Elastodynamics for Multiplate Structures

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Abstract. In this article, we derive bidimensional evolution models for linearly elastic multiplate structures from three-dimensional linearized elastodynamics. This derivation is achieved by showing that the solutions of three-dimensional elastodynamics in a thin multiplate structure of thickness $\varepsilon$ converge in an appropriate sense when $\varepsilon$ tends to 0, and by identifying the equations satisfied by these limits.

Résumé. Dans cet article, on déduit de l’élastodynamique tridimensionnelle des modèles bidimensionnels d’évolution pour des structures multi-plaques linéairement élastiques. Pour cela, on montre que les solutions de l’élastodynamique tridimensionnelle dans une telle structure d’épaisseur $\varepsilon$ petite convergent, en un certain sens, quand $\varepsilon$ tend vers 0. On identifie ensuite les équations satisfaites par les limites ainsi trouvées.

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1. Introduction. In this paper we derive bidimensional evolution models for linearly elastic thin multiplate structures. The models we obtain can be used to describe and numerically compute the dynamical response of such structures under given loads and initial data—a problem of outstanding practical interest in engineering—in a simpler fashion than with the help of a three-dimensional model. This derivation is achieved by identifying the limits (in some sense) of the solutions of the classical initial-boundary value problem of linearized three-dimensional elastodynamics in the thin 3d-structure when its thickness tends to 0. The main advantage of our approach is that we do not a priori impose any ad hoc assumption about what is going on at the junction between different plates in a given bidimensional model, but rather proceed from a widely accepted (if limited in validity from the standpoint of mechanics) three-dimensional model, and determine mathematically what happens in the zero thickness limit.

The analysis carried out in this paper relies essentially upon previous work on the derivation of evolution models for single plates by Raoult [1980] for linearly elastic materials and in the same variational setting as here, and by Blanchard & Francfort [1987] who consider the case of thermoelastic materials and who use instead a semigroup formalism. Let us also mention further work by Raoult on the derivation of more refined Morozov type plate models in the linear case [1985] and in the nonlinear case [1988]. We also make essential use of a general technique for treating junction problems in elasticity—which we describe briefly below—that has proved to be particularly effective in a variety of situations, see for instance Ciarlet, Le Dret & Nzengwa [1987, 1988] for a static 3d-2d junction, Bourquin & Ciarlet [1988, 1989] for the corresponding eigenvalue problem, Ciarlet & Le Dret [1988, 1989] for another type of 3d-2d junction that yields in the limit the classical clamping condition for a plate and Le Dret [1988a] for junctions between rods, among others. The present paper re-uses (sometimes with significant improvements) a large part of the machinery we developed for the analysis of folded plates in the static and eigenvalue cases, cf. Le Dret [1987a, 1987b, 1987c, 1988b]. However, as the relevant information is scattered throughout these papers, we have striven here towards completeness and (almost always) written whole proofs. This paper is thus essentially self-contained.

Let us detail the contents of the paper. We thus consider a standard folded plate in the sense of Le Dret [1987b, 1987c], i.e., a family of homogeneous isotropic linearly elastic bodies, \( \Omega_\varepsilon \), consisting of two square plates of thickness \( \varepsilon \) attached to one another along one of their sides and at a right angle, cf. Figure 1. We assume that the Lamé moduli of the materials vary with \( \varepsilon \) as \( (\mu_\varepsilon, \lambda_\varepsilon) = \varepsilon^{-3}(\mu, \lambda) \), that the mass density varies as \( \rho_\varepsilon = \varepsilon^{-1}\rho \) and that the structures are clamped on only one of the two plates. For each \( \varepsilon > 0 \), given loads and initial conditions for the displacement and velocity fields, we consider the solution \( u_\varepsilon \)
of the initial-boundary value problem of elastodynamics in $\Omega_\varepsilon$ on a time interval $[0, T]$, cf. equations (2.2)–(2.3). This problem is set in function spaces that vary with $\varepsilon$. In order to be able to talk about convergence of the solutions, one natural idea is to try and rescale the structure, making it independent of $\varepsilon$, and to work in fixed spaces. This idea has been successfully applied by Ciarlet & Destuynder [1979] for the equilibrium of a single linearly elastic clamped plate and in a number of subsequent works dealing with single plates and rods, see for instance Ciarlet [1990] for an extensive bibliography on the subject. Since we are interested here in a multiplate structure, this idea has to be adapted so as to take into account the effects generated by the junction between different parts of the structure. The method we use is as follows: First, rescale the different parts of the structure (here the two plates $\Omega^1_\varepsilon$ and $\Omega^2_\varepsilon$) as though they were alone. This yields two separate fixed domains, $\Omega^1$ and $\Omega^2$. The crucial point in doing so is to be sure of counting the junction region twice, once in each scaled domain. The pair of rescaled displacements $(u^1(\varepsilon), u^2(\varepsilon))$ then satisfies a set of fundamental relations, relations (3.8), in the images of the junction region that give rise to the limit junction conditions when $\varepsilon \to 0$.

We thus prove that, under appropriate boundedness assumptions on the loads and initial data, the scaled flexural displacements of the plates converge in the $L^\infty(0, T, H^1)$ weak-star sense toward functions $(\zeta^1_3, \zeta^2_3)$ that depend only on the time and in-plane variables—in which sense the limit is “bidimensional”. Moreover, the in-plane displacements of the plate that is not clamped (the “free” plate) converge toward a rigid motion that consists of a vertical translation of amount $\bar{a}^{0}_2(t)$ and a rotation of horizontal axis and angle $\bar{a}^0_1(t)$, see Figure 2. These limit unknowns are related through the following limit junction conditions (written here with the coordinate convention used throughout the paper):

i) $\zeta^1_{2\text{ junction}}(t) = -\bar{a}^0_1(t)(x_3 - 1/2) + \bar{a}^0_2(t)$, $\zeta^2_{1\text{ junction}}(t) = 0$,

ii) $\partial_t \zeta^1_{2\text{ junction}}(t) = -\partial_t \zeta^2_{1\text{ junction}}(t)$.

Relation i) says that the edge of the clamped plate follows the overall rigid motion of the free plate—the structure thus stays in one piece—and indicates a stiffening effect of the junction on both plates, i.e., the motion of the junction is rigid. Relation ii) means that the two plates stay perpendicular to each other during the motion. The limit structure is thus of rigid type, see Fayolle [1987] for a classification of possible junctions between plates. Let us emphasize that an homogeneous three-dimensional elastic structure cannot give rise to an elastic or a fortiiori free hinge type of junction. The quadruple $(\zeta^1_2, \zeta^2_1, \bar{a}^0_1, \bar{a}^0_2)$ is then shown to be the unique solution of a well-posed variational evolution problem, cf. equations (4.10)–(4.12). This problems admits an interesting interpretation. It couples classical plate evolution equations for $\zeta^1_2$ and $\zeta^2_1$ with ordinary differential equations for the overall rigid motion of the free plate $(\bar{a}^0_1, \bar{a}^0_2)$. These ODEs are very natural. Denoting
by $m_\varepsilon$ the mass of the free plate and $J_\varepsilon$ its inertia momentum with respect to the center of the junction, they assume the form
\[
\begin{align*}
\{ m_\varepsilon (\ddot{\bar{y}}_2)'' & = \text{vertical resultant force applied on } \Omega^2_\varepsilon, \\
J_\varepsilon (\ddot{\bar{a}}_1)'' & = \text{horizontal resultant moment applied on } \Omega^2_\varepsilon.
\end{align*}
\]
(this is their descaled version, i.e., equations (5.8)-(5.9)).

\textbf{Notation}

As a rule, Greek indices with the subscript 1 take their values in the set \{1, 3\} and Greek indices with the subscript 2 in the set \{2, 3\}. Let $\Omega$ be an open subset of $\mathbb{R}^n$ and let $m$ be a positive integer. We denote by
\[
\begin{align*}
\mathcal{D}(\Omega) & \text{ the space of } C^\infty\text{-functions with compact support in } \Omega, \\
\mathcal{D}'(\Omega) & \text{ the space of distributions on } \Omega, \\
L^2(\Omega) & \text{ the space of (classes) of measurable square-integrable real functions on } \Omega, \\
H^m(\Omega) & \text{ the space of functions of } L^2(\Omega) \text{ whose distributional derivatives up to the order } m \text{ belong to } L^2(\Omega), \\
H^m_0(\Omega) & \text{ the closure of } \mathcal{D}(\Omega) \text{ in } H^m(\Omega), \\
H^{-m}(\Omega) & \text{ the dual space of } H^m_0(\Omega).
\end{align*}
\]
More generally $H^m_\Gamma(\Omega)$ is the space of $H^m$-functions whose traces vanish on a part $\Gamma$ of the boundary of $\Omega$.

Next we introduce some vector-valued function spaces. Let $1 \leq p \leq +\infty$ and let $X$ be a Hilbert space. For $p < +\infty$, the space $L^p(0, T; X)$ is the space of measurable functions from $[0, T]$ into $X$ such that $\left( \int_0^T \| u(t) \|^p_X dt \right)^{1/p} < +\infty$. Similarly, $L^\infty(0, T; X)$ is the space of measurable functions from $[0, T]$ into $X$ such that $\text{ess sup}_{t \in [0, T]} \| u(t) \|_X < +\infty$. Vector-valued distributions are defined as follows: $\mathcal{D}'(0, T; X) = L(\mathcal{D}(0, T), X)$, i.e., a vector-valued distribution is a continuous linear mapping from $\mathcal{D}(0, T)$ into $X$, and the distributional derivative of $u \in \mathcal{D}'(0, T; X)$ is defined as $\langle u', \phi \rangle = -\langle u, \phi' \rangle$ for all $\phi \in \mathcal{D}(0, T)$. Then, $W^{m,p}(0, T; X)$ is the space of functions of $L^p(0, T; X)$ such that all their distributional derivatives with respect to $t$ up to the order $m$ belong to $L^p(0, T; X)$.

We will often use the following property:

\textbf{Lemma 0. Let } $u \in \mathcal{D}'(0, T; X)$. \textbf{Then, for all } $v \in X'$, $\langle u, v \rangle$ \textbf{makes sense as an element of } $\mathcal{D}'(0, T)$ \textbf{and } $\langle u', v \rangle = \langle u, v' \rangle'$.

We refer to Adams [1975], Lions & Magenes [1968a, 1968b] and Schwartz [1957] for the general properties of these spaces.
2. The three-dimensional problem. We consider a family of three-dimensional isotropic homogeneous linearly elastic bodies whose reference configurations are the sets \( \Omega_\varepsilon \) defined for \( 1 > \varepsilon > 0 \) as

\[
\Omega_\varepsilon \overset{\text{def}}{=} \Omega^1_\varepsilon \cup \Omega^2_\varepsilon,
\]

where

\[
\Omega^1_\varepsilon \overset{\text{def}}{=} \{ x \in \mathbb{R}^3, 0 < x_1, x_3 < 1, 0 < x_2 < \varepsilon \},
\]

\[
\Omega^2_\varepsilon \overset{\text{def}}{=} \{ x \in \mathbb{R}^3, 0 < x_2, x_3 < 1, 0 < x_1 < \varepsilon \}.
\]

The parameter \( \varepsilon \), which ought to be thought of as a dimensionless parameter (of the form \( \varepsilon = h/L \) where \( h \) and \( L \) are characteristic dimensions of the body), represents the thickness of the structure. For \( \varepsilon \ll 1 \) we thus consider the simplest possible model case for a multiplate structure, cf. Fig. 1.

![Figure 1](imageurl)  

Let \( T > 0 \) be given. The unknown of the problem we consider is the displacement vector field \( u^\varepsilon : [0, T] \times \Omega_\varepsilon \to \mathbb{R}^3 \), which is assumed to satisfy at any instant \( t \in [0, T] \) a boundary condition of place, \( u^\varepsilon = 0 \), on a subset \( \Gamma^1_\varepsilon \) of the boundary of \( \Omega_\varepsilon \). For definiteness, we set \( \Gamma^1_\varepsilon \) to be

\[
\Gamma^1_\varepsilon = \partial \Omega_\varepsilon \cap \{ x_1 = 1 \}.
\]

This specific choice is sufficiently general and our method applies to other cases in a straightforward fashion as well. In particular, if clamping is assumed to hold on parts of the edges of both plates \( \Omega^1_\varepsilon \) and \( \Omega^2_\varepsilon \), the calculations and results are significantly simpler than in the present case, cf. in the static case Le Dret [1987b, 1987c].
The material characteristics of the bodies are determined by three scalars, the mass density \( \rho_\varepsilon \) and the Lamé moduli \( \mu_\varepsilon \) and \( \lambda_\varepsilon \). We assume that
\[
\rho_\varepsilon = \varepsilon^{-1} \rho, \quad (\mu_\varepsilon, \lambda_\varepsilon) = \varepsilon^{-3}(\mu, \lambda),
\]
where \( \rho, \mu \) and \( \lambda \) are three strictly positive constants independent of \( \varepsilon \).

Remarks. Note that the ratios \( \mu_\varepsilon/\rho_\varepsilon \), \( \lambda_\varepsilon/\rho_\varepsilon \) are assumed to be of the order of \( \varepsilon^{-2} \) (which means in particular that the rigidity of the material is much larger than its density when the thickness goes to 0). This is essential in order that the limit dynamical plate equations be sensitive to inertia effects, see Raoult [1980] for a discussion of the possible choices for these ratios. Apart from that, the particular values of the exponents of \( \varepsilon \) in formula (2.1) are somewhat immaterial insofar as our goal is to derive a limit model when \( \varepsilon \to 0 \). Indeed, the bodies are linearly elastic and we can multiply the equations by any factor \( \varepsilon^k \) without modifying them in essence. In fact, the limit scaled models will always be the same, and their “de-scaled” counterparts will always assume the same form when expressed with the “true” physical coefficients \( \rho_\varepsilon \), \( \mu_\varepsilon \) and \( \lambda_\varepsilon \). In particular, the de-scaled limit equations in strong form inside the plates will be the classical ones, namely,
\[
\varepsilon \rho_\varepsilon \frac{\partial^2 \zeta_\varepsilon}{\partial t^2} + \varepsilon^3 \frac{E_\varepsilon}{12(1 - \nu_\varepsilon^2)} \Delta^2 \zeta_\varepsilon = F^\varepsilon,
\]
where, for \( \varepsilon > 0 \) fixed, \( \zeta_\varepsilon \) is an approximate flexural displacement solution of the limit model equations, \( E_\varepsilon = \frac{\mu_\varepsilon(3\lambda_\varepsilon + 2\mu_\varepsilon)}{\lambda_\varepsilon + 2\mu_\varepsilon} \) is the Young modulus, \( \nu_\varepsilon = \frac{\lambda_\varepsilon}{2(\lambda_\varepsilon + \mu_\varepsilon)} \) is Poisson’s ratio and \( F^\varepsilon \) is some resultant of the applied forces, cf. for instance Duvaut & Lions [1972] or Landau & Lifschitz [1970] (Note that under assumption (2.1), the coefficients in this equation turn out to be independent of \( \varepsilon \), so that if \( F^\varepsilon \) is, roughly speaking, \( O(1) \) then \( \zeta_\varepsilon \) is also \( O(1) \). This is one motivation for specifically assuming (2.1)).

For simplicity again, we assume that the structures are solely subjected to the action of given body-force densities \( f^\varepsilon \), which are supposed to be sufficiently smooth with respect to the time and space variables for all subsequent purposes, e.g., \( f^\varepsilon \in L^2(0,T;L^2(\Omega_\varepsilon)^3) \). The surface of the structure, \( \partial \Omega_\varepsilon \setminus \Gamma^1_\varepsilon \), is assumed to be traction-free. Imposing given tractions on \( \partial \Omega_\varepsilon \setminus \Gamma^1_\varepsilon \) only requires obvious modifications.

We denote the various function spaces involved as follows:
\[
V^\varepsilon \overset{\text{def}}{=} \{ v^\varepsilon \in H^1(\Omega_\varepsilon)^3, \ v^\varepsilon = 0 \text{ on } \Gamma^1_\varepsilon \}, \quad H^\varepsilon \overset{\text{def}}{=} L^2(\Omega_\varepsilon)^3.
\]
These spaces are equipped with their usual topologies. In our variational framework, the space \( V^\varepsilon \) is the space of displacements, the space \( H^\varepsilon \) is the space of velocities and
accelerations belong to \((V^\varepsilon)^0\), the dual space of \(V^\varepsilon\). Then, for all \(\varepsilon > 0\) and given any initial values \(U^\varepsilon \in V^\varepsilon\) for the displacement and \(V^\varepsilon \in H^\varepsilon\) for the velocity, it is well known that there exists a unique solution to the elastodynamics initial boundary value problem:

Find \(u^\varepsilon \in L^\infty(0,T;V^\varepsilon)\), \((u^\varepsilon)' \in L^\infty(0,T;H^\varepsilon)\), \((u^\varepsilon)'' \in L^\infty(0,T;(V^\varepsilon)^0)\),

such that

\[
\rho^\varepsilon\langle(u^\varepsilon)'' , v^\varepsilon \rangle + \int_{\Omega^\varepsilon} A^\varepsilon e(u^\varepsilon) : e(v^\varepsilon) = \int_{\Omega^\varepsilon} f^\varepsilon \cdot v^\varepsilon,
\]

(2.2)

for all \(v^\varepsilon \in V^\varepsilon\) and for almost all \(t \in [0,T]\), with

\[
u^\varepsilon(0) = U^\varepsilon, \quad (u^\varepsilon)'(0) = V^\varepsilon.
\]

(2.3)

In equation (2.2), \(A^\varepsilon\) denotes the elasticity tensor of the material under consideration, whose action on any \(3 \times 3\) symmetric tensor \(\tau\) is given by

\[A^\varepsilon\tau = 2\mu^\varepsilon\tau + \lambda^\varepsilon(\text{tr}\ \tau) \text{Id},\]

and \(e(v^\varepsilon)\) is the linearized strain tensor associated with a displacement \(v^\varepsilon\), i.e.,

\[e(v^\varepsilon) = \frac{1}{2}(\nabla v^\varepsilon + (\nabla v^\varepsilon)^T).\]

Furthermore, the brackets \(\langle \cdot , \cdot \rangle\) denote the duality pairing between \((V^\varepsilon)^0\) and \(V^\varepsilon\). We refer to Lions & Magenes [1968a, 1968b] or Duvaut & Lions [1972] for a discussion of problem (2.2)–(2.3). The strong version of this problem is indeed the classical system:

\[
\begin{aligned}
\frac{\partial^2 u^\varepsilon}{\partial t^2} &= \text{div } \sigma^\varepsilon + f^\varepsilon & \text{in } \Omega^\varepsilon \times ]0,T[, \\
\sigma^\varepsilon &= A^\varepsilon e(u^\varepsilon) & \text{in } \Omega^\varepsilon \times ]0,T[, \\
u^\varepsilon &= 0 & \text{on } \Gamma^1_\varepsilon \times [0,T], \\
\sigma^\varepsilon n^\varepsilon &= 0 & \text{on } (\partial\Omega^\varepsilon) \setminus \Gamma^1_\varepsilon \times ]0,T[, \\
u^\varepsilon(x,0) &= U^\varepsilon(x) & \text{in } \Omega^\varepsilon, \\
(u^\varepsilon)'(x,0) &= V^\varepsilon(x) & \text{in } \Omega^\varepsilon,
\end{aligned}
\]

(2.4)

where for any second-order tensor \(\tau\), \(\text{div } \tau\) is the vector whose components are: \((\text{div } \tau)_i = \partial \tau_{ij}/\partial x_j\). The notation \(\sigma^\varepsilon\) stands of course for the stress tensor that is associated with the displacement \(u^\varepsilon\).
3. Rescaling the structure. We use for the dynamical problem (2.2)–(2.3) exactly the same rescaling as that we used for the static and eigenvalue problems, cf. Le Dret [1987a, 1987b, 1987c, 1987d]. More specifically, we introduce two copies of $\mathbb{R}^3$ (temporarily called ($\mathbb{R}^3$)' and ($\mathbb{R}^3$)"") and define two rescaled plates

$$ \Omega^1 \overset{\text{def}}{=} \{ x \in (\mathbb{R}^3)', 0 < x_i < 1 \} \quad \text{and} \quad \Omega^2 \overset{\text{def}}{=} \{ x \in (\mathbb{R}^3)'', 0 < x_i < 1 \}. $$

The rescaling is more conveniently expressed in terms of its inverse

$$ \phi^\varepsilon : \Omega^1 \cup \Omega^2 \rightarrow \Omega^\varepsilon \quad \varepsilon $$

$$ x \longmapsto \begin{cases} (x_1, \varepsilon x_2, x_3) & \text{if } x \in \Omega^1, \\ (\varepsilon x_1, x_2, x_3) & \text{if } x \in \Omega^2. \end{cases} \quad (3.1) $$

The junction region $J^\varepsilon = \Omega^1 \cap \Omega^2$ thus has two preimages by $\phi^\varepsilon$, namely the sets:

$$ J^1_e \overset{\text{def}}{=} \{ x \in \Omega^1, 0 < x_1 < \varepsilon \} \quad \text{and} \quad J^2_e \overset{\text{def}}{=} \{ x \in \Omega^2, 0 < x_2 < \varepsilon \}. \quad (3.2) $$

We let

$$ \Gamma^1 \overset{\text{def}}{=} \partial \Omega^1 \cap \{ x_1 = 1 \} \quad \text{and} \quad \gamma^1 \overset{\text{def}}{=} \Gamma^1 \cap \{ x_2 = 0 \}, \quad (3.3) $$

and

$$ \omega^1 \overset{\text{def}}{=} \partial \Omega^1 \cap \{ x_2 = 0 \} \quad \text{and} \quad \omega^2 \overset{\text{def}}{=} \partial \Omega^2 \cap \{ x_1 = 0 \}. \quad (3.4) $$

As is apparent, the distinction between two copies of $\mathbb{R}^3$ is irrelevant in definition (3.4) and we denote by

$$ \gamma = \omega^1 \cap \omega^2 = \{ (0, 0, x_3), 0 < x_3 < 1 \}. \quad (3.5) $$

We will at times refer somewhat improperly to $\gamma$ as the “fold” in the “limit” structure. Next, we introduce the fixed function spaces into which the rescaled solutions will converge:

$$ V \overset{\text{def}}{=} H^1_{\Gamma^1}(\Omega^1)^3 \times H^1(\Omega^2)^3 \quad \text{and} \quad H \overset{\text{def}}{=} L^2(\Omega^1)^3 \times L^2(\Omega^2)^3 $$

and we define an operator

$$ \Theta^\varepsilon : H^\varepsilon \rightarrow H \quad \varepsilon $$

$$ v \mapsto (v_1, v_2, \varepsilon^{-1} v_3) \circ \phi^\varepsilon, (v_1, \varepsilon^{-1} v_2, \varepsilon^{-1} v_3) \circ \phi^\varepsilon. \quad (3.6) $$
We let \( H(\varepsilon) = \Theta^*H^e \subset H \) and \( V(\varepsilon) = \Theta^*V^e \subset V \) and finally we define the rescaled displacements

\[
u(\varepsilon) = (u^1(\varepsilon), u^2(\varepsilon)) = \Theta^*u^e.
\]

By construction, it is clear that \( u(\varepsilon)(t) \in V(\varepsilon) \) and \( (u(\varepsilon))'(t) \in H(\varepsilon) \). Moreover, it follows from (3.6) that any element \( v \) of \( H(\varepsilon) \) (and consequently of \( V(\varepsilon) \)) satisfies the following fundamental relations:

\[
\begin{align*}
\varepsilon v_1^1(\varepsilon x_1, x_2, x_3) &= v_1^2(x_1, \varepsilon x_2, x_3) \\
v_2^1(\varepsilon x_1, x_2, x_3) &= \varepsilon v_2^2(x_1, \varepsilon x_2, x_3) \\
v_3^1(\varepsilon x_1, x_2, x_3) &= v_3^2(x_1, \varepsilon x_2, x_3)
\end{align*}
\]  

(3.8)

for almost all \((x_1, x_2, x_3) \in ]0,1[^3\). Conversely, it is easy to see that

\[
H(\varepsilon) = \{ v \in H, v \text{ satisfies } (3.8) \} \quad \text{and} \quad V(\varepsilon) = \{ v \in V, v \text{ satisfies } (3.8) \},
\]

so that \( H(\varepsilon) \) and \( V(\varepsilon) \) are closed subspaces of \( H \) and \( V \) respectively that vary according to (3.8).

Before giving the rescaled version of equation (2.2), it is convenient to introduce the bilinear forms

\[
B^1_\varepsilon(u^1, v^1) \overset{\text{def}}{=} 2\mu e_{\alpha_1\beta_1}(u^1) e_{\alpha_1\beta_1}(v^1) + \lambda e_{\alpha_1\alpha_1}(u^1) e_{\beta_1\beta_1}(v^1) + \\
\varepsilon^{-2} [4\mu e_{\alpha_1\alpha_1}(u^1) e_{\beta_1\beta_1}(v^1) + \lambda (e_{\alpha_1\beta_1}(u^1) e_{\beta_1\beta_1}(v^1) + e_{\alpha_1\beta_1}(u^1) e_{\beta_1\beta_1}(v^1))] + \varepsilon^{-4} (2\mu + \lambda) e_{\alpha_1\alpha_1}(u^1) e_{\beta_1\beta_1}(v^1)
\]

(3.9)

and \( B^2_\varepsilon(u^2, v^2) \) by a similar formula (3.92) obtained from (3.91) by switching the upper and lower indices 1 and 2. A routine calculation (using Lemma 0) then shows that \( u(\varepsilon) \) is the unique solution of the following problem:

Find \( u(\varepsilon) \in L^\infty(0,T;V(\varepsilon)),(u(\varepsilon))' \in L^\infty(0,T;H(\varepsilon)),(u(\varepsilon))'' \in L^\infty(0,T;(V(\varepsilon))') \),

such that:

\[
\rho <u_2^1(\varepsilon),v_2^1(\varepsilon)>_{\Omega_1} + \varepsilon^2 <u_1^1(\varepsilon),v_1^1(\varepsilon)>_{\Omega_1} + \\
+ <u_2^1(\varepsilon),v_2^2(\varepsilon)>_{\Omega_2\setminus J_2} + \varepsilon^2 <u_2^2(\varepsilon),v_2^2(\varepsilon)>_{\Omega_2\setminus J_2} \\
+ \int_{\Omega_1} B^1_\varepsilon(u^1(\varepsilon), v^1(\varepsilon)) \, dx + \int_{\Omega_2\setminus J_2} B^2_\varepsilon(u^2(\varepsilon), v^2(\varepsilon)) \, dx
\]

\[
= \int_{\Omega_1} (v_2^1(\varepsilon)f_2^1(\varepsilon) + \varepsilon v_1^1(\varepsilon)f_1^1(\varepsilon)) \, dx + \int_{\Omega_2\setminus J_2} (v_2^2(\varepsilon)f_2^2(\varepsilon) + \varepsilon v_2^2(\varepsilon)f_2^2(\varepsilon)) \, dx, \quad (3.10)
\]

for all \( v(\varepsilon) \in V(\varepsilon) \), with
\[ u(\varepsilon)(0) = \Theta^\varepsilon U^\varepsilon \overset{\text{def}}{=} U(\varepsilon), \quad (u(\varepsilon))'(0) = \Theta^\varepsilon V^\varepsilon \overset{\text{def}}{=} V(\varepsilon), \]  

(3.11)

where we have let

\[ f(\varepsilon) = \varepsilon f^\varepsilon \circ \phi^\varepsilon \in L^2(0, T; H). \]

For simplicity, we assume that \( f(\varepsilon) \) does not depend on \( \varepsilon \), so that \( f(\varepsilon) = f \). This is rather restrictive in the junction region. However, what we really need is to assume that \( f(\varepsilon) \) is bounded in \( L^2(0, T; H) \) independently of \( \varepsilon \), so as to fix an order of magnitude for the loads (i.e., here \( f^\varepsilon \) is \( O(\varepsilon^{-1}) \)) that yield the desired boundedness of \( u(\varepsilon) \). In this case, the role of \( f \) is played by a weak \( L^2 \)-limit of a subsequence of \( f(\varepsilon) \). There is thus no real loss of generality—but a significant saving in notation—in assuming \( f(\varepsilon) \) independent of \( \varepsilon \). Besides, if the load \( f^\varepsilon \) happens to be of another order in a given problem, then we only have to modify the scaling (3.6) accordingly.

Remark. In equation (3.10), the brackets \( < \cdot, \cdot >_\Omega \) simply mean the \( L^2 \)-inner product over \( \Omega \). By Lemma 0 we have for instance for \( v(\varepsilon) \) smooth with compact support

\[ <u^1_2(\varepsilon), v^1_2(\varepsilon)>''_{\Omega^1} = <(u^1_2(\varepsilon))'', v^1_2(\varepsilon)>_{\Omega^1} \]

where the second bracket denotes the duality in the sense of distributions. Since we have to decompose such pairings over subdomains, the precise space to which \( (u^1_2(\varepsilon))'' \) belongs and in which \( <(u^1_2(\varepsilon))'', v^1_2(\varepsilon)>_{\Omega^1} \) makes sense for \( v(\varepsilon) \in V(\varepsilon) \) is not clear. In this respect, formulation (3.10) offers the advantage of being non ambiguous.

In order to ensure that the solutions of our dynamical problems are uniformly bounded in some sense, it is of course not sufficient to assume that the source terms are bounded. Some kind of boundedness of the initial data is also required. As in the case of a single plate, cf. Raoult [1980] or Blanchard & Francfort [1987], we thus assume:

\[ |V^1(\varepsilon)||^2_{\Omega^1, \varepsilon} + |V^2(\varepsilon)||^2_{\Omega^2, \varepsilon} + \int_{\Omega^1} B^1_\varepsilon(U^1(\varepsilon), U^1(\varepsilon)) \, dx + \int_{\Omega^2} B^2_\varepsilon(U^2(\varepsilon), U^2(\varepsilon)) \, dx \leq C \]  

(3.12)

for some constant \( C > 0 \) independent of \( \varepsilon \), where for \( n = 1, 2, m_1 = 2, m_2 = 1 \), we have let

\[ |V^n(\varepsilon)|^2_{E, \varepsilon} \overset{\text{def}}{=} \int_E \left( V^n_m(\varepsilon)^2 + \varepsilon^2 \sum_{\alpha_n} V^n_{\alpha_n}(\varepsilon)^2 \right) \, dx. \]  

(3.13)

Assumption (3.12) says, roughly speaking, that the components of the rescaled initial velocities are bounded or \( O(\varepsilon^{-1}) \) in \( L^2 \), while the rescaled elastic energy of the initial rescaled displacements is bounded. Inversion of scaling (3.7) shows that we in fact assume
that the initial kinetic energy of the structure and its initial elastic energy stay bounded as \( \varepsilon \to 0 \). The same remark as the one we made for the forces applies as well: Other orders can be treated by modifying the rescaling (3.6) accordingly. To ensure uniqueness for the limit model, we will further assume that the weak limits of the initial data whose existence will follow from assumption (3.12) are unique.

4. The limit multiplate model. In this section, we establish the weak convergence of the flexural components of \( u(\varepsilon) \) toward the solution of a well-posed 2d-2d dynamical model.

Theorem 1. There exist two functions \( a_1^\varepsilon(t) \) and \( b_2^\varepsilon(t) \) in \( W^{1,\infty}(0,T) \) such that, letting

\[
u^2(\varepsilon) = \bar{\nu}^2(\varepsilon) + \begin{pmatrix} 0 \\ -a_1^\varepsilon(t)(x_3 - \frac{1}{2}) + b_2^\varepsilon(t) \\ a_1^\varepsilon(t)x_2 \end{pmatrix}
\]

(4.1)

there exist functions \((u^1(0),\bar{u}^2(0)) \in \mathcal{V}\) and \((\bar{a}_1^0, \bar{b}_2^0) \in W^{1,\infty}(0,T)^2\) such that the following convergences as \( \varepsilon \to 0 \) hold true:

\[
u^1(\varepsilon), \bar{\nu}^2(\varepsilon) \rightharpoonup (u^1(0),\bar{u}^2(0)) \text{ in } L^\infty(0,T;\mathcal{V}),
\]

(4.2)

\[(\varepsilon a_1^\varepsilon, \varepsilon b_2^\varepsilon) \rightharpoonup (\bar{a}_1^0, \bar{b}_2^0) \text{ in } L^\infty(0,T)^2.
\]

(4.3)

Moreover, the velocities converge according to:

\[
\begin{align*}
\left( u^1_2(\varepsilon) \right)' & \rightharpoonup (u_2^1(0))' \text{ in } L^\infty(0,T;L^2(\Omega^1)), \\
\left( \bar{u}^2_1(\varepsilon) \right)' & \rightharpoonup (\bar{u}^2_1(0))' \text{ in } L^\infty(0,T;L^2(\Omega^2)), \\
\varepsilon(u^1_3(\varepsilon))' & \rightharpoonup 0 \text{ in } L^\infty(0,T;L^2(\Omega^1)), \\
\varepsilon(\bar{u}^2_3(\varepsilon))' & \rightharpoonup 0 \text{ in } L^\infty(0,T;L^2(\Omega^2)),
\end{align*}
\]

(4.4)

\[(\varepsilon(a_1^\varepsilon)'), (\varepsilon(b_2^\varepsilon)') \rightharpoonup ((\bar{a}_1^0)', (\bar{b}_2^0)') \text{ in } L^\infty(0,T)^2.
\]

(4.5)

The limit functions in Theorem 1 are characterized in Theorem 2 below as the solutions of a limit model. This model involves the following function spaces:

\[
\mathcal{V} \overset{\text{def}}{=} \{(\xi^1, \xi^2) \in H^2(\omega^1) \times H^2(\omega^2), \xi^1(1,x_3) = \partial_3 \xi^1(1,x_3) = 0, \exists (a(\xi^1), b(\xi^1)) \in \mathbb{R}^2 \text{ with } \xi^1(0,x_3) = -a(\xi^1)(x_3 - 1/2) + b(\xi^1), \}
\xi^2(0,x_3) = 0, \partial_3 \xi^2(0,x_3) = -\partial_2 \xi^2(0,x_3)\},
\]

(4.6)

\[
\mathcal{H} \overset{\text{def}}{=} L^2(\omega^1) \times L^2(\omega^2) \times \mathbb{R}^2.
\]

(4.7)

The last relations in definition (4.6) are the limit junction conditions to be satisfied by the limit displacement fields.
THEOREM 2. The limits of the flexural displacements, \( u_2^1(0) \) and \( \bar{u}_2^1(0) \), and of the overall rigid motion of the free plate, \( a_0^0, \bar{a}_0^2 \), are characterized by the following relations.

a) There exists \( (\zeta_2^1, \zeta_2^2) \in L^\infty(0, T; \mathcal{V}) \), with \(((\zeta_2^1)', (\zeta_2^1)', (\bar{a}_0^0)', (\bar{a}_0^2)') \in L^\infty(0, T; \mathcal{H}) \) such that:

\[
\begin{align*}
\{ u_2^1(0)(x_1, x_2, x_3, t) &= \zeta_2^1(x_1, x_3, t), \\
\bar{u}_2^1(0)(x_1, x_2, x_3, t) &= \zeta_2^2(x_2, x_3, t). 
\end{align*}
\]

(4.8)

Moreover,

\[
\bar{a}_1^0 = a(\zeta_2^2), \quad \bar{b}_2^0 = b(\zeta_2^2) 
\]

(4.9)

and \((\bar{a}_1^0, \bar{b}_2^0) \in W^{1, \infty}(0, T)^2 \).

b) \((\zeta_2^1, \zeta_2^2, a_0^0, \bar{a}_0^2)\) is the unique solution of the limit equation:

\[
\begin{align*}
\rho <\zeta_2^1, \zeta_2^1>'' + \rho <\zeta_2^2, \zeta_2^2>'' + \frac{5}{12} \rho (\bar{a}_1^0)'' a(\zeta_2^1) + \rho (\bar{b}_2^0)'' b(\zeta_2^1) \\
+ \int_{\omega_1} m_{\alpha_1 \beta_1}(\zeta_2^1) \partial_{\alpha_1 \beta_1} \zeta_2^1 \, dx + \int_{\omega_2} m_{\alpha_2 \beta_2}(\zeta_2^2) \partial_{\alpha_2 \beta_2} \zeta_2^2 \, dx \\
= \int_{\omega_1} \zeta_2^1 F_2 \, dx + \int_{\omega_2} \zeta_2^2 F_1 \, dx + b(\zeta_2^1) \int_{\Omega_2} f_2 \, dx \\
+ a(\zeta_2^1) \int_{\Omega_2} (x_2 f_3 - (x_3 - 1/2) f_2) \, dx 
\end{align*}
\]

(4.10)

for all \((\zeta_2^1, \zeta_2^2) \in \mathcal{V}\) with initial values

\[
\begin{align*}
\{ \zeta_2^1|_{t=0} &= U_2^1(0), \quad \zeta_2^2|_{t=0} = \bar{U}_2^2(0), \\
\bar{a}_1^0|_{t=0} &= a(U_2^1(0)), \quad \bar{b}_2^0|_{t=0} = b(U_2^1(0)), \\
\}
\]

(4.11)

\[
\begin{align*}
(\zeta_2^1)|_{t=0} &= \int_0^1 V_2^1(0) \, dx_2, \quad (\zeta_2^2)|_{t=0} = \int_0^1 V_2^2(0) \, dx_2, \\
(\bar{a}_1^0)|_{t=0} &= \frac{12}{5} \int_{\Omega_2} (x_2 \bar{V}_2^1(0) - (x_3 - 1/2) \bar{V}_2^2(0)) \, dx, \\
(\bar{b}_2^0)|_{t=0} &= \int_{\Omega_2} \bar{V}_2^2(0) \, dx, 
\end{align*}
\]

(4.12)

where we have set:

\[
m_{\alpha \beta}(\xi) = \frac{E}{12(1 - \nu^2)}[(1 - \nu)\partial_{\alpha \beta} \xi + \nu \Delta \xi \partial_{\alpha \beta}] \quad \text{(bending moments)},
\]

(4.13)

\[
F_1(x_2, x_3, t) = \int_0^1 f_1(x_1, x_2, x_3, t) \, dx_1, \quad F_2(x_1, x_3, t) = \int_0^1 f_2(x_1, x_2, x_3, t) \, dx_2,
\]

(4.14)
\[
\begin{align*}
\begin{cases}
U_1^1(0) = \lim_{\varepsilon \to 0} U_2^1(\varepsilon), & \tilde{U}_1^1(0) = \lim_{\varepsilon \to 0} \tilde{U}_1^1(\varepsilon), \\
V_2^1(0) = \lim_{\varepsilon \to 0} V_2^1(\varepsilon), & V_1^2(0) = \lim_{\varepsilon \to 0} V_1^2(\varepsilon), & \tilde{V}_2^2(0) = \lim_{\varepsilon \to 0} \varepsilon V_2^2(\varepsilon),
\end{cases}
\end{align*}
\]
(4.15)

the limits in formula (4.15) being meant in the weak \(H^1\)-sense for the displacements and weak \(L^2\)-sense for the velocities and having the property that \((U_2^1(0), \tilde{U}_2^1(0)) \in \mathcal{V}\) so that (4.11) makes sense.

We will comment upon the signification of Theorems 1 and 2 in Section 5. Let us simply note here that the coefficients \(5/12\) and 1 in formula (4.10) are of course not universal. They are related to the geometry of the free plate. In fact, \(5/12\) is the inertia momentum of \(\omega^2\) with respect to the center of \(\gamma\), and 1 is the area of \(\omega^2\). For an arbitrarily shaped plate \(\omega^2\), a cross-product term would appear as well, see the proof of Lemma 9 below.

The proof of these theorems is rather long and is therefore broken into a series of lemmas.

**Lemma 1.** There exists a constant \(C > 0\) independent of \(\varepsilon\) such that

\[
\| \int_{\Omega_1} B^1_\varepsilon(u^1(\varepsilon), u^1(\varepsilon)) \, dx + \int_{\Omega^2} B^2_\varepsilon(u^2(\varepsilon), u^2(\varepsilon)) \, dx \|_{L^\infty(0,T)} < C
\]

(4.16)

and

\[
\| \|(u^1(\varepsilon))')^3_{\Omega_1,\varepsilon} + \|(u^2(\varepsilon))')^3_{\Omega_2,\varepsilon} \|_{L^\infty(0,T)} < C.
\]

(4.17)

(Recall that the notation \(| \cdot |_{\Omega,\varepsilon}\) is defined in formula (3.13)).

**Proof.**

Let \(\varepsilon > 0\) be fixed. We use the energy inequality obtained by setting \(v^\varepsilon = (u^\varepsilon)'\) in equation (2.2) and by integrating it between 0 and \(t\). Of course, we are not allowed to do it that abruptly due to the lack of regularity of \(u^\varepsilon\). However, this argument can be carried out readily on a Galerkin approximation of (2.2) and the infinite-dimensional limit of the energy inequality in the Galerkin approximation equals exactly the outcome of the formal procedure described above. This inequality, which thus holds true, is the following:

\[
\begin{align*}
\frac{1}{2} \rho_\varepsilon \int_{\Omega^*} |(u^\varepsilon(t))'|^2 \, dx + \frac{1}{2} \int_{\Omega^*} A_\varepsilon e(u^\varepsilon(t)) : e(u^\varepsilon(t)) \, dx \leq \\
\int_0^t \int_{\Omega^*} f^\varepsilon \cdot (u^\varepsilon(s))' \, dx \, ds + \frac{1}{2} \rho_\varepsilon \int_{\Omega^*} |V^\varepsilon|^2 \, dx + \frac{1}{2} \int_{\Omega^*} A_\varepsilon e(U^\varepsilon) : e(U^\varepsilon) \, dx
\end{align*}
\]

(4.18)
for all $t$ in $[0, T]$. We then perform rescaling (3.7) in inequality (4.18), which yields:

$$
\rho||(u^1(\epsilon))'(t)||^2_{\Omega^1 \setminus J^1_\epsilon, \epsilon} + \frac{1}{2}||(u^2(\epsilon))'(t)||^2_{\Omega^2 \setminus J^2_\epsilon, \epsilon} + ||(u^2(\epsilon))'(t)||^2_{\Omega^2 \setminus J^2_\epsilon, \epsilon} + \frac{1}{2}||(u^2(\epsilon))'(t)||^2_{\Omega^2 \setminus J^2_\epsilon, \epsilon}
$$

$$+
\int_{\Omega^1 \setminus J^1_\epsilon} B^1_\epsilon(u^1(\epsilon), u^1(\epsilon))(t) \, dx + \frac{1}{2} \int_{J^1_\epsilon} B^1_\epsilon(u^1(\epsilon), u^1(\epsilon))(t) \, dx$$

$$+
\int_{\Omega^2 \setminus J^2_\epsilon} B^2_\epsilon(u^2(\epsilon), u^2(\epsilon))(t) \, dx + \frac{1}{2} \int_{J^2_\epsilon} B^2_\epsilon(u^2(\epsilon), u^2(\epsilon))(t) \, dx$$

$$\leq 2 \int_0^t \left[ \int_{\Omega^1} (u^1_1(\epsilon))' f_2 + \epsilon((u^1_1(\epsilon))' f_{\alpha_1}) \, dx + \int_{\Omega^2} (u^2_1(\epsilon))' f_1 + \epsilon((u^2_2(\epsilon))' f_{\alpha_2}) \, dx \right] \, ds$$

$$+ \rho[|V^1(\epsilon)|^2_{\Omega^1, \epsilon} + |V^2(\epsilon)|^2_{\Omega^2 \setminus J^2_\epsilon, \epsilon}] + \int_{\Omega^1} B^1_\epsilon(U^1(\epsilon), U^1(\epsilon)) \, dx + \int_{\Omega^2 \setminus J^2_\epsilon} B^2_\epsilon(U^2(\epsilon), U^2(\epsilon)) \, dx$$

(4.19)

Note the crucial trick used in the left-hand side of inequality (4.19), which consists in splitting the integrals pertaining to the junction region into two parts, one in $J^1_\epsilon$ and the other in $J^2_\epsilon$. This trick allows us to control the norms over $\Omega^1 \cup \Omega^2$. It is thus clear that there exists a constant $C > 0$ independent of $\epsilon$ such that

$$C \left(||(u^1(\epsilon))'(t)||^2_{\Omega^1, \epsilon} + ||(u^2(\epsilon))'(t)||^2_{\Omega^2, \epsilon}\right)$$

$$+ \int_{\Omega^1} B^1_\epsilon(u^1(\epsilon), u^1(\epsilon))(t) \, dx + \int_{\Omega^2} B^2_\epsilon(u^2(\epsilon), u^2(\epsilon))(t) \, dx$$

$$\leq |V^1(\epsilon)|^2_{\Omega^1, \epsilon} + |V^2(\epsilon)|^2_{\Omega^2 \setminus J^2_\epsilon, \epsilon} + \int_{\Omega^1} B^1_\epsilon(U^1(\epsilon), U^1(\epsilon)) \, dx + \int_{\Omega^2} B^2_\epsilon(U^2(\epsilon), U^2(\epsilon)) \, dx$$

(4.20)

$$+ ||f||^2_{L^2(0, T; H)} + \int_0^t \left[||(u^1(\epsilon))'(s)||^2_{\Omega^1, \epsilon} + ||(u^2(\epsilon))'(s)||^2_{\Omega^2, \epsilon}\right] \, ds.$$

Gronwall's lemma applied to inequality (4.20) and assumption (3.12) on the initial data $U(\epsilon)$ and $V(\epsilon)$ then yield estimates (4.16) and (4.17).

**Lemma 2.** Let $y^\epsilon = (1/2, \epsilon/2, 1/2)$. There exists a constant $C > 0$ independent of $\epsilon$ such that for all $1 > \epsilon > 0$ and all $(u^1(\epsilon), v^2(\epsilon)) \in V(\epsilon)$ there is a decomposition of the form:

$$v^2(\epsilon) = \tilde{v}^2(\epsilon) + a^\epsilon(v^2(\epsilon)) \wedge (x - y^\epsilon) + b^\epsilon(v^2(\epsilon)),$$

(4.21)

with

$$||\tilde{v}^2(\epsilon)||_{H^1(\Omega^2)^3} \leq C ||e(\tilde{v}^2(\epsilon))||_{L^2(\Omega^2)^3}$$

(4.22)

and

$$\epsilon|a^\epsilon_1(v^2(\epsilon))| + |a^\epsilon_2(v^2(\epsilon))| + |b^\epsilon_1(v^2(\epsilon))| + |b^\epsilon_2(v^2(\epsilon))| + \epsilon|b^\epsilon_3(v^2(\epsilon))| + |b^\epsilon_4(v^2(\epsilon))| \leq C \left( \int_{\Omega^1} B^1_\epsilon(u^1(\epsilon), v^1(\epsilon)) \, dx + \int_{\Omega^2} B^2_\epsilon(\tilde{v}^2(\epsilon), \tilde{v}^2(\epsilon)) \, dx \right)^{1/2}.$$  

(4.23)
Proof.

Let $\mathcal{R}$ be the set of infinitesimal rigid displacements over $\Omega^2$, and $P$ be the $L^2$-orthogonal projection onto $\mathcal{R}$. We set (with abbreviated notation):

$$a^\varepsilon \wedge (x - y^\varepsilon) + b^\varepsilon = P(v^2(\varepsilon)).$$

(4.24)

Then it is easily checked by use of Korn’s inequality that estimate (4.22) holds true. To prove (4.23), we must use relations (3.8). When expressed with the help of (4.21), these relations read:

$$\begin{cases}
\varepsilon v_1^1(\varepsilon)(x_1, x_2, x_3) = \tilde{v}_1^2(\varepsilon)(x_1, x_2, x_3) + a_3^\varepsilon(x_3 - 1/2) - \varepsilon a_1^\varepsilon(x_2 - 1/2) + b_1^\varepsilon,

\varepsilon v_2^1(\varepsilon)(x_1, x_2, x_3) = \varepsilon \tilde{v}_2^2(\varepsilon)(x_1, x_2, x_3) + \varepsilon a_1^\varepsilon(x_1 - 1/2) - \varepsilon a_2^\varepsilon(x_3 - 1/2) + \varepsilon b_2^\varepsilon,

\varepsilon v_3^1(\varepsilon)(x_1, x_2, x_3) = \varepsilon \tilde{v}_3^2(\varepsilon)(x_1, x_2, x_3) + \varepsilon a_1^\varepsilon(x_2 - 1/2) - \varepsilon a_2^\varepsilon(x_1 - 1/2) + b_3^\varepsilon.
\end{cases}
$$

(4.25)

Let us first estimate $|b_1^\varepsilon|$. We have ($\Omega$ denotes indifferently $\Omega^1$ or $\Omega^2$),

$$b_1^\varepsilon = \int_{\Omega} \varepsilon v_1^1(\varepsilon)(x_1, x_2, x_3) \, dx - \int_{\Omega} \tilde{v}_1^2(\varepsilon)(x_1, x_2, x_3) \, dx.
$$

(4.26)

It is clearly sufficient to estimate the second integral. The continuous inclusion $H^1(\Omega^2) \hookrightarrow C^{0,1/2}([0, T]; L^2(\omega^1))$ is valid, where $x_2$ is the distinguished variable. Therefore,

$$\left| \int_{\Omega} \tilde{v}_1^2(\varepsilon)(x_1, x_2, x_3) \, dx \right| \leq \frac{1}{\varepsilon} \int_{J^2_\varepsilon} |\tilde{v}_1^2(\varepsilon)|(x_1, x_2, x_3) \, dx
\leq C ||\tilde{v}_1^2(\varepsilon)||_{H^1(\Omega^2)}.
$$

(4.27)

For $\varepsilon < 1$, it is immediate that

$$2\mu \left( ||e(v^1(\varepsilon))||_{L^2(\Omega^1)^9}^2 + ||e(\tilde{v}^2(\varepsilon))||_{L^2(\Omega^2)^9}^2 \right)
\leq \int_{\Omega^1} B_1^2(v^1(\varepsilon), v^1(\varepsilon)) \, dx + \int_{\Omega^2} B_2^2(\tilde{v}^2(\varepsilon), \tilde{v}^2(\varepsilon)) \, dx.
$$

Thus, by Korn’s inequality in $\Omega^1$ (where $v^1(\varepsilon)$ obeys a clamping condition on $\Gamma^1$) and by (4.22), estimate (4.23) holds true for $|b_1^\varepsilon|$. It also holds true for $\varepsilon |b_2^\varepsilon|$, $|b_3^\varepsilon|$, $\varepsilon |a_1^\varepsilon|$ and $|a_2^\varepsilon|$ by the same argument (for the latter two, multiply (4.25)_1 and (4.25)_2 by $(x_3 - 1/2)$ before integrating). To estimate $|a_3^\varepsilon|$, we first differentiate (4.25)_1 with respect to $x_2$ and divide the result by $\varepsilon$. This yields:

$$a_3^\varepsilon = -\partial_2 v_1^1(\varepsilon)(x_1, x_2, x_3) + \partial_2 \tilde{v}_2^2(\varepsilon)(x_1, x_2, x_3)
= -\partial_2 v_1^1(\varepsilon)(x_1, x_2, x_3) - \partial_1 \tilde{v}_2^2(\varepsilon)(x_1, x_2, x_3) + 2e_{12}(\tilde{v}^2(\varepsilon))(x_1, x_2, x_3).
$$

(4.28)
Therefore, if we choose a function $\varphi \in \mathcal{D}([0,1])$ with $\int_0^1 \varphi(t)\,dt = 1$, multiply equation (4.28) by $\phi(x) = \prod_{i=1}^3 \varphi(x_i)$ and integrate, we obtain:

$$
|a^3_3| \leq \int_{\Omega} |\partial_2 v^1_1(\varepsilon x_1, x_2, x_3) \phi(x)| \, dx + \int_{\Omega} |\partial_1 \bar{v}^2_2(\varepsilon)(x_1, \varepsilon x_2, x_3) \phi(x)| \, dx \\
+ 2 \int_{\Omega} |e_{12}(\bar{v}^2)(x_1, \varepsilon x_2, x_3) \phi(x)| \, dx.
$$

(4.29)

The first two integrals are similar and we only have to estimate the first one. As in (4.27), we have,

$$
\int_{\Omega} |\partial_2 v^1_1(\varepsilon x_1, x_2, x_3) \phi(x)| \, dx \leq \frac{1}{\varepsilon} \int_{\Omega} |\partial_2 v^1_1(\varepsilon)(\frac{x_1}{\varepsilon}, x_2, x_3)| \, dx
$$

(4.30)

$$
\leq C\|v^1_1(\varepsilon)\|_{H^1(\Omega^1)}
$$

since $\partial_2 v^1_1(\varepsilon) \in H^1(0,1; H^{-1}(\omega^2)) \hookrightarrow C^{0,1/2}(0,1; H^{-1}(\omega^2))$ (the distinguished variable here is $x_1$). Similarly,

$$
\int_{\Omega} |\partial_1 \bar{v}^2_2(\varepsilon)(x_1, \varepsilon x_2, x_3) \phi(x)| \, dx \leq C\|\bar{v}^2_2(\varepsilon)\|_{H^1(\Omega^1)}.
$$

(4.31)

The last integral is estimated by

$$
\int_{\Omega^2} |e_{12}(\bar{v}^2(\varepsilon))(x_1, \varepsilon x_2, x_3) \phi(x)| \, dx \leq \frac{C}{\varepsilon} \left( \int_{\Omega^2} |e_{12}(\bar{v}^2(\varepsilon))|^2 \, dx \right)^{1/2} \\
\leq C \left( \int_{\Omega^2} B^2_\varepsilon(\bar{v}^2(\varepsilon), \bar{v}^2(\varepsilon)) \, dx \right)^{1/2},
$$

(4.32)

and the Lemma is proved.

\[ \square \]

From now on, we let $a^\varepsilon(t)$ and $b^\varepsilon(t)$ be the vectors associated with $u(\varepsilon)$ by Lemma 2.

**Lemma 3.** There exists a constant $C > 0$ independent of $\varepsilon$ such that

$$
\|u^1(\varepsilon, \bar{v}^2(\varepsilon))\|_{L^\infty(0,T;V)} \leq C,
$$

(4.33)

$$
\|\varepsilon (a^\varepsilon_1, b^\varepsilon_2)\|_{L^\infty(0,T)^2} \leq C,
$$

(4.34)

$$
\|(|\bar{v}^2(\varepsilon)|')_{\Omega^2,\varepsilon}\|_{L^\infty(0,T)} + \|\varepsilon (a^\varepsilon_1)'(\varepsilon(b^\varepsilon_2)')\|_{L^\infty(0,T)^2} \leq C,
$$

(4.35)
where $\bar{u}^2(\varepsilon)$ satisfies (4.1).

Proof.

First of all, since $u^1(\varepsilon) = 0$ on $\Gamma^1$, Korn's inequality and estimate (4.16) yield:

$$||u^1(\varepsilon)||_{L^\infty(0,T;H^1(\Omega^1))} \leq C. \quad (4.36)$$

Secondly, estimates (4.16), (4.22) and (4.23) together imply that

$$\begin{align*}
&||\bar{u}^2(\varepsilon)||_{L^\infty(0,T;H^1(\Omega^2))} \leq C, \\
&||\varepsilon|\alpha^2_1| + |\alpha^2_2| + |\beta^2_1| + |\beta^2_2| + |\alpha^2_3| + |\beta^2_3||_{L^\infty(0,T)} \leq C. \quad (4.37)
\end{align*}$$

Therefore, if we let

$$\bar{u}^2(\varepsilon) = \bar{u}^2(\varepsilon) + \left(\frac{\alpha^2_2(x_3 - 1/2) - \alpha^2_3(x_2 - \varepsilon/2) + \beta^2_1}{\alpha^2_2(x_1 - 1/2)} - \varepsilon \alpha^2_2/2 - \alpha^2_3(x_1 - 1/2) + \beta^2_3\right) \quad (4.38)$$

then, estimates (4.33) and (4.34) hold true. Estimate (4.35) then follows from (4.17) and the definition of $(\alpha^*, \beta^*)$. \hfill \Box

An immediate consequence of Lemma 3 is that there exists a subsequence $\varepsilon_n$ of $\varepsilon$ and functions $u^1(0)$, $\bar{u}^2(0)$, $\alpha^1_1$ and $\beta^2_2$ such that (4.2), (4.3) and (4.5) hold true for the subsequence $\varepsilon_n$. The convergence of the whole family will follow from the uniqueness of the limit (cf. Lemma 12). We will thus denote the subsequence $\varepsilon_n$ by $\varepsilon$ for brevity. In order to complete the proof of Theorem 1 (at least for the subsequence $\varepsilon_n$), it suffices to show the

**Lemma 4.** The in-plane velocities converge according to:

$$\begin{align*}
&\varepsilon(u^1_{\alpha_1}(\varepsilon))' \rightharpoonup 0 \quad \text{in} \quad L^\infty(0,T;L^2(\Omega^1)), \\
&\varepsilon(\bar{u}^2_{\alpha_2}(\varepsilon))' \rightharpoonup 0 \quad \text{in} \quad L^\infty(0,T;L^2(\Omega^2)). \quad (4.39)
\end{align*}$$

Proof.

On the one hand, by (4.2) we have,

$$\begin{align*}
&\left\{\begin{array}{ll}
(u^1_{\alpha_1}(\varepsilon))' \rightarrow (u^1_{\alpha_1}(0))' & \text{in the sense of} \quad \mathcal{D}'(0,T;H^1(\Omega^1)), \\
(\bar{u}^2_{\alpha_2}(\varepsilon))' \rightarrow (\bar{u}^2_{\alpha_2}(0))' & \text{in the sense of} \quad \mathcal{D}'(0,T;H^1(\Omega^2)).
\end{array}\right. \quad (4.40)
\end{align*}$$

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On the other hand, by (4.17) and (4.35), \( \varepsilon(u^1_\Omega(\varepsilon))' \) (resp. \( \varepsilon(\bar{u}^2_\Omega(\varepsilon))' \)) converges weakly-star in \( L^\infty(0, T; L^2(\Omega^1)) \) (resp. \( L^\infty(0, T; L^2(\Omega^2)) \)) toward some limit. By (4.40) this limit is necessarily zero. \( \square \)

Lemmas 1 to 4 establish the existence of limit displacements and velocities in a suitable sense (at least for a subsequence). Let us now study the properties of these limits. First of all, we introduce Kirchhoff-Love displacements in \( \Omega^1 \) and \( \Omega^2 \) as follows:

\[
\begin{align*}
V_{KL}^1 &= \{ v^1 \in H^1_{\gamma_1}(\Omega^1)^3; e_{2\alpha_1}(v^1) = 0 \}, \\
V_{KL}^2 &= \{ v^2 \in H^1(\Omega^2)^3; e_{1\alpha_2}(v^2) = 0 \}.
\end{align*}
\]

(4.41)

It is well-known that \( v^1 \) belongs to \( V_{KL}^1 \) if and only if there exist functions \( \xi^1_2 \in H^2_{\gamma_1}(\omega^1) \), \( \xi^3_2 \in H^2_{\gamma_1}(\omega^1) \) such that

\[
\begin{align*}
v^1(x_1, x_2, x_3) &= (\xi^1_2(x_1, x_3) - (x_2 - 1/2)\partial_1 \xi^3_2(x_1, x_3), \xi^1_2(x_1, x_3), \\
\xi^1_2(x_1, x_3) - (x_2 - 1/2)\partial_3 \xi^3_2(x_1, x_3))
\end{align*}
\]

(4.42)

and similarly for \( v^2 \in V_{KL}^2 \) with functions \( \xi^1_2 \in H^2(\omega^2) \), \( \xi^3_2 \in H^2(\omega^2) \), see for instance Ciarlet & Destuynder [1979].

**Lemma 5.** The limit displacements \( u^1(0), \bar{u}^2(0) \) satisfy

\[
u^1(0) \in L^\infty(0, T; V_{KL}^1), \quad \bar{u}^2(0) \in L^\infty(0, T; V_{KL}^2).
\]

(4.43)

In particular, the flexural components \( u^2_2(0) \) and \( \bar{u}^2_2(0) \) satisfy equation (4.8) for some functions \( (\zeta^1_2, \zeta^3_2) \in L^\infty(0, T; H^2_{\gamma_1}(\omega^1) \times H^2(\omega^2)) \).

**Proof.**

Inequality (4.16) combined with definition (3.9) imply that

\[
\lim_{\varepsilon \to 0} \sum_i \left( \| e_{2i}(u^1(\varepsilon)) \|_{L^\infty(0, T; L^2(\Omega^1))} + \| e_{1i}(\bar{u}^2(\varepsilon)) \|_{L^\infty(0, T; L^2(\Omega^2))} \right) \to 0.
\]

(4.44)

Therefore, by weak sequential lower semicontinuity of these norms,

\[
\begin{align*}
e_{2i}(u^1(0))(t) = 0 & \iff u^1(0)(t) \in V_{KL}^1 \\
e_{1i}(\bar{u}^2(0))(t) = 0 & \iff \bar{u}^2(0)(t) \in V_{KL}^2
\end{align*}
\]

for almost all \( t \in [0, T] \). Furthermore, expressing that \( (u^1(0), \bar{u}^2(0)) \in L^\infty(0, T; V) \) with the help of the representation formula (4.42), we find that (4.43) is satisfied. \( \square \)

The purpose of the next two lemmas is to identify the limit junction conditions satisfied by the pair \( (\zeta^1_2, \zeta^3_2) \).
Lemma 6. The limit flexural displacements satisfy at the fold $\gamma$

\[
\begin{align*}
&\zeta_1^2(0, x_3, t) = -\delta_1^0(t)(x_3 - 1/2) + \delta_2^0(t), \\
&\zeta_2^2(0, x_3, t) = 0,
\end{align*}
\]

for almost all $x_3, t$.

Proof.

Let us rewrite the fundamental relations (3.8) taking relation (4.1) into account. We thus get (for the flexural displacements):

\[
\begin{align*}
&\varepsilon u_1^2(\varepsilon)(x_1, x_2, x_3, t) = \bar{u}_1^2(x_1, \varepsilon x_2, x_3, t), \\
&u_2^2(\varepsilon)(x_1, x_2, x_3, t) = \varepsilon \bar{u}_2^2(x_1, \varepsilon x_2, x_3, t) - \varepsilon a_+^e(t)(x_3 - 1/2) + \varepsilon b_2^e(t).
\end{align*}
\]

for all $t \in [0, T]$. If we distinguish the $x_2$-variable in $\Omega^2$, we have that $H^1(\Omega^2) \hookrightarrow C^{0, 1/2}(0, 1; L^2(\omega^1))$. Therefore, as $\bar{u}_1^2(\varepsilon)$ is uniformly bounded (with respect to $t$) in $H^1(\Omega^2)$ independently of $\varepsilon$, there exists a constant $C > 0$ independent of $\varepsilon$ such that

\[
||u_1^2(\varepsilon)(x_1, 0, x_3, t) - \frac{1}{\varepsilon} \int_0^\varepsilon \bar{u}_1^2(\varepsilon)(x_1, s, x_3, t) ds||_{L^2(\omega^1)} \leq C \varepsilon^{1/2}
\]

for almost all $t \in [0, T]$. Now, by (4.47) we have,

\[
\begin{align*}
&\frac{1}{\varepsilon} \int_0^\varepsilon \bar{u}_1^2(\varepsilon)(x_1, s, x_3, t) ds||_{L^\infty(0, T; L^2(\omega^1))}^2 \\
&= \text{ess sup}_{t \in [0, T]} \left[ \int_{\Omega} \left( \int_0^1 \bar{u}_1^2(\varepsilon)(x_1, \varepsilon x_2, x_3, t) dx_2 \right)^2 dx_1 dx_3 \right] \\
&\leq \text{ess sup}_{t \in [0, T]} \left[ \int_{\Omega} (\bar{u}_1^2(\varepsilon)(x_1, \varepsilon x_2, x_3, t))^2 dx \right] \\
&= \varepsilon^2 \text{ess sup}_{t \in [0, T]} \int_{\Omega} (u_1^2(\varepsilon)(\varepsilon x_2, x_3, t))^2 dx \\
&\leq C \varepsilon ||u_1^2(\varepsilon)||_{L^\infty(0, T; H^1(\Omega^1))}^2.
\end{align*}
\]

On the other hand,

\[
\bar{u}_1^2(\varepsilon)_{|x_2=0} \xrightarrow{\ast} \zeta_1^2_{|x_2=0} \quad \text{in} \quad L^\infty(0, T; L^2(\omega^1))
\]

since $\delta_{x_2=0} \otimes g \in L^1(0, T; (H^1(\Omega^2))')$ for any $g \in L^1(0, T; L^2(\omega^1))$. Formulas (4.48) and (4.49) show that this convergence is in fact strong and that $\zeta_1^2_{|x_2=0} = 0$, which is the second relation in (4.46). The first relation in (4.46) follows from the same argument applied to the second relation in (4.47), and from (4.3).

The last junction condition is a relation between the normal derivatives of the functions $\zeta$ at the fold.
Lemma 7. The limit flexural displacements satisfy at the fold

\[ \partial_1 \zeta_2^1(0, x_3, t) = -\partial_2 \zeta_1^2(0, x_3, t) \]  \hspace{1cm} (4.51)

for almost all \( x_3, t \).

Proof.

Let us differentiate the first relation in (4.47) with respect to \( x_2 \). This yields:

\[ \partial_2 u_1^1(\epsilon) \epsilon_{x_1, x_2, x_3, t} = \partial_2 \bar{u}_1^2(\epsilon) \epsilon_{x_1, x_2, x_3, t} = -\partial_1 \bar{u}_2^2(\epsilon)(x_1, \epsilon x_2, x_3, t) + 2 \epsilon_{12}(\bar{u}_2^2(\epsilon))(x_1, \epsilon x_2, x_3, t). \]  \hspace{1cm} (4.52)

The terms \( \partial_2 u_1^1(\epsilon) \) and \( \partial_1 \bar{u}_2^2(\epsilon) \) are obviously similar to each other and we only need to consider the first one. If we distinguish \( x_1 \) among space variables, we have that

\[
\begin{cases}
\partial_2 u_1^1(\epsilon) \in L^\infty(0, T; L^2(0, 1; L^2(\omega^2))) \\
\partial_1 \partial_2 u_1^1(\epsilon) = \partial_2 \partial_1 u_1^1(\epsilon) \in L^\infty(0, T; L^2(0, 1; H^{-1}(\omega^2))).
\end{cases}
\]  \hspace{1cm} (4.53)

Therefore, \( \partial_2 u_1^1(\epsilon) \) is uniformly bounded in the space \( L^\infty(0, T; H^1(0, 1; H^{-1}(\omega^2))) \) and by uniqueness of weak limits,

\[ \partial_2 u_1^1(\epsilon) \overset{\text{w}}{\rightharpoonup} \partial_2 u_1^1(0) \text{ in } L^\infty(0, T; H^1(0, 1; H^{-1}(\omega^2))). \]  \hspace{1cm} (4.54)

The representation formula (4.42) implies then:

\[ \partial_2 u_1^1(\epsilon) \overset{\text{w}}{\rightharpoonup} -\partial_1 \zeta_2^1(x_1, x_3, t) \text{ in } L^\infty(0, T; H^1(0, 1; H^{-1}(\omega^2))). \]  \hspace{1cm} (4.55)

The same argument applied to \( \bar{u}_2^2(\epsilon) \) yields:

\[ \partial_1 \bar{u}_2^2(\epsilon) \overset{\text{w}}{\rightharpoonup} -\partial_2 \zeta_1^2(x_2, x_3, t) \text{ in } L^\infty(0, T; H^1(0, 1; H^{-1}(\omega^1))). \]  \hspace{1cm} (4.56)

In particular, for any \( f \in L^1(0, T; H^1_0(\omega^2)) \), the distribution \( g = \delta_{x_1=0} \otimes f \) belongs to \( L^1(0, T; (H^1(0, T; H^{-1}(\omega^2)))') \). We deduce from this and from (4.55) that

\[ \partial_2 u_1^1(\epsilon)|_{x_1=0} \overset{\text{w}}{\rightharpoonup} -\partial_1 \zeta_2^1|_{x_1=0} \text{ in } L^\infty(0, T; H^{-1}(\omega^2))). \]  \hspace{1cm} (4.57)

Similarly,

\[ \partial_1 \bar{u}_2^2(\epsilon)|_{x_2=0} \overset{\text{w}}{\rightharpoonup} -\partial_2 \zeta_1^2|_{x_2=0} \text{ in } L^\infty(0, T; H^{-1}(\omega^1))). \]  \hspace{1cm} (4.58)
Now, since $H^1(0,1; H^{-1}(\omega^2)) \hookrightarrow C^{0,1/2}(0,1; H^{-1}(\omega^2))$ and since $\partial_2 u_1^1(\varepsilon)$ is bounded in the above spaces independently of $\varepsilon$, there exists a constant $C > 0$ such that

$$
es_{\varepsilon \in [0,T]} \| \partial_2 u_1^1(\varepsilon)(\varepsilon x_1, x_2, x_3, t) - \partial_2 u_1^1(\varepsilon)|_{x_1=0} \|_{H^{-1}(\omega^2)} \leq C\varepsilon^{1/2}x_1^{1/2} \leq C\varepsilon^{1/2}.
$$

(4.59)

Let us choose four arbitrary functions $\varphi_i \in D(0,T)$, $i = 0, \ldots, 3$, multiply equation (4.52) by $\phi(x,t) = \varphi_0(t)\prod_{i=1}^3 \varphi_i(x_i)$ and integrate. We thus get:

$$
\int_0^T \varphi_0(t) \int_0^1 \varphi_1(x_1) \left( \int_{\omega^2} \partial_2 u_1^1(\varepsilon)(\varepsilon x_1, x_2, x_3, t) \varphi_2(x_2) \varphi_3(x_3) \, dx_2 \, dx_3 \right) dx_1 \, dt
$$

$$
= -\int_0^T \varphi_0(t) \int_0^1 \varphi_2(x_2) \left( \int_{\omega^1} \partial_1 u_2^1(\varepsilon)(x_1, \varepsilon x_2, x_3, t) \varphi_1(x_1) \varphi_3(x_3) \, dx_1 \, dx_3 \right) dx_2 \, dt
$$

$$
+ 2 \int_0^T \int_{\Omega} \varepsilon_1 \left( \int_{\omega^2} (\varphi_2^2(\varepsilon))(x_1, \varepsilon x_2, x_3, t) \phi(x,t) \, dx \, dt
$$

(4.60)

It follows immediately from (4.57)–(4.59) that

$$
\begin{align*}
I_1^1 & \to -\int_0^T \varphi_0(t) \int_0^1 \int_0^1 \varphi_1(x_1) \varphi_2(x_2) \left( \int_{Y} \partial_1 \zeta_2^1(0, x_3, t) \varphi_3(x_3) \, dx_3 \right) dx_1 \, dx_2 \, dt \\
I_2^1 & \to \int_0^T \varphi_0(t) \int_0^1 \int_0^1 \varphi_1(x_1) \varphi_2(x_2) \left( \int_{Y} \partial_2 \zeta_1^2(0, x_3, t) \varphi_3(x_3) \, dx_3 \right) dx_1 \, dx_2 \, dt
\end{align*}
$$

(4.61)

To conclude the proof of the lemma we only have to show that $I_3^1 \to 0$ as $\varepsilon \to 0$. Estimate (4.16) implies that

$$
I_3^1 = \varepsilon \int_0^T \int_0^1 \varphi(x,t) \phi(x,t) \, dx \, dt
$$

(4.62)

where $g(\varepsilon)$ is bounded in $L^\infty(0,T; L^2(\Omega^2))$. Thus, by Hölder’s inequality,

$$
|I_3^1| \leq C\varepsilon^{1/2}\|g(\varepsilon)\|_{L^\infty(0,T; L^2(\Omega^2))},
$$

and the proof is complete. \(\square\)

Lemmas 5 to 7 show that the pair $(\zeta_1^1, \zeta_2^1)$ belongs to the space $L^\infty(0,T; \mathcal{V})$ as defined in formula (4.6). That the quadruple $((\zeta_1^1)', (\zeta_2^1)', (\bar{a}_1^0)', (\bar{b}_2^0)')$ belongs to the space
$L^\infty(0, T; \mathcal{H})$ follows readily from estimates (4.17) and (4.35). So far we have thus proved point a) of Theorem 2. Let us now turn to proving point b), that is establishing the actual form of the limit model. First of all, we define a $3 \times 3$ symmetric tensor $\kappa^1(\epsilon)$ on $\Omega^1$ by

$$\kappa^1_{\alpha_1\beta_1}(\epsilon) = e_{\alpha_1\beta_1}(u^1(\epsilon)), \quad \kappa^1_{\alpha_1\beta_2}(\epsilon) = \epsilon^{-1}e_{\alpha_1\beta_2}(u^1(\epsilon)), \quad \kappa^1_{22}(\epsilon) = \epsilon^{-2}e_{22}(u^1(\epsilon)), \quad (4.63)$$

and $\kappa^2(\epsilon)$ on $\Omega^2$ by a similar formula. The proof of the next lemma is essentially taken from Blanchard & Francfort [1987].

**Lemma 8.** The tensors $\kappa^n(\epsilon)$ converge as $\epsilon \to 0$ according to:

$$\begin{cases}
\kappa^1(\epsilon) \rightharpoonup \kappa^1(0) & \text{in} \quad L^\infty(0, T; L^2(\Omega^1)^9), \\
\kappa^2(\epsilon) \rightharpoonup \kappa^2(0) & \text{in} \quad L^\infty(0, T; L^2(\Omega^2)^9),
\end{cases} \quad (4.64)$$

where

$$\begin{cases}
\kappa^1_{\alpha_1\beta_1}(0) = e_{\alpha_1\beta_1}(u^1(0)), \quad \kappa^1_{\alpha_12}(0) = 0, \quad \kappa^1_{22}(0) = -\frac{\lambda}{2\mu + \lambda}e_{\alpha_1\alpha_1}(u^1(0)), \\
\kappa^2_{\alpha_2\beta_2}(0) = e_{\alpha_2\beta_2}(\bar{u}^2(0)), \quad \kappa^2_{\alpha_11}(0) = 0, \quad \kappa^2_{11}(0) = -\frac{\lambda}{2\mu + \lambda}e_{\alpha_2\alpha_2}(\bar{u}^2(0)).
\end{cases} \quad (4.65)$$

**Proof.**

Inequality (4.16) implies that there exist $\kappa^1(0) \in L^\infty(0, T; L^2(\Omega^1)^9)$ and $\kappa^2(0) \in L^\infty(0, T; L^2(\Omega^2)^9)$ such that (4.64) holds true (again, up to a subsequence). It is clearly sufficient to deal with $\kappa^1(\epsilon)$ alone. First of all, by (4.2) it is immediate that $\kappa^1_{\alpha_1\beta_1}(0) = e_{\alpha_1\beta_1}(u^1(0))$. Then, for arbitrary functions $\varphi_i \in \mathcal{D}(\Omega^1)$, we define test-functions $v(\epsilon), w(\epsilon)$ by

$$\begin{cases}
v^1(\epsilon) = (v^1_1, v^1_2, 0), \quad v^2(\epsilon) = 0, \\
w^1(\epsilon) = (0, 0, v^1_3), \quad w^2(\epsilon) = 0,
\end{cases} \quad (4.66)$$

and $v^1_t(x) = \int_0^x \varphi_1(x_1, s, x_3) \, ds$. By construction $v(\epsilon)$ (resp. $w(\epsilon)$) belongs to $V(\epsilon)$ for $\epsilon$ small enough and we can use it in equation (3.10), multiplied by $\epsilon$ (resp. $\epsilon^2$), which yields in the limit:

$$\int_{\Omega^1} \kappa^1_{\alpha_12}(0) \varphi_{\alpha_1} \, dx = 0 \quad (4.67)$$

and

$$\int_{\Omega^1} \left((2\mu + \lambda)\kappa^1_{22}(0) + \lambda\kappa^1_{\alpha_1\alpha_1}(0)\right) \varphi_3 \, dx = 0 \quad (4.68)$$

for almost all $t \in [0, T]$ and the lemma is proved. \qed
Lemma 9. The quadruple \((\zeta^1, \xi^1, a_1^0, b_2^0)\) is a solution of equation (4.10).

Proof.

The proof follows the same lines as in the static case. We begin by establishing (4.10) for any \((\xi^1, \xi^2) \in \mathcal{V}\) smooth. The first step is to define a test-function \(v(\varepsilon) \in \mathcal{V}(\varepsilon)\) which is a good approximation of the Kirchhoff-Love displacements corresponding to \((\xi^1, \xi^2)\). Then we use this test-function in equation (3.10) and pass to the limit.

First of all, let us set

\[
v^2(\varepsilon) = (\tilde{v}^2(\varepsilon), \varepsilon^{-1}(-a(\xi^1)(x_3 - 1/2) + b(\xi^1)) + \tilde{v}_2^2(\varepsilon), a(\xi^1)(\varepsilon^{-1}x_2 - 1/2) + \tilde{v}_3^2(\varepsilon)) \tag{4.69}
\]

and then define \(v^1(\varepsilon)\) and \(\tilde{v}^2(\varepsilon)\). To this effect, we introduce some notation:

\[
g^1(x_1, x_3) \overset{\text{def}}{=} \xi^1(x_1, x_3) + a(\xi^1)(x_3 - 1/2) - b(\xi^1) - x_1 \partial_1 \xi^1(0, x_3) \tag{4.70}
\]

so that \(|g^1(x_1, x_3)| < Cx_1^2\) in a neighborhood of \(x_1 = 0\), and \(z_1 \overset{\text{def}}{=} x_1 - \varepsilon\). Then we let

\[
v^1(\varepsilon) = (-x_2 - 1/2) \partial_1 \xi^1(0, x_3), \xi^1(0, x_3) + (x_1 - \varepsilon/2) \partial_1 \xi^1(0, x_3), -(x_2 - 1/2) \partial_3 \xi^1(0, x_3)) \tag{4.71_1}
\]

for \(0 < x_1 < \varepsilon\),

\[
v^1(\varepsilon) = \left((-x_2 - 1/2) \partial_1 \xi^1(2z_1, x_3), \xi^1(x_1, x_3) + g^1(2z_1, x_3) - g^1(x_1, x_3)
- \varepsilon \partial_1 \xi^1(2z_1, x_3)/2, -(x_2 - 1/2) \partial_3 \xi^1(2z_1, x_3)) \right) \tag{4.71_2}
\]

for \(\varepsilon < x_1 < 2\varepsilon\) and

\[
v^1(\varepsilon) = (-x_2 - 1/2) \partial_1 \xi^1(x_1, x_3), \xi^1(x_1, x_3) - \varepsilon \partial_1 \xi^1(x_1, x_3)/2, -(x_2 - 1/2) \partial_3 \xi^1(x_1, x_3)) \tag{4.71_3}
\]

for \(x_1 > 2\varepsilon\). Similarly, we let

\[
\tilde{v}^2(\varepsilon) = ((x_2 - \varepsilon/2) \partial_2 \xi^2(0, x_3), -(x_1 - 1/2) \partial_2 \xi^2(0, x_3), 0) \tag{4.72_1}
\]

for \(0 < x_2 < \varepsilon\),

\[
\tilde{v}^2(\varepsilon) = (\xi^2(x_1, x_3) + g^2(2x_2, x_3) - g^2(x_1, x_3) - \varepsilon \partial_2 \xi^2(2x_2, x_3)/2,
- (x_1 - 1/2) \partial_2 \xi^2(2x_2, x_3), -(x_1 - 1/2) \partial_3 \xi^2(2x_2, x_3)) \tag{4.72_2}
\]

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for $\varepsilon < x_2 < 2\varepsilon$,

$$
\bar{v}^2(\varepsilon) = (\xi^1(x_2, x_3) - \varepsilon \partial_{x_2} \xi^2(x_2, x_3)/2, -(x_2 - 1/2) \partial_{x_2} \xi^2(x_2, x_3), -(x_1 - 1/2) \partial_{x_1} \xi^2(x_2, x_3))
$$

(4.72)

for $x_2 > 2\varepsilon$, with similar notation. It is easily checked that the function $v(\varepsilon)$ defined by formulas (4.70), (4.71) and (4.72) belongs to $\mathbf{V}(\varepsilon)$ (i.e., is of class $H^1$ in $\Omega^1$ and $\Omega^2$, satisfies the clamping condition on $\Gamma^1$ and satisfies relations (3.8)). Moreover we have,

$$
e_{22}(v^1(\varepsilon)) = 0, \quad e_{11}(v^2(\varepsilon)) = 0,
$$

(4.73)

and

$$
\begin{cases}
\varepsilon^{-1} e_{\alpha_1 \alpha_2}(v^1(\varepsilon)) \rightarrow -\frac{1}{4} \partial_{\alpha_1 \alpha_2} \xi^1 \text{ strongly in } L^2(\Omega^1), \\
\varepsilon^{-1} e_{\alpha_1 \alpha_2}(v^2(\varepsilon)) \rightarrow -\frac{1}{4} \partial_{\alpha_1 \alpha_2} \xi^2 \text{ strongly in } L^2(\Omega^2).
\end{cases}
$$

(4.74)

Thus, if we use $v(\varepsilon)$ as a test function in equation (3.10), the above convergences and Lemma 8 imply that the elastic energy terms in (3.10) converge according to:

$$
\begin{align*}
\int_{\Omega^1} B_1^1(\varepsilon, v^1(\varepsilon)) dx + \int_{\Omega^2 \setminus J^2_\varepsilon} B_2^2(\varepsilon, v^2(\varepsilon)) dx - \\
\int_{\omega^1} m_{\alpha_1 \beta_1}(\zeta^1_1) \partial_{\alpha_1} \xi^1 dx + \int_{\omega^2} m_{\alpha_2 \beta_2}(\zeta^2_1) \partial_{\alpha_2} \xi^2 dx
\end{align*}
$$

(4.75)

in $L^\infty(0, T)$ as $\varepsilon \rightarrow 0$. Next, let us deal with the inertia terms. First of all, it is clear that

$$
<u^1(\varepsilon), v^1(\varepsilon)>_{\Omega^1, \varepsilon} \rightarrow <\zeta^1_1, \xi^1>_{\omega^1}
$$

(4.76)

in the sense of $\mathcal{D}'(0, T)$ for instance (the last brackets denote the inner product in $L^2(\omega^1)$). Similarly,

$$
<\bar{u}^2(\varepsilon), \bar{v}^2(\varepsilon)>_{\Omega^2 \setminus J^2_\varepsilon, \varepsilon} \rightarrow <\zeta^2_1, \xi^2>_{\omega^2}
$$

(4.77)

in the sense of $\mathcal{D}'(0, T)$. All the cross-product terms tend to 0, and finally,

$$
\begin{align*}
<\varepsilon b_2^1 - \varepsilon a_1^1(x_3 - 1/2), b(\xi^1) - (x_3 - 1/2)a(\xi^1)>_{\Omega^2 \setminus J^2_\varepsilon, \varepsilon} \\
+ <\varepsilon a_1^1 x_2, (x_2 - \varepsilon/2)a(\xi^1)>_{\Omega^2 \setminus J^2_\varepsilon, \varepsilon} \\
\rightarrow b_1^0 a(\xi^1) \int_{\Omega^2} ((x_3 - 1/2)^2 + x_2^2) dx + b_2^0 b(\xi^1) \int_{\Omega^2} 1 dx
\end{align*}
$$

(4.78)
again in the sense of \( \mathcal{D}'(0, T) \). Bringing (4.76)–(4.78) together, we obtain:

\[
\begin{align*}
<u^1(\varepsilon), v^1(\varepsilon)>_{\Omega^1, x}'' + <u^2(\varepsilon), v^2(\varepsilon)>_{\Omega^2, x}'' &
\rightarrow <\zeta^1, \xi^1>_{\omega_1}'' + <\zeta^2, \xi^2>_{\omega_2}'' + \frac{5}{12}(a_1^0)''a(\xi^1) + (b_2^0)''b(\xi^1)
\end{align*}
\]  

(4.79)

in the sense of \( \mathcal{D}'(0, T) \) as \( \varepsilon \rightarrow 0 \). The next step of the proof is to examine the behavior of the force terms as \( \varepsilon \rightarrow 0 \). In view of (4.69) it is fairly clear that

\[
\int_{\Omega^1} (v_1^2(\varepsilon)f_1^1 + \varepsilon v_1^1(\varepsilon)f_1^1)\,dx + \int_{\Omega^2} (v_2^2(\varepsilon)f_2^2 + \varepsilon v_2^1(\varepsilon)f_2^2)\,dx \rightarrow \\
\int_{\omega_1} \xi^1F_2\,dx + \int_{\omega_2} \xi^2F_1\,dx + a(\xi^1)\int_{\Omega^2} (x_2f_3 - (x_3 - 1/2)f_2)\,dx + b(\xi^1)\int_{\Omega^2} f_2\,dx
\]  

(4.80)

in the sense of \( \mathcal{D}'(0, T) \) as \( \varepsilon \rightarrow 0 \). The Lemma follows from formulas (4.75)–(4.77), (4.79) and (4.80) brought together for \((\xi^1, \xi^2)\) smooth. A density argument completes the proof for any \((\xi^1, \xi^2)\) in \( \mathcal{V} \).

We next identify the initial conditions for the limit problem.

**Lemma 10.** The initial displacements \((\zeta^1_0, \zeta^2_0, a^1_1, b^0_2)\) at \( t = 0 \) satisfy relation (4.11).

**Proof.**

It follows from assumption (3.12) that \( U(\varepsilon) \) can be decomposed as in formula (4.1) and satisfies the estimates:

\[
\|
\begin{cases}
\|(U^1(\varepsilon), \bar{U}^2(\varepsilon))\|_{\mathcal{V}} \leq C \\
\|(\varepsilon a_1^\varepsilon(U^2(\varepsilon)), \varepsilon b_2^\varepsilon(U^2(\varepsilon)))\|_{L^2(\mathbb{R}^2)} \leq C
\end{cases}
\]

(4.81)

which are similar to (4.33)–(4.34). Moreover, again by (3.12) and by our assumption of uniqueness of the limits of initial data,

\[
(U^1(\varepsilon), \bar{U}^2(\varepsilon)) \longrightarrow (U^1(0), \bar{U}^2(0)) \text{ strongly in } \mathcal{H}
\]

(4.82)

with \((U^1(0), \bar{U}^2(0)) \in \mathcal{V}\) and satisfy

\[
(\varepsilon a_1^\varepsilon(U^2(\varepsilon)), \varepsilon b_2^\varepsilon(U^2(\varepsilon))) \longrightarrow (a(U^1_2(0)), b(U^1_2(0))).
\]

(4.83)

Therefore, as \((u(\varepsilon), \varepsilon a_1^\varepsilon(t), \varepsilon b_2^\varepsilon(t))\) is bounded in \( W^{1, \infty}(0, T; \mathcal{H} \times \mathbb{R}^2) \) independently of \( \varepsilon \), we can pass to the limit in its value at \( t = 0 \), which yields relations (4.11).

The identification of initial velocities is more delicate. The proof we present here is essentially taken from Raoult [1980].

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Lemma 11. The initial velocities \(((\xi_2^1)'(\xi_1^2)'),(\bar{a}_1^1)'(\bar{b}_2^0)'),(t_0)\mid t=0\) satisfy relations (4.12).

Proof.

First of all, assumption (3.12) implies that the weak \(L^2\)-limits \(V_2^1(0), V_1^1(0)\) and \(\bar{V}_2^2(0)\) exist. Relation (4.12) thus make sense. Let \((\xi_1^1, \xi_2^2) \in \mathcal{V}\) be smooth and consider the test-function \(v(\varepsilon) \in \mathcal{V}(\varepsilon)\) constructed in the proof of Lemma 9. Define

\[
\theta^\varepsilon(t) \overset{\text{def}}{=} <u^1(\varepsilon), v^1(\varepsilon)>_{\Omega_1, \varepsilon} + <u^2(\varepsilon), v^2(\varepsilon)>_{\Omega_2 \setminus J_2^1, \varepsilon}. \tag{4.84}
\]

We have seen that

\[
\theta^\varepsilon \rightharpoonup <\xi_1^1, \xi_1^1>_{\omega^1} + <\xi_2^2, \xi_2^2>_{\omega^2} + \frac{5}{12} \bar{a}_1^0 a(\xi_1^1) + \bar{b}_2^0 b(\xi_1^1) \tag{4.85}
\]

in the sense of \(\mathcal{D}'(0, T)\). Moreover, it is immediate that \(\theta^\varepsilon\) is bounded in \(W^{1, \infty}(0, T)\) by estimates (4.33)–(4.35) and (4.17). Thus the convergence above holds true in the weak-star \(W^{1, \infty}(0, T)\)-sense. Now, by equation (3.10),

\[
(\theta^\varepsilon)'' = - \int_{\Omega_1} B_2^1(u^1(\varepsilon), v^1(\varepsilon)) \, dx - \int_{\Omega_2 \setminus J_2^1} B_2^2(u^2(\varepsilon), v^2(\varepsilon)) \, dx \tag{4.86}
\]

\[+ <u^1(\varepsilon), f^1(\varepsilon)>_{\Omega_1, \varepsilon^1/2} + <u^2(\varepsilon), f^2(\varepsilon)>_{\Omega_2 \setminus J_2^1, \varepsilon^1/2}.\]

So, by (4.75) and (4.80) it follows that \((\theta^\varepsilon)''\) is bounded in \(L^\infty(0, T)\) and thus convergence (4.85) also holds true in the weak-star \(W^{2, \infty}(0, T)\)-sense. We can therefore pass to the limit in \((\theta^\varepsilon)'|_{t=0}\) and this yields:

\[
<V_2^1(0), \xi_1^1>_{\Omega_1} + <V_1^2(0), \xi_2^2>_{\Omega_2}
\]

\[+ a(\xi_1^1) \int_{\Omega_2} (x_2 V_2^1(0) - (x_3 - 1/2) \bar{V}_2^1(0)) \, dx + b(\xi_1^1) \int_{\Omega_2} \bar{V}_2^2(0) \, dx \tag{4.87}
\]

\[= <(\xi_1^1)'|_{t=0}, \xi_1^1>_{\omega^1} + <(\xi_2^2)'|_{t=0}, \xi_2^2>_{\omega^2}
\]

\[+ \frac{5}{12} (\bar{a}_1^0)'|_{t=0} a(\xi_1^1) + (\bar{b}_2^0)'|_{t=0} b(\xi_1^1)\]

for all \((\xi_1^1, \xi_2^2) \in \mathcal{V}\) smooth. Relations (4.12) follow at once from (4.87) \(\square\)

The last but crucial step in the proof of Theorem 2 consists in showing that problem (4.10) with the initial conditions (4.11) and (4.12) is well-posed. In effect, although we have proved that the limit displacements solve (4.10)–(4.12), it is conceivable at this point that we had somehow missed a junction condition and not completely identified these limits. It is thus essential to establish the uniqueness of the solution of problem (4.10)–(4.12).
Lemma 12. Problem (4.10)–(4.12) is well-posed.

Proof.

The trick is to rewrite problem (4.10)–(4.12) in standard abstract form as in Lions & Magenes [1968a, 1968b] or Duvaut & Lions [1972]. We thus equip $\mathcal{V}$ with the usual $H^2$-inner product and $\mathcal{H}$ with the following inner product:

$$<\xi, \eta>_{\mathcal{H}} = \rho <\xi^1, \xi^1>_{\Omega^1} + <\xi^2, \xi^2>_{\Omega^2} + \frac{5}{12} a(\xi^1) a(\xi^1) + b(\xi^1) b(\xi^1).$$

(4.88)

Then it is obvious that the mapping

$$\mathcal{V} \rightarrow \mathcal{H}$$

$$\xi \mapsto (\xi^1, \xi^2, a(\xi^1), b(\xi^1))$$

(4.89)

is a dense embedding, so that we can write $\mathcal{V} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{V}'$. We now define a bilinear form $A$ on $\mathcal{V} \times \mathcal{V}$ by

$$A(\xi, \eta) = \int_{\Omega^1} m_{\alpha_1, \beta_1}(\xi^1) \partial_{\alpha_1, \beta_1} \xi^1 \, dx + \int_{\Omega^2} m_{\alpha_2, \beta_2}(\xi^2) \partial_{\alpha_2, \beta_2} \xi^2 \, dx.$$  

(4.90)

The three-dimensional Korn inequality applied to the Kirchhoff-Love displacements associated with $(\xi^1, \xi^2)$ shows that $A(\cdot, \cdot)$ satisfies Gårding's inequality (in fact, a more refined argument shows that it is coercive on $\mathcal{V}$), and problem (4.10)–(4.12) may be rewritten as:

Find $\xi \in L^\infty(0, T; \mathcal{V}), \xi' \in L^\infty(0, T; \mathcal{H})$ (in the sense of embedding (4.89)) such that

$$\begin{cases}
<\xi'', \eta>_{\mathcal{H}} + A(\xi, \eta) = <F, \eta>
\xi(0) = \xi^0 \in \mathcal{V}
\xi'(0) = \xi^1 \in \mathcal{H}
\end{cases}$$

(4.91)

for all $\eta \in \mathcal{V}$, where

$$\xi^0 = (U^1_2(0), \bar{U}^2_1(0))$$

$$\xi^1 = \left( \int_0^1 V^1_2(0) \, dx_2, \int_0^1 V^2_1(0) \, dx_1, \right.$$

$$\frac{12}{5} \int_{\Omega^2} (x_2 \bar{V}^2_3(0) - (x_3 - 1/2) \bar{V}^2_2(0)) \, dx_3, \int_{\Omega^2} \bar{V}^2_2(0) \, dx_2 \right)$$

and
\[ <F, \xi> = \int_{\omega_1} \xi^1 F_2 \, dx + \int_{\omega_2} \xi^2 F_1 \, dx + a(\xi^1) \int_{\Omega_2} (x_2 f_3 - (x_3 - 1/2) f_2) \, dx \]
\[ + b(\xi^1) \int_{\Omega_2} f_2 \, dx \]

Now, it is well-known that abstract problem (4.91) has one and only one solution in the above spaces, which concludes the proof of Lemma 12.

Lemma 12 implies in particular that the weak limit of any convergent subsequence of the family \( u(\varepsilon) \) is unique. Therefore, the whole family converges as \( \varepsilon \to 0 \), which was the only thing left to prove in Theorems 1 and 2.

5. Descaling and interpretation of the limit model. We can now rewrite equation (4.10) in terms of the true material constants and applied forces, and replace the rescaled limits \((\zeta_1^1, \zeta_2^1)\) by approximate (in some sense) flexural displacements \((\zeta_1^{1, \varepsilon}, \zeta_2^{2, \varepsilon})\), which in view of (3.8) are simply equal to \((\zeta_1^1, \zeta_2^1)\) in our case. This yields:

\[
\varepsilon \rho_\varepsilon <\zeta_1^{1, \varepsilon}, \xi^1>'' + \varepsilon \rho_\varepsilon <\zeta_2^{2, \varepsilon}, \xi^2>'' \\
+ \frac{\varepsilon}{12} \rho_\varepsilon a(\zeta_1^{1, \varepsilon})'' a(\xi^1) + \varepsilon \rho_\varepsilon b(\zeta_1^{1, \varepsilon})'' b(\xi^1) \\
+ \int_{\omega_1} m^{\varepsilon}_{\alpha_1 \beta_1} (\zeta_1^{1, \varepsilon}) \partial_{\alpha_1 \beta_1} \xi^1 \, dx + \int_{\omega_2} m^{\varepsilon}_{\alpha_2 \beta_2} (\zeta_2^{2, \varepsilon}) \partial_{\alpha_2 \beta_2} \xi^2 \, dx \\
= \int_{\omega_1} \xi^1 F_2^\varepsilon \, dx + \int_{\omega_2} \xi^2 F_1^\varepsilon \, dx \\
+ a(\xi^1) \int_{\Omega_2} (x_2 f_3^\varepsilon - (x_3 - 1/2) f_2^\varepsilon) \, dx + b(\xi^1) \int_{\Omega_2} f_2^\varepsilon \, dx \\
\]

for all \( \xi \in \mathcal{V} \), where

\[
m^{\varepsilon}_{\alpha \beta}(\xi) = -\frac{\varepsilon^3 E_\varepsilon}{12(1 - \nu_\varepsilon^2)} [(1 - \nu_\varepsilon) \partial_{\alpha \beta} \xi + \nu_\varepsilon \Delta \xi \delta_{\alpha \beta}],
\]

\[
F_2^\varepsilon = \int_0^\varepsilon f_2^\varepsilon \, dx_2, \quad F_1^\varepsilon = \int_0^\varepsilon f_1^\varepsilon \, dx_1,
\]

plus initial data in \( \mathcal{V} \) and \( \mathcal{H} \).

Another way of understanding Theorems 1 and 2 in terms of convergence taking place on the actual structures is to restrict the displacements to the mid-planes of both plates (this does not work for velocities however). Then, after descaling, we see that the original flexural displacements on the mid-planes converge (in the appropriate functional
sense) toward \((\zeta_1^1, \zeta_1^2)\), while the original in-plane displacements converge toward 0 in the clamped plate and toward the rigid motion \((0, -a(\zeta_1^1)(t)(x_3 - 1/2) + b(\zeta_1^1)(t), a(\zeta_1^1)(t)x_2)\) in the free plate. This rigid motion thus consists of a vertical translation of amount \(b(\zeta_1^1)(t)\) and of an infinitesimal rotation of angle \(a(\zeta_1^1)(t)\) around the \(x_1\)-axis.

Equation (5.1) can now be considered as a model of its own right, regardless of any order of the data with respect to \(\varepsilon\). Given specific numerical values for the thickness, the mass density, the Lamé constants and the applied forces, we can solve equation (5.1) and obtain an approximate solution to the original 3d problem.

The interpretation of problem (5.1) in terms of partial differential equations is interesting per se. First of all, as we announced earlier, the inside equations are the classical plate equations:

\[
\begin{align*}
\varepsilon \rho_\varepsilon \frac{\partial^2 \zeta^{1,\varepsilon}}{\partial t^2} + \varepsilon^3 \frac{E_\varepsilon}{12(1 - \nu_\varepsilon^2)} \Delta^2 \zeta^{1,\varepsilon} &= F_2^\varepsilon \quad \text{in} \quad \omega^1, \\
\varepsilon \rho_\varepsilon \frac{\partial^2 \zeta^{2,\varepsilon}}{\partial t^2} + \varepsilon^3 \frac{E_\varepsilon}{12(1 - \nu_\varepsilon^2)} \Delta^2 \zeta^{2,\varepsilon} &= F_1^\varepsilon \quad \text{in} \quad \omega^2.
\end{align*}
\]  

\( (5.4) \)

Secondly, we have the clamping and junction conditions,

\[
\zeta^{1,\varepsilon}(1, x_3, t) = \partial_1 \zeta^{1,\varepsilon}(1, x_3, t) = 0, \quad (5.5)
\]

\[
\begin{align*}
\zeta^{1,\varepsilon}(0, x_3, t) &= -a(\zeta^{1,\varepsilon})(t)(x_3 - 1/2) + b(\zeta^{1,\varepsilon})(t), \\
\zeta^{2,\varepsilon}(0, x_3, t) &= 0, \\
\partial_1 \zeta^{1,\varepsilon}(0, x_3, t) &= -\partial_2 \zeta^{2,\varepsilon}(0, x_3, t).
\end{align*}
\]  

\( (5.6) \)

The junction conditions \((5.6)\) have the following meaning: The first two relations express the fact that the structure stays "in one piece" in the limit (recall equation \((4.9)\)). The overall rigid motion of the free plate follows that of the edge of the clamped plate. Moreover, the fold has a stiffening effect on both plates, evidenced by those two relations. The third relation means that the two plates stay perpendicular to each other throughout the motion of the structure. The limit junction is thus of "rigid" type. Figure 2 displays a typical snapshot of the structure at instant \(t\).

Thirdly, we have the traction-free condition on the remaining part of the edges of the plates (which we do not write for brevity). Next come a set of relations satisfied at the fold and obtained from formal integrations by parts in \((5.1)\). The first one expresses the transmission of bending moments across the fold,

\[
m_{11}^\varepsilon(\zeta^{1,\varepsilon}) = m_{22}^\varepsilon(\zeta^{2,\varepsilon}).
\]  

\( (5.7) \)
The two others are ordinary differential equations for the overall rigid motion of the free plate,

$$\frac{5}{12} \varepsilon \rho_\varepsilon a(\zeta^{1,\varepsilon})'' = - \int_\gamma [m_{31}(\zeta^{1,\varepsilon}) + \partial_{\beta_1} m_{\beta_1 1}(\zeta^{1,\varepsilon})(x_3 - 1/2)] \, d\gamma$$

$$+ \int_{\Omega_2^1} (x_2 f_3^x - (x_3 - 1/2)f_2^x) \, dx,$$

$$\varepsilon \rho_\varepsilon b(\zeta^{1,\varepsilon})'' = \int_\gamma \partial_{\beta_1} m_{\beta_1 1}(\zeta^{1,\varepsilon}) \, d\gamma + \int_{\Omega_2^1} f_3^x \, dx.$$  \hspace{1cm} (5.8)

These two equations are very natural. The left-hand side of equation (5.8) is equal to $\frac{5}{12} \varepsilon \rho_\varepsilon$, which is the inertia momentum of the free plate with respect to the center of the fold, times $a(\zeta^{1,\varepsilon})''$, which is the angular acceleration of the free plate. The right-hand side of equation (5.8) is equal to the resultant moment of the forces applied to the free plate. A similar interpretation is valid for equation (5.9) with the vertical acceleration ($\varepsilon \rho_\varepsilon$ is the total mass of the free plate).

Equations (5.4)–(5.9) have to be supplemented with initial conditions for $\zeta^{1,\varepsilon}$, $\zeta^{2,\varepsilon}$, $(\zeta^{1,\varepsilon})'$, $(\zeta^{2,\varepsilon})'$, $(a(\zeta^{1,\varepsilon}))'$ and $(b(\zeta^{1,\varepsilon}))'$. Initial conditions for $a(\zeta^{1,\varepsilon})$ and $b(\zeta^{1,\varepsilon})$ are supplied by the initial data for $\zeta^{1,\varepsilon}$ and $\zeta^{2,\varepsilon}$ themselves. The limit flexural displacements thus solve a rather intricate system of coupled partial and ordinary differential equations.
Concluding remarks. To begin with, let us point out that the analysis carried out in the present paper for a simple two-plates structure can of course be extended to cover more general geometries, see Le Dret [1987c] in the static case. Such extensions are however burdened by an increasingly cumbersome notation.

To conclude this article, let us mention a few points that seem to be of particular interest. First of all, the study of the controllability and stabilization of system (5.4)–(5.9), as a model for controllability of more general multiplate structures, is a problem of outstanding importance, see Lagnese & Lions [1988] for single plates. A second important point is the numerical analysis of such problems. In this respect, see Fayolle [1987], Bernadou, Fayolle & Laïné [1988] and Bernadou [1989].

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