \gamma_0 \). Then, \( C^\infty_{c, \gamma_0}(\bar{\omega}; \mathbb{R}^3) \) is dense in \( V_{1, \gamma_0} \).

**Proof.** We localize as before around \( \gamma_0 \), the interior of its complement in \( \partial \omega \) and the endpoints of \( \gamma_0 \). Clearly, for the parts localized around \( \gamma_0 \), the same argument as in the proof of Theorem 4.4 applies. Equally clearly, for the parts localized around the interior of the complement, the argument of the proof of Theorem 4.3 applies. What remains are the parts that are localized around the endpoints of \( \gamma_0 \).
Let us thus assume that 0 is such an endpoint and let us localize $u$ in a ball of radius $\varepsilon$ around this point. To this end, we introduce a function $\theta \in \mathcal{D}(B(0,1))$ such that $\theta(x) = 1$ if $|x| \leq 1/2$ and $\theta(x) = 0$ for $|x| \geq 3/4$ and let $\theta_\varepsilon(x) = \theta(x/\varepsilon)$. We want to show that $\theta_\varepsilon u$ tends to zero strongly in $W$ when $\varepsilon \to 0$, so that we can approximate $u$ by 0 in $B(0,\varepsilon/2)$.

Since $u$ and $\partial_\alpha u \cdot a_3$ vanish on an arc that has 0 as endpoint, we can apply Poincaré's inequality to both quantities to deduce that

$$\|u\|_{L^2(B(0,\varepsilon);\mathbb{R}^3)}^2 \leq \varepsilon^2 \|\nabla u\|_{L^2(B(0,\varepsilon);M_{32})}^2 \tag{4.10}$$

and

$$\|\partial_\alpha u \cdot a_3\|_{L^2(B(0,\varepsilon);\mathbb{R}^3)}^2 \leq \varepsilon^2 \|\nabla(\partial_\alpha u \cdot a_3)\|_{L^2(B(0,\varepsilon);\mathbb{R}^2)}^2. \tag{4.11}$$

By estimate (4.10), we see that $\theta_\varepsilon u \to 0$ in $H^1(\omega;\mathbb{R}^3)$. Indeed, $\partial_\alpha (\theta_\varepsilon u) = (\partial_\alpha \theta_\varepsilon) u + \theta_\varepsilon \partial_\alpha u$, and

$$\|(\partial_\alpha \theta_\varepsilon) u\|_{L^2(B(0,\varepsilon);\mathbb{R}^3)} \leq \|\nabla u\|_{L^2(B(0,\varepsilon);M_{32})} \to 0 \text{ when } \varepsilon \to 0.$$

We now note that $u \cdot a_3$ also vanishes on the same arc, so that by Poincaré’s inequality,

$$\|u \cdot a_3\|_{L^2(B(0,\varepsilon);\mathbb{R}^3)}^2 \leq \varepsilon^2 \|\nabla(u \cdot a_3)\|_{L^2(B(0,\varepsilon);\mathbb{R}^2)}^2. \tag{4.12}$$

Now, $\partial_\alpha (u \cdot a_3) = \partial_\alpha u \cdot a_3 + u \cdot \partial_\alpha a_3$ and therefore

$$\|\partial_\alpha (u \cdot a_3)\|_{L^2(B(0,\varepsilon);\mathbb{R}^3)}^2 \leq 2 \left( \|\partial_\alpha u \cdot a_3\|_{L^2(B(0,\varepsilon);\mathbb{R}^3)}^2 + \|u \cdot \partial_\alpha a_3\|_{L^2(B(0,\varepsilon);\mathbb{R}^3)}^2 \right) \leq C \varepsilon^2 \|u\|_{W(B(0,\varepsilon))}^2,$$

using estimate (4.11) for the first term and estimate (4.10) and the fact that $\partial_\alpha a_3 \in L^\infty(\omega)$ for the second term, where $\|\cdot\|_{W(B(0,\varepsilon))}$ denotes the local $W$-norm on $B(0,\varepsilon)$. Consequently, by estimate (4.12), we obtain

$$\|u \cdot a_3\|_{L^2(B(0,\varepsilon);\mathbb{R}^3)}^2 \leq C \varepsilon^4 \|u\|_{W(B(0,\varepsilon))}^2 \tag{4.13}.$$

We have

$$\partial_{\alpha\beta}(\theta_\varepsilon u) \cdot a_3 = (\partial_{\alpha\beta} \theta_\varepsilon) u \cdot a_3 + (\partial_\alpha \theta_\varepsilon)(\partial_\beta u) \cdot a_3 + (\partial_\beta \theta_\varepsilon)(\partial_\alpha u) \cdot a_3 + \theta_\varepsilon(\partial_\alpha \beta u) \cdot a_3.$$

Therefore, putting estimates (4.11) and (4.13) together, we see that

$$\|\partial_{\alpha\beta}(\theta_\varepsilon u) \cdot a_3\|_{L^2(B(0,\varepsilon);\mathbb{R}^3)} \leq C\|u\|_{W(B(0,\varepsilon))} \to 0 \text{ when } \varepsilon \to 0,$$

which shows that $\theta_\varepsilon u \to 0$ in $W$.

Finally, it is fairly clear that the elements of the sequence $u_\varepsilon$, which are reconstructed by patching together all the local approximations of $u$, belong to $C^\infty_{c,\gamma_0}(\bar{\omega};\mathbb{R}^3)$ and the theorem is proved. □

Remark 4.6. Note that the space $C^\infty_{c,\gamma_0}(\bar{\omega};\mathbb{R}^3)$ does not depend on the chart $\varphi$, whereas the space $V_{1,\gamma_0}$ does. This is useful since one of the applications we have in mind is the dependence
of the solution of Koiter’s model on the midsurface. The space $C_{\varepsilon_0}^\infty(\bar{\omega}; \mathbb{R}^3)$ is a common dense subspace of all possible $V_{1, \gamma_0}$ spaces for all possible midsurfaces. □

The case of a simply supported shell actually seems to be more difficult. We only solve it here for a domain $\omega$ of class $C^\infty$ and by resorting to classical techniques.

**Theorem 4.7.** Assume that $\omega$ of class $C^\infty$. Then, $C^\infty(\bar{\omega}; \mathbb{R}^3) \cap H_0^1(\omega; \mathbb{R}^3)$ is dense in $V_0$.

**Proof.** We proceed in a classical fashion. First localize as before. For the parts near the boundary, we can thus assume that $\omega = \{(x_1, x_2) \in \mathbb{R}^2; x_2 < \Psi(x_1)\}$ where $\Psi: \mathbb{R} \to \mathbb{R}$ is of class $C^\infty$.

Next we flatten the boundary using the $C^\infty$-diffeomorphism

$$
\begin{align*}
\Theta_1(x) &= x_1, \\
\Theta_2(x) &= x_2 - \Psi(x_1).
\end{align*}
$$

This obviously induces an isomorphism on the associated $V_0$ spaces so that we are reduced to the case $\omega = \mathbb{R} \times \mathbb{R}_+$. We now extend $u$ and $a_3$ for $x_2 > 0$ by

$$
\begin{align*}
\tilde{u}(x_1, x_2) &= -u(x_1, -x_2), \\
\tilde{a}_3(x_1, x_2) &= a_3(x_1, -x_2),
\end{align*}
$$

respectively. Clearly, $\tilde{u} \in H^1(\mathbb{R}^2; \mathbb{R}^3)$, $\tilde{a}_3 \in W^{1, \infty}(\mathbb{R}^2; \mathbb{R}^3)$ and $\tilde{u}$ is odd with respect to $x_2$.

Let us show that $\partial_{\alpha \beta} \tilde{u} \cdot \tilde{a}_3$ belongs to $L^2(\mathbb{R}^2)$, or equivalently that $\partial_{\alpha} \tilde{u} \cdot \tilde{a}_3$ is in $H^1(\mathbb{R}^2)$. For $x_2 > 0$, we have

$$
\begin{align*}
\partial_1 \tilde{u} \cdot \tilde{a}_3(x_1, x_2) &= -\partial_1 u \cdot a_3(x_1, -x_2), \\
\partial_2 \tilde{u} \cdot \tilde{a}_3(x_1, x_2) &= \partial_2 u \cdot a_3(x_1, -x_2),
\end{align*}
$$

so that $(\partial_{\alpha} \tilde{u} \cdot \tilde{a}_3)|_{R \times \mathbb{R}_+^*}$ belongs to $H^1(\mathbb{R} \times \mathbb{R}_+^*)$. It thus suffices to prove that the jump of both quantities across $x_2 = 0$ is zero. This is clear for the second one as the traces at $x_2 = 0$ of both sides of the equal sign obviously coincide.

The first quantity requires a little more attention. We know that $u \in H_0^1(\mathbb{R} \times \mathbb{R}_+^*)$. Let us separate the variables and consider all functions as functions in the variable $x_2$ with values in function spaces in the variable $x_1$. In this way, we have $u \in C^0(\mathbb{R}_-; L^2(\mathbb{R}; \mathbb{R}^3))$, with $u(0) = 0$ as an element of $L^2(\mathbb{R})$. Since the operator $\partial_1$ is linear continuous from $L^2(\mathbb{R}; \mathbb{R}^3)$ into $H^{-1}(\mathbb{R}; \mathbb{R}^3)$, it follows that $\partial_1 u \in C^0(\mathbb{R}_-; H^{-1}(\mathbb{R}; \mathbb{R}^3))$, with $\partial_1 u(0) = 0$ as an element of $H^{-1}(\mathbb{R}; \mathbb{R}^3)$. Now, we clearly have $a_3 \in C^0(\mathbb{R}_-; W^{1, q}(\mathbb{R}; \mathbb{R}^3))$ for all $q < +\infty$ (but not for $q = +\infty$!). Note that if $\psi \in H^{-1}(\mathbb{R})$ and $b \in W^{1, q}(\mathbb{R})$ with $q > 2$, then $b\psi \in H^{-1}(\mathbb{R})$ with $\|b\psi\|_{H^{-1}(\mathbb{R})} \leq C_q \|b\|_{W^{1, q}(\mathbb{R})} \|\psi\|_{H^{-1}(\mathbb{R})}$.

We infer from the previous remarks that

$$
\partial_1 u \cdot a_3 \in C^0(\mathbb{R}_-; H^{-1}(\mathbb{R})) \text{ and } \partial_1 u \cdot a_3(0) = 0 \text{ as an element of } H^{-1}(\mathbb{R}). \tag{4.14}
$$

On the other hand, we also have that $\partial_1 u \cdot a_3 \in H^1(\mathbb{R} \times \mathbb{R}_+^*)$, so that $\partial_1 u \cdot a_3 \in C^0(\mathbb{R}_-; L^2(\mathbb{R}))$ and the trace of $\partial_1 u \cdot a_3$ on $\mathbb{R} \times \{0\}$ is its value at $x_2 = 0$ as an element of $L^2(\mathbb{R})$. It follows from
this remark and from formula (4.14) that this trace is equal to zero. Consequently, $\partial_1 \tilde{u} \cdot \tilde{a}_3$ has no jump across $\mathbb{R} \times \{0\}$ and therefore $\partial_1 \tilde{u} \cdot \tilde{a}_3 \in H^1(\mathbb{R}^2)$.

We can now introduce a radial mollifier $\rho$, define $\rho_\varepsilon = \varepsilon^{-2} \rho(\cdot/\varepsilon)$ and let $u_\varepsilon = \tilde{u} * \rho_\varepsilon$. Clearly, $u_\varepsilon|_\omega \in C^\infty(\bar{\omega}; \mathbb{R}^3)$, $u_\varepsilon|_\omega \to u$ in $W$ as $\varepsilon \to 0$ by the classical Friedrichs lemma and $u_\varepsilon(x_1, 0) = 0$ by the imparity of $u$ and parity of $\rho$ with respect to $x_2$.

We can go back to the original domain by composition with the $C^\infty$-diffeomorphism $\Theta^{-1}$.

Remark 4.8. Since we are mainly interested in finding a dense subspace that does not depend on the midsurface, the above proof shows that if $\omega$ is of class $W^{2,\infty}$, then $W^{2,\infty}(\omega; \mathbb{R}^3) \cap H^1_0(\omega; \mathbb{R}^3)$ is dense in $V_0$.

Remark 4.9. The above proof also works for piecewise $C^\infty$ domains satisfying the uniform cone condition, for instance polygons, by performing adequate reflections at the angles. In this case, $C^\infty(\bar{\omega}; \mathbb{R}^3) \cap H^1_0(\omega; \mathbb{R}^3)$ is also dense in $V_0$.

4.3. CONTINUOUS DEPENDENCE ON THE MIDSURFACE

Our goal in this section is to show that the formulation of Koiter’s model we proposed in [5] and [6] provides an adequate extension of the classical formulation for $C^3$-midsurfaces to $W^{2,\infty}$-midsurfaces, at least from the mathematical point of view. The density results of the previous section were formulated for a domain that satisfies the uniform cone condition. For practical purposes in the case of shells, we will from now on only consider bounded Lipschitz domains.

Let us recall the following result, which can be extracted from Lemma 2 of [6]. This result can be viewed as a kind of a priori continuous dependence result.

**Lemma 4.10.** If $v^n \in H^1(\omega; \mathbb{R}^3)$ and $\varphi^n \in W^{2,\infty}(\omega; \mathbb{R}^3)$ are two sequences such that $v^n \to v$ strongly in $H^1(\omega; \mathbb{R}^3)$, $\varphi^n \to \varphi$ strongly in $W^{2,p}(\omega; \mathbb{R}^3)$ for all $p < +\infty$ and $\varphi^n \rightharpoonup \varphi$ weakly-* in $W^{2,\infty}(\omega; \mathbb{R}^3)$, then $\gamma_{\alpha\beta}(v^n) \to \gamma_{\alpha\beta}(v)$ strongly in $L^2(\omega)$ and $\Upsilon_{\alpha\beta}(v^n) \to \Upsilon_{\alpha\beta}(v)$ strongly in $H^{-1}(\omega)$.

Remember that the strain $\gamma_{\alpha\beta}(v^n)$ and change of curvature $\Upsilon_{\alpha\beta}(v^n)$ tensors also depend on the midsurface charts $\varphi^n$, even though the notation used here does not make this apparent. Lemma 4.10 is not entirely satisfactory as far as comparing our formulation of the Koiter model with the classical formulation is concerned. We wish to take $v^n = u^n$ to be the solution of Koiter’s model for a given sequence of applied loads, replace the assumption on $u^n$ by a corresponding assumption of convergence on the applied loads, and obtain as a result the convergence of displacements and stresses. Taking a sequence of $C^3$-midsurfaces converging in the above sense toward a given $W^{2,\infty}$-midsurface will then show that our solution is the limit in a natural sense of the classical solutions.

Example 4.11. Before proceeding further, let us consider the example of a $W^{2,\infty}$-shell made of a
plane part and a circular cylindrical part. Let us take \( \omega = \{-\pi/2, \pi/2\} \times \{0, 1\} \) and

\[
\varphi(x) = \begin{cases} 
(x_1, x_2, 0)^T & \text{for } x_1 \leq 0, \\
\sin x_1, x_2, 1 - \cos x_1)^T & \text{for } x_1 > 0.
\end{cases}
\]

The midsurface of this shell is of class \( W^{2,\infty} \) and has a curvature discontinuity across \( x_1 = 0 \). In particular, it is not \( C^3 \) and the classical formulation of Koiter’s model is not applicable. To construct a sequence of \( C^3 \)-midsurfaces that converge in all \( W^{2,p} \) and in \( W^{2,\infty} \) weak-*, it is sufficient to use standard mollification. Let us give a more explicit construction based on Hermite interpolation.

Let \( p_{\alpha i} \in P_7, \alpha = 0, 1 \) and \( i = 0, 1, 2, 3 \), be the basis polynomials for Hermite interpolation of order 3 on \([0, 1]\) defined by \( p_{\alpha i}^{(j)}(\beta) = \delta_{ij}\delta_{\alpha\beta} \). The corresponding Hermite interpolation polynomials on \([0, 1/n]\) are given by \( p_{\alpha i}^n(s) = n^{-i}p_{\alpha i}(ns) \). Let us set

\[
\begin{align*}
g_n(s) &= \frac{1}{6n^3}p_{10}^n(s) + \frac{1}{2n^2}p_{11}^n(s) + \frac{1}{n}p_{12}^n(s) + p_{13}^n(s), \\
h_n(s) &= \frac{1}{2n^2}p_{10}^n(s) + \frac{1}{n}p_{11}^n(s) + p_{12}^n(s).
\end{align*}
\]

These polynomials interpolate \( s^3/6 \) and \( s^2/2 \) respectively at \( s = 1/n \) and 0 at \( s = 0 \). It is not difficult to check that the following sequence of \( C^3 \)-charts

\[
\varphi^n(x) = \begin{cases} 
(x_1, x_2, 0)^T & \text{for } x_1 \leq 0, \\
\sin x_1, x_2, 1 - \cos x_1 - \frac{x_1^3}{6} + g_n(x_1), x_2, 1 - \cos x_1 - \frac{x_1^2}{2} + h_n(x_1))^T & \text{for } 0 < x_1 \leq 1/n, \\
\sin x_1, x_2, 1 - \cos x_1)^T & \text{for } x_1 > 1/n.
\end{cases}
\]

has the desired convergence properties since \( g_n'' \) and \( h_n'' \) are uniformly bounded with respect to \( n \). \( \square \)

Let us thus consider a sequence of midsurface charts \( \varphi^n \). All corresponding geometric quantities will from now on be indicated by an \( n \) superscript. For instance, the covariant basis vectors are denoted by \( a^n_i \), the covariant components of the first fundamental form and the area element by \( a^n_{\alpha\beta} \) and \( \sqrt{a^n} \) respectively, and the Christoffel symbols by \( \Gamma^n_{\alpha\beta\rho} \). We assume for simplicity that all shells have the same thickness and the same Lamé moduli and denote by \( a^{n,\alpha\beta\rho\sigma} \) the contravariant components of the elasticity tensor, viz. formula (4.5). Finally, for all displacements \( v \) of the shells, we denote by

\[
\gamma^n_{\alpha\beta}(v) = \frac{1}{2}(\partial_\alpha v \cdot a^n_\beta + \partial_\beta v \cdot a^n_\alpha)
\]

and

\[
\Upsilon^n_{\alpha\beta}(v) = (\partial_\alpha v - \Gamma^n_{\alpha\beta\rho} \partial_\rho v) \cdot a^n_3
\]

the covariant components of the strain and change of curvature tensors, with explicit dependence on the charts.

Let us collect all the information on convergence properties of the various geometric quantities that we will need later on in one lemma.
LEMMA 4.12. Let $\varphi^n$ be a sequence of charts such that $\varphi^n \to \varphi$ in $W^{2,p}(\omega; \mathbb{R}^3)$ strong for all $1 < p < +\infty$ and $\varphi^n \rightharpoonup \varphi$ in $W^{2,\infty}(\omega; \mathbb{R}^3)$ weak-*. Then

\begin{align*}
a^n_i \to a_i \text{ strongly in } W^{1,p}(\omega; \mathbb{R}^3) \text{ for all } 1 < p < +\infty \text{ and weakly-}^* \text{ in } W^{1,\infty}(\omega; \mathbb{R}^3),
\end{align*}

(4.15)

\begin{align*}
a^n_{\alpha\beta} \to a_{\alpha\beta}, \sqrt{a^n} \to \sqrt{a} \text{ and } a^{n,\alpha\beta\rho\sigma} \to a^{\alpha\beta\rho\sigma} \text{ in } C^0(\bar{\omega}),
\end{align*}

(4.16)

and

\begin{align*}
\Gamma^{n,\rho}_{\alpha\beta} \to \Gamma^{\rho}_{\alpha\beta} \text{ strongly in } L^p(\omega) \text{ for all } 1 < p < +\infty \text{ and weakly-}^* \text{ in } L^\infty(\omega).
\end{align*}

(4.17)

Proof. That convergence (4.15) holds true for $a^n_{\alpha}$ is trivial. By Morrey’s theorem, these vectors also converge strongly in $C^0(\bar{\omega}; \mathbb{R}^3)$. Since they are linearly independent at each point $x \in \bar{\omega}$ for all $n$, it follows that

\begin{align*}
\delta^n = \inf_{x \in \bar{\omega}} |a^n_1(x) \land a^n_2(x)| \geq \delta > 0
\end{align*}

(4.18)

for some $\delta$. The space $W^{1,\infty}(\omega)$ is a Banach algebra, therefore the previous estimate shows that $a^n_3$ is bounded in $W^{1,\infty}(\omega; \mathbb{R}^3)$, hence (4.15) for the normal vector.

Convergences (4.16) are equally trivial. Moreover, by estimate (4.18), the contravariant components of the metric tensor $a^{n,\alpha\beta}$ converge in $W^{1,p}(\omega)$ strong and $W^{1,\infty}(\omega)$ weak-* as well, which implies that the contravariant vectors $a^{n,i}$ also converge accordingly. Consequently, $\Gamma^{n,\rho}_{\alpha\beta} = a^{n,\rho} \partial_\beta a^n_3$ converge strongly in $L^p(\omega)$ and weakly-* in $L^\infty(\omega)$.

In the sequel, we will concentrate on the case of a totally clamped shell only submitted to force resultants for brevity. All results remain true—with appropriate modifications—for a simply supported shell and a partially clamped shell submitted to edge tractions and moments on the free part of the boundary. Let us thus introduce the spaces

\begin{align*}
W^n = \left\{ v \in H^1(\omega; \mathbb{R}^3), \partial_{\alpha\beta} v \cdot a^n_3 \in L^2(\omega) \right\},
\end{align*}

(4.19)

equipped with their natural norm

\begin{align*}
\|v\|_{W^n} = \left( \|v\|_{H^1(\omega; \mathbb{R}^3)}^2 + \sum_{\alpha,\beta} \|\partial_{\alpha\beta} v \cdot a^n_3\|_{L^2(\omega)}^2 \right)^{1/2},
\end{align*}

(4.20)

and

\begin{align*}
V^n_1 = \left\{ v \in W^n; v = \partial_\alpha v \cdot a^n_3 = 0 \text{ on } \partial\omega \right\},
\end{align*}

(4.21)

which is a closed subspace of $W^n$. Clearly, all these spaces depend on the midsurface under consideration.
For \( f^n \in L^2(\omega; \mathbb{R}^3) \) we let \( u^n \) be the unique solution to the variational formulation of Koiter’s model: Find \( u^n \in V_1^n \) such that
\[
\forall v^n \in V_1^n, \int_\omega (e^{a_{n\alpha\beta}}(\tau_n u^n) \gamma_n \tau(v^n) + \frac{e^2}{12} \gamma_n (\tau_n u^n) \gamma_n (\tau(v^n))) \sqrt{a_n} \, dx = \int_\omega f^n \cdot v^n \sqrt{a_n} \, dx.
\] (4.22)

Our main result is the following.

**Theorem 4.13.** Let \( \varphi^n \) be a sequence of charts such that \( \varphi^n \to \varphi \) in \( W^{2,p}(\omega; \mathbb{R}^3) \) strong for all \( 1 < p < +\infty \) and \( \varphi^n \rightharpoonup \varphi \) in \( W^2,\infty(\omega; \mathbb{R}^3) \) weak-* and \( f^n \) be a sequence of force resultant densities such that \( f^n \to f \) in \( L^2(\omega; \mathbb{R}^3) \). Then,
\[
u^n \to u \text{ in } H^1(\omega; \mathbb{R}^3) \quad \text{and} \quad \Upsilon_n^{\alpha\beta}(u^n) \to \Upsilon_{\alpha\beta}(u) \text{ in } L^2(\omega),
\] (4.23)

where \( u \) is the solution to Koiter’s model for a clamped shell with midsurface chart \( \varphi \) and applied force resultant density \( f \).

The proof is comprised of a series of lemmas. Let us begin with an analogue of Lemma 4.10 for weak convergence.

**Lemma 4.14.** If \( v^n \in H^1(\omega; \mathbb{R}^3) \) and \( \varphi^n \in W^{2,\infty}(\omega; \mathbb{R}^3) \) are two sequences such that \( v^n \to v \) weakly in \( H^1(\omega; \mathbb{R}^3) \), \( \varphi^n \to \varphi \) strongly in \( W^{2,p}(\omega; \mathbb{R}^3) \) for all \( p < +\infty \) and \( \varphi^n \rightharpoonup \varphi \) weakly-* in \( W^{2,\infty}(\omega; \mathbb{R}^3) \), then \( \gamma_n^{\alpha\beta}(v^n) \to \gamma_{\alpha\beta}(v) \) weakly in \( L^2(\omega) \) and \( \Upsilon_n^{\alpha\beta}(v^n) \to \Upsilon_{\alpha\beta}(v) \) weakly in \( H^{-1}(\omega) \).

**Proof.** First of all, since \( a^n_{\alpha} \to a_\alpha \) strongly in \( C^0(\omega; \mathbb{R}^3) \), it follows clearly that
\[
\gamma_n^{\alpha\beta}(v^n) = \frac{1}{2} (\partial_\alpha v^n \cdot a^n_\beta + \partial_\beta v^n \cdot a^n_\alpha) \to \gamma_{\alpha\beta}(v) \text{ in } L^2(\omega).
\]

The case of the change of curvature tensor is more intricate. We know that \( \partial_{\alpha\beta} v^n \to \partial_{\alpha\beta} v \) in \( H^{-1}(\omega; \mathbb{R}^3) \). Let \( \theta \) be a test-function in \( H^1(\omega) \). By the Sobolev embedding theorem, \( \theta \in L^4(\omega) \) and as \( \partial_{\alpha} a^n_3 \to \partial_{\alpha} a_3 \) strongly in \( L^4(\omega; \mathbb{R}^3) \) by (4.15) with \( p = 4 \), it follows that
\[
\begin{aligned}
&\left\{ \begin{array}{l}
\theta a^n_3 \to \theta a_3, \\
\partial_\alpha (\theta a^n_3) = (\partial_\alpha \theta) a^n_3 + \theta \partial_\alpha a^n_3 \to \partial_\alpha (\theta a_3),
\end{array} \right.
\end{aligned}
\] strongly in \( L^2(\omega; \mathbb{R}^3) \),

so that
\[
\theta a^n_3 \to \theta a_3 \text{ strongly in } H^1_0(\omega; \mathbb{R}^3).
\]

Therefore,
\[
<\partial_{\alpha\beta} v^n \cdot a^n_3, \theta> = <\partial_{\alpha\beta} v^n, \theta a^n_3> \to <\partial_{\alpha\beta} v \cdot a_3, \theta>,
\]

hence
\[
\partial_{\alpha\beta} v^n \cdot a^n_3 \to \partial_{\alpha\beta} v \cdot a_3 \text{ weakly in } H^{-1}(\omega).
\]
Let us now deal with the other part of $\Upsilon_{\alpha\beta}(v^n)$. We have that $\partial_\rho v^n \to \partial_\rho v$ weakly in $L^2(\omega; \mathbb{R}^3)$, $\Gamma^{\rho,\rho}_{\alpha\beta} \to \Gamma^{\rho}_{\alpha\beta}$ strongly in $L^p(\omega)$ and $a^n_3 \to a_3$ strongly in $W^{1,p}(\omega; \mathbb{R}^3)$ for all $p < +\infty$. It follows easily from this and Hölder’s inequality that

$$
\Gamma^{\rho,\rho}_{\alpha\beta} \partial_\rho v^n \cdot a^n_3 \to \Gamma^{\rho}_{\alpha\beta} \partial_\rho v \cdot a_3 \text{ weakly in } L^q(\omega) \text{ for all } 1 < q < 2.
$$

Now, by the two-dimensional Sobolev embedding theorem, we have $L^q(\omega) \hookrightarrow H^{-1}(\omega)$ for all $1 < q < 2$, hence the result. \hfill \Box

Let us now establish some uniform norm equivalence results.

**Lemma 4.15.** There exist two constants $0 < c < C < +\infty$ independent of $n$ such that for all $v^n \in V^n_1$,

$$
c\|v^n\|_{W^n} \leq \left\{ \sum_{\alpha\beta} \left( \|\gamma_{\alpha\beta}(v^n)\|^2_{L^2(\omega; \mathbb{R}^3)} + \|\Upsilon_{\alpha\beta}(v^n)\|^2_{L^2(\omega; \mathbb{R}^3)} \right) \right\}^{1/2} \leq C\|v^n\|_{W^n}. \quad (4.24)
$$

**Proof.** The proof is very similar to that of Lemma 11 in [6] for a single chart, which shows that the previous equivalence holds true for each value of $n$, with two constants $c_n$ and $C_n$.

Let us set

$$
\|v^n\|_n = \left\{ \sum_{\alpha\beta} \left( \|\gamma_{\alpha\beta}(v^n)\|^2_{L^2(\omega; \mathbb{R}^3)} + \|\Upsilon_{\alpha\beta}(v^n)\|^2_{L^2(\omega; \mathbb{R}^3)} \right) \right\}^{1/2}
$$

The estimate from above is trivial, we only need to prove the estimate from below. We argue by contradiction. Let us assume that $c_n \to 0$ when $n \to +\infty$. Then, there exists a sequence $v^n$ in $V^n_1$ such that

$$
\|v^n\|_{W^n} = 1 \quad \text{and} \quad \|v^n\|_{n} \to 0 \text{ when } n \to +\infty. \quad (4.25)
$$

By extracting a subsequence, still denoted $v^n$, we may assume that there exists a $v \in H^1(\omega; \mathbb{R}^3)$ such that $v^n \to v$ weakly in $H^1(\omega; \mathbb{R}^3)$ and $\partial_\alpha v^n \cdot a^n_3 \to \kappa_{\alpha\beta}$ weakly in $L^2(\omega)$. We see from the proof of Lemma 4.14 that $\kappa_{\alpha\beta} = \partial_\alpha v \cdot a_3$, so that $\partial_\alpha v^n \cdot a^n_3 \to \partial_\alpha v \cdot a_3$ weakly in $H^1(\omega)$, from which it follows that $\partial_\alpha v \cdot a_3 \in H^1(\omega)$ and thus $v \in V_1$. Moreover, still by Lemma 4.14,

$$
\gamma_{\alpha\beta}^n(v^n) \to \gamma_{\alpha\beta}(v) \quad \text{and} \quad \Upsilon_{\alpha\beta}^n(v^n) \to \Upsilon_{\alpha\beta}(v) \text{ weakly in } L^2(\omega). \quad (4.26)
$$

Since hypothesis (4.25) implies that these tensors converge strongly to zero in $L^2(\omega)$, we obtain $v = 0$ thanks to the infinitesimal rigid displacement lemma, Theorem 6 in [6]. Rellich’s lemma now implies that $v^n \to 0$ strongly in $L^2(\omega; \mathbb{R}^3)$.

Let us introduce the vector $(\vec{w}^n)_\alpha = v^n \cdot a^n_\alpha$, which is such that $\vec{w}^n \to 0$ strongly in $L^2(\omega; \mathbb{R}^2)$ by the previous remark. Let us define $2e_{\alpha\beta}(\vec{w}^n) = \partial_\beta (\vec{w}^n)_\alpha + \partial_\alpha (\vec{w}^n)_\beta$. We see that

$$
e_{\alpha\beta}(\vec{w}^n) = \gamma_{\alpha\beta}^n(v^n) + \frac{1}{2} v^n \cdot (\partial_\beta a^n_\alpha + \partial_\alpha a^n_\beta) \to 0 \text{ strongly in } L^2(\omega),$$
since \( a_{\alpha}^{n} \) is uniformly bounded in \( W^{1,\infty}(\omega;\mathbb{R}^3) \). By the two-dimensional Korn inequality, we deduce then that \( \bar{w}^n \to 0 \) strongly in \( H^1(\omega;\mathbb{R}^2) \). Consequently,

\[
\partial_{\rho} v^n \cdot a_{\alpha}^{n} = \partial_{\rho} ((\bar{w}^n)_{\alpha}) - v^n \cdot \partial_{\rho} a_{\alpha}^{n} \to 0 \text{ strongly in } L^2(\omega),
\]

(4.27)
since \( \partial_{\rho} a_{\alpha}^{n} \) is uniformly bounded in \( L^\infty(\omega;\mathbb{R}^3) \).

Moreover, since \( v^n \to 0 \) in \( H^1(\omega;\mathbb{R}^3) \), it follows that \( \partial_{\rho} v^n \cdot a_{3}^{n} \to 0 \) in \( L^2(\omega) \). On the other hand, \( \partial_{\beta} (\partial_{\rho} v^n \cdot a_{3}^{n}) = \partial_{\beta} \partial_{\rho} v^n \cdot a_{3}^{n} + \partial_{\rho} v^n \cdot \partial_{\beta} a_{3}^{n} \) \( \to 0 \) in \( L^2(\omega) \). Indeed, \( \partial_{\beta} a_{3}^{n} \) is uniformly bounded in \( L^\infty(\omega;\mathbb{R}^3) \) and converges strongly in \( L^p(\omega;\mathbb{R}^3) \) (we already know that \( \partial_{\beta} v^n \cdot a_{3}^{n} \to 0 \) weakly in \( L^2(\omega) \)). Consequently, \( \partial_{\rho} v^n \cdot a_{3}^{n} \to 0 \) weakly in \( H^1(\omega) \) and by Rellich’s lemma,

\[
\partial_{\rho} v^n \cdot a_{3}^{n} \to 0 \text{ strongly in } L^2(\omega).
\]

(4.28)

We deduce from (4.28) and (4.25) that

\[
\partial_{\alpha\beta} v^n \cdot a_{3}^{n} = \Upsilon_{\alpha\beta}^{n}(v^n) + \Gamma_{\alpha\beta}^{n} \partial_{\rho} v^n \cdot a_{3}^{n} \to 0 \text{ strongly in } L^2(\omega),
\]

(4.29)
since \( \Gamma_{\alpha\beta}^{n} \) is uniformly bounded in \( L^\infty(\omega) \), on the one hand, and on the other hand that

\[
\partial_{\rho} v^n = \left( \partial_{\rho} v^n \cdot a_{i}^{n} \right) a_{n}^{n,i} \to 0 \text{ strongly in } L^2(\omega;\mathbb{R}^3)
\]

(4.30)

by (4.27), (4.28) and since both \( a_{i}^{n} \) and \( a_{n}^{n,i} \) are uniformly bounded in \( L^\infty(\omega;\mathbb{R}^3) \). Consequently, \( v^n \to 0 \) strongly in \( H^1(\omega;\mathbb{R}^3) \). Since by (4.29), \( \partial_{\alpha\beta} v^n \cdot a_{3}^{n} \to 0 \) strongly in \( L^2(\omega) \), we see that \( \|v^n\|_{W^n} \to 0 \), which contradicts assumption (4.25) and proves the lemma.

We also need uniform positive definiteness of the elasticity tensors. By assumption, for each midsurface, there exists a constant \( \eta_n > 0 \) such that for all symmetric tensors \( \tau = (\tau_{\alpha\beta}) \) and almost all \( x \in \omega, a_{n,\alpha\beta}^{\rho\sigma}(x) \tau_{\alpha\beta} \tau_{\rho\sigma} \geq \eta_n \tau_{\alpha\beta} \tau_{\alpha\beta} \). For instance, in the case of an isotropic material, this is a statement concerning the Lamé moduli \( \mu \) and \( \lambda \), and not the geometry of the midsurface. We will concentrate here on the isotropic case.

**Lemma 4.16.** There is constant \( \eta > 0 \) independent of \( n \) such that \( \eta_n \geq \eta \) for all \( n \).

**Proof.** We know that \( a_{n,\alpha\beta}^{\rho\sigma} \) converge uniformly to \( a_{\alpha\beta}^{\rho\sigma} \). As \( \eta_n \) is the infimum of the quadratic form \( a_{n,\alpha\beta}^{\rho\sigma}(x) \tau_{\alpha\beta} \tau_{\rho\sigma} \) on the Cartesian product of the unit sphere of the space of symmetric tensors with \( \bar{\omega} \), the result is clear.

We now are in a position to establish uniform bounds for the various quantities of interest.

**Lemma 4.17.** There is constant \( M \) independent of \( n \) such that

\[
\|u^n\|_{H^1(\omega;\mathbb{R}^3)} \leq M, \quad \|\partial_{\alpha\beta} u^n \cdot a_{3}^{n}\|_{L^2(\omega)} \leq M.
\]

(4.31)
Proof. Let us take \( v^n = u^n \) as test-function in the variational formulation of Koiter’s problem (4.22). In view of Lemmas 4.15 and 4.16, we obtain

\[
\delta \min(e, e^3/12) \| u^n \|_{L^2}^2 \leq \| \sqrt{a^n f^n} \|_{L^2} \| u^n \|_{L^2},
\]

where \( \delta \) is the constant in estimate (4.18). The lemma easily follows from the above estimate. \( \square \)

**Lemma 4.18.** There exists a subsequence (still denoted by) \( u^n \) and \( u \in V_1 \) such that

\[
u^n \rightharpoonup u \text{ weakly in } H^1(\omega; \mathbb{R}^3) \quad \text{and} \quad \partial_{\alpha\beta} u^n \cdot a_3^n \rightharpoonup \partial_{\alpha\beta} u \cdot a_3 \text{ weakly in } L^2(\omega).
\]

(4.32)

**Proof.** Because of the previous bounds, we can find a subsequence \( u^n \), a function \( u \in H^1_0(\omega; \mathbb{R}^3) \) and functions \( \kappa_{\alpha\beta} \in L^2(\omega) \) such that

\[
u^n \rightharpoonup u \text{ weakly in } H^1(\omega; \mathbb{R}^3) \quad \text{and} \quad \partial_{\alpha\beta} u^n \cdot a_3^n \rightharpoonup \kappa_{\alpha\beta} \text{ weakly in } L^2(\omega).
\]

As in the proof of Lemma 4.14, we see that \( \kappa_{\alpha\beta} = \partial_{\alpha\beta} u \cdot a_3 \), so that \( u \in W \). Moreover, \( \partial_{\alpha} u^n \cdot a_3^n \rightharpoonup \partial_{\alpha} u \cdot a_3 \) in \( H^1(\omega) \), so that \( \partial_{\alpha} u \cdot a_3 \in H^1_0(\omega) \) and therefore \( u \in V_1 \). \( \square \)

Our next task is to identify the weak limit \( u \) of the above subsequence of solutions \( u^n \) as being the solution to the Koiter problem corresponding to the limit midsurface and loads.

**Lemma 4.19.** The limit \( u \) is the unique solution to: Find \( u \in V_1 \) such that

\[
\forall v \in V_1, \quad \int_\omega e a^{\alpha\beta\rho\sigma} (\gamma_{\alpha\beta}(u) \gamma_{\rho\sigma}(v) + \frac{e^2}{12} \Upsilon_{\alpha\beta}(u) \Upsilon_{\rho\sigma}(v)) \sqrt{a} \, dx = \int_\omega f \cdot v \sqrt{a} \, dx.
\]

(4.33)

The whole sequence is convergent.

**Proof.** By Theorem 4.4, we know that the spaces \( V^n_1 \) and \( V_1 \) all share a common dense subspace, namely here \( D(\omega; \mathbb{R}^3) \). Therefore, any \( \psi \in D(\omega; \mathbb{R}^3) \) is a legitimate test-function for all \( n \), as well as for the eventual limit problem (4.33). Now,

\[
a^{n,\alpha\beta\rho\sigma} \gamma_{\rho\sigma}^n(\psi) \sqrt{a^n} \to a^{\alpha\beta\rho\sigma} \gamma_{\rho\sigma}(\psi) \sqrt{a} \text{ strongly in } L^2(\omega),
\]

and

\[
a^{n,\alpha\beta\rho\sigma} \Upsilon_{\rho\sigma}^n(\psi) \sqrt{a^n} \to a^{\alpha\beta\rho\sigma} \Upsilon_{\rho\sigma}(\psi) \sqrt{a} \text{ strongly in } L^2(\omega),
\]

by Lemma 4.12. On the other hand,

\[
\gamma_{\alpha\beta}^n(u^n) \rightharpoonup \gamma_{\alpha\beta}(u) \text{ weakly in } L^2(\omega),
\]

and

\[
\Upsilon_{\alpha\beta}^n(u^n) \rightharpoonup \Upsilon_{\alpha\beta}(u) \text{ weakly in } L^2(\omega),
\]
by Lemma 4.14 and since Lemmas 4.15 and 4.17 imply that \( \Upsilon_{\alpha \beta}^n (u^n) \) is bounded in \( L^2(\omega) \). Therefore, we can pass to the limit as \( n \to +\infty \) in problem (4.22) and obtain problem (4.33) for all test-functions \( \psi \in D(\omega; \mathbb{R}^3) \). The identification of \( u \) then follows from the fact that \( D(\omega; \mathbb{R}^3) \) is dense in \( V_1 \), viz. Theorem 4.4.

The solution of problem (4.33) is unique, therefore the standard uniqueness argument shows that the whole sequence \( u^n \) converges, and not just a subsequence thereof. \( \square \)

The final step in the proof of Theorem 4.13 consists in showing that all weak convergences are actually strong. This would be a straightforward matter if the test-function spaces did not depend on \( n \). We would just take \( u - u^n \) as a test-function. There is a slight twist here since \( u \not\in V^n_1 \) and \( u^n \not\in V_1 \) so that \( u - u^n \) is not a legitimate test-function neither for problem (4.22) nor for problem (4.33). We recast the problem in abstract form to circumvent this difficulty.

Let us be given a family of Hilbert spaces \( (H^n, \| \cdot \|_n)_{n \in \mathbb{N}} \) and a Hilbert space \( (H, \| \cdot \|) \) with the following properties:

i) There is a subspace \( D \) common to all \( H^n \) and \( H \) such that \( D \) is dense in \( H \).

ii) For all \( y \in D \), \( \| y \|_n \to \| y \| \) as \( n \to +\infty \).

Let us also be given a corresponding family of continuous (on their respective spaces) symmetric bilinear forms \( a^n \) and \( a \) and a family of continuous linear forms \( l^n \) and \( l \) with the properties:

iii) There exists a constant \( \eta \) independent of \( n \) such that

\[
\forall \, y^n \in H^n, \quad a^n(y^n, y^n) \geq \eta \| y^n \|^2.
\]

iv) For all \( y, z \) in \( D \), \( a^n(y, z) \to a(y, z) \) and \( l^n(y) \to l(y) \) when \( n \to +\infty \).

**Lemma 4.20.** Assume that hypotheses i) to iv) are satisfied and let \( x^n \in H^n \) and \( x \in H \) be the solutions of the variational problems

\[
\forall \, y^n \in H^n, \quad a^n(x^n, y^n) = l^n(y^n) \quad \text{and} \quad \forall \, y \in H, \quad a(x, y) = l(y).
\]

If in addition, \( l^n(x^n) \to l(x) \), then

\[
\| x^n \|_n \to \| x \| \quad \text{when} \quad n \to +\infty.
\] (4.34)

**Proof.** For all \( y \in D \), we have

\[
a^n(x^n - y, x^n - y) = l^n(x^n - 2y) + a^n(y, y).
\]

By assumption iii), it follows that

\[
\eta \| x^n - y \|^2 \leq l^n(x^n) - 2l^n(y) + a^n(y, y).
\]
Letting $n$ tend to $+\infty$, we obtain
\[
\limsup_{n \to +\infty} \|x^n - y\|_n^2 \leq \frac{1}{\eta} (l(x) - 2l(y) + a(y, y)),
\]
by assumption iv). Therefore,
\[
\limsup_{n \to +\infty} \|x^n\|_n - \|y\|_n \leq \sqrt{\frac{1}{\eta} (l(x) - 2l(y) + a(y, y))}.
\]
Now,
\[
\|x^n\|_n - \|x\|_n \leq \|x^n\|_n - \|y\|_n + \|y\|_n - \|y\| + \|y\| - \|x\|
\]
so that, letting $n$ tend to $+\infty$, we obtain
\[
\limsup_{n \to +\infty} \|x^n\|_n - \|x\| \leq \sqrt{\frac{1}{\eta} (l(x) - 2l(y) + a(y, y))} + \|y\| - \|x\|
\]
for all $y \in D$, by assumption ii). The lemma then results from assumption i) and the fact that $a$ and $l$ are continuous on $H$. \hfill \Box

We can now apply Lemma 4.20 to our shell problem to complete the proof of Theorem 4.13.

**Lemma 4.21.** We have
\[
u^n \to u \text{ strongly in } H^1(\omega; \mathbb{R}^3) \quad \text{and} \quad \Upsilon_{\alpha\beta}^n(u^n) \to \Upsilon_{\alpha\beta}(u) \text{ strongly in } L^2(\omega).
\]

**Proof.** By the weak convergence result of Lemma 4.18, we know that
\[
\liminf_{n \to +\infty} \|u^n\|_{H^1(\omega; \mathbb{R}^3)} \geq \|u\|_{H^1(\omega; \mathbb{R}^3)} \quad \text{and} \quad \liminf_{n \to +\infty} \|\partial_{\alpha\beta} u^n \cdot a^n_3\|_{L^2(\omega)} \geq \|\partial_{\alpha\beta} u \cdot a_3\|_{L^2(\omega)}.
\]

The Hilbert spaces $V^n_1$ and $V_1$ and the bilinear and linear forms associated with the Koiter problems clearly satisfy the hypotheses of Lemma 4.20 with $D = D(\omega; \mathbb{R}^3)$. Therefore,
\[
\|u^n\|^2_{H^1(\omega; \mathbb{R}^3)} + \sum_{\alpha\beta} \|\partial_{\alpha\beta} u^n \cdot a^n_3\|^2_{L^2(\omega)} = \|u^n\|^2_{V^n_1} \to \|u\|^2_{V_1} = \|u\|^2_{H^1(\omega; \mathbb{R}^3)} + \sum_{\alpha\beta} \|\partial_{\alpha\beta} u \cdot a_3\|^2_{L^2(\omega)}
\]
which, together with the previous estimates, implies that
\[
\|u^n\|_{H^1(\omega; \mathbb{R}^3)} \to \|u\|_{H^1(\omega; \mathbb{R}^3)} \quad \text{and} \quad \|\partial_{\alpha\beta} u^n \cdot a^n_3\|_{L^2(\omega)} \to \|\partial_{\alpha\beta} u \cdot a_3\|_{L^2(\omega)}.
\]

The first convergence implies that $u^n \to u$ strongly in $H^1(\omega; \mathbb{R}^3)$, and the second convergence that $\partial_{\alpha\beta} u^n \cdot a^n_3 \to \partial_{\alpha\beta} u \cdot a_3$ strongly in $L^2(\omega)$. Both facts imply that $\partial_{\alpha} u^n \cdot a^n_3 \to \partial_{\alpha} u \cdot a_3$ strongly in $H^1(\omega)$, therefore, by the Sobolev embedding theorem and Lemma 4.12, $\Gamma_{\alpha,\beta}^n \partial_{\rho} u^n \cdot a^n_3 \to \Gamma_{\alpha,\beta}^\rho \partial_{\rho} u \cdot a_3$ strongly in $L^2(\omega)$, which completes the proof. \hfill \Box
Remarks 4.22. Note that the convergences established in Theorem 4.13 are quite natural in the sense that they imply the convergence of the displacements and of the associated strain and change of curvature tensors in their respective natural spaces. This in turn implies the strong $L^2$ convergence of the various stress resultants.

Let us close the article with the final comparison between the classical formulation of Koiter’s model and our formulation. Let us be given a sequence of charts $\varphi^n$ as in Theorem 4.13. For any displacement $v \in V^n_1$, we denote by $v^n_i = v^n \cdot a^n_i$ the covariant components of $v$ so that

$$v^n(x) = v^n_i(x)a^{n,i}(x).$$

Note that in some sense, in considering the covariant components as the basic unknown as is classically done, one mixes regularity issues concerning the displacement with regularity issues concerning the chart. This leads to the restrictive $C^3$ assumption that is made in the classical formulation. This remark may be illustrated by the following result.

**Theorem 4.23.** Let $\varphi^n$ be a sequence of $C^3$-charts such that $\varphi^n \rightarrow \varphi$ in $W^{2,p}(\omega; \mathbb{R}^3)$ strongly for all $1 < p < +\infty$ and $\varphi^n \rightharpoonup \varphi$ in $W^{2,\infty}(\omega; \mathbb{R}^3)$ weak-$\star$, with $\varphi$ piecewise $C^3$ and $\varphi \notin C^3(\omega; \mathbb{R}^3)$. Let $f^n$ be a sequence of force resultant densities such that $f^n \rightarrow f$ in $L^2(\omega; \mathbb{R}^3)$. Let $(u^n_1, u^n_2, u^n_3) \in H_0^1(\omega) \times H_0^1(\omega) \times H_0^2(\omega)$ be the solution of the classical formulation of Koiter’s problem as in [4]. Then, for all $n \in \mathbb{N}$,

$$u^n(x) = u^n_i(x)a^{n,i}(x),$$  \hspace{1cm} (4.36)

$u^n$ tends to $u$ in the sense of Theorem 4.13, but $u^n_3$ is generically unbounded in $H^2(\omega)$.

**Proof.** The fact that (4.36) holds was already noted in [6]. Clearly, $u^n_3 \rightarrow u_3$ in $L^2(\omega)$ by Theorem 4.13. If $u^n_3$ was bounded in $H^2(\omega)$, this would thus imply that $u_3 \in H^2(\omega)$. This is not the case. As was already noted in [6], for a piecewise $C^3$ midsurface, the derivatives of the second fundamental form contain Dirac masses concentrated on the interfaces between the smooth parts of the shell. The condition $\partial_\alpha u \cdot a_3 \in L^2(\omega)$ is equivalent to $\partial_\alpha u \cdot a_3 \in H^1(\omega)$, which means that the jump of $\partial_\alpha u \cdot a_3$ vanishes on each interface. In covariant components, this reads $[\partial_\alpha u_3 + b^\rho_\alpha u_{\rho}] = 0$, or equivalently $[\partial_\alpha u_3] = -[b^\rho_\alpha]u_{\rho}$, on each interface (with $u_3$ piecewise $H^2$).

Since the jump of $b^\rho_\alpha$ is nonzero for some components, this will generically induce a jump on $\partial_\alpha u_3$. The normal component $u_3$ thus cannot be in $H^2(\omega)$.  \hfill $\Box$

**Remark 4.24.** The previous result indicates that a continuous dependence analysis similar to the present one would be difficult to carry out in the classical formulation. Some extra conditions would need to be imposed on the sequence of midsurfaces in order to obtain a uniform $H^2$-bound on $u^n_3$.  \hfill $\Box$

**Example 4.25.** Let us take another look at Example 4.11 in the light of Theorem 4.23. The sequence of solutions associated with the sequence of interpolated midsurfaces falls into the classical
framework. The limit midsurface is not of class $C^3$, thus the limit displacement requires our new formulation. The interface between the smooth parts is the line $\{x_1 = 0\}$. On this interface, all mixed components of the second fundamental form are continuous except $b_{11}$ and it is easy to see that $[b_{11}] = 1$. Therefore, we obtain that $[\partial_1 u_3] = -u_1$ (and $[\partial_2 u_3] = 0$). Since in general, $u_1$ will not vanish on $\{x_1 = 0\}$, we see that $u_3$ is not in $H^2(\omega)$. This is not surprising if we remember that $u_3$ is a covariant component of $u$, hence a scalar product of $u$ with the covariant basis vector $a_3$. The (lack of) regularity of $a_3$ is thus necessarily reflected in the degree of regularity of $u_3$.  

\[\square\]

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