Unique Continuation and applications

Camille LAURENT[1] [2]

[1]CNRS, UMR 7598, Laboratoire Jacques-Louis Lions, F-75005, Paris, France
[2]UPMC Univ Paris 06, UMR 7598, Laboratoire Jacques-Louis Lions, F-75005, Paris, France, email: laurent@ann.jussieu.fr

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Chapter 1

Introduction and generalities

We can find good references about Carleman estimates and unique continuation. This course was very influenced by the very good course of Nicolas Lerner that can be found on his website http://webusers.imj-prg.fr/˜ nicolas.lerner/m2carl.pdf and the survey article by Jérôme Le Rousseau and Gilles Lebeau [9].

Other references about unique continuation are the book of Claude Zuily [14]. Chapter XXVIII of Lars Hörmander [7] gives a more general framework for what is described in Chapter 2.1.

We also refer to the very complete notes of Daniel Tataru available at https://math.berkeley.edu/˜ tataru/papers/ucpnotes.ps.

1.1 Generalities about unique continuation

1.1.1 The problem

The general problem of *unique continuation* can be set into the following form: given a differential operator P on an open set $\Omega \subset \mathbb{R}^n$, and given a small subset U of Ω, do we have a bigger set $U \subset \widetilde{U}$ strictly bigger than U so that for u regular enough

$$\left\{ \begin{array}{rcl} Pu & = & 0 \text{ in } \Omega, \\ u_{|U} & = & 0 \end{array} \right. \Longrightarrow u = 0 \text{ on } \widetilde{U}. \tag{1.1}$$

More generally, in cases where (1.1) is known to hold, we are interested in proving a quantitative version of

$$\left\{ \begin{array}{rcl} Pu & \text{small in} & \Omega, \\ u & \text{small in} & U \end{array} \right. \Longrightarrow u \text{ small in } \widetilde{U}.$$

A more tractable problem than (1.1) is the so called *local unique continuation across an hypersurface* problem: given $x^0 \in \mathbb{R}^n$ and $S = \{\Phi = 0\}$ an oriented local hypersurface containing x^0, do we have the following implication:

There is a neighborhood Ω of x^0 so that

$$\left\{ \begin{array}{rcl} Pu & = & 0 \text{ in } \Omega, \\ u & = & 0 \text{ in } \Omega \cap S^+ \end{array} \right. \Longrightarrow u = 0 \text{ in a neighborhood of } x_0. \tag{1.2}$$

where $S^+ = \{\Phi > 0\}$.

It turns out that proving (1.2) for a suitable class of hypersurface (with regards to the operator P) is in general a key step in the proof of properties of the type (1.1).

They are some very easy already known cases

- $n=1$: if P has coefficients regular enough and u is regular enough, $Pu=0$ is only an ordinary differential equation. The Cauchy-Lipschitz theorem gives directly the uniqueness since $u=0$ on $x\le 0$ implies $\frac{d^k}{dx^k}u(0)=0$ for all $k\in\mathbb{N}$.

- if $n=2$, $P=\overline{\partial}=\frac{1}{2}\left(\partial_x+i\partial_y\right)$. $Pu=0$ implies that u is an analytic function. In particular, if $u=0$ on any open set, then $u=0$.

Concerning Partial Differential equations, it seems that there could be some geometrical conditions.

Theorem 1.1.1 (Finite speed of propagation for the wave equation)**.** *Let u be a $C^2(\mathbb{R}^{1+n})$ (real valued) solution of*

$$\left\{\begin{array}{rcl}\partial_t^2 u-\Delta u & = & 0\\ (u,\partial_t u)_{|t=0} & = & (u_0,u_1)\end{array}\right.$$

Then, for any $r_0>0$, we have for any $t\in[0,r_0]$

$$E_{r_0-t}(t)\le E_{r_0}(0)$$

where $E_r(t)=\frac{1}{2}\int_{|x|\le r}(\partial_t u(t,x))^2+|\nabla_x u(t,x)|^2\ dx$ is the local energy in the ball of radius r at time t.

In particular, if $u_0(x)=u_1(x)=0$ for $|x|\le r_0$, then $u=0$ in the cone

$$C_{r_0}=\left\{(t,x)\in\mathbb{R}^{1+n}\text{ s.t. }|x|\le r_0-t,t\in[0,r_0]\right\}.$$

Proof. Multiply the equation by $\partial_t u$ to obtain $\partial_t^2 t\partial_t u-\Delta u\partial_t u=0$. First, we notice $\partial_t^2 t\partial_t u=\frac{1}{2}(\partial_t u(t,x))^2$. Moreover, by the formula $div_x(fX)=f div(X)+\nabla f\cdot X$ for f C^1 function and X C^1 vector field, we get $\Delta u\partial_t u=div_x(\nabla u)\partial_t u=div_x(\nabla_x u\partial_t u)-\nabla_x u\cdot\nabla_x\partial_t u=div_x(\nabla u\partial_t u)-\partial_t\frac{|\nabla_x u|^2}{2}$. So, denoting $e(t,x)=\frac{1}{2}\left((\partial_t u(t,x))^2+|\nabla_x u(t,x)|^2\right)$ the density of energy, we have obtained.

$$\partial_t e-div(\nabla u\partial_t u)=0.$$

The main tool will be the Stokes theorem

Lemma 1.1.1 (Stokes formula)**.** *Let $X=(X_1,\cdots,X_d)$ a C^1? vector field on a domain $\Omega\subset\mathbb{R}^d$ with boundary $\partial\Omega$ smooth by piece, with an outward pointing normal $N(x)$ smooth by piece on $\partial\Omega$. Then, we have the formula*

$$\int_\Omega div(X)\ dx=\int_{\partial\Omega}X\cdot N\ d\sigma$$

where $d\sigma$ is the ??element of area??.

We want to apply this on the truncated cone, for $t_0\le r_0$.

$$C_{r_0,t_0}=\left\{(t,x)\in\mathbb{R}^{1+n}\text{ s.t. }|x|\le r_0-t,t\in[0,t_0]\right\}.$$

$\partial C_{r_0,t_0}$ is the union of three pieces with the following normal.

- the bottom part $S_0=\left\{(0,x)\in\mathbb{R}^{1+n}\text{ s.t. }|x|\le r_0\right\}$ where the outward normal is $N(x)=(-1_t,0_x)$

- the top part $S_{t_0}=\left\{(t_0,x)\in\mathbb{R}^{1+n}\text{ s.t. }|x|\le r_0-t_0\right\}$ where the outward normal is $N(x)=(1_t,0_x)$

- the lateral boundary $M_0^{t_0} = \{(t,x) \in \mathbb{R}^{1+n} \text{ s.t. } |x| = r_0 - t, t \in [0,t_0]\}$ with the outward normal $N(t,x) = (1, x/|x|)/|(1,x/|x|)| = (1,x/|x|)/\sqrt{2}$.

We will apply this to the vector fields

- $X_1 = e(t,x)(1_t, 0_x) = e(t,x)\partial_t$ so that $div_{t,x} X_1 = \partial_t e$
- $X_2 = (0_t, \nabla_x u \partial_t u)$ so that $div_{t,x} X_2 = div_x(\nabla_x u \partial_t u)$

So, we obtain

$$\begin{aligned} 0 &= \int_{C_{r_0,t_0}} div_{t,x} X_1 + div_{t,x} X_2 \\ &= \int_{S_0} (X_1 + X_2)\cdot(-1_t, 0_x) + \int_{S_{t_0}} (X_1+X_2)\cdot(1_t,0_x) + \frac{1}{\sqrt{2}} \int_{M_0^{t_0}} (X_1+X_2)\cdot(1, x/|x|)\, d\sigma \\ &= -\int_{S_0} e(0,x) + \int_{S_{t_0}} e(t_0,x) + \frac{1}{\sqrt{2}} \int_{M_0^{t_0}} \left(e(t,x) + \partial_t u \nabla_x u \cdot \frac{x}{|x|} \right) d\sigma \end{aligned}$$

We prove that the integral on the lateral boundary is positive.

$$\left| \partial_t u \nabla_x u \cdot \frac{x}{|x|} \right| \leq |\partial_t u||\nabla_x u| \leq \frac{1}{2}\left(|\partial_t u|^2 + |\nabla_x u|^2\right) \leq e.$$

So it gives

$$-\int_{S_0} e(0,x) + \int_{S_{t_0}} e(t_0,x) \leq 0.$$

This gives the first inequality.

Now, let us prove the unique continuation property. The assumption implies that $E_{r_0}(0) = 0$. So, the inequality implies $E_{r_0-t}(t) = 0$ for $t \in [0, r_0]$ and in particular $\partial_t u = 0$ and $\nabla_x u = 0$ in the cone C_{r_0}. By connexity, this implies that $u = cste$ in C_{r_0}. This constant needs to be zero since $u(0,x) = 0$ for $|x| \leq r_0$. □

We can infer two interesting consequences for the unique continuation:

- unique continuation holds across the hypersurface $\{t = 0\}$ and actually, we have some nice local linear quantification of the unique continuation. This situation actually holds when P is said to be hyperbolic with respect to $P = 0$. We refer to ♣ for more precisions.
- unique continuation can not hold across an hypersurface tangent to the cone $|x| = t + r$.

Consider for instance the wave equation $P = \partial_t^2 - \partial_x^2$ (or more simply the transport operator $P = \partial_t - \partial_x$) on $\mathbb{R}_t \times \mathbb{R}_x$. All the regular solutions are of the form $u = f(x+t) + g(x-t)$. Take for instance $g = 0$ and f smooth with support exactly $[0,1]$. The surface $S = \{x + t = 0\}$ clearly does not satisfy the unique continuation

The first general unique continuation result of the form (1.2) is the Holmgren Theorem, stating that, for operators with analytic coefficients, unique continuation holds across any noncharacteristic hypersurface S.

Theorem 1.1.2 (Holmgren Theorem). *Let $P = \sum_{|\alpha|\leq m} a_\alpha(x) D_x^\alpha$ be a linear differential operator with analytic coefficients on some open subset $\Omega \subset \mathbb{R}^n$. Denote p_m its principal symbol.*

Let $x_0 \in \Omega$ and $\Phi \in C^1(\Omega)$. Assume that Φ is non characteristic for P at x_0, that is

$$p_m(x_0, \nabla\Phi(x_0)) \neq 0.$$

Then, there exists a neighborhood V of x_0 so that every $u \in \mathcal{D}'(\Omega)$ satisfying $Pu = 0$ on Ω and $u = 0$ in the set $\{x \in \Omega; \Phi(x) > \Phi(x_0)\}$ is zero in V.

But we would like to avoid the assumption of analyticity of the coefficients. This will require sometimes some stronger assumption say of pseudoconvexity condition (see e.g. Definition 2.1.1 below) and will be the object of Chapter 2. The following chapter 3 will deal with some intermediate case where the analyticity is with respect to only one variable (we will actually treat the simpler case where it is independant on one variable).

1.1.2 Motivation to control

The unique continuation has been pretty much studied for problems of control because it is equivalent to approximate controllability. We give the example of the heat equation. The problem of control is the following. Let Ω be a smooth domain of $\mathbb{R}^n$ and $\omega \subset \Omega$ be an open subset. The Cauchy problem is for instance well posed for initial data. Using semigroup theory (Hille Yosida) or the diagonalization of the Laplacian Δ, we get

Theorem 1.1.3. *For any $u_0 \in H_0^1(\Omega)$, $f \in L^1([0,T], H_0^1(\Omega))$, there exists a unique $u \in C([0,T], H_0^1(\Omega))$ solution of*

$$\left\{\begin{array}{rcl} \partial_t u - \Delta u = & f & \text{on } [0,T]\times\Omega \\ u_{|t=0} = & u_0 & \text{on } \Omega \\ u = & 0 & \text{on } [0,T]\times\partial\Omega \end{array}\right. \tag{1.3}$$

Moreover, this flow map can be extended for $u_0 \in L^2(\Omega)$, $f \in L^1([0,T], L^2(\Omega))$ with solutions $u \in C([0,T], L^2(\Omega))$

The problem of controllability (to zero for instance) from ω at time T is the following.

Given an initial data $u_0 \in L^2(\Omega)$, can we find a source term $f \in L^1([0,T], L^2(\omega))$ **supported in** ω so that the solution of (1.3) satisfies $u(T) = 0$. If we only want approximate controllability, we will ask that for any $\varepsilon > 0$, one can find a control so that $\|u(T)\|_{L^2(\Omega} \leq \varepsilon$.

We will precise this later, but it turns out that in many situations, the dual problem of the control problem is some observability problem for the free problem

$$\left\{\begin{array}{rcl} \partial_t u - \Delta u = & 0 & \text{on } [0,T]\times\Omega \\ u = & 0 & \text{on } [0,T]\times\partial\Omega \end{array}\right. \tag{1.4}$$

By observability, we mean the map $u_0 \mapsto u_{|[0,T]\times\omega}$. That is the map of observing the free solution on the set $[0,T]\times\omega$. In many cases (we will precise this in each case of the heat or the wave), we can expect

- the exact controllability is equivalent to some observability estimate $\|u(T)\|_{L^2}^2 \leq C \int_0^T \|u(t)\|_{L^2(\omega)}^2\, dt$ for solutions of the free equation (1.4)

- the approximate controllability is equivalent to some unique continuation of the type: $u = 0$ on $[0,T]\times\omega$ implies $u = 0$.

In particular, for evolution equation, we will be very interested in some unique continuation (eventually quantitative) from some sets of the form $[0,T]\times\omega$. It seems that these theorems will have to reflect **the way of propagation of the energy** of solutions of heat or wave.

1.1.3 Notation

We consider complex valued functions defined on $\mathbb{R}^n$.

We will denote the duality in $L^2(\mathbb{R}^n)$, denoted L^2 when there is not ambiguity, by

$$(f,g)_{L^2} = \int_{\mathbb{R}^n} f(x)\overline{g(x)}dx.$$

For any multiindex $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^n$, we define its length $|\alpha| = \alpha_1 + \cdots + \alpha_n$.
If $\zeta = (\zeta_1, \cdots, \zeta_n) \in \mathbb{C}^n$, ζ^α is defined by $\zeta_1^{\alpha_1} \cdots \zeta_n^{\alpha_n}$.
For $1 \leq j \leq n$, we denote $D^j = \frac{\partial_j}{i}$.
For $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^n$, we denote

$$\begin{aligned} \partial^\alpha &= \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} \\ D^\alpha &= \frac{\partial_\alpha}{i^{|\alpha|}}. \end{aligned}$$

This notation are made interesting by the following formula of Fourier transform

$$\widehat{D^\alpha u}(\xi) = \xi^\alpha \widehat{u}(\xi).$$

where the Fourier transform is defined with the convention

$$\widehat{u}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} u(x)dx.$$

With this definition, the Fourier inverse formula is

$$u(x) = \mathcal{F}^{-1}\widehat{u} = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \widehat{u}(\xi)d\xi.$$

while Plancherel formula is

$$(u,v)_{L^2} = \frac{1}{(2\pi)^n} (\widehat{u}, \widehat{v})_{L^2} \tag{1.5}$$

$$\|u\|_{L^2} = \frac{1}{(2\pi)^{n/2}} \|\widehat{u}\|_{L^2}. \tag{1.6}$$

With this convention, we have

$$\widehat{f * g} = \widehat{f}\widehat{g} \tag{1.7}$$

$$\widehat{fg} = \frac{1}{(2\pi)^n} \widehat{f} * \widehat{g} \tag{1.8}$$

And when we apply it to $a = \widehat{f}$, $b = \widehat{g}$ and apply $\mathcal{F}^{-1}$

$$\mathcal{F}^{-1}(ab) = \mathcal{F}^{-1}(a) * \mathcal{F}^{-1}(b) \tag{1.9}$$

$$\mathcal{F}^{-1}(a * b) = (2\pi)^n \mathcal{F}^{-1}(a)\mathcal{F}^{-1}(b) \tag{1.10}$$

We recall the Leibniz formula

$$\partial_\alpha(fg) = \sum_{\beta+\gamma=\alpha} \binom{\alpha}{\beta} (\partial_\beta f)(\partial_\gamma g)$$

where $\binom{\alpha}{\beta} = \binom{\alpha_1}{\beta_1} \cdots \binom{\alpha_d}{\beta_d}$ with $\binom{n}{k} = C_n^k = \frac{n!}{k!(n-k)!}$.

We will sometime use the notation $\|\cdot\|_{H^1(K)}$ for some subset K of $\mathbb{R}^n$. This will be defined for $f \in C_0^\infty(\mathbb{R}^n)$ by

$$\|f\|^2_{H^1(K)} = \int_K |\nabla f|^2 + \int_K |f|^2.$$

If K is complicated, the definition of $H^1(K)$ as a Banach space could lead to some difficulties, but we will not need that. We will only use the definition for smooth functions when the definition is clear.

Definition 1.1.1 (Classical differential operators)**.** *Let $m \in \mathbb{N}$. We denote* Diff^m *the set of differential operators of the form $P = \sum_{|\alpha|\leq m} p_\alpha(x) D^\alpha$ with p_α smooth and bounded as function of $x \in \mathbb{R}^n$.*

Its full symbol will be denoted $p(x,\xi) = \sum_{|\alpha|\leq m} p_\alpha(x)\xi^\alpha$. It belongs to the set of polynomial of degree m of the variable ξ, with coefficients smooth functions of $x \in \Omega$, that we denote $\Sigma^m(\Omega)$.

Respectively, if $p \in \Sigma^m$, we will denote $p(x, D)$ the operator with symbol p.

We denote $p_m(x,\xi) = \sum_{|\alpha|=m} p_\alpha(x)\xi^\alpha$ its principal symbol. It is homogeneous of degree m in ξ.

1.1.4 The general strategy of Carleman

When we want to prove unique continuation, across a surface $S = \{\Phi = 0\}$, we could have a function smooth and flat (all derivatives cancel) along. So, in some sense we need something to emphasize the local behaviour close to S.

The general idea of Carleman to prove unique continuation is to consider some weighted estimates of the form

$$\tau \left\|e^{\tau\Phi}u\right\|^2_{H^1} \leq C \left\|e^{\tau\Phi}Pu\right\|^2_{L^2} \tag{1.11}$$

for $u \in C_0^\infty(\mathbb{R}^n)$ and uniformly for $\tau \geq \tau_0$.

First, this inequality says directly that if $u \in C_0^\infty(\mathbb{R}^n)$ is solution of $Pu = 0$ on $\{\Phi \geq 0\}$, then the right hand side will tend to zero as τ tends to infinity. Therefore, the left hand side will converge to zero, which implies that u is supported in $\{\Phi \leq 0\}$.

But we want unique continuation not only for functions with compact support. Assume now the weaker assumption that $\mathrm{supp}(u) \cap \{\Phi > 0\}$ is compact. Then, applying a cutoff functions, we can arrive to the previous case. This configuration can actually happen if we replace the function Φ by another function Ψ "a little bit more convex".

Now, what will we need to prove Carleman estimates? Actually, the exponential weight is not so convenient to prove estimates and one might want to eliminate it by posing $v = e^{\tau\Phi}$. Then, we are left to study the operator $P_\Phi = e^{\tau\Phi}Pe^{-\tau\Phi}$.

We easily check that $e^{\tau\Phi}D_j e^{-\tau\Phi} = D_j + i\tau\partial_j\Phi$. Hence, we have changed D_j by another operator with one derivative and one exponent of τ. So, we could expect that if $P = \sum_\alpha p_\alpha(x)D^\alpha$, one is left to study an operator operator P_Φ with as many derivatives as exponents of τ. Yet, we want some estimates uniform in τ large. So, now we have to think τ as having the same weight as a derivative. We describe this calculus in the next section.

1.2 Operators depending on τ

In this section, we describe the setting that will be used in Chapter 2.1 and 3. The main thing is the presence of a large parameter τ for which we want to make some estimates uniform for $\tau > 0$ large. Morally, we want to think τ as having the same weight as a derivative.

H^s_τ is the H^s norm with the following norm depending on τ

$$\|u\|_{H^s_\tau} = \left\|(|D|^2+\tau^2)^{\frac{s}{2}}u\right\|_{L^2(\mathbb{R}^n)} = (2\pi)^{-n/2}\left\|(|\xi|^2+\tau^2)^{\frac{s}{2}}\widehat{u}\right\|_{L^2(\mathbb{R}^n)}.$$

In the next subsection, we will describe some important properties of operator depending on a large parameter τ. Since it is finally expected to be large, we will always assume $\tau \geq 1$ when dealing with estimates uniform in τ.

Note also that when m is an integer, this expression can be written more simply. For instance, if $m=1$,

$$\|u\|_{H^1_\tau} \approx \|u\|_{H^1} + \tau\|u\|_{L^2}.$$

1.2.1 Differential operators

Definition 1.2.1 (Differential operators depending on τ)**.** *Let $m \in \mathbb{N}$. We denote Diff^m_τ the set of differential operators of the form $P = \sum_{|\alpha|+\beta\leq m} p_{\alpha,\beta}(x)\tau^\beta D^\alpha$ with $p_{\alpha,\beta}$ smooth and bounded as function of $x \in \mathbb{R}^n$.*

Its full symbol will be denoted $p(x,\xi,\tau) = \sum_{|\alpha|+\beta\leq m} p_{\alpha,\beta}(x)\tau^\beta\xi^\alpha$. It belongs to the set of polynomial of degree m of the variable (ξ,τ), with coefficients smooth functions of $x \in \Omega$, that we denote $\Sigma^m(\Omega)$.

Respectively, if $p \in \Sigma^m$, we will denote $p(x,D,\tau)$ the operator with symbol p.

We denote $p_m(x,\xi,\tau) = \sum_{|\alpha|+\beta=m} p_{\alpha,\beta}(x)\tau^\beta\xi^\alpha$ its principal symbol. It is homogeneous of degree m in ξ,τ.

Note that is it almost the same definition as Definition 1.1.1 except with the dependance on τ which change the definition of the principal symbol.

Note that if $p \in \Sigma^m_\tau$, the inversion Fourier formula gives for $u \in \mathcal{S}(\mathbb{R}^n)$

$$p(x,D,\tau) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix\cdot\xi}p(x,\xi,\tau)\widehat{u}(\xi)d\xi \tag{1.12}$$

The following properties will be some examples of the general Heuristic (not true of course) that a pseudo differential operator acts as if it was a multiplication by $p(x,\xi,\tau)$ modulo lower order terms.

The general philosophy is that we want to get some properties of some operators only from there principal symbol.

If P, A, B of respective order m, m_1 and m_2, with respective principal symbol p, a, b, the rough summary is the following

1. P act from H^s_τ in H^{s-m}_τ
2. $A \circ B$ is of order m_1+m_2 with principal symbol ab
3. $[A,B]$ is of order m_1+m_2-1 with principal symbol $\frac{1}{i}\{a,b\}$
4. P^* is of order m with principal symbol $\overline{p}$
5. $p \geq C(\xi^2+\tau^2)^{m/2}$ implies $\mathrm{Re}(Pu,u)_{L^2} \geq C'\|u\|_{H^{m/2}_\tau}$ for large τ

Proposition 1.2.1 (Action on Sobolev spaces)**.** *Let $P \in \mathrm{Diff}^m_\tau(\mathbb{R}^n)$.*

Then, for any $s \in \mathbb{R}$, P is bounded from H^s_τ in H^{s-m}_τ uniformly for $\tau \geq 1$.

Proof. We only prove it for $s \in \mathbb{Z}$ since we will only use it in this case. Yet, the other cases can be easily obtained by interpolation. By the triangular inequality, it is sufficient to prove it for $p_{\alpha,\beta}(x)\tau^\beta D^\alpha$ with $|\alpha| + \beta \le m$. First, suppose s odd, $s = 2j+1$ with $j \in \mathbb{N}$. First, since $(|\xi|^2+\tau^2)^s \le C\left(|\xi|^{2s}+\tau^{2s}\right)$, for $u \in \mathcal{S}(\mathbb{R}^n)$, $\|u\|^2_{H^s_\tau}$ is equivalent to

$$\begin{aligned}
\int_{\mathbb{R}^n}\left(|\xi|^{2s}+\tau^{2s}\right)|\widehat{u}(\xi)|^2 d\xi &= \int_\xi ||\xi|^s\widehat{u}(\xi)|^2 d\xi + \int_{\mathbb{R}^n}\tau^{2s}|\widehat{u}(\xi)|^2 d\xi \\
&= \int_{\mathbb{R}^n} ||\xi|^{2s}\widehat{u}(\xi)\overline{\widehat{u}(\xi)}d\xi + \tau^{2s}\|u\|^2_{L^2} \\
&= \int_{\mathbb{R}^n}(-\Delta)^{2j+1}u(x)\overline{u(x)}dx + \tau^{2s}\|u\|^2_{L^2} \\
&= \int_{\mathbb{R}^n}(-\Delta)^j\nabla u)(x)\cdot((-\Delta)^j\nabla\overline{u(x)})dx + \tau^{2s}\|u\|^2_{L^2} \\
&= \left\|(-\Delta)^j\nabla u\right\|^2_{L^2} + \tau^{2s}\|u\|^2_{L^2}
\end{aligned}$$

This form easily gives that the multiplication by $p_{\alpha,\beta}$ is bounded from H^s_τ with $s \in \mathbb{N}$ odd. The even case is actually easier. Moreover, we can extend this to any $s \in \mathbb{Z}$ by duality.

So, we are left to prove that $D^\alpha\tau^\beta$ applies H^s_τ into H^{s-m}_τ for any $s \in \mathbb{Z}$ and for $|\alpha|+\beta \le m$. This is just a Fourier multiplier. We use the inequality $|\xi| \le \left(|\xi|^2+\tau^2\right)^{1/2}$ and the same for τ.

$$\begin{aligned}
\left\|D^\alpha\tau^\beta\right\|^2_{H^{s-m}_\tau} &= \int_{\mathbb{R}^n}(|\xi|^2+\tau^2)^{s-m}|\widehat{D^\alpha\tau^\beta u}(\xi)|^2 d\xi = \int_{\mathbb{R}^n}(|\xi|^2+\tau^2)^{s-m}|\xi|^{2\alpha}\tau^{2\beta}|\widehat{u}(\xi)|^2 d\xi \\
&\le \int_{\mathbb{R}^n}(|\xi|^2+\tau^2)^{s-m}\left(|\xi|^2+\tau^2\right)^{\alpha+\beta}|\widehat{u}(\xi)|^2 d\xi.
\end{aligned}$$

Moreover, since $\tau \ge 1$ and $\alpha+\beta \le m$, we have $\left(|\xi|^2+\tau^2\right)^m$. This gives the expected estimate for Schwarz functions. By density of the Schwarz functions, it gives the result. □

Proposition 1.2.2 (Composition). *Let $A = \sum_{|\alpha|+\beta\le m_1} a_{\alpha,\beta}\tau^\beta D^\alpha$, $B = \sum_{|\alpha|+\beta\le m_2} b_{\alpha,\beta}\tau^\beta D^\alpha$ be two differential operators with large parameter τ of respective order m_1 and m_2, with full symbol $a(x,\xi,\tau)$ and $b(x,\xi,\tau)$ and principal symbol $a_{m_1}(x,\xi,\tau)$ and $b_{m_2}(x,\xi,\tau)$. Then, we have that $A\circ B$ is of order m_1+m_2 in (D,τ), that is in $\mathrm{Diff}^{m_1+m_2}_\tau(\Omega)$. Moreover, it can be written*

$$A\circ B = (a_{m_1}b_{m_2})(x,D,\tau) + r(x,D,\tau)$$

with $r(x,D,\tau) \in \mathrm{Diff}^{m_1+m_2-1}_\tau(\Omega)$.

In particular, the principal symbol of $A\circ B$ is $a_{m_1}b_{m_2}$.

Proof. We prove it by iteration on $m = m_1+m_2$.

- $m = 0$: $A = f(x)$ and $B = g(x)$ so it is obvious.
- $m \mapsto m+1$: By linearity with respect to A and B, it is enough to prove it for $A = f(x)\tau^\beta D^\alpha$ and $B = g(x)\tau^{\beta'}D^{\alpha'}$. Since $m_1+m_2 = m+1$, at least one of the β, $|\alpha|$, β', $|\alpha'|$ is bigger than 1. If it is either $\beta \ge 1$, $\beta' \ge 1$ or $|\alpha'| \ge 1$, the result is direct by iteration result for m (for the composition by the right by $D^{\alpha'}$ the result is easy). If it is $|\alpha|$, take k such that $\alpha_k \ge 1$. Then $A = \widetilde{A}D_k$ with $\widetilde{A} \in \mathrm{Diff}^{m_1-1}_\tau$. Then,

  $$ABu = \widetilde{A}D_k[g(x)\tau^{\beta'}D^{\alpha'}]u = \widetilde{A}(D_k g(x))\tau^{\beta'}D^{\alpha'} + \widetilde{A}g(x)\tau^{\beta'}D^{\alpha'}D_k u$$

  The iteration result gives that the first term is in Diff^{m-1}_τ and $\widetilde{A}g(x)\tau^{\beta'}D^{\alpha'} \in \mathrm{Diff}^{m-1}_\tau$ with principal symbol $\widetilde{a}g(x)\tau^{\beta'}\xi^{\alpha'}$. We obtain easily that $\widetilde{A}g(x)\tau^{\beta'}D^{\alpha'}D_k \in \mathrm{Diff}^{m-1}_\tau$ with principal symbol $\widetilde{a}g(x)\tau^{\beta'}\xi^{\alpha'}\xi_k = ab$.

□

See Section B.1.1 for the proof. For the commutator, we will need the following notation and definition.

Definition 1.2.2 (Poisson bracket).

$$\{a,b\} := \sum_j \left[(\partial_{\xi_j} a)(\partial_{x_j} b)\right](x,\xi,\tau) - \sum_j \left[(\partial_{x_j} a)(\partial_{\xi_j} b)(x,\xi,\tau)\right]$$

We check that we have the properties

$$\{a,b\} = -\{b,a\} \tag{1.13}$$

$$\{a,bc\} = \{a,b\}c + b\{a,c\} \tag{1.14}$$

Proposition 1.2.3 (Commutation). $[A,B] = AB - BA$ *is of order* $m_1 + m_2 - 1$ *in* (D,τ), *that is in* $\mathrm{Diff}_\tau^{m_1+m_2-1}$. *Moreover, it can be written*

$$\begin{aligned}[A,B] &= \frac{1}{i}\sum_j \left[(\partial_{\xi_j} a)(\partial_{x_j} b)\right](x,D,\tau) - \sum_j \left[(\partial_{\xi_j} a)(\partial_{x_i} b)\right](x,D,\tau) + r(x,D,\tau)\\ &= \frac{1}{i}\{a,b\}(x,D,\tau) + r(x,D,\tau)\end{aligned}$$

with $r(x,D,\tau) \in \mathrm{Diff}_\tau^{m_1+m_2-2}(\Omega)$.

Proof. We first prove by iteration on m the following property:
For any $A = f(x)D_k$ and $B \in \mathrm{Diff}_\tau^m$, the conclusion of the Proposition is true.

- $m = 0$: $B = g(x)$.

  $$[A,B]u = [f(x)D_k, g(x)]u = f(x)D_k(g(x)u) - g(x)f(x)D_k u = f(x)D_k(g(x))u = \frac{1}{i}f(x)\partial_k(g(x))u.$$

  Which is is Diff_τ^0 with principal (and full) symbol $\frac{1}{i}f(x)\partial_{x_k}(g(x))$. And we check from the definition of the Poisson bracket and for the principal symbols $a_1 = f(x)\xi_k$, $b_0 = g(x)$, we have $\{a_1, b_0\} = \{a_1, b\} = f\partial_{x_k} g$.

- $m = 1$ (this is only needed as a partial result): By linearity (and the case $m = 0$), it is enough to have the result for$B = \tau$ or $B = g(x)D_l$. The first case is trivial so we just treat the second.

  $$\begin{aligned}[A,B]u &= [f(x)D_k, g(x)D_l]u = f(x)D_k\left[g(x)D_l u\right] - g(x)D_l\left[f(x)D_k u\right]\\ &= f(x)D_k(g(x))D_l u - g(x)D_l(f(x))D_k u.\end{aligned}$$

  We recognize that it belongs to Diff_τ^1 with principal (and full) symbol $\frac{1}{i}f(x)\partial_{x_k}(g(x))\xi_l - g(x)\partial_{x_l}(f(x))\xi_k$ which turns out to be equal to $\frac{1}{i}\{f(x)\xi_k, g(x)\xi_l\}$.

- $m \mapsto m+1$: the main idea is to notice that we have, similarly to (1.13) and (1.14).

  $$[A,B] = -[B,A] \tag{1.15}$$

  $$[A,BC] = [A,B]C + B[A,C]. \tag{1.16}$$

  More precisely, by linearity, it is enough to consider $B = \tau g(x)\widetilde{B}$ or $B = g(x)D_l\widetilde{B}$ with $\widetilde{B} \in \mathrm{Diff}_\tau^m$. In the first case, we have

  $$[A,B] = \tau[A, g(x)\widetilde{B}].$$

So the iteration step m gives the result. In the second case

$$[A,B]=[A,g(x)D_l\widetilde{B}]=[A,g(x)D_l]\widetilde{B}+g(x)D_l[A,\widetilde{B}].$$

The case $m=1$ and the iteration assumption then gives (after using Proposition 1.2.2) that $[A,B]\in \mathrm{Diff}_\tau^m$ with principal symbol

$$\frac{1}{i}\left(\{a,g(x)\xi_l\}\,\widetilde{b_m}+g(x)\xi_l\left\{a,\widetilde{b_m}\right\}\right)=\frac{1}{i}\left\{a,g(x)\xi_l\widetilde{b_m}\right\}=\frac{1}{i}\left\{a,g(x)\xi_l\widetilde{b_m}\right\}=\frac{1}{i}\{a,b\}$$

where we have used (1.14).

The result is now proved for any $A=f(x)D_k$ and $B\in\mathrm{Diff}_\tau^m$ (or $A\in\mathrm{Diff}_\tau^m$ $B=g(x)D_l$ by antisymmetry). The final result can then be proved easily by iteration on $m=m_1+m_2$ using the same strategy.

- $m=0$: the result is obvious since then $A=f(x)$ and $B=g(x)$.
- $m\mapsto m+1$ By linearity with respect to both variable, it is enough to get the result for $A=f(x)\tau^\beta D^\alpha$ and $B=g(x)\tau^{\beta'}D^{\alpha'}$. Since $m_1+m_2=m+1$, at least one of the β, $|\alpha|$, β', $|\alpha'|$ is bigger than 1. By symmetry of the role, we can assume that it is either $\beta\geq 1$ or $|\alpha|\geq 1$. In the first case, we apply directly the iteration result for m. If it is $|\alpha|$, take k such that $\alpha_k\geq 1$. Then $A=f(x)D_k\widetilde{A}$ where $\widetilde{A}$ is of order m_1-1. We have again

$$[A,B]=[f(x)D_k\widetilde{A},B]=f(x)D_k[\widetilde{A},B]+[f(x)D_k,B]\widetilde{A}$$

  and we conclude similarly by the iteration step m for the first term and for the second term by the previous result proved in the specific case $A=f(x)D_k$.

□

Definition 1.2.3. *Let P an operator, we will write P^* (in some cases where it is well defined) its formal dual, that is that satisfies*

$$(Pu,u)_{L^2}=(u,P^*u)_{L^2}\tag{1.17}$$

for all $u\in C_0^\infty(\mathbb{R}^n)$

Proposition 1.2.4 (Adjoint)**.** *There exists one unique operator $P(x,D,\tau)^*\in\mathrm{Diff}_\tau^m$ satisfying* (1.17)*. Moreover, $P(x,D,\tau)^*-\overline{P}(x,D,\tau)\in\mathrm{Diff}_\tau^{m-1}$ where $\overline{P}(x,D,\tau)$ is the operator with symbol $\overline{p}$.*

Proof. By linarity, it is enough to prove it for $P=a(x)\tau^\beta D^\alpha$. By looking on the Fourier side using (1.5), we have

$$(D^\alpha u,v)_{L^2}=\int_{\mathbb{R}^n}D^\alpha u\overline{v}=\int_{\mathbb{R}^n}\xi^\alpha\widehat{u}\overline{\widehat{v}}=\int_{\mathbb{R}^n}\widehat{u}\overline{\xi^\alpha\widehat{v}}=\int_{\mathbb{R}^n}u\overline{D^\alpha v}=(u,D^\alpha v)_{L^2}$$

So that

$$\left(a(x)\tau^\beta D^\alpha u,v\right)_{L^2}=\left(\tau^\beta D^\alpha u,\overline{p(x)}v\right)_{L^2}=\left(u,D^\alpha\overline{a(x)}v\right)_{L^2}.$$

So $P^*=D^\alpha\overline{a(x)}$ and we know by Theorem 1.2.2 that $P^*\in\mathrm{Diff}_\tau^m$ with principal symbol $\overline{a(x)}\tau^\beta\xi^\alpha=\overline{p_m}$. □

We will also write a weak form of Gårding estimates for operator of order 2. The general case (that we shall not need for Carleman estimates will be stated in the next subsection about pseudodifferential operators with large parameters.

Lemma 1.2.1 (An easy local Gårding inequality in some particular case)**.** *Let P be an operator of order 2 of the form*

$$P = A + \sum_{i=1}^{k} B_i \circ \frac{1}{-\Delta + \tau^2} \circ B_i \tag{1.18}$$

with $A, B_i \in \mathrm{Diff}_\tau^2$ with real principal symbol $a_2(x,\xi,\tau)$, $b_{2,i}(x,\xi,\tau)$ satisfying

$$p_2(0,\xi,\tau) = a_2(0,\xi,\tau) + \sum_{i=1}^{k} \frac{b_{2,i}^2(0,\xi,\tau)}{|\xi|^2+\tau^2} \geq C(|\xi|^2+\tau^2). \tag{1.19}$$

for all $\xi \in \mathbb{R}^n$ and $\tau \geq 0$. Then, there exist $r > 0$ and C_1, $C_2 > 0$, so that we have

$$\mathrm{Re}\,(P(x,D,\tau)u,u)_{L^2} \geq C_1 \|u\|_{H^1_\tau}^2 - C_2 \|u\|_{L^2}^2$$

for any $u \in C_0^\infty(B(0,r))$. In particular, we have, for τ large enough

$$\mathrm{Re}\,(P(x,D,\tau)u,u)_{L^2} \geq C_1 \|u\|_{H^1_\tau}^2 .$$

Proof. Denote $a(x,\xi,\tau)$, $b_i(x,\xi,\tau)$ the full symbol of A, B_i.

The idea of the proof is to "freeze" the coefficients. The theorem is an easy consequence of the following two Lemma.

Lemma 1.2.2. *Under assumption* (1.19)*, we have (with the same constant C)*

$$\mathrm{Re}\,(P_2(0,D,\tau)u,u)_{L^2} \geq C \|u\|_{H^1_\tau}^2$$

Lemma 1.2.3. *If P is of the form* (1.18) *with the above assumptions, for any $\varepsilon > 0$, there exits $C > 0$ and $r > 0$ so that*

$$|(P(x,D,\tau)u,u)_{L^2} - (P_2(0,D,\tau)u,u)_{L^2}| \leq \varepsilon \|u\|_{H^1_\tau}^2 + C \|u\|_{L^2}^2$$

for any $u \in C_0^\infty(B(0,r))$, $\tau \geq 1$.

□

Proof of Lemma 1.2.2. Since the coefficients of $P_2(0,D,\tau)$ are constant, we have $P_2\widehat{(0,D,\tau)}u(\xi) = p(0,\xi,\tau)\widehat{u}(\xi)$. So, using Parseval formula, we get

$$\begin{aligned}(P_2(0,D,\tau)u,u)_{L^2} &= \frac{1}{(2\pi)^n}\int_\xi p_2(0,\xi,\tau)\widehat{u}(\xi)\overline{\widehat{u}(\xi)}d\xi = \frac{1}{(2\pi)^n}\int_\xi p_2(0,\xi,\tau)|\widehat{u}(\xi)|^2 d\xi \geq C\int_\xi (|\xi|^2+\tau^2)|\widehat{u}(\xi)|^2 d\xi \\ &\geq C\int_\xi |\sqrt{|\xi|^2+\tau^2}\widehat{u}(\xi)|^2 d\xi = C\|u\|_{H^1_\tau}^2 .\end{aligned}$$

Note that the previous calculation actually gives that $(P_2(0,D,\tau)u,u)_{L^2}$ is actually real. □

Proof of Lemma 1.2.3. Using the estimate $\|u\|_{H^1_\tau}\|u\|_{L^2} \le \varepsilon\|u\|^2_{H^1_\tau} + C_\varepsilon\|u\|^2_{L^2}$, any bound of the remainder of the form $\|u\|_{H^1_\tau}\|u\|_{L^2}$ or $\varepsilon\|u\|^2_{H^1_\tau}$ will be sufficient.

$$(Pu,u)_{L^2} = (Au,u)_{L^2} + \left(B \circ \frac{1}{-\Delta+\tau^2} \circ Bu, u\right)_{L^2}$$

A is of order 2 with principal symbol. We denote $r = a - a_2$ of order 1.

$$(A(x,D,\tau)u,u)_{L^2} = (A_2(x,D,\tau)u,u)_{L^2} + (R(x,D,\tau)u,u)_{L^2}$$

with

$$|(R(x,D,\tau)u,u)_{L^2}| \le C\|u\|_{H^1_\tau}\|u\|_{L^2}.$$

Moreover, a_2 is a quadratic form in the variables $\zeta = (\xi,\tau)$, that can be decomposed according to the exponent in τ

$$a_2(x,\xi,\tau) = \sum_{i,j=1}^n a_2(x)\xi_i\xi_j + \sum_{i=1}^n a_{i,0}(x)\tau\xi_i + a_{0,0}(x)\tau^2$$

$$(a_{2,i,j}(x)D_iD_ju,u)_{L^2} = (a_{2,i,j}(x)D_ju, D_iu)_{L^2} - ((D_ia_{2,i,j})u, D_ju)_{L^2}$$

where the second term can be bounded by $C\|u\|_{H^1_\tau}\|u\|_{L^2}$.

$$(a_{2,i,j}(x)D_ju, D_iu)_{L^2} = (a_{2,i,j}(0)D_ju, D_iu)_{L^2} + ((a_{2,i,j}(x) - a_{2,i,j}(0))D_ju, D_iu)_{L^2}$$

But again, the second term is bounded by $\|a_{2,i,j}(x) - a_{2,i,j}(0)\|_{L^\infty(Supp(u))}\|u\|^2_{H^1_\tau}$ that can be bounded by $\varepsilon\|u\|^2_{H^1_\tau}$ if r is chosen small enough. So, at that stage, we have proved

$$(A(x,D,\tau)u,u)_{L^2} = (A_2(0,D,\tau)u,u)_{L^2} + R$$

with $|R| \le C\varepsilon\|u\|^2_{H^1_\tau} + C_\varepsilon\|u\|^2_{L^2}$.

Let us now see the other term with B.

$$\begin{aligned}
\left(B \circ \frac{1}{-\Delta+\tau^2} \circ Bu, u\right)_{L^2} &= \left(\frac{1}{-\Delta+\tau^2} \circ Bu, B^*u\right)_{L^2} \\
&= \left(\frac{1}{-\Delta+\tau^2} \circ Bu, B_2(0,D,\tau)u\right)_{L^2} + \left(\frac{1}{-\Delta+\tau^2} \circ Bu, (B^*(x,D,\tau) - B_2(0,D,\tau)u\right) \\
&= \left(\frac{1}{-\Delta+\tau^2} \circ Bu, B_2(0,D,\tau)u\right)_{L^2} + R_1
\end{aligned}$$

Using that B sends H^1_τ into H^{-1}_τ, we get

$$\left\|\frac{1}{-\Delta+\tau^2} \circ Bu\right\|_{H^1_\tau} \le C\|Bu\|_{H^{-1}_\tau} \le C\|u\|_{H^1_\tau}.$$

So, we are left to estimate $\|(B^*(x,D,\tau) - B_2(0,D,\tau))u\|_{H^{-1}_\tau}$. Since the difference is of order 1, we can replace $B(x,D,\tau)^*$ by $B_2(x,D,\tau)^*$ up to a bound by $C\|u\|_{L^2}$.

So, we need to estimate $\|(B_2^*(x,D,\tau) - B_2(0,D,\tau))u\|_{H^{-1}_\tau}$. By duality and the locality of the considered differential operators, it leads to estimate

$$\sup_{\|f\|_{H^1_\tau}=1} ((B_2^*(x,D,\tau) - B_2(0,D,\tau))u, f)_{L^2} = \sup_{\|f\|_{H^1_\tau}=1, \mathrm{supp}(f)\subset B(0,2r)} (u, (B_2(x,D,\tau) - B_2(0,D,\tau)^*)f)_{L^2}$$

Since $B_2(0,D,\tau)$ has real valued and constant coefficients $B_2(0,D,\tau)^* = B_2(0,D,\tau)$.

So, we are left to estimate, for $\|f\|_{H^1_\tau} = 1$, $\text{supp}(f) \subset B(0,2r)$

$$(u, (B_2(x,D,\tau) - B_2(0,D,\tau))f)_{L^2}$$

Similarly as before, for instance for terms of the form $b_{2,i,j}(x)D_iD_ju$

$$(u, (b_{2,i,j}(x) - b_{2,i,j}(0))D_iD_jf)_{L^2} = (D_iu, (b_{2,i,j}(x) - b_{2,i,j}(0))D_jf)_{L^2} - ((D_ib_{2,i,j})(x)u, D_jf)_{L^2}$$

The first term is bounded by $\varepsilon \|u\|_{H^1_\tau} \|f\|_{H^1_\tau}$ if $|b_{2,i,j}(x) - b_{2,i,j}(0)| \le \varepsilon$ on $B(0,2r)$. The second term is bounded by $C \|u\|_{L^2} \|f\|_{H^1_\tau}$.

The other terms of the form $\sum_{i=1}^n b_{i,0}(x)\tau\xi_i + b_{0,0}(x)\tau^2$ are treated similarly.

So, at this stage, it remains to replace $\left(\frac{1}{-\Delta+\tau^2} \circ Bu, B_2(0,D,\tau)u\right)_{L^2}$ by the same with the first B replaced by $B_2(0,D,\tau)$. That means to estimate

$$\left(\frac{1}{-\Delta+\tau^2} \circ (B(x,D,\tau) - B_2(0,D,\tau))u, B_2(0,D,\tau)u\right)_{L^2} \le \|(B(x,D,\tau) - B_2(0,D,\tau))u\|_{H^{-1}_\tau} \|u\|_{H^1_\tau}$$

This last term has already been estimated with B^* instead of B, but it works the same. □

Proposition 1.2.5 (Semiglobal Gårding inequality)**.** *Under the same notations as Lemma 1.2.1, but replacing the local assumption* (1.19) *by a semiglobal one*

$$p_2(x,\xi,\tau) \ge C(|\xi|^2 + \tau^2) \quad \forall (x,\xi,\tau) \in K \times \mathbb{R}^n \times \mathbb{R}^+ \tag{1.20}$$

for one compact K.

Then, the same estimate holds for any $u \in C_0^\infty(\mathbb{R}^n)$ supported in K.

Proof. Apply Lemma 1.2.1 for any $x_0 \in K$. Recover K by a finite number of such balls:$K \subset \cup_{i\in I} B(x_i, r_i)$. Define a smooth partition of unity $\chi_i \in C^\infty(B(x_i,r_i))$ so that

$$\sum_i \chi_i^2 = 1 \text{ on } K.$$

We have $u = \sum_i \chi_i^2 u$.

$$\begin{aligned}(Pu,u)_{L^2} &= \sum_i \left(\chi_i^2 Pu, u\right)_{L^2} = \sum_i (\chi_i Pu, \chi_i u)_{L^2} \\ &= \sum_i (P\chi_i u, \chi_i u)_{L^2} + \sum_i ([\chi_i, P]u, \chi_i u)_{L^2}\end{aligned}$$

$[\chi_i, P]$ is of order 1 for the part of P coming from A. The part coming from the B_i is also of order 1 (see the next subsection) but is not a differential operator. So, using the general fact $[T, S^{-1}] = TS^{-1} - S^{-1}T = S^{-1}[S,T]S^{-1}$ for S the differential operator $(\Delta + \tau^2)$. This allows to estimate the commutator. Moreover, applying the local Lemma, we get

$$\sum_i (P\chi_i u, \chi_i u)_{L^2} \ge \sum_i \|\chi_i u\|^2_{H^1_\tau} - C \|\chi_i u\|^2_{L^2}.$$

And with similar arguments, $\sum_i \|\chi_i u\|^2_{H^1_\tau} \ge C \|u\|^2_{H^1_\tau} - C \|u\|^2_{L^2}$. □

As an immediate consequence of we get the following result (we only proved it for $m = 2$ but the result is true more generally)

Proposition 1.2.6 (Elliptic estimates). *Let $P \in \mathrm{Diff}_\tau^m$ elliptic, in the sense that $p_m(x,\xi,\tau) \neq 0$ for all (x,ξ,τ) with $(\xi,\tau) \neq 0$ for one compact K.*

Then, we have, for τ large enough and for $u \in C_0^\infty(\mathbb{R}^n)$ supported in K,

$$\|u\|_{H_\tau^m} \leq C \|Pu\|_{L^2}.$$

Proof.

$$\|Pu\|_{L^2}^2 = \mathrm{Re}\,(Pu, Pu)_{L^2(\mathbb{R}^n)} = \mathrm{Re}\,(P^*Pu, u)_{L^2(\mathbb{R}^n)} = \mathrm{Re}\,(P^*Pu, u)_{L^2(\mathbb{R}^n)}.$$

But the principal symbol of P^*P is $|p_m|^2$ thanks to Proposition 1.2.4. Therefore, since $|p_m|^2$ is homogeneous of order $2m$ with $p_m(x,\xi,\tau) \neq 0$ for all (x,ξ,τ) with $(\xi,\tau) \neq 0$, then using the compactness of $\left\{(x,\xi,\tau)\,\middle|\, x \in K, |\xi|^2 + \tau^2 = 1\right\}$ we can find $C > 0$ so that $|p_m|^2 \geq C(\xi^2 + \tau^2)^m$ for $x \in K$. Proposition 1.2.5 allows to conclude. □

1.2.2 Pseudodifferential operators

We would like to extend formula (1.12) to some symbol $p(x,\xi,\tau)$ not necessarily polynomial. For instance, the operator $-\Delta + \tau^2$ is well defined for $\tau \neq 0$, and we would like to say that it is of order -2 with symbol $\frac{1}{|\xi|^2+\tau^2}$. One good class is the S_τ^m. Note that this theory will not be strictly necessary for obtaining Carleman estimates where the previous differential operators will be enough. Yet, we believe that it is good to know that the previous results can be set into a more general framework.

We say that $p(x,\xi,\tau)$ belongs to S_τ^m, if it is smooth in (x,ξ) and for any (α,β), there exists $C_{\alpha,\beta}$ so that

$$\left|\partial_x^\alpha \partial_\xi^\beta p(x,\xi,\tau)\right| \leq C_{\alpha,\beta} \left(1 + |(\xi,\tau)|\right)^{m-|\beta|} \quad \forall (x,\xi,\tau) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^+ \tag{1.21}$$

Note that if $p \in \Sigma^m$, then $p \in S_\tau^m$, so this definition is an extension of the differential opearators.

By mimicking formula (1.12), for $p(x,\xi,\tau) \in S_\tau^m$, we define the associated pseudodifferential operator by the formula

$$P(x,D,\tau) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^n} e^{ix\cdot\xi} p(x,\xi,\tau) \widehat{u}(\xi) d\xi.$$

that can (first) be seen to be well defined as a continuous function if $u \in \mathcal{S}(\mathbb{R}^n)$.

But it actually has much better properties similar to the differential case.

The set of such operators will be denoted Ψ_τ^m.

Proposition 1.2.7 (Action on Sobolev spaces). *Let $P \in \Psi_\tau^m(\mathbb{R}^n)$.*

Then, for any $s \in \mathbb{R}$, P is bounded from H_τ^s in H_τ^{s-m} uniformly in τ.

Since the notion of principal symbol is more complicated, we will state the results a bit differently.

Proposition 1.2.8 (Composition). *Let $A \in \Psi_\tau^{m_1}$, $B \in \Psi_\tau^{m_2}$ with full symbol $a(x,\xi,\tau)$ and $b(x,\xi,\tau)$. Then, we have the following $A \circ B$ is of order $m_1 + m_2$ in (D,τ), that is in $\mathrm{Diff}_\tau^{m_1+m_2}(\Omega)$. Moreover, it can be written*

$$A \circ B = (ab)(x,D,\tau) + \frac{1}{i} \sum_j \left[(\partial_{\xi_j} a)(\partial_{x_j} b)\right](x,D,\tau) + r(x,D,\tau)$$

with $r(x,D,\tau) \in \Psi_\tau^{m_1+m_2-2}(\Omega)$.

Proposition 1.2.9 (Commutation). $[A,B] \in \Psi_\tau^{m_1+m_2-1}$. *Moreover, it can be written*

$$[A,B] \quad = \quad \frac{1}{i}\{a,b\}(x,D,\tau) + r(x,D,\tau)$$

with $r(x,D,\tau) \in \Psi_\tau^{m_1+m_2-2}(\Omega)$.

Proposition 1.2.10 (Adjoint). $P(x,D,\tau)^* \in \Psi_\tau^m$ *and* $P(x,D,\tau)^* - \overline{P}(x,D,\tau) \in \mathrm{Diff}_\tau^{m-1}$.

Lemma 1.2.4 (Gårding). *Assume* $p \in S_\tau^2$ *real valued so that*

$$\mathrm{Re}\, p(x,\xi,\tau) \geq C(\xi^2+\tau^2)$$

then, we have

$$\mathrm{Re}\,(Pu,u)_{L^2} \geq C\,\|u\|_{H_\tau^1} - C_2\,\|u\|_{L^2}$$

for all $u \in C_0^\infty(\mathbb{R}^n)$.

Link with the semiclassical operators of the course [11] Let $p \in S_\tau^0$. Denote $h = \tau^{-1}$ and rename $p_{sc}(x,\xi,h) = p(x,\xi,\tau^{-1})$. Then, property (1.21) can be rewritten

$$\left|\partial_x^\alpha \partial_\xi^\beta p_{sc}(x,\xi,h)\right| \leq C_{\alpha,\beta}\left(1+|(\xi,h^{-1})|\right)^{-|\beta|} \leq C_{\alpha,\beta} h^{\beta|}$$

for all $(x,\xi,h) \in \mathbb{R}^n \times \mathbb{R}^n \times (0,1]$. In particular, $p_{sc}(x,\xi,h) \in S_{scl}^0$ as defined in Section 4.5 of [11].

In the case of operator of order 0,

- Proposition 1.2.7 in L^2 is a consequence of Theorem 4.5.1 of [11].
- Proposition 1.2.8 1.2.9 and 1.2.10 are consequence of Corollary 4.5.6 of [11].

The other cases can be obtained by changing p by multiplying by some power of $(\Delta+\tau^2)^s$ with the appropriate s, and sometimes change u by $(\Delta+\tau^2)^s u$.

For example, here is how to obtain Lemma 1.2.4 from Theorem 4.5.11 of [11]. Take $p \in S_\tau^2$. $r = \mathrm{Re}\, p(x,\xi,\tau) - C(\xi^2+\tau^2) \geq 0$. $a = (\Delta+\tau^2)^{-1/2} r (\Delta+\tau^2)^{-1/2}$ is of order 0. Define $a_{sc} = a(x,\xi,h^{-1})$. So, it gives

$$\begin{aligned} \mathrm{Re}\,(a_{sc}(x,D,h)u,u)_{L^2} &\geq -C_1 h\,\|u\|_{L^2} \\ \mathrm{Re}\left((\Delta+\tau^2)^{-1/2} p (\Delta+\tau^2)^{-1/2} u, u\right)_{L^2} - C\,\|u\|_{L^2} &\geq -C_1\tau^{-1}\,\|u\|_{L^2} \end{aligned}$$

If we apply it to $(\Delta+\tau^2)^{1/2} v$, it gives

$$Re\,(pv,v)_{L^2} - C\left\|(\Delta+\tau^2)^{1/2} v\right\|_{L^2} \geq -C_1\tau^{-1}\left\|(\Delta+\tau^2)^{1/2} v\right\|_{L^2}$$

which gives the result.

Chapter 2

The classical case

2.1 Main results

In this chapter, we will prove a very general result of unique continuation for operators, with real principal symbol

Before stating the general result, it is interesting to state it in the cases that will be the more useful for us, namely when they are of order 2.

Theorem 2.1.1 (Elliptic real operator of order 2). *Let Ω an open set of $\mathbb{R}^n$. Let $P = \sum_{i,j=1}^{n} a_{i,j}(x)\partial_i\partial_j + \sum_k b_k(x)\partial_k + c(x)$ be a differential operator of order 2 with $a_{i,j} \in C^\infty(\Omega)$ real valued, b_k, $c \in L^\infty(\mathbb{R}^n)$. Assume also that P is elliptic, that is there exists $C > 0$ so that*

$$\sum_{i,j=1}^{n} a_{i,j}(x)\xi_i\xi_j \geq C|\xi|^2, \text{ for all } (x,\xi) \in \Omega \times \mathbb{R}^n.$$

Let $\Phi \in C^2(\Omega)$ so that $\nabla\Phi \neq 0$ on Ω.

Let $x_0 \in \Omega$. Then, there exists V one neighborhood of x_0 so that for any $u \in C^\infty(\Omega)$,

$$\left\{ \begin{array}{rcl} Pu & = & 0 \text{ in } \Omega, \\ u & = & 0 \text{ in } \Omega \cap \{\Phi > \Phi(x_0)\} \end{array} \right. \Longrightarrow u = 0 \text{ on } V. \tag{2.1}$$

The previous theorem actually says that for elliptic operator of order 2, there is no geometric condition for unique continuation across hypersurface.

The previous theorem is a particular case of a more general theorem that we state below. For that, we will need to define some notion of convexity with respect to an operator.

Definition 2.1.1 (Strict pseudo-convexity for surfaces defined by a function). *Let $P \in \mathrm{Diff}^2$ be a (classical) differential operator with real valued principal symbol $p \in \Sigma^2$ and $\Phi \in C^\infty$ real valued.*

We say that the function Φ is pseudoconvex in the sense of surface with respect to P in Ω if it satisfies

$$p(x,\xi) = \nabla_\xi p(x,\xi) \cdot \nabla\Phi(x) = 0 \Longrightarrow \{p, \{p, \Phi\}\} > 0 \quad \forall (x,\xi) \in \Omega \times \mathbb{R}^n \setminus \{0\} \tag{2.2}$$

We can check that Definition 2.1.1 is invariant if we change the function Φ by another one $g(\Phi)$ which has the same level sets (with different value of course) and defines the same level sets. That is why this property only depends on the surface. It is a geometric property of the surface. See Subsection 2.2.1 for an interpretation as convexity with respect to the bicharacteristics curves.

Theorem 2.1.2 (Real operator of order 2)**.** *Let Ω an open set of $\mathbb{R}^n$. Let $P = \sum_{i,j=1}^n a_{i,j}(x)\partial_i\partial_j + \sum_k b_k(x)\partial_k + c(x)$ be a differential operator of order 2 with $a_{i,j} \in C^\infty(\Omega)$ **real** valued, b_k, $c \in L^\infty(\mathbb{R}^n)$. Let $\Phi \in C^2(\Omega)$ so that $\nabla\Phi \neq 0$ on Ω. Assume also that the principal symbol of P, $p(x,\xi) = -\sum_{i,j=1}^n a_{i,j}(x)\xi_i\xi_j$ satisfies*

$$p(x,\xi) = \nabla_\xi p(x,\xi)\cdot\nabla\Phi(x) = 0 \Longrightarrow \{p,\{p,\Phi\} > 0 \quad \forall (x,\xi)\in\Omega\times\mathbb{R}^n\setminus\{0\} \tag{2.3}$$

Let $x_0 \in \Omega$. Then, there exists V one neighborhood of x_0 so that for any $u \in C^\infty(\Omega)$,

$$\left\{\begin{array}{rcl} Pu &=& 0 \text{ in } \Omega, \\ u &=& 0 \text{ in } \Omega\cap\{\Phi > \Phi(x_0)\} \end{array}\right. \Longrightarrow u = 0 \text{ on } V. \tag{2.4}$$

Remark 2.1.1 (Elliptic case)**.** *The previous Theorem 2.1.2 actually includes Theorem 2.1.1. Indeed, the condition $p(x,\xi) = 0$ is never fulfilled.*

Remark 2.1.2 (The example of the wave equation)**.** *In the case of the wave equation, $P = \partial_t^2 - \Delta$, $p = -\xi_t^2 + |\xi_x|^2$, we compute (using that Φ does not depend on ξ)*

$$\begin{aligned} \{p,\Phi\} &= \nabla_\xi p\cdot\nabla_{(t,x)}\Phi = -2\xi_t\partial_t\Phi + 2\xi_x\cdot\nabla_x\Phi \\ \{p,\{p,\Phi\}\} &= \nabla_\xi p\cdot\nabla_{(t,x)}\{p,\Phi\} - \nabla_{(t,x)}p\cdot\nabla_\xi\{p,\Phi\} \\ &= \nabla_\xi p\cdot\nabla_{(t,x)}\{p,\Phi\} \\ &= -2\xi_t\partial_t\left[-2\xi_t\partial_t\Phi + 2\xi_x\cdot\nabla_x\Phi\right] + 2\xi_x\cdot\nabla_x\left[-2\xi_t\partial_t\Phi + 2\xi_x\cdot\nabla_x\Phi\right] \\ &= 4\left[\xi_t^2\partial_t^2\Phi - 2\xi_t\xi_x\cdot\nabla_x\partial_t\Phi + Hess_x(\Phi)(\xi_x,\xi_x)\right] \end{aligned}$$

If for instance, we choose $\Phi(t,x)$ of the form $-f(t)+\varphi(x)$, we can write (2.3) *(specialized in the point $(t,x) = (0,0)$) as*

$$\xi_t^2 = |\xi_x|^2 \text{ and } f'(0)\xi_t = \xi_x\cdot\nabla_x\varphi(0) \Longrightarrow -f''(0)|\xi_x|^2 + Hess_x(\varphi)(0)(\xi_x,\xi_x) > 0 \quad \forall\xi\in\mathbb{R}^n\setminus\{0\}.$$

We select several possibilities

- *$|f'(0)|^2 > |\nabla_x\varphi(0)|^2$, that is $|\partial_t\Phi|^2 > |\nabla_x\Phi|^2$: $\nabla_{t,x}\Phi$ is timelike. The first two conditions imply $|f'(0)\xi_t|^2 = |\xi_x\cdot\nabla_x\varphi(0)|^2 \leq |\xi_x|^2|\varphi(0)|^2 = |\xi_t|^2|\nabla_x\varphi(0)|^2$. This implies $\xi_t = 0$ and so $\xi_x = 0$. That means that the condition is empty.*

  Any timelike surface satisfies the unique continuation. This is very natural. Actually, the Cauchy problem is hyperbolic and indeed locally wellposed for any timelike hypersurface (like for instance the wave equation posed with initial data at $t = 0$).

- *$|f'(0)|^2 > |\nabla_x\varphi(0)|^2$, the typical example is unique continuation across an hypersurface of the form $\Phi(t,x) = \varphi(x)$, the condition is then*

  $$\xi_t^2 = |\xi_x|^2 \text{ and } \xi_x\cdot\nabla_x\varphi(0) = 0 \Longrightarrow Hess_x(\varphi)(0)(\xi_x,\xi_x) > 0 \quad \forall\xi\in\mathbb{R}^n\setminus\{0\}.$$

  Typically, if $\Phi(t,x) = |x|^2 - 1$, the condition holds when we want to prove the unique continuation from the exterior of the ball to the interior and not in the other direction. There are actually counterexamples if we allow a potential smooth in t and x (see Alinhac-Baouendi [1])

  Note also that for the $1D$ wave equation, the constraint $\xi_x\cdot\nabla_x\varphi(0)$ is much more demanding and implies $\xi_x = 0$ and $\xi = 0$ if $\xi_t^2 = |\xi_x|^2$. This is natural since we can exchange the time and space variable. So the finite speed of propagation (or a more refined version of it) implies easily the unique continuation across any non caracteristic hypersurface ($|f'(0)|^2 \neq |\partial_x\varphi(0)|^2$ in our setting).

The previous Theorem will be finally proved in section 2.3.2. The next sections are some preliminary, especially Carleman estimates that will be used for the proof of the unique continuation.

2.2 Carleman estimates

As we have seen in the Introduction in subsection 1.1.4, our goal is to get some estimates of the type of (1.11). Our main result in this section is Theorem 2.2.1. Yet, its assumptions may seem not so natural at first sight. So, before stating we describe some properties of the conjugated operator.

Hence, we will need to study the conjugated operator $P_\Phi := e^{\tau\Phi}Pe^{-\tau\Phi}$ that we will see as an operator depending on τ as defined in the previous section. We can actually compute its principal symbol.

Lemma 2.2.1 (The conjugated operator)**.** *Let $P = \sum_\alpha p_\alpha(x)D^\alpha \in \text{Diff}^m$ be a (classical) differential operator with principal symbol $p_m \in \Sigma^m$ and $\Phi \in C^\infty$ real valued.*

Then, $P_\Phi v = e^{\tau\Phi}Pe^{-\tau\Phi}v \in \text{Diff}^m_\tau$. Hence, it is a differential operator depending on τ of order m in (ξ,τ). Moreover, its principal symbol (as an operator depending on τ) is

$$p_{\phi,m} = p_m(x, \xi + i\tau\nabla\Phi) = \sum_\alpha p_\alpha(x)(\xi + i\tau\nabla\Phi)^\alpha$$

We will denote it p_ϕ for simplicity in the sequel.

Roughly speaking, the previous Lemma says that p_Φ is obtained by replacing ξ by $\xi + i\tau\nabla\Phi$ in p_m.

Note that it implies for instance that p_Φ may have a complex symbol even if p is real valued.

Proof. We easily check that $e^{\tau\Phi}D_je^{-\tau\Phi}u = D_ju + i\tau(\partial_j\Phi)u$. In particular, the conjugated operator $e^{\tau\Phi}D_je^{-\tau\Phi} \in \text{Diff}^1_\tau$ is a differential operator of order 1 is (D,τ) with principal symbol $\xi_j + i\tau\partial_j\Phi$.

Moreover,

$$\begin{aligned} e^{\tau\Phi}D_j^{\alpha_j}e^{-\tau\Phi} &= e^{\tau\Phi}D_je^{-\tau\Phi}e^{\tau\Phi}D_j\cdots e^{\tau\Phi}D_je^{-\tau\Phi} \\ &= (e^{\tau\Phi}D_je^{-\tau\Phi})(e^{\tau\Phi}D_je^{-\tau\Phi})\cdots(e^{\tau\Phi}D_je^{-\tau\Phi}). \end{aligned}$$

Therefore, using Proposition 1.2.2 $|\alpha_j|$ times, we get that it is a differential operator depending on τ of order $|\alpha_j|$ with principal symbol $(\xi_j + i\tau\partial_j\Phi)^{|\alpha_i|}$ (note that the full symbol is more complicated).

So, since $D^\alpha = D_1^{\alpha_1}\cdots D_j^{\alpha_j}\cdots D_n^{\alpha_n}$, we obtain similarly that $e^{\tau\Phi}D^\alpha e^{-\tau\Phi}$ has principal symbol

$$\prod_{i=1}^n(\xi_j + i\tau\partial_j\Phi)^{|\alpha_i|} = (\xi + i\tau\nabla\Phi)^\alpha$$

using the notation defined in section 1.1.3.

Since $P = \sum_\alpha p_\alpha(x)D^\alpha$ and p_α commutes with $e^{\tau\Phi}$, it gives the result by summing up. □

Example 1. *Take $P = -\Delta$ with symbol $|\xi|^2$. We compute*

$$\begin{aligned} e^{\tau\Phi}(-\Delta)e^{-\tau\Phi}u &= -e^{\tau\Phi}\left[\Delta(e^{-\tau\Phi})u + e^{-\tau\Phi}\Delta u + 2\nabla u\cdot\nabla(e^{-\tau\Phi})\right] \\ &= -e^{\tau\Phi}\left[-\tau(\Delta\Phi)e^{-\tau\Phi} + \tau^2|\nabla\Phi|^2e^{-\tau\Phi}u + e^{-\tau\Phi}\Delta u - 2\tau\nabla u\cdot\nabla\Phi e^{-\tau\Phi}\right] \\ &= \tau(\Delta\Phi)u - \tau^2|\nabla\Phi|^2u - \Delta u + 2\tau\nabla u\cdot\nabla\Phi \end{aligned}$$

where we have used

$$\begin{aligned} \nabla(e^{-\tau\Phi}) &= -\tau\nabla\Phi e^{-\tau\Phi} \\ \Delta(e^{-\tau\Phi}) &= div(\nabla(e^{-\tau\Phi})) = -\tau div(\nabla\Phi e^{-\tau\Phi}) = -\tau(\Delta\Phi)e^{-\tau\Phi} - \tau\nabla\Phi\cdot\nabla(e^{-\tau\Phi}) \\ &= -\tau(\Delta\Phi)e^{-\tau\Phi} - \tau^2|\nabla\Phi|^2e^{-\tau\Phi}. \end{aligned}$$

$\tau^2|\nabla\Phi|^2u$, Δu and $\tau\nabla u\cdot\nabla\Phi$ are of order 2 with respective symbol $\tau^2|\nabla\Phi|^2$, $-|\xi|^2$ and $i\tau\xi\cdot\nabla\Phi$ (remember that $\nabla u=(\partial_1 u,\cdots,\partial_n u)=i(D_1u,\cdots,D_nu)$ has complex symbol denoted for short iD).

So, denoting p_Φ the full symbol of P and $p_{\Phi,2}$ its principal symbol, we have

$$\begin{aligned} p_\Phi &= |\xi|^2-\tau^2|\nabla\Phi|^2+2i\tau\xi\cdot\nabla\Phi+\tau(\Delta\Phi)u \\ p_{\Phi,2} &= |\xi|^2-\tau^2|\nabla\Phi|^2+2i\tau\xi\cdot\nabla\Phi \end{aligned}$$

Note that we have $p_{\Phi,2}=|\xi+i\tau\nabla\Phi|^2=p_2(x,\xi+\tau\nabla\Phi)$

The right condition for obtaining Carleman estimates is the condition of pseudoconvexity.

Definition 2.2.1 (Strict pseudo-convexity for functions). *Let $P\in\mathrm{Diff}^2$ be a (classical) differential operator with real valued principal symbol $p_2\in\Sigma^2$ and $\Phi\in C^\infty$ real valued.*

We say that the function Φ is pseudoconvex with respect to P at x_0 if it satisfies

$$\{p_2,\{p_2,\Phi\}\}(x_0,\xi)>0,\quad \text{if } p_2(x_0,\xi)=0 \text{ and } \xi\neq 0; \tag{2.5}$$

$$\frac{1}{i\tau}\{\overline{p_\Phi},p_\Phi\}(x_0,\xi)>0,\quad \text{if } p_\Phi(x_0,\xi)=0 \text{ and } \tau>0, \tag{2.6}$$

where $p_\Phi(x,\xi)=p_2(x,\xi+i\tau\nabla\Phi)$.

Note that the previous definition is clearly stronger than Definition 2.1.1. Moreover, it is not only dependent on the level set of the functions. It also depends on the "convexity with respect to the level sets". This is not only a geometric quantity, contrary to Definition 2.2.1 of pseudoconvexity for surfaces. Yet, we will see later that for a surface satisfying Definition 2.1.1, there is a good choice of function satisfying the assumption of Definition 2.2.1 and leading to a unique continuation theorem.

Note also, that in some sense, we could say that for real operators, the first line is the limit of the second line as τ tends to 0, see Lemma 2.2.4 below.

Theorem 2.2.1 (The Carleman estimate). *Let $P\in\mathrm{Diff}^2$ be a (classical) differential operator with real valued principal symbol $p\in\Sigma^2$ and $\Phi\in C^\infty$ real valued, strictly pseudoconvex with respect to P at x_0, as in Definition 2.2.1.*

Then, there exists a neighborhood V of x_0, $C>0$ and $\tau_0>0$ so that we have the following estimate

$$\tau\left\|e^{\tau\Phi}u\right\|^2_{H^1_\tau}\leq C\left\|e^{\tau\Phi}Pu\right\|^2_{L^2}, \tag{2.7}$$

for any $u\in C^\infty_0(V)$ and $\tau\geq\tau_0$.

Proof. Before going further, let us notice that Lemma 2.2.1 only depends on the leading order of the operator P. More precisely, if it is true for one P , it is true for $P+R$ with $R\in\mathrm{Diff}^1$. Indeed, $e^{\tau\Phi}(P+R)e^{-\tau\Phi}=P_\phi+R_\Phi$ with $R_\Phi\in\mathrm{Diff}^1_\tau$. In particular, if (2.8) is true for P, we can write

$$\tau\left\|v\right\|^2_{H^1_\tau}\leq C\left\|P_\Phi v\right\|^2_{L^2}\leq 2C\left\|(P_\Phi+R_\Phi)v\right\|^2_{L^2}+2C\left\|R_\Phi v\right\|^2_{L^2}\leq 2C\left\|(P_\Phi+R_\Phi)v\right\|^2_{L^2}+D\left\|v\right\|^2_{H^1_\tau}$$

where we have used $R_\Phi\in\mathrm{Diff}^1_\tau$ and Proposition 1.2.1. But for τ large enough, we have $\tau-D\geq\tau/2$, which gives (2.8) with P replaced by $P+R$ and with different constants C and τ_0.

P can be written $P=\sum_{k,l}a_{k,l}D_kD_l+R_1$ with $a_{k,l}=a_{l,k}$ real valued and $R_1\in\mathrm{Diff}^1$. Moreover, since $a_{k,l}D_kD_l=D_ka_{k,l}D_l-(D_ka_{k,l})D_l$ where the second term is of order 1. So P can be written $P=\sum_{k,l}D_ka_{k,l}D_l+R_2$ with $2R_2\in\mathrm{Diff}^1$. We can assume now that P is a sum of terms $D_ka_{k,l}D_l$ with $a_{k,l}=a_{l,k}$ real valued. The previous simplification is only technical and allows to impose P formally selfadjoint.

As already described in the introduction, if we make the change of unknown $v = e^{\tau\Phi}u$, the expected inequality (2.7) is equivalent to

$$\tau \|v\|_{H^1_\tau}^2 \leq C \|P_\Phi v\|_{L^2}^2 , \tag{2.8}$$

where $P_\Phi v = e^{\tau\Phi} P e^{-\tau\Phi} v$.

Lemma 2.2.1 says that $P_\Phi \in \mathrm{Diff}^2_\tau$ with principal symbol $p(x, \xi + i\tau\nabla\Phi)$. Yet, this symbol can cancel and we cannot apply Proposition 1.2.6. In some sense, we have to go to the "next order".

Denote

$$Q_R = \frac{P_\Phi + P_\Phi^*}{2}; \quad Q_I = \frac{P_\Phi - P_\Phi^*}{2i}.$$

So that

$$P_\Phi = Q_R + iQ_I. \tag{2.9}$$

This is more or less the decomposition of P_Φ according to its real and imaginary part (in the sense of operators). This can be seen for instance at the level of principal symbols since, by Proposition 1.2.4, Q_R and Q_I are in Diff^2_τ with principal symbols

$$q_R = \frac{p_\Phi + \overline{p_\Phi}}{2} = \mathrm{Re}\, p_\Phi; \quad q_R = \frac{p_\Phi - \overline{p_\Phi}}{2i} = \mathrm{Im}\, p_\Phi.$$

Moreover, we easily check that they are formally selfadjoint: $Q_R^* = Q_R$ and $Q_I^* = Q_I$. Using, this property and (2.9), we compute for $v \in C_0^\infty(\mathbb{R}^n)$

$$\begin{aligned} \|P_\Phi v\|_{L^2}^2 &= (P_\Phi v, P_\Phi v)_{L^2} = ((Q_R + iQ_I)v, (Q_R + iQ_I)v) \\ &= (Q_R v, Q_R v) + (iQ_I v, iQ_I v) + (Q_R v, iQ_I v) + (iQ_I v, Q_R v) \\ &= \|Q_R v\|_{L^2}^2 + \|Q_R v\|_{L^2}^2 - i\,(Q_I \circ Q_R v, v) + i\,(Q_R \circ Q_I v, v) \\ &= \|Q_R v\|_{L^2}^2 + \|Q_R v\|_{L^2}^2 + (i[Q_R, Q_I]v, v)\,. \end{aligned}$$

Now, we have 2 kinds of terms

- the one with $\|Q_R v\|_{L^2}^2$ (and resp. $\|Q_I v\|_{L^2}^2$) that corresponds to $\left(Q_R^2 v, v\right)$ where Q_R^2 is of order 4 with principal symbol $(\mathrm{Re}\, p_\Phi)^2$ (resp. $(\mathrm{Im}\, p_\Phi)^2$)
- the one with $i[Q_R, Q_I]$ which is of order $2 + 2 - 1 = 3$ and principal symbol $\{\mathrm{Re}\, p_\Phi, \mathrm{Im}\, p_\Phi\}$ by Proposition 1.2.3

The first kind has stronger order but they can cancel and are therefore not sufficient to get the "coercivity" estimate. So, the idea is to consider that in the case when q_R or q_I cancel, we use the next term of commutator. But to compare them, we will need to bring them to the same order and in some sense make a "sacrifice" of this main order 4. More precisely, let $C > 0$ large to be fixed later. For τ large enough we have $\frac{1}{\tau^{1/2}} \geq \frac{C}{(|\xi|^2 + \tau^2)^{1/2}}$ for any $\xi \in \mathbb{R}^n$. So, this gives after Fourier transform,

$$\begin{aligned} \frac{1}{\tau} \|Q_R v\|_{L^2}^2 = \left\| \frac{Q_R v}{\tau^{1/2}} \right\|_{L^2}^2 &\geq \left\| \frac{C}{(-\Delta + \tau^2)^{1/2}} Q_R v \right\|_{L^2}^2 \\ &\geq \left(\frac{C^2}{(-\Delta + \tau^2)^{1/2}} Q_R v, \frac{1}{(-\Delta + \tau^2)^{1/2}} Q_R v \right) \\ &\geq \left(Q_R \frac{C^2}{(-\Delta + \tau^2)} Q_R v, v \right). \end{aligned}$$

Note that we have applied $Q_R^* = Q_R$ for some function $(-\Delta+\tau^2)^{-1}Q_R v$ which does not have compact support, but it can be easily justified for Schwarz functions also. The same applies to Q_I.

So, at that step, we have proved

$$\frac{1}{\tau}\left\|P_\Phi v\right\|_{L^2}^2 \geq (Lv, v) \tag{2.10}$$

with $L = Q_R \frac{C^2}{(-\Delta+\tau^2)} Q_R + Q_I \frac{C^2}{(-\Delta+\tau^2)} Q_I + \frac{i}{\tau}[Q_R, Q_I]$.

We would like to apply Lemma 1.2.1 to L. First, we need to remark that $\frac{i}{\tau}[Q_R, Q_I] \in \mathrm{Diff}_\tau^2$. The order are correct since $[Q_R, Q_I]$ is of order 3 while $\frac{1}{\tau}$ is "morally" of order -1. We just need to show that we can "factorize by τ".

Let us prove that Q_I (of order 2 in (ξ, τ)) can be written $\tau\widetilde{Q}_I$ with $\widetilde{Q}_I$ of order 1.

Indeed, in the beginning of the proof, we proved that it was enough to consider $p(x, \xi)$ a quadratic form. Hence, $p(x, D)$ can be decomposed as a sum of terms of the form $D_k a_{k,l}(x) D_l$ with $a_{k,l}(x)$ real valued and $a_{k,l} = a_{l,k}$. For this kind of terms, we have the following conjugate operator

$$\begin{aligned} e^{-\tau\Phi} D_k a_{k,l}(x) D_l e^{-\tau\Phi} &= (D_k + i\tau\partial_k\Phi) a_{k,l}(x)(D_l + i\tau\partial_l\Phi) \\ &= \big[D_k a_{k,l}(x) D_l - \tau^2 a_{k,l}(x)(\partial_k\Phi)(\partial_l\Phi) \\ &\quad + i\tau\left(D_k a_{k,l}(x)\partial_l\Phi + (\partial_k\Phi)a_{k,l}(x)D_l\right)\big] \end{aligned}$$

We directly recognize it can be written $P_\Phi = P + \tau M$ with $M \in \mathrm{Diff}_\tau^1$. Since by choice, $P = P^*$, we have the decomposition $Q_R = P + \tau\frac{M+M^*}{2}$ and $Q_I = \tau\frac{M-M^*}{2i}$. So, we can take $\widetilde{Q}_I = \frac{M-M^*}{2i}$.

So, we have in that case

$$L = Q_R \frac{C^2}{(-\Delta+\tau^2)} Q_R + Q_I \frac{C^2}{(-\Delta+\tau^2)} Q_I + i[Q_R, \widetilde{Q}_I]$$

with $\widetilde{Q}_I$ of order 1. So, $i[Q_R, \widetilde{Q}_I]$ is of order $2+1-1=2$ with principal symbol $\frac{1}{\tau}\{\mathrm{Re}\, p_\Phi, \mathrm{Im}\, p_\Phi\}$. We have the following Lemma which allows to conclude thanks to (2.10) and Lemma 1.2.1.

Lemma 2.2.2. *Under the assumptions of pseudoconvexity for functions, that is Definition 2.2.1, there exists C_1 and $C_2 > 0$ so that*

$$\frac{C_1}{|\xi|^2+\tau^2}\left[(\mathrm{Re}\, p_\Phi)^2 + (\mathrm{Im}\, p_\Phi)^2\right] + \frac{1}{\tau}\{\mathrm{Re}\, p_\Phi, \mathrm{Im}\, p_\Phi\} \geq C_2\left(|\xi|^2+\tau^2\right) \tag{2.11}$$

taken at the point (x_0, ξ, τ), for all $\xi \in \mathbb{R}^n$, $\tau > 0$.

□

Remark 2.2.1. *This kind of inequality is often called subelliptic estimates. Indeed, if the operator P_Φ was elliptic in the (ξ, τ) variable, we would have some estimates by below with the norm H_τ^2 instead of H_τ^1 as in Proposition 1.2.6. But here the principal symbol of P_Φ, $p_2(x, \xi + i\tau\nabla\Phi)$ can have some zero (with $(\xi, \tau) \neq 0$) even if p_2 is elliptic.*

Take for instance the Laplace operator described in Example 1. The principal symbol of p_Φ is $|\xi|^2 - \tau^2|\nabla\Phi|^2 + 2i\tau\xi\cdot\nabla\Phi$. It cancels if we take $\xi \perp \nabla\Phi$ and $\tau^2 = |\xi|^2/|\nabla\Phi|^2$, which is always possible.

Note that it can seem surprising since for fixed τ, the operator P_Φ is elliptic in the ξ variable. We could expect to have some inequalities of the form

$$\left\|u\right\|_{H^2} \leq C_\tau \left\|P_\Phi u\right\|_{L^2}.$$

It is indeed possible if P is elliptic, but the constant C_τ will then blow-up as $\tau^{1/2}$.

Proof of Lemma 2.2.2. Note first that since $\{f,f\}=0$ and $\{f,g\}=-\{g,f\}$ for any f and g, we have

$$\begin{aligned}
\frac{1}{i\tau}\{\overline{p_\Phi},p_\Phi\} &= \frac{1}{i\tau}\{\operatorname{Re}p_\Phi - i\operatorname{Im}p_\Phi, \operatorname{Re}p_\Phi + i\operatorname{Im}p_\Phi\}\\
&= \frac{1}{\tau}\{\operatorname{Re}p_\Phi,\operatorname{Im}p_\Phi\} - \frac{1}{\tau}\{\operatorname{Im}p_\Phi,\operatorname{Re}p_\Phi\}\\
&= \frac{2}{\tau}\{\operatorname{Re}p_\Phi,\operatorname{Im}p_\Phi\}.
\end{aligned}$$

Moreover, by the previous computation, we can write $\operatorname{Im}p_\Phi = \tau\widetilde{q_I}$ (note that it could be seen as a consequence of the fact that p is real and $p_\Phi = p$ on the set $\{\tau=0\}$, so we can factorize $\operatorname{Im}p_\Phi$ by Taylor expansion).

We notice that all the terms in (2.11) are homogeneous in (ξ,τ) of order 2 and continuous thanks to the previous remark. Therefore, it is enough to prove (2.11) on the set $K=\left\{(\xi,\tau)\,\middle|\,|\xi|^2+\tau^2=1;\tau\geq 0\right\}$. It is a consequence of the following Lemma with $f=(\operatorname{Re}p_\Phi)^2+(\operatorname{Im}p_\Phi)^2$ and $g=2\{\operatorname{Re}p_\Phi,\widetilde{q_I}\}$and $h=0$ (the function h will be usefull for another application later.

The Lemma 2.2.4 after, proves that actually, the first assumption in Definition 2.2.1 is the limit of the second one on the set $\{\tau=0\}$. So that the hypothesis hold, up to the set $\{\tau=0\}\cap\{|\xi|^2+\tau^2=1\}$.

Lemma 2.2.3. *Let K be a compact set and f,g,h three continuous real valued functions on K. Assume that $f\geq 0$ on K, and $g>0$ on $\{f=0\}$. Then, there exists $A_0, C>0$ such that for all $A\geq A_0$, we have $g+Af-\frac{1}{A}h\geq C$ on K.*

Lemma 2.2.4. *Let p be real valued, then $\lim_{\tau\to 0}\frac{1}{i\tau}\{\overline{p_\Phi},p_\Phi\} = 2\{p\{p,\Phi\}\}$.*

□

Proof of Lemma 2.2.3. The set $N=\{f=0\}\cap K$ is compact. Since g is continuous, its minimum on the set N is reached. So, we have $g\geq C_1>0$ on N. Since g is continuous and N compact, there exists an open neighborhood V of N so that $g\geq C_1/2$ on V. Now, $K\setminus V$ is closed in K and indeed compact. So, f reach its minimum on $K\setminus V$. But since $(K\setminus V)\cap N=\emptyset$, we have $f\neq 0$ on $K\setminus V$ and indeed $f>0$. So the minimum C_2 is actually strictly positive. Define now $C_3=\min\{g(x), x\in K\setminus V\}$ and $C_4=\max\{|h(x)|, x\in K\}$.

We are in the following situation, for some A still to be chosen:

- on V, we have $g+Af-\frac{1}{A}h\geq\frac{C_1}{2}-\frac{1}{A}C_4$.
- on $K\setminus V$, we have $g+Af-\frac{1}{A}h\geq C_3+AC_2-\frac{1}{A}C_4$.

So, we need to choose A so that it leads to positive lower bound. If we want the final estimate with $C=C_1/4$, for instance, we need

$$A\geq\tfrac{4C_4}{C_1}$$
$$A^2C_2+A\left(C_3-\tfrac{C_1}{4}\right)-C_4>0$$

Since $C_2>0$, the last case is fulfilled if A is large enough since the polynomial of order 2 converges to $+\infty$ as A goes to ∞. □

Proof of Lemma 2.2.4. We first notice that for $\tau=0$, $\{\overline{p_\Phi},p_\Phi\}=\{\overline{p},p\}$ so since p is real, $\{\overline{p_\Phi},p_\Phi\}=0$ for $\tau=0$. So, we have by Taylor expansion in τ, $\lim_{\tau\to 0}\frac{1}{\tau}\{\overline{p_\Phi},p_\Phi\}=\frac{\partial}{\partial\tau}\{\overline{p_\Phi},p_\Phi\}\big|_{\tau=0}$. Also, we easily verify $\partial_\tau(\{\overline{p_\Phi},p_\Phi\})=\{\partial_\tau\overline{p_\Phi},p_\Phi\}+\{\overline{p_\Phi},\partial_\tau p_\Phi\}$.

But since p is real, $\overline{p_\Phi} = p(x, \xi - i\tau\nabla\Phi)$, so that

$$\partial_\tau p_\Phi = i\nabla\Phi \cdot \nabla_\xi p(x, x, \xi + i\tau\nabla\Phi) = i\{p_\Phi, \Phi\}$$
$$\partial_\tau \overline{p_\Phi} = -i\nabla\Phi \cdot \nabla_\xi p(x, x, \xi - i\tau\nabla\Phi) = -i\{\overline{p_\Phi}, \Phi\}.$$

So, we get $\partial_\tau(\{\overline{p_\Phi}, p_\Phi\}) = -i\{\{\overline{p_\Phi}, \Phi\}, p_\Phi\} + i\{\overline{p_\Phi}, \{p_\Phi, \Phi\}\}$. When specified for $\tau = 0$, we get

$$\frac{\partial}{\partial\tau}\{\overline{p_\Phi}, p_\Phi\}\bigg|_{\tau=0} = -i\{\{p, \Phi\}, p\} + i\{p, \{p, \Phi\}\} = 2i\{p, \{p, \Phi\}\}.$$

This gives the result. □

The proof of Lemma 2.2.3 is left to the reader.

Remark 2.2.2. *In the elliptic case, the "trick" of factorisation by τ of the Imaginary part of Q_I can be avoided. Indeed, close to $\{\tau = 0\}$ the symbol p_Φ is actually close to p and is therefore non zero.*

Note that a more general assumption for treating the behavior of $\{\overline{p_\Phi}, p_\Phi\}$ close to $\{\tau = 0\}$ is to use the principal normality assumption

$$|\{\bar{p}, p\}| \leq C|p||\xi|^{m-1}.$$

The inequality is obviously fulfilled in the elliptic case where $|p||\xi|^{m-1} \geq C|\xi|^{2m-1}$ and for real valued real principal symbol where $\{\bar{p}, p\} = 0$.

In the more general case of principal normality, for the behavior of $\frac{1}{i\tau}\{\overline{p_\Phi}, p_\Phi\}$, close to $\tau = 0$, the inequality allows to take advantage of the term $|p_\phi|^2$for proving some inequality related to (2.11). In that case, a variant of Lemma 2.2.4 remains true, but only close to the set $\{p_\Phi = 0\}$.

We refer to Section 2.6 for the statemenent of the Theorem and to Hörmander [7] Chapter XXVIII for the proof.

2.2.1 Geometric interpretation of pseudoconvexity in the case of real symbol of order 2

2.3 Using Carleman estimates for unique continuation

2.3.1 Convexification

Up to now, we have proved some Carleman estimates under some complicated conditions of Definition 2.2.1. The main purpose of this section is to prove that for some function satisfying Definition 2.1.1, we can find some other function approriate for the Carleman estimate, that is satisfying Definition 2.2.1. We will first give a different formulation of Definition 2.1.1.

Proposition 2.3.1 (Usual pseudoconvexity for surfaces). *Let Φ satisfying the pseudoconvexity assumptions for surfaces of Definition 2.1.1 for an operator $P \in \mathrm{Diff}^2$ with real valued principal symbol. Then, it satisfies the stronger property for any $x_0 \in \Omega$*

$$\{p_2, \{p_2, \Phi\}\}(x_0, \xi) > 0, \quad \text{if } p_2(x_0, \xi) = \{p_2, \Phi\}(x_0, \xi) = 0 \text{ and } \xi \neq 0; \tag{2.12}$$

$$\frac{1}{i\tau}\{\overline{p_\Phi}, p_\Phi\}(x_0, \xi) > 0, \quad \text{if } p_\Phi(x_0, \xi) = \{p_\Phi, \Phi\}(x_0, \xi) = 0 \text{ and } \tau > 0, \tag{2.13}$$

where $p_\Phi(x, \xi) = p_2(x, \xi + i\tau\nabla\Phi)$.

Note that we have added a second property that makes it look very similar to the strict pseudoconvexity for functions of Definition 2.2.1. It is just slightly weaker because the inequality is asked in some more constrained cases because we have asked the additional conditions $\{p_2, \Phi\} = 0$ and $\{p_\Phi, \Phi\} = 0$.

The conclusion of the previous Proposition 2.3.1 are actually the usual conditions of pseudoconvexity for surfaces that are the assumptions for unique continuation for operators of higher order.

Proof. The first property is exactly the same, so , we just need to prove that the second (2.13) is automatically fulfilled in that case.

For **fixed** $\xi \in \mathbb{R}^n$, denote $f(s) = p_2(x_0, \xi + s\nabla\Phi(x_0))$. Since p is a polynomial of order 2 (on $\mathbb{R}^n$), f is a polynomial in one variable s of order at most 2, with real valued coefficients. Moreover, we have $f'(s) = \nabla\Phi(x_0) \cdot \nabla_\xi p_2(x_0, \xi + s\nabla\Phi(x_0)) = \{p_2, \Phi\}(x_0, \xi + s\nabla\Phi(x_0))$. This is actually $\{p_\Phi, \Phi\}$ taken at the point x_0, ξ and $\tau = s/i$. Therefore, the assumptions of (2.13) are equivalent to the fact that there exists one $s = i\tau$ with $\tau \in \mathbb{R}$ so that $f(s) = 0$ and $f'(s) = 0$. That means that $s = i\tau$ is a root of order 2 and s is a double root of f.

Note that since p_2 is homogeneous of order 2, Lemma A.4.1 gives

$$p_\Phi(x_0, \xi, \tau) = p_2(x_0, \xi + i\tau\nabla\Phi) = p_2(x_0, \xi) - \tau^2 p_2(x_0, \nabla\Phi) + i\tau\{p, \Phi\}(x_0, \xi). \tag{2.14}$$

We will distinguish 2 cases

- f is of order exactly 2, that is the case if $p_2(x_0, \nabla\Phi) \neq 0$, we say that the surface $\{\Phi(x) = \Phi(x_0)\}$ non characteristic for P at x_0
- f is of order at most 1, $p_2(x_0, \nabla\Phi) = 0$, we say that the surface $\{\Phi(x) = \Phi(x_0)\}$ is characteristic for P at x_0

Assume now that f is of order exactly 2. Since f is real valued, its root are either both reals or two complex conjugates. So, the double root can only be $s = 0$, which is forbidden in the assumption (actually this case refers to the assumption (2.12), see Lemma 2.2.4).

Assume now that f is of order at most 1. The fact that f has a double root implies that actually, f is the zero polynomial. That is $p_2(x_0, \xi) = p_2(x_0, \nabla\Phi) = \{p, \Phi\}(x_0, \xi) = 0$. Note that thanks to (A.22), it also implies $\{p_\Phi, \Phi\}(x_0, \xi, \tau) = 0$ for every $\tau \in \mathbb{R}$.

Since p_2 (noted p) is homogeneous of order 2, real valued , we obtain after some computations (see Lemma A.4.2) that

$$\frac{1}{i\tau}\{\overline{p_\Phi}, p_\Phi\} = 2\{p, \{p, \Phi\}\} + 2\tau^2\{p, \{p, \Phi\}\}(x, \nabla\Phi).$$

Yet, we are in some case where ξ satisfies $p_2(x_0, \xi) = \{p, \Phi\}(x_0, \xi) = 0$, so the pseudoconvexity for surface implies either $\{p, \{p, \Phi\}\} > 0$ or $\xi = 0$.

Moreover, we are in some case where $p_2(x_0, \nabla\Phi) = 0$ and, by (A.22), $\{p_\Phi, \Phi\}(x_0, \nabla\Phi, \tau) = 2R(\nabla\Phi, \nabla\Phi) + 2i\tau p(x, \nabla\Phi) = 2p(x, \nabla\Phi) + 2i\tau p(x, \nabla\Phi) = 0$. So the pseudoconvexity condition taken at the point $\nabla\Phi$ implies $\{p, \{p, \Phi\}\}(x, \nabla\Phi) > 0$.

In particular, $\frac{1}{i\tau}\{\overline{p_\Phi}, p_\Phi\}(x_0, \xi, \tau) > 0$ when $\tau > 0$.

A simpler proof, could have been made if we assume that we are in some coordinates so that $\Phi = x_1$. Actually, this is not a loss of generality since we could prove (but we did not do it yet) that the assumptions and conclusions of Proposition 2.3.1 are invariant by change of coordinates and of defining function for the surface. In that case, we can check that actually f can never be identically zero. Indeed, if it happens, we have $0 = f(s) = p_\Phi = p_2(x_0, \xi + se_1)$ and $\partial_{\xi_1} p = 0$. It gives

$$\{p, \{p, \Phi\} = \{p, \{p, x_1\} = \{p, \partial_{\xi_1} p\} = \nabla_\xi p \cdot \nabla_x \partial_{\xi_1} p - \nabla_x p \cdot \nabla_\xi \partial_{\xi_1} p = 0.$$

This is impossible since we have $p(x,\xi)=\{p,\Phi\}=0$ in the considered points. It contradict the first assumption.

Note that the cancellation of $\{p,\{p,\Phi\}$ under these assumptions is specific to the chosen coordinates. $\square$

Now, we will present a systematic way to produce some functions satisfying the stronger assumption for some function satisfying the weaker pseudoconvexity for surfaces.

Proposition 2.3.2 (Analytic convexification). *Let Φ pseudoconvex in the sense of surfaces, with $\Phi(x_0)=0$, that is satisfying the assumptions of Definition 2.1.1 (and therefore, the one of Proposition 2.3.1)*

Then there exists λ large enough so that the functions $\Psi=e^{\lambda\Phi}$ satisfies the pseudoconvexity for functions of Definition 2.2.1.

Note that the value of the level sets of Ψ are different, but the level set of Φ and Ψ are actually the same: the geometry of the level sets did not change. That means for a function that satisfy the unique conditions of the Theorem of Unique Continuation, we produce some new function admissible for the Carleman estimates, whose level sets have exactly the same geometric properties.

Proof. $p_\Psi=p(x_0,\xi+i\tau\lambda\Psi\nabla\Phi)$. We have generally

$$\begin{aligned}
\frac{1}{i\tau}\{\overline{p_\Psi},p_\Psi\}(x_0,\xi,\tau) &= \frac{1}{i\tau}\left[\nabla_\xi p(x_0,\xi-i\tau\nabla\Psi)\cdot(\nabla_x p(x_0,\xi+i\tau\nabla\Psi))\right]\\
&\quad+Hess(\Psi)\left[\nabla_\xi p(x_0,\xi-i\tau\nabla\Psi);\nabla_\xi p(x_0,\xi+i\tau\nabla\Psi)\right]\\
&\quad-\frac{1}{i\tau}\left[\nabla_x p(x_0,\xi-i\tau\nabla\Psi)\cdot(\nabla_\xi p(x_0,\xi+i\tau\nabla\Psi))\right]\\
&\quad+Hess(\Psi)\left[\nabla_\xi p(x_0,\xi-i\tau\nabla\Psi);\nabla_\xi p(x_0,\xi+i\tau\nabla\Psi)\right]\\
&= \frac{2}{\tau}\operatorname{Im}\left[\nabla_\xi p(x_0,\xi-i\tau\nabla\Psi)\cdot(\nabla_x p(x_0,\xi+i\tau\nabla\Psi))\right]\\
&\quad+2Hess(\Psi)\left[\nabla_\xi p(x_0,\xi-i\tau\nabla\Psi);\nabla_\xi p(x_0,\xi+i\tau\nabla\Psi)\right].
\end{aligned}$$

In order to simplify the notations, once x_0 is fixed, we will denote $c_\Psi(\xi,\tau)=\frac{1}{i\tau}\{\overline{p_\Psi},p_\Psi\}(x_0,\xi,\tau)$ and the same thing for c_Φ. Moreover, we will use the convention of replacing $c_\Phi(\xi,\tau)=\frac{1}{i\tau}\{\overline{p_\Phi},p_\Phi\}(x_0,\xi,\tau)$ by its limit $2\{p,\{p,\Phi\}$ if $\tau=0$. And this extension is still continuous.

We compute

$$\begin{aligned}
\partial_j\Psi &= \lambda\partial_j\Phi e^{\lambda\Phi}\\
\partial_{j,k}\Psi &= \lambda\partial_{j,k}\Phi e^{\lambda\Phi}+\lambda^2(\partial_j\Phi)(\partial_k\Phi)e^{\lambda\Phi}
\end{aligned}$$

that we can write in a shorter way

$$\begin{aligned}
\nabla\Psi &= \lambda\nabla\Phi e^{\lambda\Phi}\\
Hess(\Psi)(\xi,\widetilde{\xi}) &= \lambda Hess(\Phi)(\xi;\widetilde{\xi})e^{\lambda\Phi}+\lambda^2(\xi\cdot\nabla\Phi)(\widetilde{\xi}\cdot\nabla\Phi)e^{\lambda\Phi}
\end{aligned}$$

And taken at the point x_0, this gives, noticing several times that $e^{\lambda\Phi(x_0)}=1$

$$\begin{aligned}
\nabla\Psi(x_0) &= \lambda\nabla\Phi\\
Hess(\Psi)(x_0)(\xi,\widetilde{\xi}) &= \lambda Hess(\Phi)(x_0)(\xi;\widetilde{\xi})e^{\lambda\Phi}+\lambda^2(\xi\cdot\nabla\Phi(x_0)(\widetilde{\xi}\cdot\nabla\Phi(x_0))
\end{aligned}$$

Using the previous computations for Ψ, we get, (we drop the fact that Φ and the different derivatives of Φ are taken at x_0)

$$\begin{aligned} c_\Psi(\xi,\tau) &= \frac{1}{i\tau}\{\overline{p_\Psi},p_\Psi\}(x_0,\xi,\tau) \\ &= \frac{2}{\tau}\operatorname{Im}\left[\nabla_\xi p(x_0,\xi-i\tau\lambda\nabla\Phi)\cdot(\nabla_x p(x_0,\xi+i\tau\lambda\nabla\Phi))\right] \\ &\quad +2\lambda Hess(\Phi)\left[\nabla_\xi p(x_0,\xi-i\tau\lambda\nabla\Phi);\nabla_\xi p(x_0,\xi+i\tau\lambda\nabla\Phi)\right] \\ &\quad +2\lambda^2\left(\nabla_\xi p(x_0,\xi-i\tau\lambda\nabla\Phi)\cdot\nabla\Phi\right)\left(\nabla_\xi p(x_0,\xi+i\tau\lambda\nabla\Phi)\cdot\nabla\Phi\right) \\ &= \lambda c_\Phi(\xi,\lambda\tau)+2\lambda^2\left|\{p,\Phi\}(x_0,\xi+i\tau\lambda\nabla\Phi)\right|^2 \\ &= \lambda c_\Phi(\xi,\lambda\tau)+2\lambda^2\left|\{p_\Phi,\Phi\}(x_0,\xi,\lambda\tau)\right|^2. \end{aligned}$$

That is this additional term that comes from the convexification that will allow to get some more positivity when $\{p_\Phi,\Phi\}\neq 0$. The positivity when $\{p_\Phi,\Phi\}\neq 0$ being ensured by the assumptions on Φ.

Using Lemma 2.2.3 (combined with Lemma 2.2.4 for the limit when $\tau=0$, see the proof of Carleman estimates), Property 2.13 gives some constants C_1, C_2 so that

$$c_\Phi(\xi,\tau)+C_1\left|\{p_\Phi,\Phi\}(x_0,\xi,\tau)\right|^2+C_1\frac{|p_\Phi(x_0,\xi,\tau)|^2}{|\xi|^2+\tau^2}\geq C_2(|\xi|^2+\tau^2).$$

for any $\tau\geq 0$, $|\xi|^2+\tau^2=1$, with the convention of replacing $c_\Phi(\xi,\tau)=\frac{1}{i\tau}\{\overline{p_\Phi},p_\Phi\}(x_0,\xi,\tau)$ by its limit $2\{p,\{p,\Phi\}$ if $\tau=0$. Replacing τ by $\lambda\tau$ for $\lambda\geq 1$ and by homogeneity, it can be reformulated

$$c_\Phi(\xi,\lambda\tau)+C_1\left|\{p_\Phi,\Phi\}(x_0,\xi,\lambda\tau)\right|^2+C_1\frac{|p_\Phi(x_0,\xi,\lambda\tau)|^2}{|\xi|^2+\lambda^2\tau^2}\geq C_2(|\xi|^2+\lambda^2\tau^2).$$

for any $(\xi,\tau)\neq(0,0)$ with $\tau\geq 0$.

Moreover, we notice that $\frac{|p_\Phi(x_0,\xi,\lambda\tau)|^2}{|\xi|^2+\lambda^2\tau^2}=\frac{|p_\Psi(x_0,\xi,\tau)|^2}{|\xi|^2+\lambda^2\tau^2}$.

Taking $2\lambda\geq C_1$, it gives

$$\begin{aligned} &\frac{1}{i\tau}\{\overline{p_\Psi},p_\Psi\}(x_0,\xi,\tau)+C_1\lambda\frac{|p_\Psi(x_0,\xi,\tau)|^2}{|\xi|^2+\tau^2} \\ =\;& c_\Psi(\xi,\tau)+C_1\frac{|p_\Psi(x_0,\xi,\tau)|^2}{|\xi|^2+\tau^2} \\ =\;& \lambda c_\Phi(\xi,\lambda\tau)+2\lambda^2\left|\{p_\Phi,\Phi\}(x_0,\xi,\lambda\tau)\right|^2+C_1\lambda\frac{|p_\Psi(x_0,\xi,\tau)|^2}{|\xi|^2+\tau^2} \\ =\;& \lambda\left[c_\Phi(\xi,\lambda\tau)+2\lambda\left|\{p_\Phi,\Phi\}(x_0,\xi,\lambda\tau)\right|^2\right]+C_1\frac{|p_\Psi(x_0,\xi,\tau)|^2}{|\xi|^2+\tau^2} \\ \geq\;& \lambda\left[c_\Phi(\xi,\lambda\tau)+C_1\left|\{p_\Phi,\Phi\}(x_0,\xi,\lambda\tau)\right|^2+C_1\frac{|p_\Psi(x_0,\xi,\tau)|^2}{|\xi|^2+\lambda^2\tau^2}\right] \\ \geq\;& C_2\lambda(|\xi|^2+\lambda^2\tau^2)\geq C_2\lambda(|\xi|^2+\tau^2). \end{aligned}$$

This implies the strong convexity for functions (still with the same convention when $\tau=0$). □

Proposition 2.3.3 (Stability and Geometric convexification). *Let Ψ pseudoconvex in the sense of functions, that is satisfying the assumptions of Definition 2.2.1.*

Then there exists ε_0 small enough so that for any $0<\varepsilon<\varepsilon_0$, the functions $\Psi_\varepsilon=\Psi-\varepsilon|x-x_0|^2$ still satisfies the pseudoconvexity for functions.

Note that this step has slightly changed the level sets of Ψ. The level sets $\{\Psi_\varepsilon = 0\}$ are now slightly bended (except at x_0) into the set $\{\Psi > 0\}$ where u will be assumed to be zero. This slight change will be crucial for the unique continuation.♣ `faire un dessin`

Proof. First, we notice that we can prove as in the previous Proposition before that Definition 2.2.1 (still using Lemma 2.2.3 combined with Lemma 2.2.4 for the limit when $\tau = 0$) implies (and is actually equivalent to) the existence of an inequality of the form

$$\frac{1}{i\tau}\{\overline{p_\Psi}, p_\Psi\}(x_0, \xi, \tau) + C_1\frac{|p_\Psi(x_0, \xi, \tau)|^2}{|\xi|^2 + \tau^2} \geq C_2(|\xi|^2 + \tau^2).$$

uniformly for (ξ, τ) with $|\xi|^2 + \tau^2 = 1$, $\tau \geq 0$, still with the same convention as before when $\tau = 0$.

It can be easily checked that this estimates only involves some derivatives of Ψ up to order 2 at the point x_0. It is therefore stable by the addition of a function small for the C^2 norm around x_0. $\square$

2.3.2 Unique continuation

In this section, we give the final proof of Theorem 2.1.2.

Proof of Theorem 2.1.2. Let u solution of $Pu = 0$ in Ω so that $u = 0$ on $\Omega \cap \{\Phi > 0\}$. Φ is pseudoconvex for surfaces at x_0. Proposition 2.3.2 allows to produce a new function Ψ pseudoconvex for functions so that (up to changing Ψ by $\Psi - 1$) $\{\Psi \geq 0\} = \{\Phi \geq 0\}$ and $\{\Psi < 0\} = \{\Phi < 0\}$.

Proposition 2.3.3 proves that for some small $\varepsilon > 0$, $\Psi_\varepsilon = \Psi - \varepsilon|x - x_0|^2$ satisfies the pseudoconvexity for functions. It therefore satisfies the following properties

1. there exists $R > 0$, $C > 0$ and $\tau_0 > 0$ so that we have the following estimate

   $$\tau \left\|e^{\tau\Psi_\varepsilon} w\right\|^2_{H^1_\tau} \leq C \left\|e^{\tau\Psi_\varepsilon} Pw\right\|^2_{L^2}, \tag{2.15}$$

   for any $w \in C^\infty(B(x_0, R))$ and $\tau \geq \tau_0$.

2. there exists $\eta > 0$ so that $\Psi_\varepsilon(x) \leq -\eta$ for $x \in \{\Phi \leq 0\} \cap \{|x - x_0| \geq R/2\}$,

3. there exists a small neighborhood $V \subset B(x_0, R/2)$ of x_0 so that $\Psi_\varepsilon(x) \geq -\eta/2$ for $x \in V$.

The property 1 is a consequence of Theorem 2.2.1.

Property 2 is true thanks to the parameter ε in the geometric convexification. Indeed, for $|x-x_0| \geq R/2$, we have $\Psi_\varepsilon(x) \leq \Psi(x) - \varepsilon R^2/4$. Moreover, $\Phi(x) \leq 0$ implies $\Psi \leq 0$, so that $\Psi_\varepsilon(x) \leq -\varepsilon R^2/4$. So we can take $\eta = -\varepsilon R^2/4$.

Property 3 is only a continuity argument since $\Psi_\varepsilon(x_0) = 0$.

Pick $\chi \in C_0^\infty(B(x_0, R))$ so that $\chi = 1$ on $B(x_0, R/2)$. We want to apply the Carleman estimate to $w = \chi u$ solution of $Pw = \chi Pu + [P, \chi]u = [P, \chi]u$. Notice that $[P, \chi]$ is a classical differential operator of order 1 with coefficients supported in the set $\{\Phi \leq 0\} \cap \{\frac{R}{2} \leq |x - x_0| \leq R\}$ where we have $\Psi_\varepsilon(x) \leq -\eta$. In particular, we have $\left\|e^{\tau\Psi_\varepsilon} Pw\right\|_{L^2} \leq Ce^{-\tau\eta} \left\|u\right\|_{H^1}$.

Moreover, since $\Psi_\varepsilon(x) \geq -\eta/2$ on V, we have $e^{-\tau\eta/2} \left\|u\right\|_{L^2(V)} \leq \left\|e^{-\tau\eta/2} u\right\|_{L^2(V)} \leq \left\|e^{\tau\Psi_\varepsilon} u\right\|_{L^2(V)} \leq \left\|e^{\tau\Psi_\varepsilon} \chi u\right\|_{L^2(V)} \leq \tau^{1/2} \left\|e^{\tau\Psi_\varepsilon} w\right\|_{H^1_\tau}$. So the Carleman estimate implies

$$e^{-\tau\eta/2} \left\|u\right\|_{L^2(V)} \leq \tau^{1/2} \left\|e^{\tau\Psi_\varepsilon} w\right\|_{H^1_\tau} \leq C \left\|e^{\tau\Psi_\varepsilon} Pw\right\|_{L^2} \leq Ce^{-\tau\eta} \left\|u\right\|_{H^1}.$$

This gives $\left\|u\right\|_{L^2(V)} \leq Ce^{-\tau\eta/2} \left\|u\right\|_{H^1}$ and $u = 0$ on V by letting τ tend to infinity.

Note that when the operator P is replaced by $P + \sum_k b_k(x)\partial_k + c(x)$ with b_k, $c \in L^\infty(\mathbb{R}^n)$, the modified P does not exactly belong to the class Diff^2 because the coefficients are not smooth. For instance, the commutator could involve some derivative in the coefficients of the lower order terms. Yet, we proceed slighltly differently. We apply the same reasoning to P where P only contains the terms of order 2 that are smooth. The only difference is that we don't have any more $P\chi u = [P, \chi u]$ but $P\chi u = [P, \chi u] - \sum_k b_k(x)\partial_k(\chi u) - c(x)\chi u$.

For instance, for the 1 order terms, these terms can be estimated by

$$\begin{aligned} \left\| e^{\tau\Psi_\varepsilon} b_k(x)\partial_k w \right\|_{L^2} &\leq \|b_k\|_{L^\infty} \left\| e^{\tau\Psi_\varepsilon}\partial_k w \right\|_{L^2} \leq \|b_k\|_{L^\infty} \left[\left\| \nabla \left[e^{\tau\Psi_\varepsilon} w \right] \right\|_{L^2} + \tau \left\| (\nabla\Psi_\varepsilon) e^{\tau\Psi_\varepsilon} w \right\|_{L^2} \right] \\ &\leq C \left\| e^{\tau\Psi_\varepsilon} w \right\|_{H^1_\tau}. \end{aligned}$$

where C depends on $\|b_k\|_{L^\infty}$ and $\|\Psi_\varepsilon\|_{H^1(B(0,R))}$. This term can be made smaller than $\frac{1}{2}\tau^{1/2} \left\| e^{\tau\Psi_\varepsilon} w \right\|_{H^1_\tau}$ for τ large enough and therefore, (2.15) remains true with the constant replaced by a bigger one. We can then continue the proof similarly. $\square$

The unique continuation result that we obtain is only local. Yet, we could expect to iterate this result with a well chosen sequence of hypersurfaces. It turns out that it is not easy to do in the general case. Take for instance the wave equation. Suppose that you are given an open set ω and $T > 0$ so that a solution u satisfies

$$\begin{cases} \partial_t^2 - \Delta u &= 0 \text{ on } [0,T] \times \Omega \\ u &= 0 \text{ on } [0,T] \times \omega \end{cases}$$

The question of defining what is the "domain of dependence" for the unique continuation that can be obtained with our unique continuation using iterated pseudoconvex surfaces (as described in Remark 2.1.2) is not clear.

An easier case is the elliptic case where the pseudoconvexity condition for the surface is almost empty. This allows to get the following global result.

Theorem 2.3.1 (Global result in the elliptic case)**.** *Let Ω a connected open set. Let P satisfying the same assumptions as Theorem 2.1.1. Let u smooth solution of $Pu = 0$ on Ω, that satisfies $u = 0$ on an arbitrary open set $\omega \subset \Omega$.*

Then, $u = 0$ on Ω.

Proof. The support of u, denoted by F is a closed set. Let us prove that it is also open. Let $x \in F$. We distinguish two cases: If $x \in Int(F)$, it by definition that there exists a neighborhood of x included in F. In the other case, $x \in Fr(F) = F \setminus Int(F)$, the boundary in Ω. Define R so that $B(x, R) \subset \Omega$. Take $x_1 \in \Omega \setminus F$ with $dist(x, x_1) < R/2$ (it exists since $x \in Fr(F)$). So, we have $u(y) = 0$ in a neighborhood of x_1 by definition of the support. Define $r_1 = sup\{r \in [0, R/2]; u(x) = 0 \text{ in } B(x_1, r)\}$. We know that $r_1 > 0$.

Assume $r_1 < R/2$. So, we have $u = 0$ in $B(x_1, r)$. Moreover, since $dist(x, x_1) < R/2$ and $B(x, R) \subset \Omega$, $B(x_1, R/2) \subset \Omega$. So, we can apply Theorem 2.1.2 to any point $x_0 \in S(x_1, r_1)$ the sphere of radius r_1 and of center x_1 to get that for any $x_0 \in S(x_1, r_1)$, there exists r_{x_0} so that $u(y) = 0$ in $B(x_0, r_{x_0})$. Covering $S(x_1, r_1)$ by a finite number of such balls and using the compactness of $S(x_1, r_1)$ we get one ε so that $u(y) = 0$ on $B(x_1, r_1 + \varepsilon)$ contradicting the definition of r_1. So, we have $r_1 = R/2$.

But since $dist(x, x_1) < R/2$, there exists a neighborhood of x included in $B(x_1, R/2)$. In particular, $u = 0$ in this neighborhood. This contradicts the fact that $x \in Fr(F)$. So, $Fr(F) = \emptyset$ and F is open. So F is open and close and not Ω. So, $F = \emptyset$. $\square$

♣ `faire un dessin`

2.3.3 Quantitative estimates

In this section, we want to give some estimates that quantify the unique continuation, that is some inequality proving in some sense the implication

$$\left\{\begin{array}{ccc} Pu & \text{small in} & \Omega, \\ u & \text{small in} & U \end{array}\right. \Longrightarrow u \text{ small in } \widetilde{U}.$$

described in the introduction. More precisely, we would like to have some estimates of the kind $\|u\|_{\widetilde{U}} \leq \varphi(\|u\|_U + \|Pu\|_\omega, \|u\|_\Omega)$, with $\varphi(a,b) \to 0$ as $a \to 0$ when b is bounded. The ideal situation would be some linear estimate in a, independent on b. This is the case when the Cauchy problem is wellposed. For instance the wave operator across the surface $\{t=0\}$. Yet, in those cases, Carleman estimates are generally not the best way to get uniqueness and good estimates. We will be interested in some case where the Cauchy problem is ill-posed and linear estimates are not expected to occur.

In our situation, the estimates that we can expect are more of Hölder type, that is $\varphi(a,b) = a^\theta b^{1-\theta}$. We can obtain this kind of estimates locally, with the same generality for the operators as we proved the unique continuation.

In what follows, we will remain in the elliptic framework where "almost any reasonable" function is pseudoconvex functions for surfaces at some point x_0. It allows to simplify 2 important facts:

- we can pick some functions pseudoconvex for surfaces whose level sets are compact. The easier example being $|x - x_0|$. One advantage is that we can skip the geometric convexification.
- the globalization is much easier, as we saw in Theorem 2.3.1 for the unique continuation.

Yet, in that generality, it will be quite complicated to get global one. So, we will keep in the elliptic framework for simplicity.

We first state the local result

Theorem 2.3.2 (Local quantitative estimates for real elliptic operator of order 2)**.** *Let Ω, P be as in Theorem 2.1.2. $x_0 \in \Omega$. Let $r > 0$ so that $B(x_0, 3r) \subset \Omega$.*

Then, there exists $C > 0$, $0 < \delta < 1$ so that

$$\|u\|_{H^1(B(x_0,2r))} \leq C\left[\|u\|_{H^1(B(x_0,r)} + \|Pu\|_{L^2(B(x_0,3r))}\right]^\delta \|u\|^{1-\delta}_{H^1(B(x_0,3r))}$$

for any $u \in C^\infty(\Omega)$.

Note that in the context of complex analysis (for the usual Laplacian), this inequality is usually called 3 sphere inequality and can be proved quite differently, but only for analytic coefficients.

Proof. Denote $\Phi = -|x - x_0|$. Using compactness arguments and Proposition 2.3.2, we get that for λ large enough, $\Psi = e^{\lambda\Phi}$ is pseudoconvex for functions, uniformly for every $x_0 \in B(x_0, 3r) \setminus B(x_0, r/2)$. The "uniformly" meaning that some estimates like (2.11) are true with one fixed $C > 0$, $\tau_0 > 0$ uniformly for $x_0 \in B(x_0, 3r) \setminus B(x_0, r/2)$. Then, we get exactly by the same reasoning as before but using the "semiglobal" Gårding inequality of Proposition 1.2.5, we get the Carleman estimate

$$\tau^{1/2}\left\|e^{\tau\Psi}v\right\|_{H^1_\tau} \leq C\left\|e^{\tau\Psi}Pv\right\|_{L^2}, \tag{2.16}$$

for any $v \in C_0^\infty(B(x_0, 3r) \setminus B(x_0, r/2))$, $\tau \geq \tau_0$.

Take $\chi \in C_0^\infty(B(x_0, 3r) \setminus B(x_0, r/2))$ so that $\chi = 1$ on $B(x_0, 5r/2) \setminus B(x_0, r)$. Apply the estimate to $v = \chi u$.

We have $Pv = \chi Pu + [P, \chi]u$ where $[P, \chi]$ is of order 1 supported in two parts of $\mathbb{R}^n$

- $|x-x_0| \in [r/2,1]$, where $\Psi \leq e^{-\lambda r/2} := \rho_3$. The corresponding term is bounded by
$$\left\|e^{\tau\Phi}[P,\chi]u\right\|_{L^2(|x-x_0|\in[r/2,3r/4])} \leq Ce^{\tau\rho_3}\left\|u\right\|_{H^1(B(x_0,r))}$$
- $|x-x_0| \in [5r/2,3r]$, where $\Psi \leq e^{-\lambda 5r/2} := \rho_1$. The corresponding term is bounded by
$$\left\|e^{\tau\Phi}[P,\chi]u\right\|_{L^2(|x-x_0|\in[5r/2,3r])} \leq Ce^{\tau\rho_1}\left\|u\right\|_{H^1(B(x_0,3r))}$$

The term corresponding to χPu is bounded by $Ce^{\tau\rho_3}\left\|Pu\right\|_{L^2(B(0,3r))}$ since $\Psi \leq \rho_3$ on $Supp(\chi)$.

The first term must be estimated by below. We first write the H^1_τ norm slightly differently
$$\left\|e^{\tau\Psi}\nabla v\right\|_{L^2(\Omega)} + \tau\left\|e^{\tau\Psi}v\right\|_{L^2(\Omega)} \leq C\left\|\nabla(e^{\tau\Psi}v)\right\|_{L^2(\Omega)} + \tau\left\|\nabla\Phi e^{\tau\Psi}v\right\|_{L^2} + \tau\left\|e^{\tau\Psi}v\right\|_{L^2(\Omega)} \leq C\left\|e^{\tau\Psi}v\right\|_{H^1_\tau}.$$
where the constant C may depend on $\left\|\nabla\Phi\right\|_{L^\infty(B(x_0,3r))}$.

But since $\chi = 1$ on $|x-x_0| \in [r,2r]$, we have, for a different constant C
$$\begin{aligned}
\left\|e^{\tau\Psi}v\right\|_{H^1_\tau} &\geq C\left\|e^{\tau\Psi}\nabla v\right\|_{L^2(\Omega)} + C\left\|e^{\tau\Psi}v\right\|_{L^2(\Omega)} \\
&\geq C\left\|e^{\tau\Psi}\nabla v\right\|_{L^2(|x-x_0|\in[r,2r])} + C\left\|e^{\tau\Psi}v\right\|_{L^2(|x-x_0|\in[r,2r])} \\
&\geq C\left\|e^{\tau\Psi}\nabla u\right\|_{L^2(|x-x_0|\in[r,2r])} + C\left\|e^{\tau\Psi}u\right\|_{L^2(|x-x_0|\in[r,2r])} \\
&\geq Ce^{\tau\rho_2}\left[\left\|\nabla u\right\|_{L^2(|x-x_0|\in[r,2r])} + \left\|u\right\|_{L^2(|x-x_0|\in[r,2r])}\right]
\end{aligned}$$
where $\rho_2 := e^{-\lambda r}$ is chosen so that $e^{\tau\Psi} \geq e^{\rho_2}$ on the set $\{|x-x_0| \in [r,2r]\}$.

We finally get
$$e^{\tau\rho_2}\left\|u\right\|_{H^1(|x-x_0|\in[r,2r])} \leq Ce^{\tau\rho_1}\left\|u\right\|_{H^1(B(x_0,3r))} + Ce^{\tau\rho_3}\left[\left\|u\right\|_{H^1(B(x_0,r))} + \left\|Pu\right\|_{L^2(B(0,3r))}\right].$$
with $\rho_1 < \rho_2 < \rho_3$. This gives
$$\left\|u\right\|_{H^1(|x-x_0|\in[r,2r])} \leq Ce^{-C_1\tau}\left\|u\right\|_{H^1(B(x_0,3r))} + Ce^{C_2\tau}\left[\left\|u\right\|_{H^1(B(x_0,r))} + \left\|Pu\right\|_{L^2(B(0,3r))}\right].$$
with $C_1 = \rho_2 - \rho_1 > 0$ and $C_2 = \rho_3 - \rho_2 > 0$. Next, we apply the following Lemma of interpolation type, for which we postpone the proof.

Lemma 2.3.1. *Let C_1 , C_2, $C_3 > 0$ and $\tau_0 > 0$. Then, there exists some constants $\delta \in]0,1[$ and $C > 0$ so that:*

For any a, b, c real positive numbers that satisfies the following estimate, uniformly for $\tau \geq \tau_0$
$$\begin{aligned}
a &\leq e^{-C_1\tau}b + e^{C_2\tau}c \\
a &\leq C_3 b.
\end{aligned}$$
we also have the estimate
$$a \leq Cb^\delta c^{1-\delta}.$$

Applying this Lemma with $a = \left\|u\right\|_{H^1(|x-x_0|\in[r,2r])}/C$, $b = \left\|u\right\|_{H^1(B(x_0,3r))}$, $c = \left[\left\|u\right\|_{H^1(B(x_0,r))} + \left\|Pu\right\|_{L^2(B(0,3r))}\right]$ and noticing that $a \leq c/C$, we get, with a different constant $C > 0$,
$$\left\|u\right\|_{H^1(|x-x_0|\in[r,2r])} \leq C\left\|u\right\|^\delta_{H^1(B(x_0,3r))}\left[\left\|u\right\|_{H^1(B(x_0,r))} + \left\|Pu\right\|_{L^2(B(0,3r))}\right]^{1-\delta}.$$
Moreover, we have obvisously, if $C \geq 1$,
$$\left\|u\right\|_{H^1(B(x_0,r))} \leq C\left\|u\right\|^\delta_{H^1(B(x_0,3r))}\left[\left\|u\right\|_{H^1(B(x_0,r))} + \left\|Pu\right\|_{L^2(B(0,3r))}\right]^{1-\delta}.$$
This gives the expected result by summing up. □

Proof of the Lemma 2.3.1. We minimize in τ. The minimum is reached for $\tau = \frac{\ln\left(\frac{bC_1}{cC_2}\right)}{C_1+C_2}$. To simplify (actually, it is just changing b by bC_1), we apply the formula for $\tau_1 = \frac{\ln\left(\frac{b}{c}\right)}{C_1+C_2}$. It gives, if $\tau_1 \geq \tau_0$,

$$\begin{aligned} a &\leq e^{-\frac{C_1}{C_1+C_2}\ln\left(\frac{b}{c}\right)} b + e^{\frac{C_2}{C_1+C_2}\ln\left(\frac{b}{c}\right)} c \\ &\leq \left(\frac{b}{c}\right)^{\delta-1} b + \left(\frac{b}{c}\right)^{\delta} c = 2b^\delta c^{1-\delta}. \end{aligned}$$

where we have denoted $\delta = \frac{C_2}{C_1+C_2}$.

In the case $\tau_1 \leq \tau_0$, this means $\frac{b}{c} \leq e^{\tau_0(C_1+C_2)}$, so $b \leq C(\tau_0, C_1, C_2)c$. So, the assumption $a \leq C_3 b$ gives $a \leq C_3 b^\delta b^{1-\delta} \leq C b^\delta c^{1-\delta}$ with a new constant depending on τ_0, C_1, C_2, C_3.

This gives the expected estimate in both cases with an appropriate constant $C > 0$. □

Now, we want to globalize this estimates

Theorem 2.3.3 (Global quantitative estimates for real elliptic operator of order 2)**.** *Let Ω connected, and P be as in Theorem 2.1.2. $x_0 \in \Omega$ and $r_0 > 0$. Let K be a compact subset of Ω.*

Then, there exists $C > 0$, $0 < \delta < 1$ so that

$$\|u\|_{H^1(K)} \leq C \left[\|u\|_{H^1(B(x_0,r_0))} + \|Pu\|_{L^2(\Omega)}\right]^\delta \|u\|_{H^1(\Omega)}^{1-\delta}$$

for any $u \in C^\infty(\Omega)$.

Proof. By compactness, it is enough to prove the following similar result for any $x \in K$, there exists $0 < r_x < r_0$, δ_x so that $B(x_1, r_x)) \subset \Omega$ and

$$\|u\|_{H^1(B(x_1,r_x))} \leq C \left[\|u\|_{H^1(B(x_0,r_0))} + \|Pu\|_{L^2(\Omega)}\right]^{\delta_x} \|u\|_{H^1(\Omega)}^{1-\delta_x}. \tag{2.17}$$

Indeed, we recover K by a finite number of such balls $K \subset \cup_{i\in I} B(x_i, r_i)$. Take $\delta = \min_{i\in I} \delta_i$. Indeed, the inequality (2.17) is still true with δ_x replaced by δ. Indeed, since $\delta \leq \delta_x$, we can decompose $\|u\|_{H^1(B(x_1,r_x))} = \|u\|_{H^1(B(x_1,r_x))}^{\frac{\delta}{\delta_x}} \|u\|_{H^1(B(x_1,r_x))}^{1-\frac{\delta}{\delta_x}}$ where both exponent are positive. Using (2.17) for the first term and $\|u\|_{H^1(B(x_1,r_x))} \leq \|u\|_{H^1(\Omega)}$ for the second, we get, with eventually different constants,

$$\begin{aligned} \|u\|_{H^1(B(x_1,r_x))} &\leq C \left[\|u\|_{H^1(B(x_0,r_0))} + \|Pu\|_{L^2(\Omega)}\right]^\delta \|u\|_{H^1(\Omega)}^{\frac{\delta}{\delta_x}-\delta} \|u\|_{H^1(\Omega)}^{1-\frac{\delta}{\delta_x}} \\ &\leq C \left[\|u\|_{H^1(B(x_0,r_0))} + \|Pu\|_{L^2(\Omega)}\right]^\delta \|u\|_{H^1(\Omega)}^{1-\delta} \end{aligned}$$

By summing up and using the recovering property, we would get

$$\|u\|_{H^1(K)} \leq C \left[\|u\|_{H^1(B(x_0,r_0))} + \|Pu\|_{L^2(\Omega)}\right]^\delta \|u\|_{H^1(\Omega)}^{1-\delta}.$$

So, we are left to prove (2.17) for any $x \in K$.

We also assume $\|Pu\|_{L^2(\Omega)} \leq \|u\|_{H^1(\Omega}$, otherwise, the result is trivial because the right hand side is bigger than $\|u\|_{H^1(\Omega)}$.

We will need the following geometric Lemma that we prove later.

Lemma 2.3.2. *Under the previous assumptions. Let x_0 and $x_1 \in \Omega$ and $r_0 > 0$. Then, there exists $0 < r < r_0$, $N \in \mathbb{N}$ and a sequence of points y_k, $k = 0, \cdots, N$ so that*

- $y_0 = x_0,\ y_N = x_1$.
- $B(y_{k+1}, r) \subset B(y_k, 2r)$.
- $B(y_k, 3r) \subset \Omega$.

Assuming this Lemma, we prove recursively the following property.
There exists C_k and $\delta_k \in]0,1[$ so that

$$\|u\|_{H^1(B(y_k,r))} \leq C_k \left[\|u\|_{H^1(B(x_0,r))} + \|Pu\|_{L^2(\Omega)}\right]^{\delta_k} \|u\|_{H^1(\Omega)}^{1-\delta_k}. \tag{2.18}$$

- The property is true for $k = 0$ for $C = 1$ and any $\delta_k \in [0,1]$ since $\|u\|_{H^1(B(x_0,r))} \leq \|u\|_{H^1(\Omega)}$.
- Assume the property true for $k < N$. Theorem 2.3.2 applied at the point y_k (which can be applied since $B(y_k, 3r) \subset \Omega$) gives $C > 0$, $0 < \delta < 1$ so that

  $$\|u\|_{H^1(B(y_k,2r))} \leq C \left[\|u\|_{H^1(B(y_k,r)} + \|Pu\|_{L^2(B(y_k,3r))}\right]^{\delta} \|u\|_{H^1(B(y_k,3r))}^{1-\delta}.$$

  Since $B(y_{k+1}, r) \subset B(y_k, 2r)$ and $B(y_k, 3r) \subset \Omega$, it gives

  $$\|u\|_{H^1(B(y_{k+1},r))} \leq C \left[\|u\|_{H^1(B(y_k,r))} + \|Pu\|_{L^2(\Omega)}\right]^{\delta} \|u\|_{H^1(\Omega)}^{1-\delta}.$$

  The assumption at step k and the fact that $\delta > 0$ gives

  $$\|u\|_{H^1(B(y_{k+1},r))} \leq C \left[C_k \left[\|u\|_{H^1(B(x_0,r))} + \|Pu\|_{L^2(\Omega)}\right]^{\delta_k} \|u\|_{H^1(\Omega)}^{1-\delta_k} + \|Pu\|_{L^2(\Omega)}\right]^{\delta} \|u\|_{H^1(\Omega)}^{1-\delta}.$$

  Since we have assumed $\|Pu\|_{L^2(\Omega)} \leq \|u\|_{H^1(\Omega)}$, we have $\|Pu\|_{L^2(\Omega)} \leq \left[\|u\|_{H^1(B(x_0,r))} + \|Pu\|_{L^2(\Omega)}\right]^{\delta_k} \|u\|_{H^1(\Omega)}^{1-\delta_k}$. So, we are left with some different constant C_{k+1}

  $$\begin{aligned}\|u\|_{H^1(B(y_{k+1},r))} &\leq C_{k+1} \left[\left[\|u\|_{H^1(B(x_0,r))} + \|Pu\|_{L^2(\Omega)}\right]^{\delta_k} \|u\|_{H^1(\Omega)}^{1-\delta_k}\right]^{\delta} \|u\|_{H^1(\Omega)}^{1-\delta} \\ &\leq C_{k+1} \left[\|u\|_{H^1(B(x_0,r))} + \|Pu\|_{L^2(\Omega)}\right]^{\delta_k\delta} \|u\|_{H^1(\Omega)}^{1-\delta_k\delta}.\end{aligned}$$

  So, it gives the result with $\delta_{k+1} = \delta_{k+1}\delta$.

□

Proof of Lemma 2.3.2. Since Ω is a connected set of $\mathbb{R}^n$, it is connected by arc and we can find $\gamma(t)$ be a continuous path in Ω so that $\gamma(0) = x_0$, $\gamma(1) = x_1$.

$[0,1]$ is a compact set. Denote $d = \max_{t\in[0,1]}(dist(\gamma(t), \Omega^c)$. We fix $r = d/3$. By compactness, γ is also uniformly continuous on $[0,1]$. So, there exists $\varepsilon > 0$ so that $|t-t'| \leq \varepsilon$ implies $|\gamma(t)-\gamma(t')| \leq r/2$. We take $N = \left\lfloor \frac{1}{\varepsilon} \right\rfloor + 1$ and define

$$\begin{aligned} y_k &= \gamma(k\varepsilon) \text{ for } k = 0, \cdots N-1 \\ y_N &= x_1 = \gamma(1).\end{aligned}$$

This fulfills the expected criterium. For instance, $B(y_{k+1}, r) \subset B(y_k, 2r)$ is fulfilled if $|y_{k+1} - y_k| < r$. This works since for $k \leq N-2$, $|y_{k+1} - y_k| = |\gamma((k+1)\varepsilon) - \gamma(k\varepsilon)| \leq r/2$ by the uniform continuity assumption. For the last step, $k = N-1$, the same argument applies since $y_N = \gamma(1)$ and $y_{N-1} = \gamma(\lfloor 1/\varepsilon \rfloor \varepsilon)$. We observe that $|1 - \left\lfloor \frac{1}{\varepsilon} \right\rfloor \varepsilon| \leq \varepsilon$ because $\left|\frac{1}{\varepsilon} - \left\lfloor \frac{1}{\varepsilon} \right\rfloor\right| \leq 1$ by definition. □

2.4 The Dirichlet case

Theorem 2.4.1 (Global quantitative estimates for real elliptic operator of order 2)**.** *Let Ω connected with smooth boundary, and P be as in Theorem 2.1.2. Let $\Gamma \subset \partial\Omega$ a non empty open subset of the boundary. Let K be a compact subset of $\overline{\Omega}$.*

Then, there exists $C > 0$, $0 < \delta < 1$ so that

$$\|u\|_{H^1(K)} \leq C \left[\|\partial_\nu u\|_{L^2(\Gamma)} + \|Pu\|_{L^2(\Omega)}\right]^\delta \|u\|_{H^1(\Omega)}^{1-\delta}$$

for any $u \in C^\infty(\overline{\Omega})$ with $u = 0$ on $\partial\Omega$.

Obtaining the previous estimates follows a similar path as previously, except that we need to prove some Carleman estimates until the boundary. By some change of variables, it is always possible (see Lemma **??**) to get to the following situation.

We decompose $x \in \mathbb{R}^n$ with $x = (x', x_n)$ $x' \in \mathbb{R}^{n-1}$, $x_n \in \mathbb{R}$. The boundary $\partial\Omega$ becomes the set $\{x_n = 0\}$ and P is of the form $D_{x_n}^2 + r(x, D_{x'})$ where $r(x, D_{x'})$ is a family of operator depending on $x = (x', x_n)$, but with derivatives only in x'.

We denote $K_{r_0} = \mathbb{R}^n_+ \cap B(x_0, r_0)$ and $C_0^\infty(K_{r_0})$ is the set of functions in $C^\infty(\overline{\mathbb{R}^n_+})$ supported in $B(x_0, r_0)$. The index $+$ in the norms means that it is taken on $\mathbb{R}^n_+$.

Theorem 2.4.2 (Local Carleman estimate)**.** *Let $r_0 > 0$ and $P = D_{x_n}^2 + r(x, D_{x'})$ be a differential operator of order two on a neighborhood of K_{r_0}, with real principal part, where $r(x, D_{x'})$ is a smooth x_n family of second order operators in the (tangential) variable x'.*

Let ψ be quadratic polynomial such that $\psi'_{x_n} \neq 0$ on K_{r_0} and

$$\{p, \{p, \psi\}\}(x, \xi) > 0, \quad \text{if } p(x, \xi) = 0, \quad x \in K_{r_0}, \quad \xi \neq 0; \tag{2.19}$$

$$\frac{1}{i\tau}\{\overline{p}_\psi, p_\psi\}(x, \xi) > 0, \quad \text{if } p_\psi(x, \xi) = 0, \quad x \in K_{r_0}, \quad \tau > 0, \tag{2.20}$$

where $p_\psi(x, \xi) = p(x, \xi + i\tau\nabla\psi)$.

Then, there exist $C > 0$, $\tau_0 > 0$ such that for any $\tau > \tau_0$, we have for all $u \in C_0^\infty(K_{r_0/4})$

$$\begin{aligned} \tau\|e^{\tau\psi}u\|_{1,+,\tau}^2 &\leq C\Big(\left\|e^{\tau\Psi}Pu\right\|_{0,+}^2 + \tau^3|(e^{\tau\Psi}u)_{|x_n=0}|_0^2 \\ &\quad + \tau|\big(D(e^{\tau\Psi}u)\big)_{|x_n=0}|_0^2\Big). \end{aligned} \tag{2.21}$$

If moreover $\partial_{x_n}\psi > 0$ for $(x', x_n = 0) \in K_{r_0}$, then we have for all $u \in C_0^\infty(K_{r_0/4})$ such that $u_{|x_n=0} = 0$,

$$\tau\|e^{\tau\Psi}u\|_{1,+,\tau}^2 \leq C\left\|e^{\tau\Psi}Pu\right\|_{0,+}^2. \tag{2.22}$$

Note that the Theorem applies to real elliptic operators, but also to wave type operators with the associated pseudoconvexity condition.

We give a proof of this theorem in the appendix. The general idea is the following.

We would like to apply the same reasoning as before. Yet, we have to be more careful about the Gårding inequality in the case of boundary. One possibility is to use symbolic calculus only in the tangential variable x' where integration by parts are allowed without boundary terms. But the integration by parts for D_n^2 and its conjugated operator produce some boundary terms that we need to take into account.

How to deal with boundary terms?

In the variable x_n, the operator is D_n^2 and the conjugated operator is explicit $(D_n + i\tau(\partial_n\psi))^2 = D_n^2 - \tau^2(\partial_n\psi)^2 + 2i\tau(\partial_n\psi)D_n - (\partial_n^2\psi)$. The integration by parts can be explicitely computed. What is not a priori obvious is that there is no term of order 2 and 3.

What will save us is that fact that the real part and the imaginary part don't have the same number of derivative in x_n. Decomposing $P_\psi = Q^r + iQ^i$ as before, we check that Q^r has 2 derivatives in x_n (D_n^2) while Q^i only has one ($2i\tau(\partial_n\psi)D_n$).

Let us look at each term, using the integration by part formula $(f, D_n g) = (D_n f, g) + i(f,g)_{x_n=0}$

- integrating by part of $(Q^r u, Q^i u) = (Q^i Q^r u, u) + boundaryterms$: the worst terms should come from the integration by part of $(D_n^2 u, (\partial_n\psi)D_n u)$ and we expect the boundary term to be of the form $i(D_n^2 u, (\partial_n\psi)u)_{x_n=0}$. For instance, the boundary term corresponding to the to this should be of the form
- integrating by part of $(Q^i u, Q^r u) = (Q^r Q^i u, u) + boundaryterms$: the worst terms should come from the integration by part of

$$\begin{aligned} ((\partial_n\psi)D_n u, D_n^2 u) &= (D_n\left[(\partial_n\psi)D_n u\right], D_n u) + i((\partial_n\psi)D_n u, D_n u)_{x_n=0} \\ &= (D_n^2\left[(\partial_n\psi)u\right], D_n u) + i(D_n\left[(\partial_n\psi)D_n u\right], u)_{x_n=0} + i((\partial_n\psi)D_n u, D_n u)_{x_n=0} \end{aligned}$$

This gives the boundary terms

$$i((\partial_n\psi)D_n^2 u, u)_{x_n=0} + i((D_n\partial_n\psi)D_n u, u)_{x_n=0} + i((\partial_n\psi)D_n u, D_n u)_{x_n=0}.$$

The two terms of order 2 cancel. So we are left with some terms of order 1 that come into the boundary terms that can be handled by the Carleman method. The true computation contains some more terms, but with less derivative in x_1.

How to deal with interior terms?

The interior terms are more or less the same as in the boundaryless case. So, we could expect that their symbol satisfy the same positivity condition. Yet, we would like to use only a tangential Gårding inequality, that is only in the derivatives in the variable x' (with symbol only depending on the cotangent variable ξ'.)

The idea is to perform a kind of euclidian division of the commutator $i[Q^r, Q^i]$ by D_n. Indeed, we can factorize $i[Q^r, Q^i] = \tau\left[C_0 D_n^2 + C_1 D_n + C_2\right]$ where C_i are tangential operators (we have also used that Q^i can be written $\tau\widetilde{Q^i}$). Moreover, since Q^r contains some derivative in x_n with main coefficient D_n^2 while the main derivative of Q^i in x_n is $2i\tau(\partial_n\psi)D_n$ where $(\partial_n\psi) \neq 0$. This allows to perform a similar "euclidian division" with Q^r, Q^i which allows to write

$$i[Q^r, Q^i] = \tau D_0 Q^r + D_1 Q^i + \tau D_2.$$

Since the terms $\tau D_0 Q^r$ are in some sense weaker than $\|Q^r\|_{L^2}$ (and the same for Q^i), we are left with some tangential operator. D_2 is not always positive, but the final task is to transfer the information we have on p_ψ to this tangential operator.

2.5 Other applications

2.5.1 Spectral estimates

From the quantitative estimate of Theorem 2.3.3, we can already get some applications about spectral estimates of eigenfunctions of second order elliptic operators. We first describe the context.

We will denote $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ the n-dimensional torus. This can be seen as $[0,1]^n$ with the necessary identification of points. Functions on $\mathbb{T}^n$ can be seen as functions on $\mathbb{R}^n$ with periodic boundary conditions. Let $A = (a_{i,j})_{i,j=1}^n$ a symmetric matrix with $a_{i,j} \in C^\infty(\mathbb{T}^n)$ real valued. We define the operator $Pu = -div(A\nabla u) = -\sum_{i,j} \partial_i (a_{i,j}\partial_j u)$. Assume also that P is elliptic, that is there exists C so that

$$\sum_{i,j=1}^{n} a_{i,j}(x)\xi_i\xi_j \geq C|\xi|^2, \text{ for all } (x,\xi) \in \Omega \times \mathbb{R}^n.$$

P is also symmetric, that is $(Pu,v)_{L^2(\mathbb{T}^n)} = (u,Pv)_{L^2(\mathbb{T}^n)}$ for $u,v \in C^\infty(\mathbb{T}^n)$.

We can check that its can be extended to a positive self-adjoint operator with domain $H^2(\mathbb{T}^n)$. Therefore, since the embedding of $H^2(\mathbb{T}^n)$ into $L^2(\mathbb{T}^n)$ is compact. Therefore, the resolvent $(P+Id)^{-1}$ is well defined and compact on L^2.

All this allows to define an orthonormal basis of $L^2(\mathbb{T}^n)$. There exists some functions $\Phi_i \in C^\infty(\mathbb{T}^n)$, $\lambda_i \in \mathbb{R}$ (actually $\lambda_i \geq 0$ since P is positive) so that

- $(\Phi_i)_{i\in\mathbb{N}}$ is an orthonormal basis of the Hilbert space $L^2(\mathbb{T}^n)$
- $P\Phi_i = \lambda_i\Phi_i$.

We refer to Brézis [2] Chapter VI and IX for more details about this construction.

Remark 2.5.1. *The same construction holds for a general compact Riemannian manifold M. The metric g induces some natural volume form ω_g, definition of ∇_g, divergence and finally the Laplace-Beltrami operator defined by $\Delta_g u = div_g(\nabla_g u)$ that can be written in local coordinates as*

$$\Delta_g u = \frac{1}{\sqrt{det(g)}} \sum_{i,j} \partial_i \left(\sqrt{det(g)} g^{i,j} \partial_j u\right)$$

where $g^{i,j}$ is the inverse of the matrix $g = (g_{i,j})_{i,j=1}^n$.

We have the following Theorem which states some localization property of eigenfunctions of P

Theorem 2.5.1. *Under the previous assumptions on P and the related Φ_i, λ_i. Let ω be an open subset of $\mathbb{T}^n$. There exists C and $c > 0$ so that we have the estimate uniform in $i \in \mathbb{N}$*

$$1 = \|\Phi_i\|_{L^2(\mathbb{T}^n)}^2 \leq Ce^{c\sqrt{\lambda_j}} \|\Phi_i\|_{L^2(\omega)}^2 .$$

This is a kind of observability estimate for eigenfunctions: observing one part of the eigenfunctions gives a proportion of the energy of at least $Ce^{-c\sqrt{\lambda_j}}$.

Proof. Let $Q = -\partial_s^2 + P$ the operator defined on $\mathbb{R}_s \times \mathbb{T}_x^n$. We can easily verify that Q is elliptic with real coefficients.

Define $f_j = e^{\sqrt{\lambda_j}s}\Phi_j(x)$. We verify that $Qf_j = -\lambda_j f_j + e^{\sqrt{\lambda_j}s}P\Phi_j = 0$. We want to apply Theorem 2.3.3 to Q and f_j on an open set $\Omega =]0,3[\times\mathbb{T}^n$ and $K = [1,2] \times \mathbb{T}^n$. Note that we are not exactly in the configuration of the Theorem since $\mathbb{T}^n$ is not an open set of $\mathbb{R}^n$. But it can be easily checked that some localization in $\mathbb{R}^n$ using the periodicity gives the expected result.

If (t_0,x_0), $r > 0$ are so that $B((t_0,x_0),r) \subset]1,2[\times\omega$, the Theorem gives $C > 0$, $0 < \delta < 1$ so that

$$\|f_j\|_{H^1([1,2]\times\mathbb{T}^n)} \leq C \left[\|f_j\|_{H^1(]1,2[\times\omega)}\right]^\delta \|f_j\|_{H^1(]0,3[\times\mathbb{T}^n)}^{1-\delta} .$$

We make it even worse by

$$\|f_j\|_{L^2([1,2]\times\mathbb{T}^n)} \le C\left[\|f_j\|_{H^2(]1,2[\times\omega)}\right]^\delta \|f_j\|_{H^2(]0,3[\times\mathbb{T}^n)}^{1-\delta}.$$

But on $[1,2]$, we have $e^{\sqrt{\lambda_j}s} \ge e^{\sqrt{\lambda_j}}$. So $\|f_j\|_{L^2([1,2]\times\mathbb{T}^n)} \ge e^{\sqrt{\lambda_j}}\|\Phi_j\|_{L^2(\times\mathbb{T}^n)} = e^{\sqrt{\lambda_j}}$.

Similarly, using that P is elliptic of order 2

$$\begin{aligned}\|f_j\|_{H^2(]1,2[\times\omega)} &\le \|f_j\|_{L^2(]1,2[\times\omega)} + \left\|\partial_s^2 f_j\right\|_{L^2(]1,2[\times\omega)} + \|Pf_j\|_{L^2(]1,2[\times\omega)}\\ &\le \|f_j\|_{L^2(]1,2[\times\omega)} + \|\lambda_j f_j\|_{L^2(]1,2[\times\omega)} + \|\lambda_j f_j\|_{L^2(]1,2[\times\omega)}\\ &\le e^{2\sqrt{\lambda_j}}(1+2\lambda_j)\|\Phi_j\|_{L^2(\omega)} \le Ce^{3\sqrt{\lambda_j}}\|\Phi_j\|_{L^2(\omega)}.\end{aligned}$$

where we have used $(1+2\lambda_j) \le Ce^{\sqrt{\lambda_j}}$ for some well chosen C. Similarly

$$\begin{aligned}\|f_j\|_{H^2(]0,3[\times\mathbb{T}^n)} &\le \|f_j\|_{L^2(]0,3[\times\mathbb{T}^n)} + \left\|\partial_s^2 f_j\right\|_{L^2(]0,3[\times\mathbb{T}^n)} + \|Pf_j\|_{L^2(]0,3[\times\mathbb{T}^n)}\\ &\le Ce^{3\sqrt{\lambda_j}}(1+2\lambda_j)\|\Phi_j\|_{L^2(\mathbb{T}^n)} \le Ce^{4\sqrt{\lambda_j}}.\end{aligned}$$

We finally obtain, with some different constant C

$$e^{\sqrt{\lambda_j}} \le Ce^{3\delta\sqrt{\lambda_j}}\|\Phi_j\|_{L^2(\omega)}^\delta e^{4(1-\delta)\sqrt{\lambda_j}}.$$

Which gives the expected result. □

Remark 2.5.2. *The rate $e^{c\sqrt{\lambda_j}}$ is not always optimal, but there are some geometric situation where it is optimal. The typical example is when there is a stable geodesic. On the sphere S^2 parameterized by $\left\{(x,y,z); x^2+y^2+z^2=1\right\}$, for instance we have some exact eigenvalues of the Laplace-Beltrami operator of the form $\Phi_n = c_n \operatorname{Re}(x+iy)^n$ with eigenvalue $\lambda_n = n(n)$, with some appropriate normalization constants c_n. Since $|\Phi_n|^2(x,y,z) \le c_n(x^2+y^2)^n \le (1-z^2)^n$, we easily observe that $\Phi_n(x)/c_n$ is exponentially small outside the set $S^2\cap\{z=0\}$. Yet, we can check that $c_n \lesssim n^{1/2}$ since $\int_{-1/2}^{1/2}(1-z^2)^n dz = \int_{-1/2}^{1/2} e^{n\ln(1-z^2)}dz \ge \int_{-1/2}^{1/2} e^{-cnz^2}dz \ge n^{-1/2}\int_{-\sqrt{n}/2}^{\sqrt{n}/2} e^{-cs^2}ds$.*

There are some geometries where we can replace $c(\lambda_j) = e^{c\sqrt{\lambda_j}}$ by a constant or sometimes a power of λ_j or $\log(\lambda_j)$. The general question of making the link between the geometric properties of ω with respect to Ω and determining the appropriate $c(\lambda)$ is a big open problem in spectral theory.

Using the boundary estimates of Theorem 2.4.1, it is possible to get a more precise result. Actually, the previous result remain true not only for eigenfunctions, but also for finite sum of eigenfunction. Since now, the stability estimate is still true for an open set with boundary, we state the result for an elliptic operator P with the Dirichlet boundary conditions. The framework will be quite similar to the previous one

Let Ω be a smooth compact open set with boundary. Let $A = (a_{i,j})_{i,j=1}^n$ a symmetric matrix with $a_{i,j} \in C^\infty(\Omega)$ real valued. We define the operator $Pu = -div(A\nabla u) = -\sum_{i,j}\partial_i(a_{i,j}\partial_j u)$. We consider (and we will still denote it P the selfadjoint extension of P associated to the Drichlet boundary condition, that is $u=0$ on $\partial\Omega$. We use the same notation Φ_j and λ_j the eigenfunctions and eigenvalues.

Theorem 2.5.2. *Under the previous assumptions on P and the related Φ_i, λ_i. Let ω be an open subset of Ω. There exists C and $c>0$ so that we have the estimate uniform in λ*

$$\|u\|_{L^2(\Omega)}^2 \le Ce^{c\sqrt{\lambda}}\|u\|_{L^2(\omega)}^2.$$

for any $u = \sum_{\lambda_j\le\lambda} u_j\Phi_j$.

Proof. As before, we consider the elliptic operator $Q = -\partial_s^2 + P$ the operator defined on $\mathbb{R}_s^+ \times \Omega$.

Define $f(s,x) = \sum_{i,\lambda_j \leq \lambda} u_j \frac{\sinh(\sqrt{\lambda_j}s)}{\sqrt{\lambda_j}} \Phi_j(x)$ and we easily verify that it satisfies $Qf = 0$ and $f = 0$ on $\mathbb{R} \times \partial\Omega$ and $\{0\} \times \Omega$. Theorem 2.4.1 gives, if we take $\Gamma = \{0\} \times \omega$. Note that we can assume without loss of generality that ω is far from $\partial\Omega$, in order to avoid problems with "corners" at the points $\{0\} \times \partial\Omega$. A weak form of the inequality is then

$$\|f\|_{L^2([0,1]\times\Omega)} \leq C \|\partial_s f\|^{\delta}_{L^2(\{0\}\times\omega)} \|f\|^{1-\delta}_{H^1([0,2[\times\Omega)}$$

We have $\partial_s f(0,x) = u(x)$, so $\|\partial_s f\|_{L^2(\{0\}\times\omega)} = \|u\|_{L^2(\omega)}$.

As before, using Parseval identity

$$\begin{aligned}
\|f\|^2_{H^1([0,2[\times\Omega)} &\leq \|f\|^2_{L^2([0,2[\times\Omega)} + \|\partial_s f\|^2_{L^2([0,2[\times\Omega)} + \int_0^2 (-\Delta f, f)_{L^2(\Omega)} \\
&\leq C \int_0^2 \sum_{\lambda_j \leq \lambda} |u_j|^2 \left(\cosh(\sqrt{\lambda_j}s)^2 + \sinh(\sqrt{\lambda_j}s)^2\right) ds \\
&\leq Ce^{c\sqrt{\lambda}} \sum_{\lambda_j \leq \lambda} |u_j|^2 \leq Ce^{c\sqrt{\lambda}} \|u\|_{L^2(\Omega)}.
\end{aligned}$$

And similarly

$$\begin{aligned}
\|f\|^2_{L^2([0,1[\times\Omega)} &\geq \|f\|^2_{L^2([0,1[\times\Omega)} \\
&\geq C \int_0^1 \sum_{\lambda_j \leq \lambda} |u_j|^2 \frac{\sinh(\sqrt{\lambda_j}s)}{\lambda_j}^2 ds \\
&\geq C \sum_{\lambda_j \leq \lambda} |u_j|^2 \int_0^{\sqrt{\lambda_j}} \frac{\sinh(y)^2}{\lambda_j^{3/2}} dy \geq C' \|u\|^2_{L^2(\Omega)}.
\end{aligned}$$

So, we obtain, with some different constants C, c

$$\|u\|_{L^2(\Omega)} \leq Ce^{c\sqrt{\lambda}} \|u\|^{\delta}_{L^2(\omega)} \|u\|^{1-\delta}_{L^2(\Omega)}.$$

This gives the result. □

This type growth of the type $e^{\sqrt{\lambda}}$ is optimal whatever the geometry if $\overline{\omega} \neq \Omega$. See [9].

2.5.2 The heat equation

Our previous Theorem gives immediatly the following corollary for solutions of the heat equation at low frequency

Corollary 2.5.1. *Under the previous assumptions on P and the related Φ_i, λ_i. Let ω be an open subset of Ω. There exists C and $c > 0$ so that we have the estimate uniform in $\lambda \geq 0$ and $T > 0$*

$$\|u(T)\|^2_{L^2(\Omega)} \leq Ce^{c\sqrt{\lambda}} \frac{1}{T} \int_0^T \|u(t)\|^2_{L^2(\omega)} \, dt.$$

for any $f = \sum_{\lambda_j \leq \lambda} f_j \Phi_j$ with u solution of the heat equation

$$\left\{ \begin{aligned} \partial_t u - \Delta u &= 0 \text{ on } [0,T] \times \Omega \\ u &= 0 \text{ on } [0,T] \times \partial\Omega \\ u(0,x) &= f(x) \text{ on } \Omega \end{aligned} \right.$$

Proof. The decay of the energy gives $\|u(T)\|^2_{L^2(\Omega)} \le \|u(t)\|^2_{L^2(\Omega)}$ for any $0 \le t \le T$. Moreover, for any $t \in [0,T]$, the spectral estimates can be written $\|u(t)\|^2_{L^2(\omega)} \le Ce^{c\sqrt{\lambda}} \|u\|^2_{L^2(\omega)}$. Integrating in time gives the previous estimates and using the

$$T\|u(T)\|^2_{L^2(\Omega)} \le \int_0^T \|u(t)\|^2_{L^2(\omega)} \le Ce^{c\sqrt{\lambda}} \int_0^T \|u\|^2_{L^2(\omega)}.$$

□

We will prove some observability estimate for the heat equation on a bounded domain. It is known that some observability estimates are equivalent to some result of control.

We have chosen to prove the observability and then to deduce the related result of control.

In the original paper, the idea was the following:

- Use the "observability inequality" of Theorem 2.5.2 to get some result about control of the low frequency up to λ in time $T/4$ with a cost $\approx \frac{4}{T}e^{\sqrt{\lambda}}$. This allows to control the λ first frequency to zero.
- Use the decay of the heat equation on the remaining eigenvalues to get decay of $e^{-\lambda T/4}$ in time $T/4$.
- Iterate the result on dyadic times and tending to T.

The important fact is that the power $1/2$ in λ in $e^{c\sqrt{\lambda}}$ is strictly smaller than the exponential decay $e^{-\lambda T/4}$. We will work directly on the observability estimate, but still using the decay provided by the heat equation.

Theorem 2.5.3 (Observability for the heat equation)**.** *Let $\omega \subset \Omega$ a non empty open set and $T > 0$. Then, there exists $C > 0$ so that we have the estimate*

$$\|u(T)\|^2_{L^2} \le C\int_0^T \|u\|^2_{L^2(\omega)}$$

for any u solution of

$$\left\{\begin{array}{rcl} \partial_t u - \Delta u &=& 0 \text{ on } [0,T]\times\Omega \\ u &=& 0 \text{ on } [0,T]\times\partial\Omega \\ u(0,x) &=& u_0(x) \text{ on } \Omega \end{array}\right.$$

with $u_0 \in L^2(\Omega)$.

Proof. The idea is that our spectral estimate gives good estimates only when there are few high frequencies, that is after the decay of the wave operator have operated, that is close to times T.

We will divide the interval $[0,T]$ as the union of the intervals $[T_{k+1}, T_k]$ with $T_0 = T$, $T_{k+1} = T_k - T2^{-k}$. We check that T_k converges to $T - \sum_{k\in\mathbb{N}^*} T2^{-k} = 0$. To simplify the notations, we denote $L_k = T2^{-k}$ the length of the interval.

For each interval $[T_{k+1}, T_k]$, we will select a frequency cutoff μ_k and decompose

$$u = u_{k,L} + u_{k,H} = \sum_{\lambda_j \le \mu_k} + \sum_{\lambda_j > \mu_k}$$

We will cut $[T_{k+1}, T_k]$ in two pieces, $[T_{k+1}, T_{k+1} + L_k/2]$ where we only use the damping and $[T_{k+1} + L_k/2, T_k]$ where we observe (using that the high frequency have been damped). We apply Corollary 2.5.1 on $[T_{k+1} + L_k/2, T_k]$

$$\|u_{k,L}(T_k)\|^2_{L^2(\Omega)} \leq Ce^{c\sqrt{\mu_k}} \frac{2}{L_k} \int_{T_{k+1}+L_k/2}^{T_k} \|u_{k,L}(t)\|^2_{L^2(\omega)} \, dt. \tag{2.23}$$

So, by triangular inequality, noticing that the error we do from the cut off in frequency is small. For $\|u_{k,L}(t)\|_{L^2(\omega)} \leq \|u(t)\|_{L^2(\omega)} + \|u_{k,H}(t)\|_{L^2(\omega)}$

$$\|u_{k,H}(t)\|_{L^2(\omega)} \leq \|u_{k,H}(t)\|_{L^2(\Omega)} \leq \|u_{k,H}(T_{k+1} + L_k/2)\|_{L^2(\Omega)} \leq e^{-\mu_k L_k/2} \|u(T_{k+1})\|_{L^2(\Omega)} \tag{2.24}$$

where we have used the damping of high frequency.

Moreover, we have similarly

$$\|u_{k,H}(T_k)\|_{L^2(\omega)} \leq e^{-\mu_k L_k/2} \|u(T_{k+1})\|_{L^2(\Omega)}. \tag{2.25}$$

So, putting together (2.23), (2.24) and (2.25), we get

$$\begin{aligned} \|u(T_k)\|^2_{L^2(\Omega)} &\leq Ce^{c\sqrt{\mu_k}} \frac{2}{L_k} \int_{T_{k+1}+L_k/2}^{T_k} \|u(t)\|^2_{L^2(\omega)} \, dt + Ce^{c\sqrt{\mu_k}} e^{-\mu_k L_k/2} \|u(T_{k+1})\|_{L^2(\Omega)} \\ &\leq Ce^{c\sqrt{\mu_k}} \frac{2}{L_k} \int_0^T \|u(t)\|^2_{L^2(\omega)} \, dt + Ce^{c\sqrt{\mu_k}} e^{-\mu_k L_k} \|u(T_{k+1})\|^2_{L^2(\Omega)}. \end{aligned}$$

Now, we can choose μ_k. Recall that $L_k = T2^{-k}$ converge to zero. Pick for instance, $\mu_k = \sqrt{C_1} L_k^{-2}$ with C_1 large. We have $\mu_k L_k = \sqrt{C_1}\sqrt{\mu_k}$ and $\mu_{k+1} = 2\mu_k$. If C_1 is large enough, we have

$$Ce^{c\sqrt{\mu_k}} e^{-\mu_k L_k} \leq e^{-3c\sqrt{\mu_{k+1}}}.$$

Indeed,

$$-c\sqrt{\mu_k} + \mu_k L_k - 3c\sqrt{\mu_{k+1}} = \sqrt{\mu_k}\left(\sqrt{C_1} - c(1+3\sqrt{2})\right) = C_1 2^k \left(C_1 - c(1+3\sqrt{2})\right).$$

This can be made arbitrary arbitrary large uniformly for $k \in \mathbb{N}$. Once C_1 and μ_k are fixed, we have one constant C so that

$$e^{c\sqrt{\mu_k}} \frac{2}{L_k} \leq Ce^{2c\sqrt{\mu_k}}$$

So, we obtain

$$\|u(T_k)\|^2_{L^2(\Omega)} \leq Ce^{2c\sqrt{\mu_k}} \int_0^T \|u(t)\|^2_{L^2(\omega)} \, dt + e^{-3c\sqrt{\mu_{k+1}}} \|u(T_{k+1})\|^2_{L^2(\Omega)}.$$

$$e^{-3c\sqrt{\mu_k}} \|u(T_k)\|^2_{L^2(\Omega)} \leq Ce^{-c\sqrt{\mu_k}} \int_0^T \|u(t)\|^2_{L^2(\omega)} \, dt + e^{-3c\sqrt{\mu_{k+1}}} \|u(T_{k+1})\|^2_{L^2(\Omega)}.$$

Denoting $z_n = e^{-3c\sqrt{\mu_k}} \|u(T_k)\|^2_{L^2(\Omega)}$, we get

$$z_n - z_{n+1} \leq Ce^{-c\sqrt{\mu_k}} \int_0^T \|u(t)\|^2_{L^2(\omega)} \, dt.$$

We recognize a telescopic series and that $e^{-c\sqrt{\mu_k}}$ is summable. So, by summing up, we get with a new constant, uniform in $k \in \mathbb{N}$,

$$e^{-3c\sqrt{\mu_0}} \|u(T)\|^2_{L^2(\Omega)} - e^{-3c\sqrt{\mu_k}} \|u(T_k)\|^2_{L^2(\Omega)} \le \widetilde{C} \int_0^T \|u(t)\|^2_{L^2(\omega)} \, dt$$

Since $\|u(T_k)\|_{L^2(\Omega)}$ is bounded, $e^{-3c\sqrt{\mu_k}} \|u(T_k)\|^2_{L^2(\Omega)}$ converges to zero, which gives the result. □

Theorem 2.5.4 (Control to zero of the heat equation)**.** *Let $\omega \subset \Omega$ a non empty open set and $T > 0$. Let $u_0 \in L^2(\Omega)$. Then, there exists $g \in L^2([0,T], L^2(\omega))$ so that the solution of*

$$\begin{cases} \partial_t u - \Delta u &= g \text{ on } [0,T] \times \Omega \\ u &= 0 \text{ on } [0,T] \times \partial\Omega \\ u(0,x) &= u_0(x) \text{ on } \Omega \end{cases}$$

satisfies $u(T) = 0$.

Proof. We consider the dual to the heat equation, v is a solution of

$$\begin{cases} -\partial_t v - \Delta v &= 0 \text{ on } [0,T] \times \Omega \\ v &= 0 \text{ on } [0,T] \times \partial\Omega \\ v(T,x) &= v_T(x) \text{ on } \Omega \end{cases} \tag{2.26}$$

This is actually exactly the backward heat equation. It is made interesting, because at least for smooth solutions u and v with Dirichlet boundary conditions, we have the formula that can be easily obtained by multiplying the equation (2.26) by v (consider v real valued for simplicity), integrating over $[0,T] \times \Omega$ and integrating by parts

$$\int_\Omega u(T)v(T) - \int_\Omega u(0)v(0) = \int_0^T \int_\Omega gv.$$

The formula can also be extended to the case where $u_0 \in L^2(\Omega)$, $g \in L^2([0,T]; L^2(\Omega))$, $v_T \in L^2(\Omega)$ by a density argument.

Our hope if $u(T) = 0$ would be to get $\int_0^T gv = \int_\Omega u_0 v(0)$. Reciprocally, we can check that if $\int_0^T gv = \int_\Omega u_0 v(0)$ for any solution of (2.26) with $v_T \in L^2(\Omega)$, then $u(T) = 0$.

Now, consider the quadratic form

$$a(v_T, \widetilde{v}_T) = \int_0^T \int_\omega v\widetilde{v} dx \, dt.$$

where v, $\widetilde{v}$ are the associated solutions to (2.26). a is well defined for v_T, $\widetilde{v}_T \in L^2(\Omega)$ and defines a positive quadratic form. Our observability estimates says that it is a scalar product. Yet, it is weaker than the $L^2(\Omega)$ norm. We define the completion $\overline{H}$ of $L^2(\Omega)$ with respect to this norm.

Define the linear form

$$l(v_T) = \int_\Omega u_0 v(0).$$

Our observability estimates can be written

$$\|v(0)\|^2_{L^2(\Omega)} \le C \int_0^T \int_\omega |v|^2 dx \, dt.$$

This says exactly that l is linear continuous in $\overline{H}$, since $u_0 \in L^2(\Omega)$. By the Riesz representation (or Lax-Milgram), there exists $v_T^{u_0} \in \overline{H}$ so that

$$a(v_T, v_T^{u_0}) = l(v_T)$$

for all $v_T \in \overline{H}$.

The application $\theta : L^2(\Omega) \mapsto L^2([0,T] \times \omega)$ defined by $\theta(v_T) = v_{|[0,T]\times\omega}$ where v is solution of (2.26) is well defined in $L^2(\Omega)$, but also bounded for the norm a on $\overline{H}$. Therefore, it can be extended to $\overline{H}$.

Take $g = \theta(v_T^{u_0}) \in L^2([0,T] \times \omega)$. By choice, we have

$$\int_0^T \int_\Omega g\theta(v_T) = l(v_T)$$

for any $v_T \in \overline{H}$. If we take in particular $v_T \in L^2(\Omega)$, this gives

$$\int_0^T \int_\Omega g\theta(v_T) = \int_\Omega u_0 v(0)$$

for v solution of (2.26). This gives the expected result. □

2.6 The general Theorem of Hörmander

Theorem 2.6.1 (The Theorem of Hörmander). *Let Ω an open set of $\mathbb{R}^n$ and $x_0 \in \Omega$. Let P be a differential operator of order m with some eventually complex coefficients with $C^\infty(\Omega)$ principal symbol and all coefficients in $L^\infty_{loc}(\Omega)$. Assume that P is principally normal, that is the principal symbol p of P satisfies: for any compact K of Ω, there is $C > 0$*

$$\{\overline{p}, p\} \leq C|p||\xi|^{m-1}.$$

for all $(x,\xi) \in K \times \mathbb{R}^n$.

Let $\Phi \in C^2(\Omega)$ real valued so that $\nabla\Phi(x_0) \neq 0$. Assume that it satisfies

$$\begin{aligned} \operatorname{Re}\{\overline{p}, \{p, \Phi\}\}(x_0,\xi) > 0, &\qquad \textit{if } p(x_0,\xi) = \{p,\Phi\}(x_0,\xi) = 0 \textit{ and } \xi \neq 0; \\ \frac{1}{i\tau}\{\overline{p_\Phi}, p_\Phi\}(x_0,\xi) > 0, &\qquad \textit{if } p_\Phi(x_0,\xi) = \{p_\Phi,\Phi\}(x_0,\xi) = 0 \textit{ and } \tau > 0, \end{aligned}$$

where $p_\Phi(x,\xi) = p(x, \xi + i\tau\nabla\Phi)$ and p is the principal symbol of P.

Then, there exists V one neighborhood of x_0 in Ω so that for any $u \in H^{m-1}_{loc}(V)$,

$$\left\{ \begin{array}{rcl} Pu &=& 0 \text{ in } V, \\ u &=& 0 \text{ in } V \cap \{\Phi > \Phi(x_0)\} \end{array} \right. \Longrightarrow u = 0 \text{ on } V.$$

This is Theorem 28.3.4 of [7].

2.7 A small bibliography

The result of unique continuation have a long history going back to Carleman [4] who first, had the idea to conjugate the operator with an exponential weight to get unique continuation. He proved the result in the case of elliptic operators of order 2 in dimension 2. Calderón [3] extended the result to some operators with simple characteristics. Namely, that was in situations where $p_\phi = \{p_\phi, \phi\}$ never happens. The general version was given by Hörmander [6] for real operators and [7]. Note that other works consider the limit case where there is some higher order of cancelation. We refer to Zuily [14] for more details.

The boundary Carleman estimates where proved by Lebeau-Robbiano [10] in order to give the same application that we give in Section 2.5.2, that is the controllability of the heat equation. They also proved the spectral estimates of Section 2.5.1.

Note that there is also another proof (independently) by Fursikov-Immanuvilov [5] of the controllability of the heat equation using directly some Carleman estimates for the heat equation. Details about this and some link with the elliptic Carleman estimate are given in [9].

2.8 Further comments and problems

Many things have not been written in an optimal way in the previous theorems and can be improved:

- the fact that the $a_{i,j}$ are real valued is not necessary♣ .
- the regularity of u can be much lowered. Note also, that if the coefficients are regular enough, the regularity of u can often be recovered using classical elliptic regularity results, see Brézis [2] for instance.
- the regularity of the coefficients is not optimal. The main coefficients should actually be C^1 while the lower order terms can be in some L^p spaces.
- the fact to be an exact solution of $Pu = 0$ can be replaced by some weaker assumption like $|Pu|(x) \leq C\left(|u(x)| + |D(x)|\right)$ for almost every $x \in \Omega$.

Counterexemples of Alinhac
Rough coefficients, nonlinear problems
boundary conditions, interfaces
Systems
Global result (cf Einstein??)

Chapter 3

The wave equation with coefficients constant in time

3.1 The problem

In this section, we will specialize to some more specific operators of wave type $\partial_t^2 - Q(x, D_x)$ where Q is elliptic negative. In Remark 2.1.2, we have seen that, in the framework of regular coefficient, the unique continuation function from cylinder, that is for set of the form $\{\varphi(x) \leq 0\}$ requires some convexity assumptions of the surface. Yet, in the Holmgren Theorem 1.1.2, the condition is much weaker. For the flat wave equation, $\partial_t^2 - \Delta$, it only requires $|\partial_t \varphi|^2 \neq |\nabla_x \varphi|^2$. That means, we should not be tangent to the cone of light. More or less, this is the weaker condition that one could expect but would not contradict the finite speed of propagation. But we would like to relax the analyticity assumption. The counterexample of Alinhac-Baouendi [1] actually tell us that we can not relax this completely.

It turns out that some analyticity with respect to part of the variables can be sufficient, for instance the time in our example of the wave equation.

In the following, the variable will be $z = (t, x) \in \mathbb{R}^{1+n}$ with dual variable $\xi = (\xi_t, \xi_x) \in \mathbb{R}^{1+n}$. To keep the notation coherent with the elliptic case, we will denote $\xi_t = \xi_0$ and ξ_x will be written $\xi_x = (\xi_1, \cdots, \xi_n)$.

The main theorem of this chapter will be the following.

Theorem 3.1.1 (Wave type operator with coefficients constant in time). *Let $T > 0$ and Ω_x an open set of $\mathbb{R}^n$. Denote $\Omega =]-T, T[\times \Omega_x$.*

Let $Q = \sum_{i,j=1}^n a_{i,j}(x)\partial_i\partial_j + \sum_k b_k(x)\partial_k + c(x)$ be a differential operator of order 2 with $a_{i,j} \in C^\infty(\Omega_x)$ real valued, b_k, $c \in L^\infty(\Omega_x)$. Assume also that Q is negative elliptic, that is there exists $C > 0$ so that

$$\sum_{i,j=1}^n a_{i,j}(x)\xi_i\xi_j \geq C|\xi_x|^2, \text{ for all } (x, \xi_x) \in \Omega_x \times \mathbb{R}^n.$$

Define $P = \partial_t^2 - Q$ defined on Ω of principal symbol p.

Let $z_0 = (t_0, x_0) \in \Omega$ and $\Phi \in C^2(\Omega)$ so that $p(z_0, \nabla_{t,x}\Phi) \neq 0$, or more precisely

$$(\partial_t\Phi(z_0))^2 \neq \sum_{i,j} a_{i,j}(x_0)(\partial_i\Phi)(\partial_j\Phi).$$

Then, there exists V one neighborhood of z_0 so that for any $u \in C^\infty(\Omega)$,

$$\left\{ \begin{array}{rcl} Pu & = & 0 \text{ in } \Omega, \\ u & = & 0 \text{ in } \Omega \cap \{\Phi > \Phi(z_0)\} \end{array} \right. \Longrightarrow u = 0 \text{ on } V. \tag{3.1}$$

The main tool will be an inequality of Carleman type, but with an additional weight in the Fourier variable.

Namely, for a smooth real valued function ψ (later on, we will assume that it is polynomial of order 2), we define the operator

$$Q_{\varepsilon,\tau} u = e^{-\varepsilon \frac{|D_t|^2}{2\tau}} e^{\tau\psi}$$

where $e^{-\frac{|D_t|^2}{2\tau}}$ is the Fourier multiplier defined for $u \in \mathcal{S}(\mathbb{R}^n)$ by $\left(\widehat{e^{-\varepsilon \frac{|D_t|^2}{2\tau}} u}\right)(\xi) = e^{-\varepsilon \frac{|\xi_t|^2}{2\tau}} \widehat{u}(\xi)$ where ξ_t is the Fourier variable corresponding to the variable t.

Definition 3.1.1 (Pseudoconvexity for functions in $|\xi_t| = 0$)**.** *With the above assumptions for P, let ψ smooth real valued. We say that ψ is satisfies the pseudoconvexity for functions in $|\xi_t| = 0$ at z_0 if*

$$\{p, \{p, \psi\}\}(z_0, \xi) > 0, \quad \textit{if } p(z_0, \xi) = 0, \quad \xi_t = 0, \quad \xi \neq 0; \tag{3.2}$$

$$\frac{1}{i\tau}\{\overline{p}_\psi, p_\psi\}(z_0, \xi) > 0, \quad \textit{if } p_\psi(z_0, \xi) = 0, \quad \xi_t = 0, \quad \tau > 0, \tag{3.3}$$

where $p_\psi(x, \xi) = p(x, \xi + i\tau\nabla\psi)$.

Theorem 3.1.2 (Carleman estimate for wave type operators with coefficients constant in time)**.** *With the above assumptions for P, let ψ be quadratic real valued polynomial such that Ψ satisfies the pseudoconvexity for functions in $\xi_t = 0$ at z_0 of Definition 3.1.1.*

Then, there exists $r > 0$, $\varepsilon > 0$, $\mathsf{d} > 0$, $C > 0$, $\tau_0 > 0$ such that for all $\tau > \tau_0$ and $u \in C_0^\infty(B(x_0, r))$

$$\tau \|Q^\psi_{\varepsilon,\tau} u\|^2_{H^1_\tau} \leq C \left\| Q^\psi_{\varepsilon,\tau} P u \right\|^2_{L^2} + C e^{-\mathsf{d}\tau} \left\| e^{\tau\psi} u \right\|^2_{H^1_\tau} \tag{3.4}$$

Note that if we set $\varepsilon = 0$, this would be a classical Carleman estimate. Yet, the role of the Fourier multiplier will be to kill the high frequency in the variable t. So, we will just need to look at the very small frequency in ξ_t. That is why the pseudoconvexity assumption is only made in $\xi_t = 0$.

3.2 Proving unique continuation using the Carleman estimate

In this section, we assume that Theorem 3.1.2 is proved and we will prove Theorem 3.1.1. Some part will be similar to the classical case, that means constructing an approriate function ψ, pseudoconvex for functions in $\xi_t = 0$ from the function ϕ pseudoconvex for surfaces.

The main differences are the following:

- the pseudoconvexity is only on $\xi_t = 0$, so it requires a small adaptation of the convexification procedure. Moreover, we want ψ quadratic.
- the Carleman estimates implies an exponential weight in Fourier that change the proof of unique continuation.

3.2.1 Convexification

Quite similarly to the classical case, we will first give a different formulation of fact that Φ is non charateristic. It implies a property of pseudoconvexity quite similar to the one of Proposition 2.3.1, but on the set $\xi_t = 0$.

Proposition 3.2.1 (Usual pseudoconvexity for surfaces in $\xi_t = 0$)**.** *Let P and Φ satisfying the assumptions of Theorem 3.1.1. Then, Φ satisfies the stronger property*

$$\{p,\{p,\Phi\}\}(z_0,\xi) > 0, \quad \text{if } p(z_0,\xi) = \{p,\Phi\}(z_0,\xi) = \xi_t = 0 \text{ and } \xi \neq 0; \tag{3.5}$$

$$\frac{1}{i\tau}\{\overline{p_\Phi}, p_\Phi\}(z_0,\xi) > 0, \quad \text{if } p_\Phi(z_0,\xi) = \{p_\Phi,\Phi\}(z_0,\xi) = \xi_t = 0 \text{ and } \tau > 0, \tag{3.6}$$

where $p_\Phi(x,\xi) = p(x,\xi + i\tau\nabla\Phi)$ and p is the principal symbol of P.

Proof. The principal symbol of P is $p(t,x,\xi_t,\xi_x) = -\xi_t^2 - q$ where $q(x,\xi_x) = -\sum_{i,j} a_{i,j}(x)\xi_i\xi_j$.

So, we notice that for $\xi_t = 0$, we have $p(t,x,0,\xi_x) = -q(x,\xi_x)$. Since q is assumed to be elliptic, the assumption $p(z_0,\xi) = \xi_t = 0$ implies $\xi = 0$ and therefore, 3.5 is empty.

We will use the computations of Lemma A.4.1 (note that we have actually already proved that 3.6 is empty if $p(x,\nabla\Phi) \neq 0$ in Proposition 2.3.1). Noting R_{z_0} the symmetric quadratic form so that $R_{z_0}(\xi,\xi) = p(z_0,\xi)$, we get

$$\{p_\Phi,\Phi\}(z_0,\xi,\tau) = 2R_{z_0}(\xi,\nabla\Phi) + 2i\tau p(z_0,\nabla\Phi)$$

In particular, since by assumption, $p(z_0,\nabla\Phi) \neq 0$, this gives that $\{p_\Phi,\Phi\}$ never cancel and (3.6) is also empty. □

Next, we will follow the same previous steps of convexification as Section 2.3.1.

Proposition 3.2.2 (Analytic convexification)**.** *Let P and Φ satisfying the assumptions of Theorem 3.1.1 with $\Phi(z_0) = 0$, that is satisfying the assumptions of Definition 2.1.1 (and therefore, the one of Proposition 3.2.1)*

Then there exists λ large enough so that the functions $\Psi = e^{\lambda\Phi}$ satisfies the pseudoconvexity for functions in $\xi_t = 0$ of Definition 3.1.1.

Proof. The proof is very similar to Proposition 2.3.2.

We denote again for $\tau > 0$

$$c_\Psi(\xi,\tau) = \frac{1}{i\tau}\{\overline{p_\Psi}, p_\Psi\}(z_0,\xi,\tau).$$

Since p is real Lemma 2.2.4 still applies and we can extend $c_\Psi(\xi,\tau)$ by continuity to $\tau = 0$ by $2\{p,\{p,\Phi\}$. Then, using Lemma 2.2.3, the consequences of Proposition 3.2.1 can be reformulated by the existence of some constants C_1, $C_2 > 0$ so that

$$c_\Phi(\xi,\tau) + C_1\left[|\{p_\Phi,\Phi\}(z_0,\xi,\tau)|^2 + \frac{|p_\Phi(x_0,\xi,\tau)|^2}{|\xi|^2+\tau^2} + |\xi_t|^2\right] \geq C_2(|\xi|^2+\tau^2).$$

The same computation lead to

$$c_\Psi(\xi,\tau) = \lambda c_\Phi(\xi,\lambda\tau) + 2\lambda^2 |\{p_\Phi,\Phi\}(x_0,\xi,\lambda\tau)|^2.$$

The same arguments then lead to

$$c_\Psi(\xi,\tau) + C_1\left[\frac{|p_\Psi(x_0,\xi,\tau)|^2}{|\xi|^2+\tau^2} + |\xi_t|^2\right] \geq C_2(|\xi|^2+\tau^2).$$

for λ large enough. This implies the result. □

It remains to perform the Geometric convexification and to ensure that we can take ψ quadratic.

Proposition 3.2.3 (Geometric convexification)**.** *Let Ψ pseudoconvex in the sense of functions in $\xi_t = 0$, that is satisfying the assumptions of Definition 3.1.1. Set also $\Psi(x_0) = 0$.*

Then there exists φ quadratic that still satisfies the pseudoconvexity for functions in $\xi_t = 0$ at z_0 and fulfills the following Geometric requirements for some $R_0 > 0$, for any $0 < R < R_0$, there exists $r > 0$, $\eta > 0$ so that

1. *there exists $\eta > 0$ so that $\varphi(x) \le -\eta$ for $z \in \{\Psi \le 0\} \cap \{R/2 \le |z - z_0| \le R\}$,*
2. *$\psi(0) = 0$.*

Proof. For $\delta > 0$, we take

$$\varphi(z) = \Psi_T(x) - \delta|z - z_0|^2.$$

where

$$\Psi_T(z) = \sum_{|\alpha|\le 2} \frac{1}{\alpha!}(\partial_\alpha \Psi)(0)(z - z_0)^\alpha$$

that is Ψ_T is the Taylor expansion of Ψ at order 2. Indeed, this is almost the same construction as in the classical case, except that we have replaced Ψ by its Taylor expansion at order 2.

First, we notice that since the pseudoconvexity only involves the derivative up to order 2 at z_0, Ψ_T is also pseudoconvexity for functions in $\xi_t = 0$ at z_0. Moreover, the same stability argument as in Proposition 2.3.3 applies. So, for δ small enough, $\varphi(x)$ is pseudoconvexity for functions in $\xi_t = 0$ at z_0. We fix $\delta > 0$ sufficiently small. It remains to prove the geometric properties.

Since Ψ_T is the Taylor expansion of Ψ at order 2, there exists R_0 small enough so that $|\Psi_T - \Psi| \le |z - z_0|^2\delta/2$ for $|z - z_0| \le R_0$. Now, take $R \le R_0$.

Let $z \in \{\Psi \le 0\} \cap \{R/2 \le |z - z_0| \le R\}$. Since $\Psi(z) \le 0$, we have $\Psi_T(z) \le |z - z_0|^2\delta/2$. Therefore,

$$\varphi(z) \le -\delta|z - z_0|^2/2.$$

So, in particular since $|z - z_0|^2 \ge R^2/4$, we get $\varphi(x) \le -\delta R^2/8$ and we can take $\eta = \delta R^2/8$. ☐

3.2.2 Unique continuation

Proof of Theorem 3.1.1. Let u solution of $Pu = 0$ in Ω so that $u = 0$ on $\Omega \cap \{\Phi > 0\}$. Φ is pseudoconvex for surfaces at x_0. Proposition 3.2.2 and 3.2.3 allows to produce some quadratic function ψ that satisfies the pseudoconvexity for functions in $\xi_t = 0$ at z_0 . It therefore satisfies the following properties

1. there exists $R > 0$, $C > 0$ d, $\varepsilon > 0$ and $\tau_0 > 0$ so that we have the following estimate

$$\tau\|Q^\psi_{\varepsilon,\tau} w\|^2_{H^1_\tau} \le C \left\|Q^\psi_{\varepsilon,\tau} Pw\right\|^2_{L^2} + Ce^{-\mathsf{d}\tau} \left\|e^{\tau\psi} w\right\|^2_{H^1_\tau} \tag{3.7}$$

   for any $w \in C^\infty(B(x_0, R))$ and $\tau \ge \tau_0$.

2. there exists $\eta > 0$ so that $\psi(z) \le -\eta$ for $z \in \{\Phi \le 0\} \cap \{|x - x_0| \ge R/2\}$,
3. $\psi(z_0) = 0$.

4. $\psi(z) \leq \mathsf{d}/4$ in $B(z_0, R)$.

All the properties were already obtained. We only added 4 which can easily be obtained up to reducing R.

Pick $\chi \in C^\infty(B(x_0,R))$ so that $\chi = 1$ on $B(x_0, R/2)$. As before, we want to apply the Carleman estimate to $w = \chi u$ solution of $Pw = \chi P u + [P,\chi]u = [P,\chi]u$. Again, $[P,\chi]$ is a classical differential operator of order 1 with coefficients supported in the set $\{\Phi \leq 0\} \cap \{\frac{R}{2} \leq |x - x_0| \leq R\}$ where we have $\Psi_\varepsilon(x) \leq -\eta$. In particular, we have $\left\| Q^{\psi}_{\varepsilon,\tau} P w \right\|_{L^2} \leq \left\| e^{\tau\psi} P w \right\|_{L^2} \leq C e^{-\tau\eta} \left\| u \right\|_{H^1}$.

For the second term in the right hand side, we use Property 4 to get

$$e^{-\mathsf{d}\tau} \left\| e^{\tau\psi} w \right\|^2_{H^1_\tau} \leq e^{-\mathsf{d}\tau} e^{\mathsf{d}\tau/2} \left\| w \right\|^2_{H^1_\tau} \leq e^{-\mathsf{d}\tau/2} \tau^2 \left\| w \right\|^2_{H^1} \leq e^{-\mathsf{d}\tau/4} \left\| u \right\|^2_{H^1}$$

for τ large enough.

So, we have obtained that there exists $C > 0$, $\delta > 0$ so that for all $\tau \geq \tau_0$

$$\| Q^{\psi}_{\varepsilon,\tau} w \|_{L^2} \leq C e^{-\delta\tau}.$$

We will use the following Lemma that we prove below

Lemma 3.2.1. *Let $\psi \in C^\infty(\Omega)$ a real valued function. Let $v \in C^\infty_0(\Omega)$ so that*

$$\| Q^{\psi}_{\varepsilon,\tau} v \|_{L^2} \leq C \quad \forall \tau \geq \tau_0.$$

Then, v is supported in $\{\psi \leq 0\}$.

The Lemma gives that w is supported in the set $\{\psi \leq -\delta\}$. Yet, we have $\psi(0) = 0$. Then there exists a neighborhood V included in the set $\{\chi = 1\}$ of z_0 so that $\psi(z) \geq -\delta/2$ in V. Therefore, $w = 0$ and $u = 0$ in V. □

We need to prove Lemma 3.2.1. Note that if we had $\varepsilon = 0$, the proof would be easy. The idea is the following.

1. We make a kind of foliation along the level sets of ψ: if we want to measure u, we rather define the distribution $h_f = \psi_*(fv)$ by $\langle h_f, w \rangle_{\mathcal{E}'(\mathbb{R}), C^\infty(\mathbb{R})} = \langle fv, w(\psi) \rangle_{\mathcal{E}'(\mathbb{R}^n), C^\infty(\mathbb{R}^n)}$ and estimate it for any test function f. Heuristically, $h_f(s)$ is the integral of fu on the level set $\{\psi(x) = s\}$.

2. We notice that the Fourier transform of h_f is $\widehat{h_f}(\zeta) = \langle fv, e^{-i\zeta\psi} \rangle$ and can be extended to the complex domain if v is compactly supported. In particular, on the imaginary axis, $\widehat{h_f}(i\tau) = \langle f, v e^{\tau\psi} \rangle$. Since the estimate gives information on the norm of $e^{\tau\psi} v$ for τ large, this can be translated in some information on $\widehat{h_f}$ on the upper imaginary axis. . A Phragmén-Lindelöf type argument allows to transfer this estimate to the (almost) whole upper plan.

3. This will give some exponential bound on the upper plan. A Paley-Wiener allows to conclude.

More precisely:

Proof of Lemma 3.2.1. We will work by duality. Let $f \in \mathcal{S}(\mathbb{R}^{n+1})$ with Fourier transform $\widehat{f}$ compactly supported in $B(0, R)$ for R large. We define the distribution $h_f \in \mathcal{E}'(\mathbb{R})$ by

$$\langle h_f, w \rangle_{\mathcal{E}'(\mathbb{R}), C^\infty(\mathbb{R})} = \langle fv, w(\psi) \rangle_{\mathcal{E}'(\mathbb{R}^{n+1}), C^\infty(\mathbb{R}^{n+1})}.$$

h_f is indeed compactly supported because $Supp(h_f) \subset \{\psi(z); z \in Supp(v)\}$ which is compact.

Since $h_f \in \mathcal{E}'(\mathbb{R})$, the Fourier transform of h_f is an analytic function on $\mathbb{C}$ that can be computed with the formula

$$\widehat{h_f}(\xi_s) = \left\langle h_f, e^{-is\xi_s} \right\rangle_{\mathcal{E}'(\mathbb{R}_s), C^\infty(\mathbb{R}_s)} = \langle fv, e^{i\xi_s \psi} \rangle_{\mathcal{E}'(\mathbb{R}^{n+1}), C^\infty(\mathbb{R}^{n+1})}$$

On $\xi_s \in \mathbb{R}$, we have the general bound

$$|\widehat{h_f}(\xi_s)| \le \|f\|_{L^\infty(Supp(v))} \|v\|_{L^\infty(Supp(v))} \le C_{f,v}.$$

But our estimates gives some bound on the imaginary axis $\xi_s = i\tau$, for $\tau \ge \tau_0$,

$$\begin{aligned}
|\widehat{h_f}(i\tau)| &= \left| \langle fv, e^{\tau\psi} \rangle \right|_{\mathcal{E}'(\mathbb{R}^{n+1}), C^\infty(\mathbb{R}^{n+1})} \\
&= \left| \langle f, ve^{\tau\psi} \rangle_{\mathcal{S}'(\mathbb{R}^{n+1}), \mathcal{S}(\mathbb{R}^{n+1})} \right| \\
&= \left| \langle e^{\varepsilon \frac{|D_t|^2}{2\tau}} f, e^{-\varepsilon \frac{|D_t|^2}{2\tau}} ve^{\tau\psi} \rangle_{\mathcal{S}'(\mathbb{R}^{n+1}), \mathcal{S}(\mathbb{R}^{n+1})} \right| \\
&\le \left\| e^{\varepsilon \frac{|D_t|^2}{2\tau}} f \right\|_{L^2(\mathbb{R}^{n+1})} \left\| e^{-\varepsilon \frac{|D_t|^2}{2\tau}} ve^{\tau\psi} \right\|_{L^2(\mathbb{R}^{n+1})} \\
&\le \left\| e^{\varepsilon \frac{|\xi_t|^2}{2\tau}} \right\|_{L^\infty(Supp(\widehat{f}))} \|f\|_{L^2(\mathbb{R}^{n+1})} \left\| Q^\psi_{\varepsilon,\tau} v \right\|_{L^2(\mathbb{R}^{n+1})} \le Ce^{\frac{\varepsilon R^2}{2\tau}} \|f\|_{L^2(\mathbb{R}^{n+1})} \\
&\le C_{f,\tau_0} C.
\end{aligned}$$

Note that at that point, the term $e^{\varepsilon \frac{|D_t|^2}{2\tau}}$ was harmless because the Fourier transform of f is compactly supported. Otherwise $e^{\varepsilon \frac{|D_t|^2}{2\tau}} f$ does not have meaning, even for $f \in \mathcal{S}(\mathbb{R}^{n+1})$. That is why we had to work by duality.

Moreover, for $\tau \in [0, \tau_0]$, the estimate

$$|\widehat{h_f}(i\tau)| \le C$$

follows easily with some appropriate constant C independant on τ. Moreover, the compact support of h_f ensure $|\widehat{h_f}(z)| \le Ce^{c|z|}$ on $\mathbb{C}$.

Now, we have some nice estimates on $\mathbb{R} \cup i\mathbb{R}_+$, we will transfer them to the upper plane by the Phragmén-Lindelöf Theorem.

Lemma 3.2.2 (Phragmén-Lindelöf Theorem)**.** *Let ϕ be a holomorphic function in $Q_1 = \{x + iy; x \ge 0, y \ge 0\}$, continuous in $\bar{Q}_1$. Assume that there exist $c > 0$ and $C > 0$ such that*

$$\begin{aligned}
|\phi(z)| &\le Ce^{c|z|}, \quad z \in Q_1, \\
|\phi(z)| &\le 1 \quad , \quad z \in \partial Q_1 = \mathbb{R}_+ \cup i\mathbb{R}_+.
\end{aligned}$$

Assume moreover that there exist $C, c > 0$ so that $|\phi(z)| \le Ce^{c|z|}$.

Then $|\phi(z)| \le 1$ for all $z \in Q_1$.

This gives us (the result is also true for the up left quarter plane with the same method)

$$|\widehat{h_f}(\xi_s)| \le C \quad \forall \xi_s \in \mathbb{C}, \mathrm{Im}(\xi_s) \ge 0$$

The Paley-Wiener Theorem gives $Supp(h_f) \subset]-\infty, +]$. Therefore, we have proved that for $\chi \in C_0^\infty(\mathbb{R}^+)$,

$$0 = \langle fv, \chi(\psi) \rangle = \langle f, \chi(\psi)v \rangle$$

Since this is true for a subset of function f dense in $\mathcal{S}$, this means that $v = 0$ for $\psi > 0$. □

Proof of Lemma 3.2.2. First note that the sector Q_1 can be rotated, say to quadrant

$$Q = \{z \in \mathbb{C}, \arg(z) \in [-\frac{\pi}{4}, \frac{\pi}{4}]\}.$$

Let

$$u_\delta(z) = \phi(z) e^{-z^{\delta \frac{3}{2}}},$$

(with the principal determination of the logarithm that is if $z = re^{i\theta}$ with $-\pi < \theta < \pi$, $z^{3/2} = r^{3/2} e^{3i\theta/2}$) which is harmonic in Q.

Also, we have

$$|e^{-z^{\delta \frac{3}{2}}}| = e^{-\delta r^{3/2} \cos(3\theta/2)}$$

On Q, we have $|\theta| \leq \pi/4$ and therefore $|3\theta/2| \leq 3\pi/8 < \pi/2$ and $\cos(3\theta/2) \geq \eta > 0$.

So, the assumption on Φ gives $\limsup_{z \in Q, |z| \to \infty} u_\delta(z) = 0$. As a consequence, there exists $R > 0$ such that $|u_\delta(z)| < 1/2$ on $\{|z| \geq R\} \cap Q$. Now, on the bounded set $Q^R = Q \cap \{|z| \leq R\}$, we apply the maximum principle to the function u_δ, satisfying $|u_\delta| \leq 1$ on ∂Q^R. This yields $|u_\delta| \leq 1$ on Q^R and hence $|u_\delta| \leq 1$ on Q. Finally letting δ tend to zero, we obtain the sought result.

Note that the limit power could be $2 - \varepsilon$ with $\varepsilon > 0$: the result is false for $\varepsilon = 0$, as showed by the holomorphic function $z \mapsto e^{z^2}$ on the quarter plane Q. □

Theorem 3.2.1 (Paley-Wiener)**.** *Suppose $f \in \mathcal{S}(\mathbb{R})$. Then $f(s) = 0$ for all $s > 0$ if and only if $\widehat{f}$ can be extended to a continuous and bounded function in the closed upper half-plane $\mathbb{C}^+ = \{z = x + iy : y \geq 0\}$ with $\widehat{f}$ holomorphic in the interior.*

Proof. One implication is simpler. Assume $f(s) = 0$ for $s \geq 0$, then $\widehat{f}(\xi) = \int_{s \leq 0} e^{-is\xi} f(s) ds$. Then, is can easily be continuously extended on $\mathbb{C}^+$ with the estimate $|\widehat{f}(x + iy)| \leq \int_{s \leq 0} e^{ys} |f(s)| ds \leq \int_{s \leq 0} |f(s)| ds < +\infty$.

Let us now prove the converse. We denote $f_{\varepsilon,\delta}(s) = \frac{1}{2\pi} \int_{\xi \in \mathbb{R}} e^{is\xi} g_{\varepsilon,\delta}(\xi)\, d\xi$ the bounded continuous function with Fourier transform $g_{\varepsilon,\delta}(z) = \frac{\widehat{f}(z+i\delta)}{(1-i\varepsilon z)^2}$ well defined and holomorphic in a neighborhood of $\mathbb{C}^+$, bounded by $\frac{C}{(1+\varepsilon y)^2 + (\varepsilon x)^2}$. But we have for $s > 0$

$$f_{\varepsilon,\delta}(s) = \lim_{N+\infty} \frac{1}{2\pi} \int_{-N}^{N} e^{is\xi} g_{\varepsilon,\delta}(\xi)\, d\xi = \lim_{N+\infty} \frac{1}{2\pi} \int_{\gamma_N} e^{isz} g_{\varepsilon,\delta}(z)$$

where we have used a change of rectangular contour γ_N composed of three segment

- $[-N, -N + iN]$ where we have the estimate $|e^{isz} g_{\varepsilon,\delta}(z)| \leq \frac{C}{1+(\varepsilon N)^2}$, using that $\mathrm{Im}(z) \geq 0$ and $s \geq 0$.
- $[-N + iN, N + iN]$ where we have the estimate $|e^{isz} g_{\varepsilon,\delta}(z)| \leq \frac{Ce^{-sN}}{(1+\varepsilon N)^2}$, using that $\mathrm{Im}(z) = N$ and $s \geq 0$.
- $[N + iN, N]$ with the same estimates.

Making N converge to $+\infty$ and taking into account that the length of the path are of the order of N, we get that $f_{\varepsilon,\delta}(s) = 0$ for $s \geq 0$. By dominated convergence, we obtain first $f_{\varepsilon,0}(s) = 0$ for $s \geq 0$, and then $f = 0$ for $s \geq 0$ by dominated convergence again since $\widehat{f}$ is Schwarz on $\mathbb{R}$. □

3.3 The Carleman estimate

As in the classical case, we need to check the effect of the conjugated operator. Yet, we have to be a little careful, first because $e^{\varepsilon\frac{|D_t|^2}{2\tau}}$ is not well defined on any Sobolev space and even not on $\mathcal{S}$.

As before, we make the change of variable $v = e^{\tau\psi}u$ and for getting (3.4), we are left to prove

$$\tau\|e^{-\varepsilon\frac{|D_t|^2}{2\tau}}v\|^2_{H^1_\tau} \quad\leq\quad C\left\|e^{-\varepsilon\frac{|D_t|^2}{2\tau}}P_\psi v\right\|^2_{L^2} + Ce^{-\mathsf{d}\tau}\,\|v\|^2_{H^1_\tau}$$

Our operator P commutes with $e^{-\varepsilon\frac{|D_t|^2}{2\tau}}$ since its coefficients are independent on t. Yet, the operator $P_\psi = e^{\tau\psi}Pe^{-\tau\psi}$ may depend on t because ψ depends on t. We will take advantage of the fact that since ψ is quadratic, the principal symbol of P_ψ only involve some derivative of ψ of order at least 1 and is therefore linear. We first prove the following simple Lemma.

Lemma 3.3.1. *Let $u \in \mathcal{S}(\mathbb{R}^{n+1})$, then*

$$e^{-\varepsilon\frac{|D_t|^2}{2\tau}}(tu) = \left(t + i\varepsilon\frac{D_t}{\tau}\right)e^{-\varepsilon\frac{|D_t|^2}{2\tau}}u.$$

Proof. We have the formula

$$\widehat{tv}(\xi) = \int_{\mathbb{R}^{n+1}} e^{-it\xi_t}e^{-ix\cdot\xi_x}tv(t,x)dxdt = \int_{\mathbb{R}^{n+1}} i\partial_{\xi_t}e^{-it\xi_t}e^{-ix\cdot\xi_x}tv(t,x)dxdt = i\partial_{\xi_t}\widehat{u}(\xi).$$

$$\begin{aligned}
\left(\widehat{e^{-\varepsilon\frac{|D_t|^2}{2\tau}}(tu)}\right)(\xi) &= e^{-\varepsilon\frac{|\xi_t|^2}{2\tau}}\widehat{(tu)}(\xi) = e^{-\varepsilon\frac{|\xi_t|^2}{2\tau}}i\partial_{\xi_t}\widehat{u}(\xi) = i\partial_{\xi_t}\left[e^{-\varepsilon\frac{|\xi_t|^2}{2\tau}}\widehat{u}(\xi)\right] + i\frac{\varepsilon\xi_t}{\tau}e^{-\varepsilon\frac{|\xi_t|^2}{2\tau}}\widehat{u}(\xi)\\
&= \left[\widehat{te^{-\varepsilon\frac{|D_t|^2}{2\tau}}u}\right](\xi) + i\left[\frac{\varepsilon D_t}{\tau}\widehat{e^{-\varepsilon\frac{|D_t|^2}{2\tau}}u}\right](\xi)
\end{aligned}$$

That is

$$\begin{aligned}
e^{-\varepsilon\frac{|D_t|^2}{2\tau}}(tu) &= te^{-\varepsilon\frac{|D_t|^2}{2\tau}}u + i\varepsilon\frac{D_t}{\tau}e^{-\varepsilon\frac{|D_t|^2}{2\tau}}u\\
&= \left(t + i\varepsilon\frac{D_t}{\tau}\right)e^{-\varepsilon\frac{|D_t|^2}{2\tau}}u.
\end{aligned}$$

□

Remark 3.3.1. *Lemma 3.3.1 could easily be iterated to get the formula*

$$e^{-\varepsilon\frac{|D_t|^2}{2\tau}}(t^k u) = \left(t + i\varepsilon\frac{D_t}{\tau}\right)^k e^{-\varepsilon\frac{|D_t|^2}{2\tau}}u.$$

where the exponent k is meant in the sense of composition. For f polynomial in t, we would get

$$e^{-\varepsilon\frac{|D_t|^2}{2\tau}}(f(t)u) = f\left(t + i\varepsilon\frac{D_t}{\tau}\right)e^{-\varepsilon\frac{|D_t|^2}{2\tau}}u.$$

This means that the "formal" conjugated operator of $e^{-\varepsilon\frac{|D_t|^2}{2\tau}}$ with $f(t)$ is a differential of the same order as f.

For a general (even smooth) function $f(t)$, it seems therefore very hard to give a precise meaning to $f\left(t+i\varepsilon\frac{D_t}{\tau}\right)$.

Even in the analytic case, $f\left(t+i\varepsilon\frac{D_t}{\tau}\right)$ would be an infinite sum of differential operators, that means an operator of "infinite order". This is not clear how to define this in an exact way. Yet, some authors managed to give some meaning of an approximation of this formula. Namely, the idea is to replace $t+i\varepsilon\frac{D_t}{\tau}$ by some approximate operator $\chi\left(\frac{t}{\kappa}\right)t+i\chi\left(\frac{\varepsilon D_t}{\kappa\tau}\right)\varepsilon\frac{D_t}{\tau}$ for κ small. These operators have the advantage to be bounded and we can consider some infinite series. We refer to Hörmander [8]. Similarly, it is possible to replace the holomorphic function f with a cutoff near small x and ξ_t, see Tataru [12, 13].

We compute, if $P=\sum_{|\alpha|\le 2}p_\alpha(x)D_z^\alpha$

$$P_\psi=\sum_{|\alpha|\le 2}p_\alpha(x)(D_z+i\tau\nabla_z\psi)^\alpha.$$

Lemma 3.3.2 (The "conjugated operator")**.** *Let $P=\sum_\alpha p_\alpha(z)D_z^\alpha\in \mathrm{Diff}^m$ be a (classical) differential operator on $\mathbb{R}^{n+1}=\mathbb{R}_t\times\mathbb{R}_x$, with principal symbol $p_m\in\Sigma^m$. Assume also that **all its coefficients are independent on** t, that is $p_\alpha(z)=p_\alpha(x)$. Let ψ real valued and **quadratic** in t.*

Then, for any $\varepsilon>0$, there exists a unique $P_{\psi,\varepsilon}v\in\mathrm{Diff}_\tau^m$ so that we have

$$Q^\psi_{\varepsilon,\tau}Pu=P_{\psi,\varepsilon}Q^\psi_{\varepsilon,\tau}u.$$

for any $u\in\mathcal{S}(\mathbb{R}^{n+1})$.

Moreover, the principal symbol of $P_{\psi,\varepsilon}$ is

$$p_{\psi,m,\varepsilon}=p_m(z,\xi+i\tau\nabla\psi-\varepsilon\psi''_{t,z}\xi_t)=\sum_{|\alpha|=m}p_\alpha(x)(\xi+i\tau\nabla\Phi-\varepsilon\psi''_{t,z}\xi_t)^\alpha,$$

where we use the notation $\psi''_{t,z}\xi_t=Hess(\psi)((\xi_t,0,\cdots,0);\cdot)=\xi_tV$ with V the constant vector with coefficients $V_k=(\partial_t\partial_k\psi)$. We will denote it $p_{\psi,\varepsilon}$ for simplicity in the sequel.

Additionally, $P_{\psi,\varepsilon}$ can be decomposed $P_{\psi,\varepsilon}=P_{R,\varepsilon}+i\tau\widetilde{P_{I,\varepsilon}}$ with $(P_{R,\varepsilon})^*=P_{R,\varepsilon}$ and $(\widetilde{P_{I,\varepsilon}})^*=\widetilde{P_{I,\varepsilon}}$.

Remark 3.3.2. *The expression "conjugated operator" is a bit abusive since $e^{\varepsilon\frac{|D_t|^2}{2\tau}}$ is not well defined as an operator. Yet, we would like to write formally*

$$P_{\psi,\varepsilon}v=Q^\psi_{\varepsilon,\tau}P\left(Q^\psi_{\varepsilon,\tau}\right)^{-1}v=e^{-\varepsilon\frac{|D_t|^2}{2\tau}}e^{\tau\psi}Pe^{-\tau\psi}e^{\varepsilon\frac{|D_t|^2}{2\tau}}v=e^{-\varepsilon\frac{|D_t|^2}{2\tau}}P_\psi e^{\varepsilon\frac{|D_t|^2}{2\tau}}v.$$

Remark 3.3.3. *The previous expression is actually a consequence of the fact that for a quadratic homogeneous function f in t, we have $i\tau\nabla f\left(t+i\varepsilon\frac{D_t}{\tau}\right)=i\tau\nabla f(t)-\varepsilon\nabla f(D_t)$*

Proof. Since ψ is quadratic, for any $k=0,\cdot,n$, $\partial_k\psi$ is polynomial of order 1 and can be written

$$\partial_k\psi=f_1(x)+tf_0(x).$$

where $f_1(x)$ (resp. f_0) is polynomial in x of order 1 (resp. 0). In particular, Lemma 3.3.1 gives

$$\begin{aligned}
&e^{-\varepsilon\frac{|D_t|^2}{2\tau}}\left[(D_k+i\tau\partial_k\psi)u\right]=e^{-\varepsilon\frac{|D_t|^2}{2\tau}}\left[(D_k+i\tau(f_1(x)+tf_0(x)))u\right]\\
=&\left[D_k+i\tau\left(f_1(x)+\left(t+i\varepsilon\frac{D_t}{\tau}\right)f_0(x)\right)\right]e^{-\varepsilon\frac{|D_t|^2}{2\tau}}u\\
=&\left[(D_k+i\tau\partial_k\psi-\varepsilon f_0(x)D_t\right]e^{-\varepsilon\frac{|D_t|^2}{2\tau}}u.
\end{aligned}$$

To get an intrinsic expression, we notice that $f_0(x) = \partial_t\partial_k\psi$, so $f_0(x)D_t$ can be written $\partial_t\partial_k\psi D_t$. So, the full (and principal symbol) of $D_k + i\tau\partial_k\psi - \varepsilon f_0(x)D_t$ is

$$\xi_k + i\tau\partial_k\psi - \varepsilon(\partial_t\partial_k\psi)\xi_t = \xi_k + i\tau\partial_k\psi(x) - \varepsilon(\partial_t\partial_k\psi)\xi_t.$$

So, since $D^\alpha = D_1^{\alpha_1}\cdots D_j^{\alpha_j}\cdots D_n^{\alpha_n}$, as in Lemma 2.2.1, we obtain similarly that the "conjugated operator" for D^α has principal symbol

$$\prod_{k=1}^{n}(\xi_k + i\tau\partial_k\Phi - \varepsilon(\partial_t\partial_k\psi)\xi_t)^{|\alpha_k|} = (\xi + i\tau\nabla\Phi - \varepsilon\psi''_{t,z}\xi_t)^\alpha,$$

where we use the notation $\psi''_{t,z}\xi_t = Hess(\psi)((\xi_t, 0, \cdots, 0); \cdot)$.

Since all the functions $p_\alpha(x)$ do not depend on t, they commute with $Q^\psi_{\varepsilon,\tau}$. So, we get the conclusion of the first two statements of the Lemma.

We prove the last one for the conjugated operator of D^α by iteration on $|\alpha|$. We prove the more precise statement that it can be written $A + \tau B$ with $A^* = A$ and A with constant coefficients of order $|\alpha|$. If $|\alpha| = 0$, it is obvious. Otherwise, assume $M^\alpha = D^\alpha$ and $M^\alpha_{\psi,\varepsilon} = A + \tau B$ with A, B as before. Let $l = 0$ 1, ... or n.

$$\begin{aligned} e^{-\varepsilon\frac{|D_t|^2}{2\tau}}e^{\tau\psi}M^\alpha D_l u &= M^\alpha_{\psi,\varepsilon}e^{-\varepsilon\frac{|D_t|^2}{2\tau}}e^{\tau\psi}D_l u \\ &= (A + \tau B) \circ [D_l + i\tau\partial_l\psi(x) - \varepsilon(\partial_t\partial_l\psi)D_t]\, e^{-\varepsilon\frac{|D_t|^2}{2\tau}}e^{\tau\psi}u \end{aligned}$$

But since ψ is quadratic, $\partial_t\partial_l\psi$ is a constant and we have the expression

$$(A + \tau B) \circ [D_l + i\tau\partial_l\psi(x) - \varepsilon(\partial_t\partial_l\psi)D_t] = AD_l - \varepsilon A(\partial_t\partial_l\psi)D_t + \tau C.$$

for an appropriate C. This gives the result. The final result can also be obtained using that p_α does not depend on t. □

The important fact of the previous formula is that the principal symbol of $P_{\psi,\varepsilon}$ is actually close to the principal symbol of P_ψ if ε is small. So, we can expect that it satisfies the same subelliptic estimates.

We first write the following Lemma on p_ψ, that we have actually already used and proved in Proposition 3.2.2, using Lemma 2.2.3 and homogeneity, so we skip the proof.

Lemma 3.3.3. *Assume that ψ satisfies the pseudoconvexity for functions in $\xi_t = 0$ at z_0 of Definition 3.1.1, then there exist C_1, $C_2 > 0$ so that we have the estimate taken at the point z_0 and for any $\tau \geq 0$, $\xi \in \mathbb{R}^n$,*

$$\frac{1}{i\tau}\{\overline{p_\psi}, p_\psi\} + C_1\left[\frac{|p_\psi(z_0,\xi,\tau)|^2}{|\xi|^2 + \tau^2} + |\xi_t|^2\right] \geq C_2(|\xi|^2 + \tau^2).$$

where we have extended $\frac{1}{i\tau}\{\overline{p_\psi}, p_\psi\}$ by continuity at $\tau = 0$ with the value $2\{p, \{p, \psi\}\}$.

By perturbation, we can get a similar conclusion for the perturbated operator.

Lemma 3.3.4. *Assume that ψ satisfies the pseudoconvexity for functions in $\xi_t = 0$ at z_0 of Definition 3.1.1. Then, there exists $\varepsilon_0 > 0$ so that for any $0 \leq \varepsilon < \varepsilon_0$, there exist C_1, $C_2 > 0$ so that we have the estimate for any $\tau \geq 0$, $\xi \in \mathbb{R}^n$,*

$$\frac{1}{i\tau}\{\overline{p_{\psi,\varepsilon}}, p_{\psi,\varepsilon}\} + C_1\left[\frac{|p_{\psi,\varepsilon}(z_0,\xi,\tau)|^2}{|\xi|^2 + \tau^2} + |\xi_t|^2\right] \geq C_2(|\xi|^2 + \tau^2).$$

where we have extended $\frac{1}{i\tau}\{\overline{p_{\psi,\varepsilon}}, p_{\psi,\varepsilon}\}$ by continuity at $\tau = 0$ with the appropriate value.

Proof. The Lemma mainly follows by saying that $p_{\psi,\varepsilon}$ is a perturbation of p_ψ and using Lemma 3.3.3. Yet, we have to be a little careful because of the factor $\frac{1}{\tau}$. Noticing as before that $\frac{1}{i\tau}\{\overline{p_{\psi,\varepsilon}}, p_{\psi,\varepsilon}\} = \frac{2}{\tau}\{\operatorname{Re} p_{\psi,\varepsilon}, \operatorname{Im} p_{\psi,\varepsilon}\}$. Then, using the last part of Lemma 3.3.2, we can write $\operatorname{Im} p_{\psi,\varepsilon} = \tau \widetilde{p_{\psi,\varepsilon}}^i$. Moreover, $\widetilde{p_{\psi,\varepsilon}}^i$ and all its derivative are all continuous in ε. Hence, we can write $\frac{1}{i\tau}\{\overline{p_{\psi,\varepsilon}}, p_{\psi,\varepsilon}\} = 2\{\operatorname{Re} p_{\psi,\varepsilon}, \widetilde{p_{\psi,\varepsilon}}^i\}$. It can therefore be extended by continuity to $\tau = 0$ and the result follows by a perturbation of Lemma 3.3.3. □

We are now ready to prove a first subelliptic estimate that will be crucial for the final proof of Theorem 3.1.2

Proposition 3.3.1. *Under the previous assumptions for P, ψ, z_0, there exist $\varepsilon > 0$, a neighborhood V of z_0, $C > 0$ and $\tau_0 > 0$ so that we have the following estimate*

$$\tau \|v\|^2_{H^1_\tau} \leq C \|P_{\psi,\varepsilon} v\|^2_{L^2} + C\tau \|D_t v\|^2_{L^2}, \tag{3.8}$$

for any $v \in C_0^\infty(V)$ and $\tau \geq \tau_0$.

Proof. Using the same computations as in Theorem 2.2.1 with the decomposition $P_{\psi,\varepsilon} = P_{R,\varepsilon} + P_{I,\varepsilon} = P_{R,\varepsilon} + i\tau\widetilde{P_{I,\varepsilon}}$, we get

$$\begin{aligned}
\|P_{\psi,\varepsilon} v\|^2_{L^2} &= \|P_{R,\varepsilon} v\|^2_{L^2} + \|P_{I,\varepsilon} v\|^2_{L^2} + (i[P_{R,\varepsilon}, P_{I,\varepsilon}]v, v) \\
&= \|P_{R,\varepsilon} v\|^2_{L^2} + \|P_{I,\varepsilon} v\|^2_{L^2} + \tau \left(i[P_{R,\varepsilon}, \widetilde{P_{I,\varepsilon}}]v, v\right).
\end{aligned}$$

The same computations lead to

$$\frac{1}{\tau} \|P_{\psi,\varepsilon} v\|^2_{L^2} \geq (Lv, v)$$

with

$$\begin{aligned}
L &= P_{R,\varepsilon} \frac{C^2}{(-\Delta + \tau^2)} P_{R,\varepsilon} + P_{I,\varepsilon} \frac{C^2}{(-\Delta + \tau^2)} P_{I,\varepsilon} + \frac{i}{\tau}[P_{R,\varepsilon}, P_{I,\varepsilon}] \\
&= P_{R,\varepsilon} \frac{C^2}{(-\Delta + \tau^2)} P_{R,\varepsilon} + P_{I,\varepsilon} \frac{C^2}{(-\Delta + \tau^2)} P_{I,\varepsilon} + i[P_{R,\varepsilon}, \widetilde{P_{I,\varepsilon}}].
\end{aligned}$$

for τ large enough and C to be chosen. So, we have

$$\frac{1}{\tau} \|P_{\psi,\varepsilon} v\|^2_{L^2} + C \|D_t v\|^2_{L^2} \geq \left((L + C^2 D_t^2)v, v\right) \tag{3.9}$$

The principal symbol of $L + C^2 D_t^2$ is

$$\frac{1}{2i\tau}\{\overline{p_{\psi,\varepsilon}}, p_{\psi,\varepsilon}\} + C^2 \left[\frac{|p_{\psi,\varepsilon}(z_0, \xi, \tau)|^2}{|\xi|^2 + \tau^2} + |\xi_t|^2\right].$$

We conclude as before using Lemma 3.3.4 and Gårding inequality. □

Proof of Theorem 3.1.2. Let $\varepsilon > 0$ fixed and r_0 so that $B(z_0, r_0) \subset V$ where V is given by the previous theorem. In the proof, we consider functions $u \in C_0^\infty(B(z_0, r_0/4))$. Let $\chi \in C_0^\infty(]-r_0, r_0[)$ such that $\chi = 1$ on $]-r_0/2, r_0/2[$.

Setting $v = Q^\psi_{\varepsilon,\tau} u = e^{-\frac{\varepsilon}{2\tau}|D_a|^2}(e^{\tau\psi} u)$, we need to prove

$$\tau \|v\|^2_{H^1_\tau} \leq C \|P_{\psi,\varepsilon} v\|^2_{L^2} + C e^{-\mathsf{d}\tau} \left\|e^{\tau\psi} u\right\|^2_{H^1_\tau}.$$

Yet, v is not compactly supported in the variable t. So, we set $f = \chi(t)v(x)$ and we have $\operatorname{supp}(f) \subset B(z_0, r_0)$ so that we may apply Proposition 3.8 to f. We have $v - f = (1-\chi)Q^{\psi}_{\varepsilon,\tau}u = (1-\chi)e^{-\frac{\varepsilon}{2\tau}|D_a|^2}(\check{\chi}e^{\tau\psi}u)$ for some $\check{\chi} \in C^\infty_c(B_{\mathbb{R}_t}(0, r_0/3))$ with $\check{\chi} = 1$ in a neighborhood of $\overline{B}_{\mathbb{R}_t}(0, r_0/4)$ so that $\check{\chi}u = u$.

Lemma 3.3.5. *Let $\chi_1 \in C^\infty(\mathbb{R}^{n+1})$, $\chi_2 \in C^\infty(\mathbb{R}^{n+1})$ with all derivatives bounded such that dist*$(\operatorname{supp}(\chi_1), \operatorname{supp}(\chi_2))$ *0. Then there exist $C, c > 0$ such that for all $u \in \mathcal{S}(\mathbb{R}^n)$ and all $\lambda \geq 0$, we have*

$$\left\| \chi_1 e^{-\frac{|D_t|^2}{\lambda}}(\chi_2 u) \right\|_{H^1} \leq Ce^{-c\lambda}\left\| u \right\|_{H^1}.$$

As a consequence of Lemma 3.3.5, we have, for $\tau \geq \tau_0$

$$\|v\|_{1,\tau} \leq \|f\|_{1,\tau} + Ce^{-C\frac{\tau}{\varepsilon}}\|e^{\tau\psi}u\|_{H^1_\tau} \tag{3.10}$$

The subelliptic estimate (3.8) applied to f gives

$$\tau\left\|f\right\|^2_{H^1_\tau} \leq C\left\|P_{\psi,\varepsilon}f\right\|^2_{L^2} + C\tau\left\|D_t f\right\|^2_{L^2},$$

We need to estimate the terms on the RHS in terms of v.

Second, we estimate $\|P_{\psi,\varepsilon}f\|_{L^2} = \|P_{\psi,\varepsilon}\chi v\|_{L^2} = \|\chi P_{\psi,\varepsilon}v\|_{L^2} + \|[P_{\psi,\varepsilon},\chi]v\|_{L^2}$. For the commutator, we write $[P_{\psi,\varepsilon},\chi]v = [P_{\psi,\varepsilon},\chi]e^{-\frac{\varepsilon}{2\tau}|D_a|^2}\check{\chi}e^{\tau\psi}u$. We notice that $[P_{\psi,\varepsilon},\chi]$ is a differential operator of order 1 in (D,τ) with some coefficients supported on $\operatorname{supp}(\chi'_t)$ that is, away from $\operatorname{supp}(\check{\chi})$. In particular, Lemma 3.3.5 implies $\|[P_{\psi,\varepsilon},\chi]v\|_{L^2} \leq Ce^{-c\frac{\tau}{\varepsilon}}\left\|e^{\tau\psi}u\right\|_{H^1_\tau}$. This yields

$$\|P_{\psi,\varepsilon}f\|_{L^2} \leq \|P_{\psi,\varepsilon}v\|_{L^2} + Ce^{-c\frac{\tau}{\varepsilon}}\left\|e^{\tau\psi}u\right\|_{H^1_\tau}$$

Now, it remains to treat the term $\|D_t f\|_{L^2}$. Similarly, we obtain

$$\|D_t f\|_{L^2} = \|D_t(\chi v)\|_{L^2} \leq \|\chi D_t v\|_{L^2} + \|\chi'(t)e^{-\frac{\varepsilon}{2\tau}|D_t|^2}\check{\chi}e^{\tau\psi}u\|_{L^2} \leq \|D_t v\|_{L^2} + Ce^{-c\frac{\tau}{\varepsilon}}\left\|e^{\tau\psi}u\right\|_{L^2}$$

where we have used again Lemma 3.3.5.

Let ς a small constant to be fixed later on. We distinguish between frequencies of size smaller and bigger than $\varsigma\tau$. We get for $\tau \geq \frac{1}{\varsigma^2\varepsilon}$ large enough (so that the function $s \mapsto se^{-\frac{\varepsilon}{2\tau}s^2}$ is decreasing on $s \geq \sqrt{\frac{\tau}{\varepsilon}}$)

$$\begin{aligned}\|D_t v\|_{L^2} = \|D_t e^{-\frac{\varepsilon}{2\tau}|D_t|^2}e^{\tau\psi}u\|_{L^2} &\leq \|D_t 1_{|D_t|\leq\varsigma\tau}v\|_{L^2} + \|D_t 1_{|D_t|\geq\varsigma\tau}e^{-\frac{\varepsilon}{2\tau}|D_t|^2}e^{\tau\psi}u\|_0 \\ &\leq \varsigma\tau\|v\|_{L^2} + \varsigma\tau e^{-\frac{\tau\varsigma^2\varepsilon}{2}}\|e^{\tau\psi}u\|_{L^2}\end{aligned}$$

So, at that point, we have proved that there are some constants $c, C > 0$ so that for any $\varsigma > 0$, we have for τ large enough

$$\tau\left\|v\right\|^2_{H^1_\tau} \leq C\left\|P_{\psi,\varepsilon}v\right\|^2_{L^2} + C\varsigma^2\tau^3\|v\|^2_{L^2} + C\left(e^{-c\tau} + \varsigma^2\tau^3e^{-\tau\varsigma^2\varepsilon}\right)\left\|e^{\tau\psi}u\right\|^2_{L^2}.$$

This gives the result if ς is chosen small enough so that the term $C\varsigma^2\tau^3\|v\|^2_{L^2} \leq C\varsigma^2\tau\|v\|^2_{H^1_\tau}$ can be absorbed. □

Proof of Lemma 3.3.5. Using the Fourier transform of the Gaussian (see exercice), we have

$$(e^{-\frac{|D_t|^2}{\lambda}}f)(s) = \left(\frac{\lambda}{4\pi}\right)^{\frac{1}{2}} \int_{\mathbb{R}_s} e^{-\frac{\lambda}{4}|s-t|^2} f(s)\ ds.$$

We have

$$\begin{aligned} \chi_1 e^{-\frac{|D_t|^2}{\lambda}}(\chi_2 u)(t,x) &= \left(\frac{\lambda}{4\pi}\right)^{\frac{1}{2}} \int_{\mathbb{R}_s} \chi_1(t,x) e^{-\frac{\lambda}{4}|s-t|^2}(\chi_2 u)(s,x)\ ds \\ &= \left(\frac{\lambda}{4\pi}\right)^{\frac{1}{2}} \chi_1(t,x) \int_{s,|t-s|\geq d} e^{-\frac{\lambda}{4}|s-t|^2}(\chi_2 u)(s,x)\ ds \end{aligned}$$

where we have used the properties of support for the second equality. so that

$$\begin{aligned} |\chi_1 e^{-\frac{|D_t|^2}{\lambda}}(\chi_2 u)|(t,x) &\leq \|\chi_1\|_{L^\infty} \left(\frac{\lambda}{4\pi}\right)^{\frac{1}{2}} \int_{s,|t-s|\geq d} e^{-\frac{\lambda}{4}|s-t|^2} |\chi_2 u|(s,x)\ ds \\ &\leq \|\chi_1\|_{L^\infty} \left(\frac{\lambda}{4\pi}\right)^{\frac{1}{2}} \left(1_{|\cdot|\geq d} e^{-\frac{\lambda}{4}|\cdot|^2} *_{\mathbb{R}_s} |\chi_2 u|(\cdot,x)\right)(t). \end{aligned}$$

As a consequence, using the Young inequality, we have

$$\|\chi_1 e^{-\frac{|D_t|^2}{\lambda}}(\chi_2 u)\|_{L^2} \leq \|\chi_1\|_{L^\infty} \left(\frac{\lambda}{4\pi}\right)^{\frac{1}{2}} \left\|1_{|\cdot|\geq d} e^{-\frac{\lambda}{4}|\cdot|^2}\right\|_{L^1(\mathbb{R})} \|\chi_2 u\|_{L^2(\mathbb{R}^{n+1})},$$

and, using $\left\|1_{|\cdot|\geq d} e^{-\frac{\lambda}{4}|\cdot|^2}\right\|_{L^1(\mathbb{R})} \leq Ce^{-c\lambda}$ for some appropriate $c > 0$, we obtain

$$\|\chi_1 e^{-\frac{|D_t|^2}{\lambda}}(\chi_2 u)\|_{L^2} \leq \|\chi_1\|_{L^\infty} Ce^{-c\lambda} \|u\|_{L^2(\mathbb{R}^{n+1})},$$

which implies the result.

Note that to estimate $\left\|1_{|\cdot|\geq d} e^{-\frac{\lambda}{4}|\cdot|^2}\right\|_{L^1(\mathbb{R})}$, we could use $\int_r^{+\infty} e^{-s^2} ds \leq e^{-\frac{r^2}{2}} \int_r^{+\infty} e^{-\frac{s^2}{2}} ds \leq Ce^{-\frac{r^2}{2}}$ so that $\int_r^{+\infty} e^{-\lambda s^2} ds = \int_{\sqrt{\lambda} r}^{+\infty} e^{-y^2} dy \leq Ce^{-\lambda r^2/2}$. □

3.4 Global unique continuation and non characteristic hypersurfaces

3.4.1 Distance and metric

Let Ω_x, P be as in Theorem 3.1.1. Assume Ω_x connected. We are going to define the Riemannian distance related to the operator Q.

We can assume that $a_{i,j}(x)$ is symmetric without changing the operator P. The ellipticity and positivity assumption shows that for any $x \in \Omega_x$, we can define the matrix $(g_{i,j}) = (a_{i,j})^{-1}$ which is still positive.

For any $x \in \Omega_x$ and $\xi \in \mathbb{R}^n$, we define $\|\xi\|_{g(x)} = \sqrt{\sum_{i,j=1}^n g_{i,j}(x)\xi_i\xi_j}$. Moreover, if $\gamma : [0,1] \mapsto \Omega_x$ is a smooth path, we define

$$length(\gamma) = \int_0^1 \|\dot{\gamma}(t)\|_{g(\gamma(t))}\, dt.$$

This allows to define the Riemannian distance

$$dist(x_1, x_2) = \inf_{\gamma(0)=x_1;\gamma(1)=x_2} length(\gamma).$$

3.4.2 The global theorem

Theorem 3.4.1 (Global unique continuation)**.** *Let Ω_x, P be as in Theorem 3.1.1. Let x_0, $x_1 \in \Omega_x$. Let ω_0 be neighborhood of x_0 in Ω_x. Then, for any $T > dist(x_0, x_1)$, there exist $\varepsilon > 0$ and V_{x_1} one neighborhood of x_1 so that for any $u \in C^\infty(\Omega)$,*

$$\left\{ \begin{array}{rcl} Pu & = & 0 \text{ in }]-T,T[\times\Omega_x, \\ u & = & 0 \text{ in }]-T,T[\times\omega_0 \end{array} \right. \Longrightarrow u = 0 \text{ in }]-\varepsilon,\varepsilon[\times V_{x_1}. \tag{3.11}$$

Proof. According to Lemma 3.4.1 below, we can find local coordinates (w, x_n) near γ in which the path γ by $\gamma(s) = (0, s\ell_0)$ and the metric is given by the matrix $m(w, x_n) \in M_n(\mathbb{R})$ with

$$m(w,x_n) = \begin{pmatrix} m'(x_n) & 0 \\ 0 & 1 \end{pmatrix} + \mathcal{O}_{M_n(\mathbb{R})}(|w|), \qquad \text{for } w \in B_{\mathbb{R}^{n-1}}(0,\delta), \delta > 0, \tag{3.12}$$

with $m'(x_n) \in M_{n-1}(\mathbb{R})$ (uniformly) definite symmetric. With these coordinates in the space variable, and still using the straight time variable, the symbol of the wave operator is given by

$$p(t,w,x_n,\tau,\xi_w,\xi_n) = p(w,x_n,\tau,\xi_w,\xi_n) = -\tau^2 + \langle m(w,x_n)\xi,\xi\rangle, \quad \xi = (\xi_w,\xi_n), \tag{3.13}$$

where we have used τ for the dual of the time variable and ξ_w, ξ_n for the dual to $w \in B_{\mathbb{R}^{n-1}}(0,\delta)$ and $x_n \in [0, \ell_0]$.

We now aim to apply Theorem 3.1.1 and we need to construct appropriate non characteristic hypersurfaces.

Pick again t_0 with $\ell_0 < t_0 < T$. For $b < \delta$ small, to be fixed later on, we define

$$x_n = l, \quad x' = (t,w), \quad D = \left\{ (t,w) \,\middle|\, \left(\frac{w}{b}\right)^2 + \left(\frac{t}{t_0}\right)^2 \le 1 \right\}$$

$$G(t,w,\varepsilon) = \varepsilon\ell_0\psi\left(\sqrt{\left(\frac{w}{b}\right)^2 + \left(\frac{t}{t_0}\right)^2}\right), \quad \phi_\varepsilon(t,w,x_n) := G(t,w,\varepsilon) - x_n, \quad \varepsilon \in [0,1]$$

where ψ is such that

$$\begin{gathered} \psi \text{ even}, \quad \psi(\pm 1) = 0, \quad \psi(0) = 1, \\ \psi(s) \ge 0, \quad |\psi'(s)| \le \alpha, \text{ for } s \in [-1,1], \end{gathered}$$

with $1 < \alpha < \frac{t_0}{\ell_0}$. This is possible since $\frac{t_0}{\ell_0} > 1$.

Note also that the fact that ψ is even gives that $G(t,w,\varepsilon)$ is actually smooth.

Note also that the point $(t = 0, w = 0, x_n = \ell_0)$ corresponding in the local coordinates to x^1 belongs to $\{\phi_1 = 0\}$. We have

$$d\phi_\varepsilon(t,w,x_n) = \varepsilon\ell_0\left(\left(\frac{w}{b}\right)^2 + \left(\frac{t}{t_0}\right)^2\right)^{-1/2}\psi'\left(\sqrt{\left(\frac{w}{b}\right)^2 + \left(\frac{t}{t_0}\right)^2}\right)\left(\frac{tdt}{t_0^2} + \frac{wdw}{b^2}\right) - dx_n.$$

Given the form of the principal symbol of the wave operator in these coordinates (see (3.12)-(3.13)), we obtain

$$\begin{aligned} p(w,x_n,d\phi_\varepsilon(t,w,x_n)) &= -\varepsilon^2\ell_0^2\frac{t^2}{t_0^4}\left(\left(\frac{w}{b}\right)^2 + \left(\frac{t}{t_0}\right)^2\right)^{-1}|\psi'|^2 \\ &\quad + \ell_0^2\frac{\varepsilon^2}{b^4}\langle m'(x_n)w,w\rangle\left(\left(\frac{w}{b}\right)^2 + \left(\frac{t}{t_0}\right)^2\right)^{-1}|\psi'|^2 + 1 \\ &\quad + O(|w|^2)\left(1 + \frac{\varepsilon^2\ell_0^2}{b^4}|w|^2\left(\left(\frac{w}{b}\right)^2 + \left(\frac{t}{t_0}\right)^2\right)^{-1}|\psi'|^2\right), \end{aligned}$$

where $|\psi'|^2$ is taken at the point $\left(\sqrt{\left(\frac{w}{b}\right)^2+\left(\frac{t}{t_0}\right)^2}\right)$. Now, since $\alpha < \frac{t_0}{\ell_0}$ and $m'(x_n)$ is uniformly (for $x_n \in [0,\ell_0]$) definite positive, there is $\eta > 0$ so that for $|w| \leq b$ small enough, we have

$$\begin{aligned} 1 + O(|w|^2) &\geq \alpha^2 \frac{\ell_0^2}{t_0^2}\eta \\ \langle m'(x_n)w, w\rangle + O(|w|^2)|w|^2 &\geq \frac{1}{2}\langle m'(x_n)w, w\rangle \geq 0. \end{aligned}$$

Hence, there is a sufficiently small neighborhood (taking again b small enough) of the path (i.e. of $w = 0$), in which we have (for any $\varepsilon \in [0,1]$), and any $(t, w, x_n) \in \overline{D} \times [0,\ell_0]$

$$\begin{aligned} p(w, x_n, d\phi_\varepsilon(t,w,x_n)) &\geq -\frac{\varepsilon^2}{t_0^2}\ell_0^2\Big(\frac{t}{t_0}\Big)^2\left(\Big(\frac{w}{b}\Big)^2 + \Big(\frac{t}{t_0}\Big)^2\right)^{-1}|\psi'|^2 + \alpha^2\frac{\ell_0^2}{t_0^2} + \eta \\ &\geq -\frac{\ell_0^2}{t_0^2}|\psi'|^2 + \alpha^2\frac{\ell_0^2}{t_0^2} + \eta \geq \eta. \end{aligned}$$

So, the surface $\{\phi_\varepsilon = 0\}$ is noncharacteristic for any $\varepsilon \in [0,1]$ and, therefore, strictly pseudoconvex with respect to the wave operator.

Now, define $K_\varepsilon = \{x_n \leq G(t,w,\varepsilon)\} \cap \{x_n \geq 0\}$.

Consider $\varepsilon_0 = sup\{\varepsilon; u = 0 \text{ on } K_\varepsilon\}$. A continuity argument yields that that $u = 0$ on K_{ε_0}. A compactness argument on the compact set (taking into account the "corners") and the successive application of Theorem 3.1.1 gives the result. ♣ `un peu rapide...` □

Lemma 3.4.1. *Let $\gamma : [0,1] \to \Omega_x$ be a smooth path without self intersection of length ℓ_0 so that $\gamma(0) = x_0$ and $\gamma(1) = x_1$.*

Then, there are some coordinates $(w,l) \in B_{\mathbb{R}^{n-1}}(0,\varepsilon) \times [0,\ell_0]$ in an open neighborhood U near $\gamma([0,1])$ so that

- $\gamma([0,1]) = \{w = 0\} \times [0,\ell_0]$,
- *the metric g is of the form* $m(l,w) = \begin{pmatrix} 1 & 0 \\ 0 & m'(l) \end{pmatrix} + O_{M_n(\mathbb{R})}(|w|)$,

Proof. The path γ is of length ℓ_0 so, we can reparametrize it by $\gamma : [0,\ell_0] \to \Omega_x$ such that γ is unitary (that is $\|\dot\gamma(s)\|_{\gamma(s)} = 1$). Moreover, since γ does not have self intersection, there exist U a neighborhood in Ω_x of γ and a diffeomorphism ψ such that

- $\psi(U) \subset \{(x,y) \in \mathbb{R}^n \,|x \in [-\varepsilon, \ell_0 + \varepsilon], |y| \leq \varepsilon\}$,
- $\psi(\gamma(s)) = (s, 0)$,
- $\psi(U) = \{(x,y) \in \mathbb{R}^n, f_1(y) \leq x \leq f_2(y)\,|x \in [-\varepsilon, \ell_0+\varepsilon], |y| \leq \varepsilon\}$ for some smooth functions f_i locally defined

Then, we make some change of variable to diagonalize the metric on γ. By unitarity of the coordinates, the metric on γ has the form

$$m(x,0) = \begin{pmatrix} 1 & l(x) \\ {}^t l(x) & g(x) \end{pmatrix},$$

where l is a line vector and g is a positive definite matrix. We perform the change of variable $\Phi : (x,y) \mapsto (\widetilde{x}, \widetilde{y}) = (x - a_x \cdot y, y)$. In $y = 0$, we have $D\Phi(x,0) = \begin{pmatrix} 1 & -a_x \\ 0 & Id \end{pmatrix}$ with ${}^tD\Phi(x,0) =$

$\begin{pmatrix} 1 & 0 \\ -{}^t a_x & Id \end{pmatrix}$ (in particular, the change of variable is valid for small y) and $D\Phi(x,0)^{-1} = \begin{pmatrix} 1 & a_x \\ 0 & Id \end{pmatrix}$ with ${}^tD\Phi(x,0)^{-1} = \begin{pmatrix} 1 & 0 \\ {}^t a_x & Id \end{pmatrix}$. Moreover, in the new coordinates, the set in $\{\widetilde{y} = 0\}$ and the metric there is given by

$$
{}^tD\Phi(x,0)^{-1} m(x,0) D\Phi(x,0)^{-1} = \begin{pmatrix} 1 & l(x) + a(x) \\ {}^t l(x) +^t a(x) & * \end{pmatrix}
$$

So, we choose $a(x) = -l(x)$ so that in this new coordinates $m(x,0)$ is of the form

$$
m(x,0) = \begin{pmatrix} 1 & 0 \\ 0 & * \end{pmatrix}. \tag{3.14}
$$

The expected property of m is then obtained by the mean value theorem using the diagonal form (3.14) on γ. □

3.5 Approximate controllability

3.6 Further remarks

3.6.1 The general theorem

of Tataru (Robbiano-Zuily, Hörmander)

3.6.2 Quantitative estimates

boundary Carleman estimates

Appendix A

Appendix

A.1 Pseudodifferential operators

A.2 The Dirichlet problem for some second order elliptic operators

In this section, we shall consider a particular class of operators as described in Remark **??**, that is, with symbols the form $p_2(x, \xi) = Q_x(\xi)$ where Q_x is a smooth family of real quadratic forms. Assuming that the variables x_a are tangent to the boundary, and that the functions satisfy Dirichlet boundary conditions, we prove a counterpart of the local estimate of Theorem **??** for this boundary value problem. For this, the main goal to achieve is to prove a Carleman estimate adapted to this boundary value problem. All local, semiglobal and global results shall then follow.

This situation is of particular interest for the wave equation for which x_a is the time variable, which is always tangent to the boundary of cylindrical domains.

For the sake of simplicity, we shall further assume that the operator principal symbol of P is independent of the x_a variable (we would otherwise need to assume the coefficients of P to be analytic with respect to x_a). This allows to avoid some additional technicalities in the (already rather technical) proofs.

A.2.1 Some notation

Here, we shall always assume that the analytic variables are tangential to the boundary, that is

$$x = (x_a, x_b) \in \mathbb{R}^{n_a} \times \mathbb{R}^{n_b}_+, \quad \text{with } \mathbb{R}^{n_b}_+ = \mathbb{R}^{n_b - 1} \times \mathbb{R}_+, \quad \text{and } x_b = (x_b', x_b^n).$$

When the distinction between analytic and non-analytic variables is not essential, we shall split the variables according to

$$x = (x', x_n) \in \mathbb{R}^n_+ = \mathbb{R}^{n-1} \times \mathbb{R}_+, \quad \text{with } x' = (x_a, x_b') \in \mathbb{R}^{n_a + n_b - 1}, \quad \text{and } x_n = x_b^n \in \mathbb{R}^+.$$

We also denote by $\xi' \in \mathbb{R}^{n-1}$ the cotangential variables and ξ_n the conormal variable, by $D' = \frac{1}{i}(\partial_{x'})$ the associated tangential derivations and $D_n = \frac{1}{i}\partial_{x_n}$ the normal derivation.

For any $r_0 > 0$, we define

$$K_{r_0} = \left\{x \in \mathbb{R}^n_+ ; |x| \leq r_0\right\} = \overline{B}_{\mathbb{R}^n}(0, r_0) \cap \{x_n \geq 0\}. \tag{A.1}$$

We denote by $C^\infty_0(\mathbb{R}^n_+)$ the space of restrictions to $\mathbb{R}^n_+$ of functions in $C^\infty_0(\mathbb{R}^n)$, and by $C^\infty_0(K_{r_0})$ the space of functions $C^\infty_0(\mathbb{R}^n_+)$ supported in K_{r_0}. the trace of a function $f \in C^\infty_0(\mathbb{R}^n_+)$ at $x_n = 0$ is denoted by $f_{|x_n=0}$.

We denote by $(f,g)=\int_{\mathbb{R}^n_+} f\overline{g}$, $\|f\|_{0,+}^2=(f,f)$ the $L^2(\mathbb{R}^n_+)$ inner product and norm. For $k\in\mathbb{N}$, the norm $\|\cdot\|_{k,+}$ will denote the classical Sobolev norm on $\mathbb{R}^n_+$ and $\|\cdot\|_{k,+,\tau}$ the associated weighted norms, that is,

$$\|f\|_{k,+,\tau}^2=\sum_{j+|\alpha|\leq k}\tau^{2j}\,\|\partial^\alpha f\|_{0,+}^2\,,\quad \tau\geq 1.$$

We also define the tangential Sobolev norms, given by

$$|f|_{k,\tau}^2=\left\|(|D'|+\tau)^k f\right\|_{0,+}^2\sim\sum_{j+|\alpha|\leq k}\tau^{2j}\,\|\partial_{x'}^\alpha f\|_{0,+}^2\,,\quad \tau\geq 1.$$

We shall also use, for $f,g\in C_0^\infty(\mathbb{R}^n_+)$, the notation $(f,g)_0=\int_{\mathbb{R}^{n-1}} f_{|x_n=0}(x')g_{|x_n=0}(x')dx'$.

Finally, for $j\in\mathbb{N}$, we denote by $\mathcal{D}^k_\tau$, the space of *tangential* differential operators, i.e. operators of the form

$$P(x,D',\tau)=\sum_{j+|\alpha|\leq k}a_{j,\alpha}(x)\tau^j D'^\alpha,$$

and by

$$\sigma(P)=p(x,\xi',\tau)=\sum_{j+|\alpha|=k}a_{j,\alpha}(x)\tau^j\xi'^\alpha$$

their principal symbol.

A.2.2 The Carleman estimate

In this section, we state and prove the Carleman estimate of Theorem (2.4.2) asociated to the Dirichlet problem . Note that it applies also to elliptic operator, but also to wave type operators.

To prove Theorem 2.4.2, we define the conjugated operator $P_\psi=e^{\tau\psi}Pe^{-\tau\psi}=P(x,D+i\tau\psi')$.

When proving the theorem, we shall drop the index $+$ in the norms to lighten the notation; of course, all inner norms and integrals are meant on $\mathbb{R}^n_+$. We first need the following proposition.

Theorem A.2.1. *Under the assumptions of Theorem 2.4.2, there exist $C>0$, $\tau_0>0$ such that for any $\tau>\tau_0$ and $f\in C_0^\infty(K_{r_0})$, we have*

(A.2) $$\tau\|f\|_{1,\tau}^2\leq C\,\|P_\psi f\|_0^2+\tau^3|f_{|x_n=0}|_0^2+\tau|Df_{|x_n=0}|_0^2.$$

If moreover $\partial_{x_n}\psi>0$ for $(x',x_n=0)\in K_{r_0}$, then

(A.3) $$\tau\|f\|_{1,\tau}^2\leq C\,\|P_\psi f\|_0^2\,,\quad \text{for all } f\in C_0^\infty(K_{r_0}) \text{ such that } f_{|x_n=0}=0.$$

Proof. Defining $\tilde{Q}_2=\frac{1}{2}(P_\psi+P_\psi^*)$ and $\tilde{Q}_1=\frac{1}{2i\tau}(P_\psi-P_\psi^*)$, we have

$$P_\psi=\tilde{Q}_2+i\tau\tilde{Q}_1,$$

and denote by $\tilde{q}_j$ the principal symbol of $\tilde{Q}_j$, $j=1,2$. We have

(A.4) $$\left\{\begin{array}{lcl}\tilde{Q}_2&=&D_n^2+Q_2\\ \tilde{Q}_1&=&D_n\psi'_{x_n}+\psi'_{x_n}D_n+2Q_1,\end{array}\right.$$

where $Q_2\in\mathcal{D}^2_\tau$ and $Q_1\in\mathcal{D}^1_\tau$ with principal symbols

$$\begin{array}{lcl}q_2&=&-\tau^2(\psi'_{x_n})^2+r(x,\xi')-\tau^2 r(x,\psi'_{x'})\\ q_1&=&\tilde{r}(x_b,\xi',\psi'_{x'}),\end{array}$$

where $\tilde{r}$ is the bilinear form associated with the quadratic form r. Note that, even if it does not appear in the notation, all these operators depend upon the parameter τ.

With this notation, we hence have $p_\psi = \tilde{q}_2^0 + i\tau\tilde{q}_1^0$, so that $\frac{1}{i\tau}\{\overline{p}_\psi, p_\psi\} = 2\{\tilde{q}_2^0, \tilde{q}_1^0\}$. Assumptions (**??**) and (**??**) then translate respectively into

$$\{\tilde{q}_2^0, \tilde{q}_1^0\}(x,\xi) > 0, \quad \text{if } p(x,\xi) = 0, \quad x \in K_{r_0}, \tau = 0; \tag{A.5}$$

$$\{\tilde{q}_2^0, \tilde{q}_1^0\}(x,\xi) > 0, \quad \text{if } p_\psi(x,\xi) = 0, \quad x \in K_{r_0}, \tau > 0, \tag{A.6}$$

where the second assertion is a direct consequence of (**??**), and the first one follows from (**??**) together with the fact that, using that p is real, we have

$$\lim_{\tau\to 0^+} \frac{1}{i\tau}\{\overline{p}_\psi, p_\psi\} = \frac{\partial}{\partial\tau}\frac{1}{i}\{\overline{p}_\psi, p_\psi\}\Big|_{\tau=0} = 2\,\{p, \{p, \psi\}\}\,.$$

Next, we have the integration by parts formulæ:

$$\left\{\begin{array}{lcl} (g, \tilde{Q}_2 f) & = & (\tilde{Q}_2 g, f) - i\left[(g, D_n f)_0 + (D_n g, f)_0)_0\right], \\ (g, \tilde{Q}_1 f) & = & (\tilde{Q}_1 g, f) - 2i\left(\psi'_{x_n} g, f\right)_0. \end{array}\right. \tag{A.7}$$

So, we have for $f \in C_0^\infty(K_{r_0})$

$$\|P_\psi f\|_0 = \left\|\tilde{Q}_2 f\right\|_0^2 + \tau^2\left\|\tilde{Q}_1 f\right\|_0^2 + i\tau\left[\left(\tilde{Q}_1 f, \tilde{Q}_2 f\right) - \left(\tilde{Q}_2 f, \tilde{Q}_1 f\right)\right]. \tag{A.8}$$

So, we get, using the integration by parts formulæ (A.7)

$$\|P_\psi f\|_0 = \left\|\tilde{Q}_2 f\right\|_0^2 + \tau^2\left\|\tilde{Q}_1 f\right\|_0^2 + i\tau\left([\tilde{Q}_2, \tilde{Q}_1]f, f\right) + \tau\mathcal{B}(f), \tag{A.9}$$

with the boundary term

$$\begin{aligned} \mathcal{B}(f) &= \left[(\tilde{Q}_1 f, D_n f)_0 + (D_n\tilde{Q}_1 f, f)_0\right] - 2\left(\psi'_{x_n}\tilde{Q}_2 f, f\right)_0 \\ &= 2(\psi'_{x_n} D_n f, D_n f)_0 + (M_1 f, D_n f)_0 + (M_1' D_n f, f)_0 + (M_2 f, f)_0, \end{aligned} \tag{A.10}$$

for some tangential operator M_1 of order 1 (in ξ', τ) (note that terms of order two in D_n cancel).

Now that we have made the exact computations, we will make some estimates on the symbols of the interior part of the commutator. The idea is to tranfer the positivity assumption of the full symbol to some positivity of a tangential symbol, which will then allow to apply the tangential Gårding.

The first step is to perform a factorisation of $[\tilde{Q}_2, \tilde{Q}_1]$ with respect to $\tilde{Q}_1$ and $\tilde{Q}_2$ to have a tangential reminder. Since $[\tilde{Q}_2, \tilde{Q}_1]$ is of order 2, it can be written $i[\tilde{Q}_2, \tilde{Q}_1] = C_2 + C_1 D_n + C_0 D_n^2$ where $C_i \in \mathcal{D}^i_\tau$. But using (A.4), and $\psi'_{x_n} \neq 0$ on K_{r_0}, we can replace $D_n = \frac{1}{2\psi'_{x_n}}\tilde{Q}_1 + \mathcal{D}^1_\tau$ and $D_n^2 = \tilde{Q}_2 - Q_2$. So, in particular, we can write

$$i[\tilde{Q}_2, \tilde{Q}_1] = B_0\tilde{Q}_2 + B_1\tilde{Q}_1 + B_2. \tag{A.11}$$

where $B_i \in \mathcal{D}^i_\tau$ with real symbol b_i. Now, we need to

- use the assumption to get some positivity of the symbol $\{\overline{p}_\psi, p_\psi\}$, this is Lemma A.2.1;
- transfer this information to a tangential information on the symbol, this is Lemma A.2.2.

Lemma A.2.1. *There exist $C_1, C_2 > 0$ such that for all $(x,\xi) \in K_{r_0} \times \mathbb{R}^n$ and $\tau > 0$, we have*

$$(|\xi|^2 + \tau^2) \le C_1 \{\tilde{q}_2^0, \tilde{q}_1^0\}(x,\xi) + C_2 \left[\frac{|p_\psi(x,\xi)|^2}{|\xi|^2 + \tau^2} \right].$$

Proof. All the terms are homogeneous of order 2 in (ξ,τ) and continuous on the compact $(x,\xi,\tau) \in K_{r_0} \times \{(\xi,\tau) \in \mathbb{R}^n \times \mathbb{R}^+, |\xi|^2 + \tau^2 = 1\}$. Thus, on this set, the result is a consequence of (A.5), (A.6) and Lemma 2.2.3 applied to $f = \frac{|p_\psi(x,\xi)|^2}{|\xi|^2+\tau^2} \ge 0$, $g = \{\tilde{q}_2^0, \tilde{q}_1^0\}$ and $h = 0$. The result on the whole $K_{r_0} \times \mathbb{R}^n \times \mathbb{R}^+$ follows by homogeneity. □

Now, we set

$$\mu(x,\xi') = (q_1)^2 + (\psi'_{x_n})^2 q_2.$$

The symbol $\mu(x,\xi')$ satisfies the property that $\mu(x,\xi') = 0$ if and only if there exists ξ_n real such that $p_\psi(x,\xi',\xi_n) = 0$. This is easily seen by noticing that the zero of q_1 can only be with $\xi_n = -\frac{q_1}{\psi'_{x_n}}$.

Notice also that $\mu(x,\xi')$ is a tangential symbol of order 2.

Lemma A.2.2. *There exist $C_1, C_2 > 0$ such that for all $(x,\xi') \in K_{r_0} \times \mathbb{R}^{n-1}$ and $\tau > 0$, we have*

$$(|\xi'|^2 + \tau^2) \le C_1 b_2 + C_2 \left[\frac{[\mu(x,\xi')]^2}{|\xi'|^2 + \tau^2} \right]. \tag{A.12}$$

Proof. Note first that for any (x,ξ',ξ_n) with $\xi_n = -\frac{q_1(x,\xi')}{\psi'_{x_n}}$, we have $\tilde{q}_1(x,\xi',\xi_n) = 0$ and

$$p_\psi(x,\xi',\xi_n) = \tilde{q}_2(x,\xi',\xi_n) = (\psi'_{x_n})^{-2}\mu(x,\xi').$$

Now, assume $\mu(x,\xi') = 0$ and $\xi_a = 0$. Setting $\xi_n = -\frac{q_1(x,\xi')}{\psi'_{x_n}}$, we have $p_\psi(x,\xi',\xi_n) = 0$. Using Lemma **??**, we have $\{\tilde{q}_2, \tilde{q}_1\}(x,\xi',\xi_n) > 0$. According to the definition of B_2 in (A.11), we have $b_2(x,\xi',\xi_n) > 0$. As a consequence, we have

$$\mu(x,\xi') = 0 \Longrightarrow b_2(x,\xi',\xi_n) > 0.$$

Moreover, all terms in (A.12) are homogeneous of order 2 in the variables (ξ',τ) and continuous on $(\xi',\tau) \ne (0,0)$. Hence, applying Lemma 2.2.3 below on the compact set $K_{r_0} \times \{(\xi',\tau) \in \mathbb{R}^{n-1} \times \mathbb{R}^+, |\xi'|^2 + \tau^2 = 1\}$ yields (A.12) on that set. The conclusion follows by homogeneity. □

Taking the real part of (A.9) and using (A.11), we obtain

$$\|P_\psi f\|_0 - \tau \operatorname{Re}(\mathcal{B}(f)) = \left\|\tilde{Q}_2 f\right\|_0^2 + \tau^2 \left\|\tilde{Q}_1 f\right\|_0^2 + \tau \operatorname{Re}(B_2 f, f) + \tau \operatorname{Re}\left((B_0\tilde{Q}_2 + B_1\tilde{Q}_1)f, f\right). \tag{A.13}$$

Concerning the remainder term, we have

$$\begin{aligned} \tau \left| \operatorname{Re}\left((B_0\tilde{Q}_2 + B_1\tilde{Q}_1)f, f\right)\right| &\le \tau \|f\|_0 \|\tilde{Q}_2 f\|_0 + \tau |f|_1 \|\tilde{Q}_1 f\|_0 \\ &\le \tau^{-1/2}\left(\tau|f|_{1,\tau}^2 + \|\tilde{Q}_2 f\|_0^2 + \tau^2 \|\tilde{Q}_1 f\|_0^2\right). \end{aligned} \tag{A.14}$$

Defining now

$$\Sigma = (Q_1)^2 + (\psi'_{x_n})^2 Q_2,$$

with principal symbol μ, and for an operator G with principal symbol $\frac{\mu^\varepsilon(x,\xi')}{|\xi'|^2+\tau^2}$, the tangential Gårding inequality (that means with some derivatives only in the variable x'), in which symbols are allowed to depend smoothly upon the variable x_n yields, for τ sufficiently large,

$$|f|^2_{1,\tau} \leq C\,\mathrm{Re}\,(B_2 f, f) + \mathrm{Re}\,(\Sigma f, Gf)\,. \tag{A.15}$$

Writing $\psi'_{x_n} D_n = \frac{1}{2}(\tilde{Q}_1 - [D_n, \psi'_{x_n}]) - Q_1$ (where ψ'_{x_n} does not vanish), this allows to estimate the full norm $\|f\|_{1,\tau}$ according to

$$\|f\|_{1,\tau} \leq C(\|\tilde{Q}_1 f\|_0 + |f|_{1,\tau}). \tag{A.16}$$

Recalling the definitions of $\tilde{Q}_i$ in (A.4), we also have

$$\begin{aligned}
\Sigma &= \left(\frac{1}{2}(\tilde{Q}_1 - [D_n, \psi'_{x_n}]) - \psi'_{x_n} D_n\right)^2 \\
&\quad + (\psi'_{x_n})^2 \left(\tilde{Q}_2 - D_n^2\right) \\
&= \left(\frac{1}{2}(\tilde{Q}_1 - [D_n, \psi'_{x_n}]) - \psi'_{x_n} D_n\right) \frac{1}{2}(\tilde{Q}_1 - [D_n, \psi'_{x_n}]) \\
&\quad + (\psi'_{x_n})^2 \left(\tilde{Q}_2\right),
\end{aligned} \tag{A.17}$$

and hence

$$\Sigma \in (\psi'_{x_n})^2 \tilde{Q}_2 - \frac{1}{2}\psi'_{x_n} D_n \tilde{Q}_1 + \mathcal{D}^1_\tau \tilde{Q}_1 + \mathcal{D}^1_\tau + \mathcal{D}^0_\tau D_n.$$

We now want to estimate the term $\mathrm{Re}\,(\Sigma f, Gf)$ in (A.15). For this, integrating by parts in the tangential direction x_a, we have

$$\left|\left(\psi''_{x_n,x_a}\left((\psi'_{x_n})^2 D_n + Q_1 \psi'_{x_n}; D_a\right) f, Gf\right)\right| \leq C\|\left\langle D_a\right\rangle f\|\|f\|_{1,\tau}.$$

This yields

$$\begin{aligned}
|\,(\Sigma f, Gf)\,| &\leq C\|\tilde{Q}_2 f\|_0 \|f\|_0 + \left|\left(\frac{1}{2i}\psi'_{x_n}\tilde{Q}_1 f, Gf\right)_0\right| \\
&\quad + \|\tilde{Q}_1 f\|_0 \|f\|_{1,\tau} + \|f\|_0\|f\|_{1,\tau} + C\|\left\langle D_a\right\rangle f\|\|f\|_{1,\tau} \\
&\leq \left|\left(\frac{1}{2i}\psi'_{x_n}\tilde{Q}_1 f, Gf\right)_0\right| + C\|f\|_{1,\tau}\left(\tau^{-1}\|\tilde{Q}_2 f\|_0 + + \|\tilde{Q}_1 f\|_0 + \tau^{-1}\|f\|_{1,\tau} + \|D_a f\|_0\right)
\end{aligned} \tag{A.18}$$

According to (A.15) and (A.16) and (A.18), this now implies

$$\|f\|^2_{1,\tau} \lesssim \mathrm{Re}\,(B_2 f, f) + \|\tilde{Q}_1 f\|_0^2 + \left|\left(\frac{1}{2i}\psi'_{x_n}\tilde{Q}_1 f, Gf\right)_0\right| + \tau^{-2}\|\tilde{Q}_2 f\|_0^2 + \|D_a f\|_0^2.$$

Coming back to (A.13), we obtain, for τ large enough,

$$\begin{aligned}
\tau\|f\|^2_{1,\tau} &\lesssim \|P_\psi f\|_0^2 - \tau\,\mathrm{Re}\,(\mathcal{B}(f)) - \left\|\tilde{Q}_2 f\right\|_0^2 - \tau^2\left\|\tilde{Q}_1 f\right\|_0^2 + \tau\left|\left(\frac{1}{2i}\psi'_{x_n}\tilde{Q}_1 f, Gf\right)_0\right| \\
&\lesssim \|P_\psi f\|_0^2 - \tau\,\mathrm{Re}\,(\mathcal{B}(f)) + \tau\left|\left(\frac{1}{2i}\psi'_{x_n}\tilde{Q}_1 f, Gf\right)_0\right|.
\end{aligned}$$

Recalling te definition of $\tilde{Q}_1$, we have $\psi'_{x_n}\tilde{Q}_1 = D_n + G_1$, where $G_1 \in \mathcal{D}^1_\tau$ is a differential operator of order 1 (in (τ, D')), we finally have

$$\tau\|f\|^2_{1,\tau} \lesssim \|P_\psi f\|_0^2 - \tau\,\mathrm{Re}\,(\mathcal{B}(f)) + \tau\,|(D_n f + G_1 f, Gf)_0|\,, \tag{A.19}$$

where G a tangential pseudodifferential operator of order zero, Recalling the form of $\mathcal{B}(f)$ in (A.10) gives the bound $|\mathcal{B}(f)| \leq \tau^2 |f_{|x_n=0}|_0^2 + |Df_{|x_n=0}|_0^2$, which concludes the proof of (A.2).

Now if $f_{|x_n=0} = 0$, all tangential derivatives vanish. With (A.19) and the form of $\mathcal{B}(f)$ in (A.10), this yields

$$\tau \|f\|_{1,\tau}^2 \lesssim \|P_\psi f\|_0^2 - 2\tau(\psi'_{x_n} D_n f, D_n f)_0,$$

which proves (A.3) since $\psi'_{x_n} > 0$ for $(x', x_n = 0) \in K$. This concludes the proof of Proposition A.2.1.

□

A.3 Link between bicharacteristic flow and geodesics

A.4 Useful computation for symbols of order 2

Many computation can be simplified for symbols homogeneous of order 2 using the language of quadratic forms. Indeed, we have the following properties.

Lemma A.4.1. *Let $p(x,\xi) = \sum_{k,l} a_{k,l}(x)\xi_i\xi_j$ be a symbol of order 2. For any x, denote R_x the associated quadratic form $R_x(\xi, \widetilde{\xi}) = \sum_{k,l} \frac{a_{k,l}(x)+a_{l,k}(x)}{2} \xi_k \widetilde{\xi}_l$.*

Then, we have

$$\{p, \Phi\} = 2R_x(\xi, \nabla\Phi) \tag{A.20}$$

$$p_\Phi(x,\xi,\tau) = p(x, \xi + i\tau\nabla\Phi) = p(x,\xi) - \tau^2 p(x, \nabla\Phi) + i\tau\{p, \Phi\}(x,\xi) \tag{A.21}$$

$$\{p_\Phi, \Phi\} = 2R(\xi, \nabla\Phi) + 2i\tau p(x, \nabla\Phi). \tag{A.22}$$

Moreover, assume f only depends on x, then

$$\{f, \{p, \Phi\}\} = -\{p, f\}(x, \nabla\Phi) \tag{A.23}$$

Proof. Note that we check that $p(x,\xi) = R_x(\xi,\xi)$.

(A.20), (A.21) and (A.22) are linear in $a_{k,l}$, so it is enough to prove it for $p(x,\xi) = a(x)\xi_k\xi_l$ and $R_x(\xi, \widetilde{\xi}) = \frac{a(x)}{2}\left[\xi_k\widetilde{\xi}_l + \xi_l\widetilde{\xi}_k\right]$.

$$\{p, \Phi\} = \nabla_\xi(a(x)\xi_k\xi_l) \cdot \nabla\Phi = a(x)\left[\xi_k(\partial_l\Phi) + \xi_l(\partial_k\Phi)\right] = 2R_x(\xi, \nabla\Phi)$$

This gives (A.20).

$$\begin{aligned} p_\Phi(x,\xi,\tau) &= p(x, \xi + i\tau\nabla\Phi) = a(x)(\xi_k + i\tau\partial_k\Phi)(\xi_l + i\tau\partial_l\Phi) \\ &= a(x)\left[\xi_k\xi_l - \tau^2(\partial_k\Phi)(\partial_l\Phi) + i\tau(\partial_k\Phi)\xi_l + i\tau(\partial_l\Phi)\xi_k\right] \\ &= p(x,\xi) - \tau^2 p(x,\nabla\Phi) + 2i\tau R_x(\xi, \nabla\Phi). \end{aligned}$$

This gives (A.21). For (A.22), using (A.20), we get

$$\begin{aligned} \{p_\Phi, \Phi\} &= (\nabla_\xi p)(x, \xi + i\tau\nabla\Phi) \cdot \nabla\Phi = \{p, \Phi\}(x, \xi + i\tau\nabla\Phi) \\ &= 2R_x(\xi + i\tau\nabla\Phi, \nabla\Phi) = 2R_x(\xi, \nabla\Phi) + 2i\tau R_x(\nabla\Phi, \nabla\Phi) = 2R(\xi, \nabla\Phi) + 2i\tau p(x, \nabla\Phi). \end{aligned}$$

Finally,

$$\{f, \{p, \Phi\}\} = \{f, a(x)\left[\xi_k(\partial_l\Phi) + \xi_l(\partial_l\Phi)\right]\} = -a(x)\left[(\partial_k f)(\partial_l\Phi) + (\partial_l f)(\partial_l\Phi)\right] = -2R_x(\nabla f, \nabla\Phi) = -\{p, f\}(x, \nabla\Phi)$$

□

Lemma A.4.2. *If p is homogeneous of order 2, real valued*

$$\frac{1}{i\tau}\left\{\overline{p_\Phi}, p_\Phi\right\} = 2\left\{p,\{p,\Phi\}\right\} + 2\tau^2\left\{p,\{p,\Phi\}\right\}(x,\nabla\Phi)$$

Proof. Using (A.22) and the easy formula $\{a-b,a+b\} = 2\{a,b\}$, we have

$$\begin{aligned}\frac{1}{i\tau}\left\{\overline{p_\Phi}, p_\Phi\right\} &= 2\left\{p-\tau^2 p(x,\nabla\Phi),\{p,\Phi\}\right\} \\ &= 2\left\{p,\{p,\Phi\}\right\} - 2\tau^2\left\{p(x,\nabla\Phi),\{p,\Phi\}\right\}\end{aligned}$$

Next, using Lemma A.4.3 below,

$$\frac{1}{i\tau}\left\{\overline{p_\Phi}, p_\Phi\right\} = 2\left\{p,\{p,\Phi\}\right\} + 2\tau^2\left\{p,\{p,\Phi\}\right\}(x,\nabla\Phi)$$

□

Lemma A.4.3. *If p is of homogeneous order 2*

$$\left\{p(x,\nabla\Phi),\{p,\Phi\}\right\} = -\left\{p,\{p,\Phi\}\right\}(x,\nabla\Phi).$$

Proof. Note first that we have the general formula (true for any order), see Lemma below

$$\{p(x,\nabla\Phi),q\}(x,\nabla\Phi) + \{p,q(x,\nabla\Phi)\}(x,\nabla\Phi) = \{p,q\}(x,\nabla\Phi\}.$$

With $q = \{p,\Phi\}$, it gives

$$\left\{p(x,\nabla\Phi),\{p,\Phi\}\right\} + \left\{p,\{p,\Phi\}(x,\nabla\Phi)\right\}(x,\nabla\Phi) = \left\{p,\{p,\Phi\}\right\}(x,\nabla\Phi). \tag{A.24}$$

But for quadratic operators, (A.20) gives $\{p,\Phi\}(x,\nabla\Phi) = 2R_x(\nabla\Phi,\nabla\Phi) = 2p(x,\nabla\Phi)$, so (A.23) gives

$$\begin{aligned}\left\{p,\{p,\Phi\}(x,\nabla\Phi)\right\}(x,\nabla\Phi) &= 2\{p,p(x,\nabla\Phi)\}(x,\nabla\Phi) \\ &= -2\{p(x,\nabla\Phi),\{p,\Phi\}\}\end{aligned}$$

So, when we insert it into (A.24), we obtain

$$\{p(x,\nabla\Phi),\{p,\Phi\}\} - 2\{p(x,\nabla\Phi),\{p,\Phi\}\} = \left\{p,\{p,\Phi\}\right\}(x,\nabla\Phi).$$

□

Lemma A.4.4. *Let p, q some symbols, $\Phi(x)$ function. Then*

$$\{p(x,\nabla\Phi),q\}(x,\nabla\Phi) + \{p,q(x,\nabla\Phi)\}(x,\nabla\Phi) = \{p,q\}(x,\nabla\Phi\} \tag{A.25}$$

Proof.

$$\{p,q\}(x,\nabla\Phi) = (\nabla_\xi p)(x,\nabla\Phi)\cdot(\nabla_x q)(x,\nabla\Phi) - (\nabla_x p)(x,\nabla\Phi)\cdot(\nabla_\xi q)(x,\nabla\Phi)$$

while

$$\{p(x,\nabla\Phi),q\}(x,\xi) = -(\nabla_x p)(x,\nabla\Phi)\cdot(\nabla_\xi q)(x,\xi) - Hess\Phi\left[(\nabla_\xi p)(x,\nabla\Phi);(\nabla_\xi q)(x,\xi)\right]$$

So applied with $\xi = \nabla\Phi$

$$\{p(x,\nabla\Phi),q\}(x,\nabla\Phi) = -(\nabla_x p)(x,\nabla\Phi)\cdot(\nabla_\xi q)(x,\nabla\Phi) - Hess\Phi\left[(\nabla_\xi p)(x,\nabla\Phi);(\nabla_\xi q)(x,\nabla\Phi)\right]$$

By symmetry

$$\{p, q(x, \nabla\Phi)\}(x, \nabla\Phi) = (\nabla_x q)(x, \nabla\Phi) \cdot (\nabla_\xi p)(x, \nabla\Phi) + Hess\Phi\left[(\nabla_\xi p)(x, \nabla\Phi); (\nabla_\xi q)(x, \nabla\Phi)\right]$$

That is

$$\begin{aligned}\{p(x, \nabla\Phi), q\}(x, \nabla\Phi) + \{p, q(x, \nabla\Phi)\}(x, \nabla\Phi) &= -(\nabla_x p)(x, \nabla\Phi) \cdot (\nabla_\xi q)(x, \nabla\Phi) + (\nabla_x q)(x, \nabla\Phi) \cdot (\nabla_\xi p)(x, \nabla\Phi) \\ &= \{p, q\}(x, \nabla\Phi\}\end{aligned}$$

□

Appendix B

Correction of (some) exercices

B.1 Feuille 1

B.1.1 Exercice 1

Let $A = a_{\alpha,\beta}(x)D^\alpha\tau^\beta$. $B = b_{\alpha',\beta'}(x)D^{\alpha'}\tau^{\beta'}$ of respective order m_1 and m_2 and full symbol a and b.

$$A \circ Bu = a_{\alpha,\beta}(x)D^\alpha\tau^\beta \left[b_{\alpha',\beta'}(x)D^{\alpha'}\tau^{\beta'}u\right] = a_{\alpha,\beta}(x)\tau^{\beta+\beta'}D^\alpha \left[b_{\alpha',\beta'}(x)D^{\alpha'}u\right]$$

Using Leibniz formula

$$\partial_\alpha(fg) = \sum_{\gamma+\delta=\alpha} \binom{\alpha}{\gamma}(\partial_\gamma f)(\partial_\delta g),$$

we get

$$D^\alpha \left[b_{\alpha',\beta'}(x)D^{\alpha'}u\right] = \frac{1}{i^{|\alpha|}} \sum_{\gamma+\delta=\alpha} \binom{\alpha}{\gamma}(\partial_\gamma b_{\alpha',\beta'})(\partial_\delta D^{\alpha'}u)$$

So, we get

$$A \circ Bu = \frac{1}{i^{|\alpha|}} \sum_{\gamma+\delta=\alpha} a_{\alpha,\beta}(x)\tau^{\beta+\beta'}\binom{\alpha}{\gamma}(\partial_\gamma b_{\alpha',\beta'})(\partial_\delta D^{\alpha'}u)$$

Each term in the sum is a differential operator of order $\beta+\beta'+|\delta|+|\alpha'| \leq \beta+\beta'+|\alpha|+|\alpha'| = m_1+m_2$.

This maximum is reached only for the term $\delta = \alpha$, $\gamma = 0$, $\binom{\alpha}{\gamma} = 1$ where we have the term

$$\frac{1}{i^{|\alpha|}}a_{\alpha,\beta}(x)\tau^{\beta+\beta'}b_{\alpha',\beta'}(x)(\partial_\alpha D^{\alpha'}u) = a_{\alpha,\beta}(x)\tau^{\beta+\beta'}b_{\alpha',\beta'}(x)(D^\alpha D^{\alpha'}u) = (ab)(x, D, \tau).$$

Let us now see the terms of order $m_1 + m_2 - 1$. They are so that $\beta + \beta' + |\delta| + |\alpha'| = m_1 + m_2 - 1$, that is $|\delta| = m_1 - 1$ and $|\gamma| = 1$. Moreover, $\gamma = (1, 0, 0 \cdots, 0)$ or $\gamma = (0, 1, 0 \cdots, 0)$, etc... We denote these vectors e_j. The sum is amongst terms so that $\alpha_j \geq 1$. In each of these cases, $\binom{\alpha}{e_j} = \binom{\alpha_1}{0}\cdots\binom{\alpha_j}{1}\cdots\binom{\alpha_n}{0} = \alpha_j$.

$$\begin{aligned} &\frac{1}{i^{|\alpha|}} \sum_{j=1,\alpha_j\geq 1}^{n} a_{\alpha,\beta}(x)\tau^{\beta+\beta'}\alpha_j(\partial_j b_{\alpha',\beta'})(\partial_{\alpha-e_j}D^{\alpha'}u) \\ = \;&\frac{1}{i} \sum_{j=1,\alpha_j\geq 1}^{n} \alpha_j a_{\alpha,\beta}(x)\tau^{\beta+\beta'}(\partial_j b_{\alpha',\beta'})(D^{\alpha-e_j}D^{\alpha'}u). \end{aligned}$$

Its symbol is

$$\frac{1}{i}\sum_{j=1,\alpha_j\geq 1}^{n} a_{\alpha,\beta}(x)\tau^{\beta+\beta'}(\partial_j b_{\alpha',\beta'})\alpha_j\xi^{\alpha-e_j}\xi^{\alpha'}.$$

We ony recognize $\alpha_j\xi^{\alpha-e_j}=\partial_{\xi_j}\xi^{\alpha}$. And the formula is still true and equal to zero if $\alpha_j=0$. So, the term of order m_1+m_2-1 is therefore

$$\frac{1}{i}\sum_{j=1}^{n} a_{\alpha,\beta}(x)\tau^{\beta+\beta'}(\partial_j b_{\alpha',\beta'})(\partial_{\xi_j}\xi^{\alpha})\xi^{\alpha'}=\frac{1}{i}\sum_{j=1}^{n}(\partial_{\xi_j}a)(\partial_{x_j}b)$$

.

Now, take $A=\sum_{|\alpha|+\beta\leq m_1} a_{\alpha,\beta}(x)D^{\alpha}\tau^{\beta}$ and $B=b_{\alpha',\beta'}(x)D^{\alpha'}\tau^{\beta'}$. Decompose $A=a_{m_1}(x,D,\tau)+a_{m_1-1}(x,D,\tau)+r(x,D,\tau)$ with $a_{m_1}(x,D,\tau)$ homogeneous of order m_1, $a_{m_1-1}(x,D,\tau)$ homogeneous of order m_1-1 and $r(x,D,\tau)$ of order at most m_1-2.

The previous calculation shows

$$\begin{aligned} A\circ B &= a_{m_1}(x,D,\tau)\circ B+a_{m_1-1}(x,D,\tau)\circ B+r(x,D,\tau)\circ B\\ &= (a_{m_1}b)(x,D,\tau)+\frac{1}{i}\sum_{j=1}^{n}\left[(\partial_{\xi_j}a_{m_1})(\partial_{x_j}b)\right](x,D,\tau)+(a_{m_1-1}b)(x,D,\tau)+r(x,D,\tau)\circ B\\ &= (ab)(x,D,\tau)+\frac{1}{i}\sum_{j=1}^{n}\left[(\partial_{\xi_j}a_{m_1})(\partial_{x_j}b)\right](x,D,\tau)-(rb)(x,D,\tau)+r(x,D,\tau)\circ B \end{aligned}$$

where a and b are the full symbol of A and B (actually the coefficients greater than m_1-1 and m_1-1 are enough). So, we can write the formula in this case

$$A\circ B = (ab)(x,D,\tau)+\frac{1}{i}\sum_{j=1}^{n}\left[(\partial_{\xi_j}a_{m_1})(\partial_{x_j}b)\right](x,D,\tau)+C(x,D,\tau) \tag{B.1}$$

where C is of order at most m_1+m_2-2.

Let us now finally get to the general case, take $B=b_{m_2}(x,D,\tau)+b_{m_2-1}(x,D,\tau)+s(x,D,\tau)$ with $b_{m_1}(x,D,\tau)$ homogeneous of order m_2, $b_{m_2-1}(x,D,\tau)$ homogeneous of order m_2-1 and $s(x,D,\tau)$ of order at most m_2-2. Applying Formula (B.1) to B equal to $b_{m_2}(x,D,\tau)$ and $b_{m_2-1}(x,D,\tau)$, we get

$$A\circ B = (ab_{m_2})(x,D,\tau)+\frac{1}{i}\sum_{j=1}^{n}\left[(\partial_{\xi_j}a_{m_1})(\partial_{x_j}b_{m_2})\right](x,D,\tau)+C_1(x,D,\tau) \tag{B.2}$$

$$+(ab_{m_2-1})(x,D,\tau)+\frac{1}{i}\sum_{j=1}^{n}\left[(\partial_{\xi_j}a_{m_1})(\partial_{x_j}b_{m_2-1})\right](x,D,\tau)+C_2(x,D,\tau) \tag{B.3}$$

where C_1 is of order at most m_2-2 and C_2 m_2-3.

In particular, since $(ab_{m_2})(x,D,\tau)+(ab_{m_2-1})(x,D,\tau)=(ab)(x,D,\tau)+C_3(x,D,\tau)$ where C_3 is of order at most m_1+m_2-2 and $(\partial_{\xi_j}a_{m_1})(\partial_{x_j}b_{m_2-1})$ is of order at most m_1+m_2-2, we have the equivalent of Formula (B.1) in the general case. This also proves Proposition 1.2.2.

Note that it means that

- the symbol of order m_1+m_2 is $a_{m_1}b_{m_2}$.

- the symbol of order m_1+m_2-1 is

$$a_{m_1}b_{m_2-1}+a_{m_1-1}b_{m_2-1}+\frac{1}{i}\sum_{j=1}^{n}\left[(\partial_{\xi_j}a_{m_1})(\partial_{x_j}b_{m_2-1})\right]$$

This directly gives that $[A,B]$ is of order at most m_1+m_2-1 with principal symbol of order m_1+m_2-1

$$\frac{1}{i}\sum_{j=1}^{n}\left[(\partial_{\xi_j}a_{m_1})(\partial_{x_j}b_{m_2})\right]-\frac{1}{i}\sum_{j=1}^{n}\left[(\partial_{x_j}a_{m_1})(\partial_{\xi_j}b_{m_2})\right]=\frac{1}{i}\left\{a_{m_1},b_{m_2}\right\}.$$

B.2 Feuille 4

B.2.1 Exercice 1

1. We denote $f(s)=e^{-s^2}$ differentiate

$$\frac{d\widehat{f}(\xi)}{d\xi}=\frac{d}{d\xi}\int_{\mathbb{R}^d}e^{-is\xi}e^{-s^2}ds=-i\int_{\mathbb{R}^d}se^{-is\xi}e^{-s^2}ds=\frac{i}{2}\int_{\mathbb{R}^d}e^{-is\xi}\partial_s(e^{-s^2})ds=-\frac{\xi}{2}\int_{\mathbb{R}^d}e^{-is\xi}e^{-s^2}ds=-\frac{\xi}{2}\widehat{f}(\xi).$$

So, this equation can be explicitely solved $\widehat{f}(\xi)=\widehat{f}(0)e^{-\frac{\xi^2}{4}}$. It is still a Gaussian, but we have to find the normalization constant.

To compute $\widehat{f}(0)=\int_{\mathbb{R}}e^{-s^2}ds$, notice that $\left(\int_{\mathbb{R}}e^{-s^2}ds\right)^2=\int_{\mathbb{R}^2}e^{-(x^2+y^2)}dxdy=\int_{r>0}e^{-r^2}(2\pi r)dr=-\pi\int_{r>0}\partial_r(e^{-r^2})dr=\pi$.

So, $\int_{\mathbb{R}}e^{-s^2}ds=\sqrt{\pi}$ and $\widehat{f}(\xi)=\sqrt{\pi}e^{-\frac{\xi^2}{4}}$.

2. In higher dimension, we compute

$$\int_{\mathbb{R}^n}e^{-ix\cdot\xi}e^{-|x|^2}=\int_{\mathbb{R}_{x_1}}\cdots\int_{\mathbb{R}_{x_n}}e^{-ix_1\xi_1}\cdots e^{-ix_d\xi_d}e^{-x_1^2}\cdots e^{-x_n^2}=\widehat{f}(\xi_1)\cdots\widehat{f}(\xi_n)=\pi^{n/2}e^{-\frac{|\xi|^2}{4}}.$$

3. Now, we want to compute $\widehat{g_\lambda}(\xi)$ where $g_\lambda=e^{-\frac{|x|^2}{\lambda}}$. By scaling, we have

$$\widehat{g_\lambda}(\xi)=\int_{\mathbb{R}^n}e^{-ix\cdot\xi}e^{-\frac{|x|^2}{\lambda}}dx=\lambda^{n/2}\int_{\mathbb{R}^n}e^{-i\sqrt{\lambda}y\cdot\xi}e^{-|y|^2}dy=\lambda^{d/2}g_1(\sqrt{\lambda}\xi)=(\pi\lambda)^{n/2}e^{-\lambda\frac{|\xi|^2}{4}}.$$

That is

$$\widehat{e^{-\frac{|x|^2}{\lambda}}}(\xi)=(\pi\lambda)^{n/2}e^{-\lambda\frac{|\xi|^2}{4}}\tag{B.4}$$

4. Now, we want to give a convolution formulation for S_λ. We have for $u\in\mathcal{S}(\mathbb{R}^n)$, $\widehat{S_\lambda u}(\xi)=e^{-\frac{|\xi|^2}{\lambda}}\widehat{u}(\xi)$. So, using (1.9)

$$S_\lambda u\;=\;\mathcal{F}^{-1}\left(e^{-\frac{|\xi|^2}{\lambda}}\widehat{u}\right)=\mathcal{F}^{-1}\left(e^{-\frac{|\xi|^2}{\lambda}}\right)*u=f_\lambda*u$$

with $f_\lambda=\mathcal{F}^{-1}\left(e^{-\frac{|\xi|^2}{\lambda}}\right)=\frac{1}{(2\pi)^n}\widehat{e^{-\frac{|\cdot|^2}{\lambda}}}=\frac{(\pi\lambda)^{n/2}}{(2\pi)^n}e^{-\lambda\frac{|x|^2}{4}}=\left(\frac{\lambda}{4\pi}\right)^{n/2}e^{-\lambda\frac{|x|^2}{4}}$.

5. Assume $u \in \mathcal{S}(\mathbb{R}^n)$ and v is a smooth solution of

$$\begin{cases} \partial_t v - \Delta v &= 0 \\ v(0,x) &= u(x). \end{cases}$$

with $v(t) \in \mathcal{S}(\mathbb{R}^n)$ for $t \geq 0$. Then, applying the Fourier transform in x gives

$$\begin{cases} \partial_t \widehat{v}(t,\xi) + |\xi|^2 \widehat{v}(t,\xi) &= 0 \\ \widehat{v}(0,\xi) &= \widehat{u}(\xi). \end{cases}$$

Leading to $\widehat{v}(t,\xi) = e^{-t|\xi|^2}\widehat{u}(\xi)$ and $v(t) = e^{-t|D_x|^2}u = S_{1/t}u$. Note that we recover the heat Kernel $v(t) = K_t * u$ with $K_t(x) = \left(\frac{1}{4\pi t}\right)^{n/2} e^{-\frac{|x|^2}{4t}}$

B.2.2 Exercice 2

Formally, we want to replace t by it in the previous formula and use analytic continuation. More precisely

Take $\varphi \in \mathcal{S}$. The function $z \mapsto f_\varphi(z) = \left\langle e^{-\frac{|x|^2}{z}}, \widehat{\varphi} \right\rangle_{\mathcal{S}',\mathcal{S}}$ defines an holomorphic function on the half plane $\mathbb{C}^+ = \{z = a + ib; a > 0, b \in \mathbb{R}\}$ (use derivation under the integral). By formula B.4, it is equal to $\left\langle (\pi t)^{d/2} e^{-t\frac{|\cdot|^2}{4}}, \varphi \right\rangle_{\mathcal{S}',\mathcal{S}}$ on $\mathbb{R}^+$. But the function $g_\varphi(z) = \left\langle (\pi z)^{d/2} e^{-z\frac{|\cdot|^2}{4}}, \varphi \right\rangle_{\mathcal{S}',\mathcal{S}}$ also define an analytic function (use the branch of the square root defined on $\mathbb{C}^+$). So, since f_φ and g_φ are two holomorphic functions on $\mathbb{C}^+$ that are equal on $\mathbb{R}^+$, we have $f_\varphi = g_\varphi$ on $\mathbb{C}^+$.

Moreover, g_φ can be extended by continuity to the set $\{z = a + ib; a \geq 0, b \in \mathbb{R}^*\}$, simply by the same formula. It is also clear that f_φ can also be extended by continuity. So, both functions are equal on that set, leading to the final formula on the imaginary set $i\mathbb{R}^*$.

$$\widehat{e^{-\frac{|x|^2}{it}}}(\xi) = (\pi i t)^{d/2} e^{-it\frac{|\xi|^2}{4}}.$$

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