Mixed finite element solution of a semi-conductor problem by Dissection sparse direct solver

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Mixed finite element formulation

Find $(u,p) \in V \times Q$

$$a(u,v) + b(v,p) = (f,v) \qquad \forall v \in V,$$
$$b(u,q) = (g,q) \qquad \forall q \in Q.$$

Stokes equations

$$-\nabla \cdot 2D(u) + \nabla p = f\,, \qquad [D(u)]_{i\,j} = (\partial_j u_i + \partial_i u_j), \quad 1 \le i,j < d$$
$$\nabla \cdot u = 0\,,$$
$$u = 0 \qquad \text{on } \Gamma_D\,.$$

$$V := \{v \in H^1(\Omega)^d\,;\, v = 0 \text{ on } \Gamma_D\}, \qquad Q := L^2(\Omega).$$

$$a(u,v) = 2\int_\Omega D(u) : D(v), \qquad b(v,q) = -\int_\Omega \nabla \cdot v q.$$

Poisson equation

$$u = -\nabla p\,,$$
$$\nabla \cdot u = g\,,$$
$$u = 0 \qquad \text{on } \partial\Omega\,.$$

$$V := H(\text{div}\,;\,\Omega), \qquad Q := L^2(\Omega).$$

$$a(u,v) = 2\int_\Omega u \cdot v, \qquad b(v,q) = -\int_\Omega \nabla \cdot v q.$$

matrix form of mixed finite element method : 1/2

$\vec{u} \in \mathbb{R}^{N_u}$, $\vec{p} \in \mathbb{R}^{N_p}$

$$K \begin{bmatrix} \vec{u} \\ \vec{p} \end{bmatrix} = \begin{bmatrix} A & B^T \\ B & -\epsilon I \end{bmatrix} \begin{bmatrix} \vec{u} \\ \vec{p} \end{bmatrix} = \begin{bmatrix} \vec{f} \\ \vec{g} \end{bmatrix}$$

- A: coercive $(A\vec{x}, \vec{x}\,) > 0 \; \forall \vec{x} \neq \vec{0}$
- B^T : satisfies discrete inf-sup conditon, i.e. ker$B^T = \{\vec{0}\}$
- $\epsilon > 0$

Schur complement matrix $S = \epsilon I + BA^{-1}B^T$

$$(S\,\vec{p}, \vec{p}\,) = \epsilon(\vec{p}, \vec{p}\,) + (BA^{-1}B^T\vec{p}, \vec{p}\,) \geq \epsilon(\vec{p}, \vec{p}\,) > 0$$

- LDL^T-factorization is possible for any ordering
- we need to set appropriate regularization $\epsilon > 0$

$\sqrt{\varepsilon}$-perturbation

a regularization technique

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & \begin{matrix} 0 & \alpha \\ \beta & 0 \end{matrix} \end{bmatrix} \rightarrow \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & \begin{matrix} \sqrt{\varepsilon} & \alpha \\ \beta & 0 \end{matrix} \end{bmatrix}$$

- iterative refinement to improve accuracy of a solution
- user can/have to specify perturbation parameter for unsymmetric matrix (default = 10^{-13} for `Pardiso`)

matrix form of mixed finite element method : 2/2

$\vec{u} \in \mathbb{R}^{N_u}$, $\vec{p} \in \mathbb{R}^{N_p}$

$$K \begin{bmatrix} \vec{u} \\ \vec{p} \end{bmatrix} = \begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \vec{u} \\ \vec{p} \end{bmatrix} = \begin{bmatrix} \vec{f} \\ \vec{g} \end{bmatrix}$$

- A: coercive $(A\vec{x}, \vec{x}\,) > 0 \; \forall \vec{x} \neq 0$
- B^T : satisfies discrete inf-sup conditon, i.e. ker$B^T = \{\vec{0}\}$

Schur complement matrix $S = BA^{-1}B^T$

$$(S\,\vec{p}, \vec{p}) = (BA^{-1}B^T\vec{p}, \vec{p}) = (A^{-1}B^T\vec{p}, B^T\vec{p})$$

$$\mathrm{ker}B^T = \{\vec{0}\} \bigwedge \vec{p} \neq \vec{0} \Rightarrow (A^{-1}B^T\vec{p}, B^T\vec{p}) > 0.$$

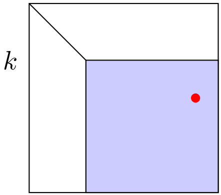
- LDL^T-factorization of K with “block” ordering as $[\vec{u}, \vec{p}]$
- not clear with node-wise $[u_1, u_2, p]$ ordering

$\Rightarrow$ postponing factorization + 2×2 pivoting

pivoting strategy

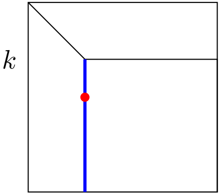
full pivoting : $A = \Pi_L^T L U \Pi_R$

find $\max_{k<i,j\le n} |A(i,j)|$



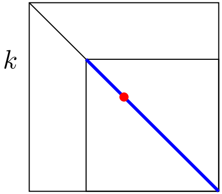
partial pivoting : $A = \Pi L U$

find $\max_{k<i\le n} |A(i,k)|$



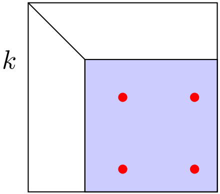
symmetric pivoting : $A = \Pi^T L D U \Pi$

find $\max_{k<i\le n} |A(k,k)|$



2×2 pivoting : $A = \Pi^T \, L \, \tilde{D} \, U \Pi$

find $\max_{k<i,j\le n} \det \begin{vmatrix} A(i,i) & A(i,j) \\ A(j,i) & A(j,j) \end{vmatrix}$



sym. pivoting is mathematically not always possible

understanding pivoting strategy by solution in subspaces

$A = \Pi^T LDU\Pi$: symmetric pivoting
D: diagonal, L: lower triangle, $L(i,i) = 1$, U: upper tri., $U(i,i) = 1$.

- index set $\{i_1, i_2, \cdots, i_m\}$
- $V_m =$ span$[\vec{e}_{i_1}, \vec{e}_{i_2}, \cdots, \vec{e}_{i_m}] \subset \mathbb{R}^N$
- $P_m : \mathbb{R}^N \rightarrow V_m$ orthogonal projection.

find $\vec{u} \in V_m \quad (A\vec{u} - \vec{f}, \vec{v}) = 0 \quad \forall \vec{v} \in V_m$.

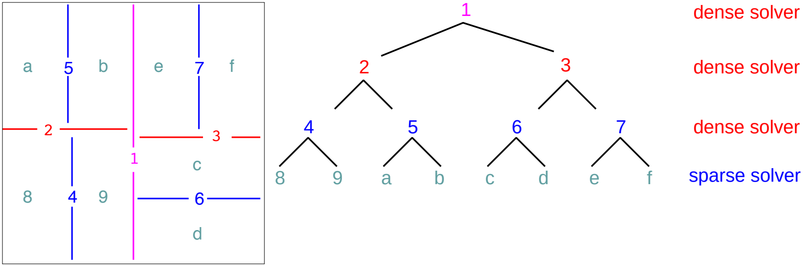
$\exists \Pi : A = \Pi^T L\, D\, U \Pi$
$\Rightarrow \exists \{i_1, i_2, \cdots, i_N\}$ s.t. $\quad P_m A\, P_m^T$: invertible on $V_m \quad 1 \leq \forall m \leq N$.

2×2 pivoting: $\quad V_{m-1}, \; V_m, \; V_{m+1}, V_{m+2}, V_{m+3},$ by skipping V_{m+1}.

J. R. Bunch, L. Kaufman. Some stable methods for calculating
inertia and solving symmetric linear systems,
Math. Comput, 31 (1977) 163–179.

R. Bank, T.-F. Chan. An analysis of the composite step biconjugate gradient
method. *Numer. Math*, 66 (1993) 295–320.

nested dissection by graph decomposition



A. George. Numerical experiments using dissection methods to solve n by n grid problems. SIAM J. Num. Anal. 14 (1977),161–179.

software package:

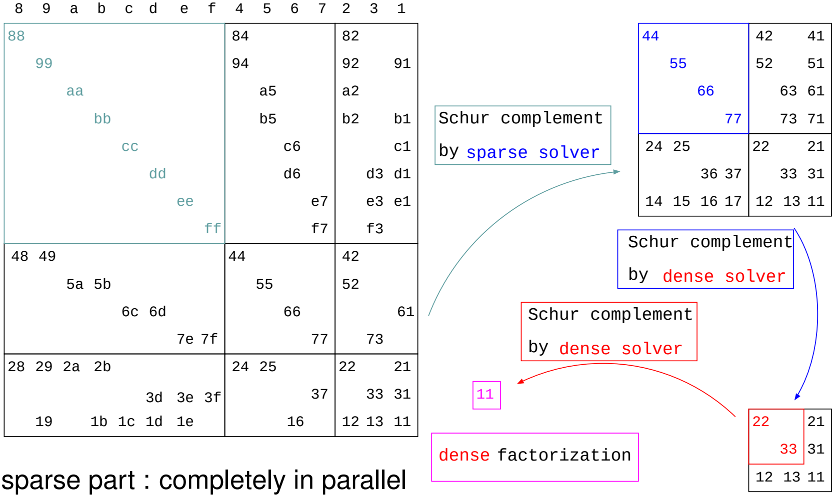
METIS : V. Kumar, G. Karypis, A fast and high quality multilevel scheme for partitioning irregular graphs. SIAM J. Sci. Comput. 20 (1998) 359–392.

SCOTCH : F. Pellegrini J. Roman J, P. Amestoy, Hybridizing nested dissection and halo approximate minimum degree for efficient sparse matrix ordering. Concurrency: Pract. Exper. 12 (2000) 69–84.

recursive generation of Schur complement

$$\begin{bmatrix} A_{11} & A_{21} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & 0 \\ A_{21} & S_{22} \end{bmatrix} \begin{bmatrix} I_1 & A_{11}^{-1} A_{12} \\ 0 & I_2 \end{bmatrix}$$

$S_{22} = A_{22} - A_{21} A_{11}^{-1} A_{12} = A_{22} - (A_{21} U_{11}^{-1}) D_{11}^{-1} L_{11}^{-1} A_{12}$: recursively computed



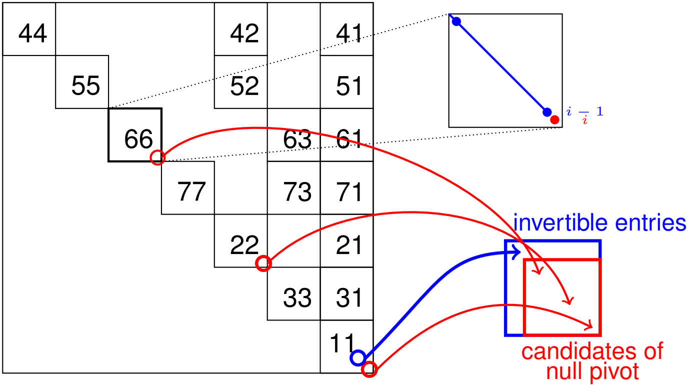
sparse part : completely in parallel

dense part : better use of BLAS 3; dgemm, dtrsm

symmetric pivoting with postponing for block strategy

- nested-dissection decomposition may produce singular sub-matrix for indefinite matrix

τ : given threshold for postponing, 10^{-2} for `MUMPS`, `Dissection`
$|A(i,i)|/|A(i-1,i-1)| < \tau \ \Rightarrow \{A(k,j)\}_{i\leq k,j}$ are postponed



Schur complement matrix from postponed pivots is computed
kernel detection algorithm : QR-factorization by `MUMPS`

Parallel performace of Dissection solver : 1/2

matrices are given from TOSHIBA memory by a joint project:
n=$569,139$, nnz=$8,338,291$, cond=$1.90090141 \cdot 10^5$

sovler	relative error	error
dissection	$7.5867639 \cdot 10^{-6}$	$4.5545655 \cdot 10^{-13}$
pardiso	$6.4318791 \cdot 10^{-2}$	$1.8114969 \cdot 10^{-7}$

	i7-6770HQ@2.60GHz		E5-2695v4@2.10GHz	
# cores	elapsed	CPU	elapsed	CPU
1	17.019	17.018	27.028	27.030
2	9.217	18.300	13.564	26.980
3	6.281	18.490	9.251	27.390
4	5.224	20.461	7.085	27.780
8			3.744	28.530
12			2.686	29.340
16			2.185	30.620

Parallel performace of Dissection solver : 2/2

n=$1,867,029$ nnz=$29,311,205$, cond=$8.93145673\cdot 10^5$

sovler	relative error	error
dissection	$1.2970417\cdot 10^{-5}$	$3.3747264\cdot 10^{-12}$
pardiso	$7.0464281\cdot 10^{-1}$	$1.6031101\cdot 10^{-5}$

E5-2695v4@2.10GHz x2

	double precision		quadruple precision	
# cores	elapsed	CPU	elapsed	CPU
1	197.58	197.89	23,988.0	23,988.0
2	102.73	200.02	12,225.0	24,007.0
4	55.22	204.66	6,195.5	24,022.0
8	30.17	208.47	3,154.5	24,016.0
16	18.08	222.63	1,567.3	24,036.0
24	14.95	250.26	1,088.9	24,059.0
32	13.47	289.29	824.5	24,083.0
36	13.38	311.15	754.1	24,124.0

quadruple arithmetic $\Leftarrow$ double-double `qd` library 2.3.17
+ Dissection `C++` template

Drift-Diffusion system at stationary state : 1/2

- φ : electrostatic potential
- n : electron concentration
- p : hole concentration

$$\begin{aligned} \text{div}(\varepsilon E) &= q(p - n + C(x)) \qquad & E &= -\nabla\varphi \\ -\text{div}J_n &= 0 & J_n &= -q(\mu_n n \nabla\varphi - \mu_n D_n \nabla n) \\ \text{div}J_p &= 0 & J_p &= -q(\mu_p p \nabla\varphi + \mu_p D_p \nabla p) \end{aligned}$$

following Maxwell-Boltzmann statistics : $p = n_i \text{exp}(\frac{\varphi_p - \varphi}{V_{th}})$

- q : positive electron charge
- ε : dielectric constant of the materials
- φ_p : quasi-Fermi level
- n_i : intrinsic concentration of the semiconductor
- $V_{\text{th}} = K_B T / q$: thermal voltage
- K_B : Boltzmann constant
- T : lattice temperature

Drift-Diffusion system at stationary state : 2/2

dimensionless system (by unit scaling)

$$-\text{div}(\lambda^2 \nabla\varphi) = p - n + C(x)$$

$$-\text{div} J_n = 0 \qquad J_n = \nabla n - n\nabla\varphi$$

$$\text{div} J_p = 0 \qquad J_p = -(\nabla p + p\nabla\varphi)$$

with boundary conditions

$$\varphi = f \text{ on } \Gamma_D \qquad \frac{\partial\varphi}{\partial\nu} = 0 \text{ on } \Gamma_N$$

$$n = g \text{ on } \Gamma_D \qquad J_n \cdot \nu = 0 \text{ on } \Gamma_N \Leftarrow \frac{\partial n}{\partial\nu} = 0$$

$$p = h \text{ on } \Gamma_D \qquad J_p \cdot \nu = 0 \text{ on } \Gamma_N \Leftarrow \frac{\partial p}{\partial\nu} = 0$$

Slotboom variables, $\eta,\ \xi$: $n = \eta e^{\varphi}$, $\quad p = \xi e^{-\varphi}$

$$\nabla n = \nabla\eta e^{\varphi} + \eta e^{\varphi}\nabla\varphi = \nabla\eta e^{\varphi} + n\nabla\varphi$$

$$J_n = \nabla n - n\nabla\varphi = e^{\varphi}\nabla\eta$$

$$e^{-\varphi} J_n = \nabla\eta \qquad e^{\varphi} J_p = -\nabla\xi$$

Finite volume discretization with Scharfetter-Gummel method

approximation of $e^{\varphi} J_p = -\nabla \xi$ in an interval $[x_i, x_{i+1}]$, $h = x_{i+1} - x_i$

$$\int_{x_i}^{x_{i+1}} e^{\varphi} dx \, J_{p\,i+1/2} = -h \nabla \xi_{i+1/2} \simeq -(\xi_{i+1} - \xi_i)$$

- φ : assumed to be linear in the interval $[x_i, x_{i+1}]$.

$$\int_{x_i}^{x_{i+1}} e^{\varphi} dx = \frac{1}{\frac{d\varphi}{dx}} \left[e^{\varphi(x)} \right]_{x_i}^{x_{i+1}} = \frac{h}{\varphi_{i+1} - \varphi_i} (e^{\varphi_{i+1}} - e^{\varphi_i})$$

$$\begin{aligned} J_{p\,i+1/2} &\simeq -(\xi_{i+1} - \xi_i) \frac{\varphi_{i+1} - \varphi_i}{h} \frac{1}{e^{\varphi_{i+1}} - e^{\varphi_i}} \\ &= -\frac{\varphi_{i+1} - \varphi_i}{h} \left(\frac{e^{-\varphi_{i+1}} \xi_{i+1}}{1 - e^{\varphi_i - \varphi_{i+1}}} - \frac{e^{-\varphi_i} \xi_i}{e^{\varphi_{i+1} - \varphi_i} - 1} \right) \\ &= -\frac{\varphi_{i+1} - \varphi_i}{h} \left(\frac{p_{i+1}}{1 - e^{\varphi_i - \varphi_{i+1}}} - \frac{p_i}{e^{\varphi_{i+1} - \varphi_i} - 1} \right) \\ &= -\frac{1}{h} \left(B(\varphi_i - \varphi_{i+1}) p_{i+1} - B(\varphi_{i+1} - \varphi_i) p_i \right) \end{aligned}$$

- $B(x) = x/(e^x - 1)$: Bernoulli function

mixed variational formulation : 1 / 2

Slotboom variable ξ : $p = \xi e^{-\varphi}$

$$\begin{aligned} \text{div}(J_p) &= 0 \quad && \text{in } \Omega, \\ e^{\varphi} J_p &= -\nabla \xi \quad && \text{in } \Omega. \end{aligned}$$

function space :

$$H(\text{div}) = \{v \in L^2(\Omega)^2 \,;\, \text{div}\, v \in L^2(\Omega)\},$$
$$\Sigma = \{v \in H(\text{div}) \,;\, v \cdot \nu = 0 \text{ on } \Gamma_N\}$$

integration by parts leads to

$$\int_\Omega e^{\varphi} J_p \cdot v = -\int_\Omega \nabla\xi \cdot v = \int_\Omega \xi \nabla \cdot v - \int_{\partial\Omega} \xi v \cdot \nu$$

$$\int_\Omega e^{\varphi} J_p \cdot v - \int_\Omega \xi \nabla \cdot v = -\int_{\Gamma_D} h e^{\varphi} v \cdot \nu - \int_{\Gamma_N} \xi v \cdot \nu \quad \forall v \in \Sigma$$

$$\int_\Omega \nabla \cdot J_p \, q = 0 \quad \forall q \in L^2(\Omega)$$

mixed variational formulation : 2 / 2

mixed-type weak formulation

find $(J_p, \xi) \in \Sigma \times L^2(\Omega)$

$$\int_\Omega e^\varphi J_p \cdot v - \int_\Omega \xi \nabla \cdot v = - \int_{\Gamma_D} h e^\varphi v \cdot \nu \qquad \forall v \in \Sigma$$

$$- \int_\Omega \nabla \cdot J_p \, q = 0 \qquad \forall q \in L^2(\Omega)$$

symmetric indefinite

replacing $\xi = e^\varphi p$ again,

find $(J_p, p) \in \Sigma \times L^2(\Omega)$

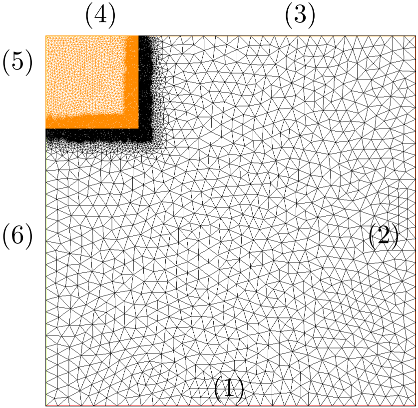
$$\int_\Omega e^\varphi J_p \cdot v - \int_\Omega e^\varphi p \nabla \cdot v = - \int_{\Gamma_D} h e^\varphi v \cdot \nu \qquad \forall v \in \Sigma$$

$$- \int_\Omega \nabla \cdot J_p \, q = 0 \qquad \forall q \in L^2(\Omega)$$

unsymmetric indefinite

hybridization of mixed formulation + mass lumping $\Rightarrow$ FVM

boundary conditions for a diode device (thanks to Dr. Sho)



N region : $x < 1/4 \wedge y > 3/4$,

$C(x,y) = n_d = 10^{17}$

P region : others

$C(x,y) = -n_a = -10^{20}$

$n\,p = n_i^2,\ n_i = 1.08 \times 10^{10}$

charge neutrality

$p - n + C(x,y) = 0$ on (1), (4).

N region :

$n = \sqrt{n_i^2 + C^2/4} + C/2 \simeq n_d$

P region :

$p = \sqrt{n_i^2 + C^2/4} - C/2 \simeq n_a$

(1)	$\varphi = -\log(\frac{n_a}{n_i}) + \frac{1}{V_{th}}\varphi_{\text{appl}}$	$n = \dfrac{n_i}{n_a}$	$p = \dfrac{n_a}{n_i}$
(2), (3), (5), (6)	$\partial_\nu \varphi = 0$	$\partial_\nu n = 0$	$\partial_\nu p = 0$
(4)	$\varphi = \log(\frac{n_d}{n_i})$	$n = \dfrac{n_d}{n_i}$	$p = \dfrac{n_i}{n_d}$

nonlinear iteration to obtain the stationary state

a variant of Gummel map
φ^m, n^m, p^m : given $\to \varphi^{m+1}, n^{m+1}, p^{m+1}$ by a fixed point method
Slotboom variable ξ^m, η^m

$$p^m = \xi^m e^{-\varphi^m} \quad n^m = \eta^m e^{\varphi^m}$$

to find a solution of the nonlinear eq. : $-\text{div}(\lambda^2 \nabla\varphi) = p - n + C(x)$

$$F(\eta^m, \xi^m ; \varphi, \psi) = \lambda^2 \int_\Omega \nabla\varphi \cdot \nabla\psi - \int_\Omega (\xi^m e^{-\varphi} - \eta^m e^{-\varphi} + C)\psi = 0$$

differential calculus with $\delta\varphi \in \{H^1(\Omega) ; \psi = 0 \text{ on } \Gamma_D\}$ leads to

$$\begin{aligned}
&F(\eta^m, \xi^m ; \varphi^m + \delta\varphi, \psi) - F(\eta^m, \xi^m ; \varphi^m, \psi) \\
&\quad = \lambda^2 \int_\Omega \nabla\delta\varphi \cdot \nabla\psi - \int_\Omega \xi^m \left(e^{-\varphi^m - \delta\varphi} - e^{-\varphi^m}\right)\psi - \eta^m \left(e^{\varphi^m + \delta\varphi} - e^{\varphi^m}\right)\psi \\
&\quad = \lambda^2 \int_\Omega \nabla\delta\varphi \cdot \nabla\psi + \int_\Omega \left(\xi^m e^{-\varphi^m} + \eta^m e^{\varphi^m}\right)\delta\varphi\,\psi \\
&\quad = \lambda^2 \int_\Omega \nabla\delta\varphi \cdot \nabla\psi + \int_\Omega (p^m + n^m)\delta\varphi\,\psi \qquad \varphi^{m+1} = \varphi^m + \delta\varphi
\end{aligned}$$

FreeFem++ script

Finite element approximation

- $J_p \in H(\text{div})$: Raviar-Thomas : $RT1(K) = (P1(K))^2 + \vec{x}P1(K)$,
- $p \in L^2(\Omega)$: piecewise linear : $P1(K)$,
- $\varphi \in H^1(\Omega)$: piecewise linear : $P1(K)$.

```
load "Element_Mixte"
load "Dissection"
defaulttoDissection();
fespace Vh(Th, RT1);       fespace Ph(Th, P1);
fespace Xh(Th, P1);
Vh [up1, up2], [v1, v2];  Ph pp, q;
Xh phi; // obtained in Gummel map
problem DDp([up1, up2, pp],[v1, v2, q],
            solver=sparsesolver,strategy=3,
            tolpivot=1.0e-2,tgv=1.0e+30) =
  int2d(Th,qft=qf9pT)(exp(phi) * (up1 * v1 + up2 * v2)
                  - exp(phi) * pp * (dx(v1) + dy(v2))
                  + q * (dx(up1) + dy(up2)))
  +int1d(Th, 1)(gp1 * exp(phi) * (v1 * N.x + v2 * N.y))
  +int1d(Th, 4)(gp4 * exp(phi) * (v1 * N.x + v2 * N.y))
  + on(2, 3, 5, 6, up1 = 0, up2 = 0);
```

matrix representation and preconditioning

linear system for hole concentration by RT1/P1

$$\begin{bmatrix} A(\vec{\varphi}) & B_1(\vec{\varphi})^T \\ -B_2 & 0 \end{bmatrix} \begin{bmatrix} \vec{u} \\ \vec{p} \end{bmatrix} = \begin{bmatrix} \vec{f}(\vec{\varphi}) \\ \vec{0} \end{bmatrix}$$

matrices and vectors weighted with exponential of electrostatic potential

$$\vec{v}^{\,T} A(\vec{\varphi})\vec{u} = \int_\Omega e^{\varphi} u \cdot v$$

$$\vec{v}^{\,T} B_1(\vec{\varphi})\vec{p} = -\int_\Omega e^{\varphi} p \nabla \cdot v$$

$$\vec{v}^{\,T} \vec{f}(\vec{\varphi}) = -\int_{\Gamma_D} h e^{\varphi} v \cdot \nu$$

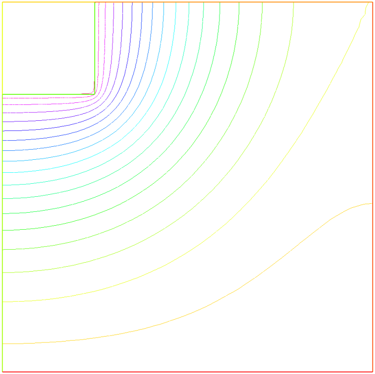
$$\vec{q}^{\,T} B_2 \vec{u} = -\int_\Omega q \nabla \cdot u$$

preconditioned system by scaling matrix $[W]_{i\,i} = 1/[A(\vec{\varphi})]_{i\,i}$

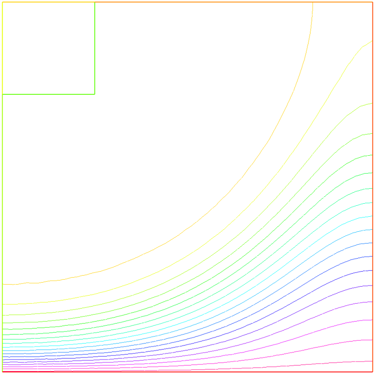
$$\begin{bmatrix} W\,A(\vec{\varphi}) & W\,B_1(\vec{\varphi})^T \\ -B_2 & 0 \end{bmatrix} \begin{bmatrix} \vec{u} \\ \vec{p} \end{bmatrix} = \begin{bmatrix} W\,\vec{f}(\vec{\varphi}) \\ \vec{0} \end{bmatrix}$$

numerical result of a semi-conductor problem

- compute thermal equilibrium with $p = n_i \text{exp}(-\varphi)V_{th}$ and $n = n_i \text{exp}(\varphi)V_{th}$
- Newton iteration for the potential equation and fixed point iteration for the whole system : a kind of Gummel map



electrostatic potential

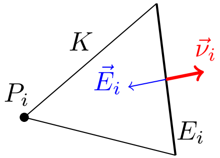


hole concentration

Raviart-Thomas finite element

$RT0(K) = (P0(K))^2 + \vec{x}P0(K) \subset (P1(K))^2$.

- K : triangle element
- $\{E_i\}$: edges of K
- $\vec{\nu}_i$: outer normal of K on E_i
- $\vec{E}_i$: normal to edge E_i



$$\vec{v} \in RT0(K) \;\Rightarrow\; \vec{v}|_{E_i} \cdot n_i \in P0(E_i), \quad \mathrm{div}\vec{v} \in P0(K)$$

finite element basis

$$\vec{\Psi}_i(\vec{x}) = \sigma_i \frac{|E_i|}{|K|}(\vec{x} - \vec{P}_i) \quad \sigma_i = \vec{E}_i \cdot \vec{\nu}_i, \quad P_i : \text{ node of } K$$

finite element vector value is continuous on the middle point on E_i.
inner finite element approximation of $H(\mathrm{div}\,;\,\Omega)$

$$\int_K e^{\varphi} \vec{\Psi}_i \cdot \vec{\Psi}_j \;\leftarrow\; \int_K e^{\varphi_1\lambda_1 + \varphi_2\lambda_2 + \varphi_3\lambda_3} \lambda_k \lambda_l \quad \text{by exact quadrature}$$

$$\int_K e^{\varphi} \vec{\Psi}_i \cdot \vec{\Psi}_j \;\simeq\; \frac{1}{|K|} \int_K e^{\sum_k \varphi_k \lambda_k} \int_K \vec{\Psi}_i \cdot \vec{\Psi}_j \quad : \text{exponential fitting}$$

$\{\lambda_1, \lambda_2, \lambda_3\}$: barycentric coordinates of K

summary

- Dissection direct solver can factorize indefinite unsymmetric matrix with symmetric pivoting
- mixed finite element formulation for Dirft-Diffusion equations with standard numerical integration instead of Schafetter-Gummel scheme / exponential fitting
- possible extension with higher order elements with $RT1(K)/P1(K)/P2(K)$

References

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