THE CONSTITUTIVE LAW OF INCOMPRESSIBLE BODIES AND EXISTENCE IN INCOMPRESSIBLE NONLINEAR ELASTICITY.

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ABSTRACT: In this paper, we first give a justification for the form of the constitutive law of incompressible materials. The existence and uniqueness of the "indeterminate" pressure is established in a general case. In the context of nonlinear incompressible elasticity, we then prove an existence and uniqueness result for the pure displacement boundary value problem with sufficiently small body forces.

RESUME: Dans cet article, on donne d'abord une justification de la forme que prend la loi de comportement d'un matériau incompressible. On établit l'existence et l'unicité de la pression "indéterminée" dans un cas général. On montre ensuite, dans le cadre de l'élasticité non linéaire incompressible, un résultat d'existence et d'unicité pour le problème aux limites de déplacement avec des forces volumiques suffisamment petites.
0. INTRODUCTION.

This paper deals with some questions relative to incompressible bodies. It is divided in two parts. Part I is devoted to a justification of the material response of incompressible bodies. Such a response has been derived (for more general internal constraints) by Ericksen - Rivlin [8] in the hyperelastic case, the hydrostatic pressure appearing as a Lagrange multiplier. This was however a rather formal approach. This point of view was completed by Le Tallec-Oden [12]. They showed that the pressure is the Lagrange multiplier for the minimization of the energy on a submanifold of a Sobolev space. For non hyperelastic materials, we find in Truesdell-Noll [17], the claim that the Cauchy stress at an incompressible elastic point must be of the form:

\[ T = -p \text{Id} + H(F), \]

(with standard notations, to be specified later) where \( p \) is not determined by \( F \). The reason they give, which is stated as a principle, is that the determined stress \( H(F) \) must be complemented by some stress which does not work in any local incompressible motion. For other aspects, see Antman [5].

Our first aim is to give some mathematical justification for this type of constitutive law — and to render it a little more precise — in a more general case (neither hyperelastic, nor even elastic). Using the principle of virtual works in an appropriate Sobolev space setting, we show that the weak form of the equilibrium equations for an incompressible body leads to a variational problem (here defined on the submanifold \( \hat{\Gamma} \) of \( H^m(\Omega)^3 \) of incompressible deformations) if only the deviatoric part of the Cauchy stress is specified in the constitutive law. To justify this definition
a posteriori, we then prove that for any given equilibrium situation (i.e., given body forces, boundary conditions and a deformation which is a solution of the variational problem on the submanifold $\Sigma$), the trace of the Cauchy stress is indeed a well determined element of $L^2(\Omega)$. This trace appears actually as some kind of Lagrange multiplier, though we do not minimize any energy. It should be noted however, that the justification of the constitutive law is based on regularity hypotheses on the deformation which might be too strong. This regularity is needed to ensure that $\Sigma$ has a manifold structure. Nevertheless, we remark that the regularity is no longer needed to prove the existence of $p$, even though we kept it for consistency. Indeed, this existence follows from the principle of virtual works expressed only with $H^1$ test functions.

Considering an incompressible elastic body in part II, we give local existence and uniqueness results around a natural state for the pure displacement problem with dead or live loads. That is, if the body forces are sufficiently small in some spaces, there exist a unique deformation $\phi \in W^{m+2,q(\Omega)}$ and a unique pressure field $p \in W^{m+1,q(\Omega)/\mathbb{R}}$ satisfying strongly the equilibrium equation. Here $m \in \mathbb{N}$, $q \in ]1 + \infty[$ and for dead loading $(m+1)q > 3$ whereas for live loading $mq > 3$. We prove this by using the inverse function theorem or the implicit function theorem. The differential of the operators occurring in this procedure have a structure similar to the one of the Stokes problem. For these linear equations, we give an existence and uniqueness result in $H^1_0(\Omega)^3 \times L^2(\Omega)/\mathbb{R}$ using variational tools, and prove regularity with index properties (cf. Agmon – Douglis – Nirenberg [3], [4], Geymonat [9]).
The idea of using the inverse function theorem in nonlinear elasticity goes back to Stopelli [15] for the traction problem, Van Buren [20] for the Dirichlet problem, both in Hölder spaces. In Sobolev spaces, this approach was taken up by Marsden-Hughes [13], Ciarlet-Desruynder [7], Valent [19]. For the incompressible elastic case, the only existence result known to us is given in Ball [6], from a very different point of view. Ball proves the existence of a minimizer for the energy in the hyperelastic case with incompressible deformations in $W^{1,p}(\Omega)^3$. Assuming additional regularity on the minimizer, Le Tallec-Oden [12] then show the existence of the pressure. Naturally our approach does not need any such a priori regularity assumption. Moreover, the solution is a $C^1$-function of the data. The main defects of this method are that the result is valid only for small enough forces, and that it is limited to special classes of boundary conditions.
1. **JUSTIFICATION OF THE CONSTITUTIVE LAW OF INCOMPRESSIBLE MATERIALS.**

1. **HYPOTHESES AND NOTATIONS.**

Let \( \Omega \) be a bounded open subset of \( \mathbb{R}^3 \) with a sufficiently smooth boundary, considered as a reference configuration of an incompressible body. We shall consider the pure displacement and mixed boundary value problems in the spaces \( H^m(\Omega)^3 \) with \( m \in \mathbb{N} \) and \( 2(m-1) > 3 \).

Let \((\partial \Omega_1, \partial \Omega_2)\) be a measurable partition of \( \partial \Omega \) with \( \text{meas } \partial \Omega_1 > 0 \), such that the space:

\[
H^1_{\partial \Omega_1}(\Omega) = \{ v \in H^1(\Omega), \ v|_{\partial \Omega_1} = 0 \}
\]

is well defined.

The set of admissible deformations of the body will be:

\[
\sum_m = \{ \phi \in \Omega^m(\Omega)^3, \ \phi|_{\partial \Omega_1} = \phi_o, \ \text{det } V\phi = 1 \}
\]

where \( \phi_o \) is a given element of \( \Omega^m(\Omega)^3 \).

We use the following notations:

\( F = V\phi \): gradient of a deformation \( \phi \).

\( \text{Adj } F = (\text{det } F)F^{-1} \) whenever \( F \) is invertible.

\( T \): the Cauchy stress tensor.

\( T_R = |\text{det } F|TF^{-1T} \): the first Piola-Kirchhoff stress tensor.

\( b \): body force density per unit volume in the reference configuration.

\( t \): surface force density per unit area in the reference configuration.

Both \( b \) and \( t \) may mean either dead or live loads.

\( b \cdot \phi \): the usual scalar product of \( \mathbb{R}^3 \).

\( A:B = \text{tr}(A^TB) \): the usual scalar product of \( \mathbb{M}_3 \times 3 \) (the space of \( 3 \times 3 \) matrices).
2. **Definition of an Incompressible Constitutive Law.**

Consider the equilibrium equations in the reference configuration:

\[
\begin{align*}
\text{div } T_R + b &= 0 \quad \text{in } \Omega, \\
T_R \cdot n &= t \quad \text{on } \partial \Omega_2,
\end{align*}
\]

(1)

where \( n \) is the unit outer normal vector to \( \Omega \). Then, if we suppose \( T_R \in H^1(\Omega)^3 \), multiplying by \( \psi \in H^1_{0, \partial \Omega_1}(\Omega)^3 \) and integrating by parts yields:

\[
(2) \quad \text{For all } \psi \in H^1_{0, \partial \Omega_1}(\Omega)^3, \quad -\int_{\Omega} T_R : \nabla \psi + \int_{\Omega} b \cdot \psi + \int_{\partial \Omega_2} \tau \cdot \psi = 0.
\]

Equations (2) are the weak form of the equilibrium equations and express the principle of virtual works. Now, by definition of the Piola-Kirchhoff stress, we can write:

\[
T_R = T^D \text{Adj } F^T + \frac{1}{3} \text{tr } T \text{ Adj } F^T.
\]

where:

\[
T^D = T - \frac{1}{3} \text{tr } T I_d.
\]

is the deviatoric part of \( T \). Thus we have:

\[
(2') \quad -\int_{\Omega} T^D \text{Adj } F^T : \nabla \psi - \frac{1}{3} \int_{\Omega} \text{tr } T \text{ Adj } F^T : \nabla \psi + \int_{\Omega} b \cdot \psi + \int_{\partial \Omega_2} \tau \cdot \psi = 0
\]

for all \( \psi \) in \((H^1_{0, \partial \Omega_1}(\Omega))^3\).

Since we are looking for deformations in \( H^m(\Omega)^3 \) and since \( 2(m-1) > 3 \), \( \Sigma_m \) is a submanifold of \( H^m(\Omega)^3 \) as will be seen later. We want to generalize variational problems to such manifolds. A classical variational problem takes the following form: let \( E \) be a Banach space and \( A \) be a mapping \( A : E \to E' \). We usually want to solve the equation:

\[
Au = 0, \text{ or equivalently, } \forall \psi \in E, \quad <Au, \psi> = 0.
\]
Now, suppose our unknowns lie on a manifold \( M \). A natural generalization of this kind of problem is: let \( A : M \to T^* M \) be a section of \( T^* M \), i.e. \( \forall u \in M, Au \in T_u^* M \), \( T_u^* M \) is the cotangent space to \( M \) at \( u \) (the dual space of the tangent space \( T_u M \)) and \( T^* M = \bigcup_{u \in M} T_u^* M \) is the cotangent bundle.

The variational problem will be to solve:

\[
Au = 0 \quad \text{or} \quad \forall v \in T_u M, \quad <Au, v> = 0.
\]

When there are no internal constraints, the principle of virtual works together with a constitutive law actually takes this form. We want to keep it in our case. To do this, we must restrict equations (2) to test functions \( \psi \) such that:

\[
\forall \psi \in T\phi \sum_m \Delta_m, \text{ i.e. } \psi \in H^m(\Omega)^3, \psi|_{\partial \Omega_1} = 0, \text{ Adj } F^T : \nabla \varphi = 0 \text{ (cf corollary 3.3).}
\]

Hence, equation (2') implies:

\[
(3) \quad \forall \psi \in T\phi \sum_m \sum_{\Omega} T^d \text{ Adj } F^T : \nabla \psi + \int_{\Omega} b \cdot \psi + \int_{\partial \Omega_2} t^* \psi = 0.
\]

Therefore, the only part of \( T \) to be given by a constitutive law in order to keep the variational form must be the deviator \( T^d \). We are thus led to the:

2.1. Definition: A constitutive law for an incompressible body is a mapping \( H^D \) from the space of deformations to the space of \( 3 \times 3 \) symmetric matrices with zero trace such that the deviator of the Cauchy stress in a deformation \( \phi \) at a point \( \phi(x) \) verifies:

\[
T^d = H^D(x, \phi)
\]

Remarks: In the dynamical case one should indeed consider the history of the deformation instead of the deformation itself. The function \( H^D \) must satisfy the principle of frame indifference.
Actually, we shall prove that (3) (which we call the restricted principle of virtual works), with a constitutive law as in 2.1., determines (in the Dirichlet case, almost) uniquely \( p = \frac{\text{tr} \ T}{3} \in L^2(\Omega) \) such that:

\[
T_R = H^D(x, \phi) \ Adj F^T + p(x) \ Adj F^T.
\]

satisfies the "full" principle of virtual works (2). This argument could be carried out for other internal constraints such as those considered in [8] or [21], yielding results in the same spirit as those given there. Finally, the stress tensor of an incompressible elastic body should depend "as much as possible" on the deformation gradient only. Thus we pose:

2.2. DEFINITION: We call incompressible elastic body, an incompressible body the constitutive law of which takes the form:

\[
H^D(x, \phi) = \tilde{H}^D(x, \nabla \phi(x))
\]

where:

\[
\tilde{H}^D : \Omega \times M_3 \times 3 \rightarrow M_3 \times 3.
\]

Of course, this definition is equivalent to the usual one, while stated in a perhaps more precise form.

3. EXISTENCE AND UNIQUENESS OF THE HYDROSTATIC PRESSURE FIELD.

Given an incompressible body with its constitutive law, forces \( \mathbf{b} \) and \( \mathbf{t} \), let there be given a deformation \( \overline{\phi} \in \Sigma_m \) such that (3) holds. Equations (3) are of course necessary conditions for equilibrium. We show then that \( \text{tr} \ T \) is determined. To simplify the proofs, we shall assume that \( \overline{\phi} \) is a global \( C^1 \)-diffeomorphism of \( \overline{\Omega} \) to \( \overline{\varphi(\Omega)} \) (indeed, since \( \overline{\phi} \in \Sigma_m \) and because of the Sobolev imbeddings, \( \overline{\phi} \) is at least a local \( C^1 \)-diffeomorphism). We can then consider \( \overline{\varphi(\Omega)} \) as a new reference configuration, and restrict ourselves to the case where \( \overline{\phi} = \phi_0 = \text{Id} \). Then, the Cauchy stress and the Piola - Kirchhoff
Let us consider the problem:

\[
\begin{align*}
\Delta \theta &= g \quad \text{in } \Omega, \\
\theta &\in H^1_0(\Omega).
\end{align*}
\]

Then, by the regularity property, \( \nabla \theta \in H^m(\Omega)^3 \) and:

\[
\int_{\partial \Omega} \nabla \theta \cdot v = \int_{\Omega} \Delta \theta = \int_{\Omega} g = 0.
\]

Since \( \nabla \theta \big|_{\partial \Omega} \in H^{m-1/2}(\Omega)^3 \), by [10, remark 3.4.], there exists a function \( w \in H^m(\Omega)^3 \) such that:

\[
div w = 0 \quad \text{and} \quad w|_{\partial \Omega} = -\nabla \theta|_{\partial \Omega}.
\]

So, if we put: \( v = w + \nabla \theta \), then:

\[
\begin{align*}
\text{div } g, \\
\text{and } v|_{\partial \Omega} &= 0.
\end{align*}
\]

Proof of ii): The space \( \text{Ker } D_{\text{Id}}^m G \) is closed and has a closed complement, since we are working in Hilbert spaces.

Finally, we have [1 1]:

\[
T_1^m \sum_2 \text{Ker } D_{\text{Id}}^m G = \{ \psi \in (H^m(\Omega) \cap H^1_0(\Omega))^3, \text{ div } \psi = 0 \}.
\]

b) \( \partial \Omega \neq \emptyset \) (mixed displacement–traction boundary value problem).

We only have to check i). Let:

\[
g \in H^{m-1}(\Omega), \quad g_1 = g - \frac{1}{|\Omega|} \int_{\Omega} g.
\]

Then, there exists a function \( v \in (H^m(\Omega) \cap H^1_0(\Omega))^3 \) such that \( \text{div } v = g_1 \).

Thus, given \( \lambda \in \mathbb{R} \), we must find a function \( \hat{v} \) in \( H^m(\Omega)^3 \) which satisfies:

\[
\hat{v}|_{\partial \Omega} = 0 \quad \text{and} \quad \text{div } \hat{v} = \lambda.
\]
We consider again the problem:
\[
\begin{cases}
\Delta \theta = \lambda & \text{in } \Omega, \\
\theta \in H^1_0(\Omega).
\end{cases}
\]

Then we have: \( \theta \in H^{m+1}(\Omega) \) and \( \text{div} (\nabla \theta) = \lambda \). Let us choose a cut-off function \( \eta \) on \( \partial \Omega \) in such a way that:
\[
\eta \equiv 1 \text{ on } \partial \Omega_1, \quad \eta \equiv 0 \text{ on } \Gamma \subset \partial \Omega_2 \quad \text{and} \quad \eta \in C^\infty(\partial \Omega, \mathbb{R}) .
\]

Let us choose another function \( \eta' \in C^\infty(\partial \Omega, \mathbb{R}) \) such that:
\[
\text{supp } \eta' \subset \Gamma \text{ and } \int_{\partial \Omega} \eta' = 1 .
\]

Then let \( \alpha = -\int_{\partial \Omega} \eta (\nabla \theta \cdot \nu) \) and \( h = -\eta \nabla \theta - \alpha \eta' \nu \).

By construction, \( h \in C^\infty(\partial \Omega, \mathbb{R}^3) \) and:
\[
\int_{\partial \Omega} h \cdot \nu = \alpha - \alpha \int_{\partial \Omega} \eta' (\nu \cdot \nu) = 0 .
\]

So, always by [10], there exists a \( w \in H^m(\Omega)^3 \) such that:
\[
\begin{cases}
\text{div} \, w = 0 & \text{in } \Omega, \\
w = h & \text{on } \partial \Omega.
\end{cases}
\]

If we pose \( \tilde{\nu} = w + \nabla \theta \), then:
\[
\begin{cases}
\text{div} \, \tilde{\nu} = \lambda & \text{in } \Omega, \\
\tilde{\nu}|_{\partial \Omega_1} = h|_{\partial \Omega_1} + \nabla \theta|_{\partial \Omega_1} = 0 \quad \text{and} \quad \tilde{\nu} \in (H^m(\Omega))^3 .
\end{cases}
\]

3.3. COROLLARY. Let \( \phi \in \sum_m \). If \( \phi \) is globally invertible, then:
\[
T_\phi \sum_m = \{ \psi \in H^m(\Omega)^3, \quad \psi|_{\partial \Omega_1} = 0, \quad \text{Adj } F^T : \nabla \psi = 0 \} .
\]

Proof: Compose the result of 3.2. with \( \phi^{-1} \) to get the result.
To proceed further, let us introduce some notations:

\[ Y = \{ v \in H^m(\Omega)^3, v|_{\partial \Omega_1} = 0 \} \]

\[ P \in Y', \text{ defined by } P \cdot v = -\int_{\Omega} T^D(\text{Id}) : \nabla v + \int_{\Omega} b \cdot v + \int_{\partial \Omega_2} t \cdot v. \]

Then (3) may be written:

\[ \forall \psi \in T^D_{\text{Id}} \sum_{\mathbb{m}} P \cdot \psi = 0. \]

3.4. **THEOREM**: There exists a unique \( p \in L^2(\Omega) \) (\( p \in L^2(\Omega)/\mathbb{R} \) in the Dirichlet problem) such that:

\[ \forall v \in Y, P \cdot v = \int_{\Omega} p \nabla \cdot v. \]

**Proof**:

We see that \( P \in (H^{-1}(\Omega))^3 \). Since:

the space \( \mathcal{V} = \{ v \in \mathcal{D}(\Omega)^3, \nabla \cdot v = 0 \} \) is dense in the space

\( \mathcal{V} = \{ v \in H^1_0(\Omega)^3, \nabla \cdot v = 0 \} \) and since \( \mathcal{V} \subset T^D_{\text{Id}} \sum_{\mathbb{m}} \), we obtain:

\[ \forall v \in \mathcal{V}, P \cdot v = 0. \]

Then, by [10], there exists one and only one \( p \in L^2(\Omega)/\mathbb{R} \) such that:

\[ \forall v \in H^1_0(\Omega)^3, P \cdot v = \int_{\Omega} p \nabla \cdot v. \]

Now, if \( \partial \Omega_2 = \emptyset \) (the Dirichlet case), we are done, since

\[ Y = (H^m(\Omega) \cap H^1_0(\Omega))^3. \]

If \( \partial \Omega_2 \neq \emptyset \), choose a representant of \( p \) such that \( \int_{\Omega} p = 0 \), and consider the space:

\[ M = \{ v \in H^1(\Omega)^3, v = 0 \quad \text{on} \quad \partial \Omega_1, \quad \int_{\partial \Omega_2} v \cdot v = 0 \}. \]

Let \( v \in M \). Then, by [10], there exists \( v_1 \in H^1(\Omega)^3 \) such that:
\[
\begin{align*}
\begin{cases}
\text{div } v_1 = 0 & \text{ in } \Omega, \\
v_1|\partial\Omega = v|\partial\Omega.
\end{cases}
\end{align*}
\]

So \( v - v_1 \in H^1_0(\Omega)^3 \) and we have, since \( v_1 \in \text{Ker } P \):

\[
P \cdot (v - v_1) = P \cdot v - P \cdot v_1 = P \cdot v = \int_{\Omega} p \text{ div}(v - v_1) = \int_{\Omega} p \text{ div } v.
\]

Hence, for all \( v \in M \), we have:

\[
P \cdot v = \int_{\Omega} p \text{ div } v.
\]

Now, \( M \) is a closed hyperplane of the space \( Y_1 = H^1_{0,\partial\Omega}(\Omega)^3 \). If we pose,

for \( v \in Y_1 \), \( L \cdot v = P \cdot v - \int_{\Omega} p \text{ div } v \), then \( L \in Y'_1 \) and \( M \subset \text{Ker } L \).

Therefore, \( L \) is determined by its action on the orthogonal complement of \( M \) in \( Y_1 \). We shall identify \( Y'_1 \) with \( Y_1 \) via the scalar product

\[
\langle v_u, v_v \rangle^2 = \int_{\Omega} v_u : v_v. \text{ Let } u \in M^1, \text{ then by definition :}
\]

\[
\forall v \in M, \int_{\Omega} v_u : v_v = 0.
\]

Since \( D (\Omega)^3 \supseteq M \), this implies that (considering \( \Delta u \) as a distribution):

\[
\forall v \in D (\Omega)^3, \quad \langle \Delta u, v \rangle = 0, \text{ or } \Delta u = 0.
\]

But \( \Delta u = \text{div}(v_u) = 0 \), so \( v_u \in H(\text{div},\Omega)^3 \) and the following Green's formula holds (which slightly generalizes the one given in [10]):

\[
\int_{\Omega} v_u : v_v = -\int_{\Omega} \Delta u \cdot v + \langle v_u, (v_u)^T \cdot v \rangle_{\partial\Omega}, \text{ for all } v \in H^1(\Omega)^3,
\]

the last pairing being the one between \( (H^{1/2}(\partial\Omega))^3 \) and \( (H^{-1/2}(\partial\Omega))^3 \).

So, for all \( v \) in \( M \), we have:

\[
\langle v, v_u \cdot v \rangle_{\partial\Omega} = 0,
\]

and for all \( v \) in \( Y_1 \), we have:

\[
(4) \quad \langle v_u, v_v \rangle^2 = \langle v, v_u \cdot v \rangle_{\partial\Omega}.
\]
Hence, there exists one and only one scalar $\lambda_1$ such that for all $v \in Y_1$:

$$L \cdot v = \lambda_1 < v, vu^T \cdot v >_{\partial \Omega}$$

We have to identify $vu^T \cdot v$. To do this, consider the space:

$$M_1 = \{ v \in H^{1/2}(\partial \Omega)^3, v|_{\partial \Omega_1} = 0, \int_{\partial \Omega} v \cdot v = 0 \}.$$

Then $M_1$ is a closed hyperplane of the space $\{ v \in H^{1/2}(\partial \Omega)^3, v|_{\partial \Omega_1} = 0 \}$.

By formula (4), we see that $vu^T \cdot v \in M_1$. But $M_1$ being the orthogonal complement of a hyperplane is one dimensional, and we know one of its elements, by the definition of $M_1$: namely $v$. So, there exists one and only one scalar $\lambda_2$ such that:

$$vu^T \cdot v = \lambda_2 v.$$

Finally, if we let $\lambda = \lambda_1 \lambda_2$, we have for all $v \in Y_1$:

$$L \cdot v = \lambda < v, v >_{\partial \Omega} = \lambda \int_{\partial \Omega} v \cdot v.$$

And:

$$P \cdot v = \int_{\Omega} p \text{ div } v + \lambda \int_{\partial \Omega} v \cdot v = \int_{\Omega} (p + \lambda) \text{ div } v = \int_{\Omega} (p + \lambda) \text{ Id } \cdot v.$$

3.5 INTERPRETATION:

Let Id be an equilibrium configuration. Equations (3) imply that:

$$\mathbb{W} v \in Y_1, -\int_{\Omega} (T^D + (p + \lambda) \text{ Id }) : \nabla v + \int_{\Omega} b \cdot v + \int_{\partial \Omega_2} t \cdot v = 0.$$

Hence, the uniqueness of $p' = p + \lambda$ together with (2) show that

$$\text{tr } T = 3 p'.$$

So, the so-called "indeterminate" pressure appears as a kind of Lagrange multiplier, associated with the restricted principle of virtual works.
In the case of a hyperelastic body and conservative forces, our result reduces to the one of Le Tallec – Oden [12], where \( p' \) is a genuine Lagrange multiplier. Note however that we have not specified what kind of load \( b \) and \( t \) represent. We only need that they define a continuous linear form on \( H^1(\Omega)^3 \), for a fixed deformation \( \phi \), e.g. that \( b \in L^2(\Omega)^3 \), \( t \in L^2(\partial \Omega)^3 \). Hence, either dead or live, conservative or not conservative body forces are allowed, together with boundary conditions such as one of pressure. Note also that it is not necessary to assume that the body is elastic. The existence and uniqueness of \( p' \) follows only from equations (3).

**Remark:** In the Dirichlet case, the pressure need only be defined up to an additive constant. But, if \( \partial \Omega \neq \emptyset \), the value of the pressure must be completely determined, since it appears in the actual traction acting on the boundary. Hence, the two steps in the proof are necessary.

After having justified, though under perhaps unqualified regularity hypotheses, the form of the material response of an incompressible body, we now turn to existence results.

**II. LOCAL EXISTENCE RESULTS FOR THE PURE DISPLACEMENT PROBLEM.**

**I. EQUATIONS AND HYPOTHESES.**

Let \( m \in \mathbb{N} \). We suppose that for dead loading \( 2(m + 1) > 3 \) and for live loading \( 2m > 3 \). We suppose that \( T^D \) \( \text{Adj} \ F^T \) is a given function \( A(x, F) \) with \( A(.,\cdot) \in C^{m+2}(\Omega \times \mathbb{R}^3, \mathbb{R}^3) \). Then, the Nemytsky operator \( \phi \rightarrow A(.,\nabla \phi(.)) \) is a \( C^1 \)-mapping from \( H^{m+2}(\Omega)^3 \) to \( H^{m+1}(\Omega)^3 \), see Valent [18]. Given \( f \in H^m(\Omega)^3 \) for dead loading or \( f \in C^m_B(\mathbb{R}^3)^3 \) (the Banach space of \( m \) times continuously differentiable functions with bounded derivatives) for live
loading, the problem is (we denote by $H^{m+1,0}(\Omega)$ the space of $H^{m+1}(\Omega)$ functions with zero integral):

$$\begin{align*}
\text{Find } (\phi, p) \in \prod_{m+2} \times H^{m+1,0}(\Omega) \text{ such that:} \\
\begin{cases}
\text{div } A(x, \nabla \phi(x)) - \text{div}(p(x)(\text{Adj}\nabla \phi)^T(x)) + f(x) = 0 & \text{in } \Omega, \\
\phi = \text{Id} & \text{on } \partial \Omega.
\end{cases}
\end{align*}$$

(5.d)

for dead loads.

$$\begin{align*}
\text{div } A(x, \nabla \phi(x)) - \text{div}(p(x)(\text{Adj}\nabla \phi)^T(x)) + f(\phi(x)) = 0 & \text{in } \Omega, \\
\phi = \text{Id} & \text{on } \partial \Omega.
\end{align*}$$

(5.e)

for live loads.

As it has been remarked previously, any deformation $\phi$ satisfying (5.d) or (5.e) is automatically a $C^1$-diffeomorphism of $\Omega$.

We assume that the reference configuration is natural, i.e. $\Psi \times \in \Omega$, $A(x, \text{Id}) = 0$, so that for $f = 0$, $(\text{Id}, 0)$ is a solution of (5.d) and (5.e).

We make on the function $A$ the following hypotheses. Denote by $L(.)$ the differential of $A$ (considered as a Nemytskii operator) at $\phi = \text{Id}$. From the principle of material indifference, the argument of $L$ is $c(u) = \frac{1}{2}(\Psi u + \Psi u^T)$ and we let:

$$L(\varepsilon(u))_{ij} = c_{ijkl}(x) \varepsilon_{kl}(u).$$

We suppose that the coefficients $c_{ijkl}$ are sufficiently regular. We make on $c_{ijkl}$ the following assumptions:

H1 : $c_{ijkl} = c_{klji}$ (which means that the linearized material is hyperelastic).

H2 : There exists $\alpha > 0$ such that for all $x \in \Omega$, $c \in M_3 \times 3$ with $\varepsilon = \varepsilon^T$ and $\text{tr } \varepsilon = 0$:

$$c_{ijkl}(x) \varepsilon_{ij} \varepsilon_{kl} \geq \alpha \varepsilon : \varepsilon = \alpha \| \varepsilon \|^2.$$

(ellipticity).
Remark: from the symmetry of $L$ and $\varepsilon$, we deduce that:

$$c_{ijk\ell} = c_{ij\ell k} \quad \text{and} \quad c_{ijk\ell} = c_{jik\ell}.$$ 

Consider the following nonlinear operators:

$$R_D : \sum_{m+2} H^{m+1,0}(\Omega) \to H^m(\Omega)^3$$

$$\begin{array}{c}
\phi, p \mapsto \text{div} (A - p \text{Adj } F^T).
\end{array}$$

$$R_L : \sum_{m+2} H^{m+1,0}(\Omega) \times C^m_0(R^3) \to H^m(\Omega)^3$$

$$\begin{array}{c}
\phi, p, f \mapsto \text{div} (A - p \text{Adj } F^T) + f(\phi).
\end{array}$$

with abbreviated notations and where $R_D$ stands for dead loads while $R_L$ stands for live loads. From the hypotheses on $m$, $R_D$ and $R_L$ are of class $C^1$, [18].

To apply the inverse function theorem or the implicit function theorem, we have to compute the:

2. DIFFERENTIALS OF $R_L$ AND $R_D$ AT $(\text{Id}, 0)$.

2.1. Remark: We have that: \( \text{div} (p \text{Adj } F^T) = \text{Adj } F^T \, \nabla p \).

Proof: Let $a$ be a scalar $C^\infty$ function on $\Omega$ and $B$ a matrix valued $C^\infty$ function on $\Omega$. Then, by direct computation:

$$\text{div} (aB) = B \cdot \nabla a + a \text{ div } B.$$ 

Since $H^{m+1}(\Omega)$ is a Banach algebra, this identity holds in $H^{m+1}(\Omega)$ by a density argument, and we can take $a = p$ and $B = \text{Adj } F^T$. This yields:

$$\text{div} (p \text{Adj } F^T) = \text{Adj } F^T \text{ div } p + p \text{ div } (\text{Adj } F^T).$$

Since $\text{div} (\text{Adj } F^T) = 0$ (which is the Piola identity), we have the result.

For $R_D$, we need the differential at $(\text{Id}, 0)$ and for $R_L$, the partial differential with respect to $(\phi, p)$ at $(\text{Id}, 0, 0)$. 
2.2. **PROPOSITION** : These two differentials are identical. If we denote their common expression by $DR$, we have:

$$DR : T_{Id} \sum L_{m+2} \times H^{m+1,0}(\Omega) \rightarrow H^m(\Omega)^3$$

$$(u, q) \mapsto \text{div} L (\varepsilon(u)) - \nabla q.$$  

**Proof** : Easy verification, recalling that we differentiate at $p = 0$, $f = 0$. 

In order to proceed further, we have to prove that this operator is an isomorphism between these spaces. This problem can be rewritten as a linear partial differential equation in the following way:

Given $g \in H^m(\Omega)^3$, find $(u, q) \in (H^{m+2}(\Omega) \cap H_0^1(\Omega))^3 \times H^{m+1,0}(\Omega)$ such that:

$$
\begin{cases}
\text{div} L (\varepsilon(u)) - \nabla q = g & \text{in } \Omega, \\
\text{div} u = 0 & \text{in } \Omega.
\end{cases}
$$

(6)

and show that the solution is unique.

We recognize here a "Stokes-like" problem: the role of the fluid velocity is played by the linearized displacement. The question breaks down in two parts: existence and uniqueness in $H^1_0(\Omega)^3 \times L^2_0(\Omega)$ by variational methods, and regularity using index properties [4], [9]. To deal with the second aspect, we introduce the following operators:

$$\forall \, m \in \mathbb{Z}, \, p \in [1, +\infty],$$

$$T_{m, p} : \mu^{m+2, p}(\Omega)^3 \times \mu^{m+1, p}(\Omega) \rightarrow \mu^m, p(\Omega)^3 \times \mu^{m+1, p}(\Omega) \times \mu^{m+2-1/p, p}(\partial \Omega)^3,$$

$$
\begin{cases}
\text{div} L(\varepsilon(v)) - \nabla q = g & \text{in } \Omega, \\
\text{div} v = h & \text{in } \Omega, \\
v|_{\partial \Omega} = f & \text{on } \partial \Omega.
\end{cases}
$$

(7)
3. KERNEL, IMAGE AND INDEX OF $T_{-1,2}$.

Denote the index of $T_{m,p}$ by $\chi(T_{m,p})$.

3.1. PROPOSITION: We have the following characterizations:

$$\text{Im} \, T_{-1,2} = \{ (g,h,f) \in H^{-1}(\Omega)^3 \times L^2(\Omega) \times H^{1/2}(\partial \Omega)^3, \int_{\Omega} h = \int_{\partial \Omega} f \cdot \nu \}. $$

$$\text{Ker} \, T_{-1,2} = \{ 0 \} \times \mathbb{R}.$$ 

$$\chi(T_{-1,2}) = 0 .$$

where we identify the space of constant functions on $\Omega$ with $\mathbb{R}$.

Proof:

The characterization of $\text{Im} \, T_{-1,2}$ is obviously necessary by Green's formula. Suppose given a triple $(g,h,f)$ which satisfies it. We want to show that $(g,h,f)$ is in the image of $T_{-1,2}$. We first reduce ourselves to the case $f = 0$ by setting:

$$\nabla \cdot L(\v') - Vq = g - \nabla \cdot L(\v(f)) = g' \quad \text{in } \Omega ,$$

$$\v' = \nu - f = \left\{ \begin{array}{l}
\nabla \v' = h - \nabla f = h' \\
\nu' = 0
\end{array} \right. \quad \text{in } \Omega ,$$

$$\text{on } \partial \Omega .$$

where we have identified $f$ with some $H^1$-function having a trace equal to $f$. Hence:

$$\int_{\Omega} h' = \int_{\Omega} h - \int_{\Omega} \nabla f = \int_{\Omega} h - \int_{\partial \Omega} f \cdot \nu = 0 .$$

Therefore $h' \in L^2(\Omega)$. But there exists a continuous linear lift of $L^2(\Omega)$ into $H^1(\Omega)^3$ which is a right inverse of the mapping $\nabla \cdot$, see [10], since $\nabla$ is an isomorphism of $V^1$ onto $L^2(\Omega)$ ($V$ is defined in I.3.4.). Hence, we have, with $v'' = \nu' - \nabla^{-1} h'$:
\[ s_i = 2, \quad i = 1, 2, 3, \quad s_4 = 1, \]
\[ t_j = 0, \quad j = 1, 2, 3, \quad t_4 = -1, \]
\[ q = 0, \quad q = 1, 2, 3, 4. \]

Then: \( L \equiv \max(0, \sigma_1 + 1, \sigma_2 + 1, \sigma_3 + 1, \sigma_4 + 1) = 1. \)

And the system is elliptic in the sense of Agmon - Douglis - Nirenberg \[4\] if:

\[
\begin{vmatrix}
\epsilon_{jkl} \xi_j \xi_k & \xi_i \\
\xi_j & 0
\end{vmatrix} \neq 0.
\]

(i is the row index, \( \ell \) the column index).

4.1. **PROPOSITION**: Hypothesis H2 implies the ellipticity in the sense of Agmon - Douglis - Nirenberg.

**Proof**: Given \( \xi \in \mathbb{R}^3 \setminus \{0\} \), let:

\[ r_{i\ell} = c_{ijkl} \xi_j \xi_k. \]

We consider \( \eta \in \mathbb{R}^3 \setminus \{0\} \) such that \( \eta \cdot \xi = 0 \). Then, the matrix

\[ \epsilon = \frac{1}{2}(\xi \otimes \eta + \eta \otimes \xi) \]

(where \( \otimes \) denotes the usual tensor product) is symmetric and:

\[ \text{tr} \epsilon = \xi \cdot \eta = 0. \]

We can then apply H2 to \( \epsilon \) to get:

\[ c_{ijkl} \epsilon_{ij} \epsilon_{kl} = \frac{1}{4} c_{ijkl} (\xi_k \eta_k + \xi_k \eta_k) (\xi_i \eta_j + \xi_j \eta_i) \]

\[ = c_{ijkl} \xi_k \eta_i \eta_j. \]

and

\[ c_{ijkl} \xi_k \eta_i \eta_j > \alpha \| \epsilon \|^2. \]
But, by definition, $\|c\|^2 = \text{tr} \, c^T c = \text{tr} \, \epsilon^2$, and:

$$\epsilon^2 = \frac{1}{4} \left[ (\xi \otimes \eta)^2 + (\eta \otimes \xi)^2 + (\eta \otimes \xi) \cdot (\xi \otimes \eta) + (\xi \otimes \eta) \cdot (\eta \otimes \xi) \right].$$

Now, recall that $\xi \cdot \eta = 0$, which implies $(\xi \otimes \eta)^2 = (\eta \otimes \xi)^2 = 0$.

So we have:

$$\|\epsilon\|^2 = \frac{1}{2} \|\xi \otimes \eta\|^2 = \frac{1}{2} \|\xi\|^2 \|\eta\|^2.$$

Finally we have the following inequality:

$$\forall \eta \in \mathbb{R}^3 \text{ with } \eta \cdot \xi = 0, \quad \tau_{il} \eta_i \eta_l \geq c \|\eta\|^2 \quad \text{with } c > 0.$$

We consider now the endomorphism of $\mathbb{R}^3 \times \mathbb{R}$ whose matrix in the canonical basis is:

$$M_{\xi} = \begin{pmatrix} \tau_{il} & \xi_l \\ \xi_i & 0 \end{pmatrix}.$$

To prove that $D(\xi) \neq 0$, we only have to show that $M_{\xi}$ is one-to-one, that is to solve:

$$\text{find } (\zeta, p) \in \mathbb{R}^3 \times \mathbb{R}, \quad M_{\xi} \begin{pmatrix} \zeta_i \\ p \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

or equivalently:

$$\begin{cases} \tau_{il} \zeta_l + p \xi_i = 0, \\
\xi_l \zeta_l = 0. \end{cases}$$

From the last equation, we get that $\xi \cdot \zeta = 0$, and taking the scalar product of the first equations against $\zeta$ yields:

$$\tau_{il} \zeta_l \zeta_i = 0 \geq c \|\zeta\|^2,$$

which implies $\zeta = 0$ and so $p = 0$. \QED

In order to apply [9], we only need the hypothesis:

$H3: \text{The complementary condition of Agmon - Douglis - Nirenberg}$ [4 ]

holds.

Hence we have:
4.2. **PROPOSITION**: The same characterizations as in 3.1. hold:

\[
\text{Ker } T_{m,p} = \{0\} \times \mathbb{R}.
\]

\[
\text{Im } T_{m,p} = \{(g, h, f) \in \mathcal{W}^{m, p(\Omega)} \times \mathcal{W}^{m+1, p(\Omega)} \times \mathcal{W}^{m+2-1/p, p(\Omega)} : h = \int_{\Omega} f \cdot \nu \}\.
\]

for \( m \in \mathbb{Z} \), \( m \geq -1 \) and \( p \in \]1, +\infty[\).

**Proof**: Using the results of Geymonat [9], we infer that \( \text{Ker } T_{m,p} \) and \( \chi(T_{m,p}) \) are independent of \( m \) and \( p \), for \( m \in \mathbb{Z} \), \( m+2 \geq \ell = 1 \), and \( 1 < p < +\infty \). Hence, \( \text{codim } \text{Im } T_{m,p} = 1 \). But, by Green's formula, we have:

\[
\text{Im } T_{m,p} \subset \{(g, h, f) : \int_{\Omega} h = \int_{\partial \Omega} f \cdot \nu\}.
\]

The space of the right hand side is a hyperplane, just like \( \text{Im } T_{m,p} \).

Therefore we have:

\[
\text{Im } T_{m,p} = \{(g, h, f) : \int_{\Omega} h = \int_{\partial \Omega} h \cdot \nu\}.
\]

4.3. **COROLLARY**: Equations (6) define an isomorphism between the spaces

\[
\mathcal{W}^{m+2, p(\Omega)} \times \mathcal{W}^{m+1, p, 0(\Omega)} \text{ and } \mathcal{W}^{m, p(\Omega)} \text{, for all } m \in \mathbb{Z} \text{, } m \geq -1, p \in \]1, +\infty[.
\]

**Proof**: obvious.

5. **A SPECIAL CASE: THE ISOTROPIC HOMOGENEOUS BODY.**

5.1. **PROPOSITION**: If the body is isotropic and homogeneous, then equations (6) become:

\[
\begin{cases}
\mu \Delta v - \nu q = g & \text{in } \Omega, \\
d \nu v = 0 & \text{in } \Omega.
\end{cases}
\]

which is the classical Stokes problem.
Proof: By assumption, \( T^D \) must be such that: \( T^D(F) = T^D(B) \) where \( B = FF^T \), see [21]. In addition, \( T^D \) must be an isotropic function, so that linearizing about the identity, we get:

\[
L(\varepsilon(v)) = 2\mu \varepsilon(v) - \frac{2\mu}{3} \text{ div } v \text{ Id }.
\]

where \( \mu \) is the second Lamé coefficient. We assume that \( \mu > 0 \), which is a realistic assumption for actual materials. For \( v \in V \), we have:

\[
\text{div } L(\varepsilon(v))_h = \mu a_k(\varepsilon_k v_h + \varepsilon_h v_k) = \mu a_{kk} v_k h + \mu a_{kh} v_k.
\]

But, since \( v \in V \), \( \text{div } v = 0 \) and:

\[
a_{kh} v_k = a_h(\varepsilon_k v_k) = \varepsilon_h(\text{div } v) = 0.
\]

Therefore we have:

\[
\text{div } L(\varepsilon(v))_h = \mu \Delta v_h \quad \text{or } \quad \text{div } L(\varepsilon(v)) = \mu \Delta v.
\]

Remark: In this particular case, hypotheses H1, H2 and H3 are satisfied, see Temam [16].

6. CONCLUSIONS.

6.1. THEOREM: There exist a neighborhood \( W \) of \( 0 \) in \( H^m(\Omega)^3 \), and a neighborhood \( U \) of \( (\text{Id}, 0) \) in \( \sum_{m+2} \times H^{m+1,0}(\Omega) \) such that, for \( f \in W \), the dead loading pure displacement problem has a unique solution \( (\phi, p) \in U \). Moreover, \( (\phi, p) \) is a \( C^1 \) function of \( f \).

Proof: Apply the inverse function theorem to \( R_D \) at \( (\text{Id}, 0) \) between \( \sum_{m+2} \times H^{m+1,0}(\Omega) \) and \( H^m(\Omega)^3 \). The differential \( D_{\text{Id}} R_D \) is an isomorphism; therefore the result holds.

6.2. THEOREM: There exist a neighborhood \( W \) of \( 0 \) in \( \mathcal{C}^m_b(R^3)^3 \), a neighborhood \( U \) of \( (\text{Id}, 0) \) in \( \sum_{m+2} \times H^{m+1,0}(\Omega) \) and a \( C^1 \) mapping \( \mathcal{E} \) from \( W \) to \( U \), such that for \( f \in W \), there exists a unique solution to the live loading pure displacement problem, which is \( \mathcal{E}(f) \).
Proof: Apply the implicit function theorem to $\mathbb{R}$ at $(\text{Id}, 0, 0)$.

6.3. Remark: The two results above hold in $W^{m+2,p}(\Omega)^3 \times W^{m+1,p,0}(\Omega)$ for $(m+1) p > 3$ in the dead loading case and $mp > 3$ in the live loading case.

Proof: It remains to show that the set:

$$\sum_{m+2,p} = \{ \phi \in W^{m+2,p}(\Omega)^3, \phi = \text{Id} \text{ on } \partial \Omega, \det V\phi = 1 \},$$

is a manifold for $(m + l) p > 3$. By the transversality properties, this is implied by the surjectivity of the operator $\text{div}$ from $(W^{m+2,p}(\Omega) \cap W^{1,p,0}(\Omega))^3$ onto $W^{m+1,p,0}(\Omega)$, (condition i) of prop. I.3.2.), and by the existence of a continuous right inverse of $\text{div}$ (condition ii) of prop. I.3.2.). These two facts are easy consequences of proposition II.4.2. The rest of the proof goes the same as above.

When we replace the spaces $H^{m+2}(\Omega)^3$, $m \geq 1$, by spaces $W^{m+2,p}(\Omega)^3$, the optimal order of differentiation of the solution is decreased by one. Indeed, if we take $p > 3$, then $m > 0$ suffices. This corresponds to a solution $(\phi, p)$ in the spaces $W^2,p(\Omega)^3 \times W^1,p(\Omega)$, instead of $H^3(\Omega)^3 \times H^2(\Omega)$, and admissible loads in $L^p(\Omega)^3$, instead of $H^1(\Omega)^3$. 
- BIBLIOGRAPHY -


