An example of $H^1$-unboundedness of solutions to strongly elliptic systems of partial differential equations in a laminated geometry

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Synopsis

In this paper, a counterexample is given to the $H^1$-boundedness of solutions to a sequence of systems of linear partial differential equations uniformly satisfying a strict Legendre–Hadamard condition and whose coefficients depend on one direction only. This counterexample is relevant for the theory of homogenisation of laminated elastic materials.

1. Introduction

In this paper, we describe an example of a sequence of systems of linear partial differential equations which are uniformly strictly strongly elliptic and whose coefficients depend on one direction only, but whose solutions are not bounded in the $H^1$ norm. This example is relevant for the theory of homogenisation of laminated linear elastic materials.

Let $S_n$ be the set of $n \times n$ symmetric matrices and let $\Omega$ be a open subset of $\mathbb{R}^n$ with a sufficiently smooth boundary. We consider a family of elasticity tensors $A^\varepsilon$ for $\varepsilon \in \mathbb{R}$, i.e. a family of mappings:

$$A^\varepsilon : \Omega \times S_n \to S_n,$$

which are measurable in the first variable and linear in the second variable. Let $u$ be a displacement, i.e. an $H^1$-mapping from $\Omega$ to $\mathbb{R}^n$, and let $e(u) = (\nabla u + \nabla u^T)/2$ be the linearised strain tensor. Then the tensor $A^\varepsilon(e(u))$ is the associated stress tensor, and the equations of equilibrium of the corresponding linear elastic body under the action of a load $f \in L^2(\Omega)^n$ are:

$$-\text{div} A^\varepsilon(e(u)) = f. \tag{1}$$

together with appropriate boundary conditions. We assume that the tensors $A^\varepsilon$ satisfy a uniform strict Legendre–Hadamard inequality (also called strict strong ellipticity condition), i.e. there exists a constant $\alpha > 0$ such that:

for all $\varepsilon \in \mathbb{R}$, for almost every $x \in \Omega$, and for all $(\lambda, \mu) \in \mathbb{R}^n \times \mathbb{R}^n$:

$$A^\varepsilon_{ijk}(x)\lambda_i\lambda_j\mu_k \geq \alpha\lambda_i\lambda_j\mu_k, \tag{2}$$

We will use the Einstein summation convention throughout the paper. We also assume that the coefficients $A^\varepsilon_{ijk}(x)$ satisfy:

$$|A^\varepsilon_{ijk}(x)| \leq M, \tag{3}$$
for some constant $M$ independent of $x$ and $\varepsilon$. An elastic material is said to be laminated if its elasticity tensor depends only on one direction (for example $x_1$).

The aim of this paper is to prove the following:

**Theorem 1.** Let $\Omega = ]-1, 1[^n$. There exist a function $f \in L^2(\Omega)^n$ and a sequence of isotropic elasticity tensors $A^e(x_1)$ satisfying (2) and (3) such that there exists a sequence $u^e \in H^1_0(\Omega)^n$ of solutions to (1), that is unbounded in $H^1_0(\Omega)^n$.

**Remark 1.** We will also obtain the same result with a variable right-hand side in (1), i.e. with a load $f^e$ bounded in $L^2(\Omega)^n$.

**Remark 2.** When, instead of (2), $A^e$ satisfy the stronger inequality (very strong ellipticity condition):

$$A^e_{ij}(x)b_{ij} \geq c b_{ij}, \quad (c > 0),$$

it is known (and easy to prove) that the solutions to (1), with Dirichlet boundary conditions are bounded in $H^1(\Omega)^n$. Notice that in this case the corresponding energy function is convex. The boundedness result also holds true when $A^e$ satisfy (2) only and have constant coefficients (in space). This follows by multiplication of the equation by $u^e_i$, integration by parts and use of Fourier transforms. It equally holds true, under (2) alone, when attention is a priori restricted to functions $u^e$ depending on $x_1$ only, with boundary conditions of Neumann type.

**Remark 3.** When the coefficients are continuous, condition (2) is known to imply the sequential weak lower semicontinuity of the corresponding quadratic energy functional for $\varepsilon$ fixed, in the following sense: let $u^\varepsilon$ be a sequence such that

$$u^\varepsilon \rightarrow u \text{ in } H^1(\Omega)^n \text{ weakly}$$

then, for any smooth nonnegative test function $\varphi$ with compact support in $\Omega$,

$$\liminf_{\varepsilon \rightarrow +0} \int_{\Omega} \varphi(x) |A^\varepsilon(x_1) e(u^\varepsilon)(x)| e(u^\varepsilon)(x) \, dx \geq \int_{\Omega} \varphi(x) |A^e(x_1) e(u)(x)| e(u)(x) \, dx,$$

see e.g. Murat [6, Section 4]. This weak lower semicontinuity is known to be equivalent to the rank-1-convexity of the corresponding quadratic energy function for $x_1$ fixed, see [6]. Notice that there is no relationship between this weak lower semicontinuity and any boundedness from below for the functional, see Lemma 1 below.

**Remark 4.** Theorem 1 is especially relevant for the theory of homogenisation of layered elastic media. This theory is usually developed under assumption (2)' of uniform very strong ellipticity of the elasticity tensors. It is then known that for a given family $A^e$ of elasticity tensors satisfying (2)' and (3), there exists a subsequence $\varepsilon'$ and an elasticity tensor $A^0$ still satisfying (2)' and (3) such that for all $f$ in $L^2(\Omega)^n$ the solutions $u^{e\varepsilon}$ of (1), with $u^e$ in $H^1_0(\Omega)$ converge weakly in $H^1_0(\Omega)^n$ to a displacement $u^0$ which is a solution to the same problem with elasticity tensor $A^0$ (see [2]).
In the case where assumption (2') is dropped but only (2) is supposed to hold, it was noticed by Tartar [7, Remark 6], that for laminated materials, the homogenisation process can still be achieved provided that an a priori $H^1$-bound holds. Theorem 1 shows that such an assumption is not reasonable in the case of Dirichlet boundary conditions. It should be noted that the sequence $A^e$ whose existence is stated in Theorem 1 is of the type $A^e(x_1) = A(x_1) + \varepsilon B$ for $\varepsilon$ in a neighbourhood of a certain $\varepsilon_0$, with $A(x_1)$ continuous in $x_1$. In particular the modulus of continuity of the coefficients of $A^e$ is independent of $\varepsilon$, and the convergence is very strong, an a priori much more favourable situation than the ones usually encountered in homogenisation.

2. Proof of Theorem 1

We will divide the proof into several lemmas. Let $\Omega = ]-1, 1[^n$.

**Lemma 1.** There exists an isotropic linear elasticity tensor $A$ with $C^\infty$ coefficients depending only on the variable $x_1$, such that $A$ satisfies (2) but:

$$\inf_{u \in H^1(\Omega)} \int_{\Omega} A e(u) : e(u) \, dx = -\infty,$$

**Proof.** Let $F$ be an $n \times n$ matrix and define $E = \text{Sym} F = (F + F^T)/2$. The stored energy function of a linear isotropic elastic material takes the form:

$$2W(F) = 2\mu(x) \text{tr} (E^2) + \lambda(x)(\text{tr} E)^2,$$

where $\lambda(x)$ and $\mu(x)$ are the Lamé coefficients. The elasticity tensor is then obtained by the formula $A = \partial W/\partial E$, so that $AE = 2\mu(x)E + \lambda(x)(\text{tr} E) Id$. It is known, cf. [3], that the function $W$ is uniformly strictly rank-1-convex (or equivalently that $A$ satisfies (2)), if and only if the following inequalities hold:

$$\mu(x) \geq \alpha > 0 \quad \text{and} \quad 2\mu(x) + \lambda(x) \geq \alpha > 0,$$

($\alpha$ to be chosen sufficiently small in the sequel) and is not convex if furthermore:

$$2\mu(x) + n\lambda(x) < 0.$$  \hspace{1cm} (5)

Let us choose a constant $\mu_0 > \alpha$ and let us define $\lambda = -2\mu_0 + \eta$ for $\eta$ real. Then the couple $(\mu_0, \lambda)$ satisfies inequalities (4) and (5) if $\alpha < \eta < 2\mu_0(1-1/n)$. We will select an appropriate value of $\eta$ in the course of the proof. We rewrite the stored energy function of the corresponding homogeneous material as:

$$2W_0(F) = 2\mu_0[\text{tr} (E^2)^2 - (\text{tr} E)^2] + \eta(\text{tr} E)^2.$$  \hspace{1cm} (6)

For $1 > \delta > 0$ and $1 > \gamma > 0$ (to be chosen later), let us set $\Omega_{\delta} = \{ x \in \Omega, |x_1| < \delta \}$ and let us consider a function $g \in C^0([-1, 1])$ such that:

$$g(z) = 1 \quad \text{for} \quad |z| < \delta, \quad g(z) = \gamma \quad \text{for} \quad |z| > \delta + \gamma, \quad \text{for} \quad |z| > \delta + \gamma, \quad (7)$$

(we assume $\delta + \gamma < 1$).

We consider the laminated elastic material filling $\Omega$ whose stored energy
The function is:

$$W(x, F) = g(x_1)W_0(F).$$  \quad (8)

This material is homogeneous in the layer $\Omega_\delta$ with Lamé coefficients $\lambda$ and $\mu_0$, and also outside the layer $\Omega_{\delta+y}$ with Lamé coefficients $\gamma\lambda$ and $\gamma\mu_0$. The body therefore consists, roughly speaking, of three layers. (For the proof of the lemma, we could as well have chosen a material consisting exactly of three homogeneous layers, that is with only $L^\infty$-coefficients. However the regularity of the coefficients will be used in the sequel.) We will now construct a deformation $u \in H_0^1(\Omega)'$ whose total energy is strictly negative, and we begin by defining $u$ on $\Omega_\delta$. Let $\Omega' = \{x \in \Omega, x_1 = 0\}$ and let $x'$ denote $(x_2, \ldots, x_n)$. For $x$ in $\Omega_\delta$, we set:

$$\begin{align*}
  u_1(x) &= x_1h(x'), \\
  u_2(x) &= k(x'), \\
  u_3(x) &= \ldots = u_n(x) = 0,
\end{align*}$$

where $h(x')$ and $k(x')$ belong to $C^0(\Omega')$ and will be specified in the sequel. By construction it is clear that for $x$ in $\partial\Omega \cap \Omega_\delta$, $u(x) = 0$. Let us compute the elastic energy of $u$ stored in the layer $\Omega_\delta$. We have:

$$\nabla u = \begin{pmatrix}
  h(x') & x_1\partial_2 h(x') & \cdots & x_1\partial_n h(x') \\
  0 & \partial_2 k(x') & \cdots & \partial nk(x') \\
  \vdots & 0 & \cdots & 0 \\
  0 & 0 & \cdots & 0
\end{pmatrix}
$$

and therefore:

$$e(u) = \begin{pmatrix}
  h(x') & x_1\partial_2 h(x')/2 & \cdots & x_1\partial_n h(x')/2 \\
  x_1\partial_2 h(x')/2 & \partial_2 k(x') & \cdots & \partial nk(x')/2 \\
  \vdots & 0 & \cdots & 0 \\
  x_1\partial_n h(x')/2 & \partial nk(x')/2 & \cdots & 0
\end{pmatrix}.
$$

Insertion into the expression for the energy (6) yields:

$$2\int_{\Omega_\delta} W(x, \nabla u) \, dx = \mu_0\int_{\Omega_\delta} \left( x_1^2 \left( \sum_{i=2}^n \partial_i h(x')^2 \right) + \sum_{i=2}^n \partial_i k(x')^2 - 4h(x')\partial_2 k(x') \right) \, dx$$

$$+ \eta\int_{\Omega_\delta} (\text{tr} \, e(u))^2 \, dx,$$

or, after integration in $x_1$:

$$2\int_{\Omega_\delta} W(x, \nabla u) \, dx = \frac{3}{2}\mu_0\delta^3\int_{\Omega_\delta} \left( \sum_{i=2}^n \partial_i h(x')^2 \right) \, dx' + 2\mu_0\delta \int_{\Omega} \left( \sum_{i=3}^n \partial_i k(x')^2 \right. $$

$$\left. - 4h(x')\partial_2 k(x') \right) \, dx' + \eta\int_{\Omega_\delta} (\text{tr} \, e(u))^2 \, dx. \quad (10)$$
Let us choose \( k \) to be nonzero. Then, there exists a function \( h \) such that:

\[
\int_{\Omega'} \left( \sum_{i=3}^{n} \partial_i k(x')^2 - 4h(x') \partial_2 k(x') \right) dx' < 0.
\]  

(11)

To show that inequality (11) holds, it suffices to see that there exists a function \( h_1 \) in \( C_c^\infty(\Omega') \) such that:

\[
\int_{\Omega'} h_1(x') \partial_2 k(x') \, dx' \neq 0,
\]

(12)

and to multiply such an \( h_1 \) by an appropriate constant. Assume that (12) does not hold, then \( \partial_2 k = 0 \) and therefore \( k = 0 \), which contradicts our choice of \( k \). As inequality (11) holds, inspection of formula (10) shows clearly that we can choose first \( \delta_0 \) and then \( \eta_0 \) sufficiently small so that:

\[
\int_{\Omega_{\delta_0}} W(x, \nabla u) \, dx < 0.
\]

We now extend the deformation \( u \) to the whole \( \Omega \) in such a way that \( u \) belongs to \( H^1_0(\Omega) \), which is clearly possible. Then, by formul\( e (7) \) and (8), we can choose the constant \( \gamma \) sufficiently small but strictly positive so that:

\[
\int_{\Omega} W(x, \nabla u) \, dx < 0.
\]

(13)

All the parameters have now been chosen and it is a straightforward matter to verify that the corresponding elasticity tensor \( A = \partial W/\partial E \) meets all our requirements. To conclude the proof, we finally remark that, since the energy is quadratic, Lemma 1 follows from (13) and multiplication of \( u \) by an arbitrarily large constant \( \tau \).

**Lemma 2.** There exists an isotropic linear elasticity tensor satisfying (2) and depending only on \( x \), such that the associated differential operator has a nontrivial kernel in \( H^1_0(\Omega) \).

**Remark.** Such an example is given explicitly in [1] for the case of a spherical shell. In our present laminated geometry, it does not seem to be possible to construct it explicitly (i.e. by separation of the variables).

**Proof.** Let \( A(x_1) = \partial W/\partial E \) and consider:

\[
A^\varepsilon(x_1) \cdot E = A(x_1) \cdot E + 2\varepsilon E + \varepsilon \text{ tr } E \cdot \text{ Id}.
\]

The family of operators defined on \( L^2(\Omega)^n \) with domain \( H^2(\Omega)^n \cap H_0^1(\Omega)^n \) by:

\[
A^\varepsilon : u \mapsto \text{ div } A^\varepsilon(e(u)),
\]

is holomorphic of type (A) for \( \text{ Re } \varepsilon > 0 \), cf. [4, p. 375]. Since the tensors \( A^\varepsilon \) satisfy (2) and are \( C^\infty \) with respect to \( x \), the operators \( A^\varepsilon \) have a discrete spectrum which is bounded below with eigenvalues of finite multiplicity, and satisfy the Fredholm alternative, cf. [5, Theorem 6.5.3, p. 254]. Moreover, for \( \varepsilon = 0 \), at least one of the eigenvalues is strictly negative, otherwise the corresponding energy
would be positive for all \( u \), thus contradicting Lemma 1. Let \( \lambda_0 = \eta_0 - 2\mu_0 \). Then, for \( \varepsilon > -(2\mu_0 + n\lambda_0)/(2 + n) \), the energy becomes positive and therefore, all the eigenvalues also become positive. Since the eigenvalues depend holomorphically on \( \varepsilon \) for \( \text{Re} \varepsilon > 0 \), cf. [4, p. 392], and are real for \( \varepsilon \) real since the operators are selfadjoint, there exists \( \varepsilon_0 \) such that 0 is an eigenvalue of \( A^\varepsilon \), which completes the proof. \( \square \)

**Proof of Theorem 1.** We consider the family \( A^\varepsilon \) constructed in the proof of Lemma 2 for \( \varepsilon \) in a neighbourhood of \( \varepsilon_0 \). Let \( \lambda(\varepsilon) \) be an eigenvalue of \( A^\varepsilon \) that tends to 0 as \( \varepsilon \) tends to \( \varepsilon_0 \) and \( \phi(\varepsilon) \) a corresponding eigenfunction normalised in \( H_0^0(\Omega)^n \). Let us begin with the case of a variable right-hand side \( f^\varepsilon \) bounded in \( L^2(\Omega) \). We simply take \( f^\varepsilon = \phi(\varepsilon) \). Then \( u^\varepsilon = \phi(\varepsilon)/\lambda(\varepsilon) \) is not bounded in \( H_0^0(\Omega)^n \). Let us now turn to the case of a fixed left-hand side. We take a function \( f \) that does not belong to \( (\text{Ker} A^\varepsilon)^\perp \) (orthogonality is taken in the \( L^2 \) sense). For \( \varepsilon \) in a neighbourhood of \( \varepsilon_0 \) but different from \( \varepsilon_0 \), 0 is not an eigenvalue of \( A^\varepsilon \) (in fact, the eigenvalues are strictly increasing with \( \varepsilon \)) and therefore by the Fredholm alternative we can solve uniquely in \( H_0^0(\Omega)^n \):

\[
-\text{div}(A^\varepsilon e(u^\varepsilon)) = f.
\]

Let us assume that \( u^\varepsilon \) is uniformly bounded in \( H_0^0(\Omega)^n \). Then we can extract a subsequence \( \varepsilon' \) such that \( u^\varepsilon \) converges weakly to some \( u^0 \) in \( H_0^0(\Omega)^n \), and:

\[
-\text{div}(A^\varepsilon e(u^\varepsilon)) = -\text{div}(A^\varepsilon e(u^\varepsilon)) - (\varepsilon' - \varepsilon_0) \text{div}(\text{Bet}(u^\varepsilon)),
\]

where \( B \) is the elasticity tensor whose Lamé coefficients are \( (\lambda, \mu) = (1, 1) \). Therefore we see that:

\[
f = -\text{div}(A^\varepsilon e(u^\varepsilon)) \rightarrow -\text{div}(A^\varepsilon e(u^0)) \quad \text{as} \quad \varepsilon' \rightarrow \varepsilon_0,
\]

which contradicts the choice of \( f \) by the Fredholm alternative.

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**References**


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