JUNCTIONS BETWEEN THREE-DIMENSIONAL AND TWO-DIMENSIONAL LINEARLY ELASTIC STRUCTURES

By Philippe G. CIARLET, Hervé LE DRET and Robert NZENGWA

ABSTRACT. — We consider a problem in three-dimensional linearized elasticity, posed over a domain consisting of a plate with thickness $2\varepsilon$, inserted into a solid whose Lamé constants are independent of $\varepsilon$. If the Lamé constants of the material constituting the plate vary as $\varepsilon^{-3}$, we show that the solution of the three-dimensional problem converges, as $\varepsilon$ approaches zero, to the solution of a coupled "pluri-dimensional" problem of new type, posed simultaneously over a three-dimensional open set with a slit and a two-dimensional open set.

RÉSUMÉ. — On considère un problème d'élasticité linéarisée tridimensionnelle, posé sur un domaine constitué d'une plaque d'épaisseur $2\varepsilon$, insérée dans un solide dont les constantes de Lamé sont indépendantes de $\varepsilon$. Si les constantes de Lamé du matériau constituant la plaque varient comme $\varepsilon^{-3}$, on montre que la solution du problème tridimensionnel converge, lorsque $\varepsilon$ tend vers zéro, vers la solution d'un problème couplé, "pluri-dimensionnel" d'un nouveau type, posé simultanément sur un ouvert tri-dimensionnel avec une fente et sur un ouvert bi-dimensionnel.

1. Introduction

The modeling of junctions between elastic structures of possibly different dimensions is a problem of outstanding practical importance, since junctions are very commonly found in actual structures: Consider for instance the structure formed by an H-shaped beam inserted into an elastic foundation (Fig. 1), or a satellite (Fig. 2). Nevertheless, this problem does not seem to have been so far investigated from a mathematical viewpoint.

By contrast, the mathematical modeling and analysis of linearly and nonlinearly elastic structures without junctions are well established: Three-dimensional structures are extensively discussed in Truesdell & Noll [1965], Duvaut & Lions [1972], Fichera [1972], Germain [1972], Gurtin [1972, 1981], Wang & Truesdell [1973], Marsden & Hughes [1983], Ciarlet [1988]. The modeling of two-dimensional structures (plates, shells) and of one-dimensional structures (rods, strings, arches) has been likewise extensively analyzed, notably by Friedrichs & Dressler [1961], Goldeneizer [1962], John [1971] (who was the first to mathematically justify the Kirchhoff-Love approximation), Rigolot [1972, 1976], Caillerie [1980]; particular mention should be also made of the geometrically exact plate, shell, and rod, theories of Antman [1972, 1990], which successfully account for the

We consider here the modeling of the junction between a three-dimensional elastic structure and a plate. Our approach is of wide applicability, since it can be also used for modeling junctions between plates (folded plates, possibly with corners; cf. Le Dret [1987, 1989 a, 1989 b], junctions between plates and rods (cf. Ciarlet & Gruais [1989]),
juncti0ns between rods (cf. Le Dret [1989 c]), and the corresponding eigenvalue problems (cf. Bourquin & Ciarlet [1989]). See also Ciarlet [1988] for an overview of this approach.

In each instance, one or several portions of the whole three-dimensional structure has a “small” thickness, or diameter of a cross-section, which is proportional to a dimensionless parameter $\varepsilon$. If the various data (Lamé constants, applied body and surface force densities) behave as specific powers of $\varepsilon$ as $\varepsilon \to 0$, one can establish the $H^1$–convergence of the (appropriately scaled) components of the displacement vector field towards the solution of a “limit” variational problem of a new type, posed simultaneously over an open subset of $\mathbb{R}^n$ and an open subset of $\mathbb{R}^s$, with $1 \leq m, n \leq 3$. The crucial idea for treating the junction consists in scaling the different parts of the full structure independently of each other (for instance, the plate is scaled as is usually done in “single plate” theory), but counting the junction twice, once in each portions that it connects. The scaled components of the displacement, which are defined in this fashion on two separate domains, thus contain the information about the junction twice. That they correspond to the same displacement of the whole structure then yields the “junction conditions” that the solution of the limit problem must satisfy.

The results of this paper have been announced in Ciarlet, Le Dret & Nzengwa [1987].

Remark. — Structures comprising “many” junctions between plates, or between rods, are also amenable to a completely different approach, based on the techniques of homogenization theory. The limit, “homogenized” problems obtained in this fashion are thus models of structures with “infinitely many” junctions. In this direction, see notably the works of Caillerie [1980, 1984, 1987], Kohn & Vogelius [1984, 1985, 1986], Cioranescu & Saint-Jean Paulin [1986, 1988 a, 1988 b].

2. The three-dimensional problem

Latin indices take their values in the set $\{1, 2, 3\}$ and Greek indices take their values in the set $\{1, 2\}$; the repeated index convention for summation is systematically used in conjunction with the above rules. Vector-valued function and their associated function spaces are denoted by boldface letters.

We are given constants $a_1, b_1, a_2, a_3, b_3, \beta$ which are all $> 0$, and we assume that $\beta < b_1$. For each $\varepsilon > 0$, we let (cf. Fig. 3):

\[ \omega = \{ (x_1, x_2) \in \mathbb{R}^2; 0 < x_1 < b_1, |x_2| < a_2 \}, \quad \Omega^\varepsilon = \omega \times \varepsilon, \varepsilon, \]
\[ \gamma_0 = \{ (b_1, x_2) \in \mathbb{R}^2; |x_2| \leq a_2 \}, \quad \Gamma^\varepsilon_0 = \gamma_0 \times \varepsilon, \varepsilon, \]
\[ \omega_\beta = \{ (x_1, x_2) \in \mathbb{R}^2; 0 < x_1 < \beta, |x_2| < a_2 \}, \quad \Omega^\varepsilon_\beta = \omega_\beta \times \varepsilon, \varepsilon, \]
\[ O = \{ (x_1, x_2, x_3) \in \mathbb{R}^3; -a_1 < x_1 < \beta, |x_2| < a_2, -a_3 < x_3 < b_3 \}, \quad O^\varepsilon_0 = O \cup \Omega^\varepsilon_\beta, \quad S^\varepsilon = O \cup \Omega^\varepsilon, \]

and we denote by $x^\varepsilon = (x^\varepsilon_i)$ a generic point in the set $S^\varepsilon$ and by $\partial / \partial x^\varepsilon_i$ the partial derivative.
Remark. — Since $\varepsilon$ is to be understood as a dimensionless parameter, the thickness of the thin structure should be written as $2\varepsilon h$, for some fixed length $h>0$. We assume here that $h=1$, thus saving another notation. □

The set $S^\varepsilon$ is the reference configuration of a linearly elastic structure comprising two parts, glued together along their common boundary: The set $\Omega^\varepsilon_h$ corresponds to an elastic body whose Lamé constants $\lambda^\varepsilon>0$ and $\mu^\varepsilon>0$ are assumed to be independent of $\varepsilon$; the set $\Omega^\varepsilon$ corresponds to an elastic body whose Lamé constants $\lambda^\varepsilon$, $\mu^\varepsilon$ are assumed to be of the form

\begin{equation}
\lambda^\varepsilon = \varepsilon^{-3} \lambda, \quad \mu^\varepsilon = \varepsilon^{-3} \mu,
\end{equation}

where $\lambda>0$ and $\mu>0$ are two constants independent of $\varepsilon$. The unknown is the displacement vector field $u^\varepsilon=(u^\varepsilon)$: $S^\varepsilon \rightarrow \mathbb{R}^3$, which is assumed to satisfy a boundary condition of place $u^\varepsilon=0$ on $\Gamma_0$. In linearized elasticity, $u^\varepsilon$ is solution of the following variational equations:

\begin{equation}
\int_{\Omega^\varepsilon_h} \left\{ \lambda^\varepsilon e_{pp}(u^\varepsilon) e_{qq}(v^\varepsilon) + 2\mu^\varepsilon e_{ij}(u^\varepsilon) e_{ij}(v^\varepsilon) \right\} dx^3 - \int_{\partial \Omega^\varepsilon_h} f^\varepsilon \cdot v^\varepsilon dx^2
\end{equation}

\begin{equation}
+ \int_{\Omega^\varepsilon} \left\{ \lambda^\varepsilon e_{pp}(u^\varepsilon) e_{qq}(v^\varepsilon) + 2\mu^\varepsilon e_{ij}(u^\varepsilon) e_{ij}(v^\varepsilon) \right\} dx^3 - \int_{\partial \Omega^\varepsilon} f^\varepsilon \cdot v^\varepsilon dx^2 = 0 \quad \text{for all } v \in V^\varepsilon,
\end{equation}

where the space $V^\varepsilon$ is defined by

\begin{equation}
V^\varepsilon = \{ v^\varepsilon=(v^\varepsilon) \in H^1(S^\varepsilon); v^\varepsilon=0 \text{ on } \Gamma_0 \},
\end{equation}
where \( e_{ij}^{v} = 1/2(\partial_i^{v} v_j^{v} + \partial_j^{v} v_i^{v}) \) denote the components of the linearized strain tensor \( e^{v} \), and where the vector field \( f^{v} \in L^2(S^v) \) represents the given applied body force density.

Equations (2.2) form a coercive variational problem (by Korn’s inequality in the space \( H^1(S^v) \), combined with the assumed boundary condition of place), which has thus one and only one solution \( u^v \). This solution can be also characterized as the unique solution of the minimization problem: Find

\[
(2.4) \quad u^v \in V^v \quad \text{such that} \quad J^v(u^v) = \inf_{v \in V^v} J^v(v),
\]

with

\[
(2.5) \quad J^v(v) = \frac{1}{2} \int_{\partial \Omega^v} \tilde{A} e^{v} : e^{v} dx^v - \int_{\partial \Omega^v} f^v : v^v dx^v + \frac{1}{2} \int_{\Omega^v} \bar{A} e^{v} : e^{v} dx^v - \int_{\Omega^v} f^v : v^v dx^v,
\]

where we have let

\[
\tilde{A} = (\lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk})), \quad A^v = \varepsilon^{-3} A, \quad A = (\lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk})).
\]

\[
(B e)_{ij} = b_{ijkl} e_{kl} \quad \text{if} \quad B = (b_{ijkl}), \quad e = (e_{ij}), \quad a : b = a_{ij} b_{ij} \quad \text{if} \quad a = (a_{ij}), \quad b = (b_{ij}), \quad f : v = f_{i} v_{i} \quad \text{if} \quad f = (f_{i}), \quad v = (v_{i}).
\]

The function \( u^v \) is also, at least formally, solution of a classical “transmission problem” of three-dimensional elasticity, which takes here the following form:

\[
(2.6) \quad -\text{div}^v \{ \tilde{A} e(u^v) \} = f^v \quad \text{in} \ \Omega^v,
\]

\[
(2.7) \quad -\text{div}^v \{ A^v e(u^v) \} = f^v \quad \text{in} \ \Omega^v,
\]

\[
(2.8) \quad u^v = 0 \quad \text{on} \ \Gamma_b,
\]

\[
(2.9) \quad \tilde{A} e(u^v) \tilde{n} = 0 \quad \text{on} \ \partial \Omega^v \setminus \partial \Omega^v_b,
\]

\[
(2.10) \quad A^v e(u^v) n^v = 0 \quad \text{on} \ \partial \Omega^v \setminus \partial \Omega^v_b,
\]

\[
(2.11) \quad (u^v|_{\partial \Omega^v})|_{\partial \Omega^v_b \cap \partial \Omega^v} = (u^v|_{\partial \Omega^v})|_{\partial \Omega^v_b \cap \partial \Omega^v}.
\]

\[
(2.12) \quad \tilde{A} e(u^v) \tilde{n}^v + A^v e(u^v) n^v = 0 \quad \text{on} \ \partial \Omega^v \cap \partial \Omega^v_b,
\]

where

\[
(\text{div}^v a^v) = \partial_j^v a_{ij} \quad \text{if} \quad a^v = (a_{ij}).
\]
\( \tilde{n} \) and \( n \) denote the unit outer normal vectors along the boundaries of the sets \( \Omega_0 \) and \( \Omega, \) respectively, and \( w|_B \) denotes the restriction of a function \( w \) to a set \( B. \)

Relations (2.11) and (2.12), which formally expresses the continuity of the displacement vectors and of the stress vectors along the common portion of the two boundaries, are called transmission conditions; details about such transmission problems are found in Dautray & Lions [1984], p. 1245. Relation (2.11) shows that we are modeling a situation where the inserted portion of the plate is glued to the three-dimensional structure; we are thus excluding here situations where the inserted portion could slide along, or part away from, the three-dimensional structure.

3. Equivalent formulation of the three-dimensional problem

over two open sets independent of \( \varepsilon \)

With the sets \( \Omega, \) which overlap over the "inserted" part \( \Omega_0 \) of the "thin" part \( \Omega, \) we associate two disjoint sets \( \bar{\Omega} \) and \( \{ \Omega \}^- \), as follows: First, as in the case of a single plate (cf. Ciarlet & Destuynder [1979a]), we let \( \Omega = \omega \times [-1, 1[ \); with each point \( x' = (x_1', x_2', x_3') \in \bar{\Omega}, \) we associate the point \( x = (x_1', x_2', \varepsilon^{-1} x_3') \in \bar{\Omega} \) (cf. Fig. 4); finally we have the restriction (still denoted) \( u = (u): \bar{\Omega} \to \mathbb{R}^3 \) of the unknown \( u \) to the set \( \bar{\Omega} \), we associate the function \( u(x) = (u_x(x)): \{ \Omega \}^- \to \mathbb{R}^3 \) defined by the scalings

\[ u_x(x') = \varepsilon^2 u_x(x) \quad \text{for all} \quad x' \in \bar{\Omega} ; \]

\[ u_x(x') = \varepsilon u_x(x) \quad \text{for all} \quad x' \in \Omega. \]

Secondly, we define the translated set \( \bar{\Omega} = \Omega + t, \) the vector \( t \) being such that \( \{ \Omega \}^- \cap \bar{\Omega} = \emptyset; \) with each point \( x \in \bar{\Omega}, \) we associate the translated point \( \tilde{x} = (x + t) \in \{ \Omega \}^- \) (cf. Fig. 4); finally, with the restriction (still denoted) \( u = (u): \bar{\Omega} \to \mathbb{R}^3 \) of the unknown \( u \) to the set \( \bar{\Omega}, \) we associate the function \( \tilde{u}(x) = (\tilde{u}_x(x)): \{ \tilde{\Omega} \}^- \to \mathbb{R}^3 \) defined by the scalings

\[ u_x(x') = \varepsilon \tilde{u}_x(x) \quad \text{for all} \quad x' \in \tilde{\Omega} \]

The function \( u \in V ' \), where \( V ' \) is the space defined in (2.3), is thus mapped through relations (3.1)-(3.3) into a pair \((\tilde{u} \in V, u \in H^1)\) which belongs to the space \( H^1(\bar{\Omega}) \times H^1(\Omega), \) which satisfies the boundary condition \( u = 0 \) on \( \Gamma_0 = \{ x_0 \times ] - 1, 1[ \), and which satisfies the function conditions for the three-dimensional problem:

\[ \tilde{u}_x(x) = \varepsilon u_x(x), \]

\[ \tilde{u}_3(x) = u_3(x), \]

at all corresponding points \( x \in \bar{\Omega} = \Omega_0 + t \) and \( x \in \Omega = \omega \times ] - 1, 1[, \) i.e., that correspond to the same point \( x \in \Omega_0 \) (Fig. 4).
Fig. 4. — The sets $\bar{\Omega}$ and $\Omega$, which are respectively occupied by the “thin” part and the “three-dimensional” part of the elastic structure, are mapped into two disjoint sets $\tilde{\Omega}$ and $[\tilde{\Omega}]^\epsilon$. The “inserted” part $\tilde{\Omega}_1^\epsilon$ of the thin part $\bar{\Omega}$ is thus mapped twice, once onto $\tilde{\Omega}_2 = \bar{\Omega}$ and once onto $[\tilde{\Omega}]^\epsilon = [\bar{\Omega}]^\epsilon$.

Finally, we assume that there exist functions $f_1 \in L^2(\Omega)$ and $f_2 \in L^2(\bar{\Omega})$ independent of $\epsilon$ such that

\begin{align}
(3.6) & \quad f_2^\epsilon(x^\epsilon) = \epsilon^{-1} f_2(x) \quad \text{for all } x^\epsilon \in \Omega^\epsilon; \\
(3.7) & \quad f_3^\epsilon(x^\epsilon) = f_3(x) \quad \text{for all } x^\epsilon \in \Omega^\epsilon; \\
(3.8) & \quad f_1^\epsilon(x^\epsilon) = \epsilon f_1(x) \quad \text{for all } x^\epsilon \in \Omega_0^\epsilon.
\end{align}

Using the assumptions (2.1) and (3.6)-(3.8) made on the data, and the scalings (3.1)-(3.3), we can thus re-formulate the variational problem (2.2) in the following equivalent
form: the pair \((\tilde{u}(\varepsilon), u(\varepsilon))\) constructed in (3.1)-(3.3) belongs to the space

\[(3.9) \quad V(\varepsilon) = \{ (\tilde{v}, v) \in H^1(\tilde{\Omega}) \times H^1(\Omega), v = 0 \text{ on } \Gamma_0, \]
\[\tilde{v}_{\varepsilon}(\tilde{x}) = \varepsilon \epsilon_{x_1}(x), \quad \tilde{v}_{\varepsilon}(\tilde{x}) = \varepsilon_{x_3}(x) \text{ at all corresponding points } \tilde{x} \in \tilde{\Omega}_\varepsilon \text{ and } x \in \Omega_\varepsilon \},\]

and it satisfies the variational equations:

\[(3.10) \quad \int_\Omega \chi(\tilde{\Omega}_\varepsilon) \{ \lambda \varepsilon_{x_3}(u(\varepsilon)) e_{x_3}(\tilde{v}) + 2 \mu \varepsilon e_{x_3}(u(\varepsilon)) e_{x_3}(\tilde{v}) \} d\tilde{x} - \int_\tilde{\Omega} \chi(\tilde{\Omega}_\varepsilon) \tilde{v} d\tilde{x} \]
\[+ \int_\Omega \left\{ \lambda e_{x_3}(u(\varepsilon)) e_{x_3}(v) + 2 \mu e_{x_3}(u(\varepsilon)) e_{x_3}(v) \right\} dx - \int_\Omega f(v) dx \]
\[+ \frac{1}{\varepsilon^2} \int_\Omega \left\{ \lambda e_{x_1}(u(\varepsilon)) e_{x_3}(v) + e_{x_3}(u(\varepsilon)) e_{x_3}(v) \right\} dx \]
\[+ \frac{1}{\varepsilon^2} \int_\Omega \left( \lambda + 2 \mu \right) e_{x_3}(u(\varepsilon)) e_{x_3}(v) dx = 0 \quad \text{for all } (\tilde{v}, v) \in V(\varepsilon),\]

where \(\chi(A)\) denotes in general the characteristic function of a set \(A\), and \(\tilde{\Omega}_\varepsilon = \Omega_\varepsilon + \varepsilon t.\)

As in (2.4)-(2.5), the pair \((\tilde{u}(\varepsilon), u(\varepsilon))\) can be also characterized as the unique solution of a minimization problem, viz.: Find

\[(3.11) \quad (\tilde{u}(\varepsilon), u(\varepsilon)) \in V(\varepsilon) \quad \text{such that } J(\varepsilon)(\tilde{u}(\varepsilon), u(\varepsilon)) = \inf_{(\tilde{v}, v) \in V(\varepsilon) \times V(\varepsilon)} J(\varepsilon)(\tilde{v}, v),\]

where (the notations are as in (2.5)):

\[(3.12) \quad J(\varepsilon)(\tilde{v}, v) = \frac{1}{2} \int_\tilde{\Omega} \chi(\tilde{\Omega}_\varepsilon) \lambda \varepsilon_{x_3}(\tilde{v}) e_{x_3}(v) d\tilde{x} - \int_\tilde{\Omega} \chi(\tilde{\Omega}_\varepsilon) \tilde{v} d\tilde{x} \]
\[+ \frac{1}{2} \int_\Omega \left\{ 2 \mu e_{x_3}(v) e_{x_3}(v) + \lambda e_{x_3}(v) e_{x_3}(v) \right\} dx - \int_\Omega f(v) dx \]
\[+ \frac{1}{\varepsilon^2} \int_\Omega \left\{ 2 \mu e_{x_1}(v) e_{x_3}(v) + \lambda e_{x_3}(v) e_{x_3}(v) \right\} dx \]
\[+ \frac{1}{\varepsilon^2} \int_\Omega \left( \lambda + 2 \mu \right) e_{x_3}(v) e_{x_3}(v) dx.\]

4. Convergence of \((\tilde{u}(\varepsilon), u(\varepsilon))\) as \(\varepsilon \to 0\)

We use the following notation: the norms of the space \(L^2(A)\) and of the Sobolev spaces \(H^m(A)\), \(m \geq 1\), where \(A\) is an open subset in \(\mathbb{R}^n\), are respectively denoted \(| \cdot |_{L^2(A)}\)

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and \( \| \cdot \|_{m,A} \); the same notations are also used for the norms of the spaces \( L^2(A) \) and \( H^m(A) \), whose elements are vector-valued functions. Strong and weak convergences are respectively denoted \( \to \) and \( \rightharpoonup \).

We now show that the family \((\tilde{u}(e), u(e))\) converges to a limit \((\tilde{u}, u)\) in the space \( \mathbf{H}^1(\Omega) \times \mathbf{H}^1(\Omega) \) as \( e \to 0 \); in addition, we identify a "limit" variational problem that \((\tilde{u}, u)\) solves. In (4.3) and subsequently, \( \widetilde{\omega} \) denotes the translated set \( (\omega, e) \); \( W_1 \) denotes the trace of a function \( w \) on the set \( A \) in the sense of Sobolev spaces (for instance, the trace \( \nabla \eta_{31} \) is to be understood as a function in the space \( \mathbf{H}^{1/2}(\partial \Omega) \), etc.); the equality \( \nabla \eta_{31} = \nabla \eta_{31} \) is to be understood as holding up to a translation by the vector \( t \); finally, \( \partial_1 \), \( \partial_2 \), \( \partial_3 \) denotes the outer normal derivative operator along \( \partial \omega \).

**Theorem 1.** As \( e \to 0 \), the family \((\tilde{u}(e), u(e))_{e>0}\) converges strongly in the space \( \mathbf{H}^1(\Omega) \times \mathbf{H}^1(\Omega) \) towards an element \((\tilde{u}, u)\) that satisfies the following relations:

(a) The limit \( u \in \mathbf{H}^1(\Omega) \) vanishes on \( \Gamma_0 = \gamma_0 \times [-1, 1] \) and is a Kirchhoff-Love field in \( \Omega \), i.e., there exist functions \( \xi_0 \in \mathbf{H}^1(\omega) \) and \( \xi_3 \in \mathbf{H}^2(\omega) \) satisfying \( \xi_0 = \partial_1 \xi_3 = 0 \) on \( \gamma_0 \) such that

\[
\begin{align*}
(4.1) & \quad u_3(x_1, x_2, x_3) = \xi_0(x_1, x_2) \quad \text{for all } (x_1, x_2, x_3) \in \Omega, \\
(4.2) & \quad u_4(x_1, x_2, x_3) = \xi_3(x_1, x_2) - x_3 \partial_3 \xi_3(x_1, x_2) \quad \text{for all } (x_1, x_2, x_3) \in \Omega.
\end{align*}
\]

(b) The pair \((\tilde{u}, \xi_3)\) belongs to the space

\[
(4.3) \quad [\mathbf{H}^1(\Omega) \times \mathbf{H}^2(\omega)]_0 = \{ (\tilde{\eta}, \eta_{3}) \in \mathbf{H}^1(\Omega) \times \mathbf{H}^2(\omega); \eta_{3|\gamma_0} = \partial_1 \eta_{3|\gamma_0} = 0, \tilde{\omega}_{3|\omega} = \eta_{3|\omega}, \tilde{\omega}_{3|\omega} = 0 \},
\]

and it solves the variational equations:

\[
(4.4) \quad \int_{\Omega} \left\{ \tilde{\eta}_e_{pp}(\tilde{u}) e_{\omega e}(\tilde{v}) + 2 \tilde{\mu} e_{ij}(\tilde{u}) e_{ij}(\tilde{v}) \right\} d\tilde{x} - \int_{\Omega} f e_{ij} d\tilde{x} + \int_{\Omega} m_{ab}(\xi_3) \partial_{ab} \eta_{3} d\omega - \left( \int_{\Omega} \left\{ \int_{-1}^{1} f_3 d\omega \right\} \eta_{3} d\omega + \int_{\Omega} \left\{ \int_{-1}^{1} x_3 f_3 d\omega \right\} \partial_3 \eta_{3} d\omega \right) = 0
\]

for all \((\tilde{\eta}, \eta_{3}) \in [\mathbf{H}^1(\Omega) \times \mathbf{H}^2(\omega)]_0\), where

\[
(4.5) \quad m_{ab}(\xi_3) \overset{\text{def}}{=} \frac{4 \mu + \lambda}{3} \left\{ \partial_{ab} \xi_3 + \frac{\lambda}{\lambda + 2 \mu} \Delta \xi_3 \partial_{ab} \right\}
\]

(c) The pair \((\xi_1, \xi_2)\) belongs to the space

\[
(4.6) \quad \mathbf{H}(\omega) = \{ (\eta_1, \eta_2) \in \mathbf{H}^1(\omega); \eta_2 = 0 \text{ on } \gamma_0 \},
\]

and it solves the variational equations:

\[
(4.7) \quad \int_{\Omega} \eta_{3} \partial_3 \eta_{3} d\omega - \int_{\Omega} \left\{ \int_{-1}^{1} f_3 d\omega \right\} \eta_{3} d\omega = 0 \quad \text{for all } (\eta_1, \eta_2) \in \mathbf{H}(\omega),
\]

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where

\[ n_{\beta}(\zeta_1, \zeta_2) \overset{\text{def}}{=} 4 \mu \left\{ \frac{\partial_x \zeta_\beta + \partial_y \zeta_\beta}{2} + \frac{\lambda}{\lambda + 2 \mu} \partial_z \zeta_\gamma \delta_{\gamma \beta} \right\}. \]

The proof of Theorem 1 is long and technical, and, for this reason, is broken into a series of ten lemmas. We first show (Lemma 1) that the semi-norm

\[ |(\vec{v}, v)| \overset{\text{def}}{=} \left\{ |e(\vec{v})|_{\alpha, \delta}^2 + |e(v)|_{\beta, \alpha}^2 \right\}^{1/2}, \]

where

\[ e(\vec{v}) = (e_{ij}(\vec{v})), \quad e_{ij}(\vec{v}) = -\frac{1}{2} (\partial_i \vec{v}_j + \partial_j \vec{v}_i), \quad \partial_i = \frac{\partial}{\partial x_i}, \]

\[ e(v) = (e_{ij}(v)), \quad e_{ij}(v) = -\frac{1}{2} (\partial_i v_j + \partial_j v_i), \quad \partial_i = \frac{\partial}{\partial x_i}, \]

is a norm over the space \( V(\varepsilon) \) of (3.9), which is in addition uniformly (which respect to \( \varepsilon \)) equivalent to the norm

\[ \|(\vec{v}, v)\| = \left\{ \|\vec{v}\|_{\alpha, \delta}^2 + \|v\|_{\beta, \alpha}^2 \right\}^{1/2}. \]

This property will in turn be used for showing that the family \( ((\vec{u}(\varepsilon), u(\varepsilon)))_{\varepsilon > 0} \) is bounded in the space \( H^1(\Omega) \times H^1(\Omega) \) (Lemma 2).

**Lemma 1.** There exists a constant \( C \) independent of \( \varepsilon \) such that

\[ \|(\vec{v}, v)\| \leq C |(\vec{v}, v)| \quad \text{for all } (\vec{v}, v) \in V(\varepsilon). \]

**Proof.** With an arbitrary function \((\vec{v}, v) \in V(\varepsilon)\), we associate the function \( v^* \in V^* \) defined by the same formulas as in (3.1)-(3.3), viz.,

\[ e_3^* (x^*) = \varepsilon e_3 (x) \quad \text{and} \quad \varepsilon x_3^* (x^*) = \varepsilon x_3 (x) \quad \text{for all } x^* \in \Omega^*, \]

\[ e_j^* (x^*) = \varepsilon e_j (x) \quad \text{for all } x^* \in \widehat{\Omega}. \]

In this fashion, the components of the tensors \( e(v^*) \) and \( e(v) \) are related by:

\[ e_{ab} (v^*) (x^*) = \varepsilon^2 e_{ab} (v) (x) \]
\[ e_{33} (v^*) (x^*) = \varepsilon e_{33} (v) (x) \]
\[ e_{ij} (v^*) (x^*) = \varepsilon e_{ij} (v) (x) \]

for all \( x^* \in \Omega^* \).

Hence \( |(\vec{v}, v)| = 0 \) implies that \( e_{ij}(v^*) = 0 \) on \( S^* \). Since \( v \to |e(v)|_{0, \infty} \) is a norm on the space \( V^* \) (by Korn's inequality and the boundary condition on \( \Gamma_0^* \)), we conclude that the mapping \( (\vec{v}, v) \to |(\vec{v}, v)| \) is a norm on the space \( V(\varepsilon) \).
If inequality (4.11) is false, there exist sequences \((\alpha_k, \beta_k)\) and \((\mathbf{v}_k^\alpha, \mathbf{v}_k^\beta) \in V(\alpha_k)\) such that:

\[
\alpha_k \to 0, \quad \beta_k \to 0.
\]

(4.14)

\[
(\mathbf{v}_k^\alpha, \mathbf{v}_k^\beta) \to 1 \text{ for all } k,
\]

(4.15)

\[
\| \mathbf{v}_k^\alpha, \mathbf{v}_k^\beta \| \to 0.
\]

(4.16)

Since \(|e(\mathbf{v}_k^\alpha)|_{0, \Omega} \to 0\) by (4.16), and since the functions \(\mathbf{v}_k^\alpha\) vanish on \(\Gamma_0\), we have

\[
\| \mathbf{v}_k^\alpha \|_{1, \Omega} \to 0,
\]

(4.17)

by Korn's inequality, and thus

\[
\mathbf{v}_k^\alpha \big|_{\partial \Omega} \to 0 \text{ in } H^{1/2}(\partial \Omega),
\]

(4.18)

on the one hand. The relation \(|e(\mathbf{v}_k^\beta)|_{0, \Omega} \to 0\) implies on the other hand (e.g., by invoking the same kind of arguments as in Duvaut & Lions [1972], Th. 3.4, p. 117) that there exist vectors \(\mathbf{a}_k^\beta, \mathbf{b}_k^\beta \in \mathbb{R}^3\) and functions \(\mathbf{w}_k^\beta \in H^1(\Omega)\) such that

\[
\mathbf{v}_k^\beta(\mathbf{x}) = \mathbf{a}_k^\beta + \mathbf{b}_k^\beta \cdot \mathbf{n} + \mathbf{w}_k^\beta(\mathbf{x}) \text{ for almost all } \mathbf{x} \in \Omega,
\]

(4.19)

\[
\| \mathbf{w}_k^\beta \|_{1, \Omega} \to 0.
\]

(4.20)

By (4.12)-(4.13),

\[
\mathbf{v}_k^\beta(\mathbf{x}) = \epsilon_k^k \mathbf{u}_k^\beta(\mathbf{x}) \quad \text{ and } \quad \mathbf{v}_k^\beta(\mathbf{x}) = \epsilon_k^k \mathbf{z}_k^\beta(\mathbf{x})
\]

(4.21)

at all corresponding points \(\mathbf{x} \in \Omega_k^\beta\) and \(\mathbf{x} \in \Omega_k^\beta\). Combining (4.14), (4.18), and (4.21), we thus conclude that

\[
\mathbf{v}_k^\beta \big|_{\partial \Omega} \to 0 \text{ in } H^{1/2}(\partial \Omega),
\]

(4.22)

Since the functions \((\mathbf{a}_k^\beta + \mathbf{b}_k^\beta \cdot \mathbf{n} \mathbf{x})\) all belong to the same finite-dimensional vector space and are bounded independently of \(k\), there exists a subsequence, still indexed by \(k\) for convenience, that converges to a function of the form \((\mathbf{a} + \mathbf{b} \cdot \mathbf{n} \mathbf{x})\). Then (4.19), (4.20), and (4.22), together imply that \(\mathbf{a} + \mathbf{b} \cdot \mathbf{n} \mathbf{x} = 0\) on \(\partial \Omega\), and thus \(\mathbf{a} = \mathbf{b} = 0\). Hence we have proved that

\[
\| \mathbf{v}_k^\beta \|_{1, \Omega} \to 0,
\]

(4.17)

which, together with (4.17), contradicts (4.15). Therefore an inequality of the form (4.11) holds.

Lemma 2. — The family \((\mathbf{u}(\epsilon), \mathbf{u}(\epsilon))_{\epsilon > 0}\) is bounded in the space \(H^1(\Omega) \times H^1(\Omega)\).

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Thus there exist a subsequence, still denoted $((\tilde{u}(\varepsilon), u(\varepsilon)))_{\varepsilon > 0}$ for notational convenience, and a pair $(\tilde{u}, u) \in H^1(\Omega) \times H^1(\Omega)$ such that

\begin{align*}
(4.23) & \quad \tilde{u}(\varepsilon) \rightharpoonup \tilde{u} \text{ in } H^1(\Omega) \text{ as } \varepsilon \to 0, \\
(4.24) & \quad u(\varepsilon) \rightharpoonup u \text{ in } H^1(\Omega) \text{ as } \varepsilon \to 0, \quad \text{and } u = 0 \text{ on } \Gamma_0.
\end{align*}

\textbf{Proof.} — The trick consists in splitting the integral over the set $\Omega_\beta^\varepsilon$ appearing in the quadratic part of the functional $J^\varepsilon$ of (2.5) into two equal (for definiteness) parts; one part is then mapped as an integral over the set $\bar{\Omega}_\beta^\varepsilon$, and the other is mapped as an integral over the set $\Omega_\beta$. In this fashion, we obtain the following equivalent expression of the functional $J(\varepsilon)$ of (3.12) (the notations are as in (2.5))

\begin{align*}
(4.25) & \quad J(\varepsilon)(\bar{\nu}, \bar{v}) = \frac{1}{2} \int_{\bar{\Omega}} \chi(\bar{\Omega}_\beta^\varepsilon) \bar{A} e(\bar{\nu}) : e(\bar{v}) \, d\bar{x} - \int_{\bar{\Omega}} \chi(\bar{\Omega}_\beta^\varepsilon) \bar{\nu} \cdot \bar{v} \, d\bar{x} \\
& \quad + \frac{1}{4 \varepsilon} \int_{\bar{\Omega}} \chi(\bar{\Omega}_\beta^\varepsilon) \bar{A} e(\bar{\nu}) : e(\bar{v}) \, d\bar{x} \\
& \quad + \frac{1}{2} \int_{\bar{\Omega}} \left\{ \frac{1}{2} \chi(\Omega_\beta) + \chi(\Omega - \Omega_\beta) \right\} \bar{K}(v) : K(v) \, dx - \int_{\Omega} f \cdot v \, dx,
\end{align*}

where the tensor $K(v) = (K_{ij}(v))$ is defined by

\begin{align*}
(4.26) & \quad K_{\alpha\beta}(v) = e_{\alpha\beta}(v), \quad K_{33}(v) = \frac{1}{\varepsilon} e_{33}(v), \quad K_{33}(v) = \frac{1}{\varepsilon^2} e_{33}(v).
\end{align*}

Since there exists a constant $c = c(\lambda, \mu, \tilde{\lambda}, \tilde{\mu})$ such that

\begin{align*}
(4.27) & \quad c > 0 \quad \text{and} \quad \bar{A} e : e \geq c e : e \quad \text{and} \quad A e : e \geq c e : e,
\end{align*}

the inequality $J(\varepsilon)(\tilde{u}(\varepsilon), u(\varepsilon)) \leq 0$ (which follows from the minimization property (3.11)), together with (4.25)-(4.27), implies that

\begin{align*}
& \quad c \int_{\bar{\Omega}} \left\{ \frac{1}{2} \chi(\bar{\Omega}_\beta^\varepsilon) + \frac{1}{2 \varepsilon^2} \chi(\bar{\Omega}_\beta^\varepsilon) \right\} e(\tilde{u}(\varepsilon)) : e(\tilde{u}(\varepsilon)) \, d\bar{x} \\
& \quad + c \int_{\Omega} \left\{ \frac{1}{2} \chi(\Omega_\beta) + \chi(\Omega - \Omega_\beta) \right\} K(\varepsilon) : K(\varepsilon) \, dx \\
& \quad \leq 2 \int_{\bar{\Omega}} \bar{\nu} \cdot \tilde{u}(\varepsilon) \, d\bar{x} + 2 \int_{\Omega} f \cdot u(\varepsilon) \, dx,
\end{align*}

where $K(\varepsilon) = K(u(\varepsilon))$.

Without loss of generality, we may restrict ourselves to values of $\varepsilon$ that are $\leq 1$, in which case the last inequality implies that

\begin{align*}
(4.28) & \quad |e(\tilde{u}(\varepsilon))|_{0, \tilde{\alpha}} + |e(u(\varepsilon))|_{0, \alpha} \leq |e(\tilde{u}(\varepsilon))|_{0, \tilde{\alpha}} + |K(\varepsilon)|_{0, \alpha} \\
& \quad \leq 4 c^{-1} \{|f|_{0, \alpha} + |\tilde{u}(\varepsilon)|_{0, \tilde{\alpha}} + |u(\varepsilon)|_{0, \alpha} -
\end{align*}
Since there exists a constant $C$ independent of $\varepsilon$ such that
\[
\| \tilde{u}(\varepsilon) \|_{1, \tilde{\alpha}} + \| u(\varepsilon) \|_{1, \alpha} \leq C \sqrt{\varepsilon} \left\{ \| e(\tilde{u}(\varepsilon)) \|_{2, \tilde{\alpha}} + \| e(u(\varepsilon)) \|_{2, \alpha} \right\}
\]
by Lemma 1, we conclude from inequation (4.28) that the family $(\tilde{u}(\varepsilon), u(\varepsilon))_{\varepsilon > 0}$ is bounded independently of $\varepsilon$ in the space $H^1(\tilde{\Omega}) \times H^1(\Omega)$. The other conclusions of Lemma 2 then follow from this property.

As in the case of "single" plates (see Destuynder [1981, 1986], or Ciarlet & Kesavan [1981]), we next show that the weak limit $u \in H^1(\Omega)$ found in (4.24) is a Kirchhoff-Love vector field over the set $\Omega$, in the sense that it belongs to the vector space $V_{KL}(\Omega)$ defined in the next lemma. Notice that the space $V_{KL}(\Omega)$ is closed and strictly contained in the space $\{ u \in H^1(\Omega); \nabla = 0 \text{ on } \Gamma_0 \}$.

**Lemma 3.** — The function $u$ belongs to the space
\[
V_{KL}(\Omega) = \{ v \in H^1(\Omega); e_{i,3}(v) = 0 \text{ in } \Omega, v = 0 \text{ on } \Gamma_0 \},
\]
which can be also defined as
\[
V_{KL}(\Omega) = \{ v = (v_1, v_2) \in H^1(\Omega); v_2 = \eta_2 - x_3 \partial_{\eta_3} \eta_3, v_3 = \eta_3, \}
\]
with $\eta_2 \in H^1(\omega), \eta_3 \in H^2(\omega), \eta_{1,1} = 0, \eta_3 = 0$.

**Proof.** — The second inequality in (4.28) shows that the sequence $(K(e))_{\varepsilon > 0}$ is bounded in the space $L^2(\Omega)$. Hence (cf. (4.26)) there exists in particular a constant $C$ independent of $\varepsilon$ such that
\[
| e_{i,3}(u) |_{0, \Omega} \leq C \varepsilon, \quad | e_{33}(u) |_{0, \Omega} \leq C \varepsilon^2.
\]
The weak lower semi-continuity of the norm implies that
\[
| e_{i,3}(u) |_{0, \Omega} \leq \liminf_{\varepsilon \to 0} | e_{i,3}(u(\varepsilon)) |_{0, \Omega} = 0;
\]
consequently, $u$ belongs to the space $V_{KL}(\Omega)$ defined in (4.29). The equivalence between the definitions (4.29) and (4.30) is established as in Ciarlet & Destuynder [1979a].

We next identify (cf. (4.32)) the function conditions that the pair $(\tilde{u}, u)$ must satisfy. Note that, in (4.32), the equality $u_3 |_{\tilde{\alpha}} = u_3 |_{\alpha}$ is to be understood up to a translation by the vector $t$.

**Lemma 4.** — The weak limit $(\tilde{u}, u)$ satisfies
\[
\tilde{u}_3 |_{\tilde{\alpha}} = 0 \quad \text{and} \quad u_3 |_{\alpha} = u_3 |_{\alpha},
\]

**Proof.** — By definition of the space $V(e)$ (cf. (3.9)),
\[
\tilde{u}_n(x) = e u_4(\varepsilon)(x) \quad \text{and} \quad \tilde{u}_3(x) = u_3(x)
\]

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at all corresponding points \( \bar{x} \in \bar{\Omega}_p \) and \( x \in \Omega_p \). Hence

\[
(4.33) \quad \tilde{u}_3(\varepsilon)|_{\partial \Omega} = \varepsilon u_3(\varepsilon)|_{\partial \Omega} \quad \text{and} \quad \tilde{u}_3(\varepsilon)|_{\partial \Omega} = u_3(\varepsilon)|_{\partial \Omega} \quad \text{for each } \varepsilon > 0.
\]

Since

\[
(4.34) \quad \tilde{u}(\varepsilon)|_{\partial \Omega} \rightharpoonup u|_{\partial \Omega} \text{ in } H^{1/2}(\partial \Omega_p) \quad \text{and} \quad u(\varepsilon)|_{\partial \Omega} \rightharpoonup u|_{\partial \Omega} \text{ in } H^{1/2}(\partial \Omega_p)
\]

(the trace operators from \( H^1(\bar{\Omega}) \) onto \( H^{1/2}(\partial \Omega_p) \) and from \( H^1(\Omega) \) onto \( H^{1/2}(\partial \Omega_p) \) are strongly continuous, and a linear mapping that is strongly continuous is also continuous with respect to the weak topologies; cf. e.g. Brezis [1983, p. 39]), the second equality in (4.32) follows from the second equality in (4.33) and from (4.34).

Since \((u_3(\varepsilon)|_{\partial \Omega})_{\varepsilon > 0}\) is a weakly convergent sequence, it is bounded; therefore the sequence \((\varepsilon u_3(\varepsilon)|_{\partial \Omega})_{\varepsilon > 0}\) converges strongly to 0 in the space \( H^{1/2}(\partial \Omega_p) \). This fact, combined with the first equality of (4.33) and with (4.34), implies that the first equality of (4.32) holds.

We next prove two independent technical results, which play a key rôle in the identification of the "limit" variational problem solved by the pair \((\tilde{u}, u)\).

**Lemma 5.** There exists a subsequence, still indexed by \( \varepsilon \) for notational convenience, of the sequence \((u(\varepsilon))_{\varepsilon > 0}\) such that

\[
(4.35) \quad K_{33}(\varepsilon) = \frac{1}{\varepsilon} e_{33}(u(\varepsilon)) \rightharpoonup 0 \text{ in } L^2(\Omega),
\]

\[
(4.36) \quad K_{33}(\varepsilon) = \frac{1}{\varepsilon^2} e_{33}(u(\varepsilon)) \rightharpoonup - \frac{\lambda}{(\lambda + 2\mu)} e_{pp}(u) \text{ in } L^2(\Omega).
\]

**Proof.** Since the sequence \((K(\varepsilon))_{\varepsilon > 0}\) is bounded in the space \( L^2(\Omega) \) (cf. (4.28)), there exists a subsequence, still denoted \((K(\varepsilon))_{\varepsilon > 0}\) for convenience, and \( K = (K_{ij}) \in L^2(\Omega) \) such that \( K(\varepsilon) \rightharpoonup K \) in \( L^2(\Omega) \) as \( \varepsilon \to 0 \). In particular,

\[
(4.37) \quad K_{33}(\varepsilon) = \frac{1}{\varepsilon} e_{33}(u(\varepsilon)) \rightharpoonup K_{33} \text{ in } L^2(\Omega),
\]

\[
(4.38) \quad K_{33}(\varepsilon) = \frac{1}{\varepsilon^2} e_{33}(u(\varepsilon)) \rightharpoonup K_{33} \text{ in } L^2(\Omega).
\]

As in Ciarlet & Destuynder [1979], we define the *scaled stresses*:

\[
(4.39) \quad \sigma_{33}(\varepsilon) \overset{\text{def}}{=} \lambda K_{pp}(\varepsilon) \delta_{33} + 2\mu K_{33}(\varepsilon),
\]

\[
(4.40) \quad \sigma_{31}(\varepsilon) \overset{\text{def}}{=} \varepsilon^{-1} \{ 2\mu K_{33}(\varepsilon) \},
\]

\[
(4.41) \quad \sigma_{33}(\varepsilon) \overset{\text{def}}{=} \varepsilon^{-2} \{ \lambda K_{pp}(\varepsilon) + 2\mu K_{33}(\varepsilon) \}.
\]
which are related to the actual stresses in $\Omega$:

\[(4.42)\]
\[
\sigma_{ij}^0 = \lambda^0 e_{pp}(u^0) \delta_{ij} + 2 \mu^0 e_{ij}(u^0) = (A^0 e(u^0))_{ij}
\]

through the scalings

\[(4.43)\]
\[
\sigma_{ap}^e = \varepsilon^{-1} \sigma_{ap}(e), \quad \sigma_{a3}^e = \sigma_{a3}(e), \quad \sigma_{33}^e = \varepsilon \sigma_{33}(e),
\]

themselves induced by the scalings on the components of the displacement field $u^e$. The scalings (4.43) have the property that the scaled stresses $\sigma_{ij}^e$ satisfy

\[(4.44)\]
\[
- \partial_j \sigma_{ij}(e) = f_i \quad \text{in} \quad \Omega^e,
\]

i.e., equations that are similar to the equations $- \partial_j \sigma_{ij} = f_i^0$ satisfied in $\Omega^0$ by the actual stresses $\sigma_{ij}^0$ (cf. 2.7).

The first two equations in (4.44) can be rewritten as

\[
\partial_3 \sigma_{a3}(e) = - \partial_p \sigma_{ap}(e) - f_p,
\]

which shows that $(\partial_3 \sigma_{a3}(e))_{k>0}$ is a bounded sequence in the space $L^2(-1,1; H^{-1}(\omega))$ (the sequence $(\sigma_{ap}(e))_{k>0}$ is bounded in the space $L^2(\Omega)$ since it weakly converges in that space, and the functions $f_p$ belong to the space $L^2(\Omega)$ by assumption). Since $\partial_3 K_{a3}(e) = (\varepsilon/2 \mu) \partial_3 \sigma_{a3}(e)$, it follows that

\[(4.45)\]
\[
\partial_3 K_{a3}(e) \to 0 \quad \text{in} \quad L^2(-1,1; H^{-1}(\omega)).
\]

Since $(K_{a3}(e))_{k>0}$ converges weakly in $L^2(\Omega)$ by (4.37), whence a fortiori in the space $L^2(-1,1; H^{-1}(\omega))$, we conclude that the sequence $(K_{a3}(e))_{k>0}$ converges strongly in the space $H^1(-1,1; H^{-1}(\omega))$ (details about such functional spaces may be found in Brezis [1973], Lions & Magenes [1968]).

We next state a result that will be used several times in the remainder of this proof (its proof may be found in Temam [1977, p. 9] for domains with smooth boundaries, and in Girault & Raviart [1986, p. 27] for domains with Lipschitz-continuous boundaries; the extension used here for cylindrical domains offers no difficulty): Let $U$ be an open subset of $R^3$ of the form $U = \Gamma \times [a, b]$, where $\Gamma$ is a bounded open connected subset of $R^2$ with a Lipschitz-continuous boundary. Then there exists a continuous linear mapping

\[(4.46)\]
\[
\gamma: H(\text{div}; U) = \{ T \in L^2(U); \text{div} T \in L^2(U) \} \to H^{-1/2}(\Gamma \times \{ a \})
\]

such that, for a smooth enough tensor field $T$,

\[(4.47)\]
\[
\gamma T = T e_3 \quad \text{on} \quad \Gamma \times \{ a \}.
\]

Let us denote by $\Gamma_+$ the upper face of the set $\Omega$, and by $\Gamma^-_p$, the upper face of the set $\Omega_p$. Then the boundary condition (2.10) implies that

\[(4.48)\]
\[
K_{a3}(e) = 0 \quad \text{in} \quad H^{-1/2}(\Gamma_+ - \Gamma^-_p),
\]
and the transmission condition (2.12) implies that

$$K_{a 3}(e) = \frac{\mu}{\varepsilon^2} e_{a 3} (\tilde{u}(e)) \quad \text{in} \quad H^{-1/2} (\Gamma_{\beta^+}).$$

(4.49)

Note that both conditions make an essential use of the mapping defined in (4.46), and that the equality (4.49) is to be understood as holding the corresponding points $x \in \Gamma_{\beta^+}$ and $\tilde{x} \in \tilde{\Gamma}_{\beta^+}$, where $\tilde{\Gamma}_{\beta^+}$ is the upper face of the set $\tilde{\Omega}^\beta$ translated by the vector $t$.

Let $\tilde{\Gamma}^\beta = \Gamma_{\beta^+} \times \{ \varepsilon \}$, $e + b$, where $b$ is so chosen that $\tilde{\Omega}^\beta = \tilde{\Omega}_0^\beta$ for all the values of $\varepsilon$ that are considered. Because the sets $\tilde{\Omega}^\beta$ are translations of the set $\tilde{\Omega}_0^\beta$, the norm of the associated operators $\tilde{\gamma}': H(\text{div}; \tilde{\Gamma}^\beta) \to H^{-1/2} (\tilde{\Gamma}_{\beta^+})$, as defined in (4.46), is independent of $\varepsilon$; besides,

$$2 \mu e_{a 3} (\tilde{u}(e)) = (\tilde{\gamma}' (A e (\tilde{u}(e))))_\varepsilon.$$

By (4.28), the sequence $(e (\tilde{u}(e)))_{\varepsilon > 0}$ is bounded in the space $L^2 (\tilde{\Omega})$; hence the norms $|A e (\tilde{u}(e))|_0, \tilde{\gamma}^\beta$ are bounded independently of $\varepsilon$. Since

$$- \text{div} A e (\tilde{u}(e))|_{\tilde{\gamma}^\beta} = T|_{\tilde{\gamma}^\beta}$$

where $\text{div} a = (\tilde{\varepsilon}_j a_j)$, the norms $|\text{div} A e (\tilde{u}(e))|_0, \tilde{\gamma}^\beta$ are also bounded independently of $\varepsilon$. At this point, we note that

$$K_{a 3}(e) = \chi (\Gamma_+ - \Gamma_{\beta^+}) K_{a 3}(e) + \chi (\Gamma_{\beta^+}) K_{a 3}(e) \in H^{-1/2} (\Gamma_+),$$

(4.50)

since a distribution in $H^{-1/2} (\Gamma_+)$ whose support is contained in a segment is equal to 0. Therefore we conclude from (4.48)-(4.50) and from the uniform continuity of the operators $\gamma'$ that

$$K_{a 3}(e) \rightharpoonup 0 \quad \text{in} \quad H^{-1/2} (\Gamma_+) \subset H^{-1} (\Gamma_+).$$

(4.51)

It follows immediately from (4.45) and (4.51) that

$$K_{a 3}(e) \rightharpoonup 0 \quad \text{in} \quad H^1 (-1,1; H^{-1} (\omega)).$$

(4.52)

Since $K_{a 3}(e) \rightarrow K_{a 3}$ in $L^2 (\Omega) \subset L^2 (-1,1; H^{-1} (\omega))$ (cf. (4.37)) we thus infer that $K_{a 3} = 0$, and (4.35) is proved.

The last equation in (4.44) can be rewritten as

$$\partial_3 \sigma_{33}(e) = - \partial_3 \sigma_{3p}(e) - f_3,$$

and thus, by (4.41),

$$\partial_3 \left( \lambda K_{r p}(e) + 2 \mu K_{a 3}(e) \right) = - 2 \mu \partial_3 \partial_3 K_{a 3}(e) - \varepsilon^2 f_3.$$

(4.35)

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which shows that (compare with (4.45))

\[ \varepsilon_3 \left\{ \lambda K_{pp}(\varepsilon) + 2\mu K_{33}(\varepsilon) \right\} \to 0 \quad \text{in} \quad L^2 \left(-1,1; H^{-1}(\omega) \right). \]

Therefore the sequence \( \left( \lambda K_{pp}(\varepsilon) + 2\mu K_{33}(\varepsilon) \right)_{\varepsilon > 0} \) converges strongly in the space \( H^1 \left(-1,1; H^{-1}(\omega) \right). \)

The boundary condition (2.10) implies

\[ \lambda K_{pp}(\varepsilon) + 2\mu K_{33}(\varepsilon) = 0 \quad \text{in} \quad H^{-1/2} \left( \Gamma_+, -\Gamma_{\beta+} \right), \]

and the transmission condition (2.12) implies

\[ \lambda K_{pp} + 2\mu K_{33}(\varepsilon) = \varepsilon^2 \left\{ \tilde{X} e_{pp}(\tilde{u}(\varepsilon)) + 2\mu \tilde{e}_{33}(\tilde{u}(\varepsilon)) \right\} \quad \text{in} \quad H^{-1/2} (\Gamma_{\beta+}). \]

(compare with (4.48) and (4.49)). Noting that

\[ \tilde{X} e_{pp}(\tilde{u}(\varepsilon)) + 2\mu \tilde{e}_{33}(\tilde{u}(\varepsilon)) = \tilde{\gamma}^+ \left\{ \tilde{X} e(\tilde{u}(\varepsilon)) \right\}, \]

we conclude from (4.54) and (4.55) that (compare with (4.50))

\[ \lambda K_{pp}(\varepsilon) + 2\mu K_{33}(\varepsilon) \to 0 \quad \text{in} \quad H^{-1/2} (\Gamma_+), \]

and thus inequality (4.51), combined with (4.53) and (4.56), shows that (compare with (4.52)):

\[ \lambda K_{pp}(\varepsilon) + 2\mu K_{33}(\varepsilon) \to 0 \quad \text{in} \quad H^1 \left(-1,1; H^{-1}(\omega) \right), \]

and thus (4.36) is proved. ■

It follows from Lemmas 2,3 and 4 that the weak limit \((\tilde{u}, u)\) belongs to the space

\[ \mathcal{H}^1(\tilde{\Omega}) \times V_{KL}(\Omega) \]

\[ \overset{\text{def}}{=} \{ (\tilde{v}, v) \in H^1(\tilde{\Omega}) \times V_{KL}(\Omega); \tilde{v}_3|_{\tilde{\partial}b} = 0, v_3|_{\partial b} = v_3|_{\partial b} \}. \]

We now show that any function lying in two particular subspaces of the space \( \mathcal{H}^1(\tilde{\Omega}) \times V_{KL}(\Omega) \) can be approximated as well as we please by functions \((\tilde{v}(\varepsilon), v(\varepsilon))\) in the space \( V(\varepsilon) \), whose component \( v(\varepsilon) \) lie in addition in the space \( V_{KL}(\Omega) \). By taking limits as \( \varepsilon \to 0 \), this density property will later enable us to find the variational equations satisfied by \( (\tilde{u}, u) \). In what follows, \( \tilde{\omega} \) designates the intersection of the set \( \tilde{\Omega} \) by the plane that contains the set \( \tilde{\partial}B \) and we are assuming that the origin \( \tilde{\Omega} \) for the points \( \tilde{x} \in \tilde{\Omega} \) belongs to the left edge of the set \( \tilde{\omega} \) (cf. Fig. 4); the spaces \( V(\varepsilon) \) and \( V_{KL}(\Omega) \) have been defined in (3.9) and (4.29).

**Lemma 6.** — Let \((\tilde{v}, v)\) be a function in the space \( \mathcal{H}^1(\tilde{\Omega}) \times V_{KL}(\Omega) \) of (4.57) such that, either \( \text{supp} \tilde{v} \) is contained in the set \( \{ \tilde{x} = (\tilde{x}_1) \in \tilde{\Omega}; \tilde{x}_1 \leq 0 \} \) and \( v = 0 \), or \( \tilde{v}|_{\tilde{b}} \in H^1(\tilde{\omega}) \). Then there exists a sequence \((\tilde{v}(\varepsilon), v(\varepsilon))\) such that

\[ (\tilde{v}(\varepsilon), v(\varepsilon)) \in V(\varepsilon) \quad \text{for all} \quad \varepsilon > 0. \]
\{(4.59) \quad v(\varepsilon) \in V_{KL}(\Omega) \quad \text{for all} \quad \varepsilon > 0, \nn (4.60) \quad \| v(\varepsilon) - v \|_{1,\alpha} \to 0, \quad \varepsilon \to 0 \nn (4.61) \quad \| \tilde{v}(\varepsilon) - \tilde{v} \|_{1,\beta} \to 0, \quad \varepsilon \to 0 \}

Proof. — If \text{supp} \tilde{\nu} = \{ \tilde{x} = (\tilde{x}_i) \in \tilde{\Omega}; \ x_i \leq 0 \} and \nu = 0, it suffices to let \tilde{v}(\varepsilon) = \tilde{v} and \nu(\varepsilon) = 0 for all \varepsilon > 0. Assume that we are given a function (\tilde{v}, \nu) \in [H^1(\tilde{\Omega}) \times V_{KL}(\Omega), b] such that \tilde{v}_3 | \alpha \in H^1(\tilde{\omega}) (note that an arbitrary function (\tilde{v}, \nu) in the space [H^1(\tilde{\Omega}) \times V_{KL}(\Omega), b] \text{a priori} only satisfies \tilde{v}_3 | \alpha \in H^{1/2}(\tilde{\omega}) \cap H^{1/2}(\tilde{\omega}), \text{and} \tilde{v}_3 | \alpha \in H^{1/2}(\tilde{\omega}) \cap H^{1/2}(\tilde{\omega}), \text{cf. Lemmas 3 and 4}).

Since \nu \in V_{KL}(\Omega), Lemma 3 implies that there exist functions \eta_3 \in H^1(\omega) and \eta_3 \in H^2(\omega) such that
\begin{equation}
(4.62) \quad v_3 = \eta_3 \text{ in } \Omega.
\end{equation}

Let \tilde{\eta}_i(\tilde{x}) = \eta_i(x) at all the corresponding points \tilde{x} \in \tilde{\omega} and x \in \omega. Since the set \tilde{\omega} has a Lipschitz-continuous boundary, the functions \tilde{\eta}_3 \in H^1(\tilde{\omega}) and \tilde{\eta}_3 \in H^2(\tilde{\omega}) can be extended to functions (still denoted) \tilde{\eta}_3 \in H^1(\tilde{\omega}) and \tilde{\eta}_3 \in H^2(\tilde{\omega}) (see e.g. Nečas [1967, p. 80]). We then let
\begin{equation}
(4.63) \quad v(\varepsilon) = \tilde{v} \quad \text{in } \Omega \quad \text{for each } \varepsilon > 0,
\end{equation}

so that conditions (4.59) and (4.60) are certainly satisfied, and we define a function \tilde{v}(\varepsilon) in \tilde{\Omega} by letting:
\begin{equation}
(4.64) \quad \tilde{v}_3(\varepsilon) = \tilde{v}_3(\varepsilon) = e \tilde{v}_3 - \tilde{v}_3 \tilde{v}_3 + \tilde{v}_3 \tilde{v}_3 + \tilde{v}_3 \tilde{v}_3 \quad \text{in } \tilde{\Omega},
\end{equation}
\begin{equation}
(4.65) \quad \tilde{v}_3(\varepsilon) = \left[ e - \frac{1}{\varepsilon} \right] \tilde{v}_3 + \left( e - \frac{1}{\varepsilon} \right) \tilde{v}_3 \quad \text{in } \tilde{\Omega}.
\end{equation}

where
\begin{equation}
(4.66) \quad \tilde{v}(\varepsilon) = \tilde{v} \quad \text{in } \tilde{\Omega} - \tilde{\Omega}^2.
\end{equation}

Since the function \tilde{v}_3 \in H^1(\tilde{\omega}) by assumption, the function \tilde{v}(\varepsilon) constructed in (4.64)-(4.66) belongs to the space \tilde{H}^1(\tilde{\Omega}) (the assumption \tilde{v}_3 \in H^1(\tilde{\omega}) is thus crucially used here); besides, a simple computation shows that
\begin{equation}
(4.67) \quad \tilde{v}_3(\varepsilon)(\tilde{x}) = \tilde{v}_3(\tilde{x} \varepsilon) \quad \text{and} \quad \tilde{v}_3(\varepsilon)(\tilde{x}) = \tilde{v}_3(x)
\end{equation}
at all corresponding points \tilde{x} \in \tilde{\Omega}^2 and x \in \omega. Hence the function (\tilde{v}(\varepsilon), v(\varepsilon)) constructed in (4.63)-(4.66) belongs to the space (4.67) and (4.58) is satisfied. It thus remains to prove that \| \tilde{v}(\varepsilon) - \tilde{v} \|_{1,\beta} \to 0 as \varepsilon \to 0.

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To begin with, Lebesgue's dominated convergence theorem shows that
\[ |\tilde{y}(\epsilon) - \tilde{y}|_0, \tilde{y} \to 0, |\tilde{y}_j(\epsilon) - \tilde{y}_j|_0, \tilde{y} \to 0, \epsilon \to 0 \]
\[ |\tilde{\xi}_j \tilde{y}_j(\epsilon) - \tilde{\xi}_j \tilde{y}_j|_0, \tilde{y} \to 0, \epsilon \to 0 \]

since no factor \(1/\epsilon\) is introduced by partial differentiation with respect to \(\tilde{x}_\alpha\), nor by partial differentiation with respect to \(\tilde{x}_3\) in the set \(\tilde{\Omega}\) (the assumption \(\tilde{v}\) is in \(H^1(\tilde{\Omega})\) is again crucially used here). It follows from (4.65) that
\[
\tilde{\xi}_3 \tilde{v}_3(\epsilon) - \tilde{\xi}_3 \tilde{v}_3 = \begin{cases}
-\tilde{\xi}_3 + 2 \left(\frac{\tilde{x}_3 - 2 \epsilon}{\epsilon}\right) \tilde{\xi}_3 \tilde{v}_3 + 2 \left(\frac{\tilde{x}_3 - 2 \epsilon}{\epsilon}\right) \tilde{\xi}_3 \tilde{v}_3 + \frac{1}{\epsilon} (\tilde{v}_3 - \tilde{v}_3|_\Omega) \\
\tilde{\xi}_3 - 2 \left(\frac{\tilde{x}_3 + 2 \epsilon}{\epsilon}\right) \tilde{\xi}_3 \tilde{v}_3 - 2 \left(\frac{\tilde{x}_3 + 2 \epsilon}{\epsilon}\right) \tilde{\xi}_3 \tilde{v}_3 - \frac{1}{\epsilon} (\tilde{v}_3 - \tilde{v}_3|_\Omega)
\end{cases}
\]

if \(\tilde{x}_3 > 0\),
\[ \tilde{\xi}_3 \tilde{v}_3(\epsilon) - \tilde{\xi}_3 \tilde{v}_3 = \begin{cases}
\tilde{\xi}_3 + 2 \left(\frac{\tilde{x}_3 - 2 \epsilon}{\epsilon}\right) \tilde{\xi}_3 \tilde{v}_3 + \frac{1}{\epsilon} (\tilde{v}_3 - \tilde{v}_3|_\Omega) \\
\tilde{\xi}_3 - 2 \left(\frac{\tilde{x}_3 + 2 \epsilon}{\epsilon}\right) \tilde{\xi}_3 \tilde{v}_3 - \frac{1}{\epsilon} (\tilde{v}_3 - \tilde{v}_3|_\Omega)
\end{cases}
\]

if \(\tilde{x}_3 < 0\),

in the set \(\tilde{\Omega} - \tilde{\Omega}\). Hence it remains to prove that
\[ (4.67) \]
\[ \frac{1}{\epsilon^2} \int_{\tilde{\Omega}^1 - \tilde{\Omega}} |\tilde{v} - \tilde{v}|_0|^2 d\tilde{x} \to 0, \epsilon \to 0 \]

since the other terms found in the differences \((\tilde{\xi}_3 \tilde{v}_3(\epsilon) - \tilde{\xi}_3 \tilde{v}_3)\) can be again handled by the Lebesgue dominated convergence theorem. If \(\tilde{v}\) is a smooth function,
\[ |\tilde{v}(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) - \tilde{v}(\tilde{x}_1, \tilde{x}_2, \tilde{0})|^2 = \int_0^{\tilde{x}_3} |\tilde{v}_3(\tilde{x}_1, \tilde{x}_2, \tilde{s})|^2 d\tilde{s} \]
\[ \leq |\tilde{x}_3| \int_0^{\tilde{x}_3} \frac{|\tilde{v}_3(\tilde{x}_1, \tilde{x}_2, \tilde{s})|^2 d\tilde{s}}{|\tilde{x}_3|} \leq |\tilde{x}_3| \int_0^{\tilde{x}_3} |\tilde{v}_3(\tilde{x}_1, \tilde{x}_2, \tilde{s})|^2 d\tilde{s} \]
and thus
\[ \int_{\tilde{\Omega}} \frac{|\tilde{v}(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) - \tilde{v}(\tilde{x}_1, \tilde{x}_2, \tilde{0})|^2 d\tilde{x}_3 d\tilde{x}_2 \leq |\tilde{x}_3| |\tilde{v}|^2_{L^2(\tilde{\Omega})} \]

which in turn implies that, for any function \(\tilde{v} \in H^1(\tilde{\Omega})\),
\[ \int_{\tilde{\Omega}^1 - \tilde{\Omega}} |\tilde{v} - \tilde{v}|_0|^2 d\tilde{x} \leq 3 \epsilon^2 \|\tilde{v}\|^2_{H^1(\tilde{\Omega})}. \]
This last inequality then implies inequality (4.67) since \( \| \widetilde{\nu} \|_{1, \xi} \leq \epsilon \rightarrow 0 \). □

**Remark.** — The introduction of the functions \( \tilde{e} (\varepsilon) \) and the proof of relation (4.67) are based on an idea of Caillerie [1980]. □

As a first step towards identifying the "limit" variational problem solved by the weak limit \( (\tilde{u}, \tilde{u}) \), we obtain the variational equations that this weak limit should satisfy when the test functions \( (\tilde{\nu}, \tilde{v}) \) are subjected to the same restrictions as in Lemma 6.

**Lemma 7.** — Let \( \tilde{v}, \tilde{v} \) be a function in the space \( [H^1 (\tilde{\Omega}) \times V_{KL} (\Omega)]_\beta \) of (4.57) such that, either supp \( \tilde{v} \) is contained in the set \( \{ \tilde{x} = (\tilde{x}) \in \tilde{\Omega}; \tilde{x}_1 \leq 0 \} \) and \( \nu = 0 \), or \( \tilde{v} \in H^1 (\tilde{\omega}) \).

Then the weak limit \( (\tilde{u}, \tilde{u}) \in [H^1 (\tilde{\Omega}) \times V_{KL} (\Omega)]_\beta \) satisfies

\[
(4.68) \quad \int_{\tilde{\Omega}} \left( \begin{array}{c}
\tilde{\alpha} (\tilde{u}) : \tilde{e} (\tilde{v}) \\
\tilde{g} (\tilde{v})
\end{array} \right) d\tilde{x} = \int_{\tilde{\Omega}} \left( \begin{array}{c}
\tilde{\alpha} (\tilde{u}) : \tilde{e} (\tilde{v}) \\
\tilde{g} (\tilde{v})
\end{array} \right) d\tilde{x}
\]

with (cf. Lemma 3)

\[
(4.69) \quad \left\{ \begin{array}{rl}
\nu_a = \zeta_a - x_x \zeta_a, & \nu_a = \zeta_a, \\
\zeta_a \in H^1 (\omega), & \zeta_a \in H^1 (\omega), \\
\zeta_2 = \zeta_2, & \zeta_2 |_{\gamma_0} = \partial_n \zeta_2 |_{\gamma_0} = 0,
\end{array} \right.
\]

\[
(4.70) \quad \left\{ \begin{array}{rl}
\nu_a = \eta_a - x_x \zeta_a, & \nu_a = \eta_a, \\
\eta_a \in H^1 (\omega), & \eta_a \in H^1 (\omega), \\
\eta_3 = \eta_3, & \eta_3 |_{\gamma_0} = \partial_n \eta_3 |_{\gamma_0} = 0.
\end{array} \right.
\]

**Proof.** — We use the functions \( (\tilde{v}, \tilde{v}) \) constructed in Lemma 6 for approximating the function \( (\tilde{v}, \tilde{v}) \), as test functions in the variational equations (3.10). Since \( e_{\gamma} (\nu (e)) = 0 \) in \( \Omega \) by construction, these equations reduce to

\[
(4.71) \quad \int_{\tilde{\Omega}} \chi (\tilde{\Omega}) \left( \tilde{\alpha} (\tilde{u}) : \tilde{e} (\tilde{v}) \right) d\tilde{x} = \int_{\tilde{\Omega}} \left( \tilde{\alpha} (\tilde{u}) : \tilde{e} (\tilde{v}) \right) d\tilde{x}
\]

\[
+ \int_{\tilde{\Omega}} \left\{ \begin{array}{c}
2 \mu e_{ab} (u (e)) e_{ab} (v (e)) + \lambda \left[ e_{aa} (u (e)) + K_{ab} (e) \right] e_{bb} (v (e)) dx
\end{array} \right.
\]

\[
= \int_{\tilde{\Omega}} f (\nu (e)) d\tilde{x} = 0,
\]

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where $K_{33} (e) = e^{-2} e_{33} (u (e))$ (cf. (4.26)). Let then $e$ approach 0 in equations (4.71). Since (cf. Lemmas 2, 5, 6)

$$
u (e) \to \nu \text{ in } H^1 (\Omega), \quad u (e) \to u \text{ in } H^1 (\Omega),$$

$$K_{33} (e) \to - \frac{\lambda}{(\lambda + 2 \mu)} e_{33} (u) \text{ in } L^2 (\Omega)$$

$$\tilde{v} (e) \to v \text{ in } H^1 (\tilde{\Omega}), \quad v (e) \to v \text{ in } H^1 (\Omega),$$

we can pass to the limit in equation (4.71) (whenever $B$ is a strongly continuous bilinear form, $u_n \to u$ and $v_n \to v$ implies $B(u_n, v_n) \to B(u, v)$). We obtain in this fashion equation (4.68), after replacing the components of $u$ and $v$ by their expressions (4.69)-(4.70).

**Remark.** — The weak convergence of the sequence $(K_{33} (e))$ is thus needed here; the weak convergence of the sequence $(K_{33} (e))$ will not be used until Lemma 10.

It turns out that the limit problem consists of two independent problems, one with $(u, \zeta_3)$ as the unknown, the other one with $(\xi_1, \zeta_2)$ as the unknown. Accordingly, our identification of the limit problem comprises two stages (Lemmas 8 and 9).

**Lemma 8.** — The pair $(\tilde{u}, \zeta_3)$ belongs to the space

$$[H^1 (\tilde{\Omega}) \times H^2 (\omega)]_p = \{ (\tilde{v}, \eta_3) \in H^1 (\tilde{\Omega}) \times H^2 (\omega); \eta_3 \mid r_0 = \tilde{\eta}_3 \mid r_0 = 0, \tilde{v}_3 \mid \tilde{a}_p = \eta_3 \mid a_0, \tilde{v}_3 \mid \tilde{a}_p = 0 \},$$

and it satisfies

$$\int_\Omega \tilde{A} e (\tilde{u}) : e (\tilde{v}) \, d\tilde{x} - \int_\Omega \tilde{f} \cdot \tilde{v} \, d\tilde{x} + \int_\gamma \frac{d}{\lambda + 2 \mu} \left( \mu \partial_{\tilde{a}_\alpha} \zeta_3 + \frac{\lambda \mu}{\lambda + 2 \mu} \Delta \zeta_3 \partial_{\tilde{a}_\beta} \right) \partial_{\tilde{a}_\beta} \eta_3 \, d\omega \geq \int \left( \int_{-1}^1 f_3 \, dx_3 \right) \eta_3 \, d\omega - \int_\gamma \left( \int_{-1}^1 x_3 f_3 \, dx_3 \right) \partial_{\tilde{a}_\alpha} \eta_3 \, d\omega$$

for all $(\tilde{v}, \eta_3) \in [H^1 (\tilde{\Omega}) \times H^2 (\omega)]_p$.

The variational problem formed by equations (4.72) is coercive, and thus $(\tilde{u}, \zeta_3)$ is its unique solution.

**Proof.** — By Lemma 7, the variational equations (4.68) are satisfied in particular by any function of the form $(\tilde{v}, (-x_3 \partial_1 \eta_3, -x_3 \partial_2 \eta_3, \eta_3))$, such that $(\tilde{v}, \eta_3) \in [H^1 (\tilde{\Omega}) \times H^2 (\omega)]_p$ and either supp $\tilde{v}$ is contained in the set $\{ \tilde{x} = (\tilde{x}_i) \in \tilde{\Omega}; \tilde{x}_i \leq 0 \}$ and $\eta_3 = 0$, or $\tilde{v}_3 \in H^1 (\omega)$, in which cases they reduce to equations (4.72).

Given an arbitrary function $(\tilde{v}, \eta_3) \in [H^1 (\tilde{\Omega}) \times H^2 (\omega)]_p$, let $\tilde{\eta}_3 \in H^2 (\omega)$ denote an extension of $\tilde{\eta}_3 \mid \tilde{a}_p$ and let $\tilde{\omega}_3 (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) = \tilde{\eta}_3 (\tilde{x}_1, \tilde{x}_2)$ for all $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) \in \tilde{\Omega}$. Since the
function \((\tilde{w}^*, \eta_3)\)\(^{\text{def}}\) \(=((0, 0, \tilde{w}_3), \eta_3)\) belongs to the space \([H^1(\tilde{\Omega}) \times H^2(\tilde{\omega})]_b\) and satisfies \(\tilde{w}^* \in H^2(\tilde{\omega}) \subseteq H^1(\tilde{\omega})\), it satisfies the variational equation (4.72). Since equations (4.72) are linear with respect to \((\tilde{v}, \eta_3)\), it thus suffices to show that they are satisfied for all pairs of the form \((\tilde{v}, 0) \in [H^1(\tilde{\Omega}) \times H^2(\tilde{\omega})]_b\), with functions \(\tilde{v} = (\tilde{v}) \in H^1(\tilde{\Omega})\) satisfying \(\tilde{v}_{\mid_{\partial \Omega}} = 0\).

To this end, we show that, given any function \(\tilde{v} \in H^1(\tilde{\Omega})\) that satisfies

\[
(4.73) \quad \tilde{v}_{\mid_{\partial \Omega}} = 0,
\]

there exist functions \(\tilde{\rho}^*\) and \(\tilde{\rho}^*\) with the following properties:

\[
(4.74) \quad \tilde{\rho}^* \in H^1(\tilde{\Omega}) \quad \text{and} \quad \tilde{\rho}^*_{\mid_{\partial \Omega}} \in H^1(\tilde{\omega}),
\]

\[
(4.75) \quad \tilde{\rho}^* \in H^1(\tilde{\Omega}) \quad \text{and} \quad \supp \tilde{\rho}^* \subset \{ \tilde{x} = (\tilde{x}) \in \tilde{\Omega}; \tilde{x}_1 \leq 0 \},
\]

\[
(4.76) \quad (\tilde{\rho}^* + \tilde{\rho}^*) \rightarrow \tilde{v} \quad \text{in} \quad H^1(\tilde{\Omega}).
\]

Since the variational equations (4.72) are separately satisfied by the functions \((\tilde{\rho}^*), 0\) and \((\tilde{\rho}^*), 0\), and since they are linear and continuous with respect to \(\tilde{v} \in H^1(\tilde{\Omega})\), the conclusion will then follow.

Given \(\tilde{v} \in H^1(\tilde{\Omega})\) that satisfies (4.73), let the function \(\tilde{\rho}^* \in H^1(\Omega)\) be defined for each \(n \geq 1\) by

\[
\tilde{\rho}^*(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) = \begin{cases} 
\tilde{v}(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) & \text{for } \tilde{x}_1 \leq -\frac{2}{n}, \\
\tilde{v}(\tilde{x}_1 + \frac{1}{n}, \tilde{x}_2, \tilde{x}_3) & \text{for } -\frac{2}{n} \leq \tilde{x}_1 \leq -\frac{1}{n}, \\
\tilde{v}(\tilde{x}_1 + \frac{1}{n}, \tilde{x}_2, \tilde{x}_3) & \text{for } -\frac{1}{n} \leq \tilde{x}_1 \leq \beta - \frac{2}{n}, \\
\tilde{v}(\tilde{x}_1 + \frac{1}{n} + \beta, \tilde{x}_2, \tilde{x}_3) & \text{for } \beta - \frac{2}{n} \leq \tilde{x}_1 \leq \beta,
\end{cases}
\]

and let \(\tilde{\rho}^*\) be a smooth enough function of the variable \(\tilde{x}_1\) that satisfies

\[
\tilde{\rho}^*(\tilde{x}_1) = 1 \quad \text{for } \tilde{x}_1 \leq -\frac{1}{n} \quad \text{and} \quad \tilde{\rho}^*(\tilde{x}_1) = 0 \quad \text{for } \tilde{x}_1 \leq 0.
\]

Then the functions

\[
\tilde{\rho}^* = (1 - \tilde{\rho}^*) \tilde{\rho}^*, \quad \tilde{\rho}^* = \tilde{\rho}^* \tilde{\rho}^*
\]
clearly satisfy relations (4.74) (since \( \overline{\Omega} \setminus \Omega = 0 \)) and (4.75); it remains to show that (4.76) holds, i.e., that \( \| \overline{\nu} - \overline{\nu} \|, |n| \to 0 \). We have

\[
(4.77) \quad | \overline{\nu} - \overline{\nu} |, \Omega = \int_{\Omega (x \in \tilde{\Omega}, -2/n \leq x_1 \leq -1/n)} \left| \overline{\nu} \left( \frac{1}{n} \left( x_1 + \frac{1}{n} \right), x_2, x_3 \right) - \overline{\nu} \left( \tilde{x}_1, x_2, x_3 \right) \right|^2 d\tilde{x} 
\]

\[
+ \int_{\Omega (x \in \tilde{\Omega}, -1/n \leq x_1 \leq -2/n)} \left| \overline{\nu} \left( \frac{1}{n} \left( x_1 + \frac{1}{n} \right), x_2, x_3 \right) - \overline{\nu} \left( \tilde{x}_1, x_2, x_3 \right) \right|^2 d\tilde{x} 
\]

\[
+ \int_{\Omega (x \in \tilde{\Omega}, -2/n \leq x_1 \leq -1/n)} \left| \overline{\nu} \left( \frac{1}{n} \left( \frac{1}{2} \left( x_1 + \frac{1}{n} \right) \right), x_2, x_3 \right) - \overline{\nu} \left( \tilde{x}_1, x_2, x_3 \right) \right|^2 d\tilde{x}.
\]

Since

\[
\int_{\Omega (x \in \tilde{\Omega}, -2/n \leq x_1 \leq -1/n)} \left| \overline{\nu} \left( \frac{1}{n} \left( x_1 + \frac{1}{n} \right), x_2, x_3 \right) \right|^2 d\tilde{x} = \frac{1}{2} \int_{\Omega (x \in \tilde{\Omega}, -2/n \leq x_1 \leq 0)} \left| \overline{\nu} \left( \tilde{x}_1, x_2, x_3 \right) \right|^2 d\tilde{x}
\]

converges to zero as \( n \to \infty \), the first integral in (4.77) converges to zero, as well as the third; the second integral in (4.77) converges to zero since the translation operator is continuous in \( L^2 (\tilde{\Omega}) \) (see e.g. Nečas [1967, p. 57]). The norms \( | \partial_1 \overline{\nu} - \partial_3 \overline{\nu} |, \tilde{\Omega} \) likewise converge to zero as \( n \to \infty \), since the only effect of differentiating with respect to \( \tilde{x}_1 \) is to introduce \( 1/2 \) or 2 as factors.

In order to show that the variational equations (4.72) are satisfied, it suffices to observe that the mapping

\[
(\overline{\nu}, \eta_3) \to \left( \| \overline{\nu} \|, \tilde{\Omega}, \eta_3 + \sum_{n, \beta} \delta_{n, \beta} \eta_3 \|, \tilde{\Omega}, \eta_3 \right)^{1/2}
\]

is a norm over the space \( [H^1 (\tilde{\Omega}) \times H^2 (\omega)]_p \), equivalent to the norm

\[
(\overline{\nu}, \eta_3) \to \left( \| \overline{\nu} \|, \alpha + \eta_3 \|, \tilde{\Omega}, \eta_3 \right)^{1/2},
\]

the proof of this last result is similar to that of Lemma 1 and, for this reason, is omitted.

**Lemma 9.** The pair \( (\zeta_1, \zeta_2) \) belongs to the space

\[
\mathcal{H} (\omega) = \left\{ (\eta_1, \eta_2) \in H^1 (\omega) \times H^1 (\omega); \eta_0 = \eta_0 \text{ on } \gamma_0 \right\},
\]
and it satisfies

\[
(4.78) \quad \int_\Omega \left\{ \frac{\mu (\partial_{\alpha} \xi_\alpha + \partial_{\beta} \xi_\beta)}{2} + \frac{\lambda \mu}{\lambda + 2 \mu} \partial_{\gamma} \xi_{\gamma} \partial_{\alpha} \xi_{\beta} \right\} \xi_\beta \eta_\beta \, d\omega
\]

\[
\quad = \int_\Omega \left\{ \int_{-1}^1 f_\alpha \, dx_3 \right\} \eta_\alpha \, d\omega \quad \text{for all } (\eta_1, \eta_2) \in H(\omega).
\]

The variational problem formed by equations (4.78) is coercive, and thus \((\xi_1, \xi_2)\) is its unique solution.

Proof. — By Lemma 7, the variational equations (4.68) are satisfied in particular by any function of the form \((\tilde{\Theta}, (\eta_1, \eta_2, 0))\) such that \((\eta_1, \eta_2) \in H(\omega)\) (since \(\tilde{\Theta} \in H^1(\tilde{\omega})\)), in which case they reduce to equations (4.78). The coerciveness of the associated variational problem is a simple consequence of the two-dimensional Korn inequality.

The next lemma is the final step of the proof of Theorem 1.

**Lemma 10.** — The whole family \( (\bar{u}(\varepsilon), u(\varepsilon))_{\varepsilon > 0} \) converges strongly to \((\bar{u}, u)\) in the space \( H^1(\tilde{\Omega}) \times H^1(\Omega) \).

Proof. — By Lemmas 8 and 9, the weak limit \((\bar{u}, u)\) is unique; hence the whole family \( (\bar{u}(\varepsilon), u(\varepsilon))_{\varepsilon > 0} \) converges weakly to \((\bar{u}, u)\) in the space \( H^1(\tilde{\Omega}) \times H^1(\Omega) \), and strongly to \((\bar{u}, u)\) in the space \( L^2(\tilde{\Omega}) \times L^2(\Omega) \), by the Rellich-Kondrašov theorem. It thus suffices to show that the family \( (e(\bar{u}(\varepsilon)), e(u(\varepsilon)))_{\varepsilon > 0} \) strongly converges in the space \( L^2(\tilde{\Omega}) \times L^2(\Omega) \), as the conclusion will then follow from Korn’s inequality, applied in the spaces \( H^1(\tilde{\Omega}) \) and \( H^1(\Omega) \); in fact, we will even prove a sharper result: the strong convergence of the family \( (e(\bar{u}(\varepsilon)), e(u(\varepsilon)))_{\varepsilon > 0} \) to \( (e(\bar{u}), e(u)) \) in the space \( L^2(\tilde{\Omega}) \times L^2(\Omega) \), where the tensor \( K(\varepsilon) = (K_{ij}(\varepsilon)) \) is defined for each \( \varepsilon > 0 \) as in (4.26), and the components of the tensor \( K = (K_{ij}) \) are defined by

\[
(4.79) \quad K_{ab} = e_{ab}(u), \quad K_{a8} = 0, \quad K_{33} = -\frac{\lambda}{(\lambda + 2\mu)} e_{uu}(u).
\]

We recall that the weak convergence

\[
(e(\bar{u}(\varepsilon)), K(\varepsilon)) \rightharpoonup (e(\bar{u}), K)
\]

has been established in Lemmas 2 and 5.

By inequalities (4.27), there exists a constant \( c > 0 \) such that

\[
(4.80) \quad c \left( |e(\bar{u}(\varepsilon)) - e(\bar{u})|_{\tilde{\Omega}} \right)^2 + |K(\varepsilon) - K|_{\tilde{\Omega}}^2
\]

\[
\leq \int_{\tilde{\Omega}} \chi(\tilde{\Omega}) A(e(\bar{u}(\varepsilon)) - e(\bar{u}) : (e(\bar{u}(\varepsilon)) - e(\bar{u})) \, d\tilde{x}
\]

\[
+ \int_{\tilde{\Omega}} \chi(\tilde{\Omega}) A(e(\bar{u}(\varepsilon)) - e(\bar{u}) : (e(\bar{u}(\varepsilon)) - e(\bar{u})) \, d\tilde{x}
\]

\[
+ \int_{\tilde{\Omega}} A(K(\varepsilon) - K) : (K(\varepsilon) - K) \, dx.
\]
Let us then examine the behavior of the right-hand side of (4.80) as $\varepsilon \to 0$.

First, a simple computation, based on the junction conditions (3.4)-(3.5) for the three-dimensional problem, shows that

$$
\int_{\mathcal{\Gamma}_p} \chi(\mathcal{\Gamma}_p^0) A e(\tilde{u}(\varepsilon)) : e(\tilde{u}(\varepsilon)) \, d\tilde{x} \to 0 \quad \varepsilon \to 0
$$

since the weakly convergent family $(K(\varepsilon))_{\varepsilon > 0}$ is bounded in the space $L^2(\Omega)$. Further,

$$
\int_{\mathcal{\Gamma}_p} \chi(\mathcal{\Omega}_p^0) A e(\tilde{u}(\varepsilon)) : (e(\tilde{u}(\varepsilon)) - e(\tilde{u})) \, d\tilde{x} \to 0 \quad \varepsilon \to 0
$$

since the family $(\chi(\mathcal{\Omega}_p^0) A e(\tilde{u}))_{\varepsilon > 0}$ converges strongly to 0 in $L^2(\Omega)$, the family $(e(\tilde{u}(\varepsilon)) - e(\tilde{u}))_{\varepsilon > 0}$ converges weakly to 0 in $L^2(\Omega)$, and the inner product in the space $L^2(\Omega)$ is a continuous bilinear form (this argument will be used as several later places, but will not be repeated). Finally,

$$
\int_{\mathcal{\Gamma}_p} \chi(\mathcal{\Omega}_p^0) A e(\tilde{u}(\varepsilon)) \, dx \to 0 \quad \varepsilon \to 0
$$

since the $d\tilde{x}$-measure of the set $\mathcal{\Omega}_p^0$ approaches 0 as $\varepsilon \to 0$. Hence

(4.81) $\int_{\mathcal{\Omega}_p} \chi(\mathcal{\Omega}_p^0) A (e(\tilde{u}(\varepsilon)) - e(\tilde{u})) : (e(\tilde{u}(\varepsilon)) - e(\tilde{u})) \, d\tilde{x} \to 0$.

The remaining terms in the right-hand side of (4.80) can be rewritten as

(4.82) $\int_{\mathcal{\Omega}_p} \chi(\mathcal{\Omega}_p^0) \bar{A}(e(\tilde{u}(\varepsilon)) - e(\tilde{u})) : (e(\tilde{u}(\varepsilon)) - e(\tilde{u})) \, d\tilde{x}$

$$
+ \int_{\Omega} A(K(\varepsilon) - K) : (K(\varepsilon) - K) \, dx
$$

$$
= \int_{\mathcal{\Gamma}_p} \chi(\mathcal{\Gamma}_p^0) A e(\tilde{u}) : (e(\tilde{u}) - 2e(\tilde{u}(\varepsilon))) \, d\tilde{x} + \int_{\Omega} AK : (K - 2K(\varepsilon)) \, dx
$$

$$
+ \int_{\mathcal{\Gamma}_p} \chi(\mathcal{\Gamma}_p^0) \bar{A} e(\tilde{u}(\varepsilon)) : e(\tilde{u}(\varepsilon)) \, d\tilde{x} + \int_{\Omega} AK(\varepsilon) : K(\varepsilon) \, dx.
$$

First, we clearly have

(4.83) $\int_{\mathcal{\Gamma}_p} \chi(\mathcal{\Gamma}_p^0) \bar{A} e(\tilde{u}) : (e(\tilde{u}) - 2e(\tilde{u}(\varepsilon))) \, d\tilde{x} \to \left\{ - \int_{\mathcal{\Gamma}_p} \bar{A} e(\tilde{u}) : e(\tilde{u}) \, d\tilde{x} \right\} \varepsilon \to 0$

(4.84) $\int_{\Omega} AK : (K - 2K(\varepsilon)) \, dx \to \left\{ - \int_{\Omega} AK : K \, dx \right\}$.
and secondly, expressing that the variational equations (3.10) are satisfied in particular by the pair \((\tilde{u}(\varepsilon), u(\varepsilon)) \in V(\varepsilon)\), we find that

\[
(4.85) \quad \int_{\Omega} \chi(\bar{\Omega}) \bar{A} \varepsilon(\tilde{u}(\varepsilon)) : \varepsilon(\tilde{u}(\varepsilon)) \, d\tau + \int_{\Omega} AK(\varepsilon) : K(\varepsilon) \, dx = \int_{\Omega} \chi(\bar{\Omega}) \tilde{f} \, d\tau + \int_{\Omega} f \cdot u \, dx \rightarrow \int_{\Omega} \tilde{f} \cdot \tilde{u} \, d\tau + \int_{\Omega} f \cdot u \, dx.
\]

We thus infer from (4.83), (4.84), and (4.85), that the remaining terms (4.82) in the right-hand side of (4.80) converge to

\[
(4.86) \quad \text{def} \quad L = -\left\{ \int_{\Omega} \bar{A} \varepsilon(\tilde{u}) : \varepsilon(\tilde{u}) \, d\tau + \int_{\Omega} AK : K \, dx \right\} + \int_{\Omega} \tilde{f} \cdot \tilde{u} \, d\tau + \int_{\Omega} f \cdot u \, dx
\]

Using the special form of the field \(u\) found in Lemma 3 (that is, \(u = x_3 - \partial u_3, u_4 = \zeta_2\)), the special form of the weak limit \(K = (K, K)\) found in Lemma 5 (that is, \(K = \varepsilon K(\varepsilon)\), \(K_{x_3} = 0, K_{x_3} = -\lambda/(\lambda + 2\mu)\) \(E_{yy}(\varepsilon)\), and expressing that the variational equations (4.72) of Lemma 8 and the variational equations (4.78) of Lemma 9 are satisfied in particular by the pairs \((\tilde{u}, \zeta_2)\) and \((\zeta_1, \zeta_2)\), respectively, we then easily verify that the limit \(L\) found in (4.86) vanishes. Therefore the proof is complete. \(\square\)

Remark. – The proof of Theorem 1 contains the convergence proof of the scaled displacements for a “single” plate, already established in Destuynder [1981] and in Ciarlet & Kesavan [1981]. Destuynder [1981] has in addition established the stronger result \(\|u(\varepsilon) - u\|_{1, \Omega} = O(\sqrt{\varepsilon})\). \(\square\)

5. Interpretation of the limit problem as a boundary value problem

It remains to describe the boundary value problem that is, at least formally, associated with the variational equations (4.4) and (4.7). To begin with, we define the open set

\[
(5.1) \quad \bar{\Omega}_{\varepsilon} = \bar{\Omega} - \{ \tilde{\omega}_\varepsilon \}^-,
\]

which is thus a three-dimensional open set with a two-dimensional slit, and we let \(\tilde{\omega}_\varepsilon^+\) and \(\tilde{\omega}_\varepsilon^-\) denote the upper and lower faces of the slit. When viewed as sets, these faces are fictitiously distinguished, since they coincide with the set \(\tilde{\omega}_\varepsilon\); on the other hand, the introduction of different notations allows for a convenient distinction between the trace “from above” and the trace “from below” of a function defined over the set \(\bar{\Omega}_{\varepsilon}\), as in equ. (5.5). Finally, we denote by \(\vec{n} = (\vec{n}, \nu)\) the unit outer normal vector along the set \(\partial \bar{\Omega}_{\varepsilon} - \{ \tilde{\omega}_\varepsilon \}^-\); by \((\nu_\varepsilon)\) and \((\zeta_\varepsilon)\) the unit outer normal and unit tangential vectors along \(\partial \bar{\omega}\); and by \(\tilde{\tau}_t\), the tangential derivative operator along \(\tilde{\omega}\).
THEOREM 2. — A smooth enough solution $(\tilde{u}, \zeta_3)$ of equations (4.4) solves the following equations:

(a) in the set $\bar{\Omega}_p$:

\begin{align}
-\partial_j \sigma_{ij}(\tilde{u}) &= \tilde{f}_i \quad \text{in } \bar{\Omega}_p, \\
\sigma_{ij}(\tilde{u}) \tilde{n}_j &= 0 \quad \text{on } \partial \bar{\Omega}_p = \{ \tilde{\omega}_p \}^-,
\end{align}

where $\partial_i = \partial_j \partial \tilde{x}_p$ and

\begin{equation}
\sigma_{ij}(\tilde{u}) = \lambda \varepsilon_{pp}(\tilde{u}) \delta_{ij} + 2 \mu \varepsilon_{ij}(\tilde{u});
\end{equation}

(b) in the set $\omega$:

\begin{equation}
\frac{8 \mu (\lambda + \mu)}{3(\lambda + 2 \mu)} \Delta^2 \zeta_3 = \int_{-1}^{1} f_3 \, dx_3
\end{equation}

\begin{equation}
+ \int_{-1}^{1} x_3 \partial_a f_a \, dx_3 + \chi(\omega_p) \left( \tilde{\sigma}_{33}(\tilde{u}) |_{\partial \omega} - \tilde{\sigma}_{33}(\tilde{u}) |_{\partial \omega} \right) \quad \text{in } \omega,
\end{equation}

\begin{equation}
\zeta_3 = \partial_\nu \zeta_3 = 0 \quad \text{on } \gamma_0,
\end{equation}

\begin{equation}
m_{ap}(\zeta_3) \nu_a \nu_p = 0 \quad \text{on } (\partial \omega - \gamma_0)
\end{equation}

\begin{equation}
\partial_t \left\{ m_{ap}(\zeta_3) \nu_a \gamma_p \right\} + \partial_\nu m_{ap}(\zeta_3) \nu_p = \left\{ \int_{-1}^{1} x_3 f_a \, dx_3 \right\} \nu_a \quad \text{on } \partial \omega - \gamma_0,
\end{equation}

where $m_{ap}(\zeta_3)$ is defined as in (4.5);

(c) at the "junction" between the sets $\bar{\Omega}_p$ and $\omega$:

\begin{equation}
\tilde{u}_3 |_{\partial \omega} = \tilde{u}_3 |_{\partial \omega} = \zeta_3 |_{\partial \omega},
\end{equation}

\begin{equation}
\tilde{u}_a |_{\partial \omega} = \tilde{u}_a |_{\partial \omega} = 0.
\end{equation}

A smooth enough solution $(\zeta_1, \zeta_2)$ of equations (4.7) solves the following equations:

\begin{equation}
-\partial_p n_{ap}(\zeta_1, \zeta_2) = \int_{-1}^{1} f_a \, dx_3 \quad \text{in } \omega,
\end{equation}

\begin{equation}
\zeta_1 = \zeta_2 = 0 \quad \text{on } \gamma_0,
\end{equation}

\begin{equation}
n_{ap}(\zeta_1, \zeta_2) \nu_a = 0 \quad \text{on } (\partial \omega - \gamma_0),
\end{equation}

where $n_{ap}(\zeta_1, \zeta_2)$ is defined as in (4.8).

Proof. — In the variational equations (4.4), let $\eta_3 = 0$ and let $\tilde{v}$ vary in the space $\{ \tilde{v} \in \mathcal{C}^\infty \left( \{ \bar{\Omega} \}^- \right); \tilde{v} = 0 \text{ in a neighborhood of } \{ \tilde{\omega}_p \} \}$; this shows that eqs. (5.2) and (5.3) are satisfied. If $\tilde{v} \in \mathcal{H}^1(\Omega)$ satisfies $\tilde{v}_a \tilde{n}_p = \tilde{v}_a \tilde{n}_p = 0$, and if $\tilde{u}$ is smooth enough (as is customarily assumed when variational equations are interpreted as a boundary value.
where

\begin{equation}
\mathcal{E}_{ab}^{(e)}(\zeta_3) = \frac{4 \mu^e e^3}{3} \left\{ \varepsilon_{ab} \varepsilon_{3}^{e} + \frac{\lambda^e}{\lambda^e + 2 \mu^e} \Delta_{ab} \varepsilon_{3}^{e} \right\},
\end{equation}

\((f_1)\) : \Omega^e \to \mathbb{R}^3 is the "true" applied body force density acting on the "thin" part of the structure, and \(\lambda^e, \mu^e\) are its "true" Lamé constants; further, the pair \((\vec{u}, \zeta_3)\) satisfy the "junction conditions":

\begin{align}
&\vec{u}_3 \mid_{\omega_3} = \vec{u}_3 \mid_{\omega_3} = \zeta_3 \mid_{\omega_3} \\
&\vec{u}_3 \mid_{\omega_3} = \vec{u}_3 \mid_{\omega_3} = 0;
\end{align}

finally, we find that the pair \((\zeta_3^1, \zeta_3^2)\) solves

\begin{align}
&-\partial_b n_{ab}^{(e)}(\zeta_3^1, \zeta_3^2) = \int_{-e}^{e} f_a \, dx^3 \text{ in } \omega_3 \\
&\zeta_3^1 = \zeta_3^2 = 0 \text{ on } \gamma_0, \\
&n_{ab}^{(e)}(\zeta_3^1, \zeta_3^2) \nu_a = 0 \text{ on } (\partial \omega_3 - \gamma_0),
\end{align}

where

\begin{equation}
n_{ab}^{(e)}(\zeta_3^1, \zeta_3^2) = 4 \mu^e e^3 \left\{ \varepsilon_{ab} \varepsilon_{3}^{e} + \frac{\lambda^e}{\lambda^e + 2 \mu^e} \Delta_{ab} \varepsilon_{3}^{e} \right\}.
\end{equation}

(ii) One major conclusion is that the pair \((\vec{u}, \zeta_3)\) solves a coupled, pluridimensional, variational problem of a new type, posed over a subspace of \(H^1(\Omega_3) \times H^2(\omega)\), whose elements satisfy the junction conditions (6.12)-(6.13). Such a problem is interesting per se, and it would be worthwhile to study, in particular: the singularity of the unknown \(\vec{u}\) around the inner edge of the inserted part \(\omega_3\); the numerical analysis of the associated "limit" eigenvalue problem (as identified and justified in Bourquin and Ciarlet [1988]); especially by methods adapted to its pluri-dimensional character; the associated time-dependent problem (whose identification and justification should rely on the results of Raoult [1985] for a single plate); the controllability of the limit problem, by the Hilbert Uniqueness Method of Lions [1987, 1988] (see also Lagnese & Lions [1988]; the controllability of structures with junctions is a problem of outstanding practical interest).

(iii) The function

\begin{equation}
h_3^{\text{det}} = \chi(\omega_p) \left\{ \sigma_{33}^{(e)}(\vec{u}) \mid_{\omega_3} - \sigma_{33}^{(e)}(\vec{u}) \mid_{\omega_3} \right\}
\end{equation}

appearing in the right-hand side of the otherwise familiar two-dimensional plate equation (6.7) is nothing but the Lagrange multiplier associated with the equality constraints (in the sense of optimization theory) imposed to the "trial" functions \((\vec{v}, \eta_3)\) in the form of the junction conditions (5.9)-(5.10).
(iv) The mechanical interpretation of the limit problem is natural: The function $\widetilde{u}^*$ solves the standard equations (6.4)-(6.6) of three-dimensional linearized elasticity, while the functions $\zeta_1^*$ and $(\zeta_1^*, \zeta_2^*)$ solve the standard equations (6.7)-(6.11) and (6.14)-(6.17) satisfied in two-dimensional plate theory by the transverse and in-plane displacements, respectively (cf. Germain [1986, Sect. VIII. 7]). Besides, the function $h_5^*(\text{cf. } (6.18))$ that is added to the right-hand side of the biharmonic equation (6.7) balances, as expected, the vertical resultant of the forces that act on the three-dimensional part $O_B$ in the sense that

\begin{equation}
\int_0^\pi h_5^* \, n_0 \, ds = \int_0^\pi f_5^* \, ds^*
\end{equation}

(to see this, simply let $\widetilde{v}_5 = 0$ and $\widetilde{v}_3 = 1$ in (5.14) and use formulas (3.3) and (3.8)).

(v) While the junction conditions $\tilde{u}_3^* |_{n_0} = \tilde{u}_3^* |_{n_0} = \zeta_3^* |_{n_0}$ express the continuity of the vertical displacement along the inserted portion of the plate, the other junction conditions $\tilde{u}_2^* |_{n_0} = \tilde{u}_2^* |_{n_0} = 0$ do not involve the functions $\zeta_2^* |_{n_0}$ which do not vanish in general (except in some special cases, such as if $f_5^* = 0$ in). This is only an apparent paradox, for the convergence result obtained in Theorem 1 implies that

\begin{equation}
(u_5^* |_{n_0} = \varepsilon \zeta_3 + o(\varepsilon) \quad \text{in } H^{1/2}(\omega_B),
\end{equation}

\begin{equation}
(iu_1^* |_{n_0} = \varepsilon \zeta_3 + o(\varepsilon) \quad \text{in } H^{1/2}(\omega_B);
\end{equation}

hence the first-order term (with respect to $\varepsilon$) of the horizontal components of the displacement of the three-dimensional structure should also vanish in $H^{1/2}(\omega_B)$ if these "limit" horizontal components are to be continuous; but this is exactly what is implied by the conditions (6.13) and the scalings $u_2^*(X) = \varepsilon u_2^*(x)$ of (6.1).

(vi) Relations (2.1) express that the rigidity of the material constituting the thin portion of the structure should increase as $\varepsilon^{-3}$ when $\varepsilon \to 0$. That such asymptotic orders, as well as the assumed asymptotic orders (3.6)-(3.8) on the applied forces, are inevitable assumptions in order that a limit problem exist, has already been observed by Caillerie [1980] and Ciarlet [1980] in the case of a "single" plate. The reader is refered to Ciarlet [1980, 1987, 1989, 1990] for more detailed discussions of the meaning of such asymptotic orders.

(vii) In the spirit of the present work, we may assume that a boundary condition of place is satisfied along a portion of the boundary of the three-dimensional part (in which case no boundary condition of place is needed along the boundary of the thin part), and that the Lamé constants of the three-dimensional part also converge in an appropriate manner to $+\infty$ as $\varepsilon \to 0$ (here, they were assumed to be independent of $\varepsilon$). As shown by Ciarlet & Le Dret [1989], the junction conditions found in the corresponding limit problem represent the genuine boundary conditions that a two-dimensional clamped plate model should satisfy at the junction with the three-dimensional support. There are indeed very few other works, where the elastic equilibrium of a body is studied together with that of the interacting surrounding elastic bodies; see however Batra [1972],

(viii) While the "full" three-dimensional problem is well defined for any \( \varepsilon > 0 \) if \( \beta = 0 \), our approach does not yield a coupled limit problem in this case: Even if a boundary condition of place is satisfied along a portion of the boundary of the three-dimensional part (in order to "fix" this part), the limit problem consists of two unrelated problems, \textit{i.e.}, there is no longer any junction condition in the limit problem when \( \beta = 0 \). This difficulty can be overcome by the approach of Sanchez-Palencia [1988].

(ix) Satisfactory numerical results have been obtained with the limit problem; cf. Aufranc [1989] (some of them have also been reported in Ciarlet [1988]). It remains however to assess the range of validity of the limit problem, as has been done by Miara [1987] for a single plate.


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P. G. Ciarlet,
H. Le Dret,
Laboratoire d'Analyse Numérique,
Université Pierre-et-Marie-Curie,
4, place Jussieu,
75005 Paris

R. Nzungwa,
Département de Génie Civil
École Nationale Supérieure Polytechnique
B. P. 728
Yaoundé, Cameroun