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CONTROLABILIDAD DE ALGUNAS ECUACIONES EN DERIVADAS PARCIALES NO LINEALES DE TIPO PARABÓLICO E HIPERBÓLICO

Memoria presentada por
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Índice general

Introducción

En esta memoria se presentan varios resultados de controlabilidad para sistemas asociados a algunas ecuaciones en derivadas parciales de tipo parabólico e hiperbólico. En lo que a problemas parabólicos se refiere, vamos a considerar la ecuación del calor con condiciones de contorno de tipo Fourier (lineales y no lineales), sistemas de Navier-Stokes con condiciones de contorno de tipo Dirichlet y de tipo Navier y el sistema de Boussinesq con condiciones de Dirichlet. Por otro lado, como ejemplo no trivial de problema hiperbólico, trataremos también el sistema de elasticidad de Lamé anisotrópico.

A lo largo de este trabajo, $\Omega \subset \mathbf{R}^N$ $(N \geq 1)$ denotará un abierto acotado, $\omega, \mathcal{O} \subset \Omega$ serán abiertos no vacíos (los dominios de control), $\mathbb{1}_U$ denotará la función característica de U y $T > 0$ será el tiempo final de evolución del sistema.

En general, para un sistema como los anteriores, la tarea consistirá en encontrar un control v que, actuando de algún modo sobre el sistema, haga que alguna solución de éste tenga un comportamiento deseado en el instante final de tiempo T. Diremos que tenemos la *controlabilidad aproximada* del sistema si la solución puede conducirse arbitrariamente cerca (en cierta norma) de un estado deseado arbitrario. Por otra parte, la *controlabilidad exacta* indicará que la solución puede llevarse exactamente a todo estado deseado. Como caso particular de controlabilidad exacta, se dirá que el sistema posee la propiedad de *controlabilidad nula* si, partiendo de un estado inicial arbitrario, puede siempre lograrse conducir la solución a cero. Finalmente, otro ejemplo interesante de controlabilidad exacta es la *controlabilidad exacta a trayectorias*, que indica que podemos hacer que una solución de nuestro sistema controlado coincida con una trayectoria del mismo sistema, es decir, con una solución no controlada.

En los últimos tiempos se ha progresado considerablemente en la controlabilidad de problemas como los que preceden. Con respecto a resultados de controlabilidad aproximada, destacamos [13], [8], [15] y [46]. Algunos trabajos relevantes sobre controlabilidad exacta o controlabilidad nula son [52], [27], [25] y [35].

Para probar resultados de controlabilidad de problemas no lineales, demostraremos primero la controlabilidad nula de problemas linealizados adecuados. La herramienta principal para establecer el control nulo son las llamadas *desigualdades globales de Carleman* para los correspondientes problemas adjuntos asociados. Estas desigualdades han sido aplicadas en el contexto de la controlabilidad en muchos trabajos, entre otros: [34], [14], [49], [37] y [36]. Para otros aspectos y consecuencias de las desigualdades de Carleman, véase [33]. Las desigualdades de Carleman son estimaciones de normas L^2 ponderadas y, en su forma más simple, obedecen a la estructura

$$\iint_{\Omega\times(0,T)} \rho_1^2(x,t)|\varphi|^2\,dx\,dt \le C \iint_{\omega\times(0,T)} \rho_2^2(x,t)|\varphi|^2\,dx\,dt,$$

donde φ es la solución de un problema de evolución retrógrado con condición final $\varphi(T) = \varphi^0$ (el *problema adjunto*) y la constante $C > 0$ y las funciones positivas ρ_1 y ρ_2 son independientes de φ^0.

En lo que a la ecuación del calor se refiere, el control exacto a cero del problema lineal

$$\begin{cases} y_t - \Delta y = v1\!\!1_\omega & \text{en } Q = \Omega\times(0,T), \\ y = 0 & \text{sobre } \Sigma = \partial\Omega\times(0,T), \\ y(0) = y^0 & \text{en } \Omega \end{cases} \tag{1}$$

fue probado en [34] y [43]. En el primero de estos trabajos, la herramienta fundamental es una desigualdad de tipo Carleman para el problema adjunto asociado a (1). En [43] se usa otro método que aprovecha el carácter disipativo de (1) y sólo tiene validez en el contexto de la EDP clásica del calor. Además, desde el punto de vista de la metodología, los argumentos en [34] son puramente Hilbertianos mientras que los argumentos utilizados en [43] se basan en el Análisis de Fourier.

Cuando aparecen términos no lineales en la ecuación del calor, la tarea es mucho más complicada. En [25] se prueba que, incluso para algunas no linealidades "explosivas" que sólo dependen del estado, el sistema

$$\begin{cases} y_t - \Delta y + f(y) = v1\!\!1_\omega & \text{en } Q, \\ y = 0 & \text{sobre } \Sigma, \\ y(0) = y^0 & \text{en } \Omega, \end{cases}$$

es controlable. Más precisamente, se prueba el control exacto a trayectorias de este sistema para funciones f localmente lipschitzianas que cumplen $f(0) = 0$ y

$$\lim_{|s|\to\infty} \frac{f(s)}{s\log^{3/2}(1+|s|)} = 0.$$

Más recientemente, como aplicación de los resultados probados en [37], se prueba en [11] que de nuevo tenemos el control exacto a trayectorias del sistema

$$\begin{cases} y_t - \Delta y + F(y,\nabla y) = v1\!\!1_\omega & \text{en } Q, \\ y = 0 & \text{sobre } \Sigma, \\ y(0) = y^0 & \text{en } \Omega, \end{cases}$$

siempre que F sea localmente lipschitziana y verifique $F(0,0) = 0$ y, esencialmente, que la derivada parcial de F respecto de su primera variable (resp. sus N últimas variables) crezca más lentamente en el infinito que $\log^{3/2}(1+|s|+|p|)$ (resp. $\log^{1/2}(1+|s|+|p|)$).

Todo lo anterior es válido cuando las condiciones de contorno son de tipo Dirichlet. Cuando hablamos de condiciones de contorno de tipo Fourier, la tarea es más complicada. A. V. Fursikov y O. Yu. Imanuvilov probaron en [27] que, si $\beta \in L^\infty(\Sigma)$ y $\beta_t \in L^\infty(\Sigma)$, entonces el sistema

$$\begin{cases} y_t - \Delta y = v \mathbb{1}_\omega & \text{en } Q, \\ \dfrac{\partial y}{\partial n} + \beta\, y = 0 & \text{sobre } \Sigma, \\ y(0) = y^0 & \text{en } \Omega \end{cases} \tag{2}$$

es controlable a cero. Como consecuencia de este resultado, en [10] se prueba el control local a cero del sistema no lineal

$$\begin{cases} y_t - \Delta y = v \mathbb{1}_\omega & \text{en } Q, \\ \dfrac{\partial y}{\partial n} + g(y) = 0 & \text{sobre } \Sigma, \\ y(0) = y^0 & \text{en } \Omega, \end{cases} \tag{3}$$

donde $g : \mathbf{R} \longmapsto \mathbf{R}$ es una función de clase C^3 y $g(0) = 0$.

En los dos primeros capítulos de esta memoria, consideraremos los sistemas (2) y (3) respectivamente con β sólo en $L^\infty(\Sigma)$ y g sólo globalmente lipschitziana. El objetivo es mejorar los resultados mencionados y probar el control global del sistema no lineal (3). Como explicaremos más adelante, esto pasará por eliminar la condición $\beta_t \in L^\infty(\Sigma)$.

En el primer capítulo, consideraremos el sistema lineal

$$\begin{cases} y_t - \Delta y + B \cdot \nabla y + a\, y = v \mathbb{1}_\omega & \text{en } Q, \\ \dfrac{\partial y}{\partial n} + \beta\, y = 0 & \text{sobre } \Sigma, \\ y(0) = y^0 & \text{en } \Omega \end{cases} \tag{4}$$

y probaremos que, si a, B y β son de clase L^∞, (4) es exactamente controlable a cero en el instante T, es decir, para cada $T > 0$ y cada $y^0 \in L^2(\Omega)$, existen controles $v \in L^2(\omega \times (0,T))$ tales que la correspondiente solución de (4) verifica

$$y(T) = 0 \ \text{ en } \Omega.$$

Además, seremos capaces de construir el control dependiendo continuamente de y^0 y verificando una estimación

$$\|v\|_{L^2(\omega\times(0,T))} \le C \|y^0\|_{L^2(\Omega)},$$

con una constante C conocida explícitamente en función de T, $\|a\|_\infty$, $\|B\|_\infty$ y $\|\beta\|_\infty$. Este capítulo corresponde al artículo [17].

El segundo capítulo de la memoria está dedicado al estudio de las propiedades de controlabilidad del problema no lineal (5). En primer lugar probamos la existencia de controles en $L^\infty(Q)$ que conducen el sistema (4) a cero. Además obtenemos una estimación explícita de la norma L^∞ del control respecto de T y de la norma L^∞ de los coeficientes. Como consecuencia, deducimos el control global a trayectorias del sistema

$$\begin{cases} y_t - \Delta y + F(y, \nabla y) = v\mathbb{1}_\omega & \text{en } Q, \\ \dfrac{\partial y}{\partial n} + f(y) = 0 & \text{sobre } \Sigma, \\ y(0) = y^0 & \text{en } \Omega, \end{cases} \tag{5}$$

bajo condiciones adecuadas sobre F y f (ver (31)–(33) más abajo). Concretamente, dada $\overline{y}$ una solución 'regular' de (5) con $v \equiv 0$ y con condición inicial $\overline{y}^0$, probamos la existencia de controles v tal que $y(\cdot, T) = \overline{y}(\cdot, T)$ en Ω, para todo $\overline{y}^0$.

Todo esto se encuentra desarrollado en el trabajo [18].

Con respecto a sistemas de tipo Navier-Stokes, mencionemos que los primeros resultados de controlabilidad se encuentran en [24] y [16]. En estos trabajos, se prueba que el espacio vectorial generado por los estados finales alcanzables es denso (en el sentido de la norma L^2) en el espacio de Hilbert H constituido por los $v \in L^2(\Omega)^N$ que verifican $\nabla \cdot v = 0$ en Ω y $v \cdot n = 0$ sobre $\partial\Omega$. Por otro lado, la cuestión fue considerada por A. V. Fursikov y O. Yu. Imanuvilov (véanse por ejemplo [26] y [28]), donde las condiciones de contorno son de tipo Navier o periódicas, lo que hace el problema más sencillo. La continuación única para problemas de tipo Stokes fue establecida por C. Fabre y G. Lebeau en [14] a partir de *desigualdades de Carleman locales* adecuadas. De esta propiedad, C. Fabre dedujo la controlabilidad aproximada. En el caso de condiciones de Navier, el resultado más interesante fue probado en [8] usando el método de retorno.

Por último, el control local exacto a trayectorias para el sistema de Navier-Stokes

$$\begin{cases} y_t - \Delta y + (y, \nabla)y + \nabla p = v\mathbb{1}_\omega & \text{en } Q, \\ \nabla \cdot y = 0 & \text{en } Q, \\ y = 0 & \text{sobre } \Sigma, \\ y(0) = y^0 & \text{en } \Omega \end{cases} \tag{6}$$

fue establecido por O. Yu. Imanuvilov en [35] a partir de desigualdades de Carleman globales. Una mejora de este resultado se ha establecido en el reciente trabajo [21].

Basándonos en esta última referencia, probamos en el capítulo 3 en primer lugar el control local exacto a trayectorias del sistema N-dimensional (6) con $N-1$ controles escalares. En concreto, bajo ciertas hipótesis sobre el dominio de control ω y suponiendo que partimos de una condición inicial y^0 cercana (en cierta norma) a una trayectoria de (6), se prueba la existencia de controles v con al menos una componente nula de modo que y coincide con dicha trayectoria en el instante final T.

También recientemente (y siguiendo las ideas de [21]), se ha probado en [31] el control local a trayectorias del sistema de Boussinesq

$$\begin{cases} y_t - \Delta y + (y,\nabla)y + \nabla p = v_1 \mathbb{1}_\omega + \theta\, e_N & \text{en } Q, \\ \nabla\cdot y = 0 & \text{en } Q, \\ \theta_t - \Delta\theta + y\cdot\nabla\theta = v_2\mathbb{1}_\omega & \text{en } Q, \\ y = 0 & \text{sobre } \Sigma, \\ y(0) = y^0 & \text{en } \Omega. \end{cases} \tag{7}$$

Aquí, hemos denotado e_N el N-ésimo vector de la base canónica de $\mathbf{R}^N$. Usando este resultado, establecemos en el capítulo 3 un resultado de controlabilidad similar pero con ayuda de sólo $N-1$ controles escalares.

Por último, en este capítulo se recoge un resultado de controlabilidad global de un sistema truncado de Navier-Stokes bidimensional:

$$\begin{cases} y_t - \Delta y + (y,\nabla)\mathbf{T}_M(y) + \nabla p = v\mathbb{1}_\mathcal{O} & \text{en } Q, \\ \nabla\cdot y = 0 & \text{en } Q, \\ \nabla\times y = 0,\ y\cdot n = 0 & \text{sobre } \Sigma, \\ y(0) = y^0 & \text{en } \Omega, \end{cases} \tag{8}$$

con $M > 0$. Aquí, $\mathbf{T}_M(y) = (T_M(y_1), T_M(y_2))$ y

$$T_M(s) = \begin{cases} -M & \text{si } s \le -M, \\ s & \text{si } -M < s < M, \\ M & \text{si } s \ge M. \end{cases}$$

Obsérvese que las condiciones de contorno aquí utilizadas son de tipo Navier (véanse [29] y [26] por ejemplo). Al igual que en [8], [26] and [28], el estudio de la controlabilidad de este sistema es menos complicado que el de (6), debido a las condiciones de contorno. En concreto, las condiciones de contorno de (8) nos dicen que la vorticidad ($\omega = \nabla\times y$) verifica una ecuación del calor con condiciones de contorno de tipo Dirichlet.

Mediante la aplicación de un punto fijo sobre el sistema (8), se deduce el control a cero (global) del mismo. Así pues, este resultado supone una primera aproximación al intento de controlar el sistema de Navier-Stokes de forma global (es decir, sin suponer que partimos de un estado cercano a la trayectoria que queremos alcanzar). Todo esto es motivo del artículo [23].

Para mayor claridad, hemos incluido un Apéndice al final de la memoria donde presentamos un resumen de la demostración de la desigualdad de Carleman para problemas de tipo Stokes con condiciones de Dirichlet.

En cuanto al sistema de elasticidad se refiere, la gran mayoría de los trabajos de controlabilidad toman como modelo el sistema de Lamé isotrópico. De manera general, el sistema de

Lamé tridimensional se escribe como sigue

$$\rho\partial_{x_0}^2 u_i - \sum_{j=1}^{3} \partial_{x_j}(\sigma_{ij}) = f, \quad 1 \leq i \leq 3. \tag{9}$$

En (9), x_0 denota la variable temporal, $x' = (x_1, x_2, x_3)$ es la variable espacial y σ es el tensor de esfuerzos, que tiene la forma

$$\begin{aligned}\sigma = R + (\nabla u)R + \lambda(tr\epsilon)I + 2\mu\epsilon + \beta_1(tr\epsilon)(trR)I + \beta_2(trR)\epsilon \\ + \beta_3((tr\epsilon)R + tr(\epsilon R)I) + \beta_4(\epsilon R + R\epsilon),\end{aligned} \tag{10}$$

donde $R = R(x)$ es un tensor simétrico que verifica $\nabla \cdot R = 0$ y $\epsilon = \frac{1}{2}(\nabla u + \nabla^t u)$ es el tensor de deformaciones.

En (10), $\rho = \rho(x')$ es la densidad y $\lambda = \lambda(x')$ y $\mu = \mu(x')$ son los coeficientes de Lamé. El sistema de Lamé isotrópico corresponde a tomar $R \equiv 0$ en (10). Esto proporciona propiedades simplificadoras, pues las variables $z_0 = \nabla \cdot u$ y $z' = \nabla \times u$ verifican entonces ecuaciones independientes. Las herramientas fundamentales más utilizadas para obtener resultados de controlabilidad para este sistema han sido (de nuevo) desigualdades de Carleman. Para desplazamientos u con soporte compacto, estas desigualdades fueron obtenidas en [9] y [51] para el sistema estacionario y en [47], [44] y [42] para el sistema de evolución. Como consecuencia, en estos trabajos se prueban resultados de continuación única y de control aproximado. Para desplazamientos u verificando (9) (con $R \equiv 0$) y completados con condiciones de contorno de tipo Dirichlet, véase [41] en el caso estacionario y [38]–[40] en el caso de evolución.

En el capítulo 4 presentamos el sistema de Lamé anisotrópico. Por simplicidad, hemos supuesto que $\beta_j \equiv 0$ $(1 \leq j \leq 4)$ en (10). En el caso general el análisis sería similar, debiéndose imponer hipótesis suplementarias a los coeficientes de Lamé y al tensor residual R (véase (68)).

En primera instancia, la ecuación (9) se completa con condiciones de contorno de tipo Dirichlet y condiciones iniciales. Por tanto, el sistema de control queda como sigue:

$$\begin{cases} \rho(x')\partial_{x_0}^2 u - \mu\Delta u - (\lambda+\mu)\nabla(\nabla\cdot u) + [(R,\nabla)\nabla^t]u = f + v\chi_\omega & \text{en } Q, \\ u = 0 & \text{sobre } \Sigma, \\ u(0) = u_0, \quad \partial_{x_0}u(0) = u_1 & \text{en } \Omega, \end{cases} \tag{11}$$

donde las funciones ρ, μ, λ y R son dadas. Con respecto a este tipo de sistemas, se prueba en [1] un resultado de controlabilidad exacta frontera basado en el método de los multiplicadores, utilizado previamente en [45] también para el sistema de Lamé. Este resultado se establece para un tiempo suficientemente grande y es válido para operadores no necesariamente isotrópicos pero con tensores R 'pequeños'.

En este último capítulo probamos el control exacto de (11), como consecuencia de una desigualdad de Carleman global. Concretamente, fijados dos estados finales u_2, u_3, se prueba la existencia de un control v tal que la solución u de (11) verifica

$$u(\cdot, T) = u_2 \quad \text{y} \quad u_{x_0}(\cdot, T) = u_3 \quad \text{en } \Omega$$

para todo tiempo $T > 0$ que permita la existencia de una ciertas funciones peso (véanse Condiciones A y B en el último capítulo).

El contenido de este capítulo corresponde al artículo [32].

A continuación, vamos a explicar con más detalle el desarrollo de cada uno de los capítulos de esta memoria.

Controlabilidad a cero de la ecuación del calor lineal con condiciones de contorno de Fourier

En la primera parte de esta memoria trabajaremos con el sistema

$$\begin{cases} y_t - \Delta y + B \cdot \nabla y + a\, y = v 1\!\!1_\omega & \text{en } Q, \\ \dfrac{\partial y}{\partial n} + \beta\, y = 0 & \text{sobre } \Sigma, \\ y(0) = y^0 & \text{en } \Omega \end{cases} \tag{12}$$

y probaremos la controlabilidad a cero del mismo cuando $y^0 \in L^2(\Omega)$ y los coeficientes a, B y β son de clase L^∞.

El primer resultado importante que demostramos es una desigualdad global de Carleman para las soluciones débiles de problemas retrógrados de la forma

$$\begin{cases} -\varphi_t - \Delta\varphi = f_1 + \nabla \cdot f_2 & \text{en } Q, \\ (\nabla\varphi + f_2) \cdot n = f_3 & \text{sobre } \Sigma, \\ \varphi(T) = \varphi^0 & \text{en } \Omega, \end{cases} \tag{13}$$

con $f_1 \in L^2(Q)$, $f_2 \in L^2(Q)^N$ y $f_3 \in L^2(\Sigma)$. Obsérvese que se le puede dar 'a priori' un sentido a la condición frontera, puesto que la solución débil de (13) verifica $\nabla\varphi + f_2 \in L^2(Q)^N$ y $\nabla \cdot (\nabla\varphi + f_2) \in H^{-1}(0, T; L^2(\Omega))$.

Esta desigualdad es la siguiente:

Teorema 1 *Bajo las condiciones anteriores, existen constantes $\overline{\lambda}$, σ_1, σ_2 y C dependientes de Ω y ω tales que, para todo $\lambda \geq \overline{\lambda}$, para todo $s \geq \overline{s} = \sigma_1(e^{\sigma_2 \lambda} T + T^2)$ y para todo $\varphi^0 \in L^2(\Omega)$, la solución débil de* (13) *satisface*

$$\begin{aligned} &\iint_Q e^{-2s\alpha}(s\,\lambda^2 \xi\, |\nabla\varphi|^2 + s^3\, \lambda^4\, \xi^3\, |\varphi|^2) dx\, dt \\ &\qquad\qquad + s^2 \lambda^3 \iint_\Sigma e^{-2s\alpha}\, \xi^2\, |\varphi|^2\, d\sigma\, dt \\ &\leq C \left(\iint_Q e^{-2s\alpha}(|f_1|^2 + \ s^2\, \lambda^2\, \xi^2\, |f_2|^2) dx\, dt \right. \\ &\left. + s\lambda \iint_\Sigma e^{-2s\alpha}\, \xi\, |f_3|^2\, d\sigma\, dt + s^3 \lambda^4 \iint_{\omega \times (0,T)} e^{-2s\alpha}\, \xi^3\, |\varphi|^2\, dx\, dt \right). \end{aligned} \tag{14}$$

Aquí, los pesos $\xi = \xi(x,t)$ y $\alpha = \alpha(x,t)$ vienen dados por

$$\xi(x,t) = \frac{e^{\lambda \eta^0(x)}}{t(T-t)}\,, \quad \alpha(x,t) = \frac{e^{2\lambda \|\eta^0\|_\infty} - e^{\lambda \eta^0(x)}}{t(T-t)}\,, \tag{15}$$

donde $\eta^0 = \eta^0(x)$ verifica

$$\begin{gathered} \eta^0 \in C^2(\overline{\Omega}), \quad \eta^0(x) > 0 \text{ en } \Omega, \quad \eta^0(x) = 0 \text{ sobre } \partial\Omega, \\ |\nabla \eta^0(x)| > 0 \text{ en } \overline{\Omega \setminus \omega'} \end{gathered} \tag{16}$$

y $\omega' \subset\subset \omega$ es un abierto no vacío.

La prueba de este resultado está inspirada en las ideas de [37]. Más concretamente, seguimos una estrategia de dualidad que nos permite relajar las hipótesis de regularidad impuestas sobre los datos. Esto nos permite imponer sólo la hipótesis L^∞ sobre el coeficiente $\beta = \beta(x,t)$.

Para utilizar esta técnica, hay que basarse en una desigualdad de Carleman para un sistema *homogéneo* asociado a (13). En concreto, se puede probar que las soluciones del problema

$$\begin{cases} -q_t - \Delta q = g & \text{en } Q, \\ \dfrac{\partial q}{\partial n} = 0 & \text{sobre } \Sigma, \\ q(T) = q^0 & \text{en } \Omega, \end{cases}$$

con $g \in L^2(Q)$ y $q^0 \in L^2(\Omega)$, satisfacen

$$I_{s,\lambda}(q) \le C \left(\iint_Q e^{-2s\alpha} |g|^2 \, dx\, dt + s^3 \lambda^4 \iint_{\omega \times (0,T)} e^{-2s\alpha} \xi^3 |q|^2 \, dx\, dt \right), \tag{17}$$

para elecciones de s y de λ apropiadas. El término $I_{s,\lambda}(q)$ está dado por

$$I_{s,\lambda}(q) = \iint_Q e^{-2s\alpha} \left((s\xi)^{-1} (|q_t|^2 + |\Delta q|^2) + s\lambda^2 \xi |\nabla q|^2 + s^3 \lambda^4 \xi^3 |q|^2 \right) \, dx\, dt.$$

Una vez establecida esta desigualdad, miramos φ como la solución por transposición de (13), es decir, la única función de $L^2(Q)$ que satisface

$$\begin{cases} \displaystyle\iint_Q \varphi\, h \, dx\, dt = \iint_Q f_1(x,t)\, z \, dx\, dt - \iint_Q f_2(x,t) \cdot \nabla z \, dx\, dt \\ \displaystyle\quad + \iint_\Sigma f_3(x,t)\, z \, d\sigma\, dt + \int_\Omega \varphi^0(x)\, z(x,T)\, dx \quad \forall h \in L^2(Q), \end{cases} \tag{18}$$

donde z es, para cada $h \in L^2(Q)$, la solución del problema

$$\begin{cases} z_t - \Delta z = h & \text{en } Q, \\ \dfrac{\partial z}{\partial n} = 0 & \text{sobre } \Sigma, \\ z(0) = 0 & \text{en } \Omega. \end{cases}$$

Consideramos ahora el siguiente problema de mínimos:

$$\left\{\begin{array}{l} \text{Minimizar } \dfrac{1}{2}\left(\displaystyle\iint_Q e^{2s\alpha}\,|z|^2 + s^{-3}\lambda^{-4}\iint_{\omega\times(0,T)} e^{2s\alpha}\,\xi^{-3}\,|v|^2\,dx\,dt\right) \\ \text{sujeto a } v\in L^2(Q) \text{ y} \\ \quad\left\{\begin{array}{ll} z_t-\Delta z = s^3\lambda^4 e^{-2s\alpha}\xi^3\varphi + v\mathbb{1}_\omega & \text{en } Q, \\ \dfrac{\partial z}{\partial n}=0 & \text{sobre } \Sigma, \\ z(0)=0, \quad z(T)=0 & \text{en } \Omega. \end{array}\right. \end{array}\right. \tag{19}$$

Siguiendo las ideas de [37], llegamos al sistema de optimalidad

$$\left\{\begin{array}{ll} \mathcal{L}(e^{-2s\alpha}\,\mathcal{L}^*p) + s^3\lambda^4 e^{-2s\alpha}\xi^3 p 1_\omega = s^3\lambda^4 e^{-2s\alpha}\xi^3\varphi & \text{en } Q, \\ \dfrac{\partial p}{\partial n}=0, \quad \dfrac{\partial}{\partial n}(e^{-2s\alpha}\,\mathcal{L}^*p)=0 & \text{sobre } \Sigma, \\ (e^{-2s\alpha}\mathcal{L}^*p)_{|t=0} = (e^{-2s\alpha}\mathcal{L}^*p)_{|t=T}=0 & \text{en } \Omega, \end{array}\right. \tag{20}$$

donde $\mathcal{L}$ y $\mathcal{L}^*$ son los operadores $\mathcal{L}=\partial_t-\Delta$ y $\mathcal{L}^*=-\partial_t-\Delta$. Entonces, debido a la desigualdad de Carleman (17), podemos probar que (20) (y por tanto (19)) admite una única solución p y que

$$\widehat{v} = -s^3\,\lambda^4\,e^{-2s\alpha}\,\xi^3\,p\mathbb{1}_\omega \quad \text{y} \quad \widehat{z}=e^{-2s\alpha}\mathcal{L}^*p$$

es la solución de (19). Se prueba que esta solución verifica la desigualdad

$$\begin{array}{l} \displaystyle\iint_Q e^{2s\alpha}\,|\widehat{z}|^2\,dx\,dt + s^{-2}\lambda^{-2}\iint_Q e^{2s\alpha}\,\xi^{-2}\,|\nabla\widehat{z}|^2\,dx\,dt \\ \displaystyle +s^{-1}\lambda^{-1}\iint_\Sigma e^{2s\alpha}\xi^{-1}\,|\widehat{z}|^2\,d\sigma\,dt + s^{-3}\lambda^{-4}\iint_{\omega\times(0,T)} e^{2s\alpha}\,\xi^{-3}\,|\widehat{v}|^2\,dx\,dt \\ \displaystyle \qquad \le C(\Omega,\omega)\,s^3\lambda^4\iint_Q e^{-2s\alpha}\,\xi^3\,|\varphi|^2\,dx\,dt, \end{array} \tag{21}$$

para ciertos s y λ.

Elegimos ahora

$$h = s^3\lambda^4 e^{-2s\alpha}\xi^3\varphi + \widehat{v}\mathbb{1}_\omega$$

en (18) y, teniendo en cuenta (21), deducimos que

$$\begin{array}{l} \displaystyle s^3\lambda^4\iint_Q e^{-2s\alpha}\,\xi^3\,|\varphi|^2\,dx\,dt \le C(\Omega,\omega)\left(\iint_Q e^{-2s\alpha}\,|f_1|^2\,dx\,dt\right. \\ \displaystyle +s^2\lambda^2\iint_Q e^{-2s\alpha}\,\xi^2\,|f_2|^2\,dx\,dt + s\lambda\iint_\Sigma e^{-2s\alpha}\,\xi\,|f_3|^2\,d\sigma\,dt \\ \displaystyle \left. +s^3\lambda^4\iint_{\omega\times(0,T)} e^{-2s\alpha}\,\xi^3\,|\varphi|^2\,dx\,dt\right) \end{array} \tag{22}$$

para s y λ apropiados. Por último, deducimos (14) directamente a partir de (22).

Como consecuencia del teorema 1, se deduce una *desigualdad de observabilidad* para el sistema adjunto asociado a (12), esto es,

$$\begin{cases} -\varphi_t - \Delta\varphi - \nabla \cdot (\varphi B) + a\,\varphi = 0 & \text{en } Q, \\ (\nabla\varphi + \varphi B)\cdot n + \beta\,\varphi = 0 & \text{sobre } \Sigma, \\ \varphi(T) = \varphi^0 & \text{en } \Omega. \end{cases} \tag{23}$$

De hecho, las soluciones de (23) verifican

$$\iint_{\Omega\times(T/4,3T/4)} |\varphi|^2\,dx\,dt \le K \iint_{\omega\times(0,T)} |\varphi|^2\,dx\,dt, \tag{24}$$

con una constante K de la forma

$$K = \exp\{C(1 + \frac{1}{T} + \|a\|_\infty^{2/3} + \|B\|_\infty^2 + \|\beta\|_\infty^2)\}. \tag{25}$$

El segundo resultado importante de este capítulo es la controlabilidad nula del sistema (12) :

Teorema 2 *Supongamos que $a \in L^\infty(Q)$, $B \in L^\infty(Q)^N$ y $\beta \in L^\infty(\Sigma)$. Entonces* (12) *es controlable a cero para todo $T > 0$ con controles v tales que*

$$\|v\|_{L^2(\omega\times(0,T))} \le \widetilde{K}\,\|y^0\|_{L^2(\Omega)}\,, \tag{26}$$

con una constante $\widetilde{K}$ de la forma

$$\widetilde{K} = \exp\left\{C\left(1 + \frac{1}{T} + \|a\|_\infty^{2/3} + \|B\|_\infty^2 + \|\beta\|_\infty^2 + T(\|a\|_\infty + \|B\|_\infty^2 + \|\beta\|_\infty^2)\right)\right\}, \tag{27}$$

donde $C = C(\Omega,\omega)$ es una constante positiva.

La deducción de este teorema a partir de la desigualdad de observabilidad (24) es “standard”.

En el segundo capítulo de la memoria presentamos resultados de controlabilidad exacta a trayectorias para una ecuación del calor no lineal con condiciones de contorno de tipo Fourier no lineales. Este constituye una continuación natural del primero. De hecho, los resultados presentados en los teoremas 1 y 2 serán cruciales.

Controlabilidad exacta a las trayectorias de la ecuación del calor no lineal con condiciones de contorno de tipo Fourier no lineales

Con ayuda de los resultados enunciados en el capítulo anterior, vamos a deducir la controlabilidad exacta (global) a las trayectorias del sistema

$$\begin{cases} y_t - \Delta y + F(y, \nabla y) = v\mathbb{1}_\omega & \text{en } Q, \\ \dfrac{\partial y}{\partial n} + f(y) = 0 & \text{sobre } \Sigma, \\ y(0) = y^0 & \text{en } \Omega, \end{cases} \tag{28}$$

bajo ciertas condiciones de crecimiento en el infinito para las funciones F y f. En esta situación, consideraremos datos iniciales $y^0 \in L^\infty(\Omega)$. Algunos resultados de existencia, unicidad, regularidad y otras propiedades de las soluciones de problemas con la estructura de (28) pueden ser encontrados, por ejemplo, en [2], [3] y [12].

Fijamos una trayectoria no controlada del sistema (28), es decir, una solución $\overline{y}$ de

$$\begin{cases} \overline{y}_t - \Delta\overline{y} + F(\overline{y}, \nabla\overline{y}) = 0 & \text{en } Q, \\ \dfrac{\partial\overline{y}}{\partial n} + f(\overline{y}) = 0 & \text{sobre } \Sigma, \\ \overline{y}(0) = \overline{y}^0 & \text{en } \Omega. \end{cases} \tag{29}$$

Imponemos las siguientes hipótesis:

$$\overline{y} \in L^2(0,T;H^1(\Omega)) \cap C^0([0,T];L^2(\Omega)) \cap L^\infty(Q), \quad \overline{y}^0 \in L^\infty(\Omega). \tag{30}$$

El resultado principal de este trabajo es el siguiente:

Teorema 3 *Supongamos que F y f son funciones localmente lipschitzianas y verifican*

$$\lim_{|s|\to\infty} \frac{|F(s,p) - F(r,p)|}{|s-r|\log^{3/2}(1+|s-r|)} = 0, \tag{31}$$

uniformemente en $(r,p) \in [-K,K] \times \mathbf{R}^N$ para cada $K > 0$,

$$\begin{cases} \forall L > 0,\ \exists M > 0 \text{ tal que} \\ |F(s,p) - F(r,p)| \leq M|s-r|, \quad |F(s,p) - F(s,q)| \leq M|p-q| \\ \forall (s,r,p,q) \in [-L,L]^2 \times \mathbf{R}^N \times \mathbf{R}^N \end{cases} \tag{32}$$

y

$$\lim_{|s|\to\infty} \frac{|f(s) - f(r)|}{|s-r|\log^{1/2}(1+|s-r|)} = 0 \tag{33}$$

uniformemente en $r \in [-K,K]$ para todo $K > 0$. Entonces, fijado $T > 0$, tenemos la controlabilidad exacta a las trayectorias $\overline{y}$ que verifican (30), con controles $v \in L^\infty(\omega \times (0,T))$.

Antes de resumir la prueba del teorema 3, necesitaremos un resultado de controlabilidad nula para el sistema lineal (12) con controles en $L^\infty(\omega \times (0,T))$:

Proposición 1 *Sea $T > 0$. Entonces, el sistema (12) es exactamente controlable a cero con controles $v \in L^\infty(\omega \times (0,T))$. Además, podemos encontrar controles v satisfaciendo*

$$\|v\|_{L^\infty(\omega\times(0,T))} \leq e^{C(\Omega,\omega)K(T,\|a\|_\infty,\|B\|_\infty,\|\beta\|_\infty)} \|y^0\|_{L^2(\Omega)}, \tag{34}$$

donde

$$K = 1 + 1/T + \|a\|_\infty^{2/3} + \|B\|_\infty^2 + \|\beta\|_\infty^2 + T(1 + \|a\|_\infty + \|B\|_\infty^2 + \|\beta\|_\infty^2). \tag{35}$$

La demostración que presentamos de este resultado está inspirada en la técnica utilizada en [7] para regularizar controles. Indicaremos a continuación las ideas principales de este argumento.

Sea $y^0 \in L^2(\Omega)$ y sean ω' y ω'' dos abiertos con $\omega'' \subset\subset \omega' \subset\subset \omega$. Entonces, utilizando el teorema 2 con dominio de control ω'', deducimos que existe $\widetilde{v} \in L^2(\omega'' \times (0,T))$ tal que la solución $\widetilde{y}$ de (12) verifica $\widetilde{y}(T) = 0$ y tenemos la estimación

$$\|\widetilde{v}\|_{L^2(\omega''\times(0,T))} \le e^{C(\Omega,\omega)\, K(T,\|a\|_\infty,\|B\|_\infty,\|\beta\|_\infty)}\, \|y^0\|_{L^2(\Omega)}, \tag{36}$$

con K de la forma (35). Definimos ahora una función truncante $\eta \in C^\infty([0,T])$ cumpliendo

$$\eta(t) = 1 \text{ en } (0,T/4), \quad \eta(t) = 0 \text{ en } (3T/4,T), \quad 0 \le \eta(t) \le 1 \text{ en } (0,T).$$

Consideremos la solución χ del sistema

$$\begin{cases} \chi_t - \Delta\chi + a\,\chi + B\cdot\nabla\chi = 0 & \text{en } Q, \\ \dfrac{\partial\chi}{\partial n} + \beta\,\chi = 0 & \text{sobre } \Sigma, \\ \chi(0) = y^0 & \text{en } \Omega. \end{cases}$$

Entonces $\widetilde{w} = \widetilde{y} - \eta\chi$ cumple

$$\begin{cases} \widetilde{w}_t - \Delta\widetilde{w} + a\,\widetilde{w} + B\cdot\nabla\widetilde{w} = -\eta'(t)\,\chi + \widetilde{v}\mathbb{1}_{\omega''} & \text{en } Q, \\ \dfrac{\partial\widetilde{w}}{\partial n} + \beta\,\widetilde{w} = 0 & \text{sobre } \Sigma, \\ \widetilde{w}(0) = 0, \quad \widetilde{w}(T) = 0 & \text{en } \Omega. \end{cases} \tag{37}$$

Introduzcamos ahora un abierto ω_0 con $\omega' \subset\subset \omega_0 \subset\subset \omega$ y una nueva función cortante ξ que verifica

$$\xi \in C_0^2(\omega_0), \quad \xi \equiv 1 \text{ en } \omega'.$$

Entonces la función $w = (1-\xi)\,\widetilde{w}$ verifica

$$\begin{cases} w_t - \Delta w + a\,w + B\cdot\nabla w = -\eta'(t)\,\chi + v\mathbb{1}_{\omega} & \text{en } Q, \\ \dfrac{\partial w}{\partial n} + \beta\,w = 0 & \text{sobre } \Sigma, \\ w(0) = 0, \quad w(T) = 0 & \text{en } \Omega, \end{cases}$$

con

$$v = \eta'\,\xi\,\chi + 2\nabla\xi\cdot\nabla\widetilde{w} + \Delta\xi\,\widetilde{w} - B\cdot\nabla\xi\,\widetilde{w}. \tag{38}$$

Usando resultados de regularidad local para las soluciones del sistema (37), no es difícil deducir que $v \in L^\infty(\omega\times(0,T))$ y que la función $y = w + \eta\chi$ resuelve (con este control v) el problema de controlabilidad a cero de (12) del modo indicado en la proposición 1.

Otro resultado que se utilizará para demostrar el teorema 2 es una estimación L^∞ de las soluciones del sistema (12):

Proposición 2 *Supongamos que* $v \in L^\infty(Q)$, $y^0 \in L^\infty(\Omega)$ *y los coeficientes* a, B, β *están en* L^∞. *Entonces* $y \in L^\infty(Q)$ *y verifica*

$$\|y\|_\infty \leq e^{CT\left(1+\|a\|_\infty+\|B\|_\infty^2+\|\beta\|_\infty^2\right)}\left(\|y^0\|_\infty + \|f\|_\infty\right). \tag{39}$$

con una constante $C > 0$ *sólo dependiente de* Ω.

Volvamos a la demostración del teorema 2. Consideremos el sistema (28) que, tras el cambio de variable $y = \overline{y} + w$, puede escribirse en la forma

$$\begin{cases} w_t - \Delta w + F_1(w, \nabla w; x,t)w + F_2(\nabla w; x,t)\cdot \nabla w = v1\!\!1_\omega & \text{en } Q, \\ \dfrac{\partial w}{\partial n} + F_3(w;x,t)w = 0 & \text{sobre } \Sigma, \\ w(0) = y^0 - \overline{y}(0) & \text{en } \Omega, \end{cases} \tag{40}$$

donde

$$F_1(s,p;x,t) = \frac{F(\overline{y}(x,t)+s, \nabla\overline{y}(x,t)+p) - F(\overline{y}(x,t), \nabla\overline{y}(x,t)+p)}{s}, \tag{41}$$

$$F_2 = (F_{21},\ldots,F_{2N}), \quad F_{2j}(p;x,t) = \int_0^1 \frac{\partial F}{\partial p_j}(\overline{y}(x,t), \nabla\overline{y}(x,t)+\lambda p)\, d\lambda \tag{42}$$

y

$$F_3(s;x,t) = \frac{f(\overline{y}(x,t)+s) - f(\overline{y}(x,t))}{s} \tag{43}$$

para $s \in \mathbf{R}$ y $p \in \mathbf{R}^N$.

De este modo, nuestra tarea consiste en demostrar la controlabilidad nula del sistema (40) y, para hacer esto, nos restringimos (por densidad) al caso en el que las funciones F_j tienen la primera derivada continua.

La idea de la prueba es clásica y se basa en un argumento de punto fijo en el espacio $Z = L^2(0,T;H^1(\Omega))$. Estas técnicas fueron introducidas en [52], en el contexto de la controlabilidad exacta de la ecuación de ondas. Desde entonces, han sido aplicadas a un gran número de problemas diferentes. En nuestro caso, para cada $R > 0$, consideramos la función M_R, con

$$M_R(s) = \begin{cases} -R & \text{si } s < -R, \\ s & \text{si } -R \leq s \leq R, \\ R & \text{si } s > R \end{cases}$$

y los coeficientes $a_{R,z}$, B_z y $\beta_{R,z}$, con

$$a_{R,z}(x,t) = F_1(M_R(z(x,t)), \nabla z(x,t); x,t),$$

$$B_z(x,t) = F_2(\nabla z(x,t); x,t)$$

y

$$\beta_{R,z}(x,t) = F_3(M_R(z(x,t)); x,t).$$

Sabemos que el problema de controlabilidad nula

$$\begin{cases} w_t - \Delta w + a_{R,z}\, w + B_z \cdot \nabla w = v \mathbb{1}_\omega & \text{en } Q, \\ \dfrac{\partial w}{\partial n} + \beta_{R,z}\, w = 0 & \text{sobre } \Sigma, \\ w(0) = y^0 - \overline{y}(0), \quad w(T) = 0 & \text{en } \Omega, \end{cases} \tag{44}$$

tiene solución, en virtud de la proposición 1.

A continuación, elegimos una solución particular del problema (44), procediendo como en [25], es decir, trabajando en un intervalo temporal $[0, T_z]$ que hace que el control y el estado asociado verifiquen

$$\|v_{R,z}\|_{L^\infty(\omega\times(0,T))} \leq C_R\, \|w^0\|_{L^2(\Omega)}, \tag{45}$$

$$\|w_{R,z}\|_Z \leq C_R\, \|w^0\|_{L^2(\Omega)}, \tag{46}$$

y

$$\|w_{R,z}\|_\infty \leq C_R\, \|w^0\|_{L^\infty(\Omega)}, \tag{47}$$

con

$$C_R = \exp\left\{ C(\Omega, \omega, T) \left(1 + a_R^{2/3} + \overline{B}^2 + \beta_R^2\right)\right\},$$

donde

$$a_R = \sup_{|s|\leq R,\ p\in\mathbf{R}^N} \operatorname*{ess\,sup}_{(x,t)\in Q} |F_1(s, p; x, t)|,$$

$$\overline{B} = \sup_{p\in\mathbf{R}^N} \operatorname*{ess\,sup}_{(x,t)\in Q} |F_2(p; x, t)|$$

y

$$\beta_R = \sup_{|s|\leq R} \operatorname*{ess\,sup}_{(x,t)\in \Sigma} |F_3(s; x, t)|.$$

Finalmente, definimos la aplicación multivaluada Λ_R que a cada $z \in Z$ asocia $\Lambda_R(z)$ que es el conjunto de estados $w_{R,z}$ que satisfacen las estimaciones (46) y (47) correspondientes a los controles $v_{R,z} \in L^\infty(\omega \times (0, T))$ que verifican (45) y hacen que $w_{R,z}(T) = 0$. Para terminar la demostración, basta probar la existencia de puntos fijos de Λ_R. Dividimos la prueba en dos etapas:

- Demostramos primero que, para cada $R > 0$, Λ_R tiene al menos un punto fijo w_R. Para ello, utilizamos el teorema de Kakutani (véase [4]). La parte más delicada de la aplicación de este resultado consiste en comprobar que existe un compacto K (que depende de R) tal que $\Lambda_R(z) \subset K$, para cada $z \in Z$.

- Por último, demostramos la existencia de un R suficientemente grande tal que $M_R(w_R) = w_R$. Esto probaría el resultado.

Controlabilidad de sistemas N-dimensionales de Navier-Stokes y de Boussinesq con $N-1$ controles escalares

En la tercera parte de esta memoria, probaremos diversos resultados de controlabilidad para problemas no lineales de tipo Navier-Stokes y Boussinesq.

- En primer lugar, trabajamos con el sistema controlado de Navier-Stokes

$$\begin{cases} y_t - \Delta y + (y,\nabla)y + \nabla p = v 1\!\!1_{\mathcal{O}}, \quad \nabla \cdot y = 0 & \text{en } Q, \\ y = 0 & \text{sobre } \Sigma, \\ y(0) = y^0 & \text{en } \Omega \end{cases} \tag{48}$$

y probamos el control local exacto a trayectorias del mismo con la ayuda de $N-1$ controles escalares.

Para poder imponer las hipótesis de regularidad necesarias sobre las condiciones iniciales, definimos los espacios funcionales H, E y V, dados por:

$$H = \{\, w \in L^2(\Omega)^N : \nabla \cdot w = 0 \text{ en } \Omega,\ w \cdot n = 0 \text{ sobre } \partial\Omega \,\}, \tag{49}$$

$$E = \begin{cases} H & \text{si } N = 2, \\ L^4(\Omega)^3 \cap H & \text{si } N = 3 \end{cases}$$

y

$$V = \{ w \in H_0^1(\Omega)^N : \nabla \cdot w = 0 \text{ en } \Omega \}.$$

También deberemos suponer que el dominio de control $\mathcal{O}$ sea adyacente a la frontera $\partial\Omega$, es decir:

$$\exists x^0 \in \partial\Omega,\ \exists \varepsilon > 0 \ \text{ tal que } \ \overline{\mathcal{O}} \cap \partial\Omega \supset B(x^0;\varepsilon) \cap \partial\Omega \tag{50}$$

($B(x^0;\varepsilon)$ es la bola de centro x^0 y radio ε).

Recientemente se ha probado que, con N controles escalares, se puede conseguir la controlabilidad local exacta a trayectorias $(\overline{y},\overline{p})$ del sistema (48) que verifican

$$\overline{y} \in L^\infty(Q)^N, \quad \overline{y}_t \in L^2(0,T;L^\sigma(\Omega)^N) \quad \begin{pmatrix} \sigma > 1 & \text{si } N = 2 \\ \sigma > 6/5 & \text{si } N = 3 \end{pmatrix} \tag{51}$$

(véase [20] y [21]). Lógicamente, estas hipótesis también deberán ser impuestas aquí.

Más precisamente, probaremos que para cualquier par $(\overline{y},\overline{p})$ que verifique

$$\begin{cases} \overline{y}_t - \Delta \overline{y} + (\overline{y},\nabla)\overline{y} + \nabla \overline{p} = 0, \quad \nabla \cdot \overline{y} = 0 & \text{en } Q, \\ \overline{y} = 0 & \text{sobre } \Sigma, \end{cases} \tag{52}$$

y (51), existe $\delta > 0$ tal que, para cualquier $y^0 \in E$ satisfaciendo

$$\|y^0 - \overline{y}(0)\|_E \le \delta,$$

encontramos controles v de clase L^2 con $v_k \equiv 0$ para al menos un k y estados asociados (y, p) verificando

$$y(T) = \overline{y}(T) \ \text{ en } \Omega. \tag{53}$$

Obsérvese que, en esta situación, tras el tiempo $t = T$ podemos dejar evolucionar el sistema libremente; en efecto, éste seguirá la trayectoria 'ideal' $(\overline{y}, \overline{p})$.

Para mayor claridad, enunciamos el resultado a continuación:

Teorema 4 *Supongamos que $\mathcal{O}$ satisface* (50). *Entonces, para todo $T > 0$,* (48) *es localmente exactamente controlable en el tiempo T a las trayectorias $(\overline{y}, \overline{p})$ que verifican* (51), *con controles $v \in L^2(\mathcal{O} \times (0, T))^N$ tales que $v_k \equiv 0$ para al menos un k.*

La demostración sigue los mismos pasos del resultado principal de [21] y se basa principalmente en una desigualdad de Carleman adecuada para el sistema adjunto asociado a una linealización de (48):

$$\begin{cases} -\varphi_t - \Delta\varphi - (D\varphi)\,\overline{y} + \nabla\pi = g, \quad \nabla \cdot \varphi = 0 & \text{en } Q, \\ \varphi = 0 & \text{sobre } \Sigma, \\ \varphi(T) = \varphi^0 & \text{en } \Omega, \end{cases} \tag{54}$$

con $D\varphi = \nabla\varphi + \nabla\varphi^t$. Recordemos que fue probado en [21] que, si $\overline{y}$ cumple (51), existe una constante $C > 0$ tal que la siguiente desigualdad es cierta para todo $\varphi^0 \in L^2(\Omega)^N$:

$$\begin{cases} \displaystyle\iint_Q e^{\frac{-2\overline{\alpha}}{t^4(T-t)^4}} \left(t^{-12}(T-t)^{-12}\,|\varphi|^2 + t^{-4}(T-t)^{-4}\,|\nabla\varphi|^2\right) dx\,dt \\ \quad \displaystyle + \iint_Q e^{\frac{-2\overline{\alpha}}{t^4(T-t)^4}}\, t^4(T-t)^4\, \left(|\Delta\varphi|^2 + |\varphi_t|^2\right) dx\,dt \\ \quad \displaystyle \leq C \left(\iint_Q e^{\frac{-4\widetilde{\alpha}+2\overline{\alpha}}{t^4(T-t)^4}} t^{-30}(T-t)^{-30}\,|g|^2\,dx\,dt \right. \\ \quad \displaystyle \left. + \iint_{\mathcal{O}_0 \times (0,T)} e^{\frac{-8\widetilde{\alpha}+6\overline{\alpha}}{t^4(T-t)^4}}\, t^{-64}(T-t)^{-64}\,|\varphi|^2\,dx\,dt \right) \end{cases} \tag{55}$$

($\overline{\alpha}$ y $\widetilde{\alpha}$ son dos constantes positivas). Aquí, hemos debido elegir $\mathcal{O}_0 \subset \mathcal{O}$, de modo que para todo punto $x \in \mathcal{O}_0$, la recta $\{x + \mathbf{R}e_k\}$ corte a $\partial\Omega$ en un punto de $\partial\mathcal{O}_0$ para cierto $k \in \{1, \cdots, N\}$. Obsérvese que esto es posible gracias a (50).

La desigualdad (55) usada en combinación con la condición de incompresibilidad de φ, permite probar una nueva desigualdad de Carleman donde no aparece el término $|\varphi_k|^2$ a la derecha (véase el lema 4 del capítulo 3).

Con la ayuda de esta última desigualdad de Carleman, podemos probar la controlabilidad nula de un sistema linealizado con segundo miembro no necesariamente nulo:

$$\begin{cases} y_t - \Delta y + (\overline{y}, \nabla)y + (y, \nabla)\overline{y} + \nabla p = f + v\mathbb{1}_{\mathcal{O}}, \quad \nabla \cdot y = 0 & \text{en } Q, \\ y = 0 & \text{sobre } \Sigma, \\ y(0) = y^0 & \text{en } \Omega. \end{cases} \tag{56}$$

En concreto, en la proposición 5 del capítulo 3 se prueba que, fijados f e y^0 adecuados, tenemos el control nulo de (56) con controles v cuya k-ésima componente es nula. Además, se consigue en este resultado que el campo de velocidades asociado y decrezca exponencialmente a cero cuando $t \to T^-$. Para esto último se usan la desigualdad de Carleman probada en el lema 4 del capítulo 3 y ciertos argumentos debidos a O. Yu. Imanuvilov (véanse por ejemplo [35] y [21]).

Por último, con un argumento local basado en un teorema de la función inversa (véase el teorema 8 del capítulo 3), se demuestra la controlabilidad local a trayectorias enunciada en el teorema 4.

• En la segunda parte de este capítulo estudiamos las propiedades de controlabilidad del sistema de Boussinesq:

$$\begin{cases} y_t - \Delta y + (y, \nabla)y + \nabla p = v\mathbb{1}_{\mathcal{O}} + \theta\, e_N, \quad \nabla \cdot y = 0 & \text{en } Q, \\ \theta_t - \Delta\theta + y \cdot \nabla\theta = h\mathbb{1}_{\mathcal{O}} & \text{en } Q, \\ y = 0, \quad \theta = 0 & \text{sobre } \Sigma, \\ y(0) = y^0, \quad \theta(0) = \theta^0 & \text{en } \Omega \end{cases} \tag{57}$$

en dimensión $N = 2$ y en dimensión $N = 3$.

Probaremos un resultado análogo al teorema 4. Así, fijemos una trayectoria no controlada $(\overline{y}, \overline{p}, \overline{\theta})$, esto es, una solución de

$$\begin{cases} \overline{y}_t - \Delta\overline{y} + (\overline{y}, \nabla)\overline{y} + \nabla\overline{p} = \overline{\theta}\, e_N, \quad \nabla \cdot \overline{y} = 0 & \text{en } Q, \\ \overline{\theta}_t - \Delta\overline{\theta} + \overline{y} \cdot \nabla\overline{\theta} = 0 & \text{en } Q, \\ \overline{y} = 0, \quad \overline{\theta} = 0 & \text{sobre } \Sigma, \\ \overline{y}(0) = \overline{y}^0, \quad \overline{\theta}(0) = \overline{\theta}^0 & \text{en } \Omega. \end{cases} \tag{58}$$

Al igual que antes, supondremos que $\overline{y}$ verifica (51). Supondremos también que

$$\overline{\theta} \in L^\infty(Q), \quad \overline{\theta}_t \in L^2(0, T; L^\sigma(\Omega)) \quad \begin{pmatrix} \sigma > 1 & \text{if } N = 2 \\ \sigma > 6/5 & \text{if } N = 3 \end{pmatrix}. \tag{59}$$

El resultado que probamos es el siguiente:

Teorema 5 *Supongamos que $\mathcal{O}$ satisface* (50) *con $n_k(x^0) \neq 0$ para cierto $k < N$. Entonces, para cualquier $T > 0$,* (57) *es localmente exactamente controlable en el instante T a las trayectorias $(\overline{y}, \overline{p}, \overline{\theta})$ que cumplen* (51) *y* (59)*, con controles v y h de clase L^2 tales que $v_k \equiv v_N \equiv 0$. En particular, si $N = 2$, tenemos el control exacto local a trayectorias con controles $v \equiv 0$ y $h \in L^2(\mathcal{O} \times (0, T))$.*

Análogamente a como ocurre en el parágrafo anterior, todo reposa sobre una desigualdad de Carleman 'apropiada' para el problema adjunto

$$\begin{cases} -\varphi_t - \Delta\varphi - (D\varphi)\,\overline{y} + \nabla\pi = g + \overline{\theta}\,\nabla\psi, \quad \nabla \cdot \varphi = 0 & \text{en } Q, \\ -\psi_t - \Delta\psi - \overline{y} \cdot \nabla\psi = q + \varphi_N & \text{en } Q, \\ \varphi = 0, \quad \psi = 0 & \text{sobre } \Sigma, \\ \varphi(T) = \varphi^0, \quad \psi(T) = \psi^0 & \text{en } \Omega. \end{cases} \tag{60}$$

De hecho, se probó en [31] que si $\overline{y}$ y $\overline{\theta}$ verifican respectivamente (51) y (59), entonces existe una constante $C > 0$ de modo que, para todo $(\varphi^0, \psi^0) \in L^2(\Omega)^N \times L^2(\Omega)$, se tiene:

$$\begin{aligned}
&\iint_Q e^{\frac{-2\overline{\alpha}}{t^4(T-t)^4}} t^{-12}(T-t)^{-12}\left(|\varphi|^2 + |\psi|^2\right) dx\,dt \\
&+ \iint_Q e^{\frac{-2\overline{\alpha}}{t^4(T-t)^4}} t^{-4}(T-t)^{-4}\left(|\nabla\varphi|^2 + |\nabla\psi|^2\right) dx\,dt \\
&+ \iint_Q e^{\frac{-2\overline{\alpha}}{t^4(T-t)^4}}\, t^{4}(T-t)^{4}\left(|\Delta\varphi|^2 + |\Delta\psi|^2 + |\varphi_t|^2 + |\psi_t|^2\right) dx\,dt \\
&\qquad \le C\Bigg(\iint_Q e^{\frac{-4\tilde{\alpha}+2\overline{\alpha}}{t^4(T-t)^4}} t^{-30}(T-t)^{-30}\left(|g|^2 + |q|^2\right) dx\,dt \\
&\qquad + \iint_{\mathcal{O}_0\times(0,T)} e^{\frac{-8\tilde{\alpha}+6\overline{\alpha}}{t^4(T-t)^4}}\, t^{-64}(T-t)^{-64}\left(|\varphi|^2 + |\psi|^2\right) dx\,dt\Bigg).
\end{aligned} \tag{61}$$

En esta desigualdad, se puede elegir de nuevo el mismo abierto $\mathcal{O}_0$ de antes. Luego, usando (50), podemos obtener una desigualdad análoga, sin el término $|\varphi_k|^2$ a la derecha. Además, en virtud de la ecuación del calor que aparece en (60), podemos poner la integral de $|\varphi_N|^2$ en función de integrales de $|\psi|^2$ y de términos dependientes de φ_N que pueden ser compensados con lo que hay a la izquierda. La conclusión es una desigualdad en la cual sólo aparecen como términos locales $|\psi|^2$ y (en dimensión 3) $|\varphi_2|^2$ (si $k = 1$) o $|\varphi_1|^2$ (si $k = 2$) (véase el lema 7 del capítulo 3).

Ahora podemos seguir el mismo razonamiento que para el sistema de Navier-Stokes y deducir, en primer lugar, que tenemos la controlabilidad nula de un sistema linealizado asociado a (57) (proposición 6) y, por último, la controlabilidad local exacta a trayectorias del sistema (57).

• El último resultado de este capítulo concierne al siguiente sistema truncado de Navier-Stokes con $N = 2$:

$$\begin{cases} y_t - \Delta y + (y,\nabla)\mathbf{T}_M(y) + \nabla p = v\mathbb{1}_{\mathcal{O}}, \quad \nabla\cdot y = 0 & \text{en } Q, \\ y\cdot n = 0, \quad \nabla\times y = 0 & \text{sobre } \Sigma, \\ y(0) = y^0 & \text{en } \Omega, \end{cases} \tag{62}$$

donde $M > 0$ es una constante.

El objetivo es probar la controlabilidad nula global para este sistema con controles en un cierto espacio de Hilbert.

En concreto, para cada $y^0 \in H$, probaremos que existen controles $v\mathbb{1}_{\mathcal{O}}$, con v perteneciendo al espacio

$$W = \{\, \nabla\times z = (\partial_2 z, -\partial_1 z) : z \in L^2(0,T;H^1(\mathcal{O}))\,\},$$

tales que las soluciones asociadas (y,p) verifican

$$y(T) = 0 \ \text{ en } \Omega. \tag{63}$$

Por el momento, no se sabe si es cierto este mismo resultado con condiciones de contorno de Dirichlet.

El resultado que probamos es el siguiente:

Teorema 6 *Sea $N = 2$. Entonces, para todo $T > 0$ y todo $M > 0$, (62) es exactamente controlable a cero en el tiempo T con controles de la forma $v\mathbb{1}_{\mathcal{O}}$, donde $v \in W$.*

La estrategia que seguimos para probar este resultado difiere de las utilizadas en las demostraciones de los teoremas 4 y 5, pues en este caso resolveremos el problema no lineal a través de un argumento de punto fijo. Más concretamente, consideremos para cada $\overline{y} \in L^\infty(Q)^2$ el sistema

$$\begin{cases} y_t - \Delta y + (y, \nabla)\overline{y} + \nabla p = v\mathbb{1}_{\mathcal{O}}, \quad \nabla \cdot y = 0 & \text{en } Q, \\ y \cdot n = 0, \quad \nabla \times y = 0 & \text{sobre } \Sigma, \\ y(0) = y^0 & \text{en } \Omega \end{cases} \tag{64}$$

y su adjunto, escrito en la forma

$$\begin{cases} -\rho_t - \Delta\rho - \nabla \times ((\overline{y} \cdot \nabla \times)\nabla\gamma) = 0, \quad \Delta\gamma = \rho & \text{en } Q, \\ \gamma = 0, \quad \rho = 0 & \text{sobre } \Sigma, \\ \rho(T) = \rho^0 & \text{en } \Omega. \end{cases} \tag{65}$$

Para las soluciones de este problema se puede de nuevo probar una desigualdad de tipo Carleman (véase el lema 10 del capítulo 3), que nos conduce a una desigualdad de observabilidad:

$$\|(\nabla\gamma)(0)\|_{L^2}^2 \le C \iint_{\mathcal{O}\times(0,T)} |\nabla\gamma|^2 \, dx \, dt. \tag{66}$$

Basándonos en esta desigualdad, se prueba la controlabilidad nula de (64). Además, la solución asociada tiene la regularidad siguiente:

$$y \in L^2(0,T; H^1(\Omega)^2) \cap H^1(0,T; H^{-1}(\Omega)^2)$$

(proposición 7 del capítulo 3). Esto permite probar que una adecuada ecuación de punto fijo en $L^2(Q)^2$ posee soluciones. En consecuencia, nuestro problema de controlabilidad nula para (62) posee solución.

Controlabilidad del sistema de Lamé anisotrópico

En la cuarta y última parte de esta memoria, probaremos la controlabilidad exacta del sistema de Lamé tridimensional anisotrópico siguiente:

$$\begin{cases} \rho(x')\partial_{x_0}^2 u - \mu\Delta u - (\lambda + \mu)\nabla(\nabla \cdot u) + [(R, \nabla)\nabla^t]u = f + v\mathbb{1}_\omega & \text{en } Q, \\ u = 0 & \text{sobre } \Sigma, \\ u(0) = u_0, \quad \partial_{x_0} u(0) = u_1 & \text{en } \Omega, \end{cases} \tag{67}$$

donde las funciones f, u_0 y u_1 son dadas y $\omega \subset \Omega$ es un abierto no vacío.

Como se explicó en la introducción, este sistema proviene del más general (9)–(10) donde se supone que los $\beta_j = 0$ y por tanto

$$\sigma = R + (\nabla u)R + \lambda(tr\epsilon)I + 2\mu\epsilon.$$

En este trabajo, necesitamos imponer varias hipótesis de regularidad y positividad de R y los coeficientes de Lamé. En concreto, supondremos que

$$\rho,\ \lambda,\ \mu,\ R_{ij} \in C^2(\overline{\Omega}),\ \nabla \cdot R = 0,\ \rho > 0,\ \mu - R_{33} > 0,\ \text{ y }\ \lambda + 2\mu - R_{33} > 0 \ \text{ en } \Omega, \tag{68}$$

para $i, j = 1, 2, 3$.

La herramienta clave para establecer este resultado de controlabilidad es una desigualdad de Carleman para las soluciones de sistemas análogos de la forma

$$\begin{cases} \rho(x')\partial_{x_0}^2 u - \mu\Delta u - (\lambda+\mu)\nabla(\nabla\cdot u) + [(R,\nabla)\nabla^t]u = f & \text{en } Q, \\ u = 0 & \text{sobre } \Sigma, \\ u(0) = \partial_{x_0}u(0) = u(T) = \partial_{x_0}u(T) = 0 & \text{en } \Omega. \end{cases} \tag{69}$$

A continuación destacamos los puntos fundamentales de la prueba de esta estimación.

Definimos la función peso

$$\phi(x) = e^{\tau\psi(x)},$$

donde $\tau > 0$ y suponemos que ψ verifica las hipótesis detalladas en la Condición A que aparece al principio de la segunda sección del capítulo 4. En la práctica, esta condición juega un papel análogo al desempeñado por la condición geométrica de Bardos, Lebeau y Rauch para la ecuación de ondas (véase [6]). Tenemos el siguiente resultado:

Teorema 7 *Sea $f \in H^1(Q)^3$ y supongamos que existe una función $\psi = \psi(x)$ con $\frac{\partial\psi}{\partial n}$ suficientemente grande verificando la Condición A y que los coeficientes de Lamé verifican* (68)*. Entonces existe $\tau^* > 0$ tal que para todo $\tau > \tau^*$, existe $s^* > 0$ de modo que toda solución $u \in L^2(0,T;H^2(\Omega)^3) \cap H^1(Q)^3$ de* (69) *verifica*

$$\|u\|_{\mathcal{Y}_\phi(Q)} \le C(\|e^{s\phi}f\|_{H^{1,s}(Q)^3} + \|u\|_{\mathcal{B}_\phi(Q_\omega)}) \quad \forall s > s^* \tag{70}$$

para cierta constante $C > 0$ independiente de s.

En el teorema 7, hemos utilizado la notación Q_ω para designar el dominio de control ($Q_\omega = \omega \times (0,T)$ con $\omega \subset \Omega$ abierto no vacío). También hemos denotado

$$\|u\|^2_{H^{1,s}(Q)} = s^2 \int_Q |u|^2\,dx + \int_Q |\nabla u|^2\,dx,$$

$$\begin{aligned} \|u\|^2_{\mathcal{B}_\phi(Q_\omega)} = \int_{Q_\omega} e^{2s\phi} \Big(& \textstyle\sum_{|\alpha|=0}^2 s^{4-2|\alpha|}|D^\alpha u|^2 + s|\nabla(\nabla\times u)|^2 \\ & + s^3|\nabla\times u|^2 + s|\nabla(\nabla\cdot u)|^2 + s^3|\nabla\cdot u|^2\Big)\,dx \end{aligned} \tag{71}$$

y

$$\|u\|^2_{\mathcal{Y}_\phi(Q)} = \|u\|^2_{\mathcal{B}_\phi(Q)} + s\left\|e^{s\phi}\frac{\partial u}{\partial n}\right\|^2_{H^{1,s}(\Sigma)} + s\left\|e^{s\phi}\frac{\partial^2 u}{\partial n^2}\right\|^2_{L^2(\Sigma)}. \tag{72}$$

Explicaremos a continuación cuál es la idea de la demostración del teorema 7. Empezaremos escribiendo las ecuaciones que verifican las funciones $\nabla \times u$ y $\nabla \cdot u$:

$$\begin{aligned} P(x,D)(\nabla \times u) &\equiv \partial^2_{x_0}(\nabla \times u) - \mu\Delta(\nabla \times u) + (R,\nabla)\nabla^T(\nabla \times u) \\ &= \nabla \times f + P_1 u \end{aligned} \tag{73}$$

y

$$\begin{aligned} P(x,D)(\nabla \cdot u) &\equiv \partial^2_{x_0}(\nabla \cdot u) - (\lambda + 2\mu)\Delta(\nabla \cdot u) + (R,\nabla)\nabla^T(\nabla \cdot u) \\ &= \nabla \cdot f + P_2 u, \end{aligned} \tag{74}$$

con P_1 y P_2 operadores diferenciales de segundo orden.

Obsérvese que no tenemos condiciones de contorno para $\nabla \times u$ ni $\nabla \cdot u$. Utilizaremos una desigualdad de Carleman probada en [49] que puede ser aplicada a las ecuaciones (73) y (74). Esta desigualdad, escrita para $\nabla \times u$ y para $\nabla \cdot u$, nos dice que

$$\begin{aligned} s\|e^{s\phi}(\nabla \times u)\|^2_{H^{1,s}(Q)^3} + s\|e^{s\phi}(\nabla \cdot u)\|^2_{H^{1,s}(Q)} \le C \Bigg(\|e^{s\phi} f\|^2_{H^{1,s}(Q)^3} \\ + s\left\| e^{s\phi}\frac{\partial u}{\partial n}\right\|^2_{H^{1,s}(\Sigma)^3} + s\left\| e^{s\phi}\frac{\partial^2 u}{\partial n^2}\right\|^2_{L^2(\Sigma)^3} + \|u\|^2_{\mathcal{B}_\phi(Q_\omega)}\Bigg) \quad \forall s \ge s_0. \end{aligned} \tag{75}$$

Usando ahora la identidad

$$\Delta u = -\nabla \times (\nabla \times u) + \nabla(\nabla \cdot u)$$

y la condición de contorno $u = 0$ sobre Σ, deducimos directamente que

$$\begin{aligned} \|u\|^2_{\mathcal{Y}_\phi(Q)} \le C\Bigg(\|e^{s\phi} f\|^2_{H^{1,s}(Q)^3} + s\left\| e^{s\phi}\frac{\partial u}{\partial n}\right\|^2_{H^{1,s}(\Sigma)^3} + s\left\| e^{s\phi}\frac{\partial^2 u}{\partial n^2}\right\|^2_{L^2(\Sigma)^3} \\ + \|u\|^2_{\mathcal{B}_\phi(Q_\omega)}\Bigg), \quad \forall s \ge s_0. \end{aligned} \tag{76}$$

Para obtener (70), será suficiente estimar los términos frontera de la expresión anterior. Con este objetivo, definimos una nueva función peso φ que verifica $\varphi = \phi$ sobre Σ:

$$\varphi = e^{\tau\tilde{\psi}}, \quad \text{con} \quad \tilde{\psi} = \psi - \frac{1}{Z^2}\ell_1 + Z\ell_1^2,$$

donde Z es un número suficientemente grande y ℓ_1 es una función regular que verifica

$$\ell_1 = 0 \text{ sobre } \partial\Omega, \quad \ell_1 > 0 \text{ en } \Omega \quad \text{y} \quad \nabla \ell_1 \ne 0 \text{ sobre } \partial\Omega.$$

En concreto, si denotamos Ω_{1/Z^2} al conjunto de los puntos de Ω que se encuentran a una distancia de $\partial\Omega$ inferior a $1/Z^2$, podemos suponer que

$$\varphi(x) < \phi(x) \quad \forall x \in \Omega_{1/Z^2} \times (0,T). \tag{77}$$

Para una función peso de este tipo podemos probar el siguiente resultado:

Lema 1 *Con la notación anterior, tenemos que*

$$\|u\|_{\mathcal{Y}_\varphi(Q)} \leq C\left(\|e^{s\varphi} f\|_{H^{1,s}(Q)^3} + \|u\|_{\mathcal{B}_\varphi(Q_\omega)}\right) \quad \forall s \geq s_0, \tag{78}$$

para las soluciones u de (69) *con soporte en* $\overline{\Omega}_{1/Z^2} \times [0,T]$.

Supongamos por un momento que hemos probado el lema 1 y comprobemos que esto basta para concluir la prueba del teorema 7.

Tomemos $\varepsilon \in (0, 1/Z^2)$, con Z tal que se cumple (77). Entonces

$$\varphi(x) < \phi(x) \quad \forall x \in (\overline{\Omega_\varepsilon \setminus \Omega_{\varepsilon/2}}) \times [0,T]. \tag{79}$$

Definimos ahora una función truncante $\theta \in C_c^2(\Omega_\varepsilon)$ verificando $\theta = 1$ en $\Omega_{\varepsilon/2}$. Obtenemos que la función θu verifica

$$\begin{cases} P(\theta u) \equiv \theta f + [P,\theta]u & \text{en } Q, \\ \theta u = 0 & \text{sobre } \Sigma, \\ (\theta u)(T) = \partial_{x_0}(\theta u)(T) = (\theta u)(0) = \partial_{x_0}(\theta u)(0) = 0 & \text{en } \Omega. \end{cases}$$

Aplicamos pues la desigualdad (78) a θu y deducimos que

$$\begin{aligned} s\left\|e^{s\phi}\frac{\partial u}{\partial n}\right\|^2_{H^{1,s}(\partial\Omega\times(0,T))^3} &+ s\left\|e^{s\phi}\frac{\partial^2 u}{\partial n^2}\right\|^2_{L^2(\partial\Omega\times(0,T))^3} \\ &\leq C\left(\|e^{s\varphi} f\|^2_{H^{1,s}(Q)^3} + \|e^{s\varphi}[P,\theta]\|^2_{H^{1,s}(Q)^3} + \|u\|^2_{\mathcal{B}_\varphi(Q_\omega)}\right) \quad \forall s \geq s_0, \end{aligned} \tag{80}$$

pues $\varphi = \phi$ sobre la frontera. Como el soporte de $[P,\theta]u$ está contenido en $\overline{\Omega_\varepsilon \setminus \Omega_{\varepsilon/2}} \times [0,T]$ y $\varphi < \phi$ en este conjunto (véase (77)), tenemos que

$$\|e^{s\varphi}[P,\theta]u\|_{H^{1,s}(Q)^3} \leq C \sum_{|\alpha|=0}^{2} \|e^{s\phi} D^\alpha u\|_{L^2(Q)^3} \quad \forall s > 0.$$

Esto termina la prueba del teorema 7.

Por lo tanto, en la prueba del lema 1 podemos restringirnos a la situación supp $u \subset B_\delta \cap (\overline{\Omega}_{1/Z^2} \times [0,T])$. En primer lugar, realizamos el cambio de variables

$$\begin{cases} y_1 = x_1, \\ y_2 = x_2, \\ y_3 = x_3 - \ell(x_1, x_2), \end{cases}$$

que nos permite situarnos en un pequeño entorno del punto $y^* = (y_0, 0, 0, 0)$. Escribimos ahora el símbolo principal de las ecuaciones satisfechas por $\nabla \times u$ y $\nabla \cdot u$ en las nuevas coordenadas:

$$\begin{aligned} p_\beta(y,\xi) = -\xi_0^2 + (\beta - R_{11})\xi_1^2 + (\beta - R_{22})\xi_2^2 + \{[(\beta E_3 - R)G^t]G\}\xi_3^2 \\ -2R_{12}\xi_1\xi_2 - 2\textstyle\sum_{j=1}^2 (R_{j3} + \ell_{y_j}(\beta - R_{jj}) - \ell_{y_{3-j}} R_{12})\xi_j\xi_3, \end{aligned} \tag{81}$$

con

$$\beta = \begin{cases} \mu & \text{para } \nabla \times u, \\ \lambda + 2\mu & \text{para } \nabla \cdot u \end{cases}$$

y

$$G = (-\ell_{y_1}, -\ell_{y_2}, 1)^t. \tag{82}$$

Consideremos ahora un recubrimiento finito

$$\{\zeta \in S^3 : |\zeta - \zeta^*_\nu| < \delta_1\}_{1 \le \nu \le M(\delta_1)},$$

de la esfera unidad de $\mathbf{R}^4$

$$S^3 = \{\zeta = (s, \xi') : s^2 + \xi_0^2 + \xi_1^2 + \xi_2^2 = 1\}.$$

Aquí, ζ^*_ν representa un punto de S^3. A este recubrimiento le asociamos una partición de la unidad $\{\chi_\nu\}_{1 \le \nu \le M}$, extendiendo χ_ν fuera de S^3 por una función homogénea de orden 0 con soporte en

$$\mathcal{O}(\delta_1) = \left\{\zeta : \left|\frac{\zeta}{|\zeta|} - \zeta^*_\nu\right| < \delta_1\right\}.$$

El lema 1 es entonces consecuencia del lema siguiente:

Lema 2 *Sea $\gamma^* = (y^*, \zeta^*) \in \partial\mathcal{G} \times S^3$. Entonces, si δ y δ_1 son suficientemente pequeños, tenemos*

$$\begin{aligned} s\|z_\nu\|^2_{H^{1,s}(\mathcal{G})^4} + s(\|z_\nu\|^2_{H^{1,s}(\partial\mathcal{G})^4} + \|\partial_{y_3} z_\nu\|^2_{L^2(\partial\mathcal{G})^4}) \\ \le C(\|e^{s\varphi} f\|^2_{H^{1,s}(\mathcal{G})^3} + \|z\|^2_{H^{1,s}(\mathcal{G})^4}). \end{aligned} \tag{83}$$

En esta desigualdad, hemos introducido el conjunto $\mathcal{G} = \mathbf{R}^3 \times [0, 1/Z^2]$ y la función $z_\nu = \chi_\nu(s, D')z$, donde $z = e^{s\varphi}\mathbf{w}$. y $\mathbf{w} = (w', w_0)$ es el par $(\nabla \times u, \nabla \cdot u)$ escrito en la nueva variable y.

Para demostrar el lema 2, distinguimos varios casos, según que los símbolos principales de $\nabla \times u$ y $\nabla \cdot u$ (definidos en (81)) se anulen o no en γ^*. En cada uno de los casos, las desigualdades buscadas son, esencialmente, consecuencia de la desigualdad de Gärding (véase, por ejemplo, [50]). Todos los detalles de la prueba se encuentran en la segunda sección del capítulo 4.

Explicaremos ahora brevemente cómo se puede deducir el lema 1 a partir del lema 2. De la definición de z_ν, tenemos directamente que

$$\begin{aligned} & s\|z\|^2_{H^{1,s}(\mathcal{G})^4} + s(\|z\|^2_{H^{1,s}(\partial\mathcal{G})^4} + \|\partial_{y_3} z\|^2_{L^2(\partial\mathcal{G})^4}) \\ & \le Cs \sum_{\nu=1}^{M} (\|z_\nu\|^2_{H^{1,s}(\mathcal{G})^4} + \|z_\nu\|^2_{H^{1,s}(\partial\mathcal{G})^4} + \|\partial_{y_3} z_\nu\|^2_{L^2(\partial\mathcal{G})^4}) \\ & \le C(\|e^{s\varphi} f\|^2_{H^{1,s}(\mathcal{G})^3} + \|e^{s\varphi} u\|^2_{H^{2,s}(\mathcal{G})^3}). \end{aligned} \tag{84}$$

Ahora usamos el resultado enunciado en la proposición 4.2 de [39] y encontramos que

$$Z \sum_{|\alpha|=0}^{2} s^{4-2|\alpha|} \|e^{s\varphi} D^\alpha_{y'} u\|^2_{L^2(\mathcal{G})^3} \le C(\|e^{s\varphi} f\|^2_{H^{1,s}(\mathcal{G})^3} + \|e^{s\varphi} u\|^2_{H^{2,s}(\mathcal{G})^3}),$$

con C una constante positiva independiente de s y de Z.

Usando la ecuación diferencial en (69) junto con desigualdades de interpolación, deducimos que las normas $\|e^{s\varphi}\partial^2_{y_0}u\|^2_{L^2(\mathcal{G})^3}$ y $\|e^{s\varphi}\partial^2_{y_0y_1}u\|^2_{L^2(\mathcal{G})^3}$ pueden añadirse a la izquierda de la última desigualdad. Por tanto, combinando esto con (84), obtenemos que

$$\sum_{|\alpha|=0}^{2} s^{4-2|\alpha|}\|e^{s\varphi}D^\alpha u\|^2_{L^2(\mathcal{G})^3} + s(\|z\|^2_{H^{1,s}(\partial\mathcal{G})^4} + \|\partial_{y_3}z\|^2_{L^2(\partial\mathcal{G})^4}) \leq C\|e^{s\varphi}f\|^2_{H^{1,s}(\mathcal{G})^3}. \tag{85}$$

Finalmente, veamos que la desigualdad (78) se puede deducir a partir de (85), es decir, que podemos acotar apropiadamente los términos frontera:

$$s\,\|e^{s\varphi}\partial_{y_3}u\|^2_{H^{1,s}(\partial\mathcal{G})^3} \quad \text{y} \quad s\,\left\|e^{s\varphi}\partial^2_{y_3}u\right\|^2_{L^2(\partial\mathcal{G})^3}. \tag{86}$$

Para ello, sólo usaremos la definición de las variables y_j y las condiciones de contorno de tipo Dirichlet para u, En primer lugar, tenemos que

$$|\partial_{y_3}u_j| \leq |(\nabla\times u)_{3-j}| + \varepsilon(\delta)|\partial_{y_3}u| \quad \text{sobre } \partial\mathcal{G} \quad j=1,2,$$

$$|\partial_{y_3}u_3| \leq |\nabla\cdot u| + \varepsilon(\delta)|\partial_{y_3}u| \quad \text{sobre } \partial\mathcal{G},$$

lo cual nos dice que

$$|e^{s\varphi}\partial_{y_3}u| \leq |z_1| + |z_2| + |z_4| \quad \text{sobre } \partial\mathcal{G}. \tag{87}$$

Además, del mismo modo deducimos que

$$|\partial^2_{y_jy_3}u_k| \leq |\partial_{y_j}(\nabla\times u)_{3-k}| + \varepsilon(\delta)|\partial_{y_jy_3}u| \quad \text{sobre } \partial\mathcal{G} \quad j=0,1,2,\ k=1,2$$

y

$$|\partial^2_{y_jy_3}u_3| \leq |\partial_{y_j}(\nabla\cdot u)| + \varepsilon(\delta)|\partial_{y_jy_3}u| \quad \text{sobre } \partial\mathcal{G} \quad j=0,1,2.$$

Luego

$$|e^{s\varphi}\nabla^{tg}_y\partial_{y_3}u| \leq |\nabla^{tg}_y z_1| + |\nabla^{tg}_y z_2| + |\nabla^{tg}_y z_4| \quad \text{sobre } \partial\mathcal{G}. \tag{88}$$

Por último,

$$|\partial^2_{y_3}u_j| \leq |\partial_{y_3}(\nabla\times u)_{3-j}| + |\partial_{y_j}(\nabla\cdot u)| + \varepsilon(\delta)|\partial_{y_3}\nabla_y u| \quad \text{sobre } \partial\mathcal{G} \quad j=1,2$$

y

$$|\partial^2_{y_3}u_3| \leq |\partial_{y_1}(\nabla\times u)_2| + |\partial_{y_2}(\nabla\times u)_1| + |\partial_{y_3}(\nabla\cdot u)| + \varepsilon(\delta)|\partial_{y_3}\nabla_y u| \quad \text{sobre } \partial\mathcal{G},$$

lo cual conduce a la desigualdad

$$|e^{s\varphi}\partial^2_{y_3}u| \leq |\nabla_y z_1| + |\nabla_y z_2| + |\nabla_y z_4| + \varepsilon(\delta)|e^{s\varphi}\partial_{y_3}\nabla^{tg}_y u| \quad \text{sobre } \partial\mathcal{G}. \tag{89}$$

Podemos ver ya directamente que las estimaciones (87)–(89) implican la desigualdad deseada para los términos (86).

Con esto, queda justificada la prueba del lema 1 a partir de la del lema 2.

Podemos ahora formular el resultado principal de este capítulo:

Teorema 8 *En las condiciones del teorema 7, si la función* ψ *verifica además la Condición B (véase la sección 3 del capítulo 4), se tienen las propiedades de controlabilidad siguientes:*
• *Dados* $f \in L^2(Q)$, $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ *y* $(u_2, u_3) \in H_0^1(\Omega) \times L^2(\Omega)$, *existe un control* $v \in L^2(\omega \times (0,T))$ *tal que la correspondiente solución* u *de* (67) *verifica*

$$u(T) = u_2, \quad \partial_{x_0} u(T) = u_3 \quad en\ \Omega. \tag{90}$$

• *Dados* $f \in H^{-1}(Q)$, $(u_0, u_1) \in L^2(\Omega) \times H^{-1}(\Omega)$ *y* $(u_2, u_3) \in L^2(\Omega) \times H^{-1}(\Omega)$, *existe un control* $v \in H^{-1}(\omega \times (0,T))$ *tal que la correspondiente solución de* (67) *verifica* (90).

La demostración del teorema 8 reposa sobre una desigualdad de observabilidad adecuada que es consecuencia directa del teorema 7. Para más detalles, véase el capítulo 4.

Otros resultados

En esta sección mencionaremos otros resultados relacionados con los que preceden (aunque distintos), en cuya obtención también ha intervenido el autor de la memoria.

En [19], se presenta una panorámica actualizada del papel que juegan las desigualdades globales de Carleman en el contexto de la controlabilidad de ecuaciones en derivadas parciales. Se presta especial atención a las demostraciones de dichas desigualdades para funciones que verifican ecuaciones parabólicas de tipo calor o Stokes. Se tratan distintos casos según la regularidad del segundo miembro y las condiciones de contorno impuestas sobre la solución. También se recuerdan algunos de los últimos avances con respecto a la controlabilidad de dichas ecuaciones.

En el trabajo [21] se prueba el control local a las trayectorias del sistema de Navier-Stokes. Los resultados principales se habían expuesto previamente en [20]. Estos trabajos constituyen un paso previo a los resultados de [23] (que fueron anunciados en la Nota [22]), recogidos como se ha dicho en el capítulo 3.

También, como etapa previa a uno de los resultados establecidos en [23], se prueba en [31] el control local a las trayectorias del sistema de Boussinesq con ayuda de $N+1$ controles escalares.

Por último, otro trabajo a reseñar es [30], donde el autor prueba el control local a las trayectorias de un sistema de Navier-Stokes con condiciones de contorno de tipo Navier no lineales.

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Capítulo 1

Null controllability of the heat equation with Fourier boundary conditions: The linear case

Null controllability of the heat equation with Fourier boundary conditions: The linear case

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Abstract

In this paper, we prove the global null controllability of the linear heat equation completed with linear Fourier boundary conditions of the form $\frac{\partial y}{\partial n} + \beta\, y = 0$. We consider distributed controls with support in a small set and nonregular coefficients $\beta = \beta(x,t)$. For the proof of null controllability, a crucial tool will be a new Carleman estimate for the weak solutions of the classical heat equation with nonhomogeneous Neumann boundary conditions.

1. Introduction

Let $\Omega \subset \mathbf{R}^N$ be a bounded connected open set whose boundary $\partial\Omega$ is regular enough ($N \geq 1$). Let $\omega \subset \Omega$ be a (small) nonempty open subset and let $T > 0$. We will use the notation $Q = \Omega \times (0,T)$ and $\Sigma = \partial\Omega \times (0,T)$ and we will denote by $n(x)$ the outward unit normal to Ω at $x \in \partial\Omega$. On the other hand, we will denote by C, C_1, C_2, ... generic positive constants (usually depending on Ω and ω).

We will consider the linear heat equation with linear Fourier (or Robin) conditions

$$\begin{cases} y_t - \Delta y + B(x,t)\cdot\nabla y + a(x,t)\, y = v(x,t) 1_\omega & \text{in } Q, \\ \dfrac{\partial y}{\partial n} + \beta(x,t)\, y = 0 & \text{on } \Sigma, \\ y(x,0) = y^0(x) & \text{in } \Omega. \end{cases} \tag{1.1}$$

Here, it will be assumed that the coefficients a, B and β satisfy

$$a \in L^\infty(Q), \quad B \in L^\infty(Q)^N, \quad \beta \in L^\infty(\Sigma). \tag{1.2}$$

On the other hand, we suppose that $v \in L^2(\omega \times (0,T))$, 1_ω is the characteristic function of ω and $y^0 \in L^2(\Omega)$. In (1.1), $y = y(x,t)$ is the state and $v = v(x,t)$ is the control. It is assumed that we can act on the system only through $\omega \times (0,T)$.

An illustrative interpretation of the data and variables in (1.1) is the following. The function y can be viewed as the relative temperature of a body (with respect to the exterior surrounding air). The parabolic equation in (1.1) means, among other things, that a heat source $v 1_\omega$ acts on a part of the body. On the boundary, $-\frac{\partial y}{\partial n}$ must be viewed as the *normal heat flux*, directed inwards, up to a positive coefficient. Thus, the equality

$$-\frac{\partial y}{\partial n} = \beta\, y$$

means that this flux is a linear function of the temperature. Thus, it is reasonable to suppose that $\beta \geq 0$ (although this assumption will not be imposed in this paper).

The main goal of this paper is to analyze the controllability properties of (1.1). It will be said that this system is *null controllable* at time T if, for each $y^0 \in L^2(\Omega)$, there exists $v \in L^2(\omega \times (0,T))$ such that the associated solution satisfies

$$y(x,T) = 0 \quad \text{in} \quad \Omega. \tag{1.3}$$

The null controllability of linear parabolic equations has been intensively studied these last years; see for instance [8], [7], [5], [4] and [1].

In this paper, we will be concerned with (1.1), where the main difficulties arise from the particular form of the boundary condition. Indeed, it has been shown in [5] and [2] that this is more difficult to analyze than the case of Dirichlet boundary conditions, considered in [7], [5] and [4].

More precisely, what has been proved until now is that (1.1) is null controllable with $B \equiv 0$ under the assumptions (1.2) whenever $\beta_t \in L^\infty(\Sigma)$. This was shown in [5]. However, it would be important to prove the null controllability of (1.1) without this regularity hypothesis on β_t in view of applications to control systems with *nonlinear* boundary conditions.

The first main result in this paper concerns a Carleman inequality for a general (adjoint) system of the form

$$\begin{cases} -\varphi_t - \Delta\varphi = f_1(x,t) + \nabla \cdot f_2(x,t) & \text{in } Q, \\ (\nabla\varphi + f_2(x,t)) \cdot n = f_3(x,t) & \text{on } \Sigma, \\ \varphi(x,T) = \varphi^0(x) & \text{in } \Omega, \end{cases} \tag{1.4}$$

where $f_1 \in L^2(Q)$, $f_2 \in L^2(Q)^N$ and $f_3 \in L^2(\Sigma)$. Observe that, as long as $\varphi \in L^2(Q)$, $\nabla\varphi + f_2 \in L^2(Q)^N$ and $\nabla \cdot (\nabla\varphi + f_2) \in H^{-1}(0,T;L^2(\Omega))$, we can give a sense to the boundary condition in the space $H^{-1}(0,T;H^{-1/2}(\partial\Omega))$.

We present now this result:

Theorem 1 *Under the previous assumptions on f_1, f_2 and f_3, there exist $\overline{\lambda}$, σ_1, σ_2 and C, only depending on Ω and ω, such that, for any $\lambda \geq \overline{\lambda}$, any $s \geq \overline{s} = \sigma_1(e^{\sigma_2\lambda}T + T^2)$ and any*

$\varphi^0 \in L^2(\Omega)$, *the weak solution to* (1.4) *satisfies*

$$
\begin{aligned}
&\iint_Q e^{-2s\alpha}(s\,\lambda^2\xi\,|\nabla\varphi|^2 + s^3\,\lambda^4\,\xi^3\,|\varphi|^2)dx\,dt \\
&+s^2\lambda^3 \iint_\Sigma e^{-2s\alpha}\,\xi^2\,|\varphi|^2\,d\sigma\,dt \\
&\le C\left(\iint_Q e^{-2s\alpha}(|f_1|^2 + \; s^2\,\lambda^2\,\xi^2\,|f_2|^2)dx\,dt\right. \\
&\left.+s\lambda \iint_\Sigma e^{-2s\alpha}\,\xi\,|f_3|^2\,d\sigma\,dt + s^3\lambda^4 \iint_{\omega\times(0,T)} e^{-2s\alpha}\,\xi^3\,|\varphi|^2\,dx\,dt\right).
\end{aligned}
\tag{1.5}
$$

Here, $\alpha = \alpha(x,t)$ and $\xi = \xi(x,t)$ are appropriate positive functions, again only depending on Ω and ω. They are given below; see (1.13)–(1.14).

As a consequence of theorem 1, we can deduce an *observability inequality* for the adjoint system associated to (1.1). More precisely, let us consider the backward in time system

$$
\begin{cases}
-\varphi_t - \Delta\varphi - \nabla\cdot(\varphi B(x,t)) + a(x,t)\,\varphi = 0 & \text{in } Q, \\
(\nabla\varphi + \varphi B(x,t))\cdot n + \beta(x,t)\varphi = 0 & \text{on } \Sigma, \\
\varphi(x,T) = \varphi^0(x) & \text{in } \Omega,
\end{cases}
\tag{1.6}
$$

where $\varphi^0 \in L^2(\Omega)$. It will be seen that, for some K of the form

$$
K = e^{C(1+\frac{1}{T}+\|a\|_\infty^{2/3}+\|B\|_\infty^2+\|\beta\|_\infty^2)},
\tag{1.7}
$$

the solutions of (1.6) satisfy

$$
\iint_{\Omega\times(T/4,3T/4)} |\varphi|^2\,dx\,dt \le K \iint_{\omega\times(0,T)} |\varphi|^2\,dx\,dt.
\tag{1.8}
$$

Remark 1 In fact, (1.8) is not the unique way of saying that (1.6) is observable. It is indeed more frequent to use other inequalities of the form

$$
\|\varphi(\cdot,0)\|_{L^2(\Omega)}^2 \le C \iint_{\omega\times(0,T)} |\varphi|^2\,dx\,dt
\tag{1.9}
$$

for some C. The estimates (1.9) can be easily deduced from (1.8) and the energy inequalities satisfied by φ.

The second main result in this paper concerns the *null controllability* of (1.1). It is the following:

Theorem 2 *Let us assume that* (1.2) *is satisfied. Then, for each* $T > 0$, (1.1) *is null controllable at time* T *with controls* $v \in L^2(\omega\times(0,T))$. *Moreover, one can find* v *such that*

$$
\|v\|_{L^2(\omega\times(0,T))} \le H\,\|y^0\|_{L^2}\,,
\tag{1.10}
$$

with a constant H of the form

$$H = e^{C(1+\frac{1}{T}+\|a\|_\infty^{2/3}+\|B\|_\infty^2+\|\beta\|_\infty^2+T(\|a\|_\infty+\|B\|_\infty^2+\|\beta\|_\infty^2))} \tag{1.11}$$

for some $C = C(\Omega, \omega)$.

In the proof of theorem 2, the main tool is the estimate (1.8). This arises from a general principle that asserts that the null controllability of (1.1) with controls in $L^2(\omega \times (0,T))$ (depending continuously on the data) is equivalent to the observability of (1.6). More details will be given below.

In a second part of this work, which will appear in a forthcoming paper, we will consider controllability questions for semilinear heat equations completed with *nonlinear* Fourier boundary conditions of the form

$$\frac{\partial y}{\partial n} + f(y) = 0 \quad \text{on} \quad \Sigma,$$

where $f : \mathbf{R} \mapsto \mathbf{R}$ is locally Lipschitz-continuous. For the analysis of these systems, theorems 1 and 2 of the present paper will be crucial.

The rest of this paper is organized as follows. Section 2 is devoted to the proof of theorem 1. In section 3, we deduce the observability inequality (1.8) and we prove theorem 2. For completeness, we have included an Appendix, where we give a detailed proof of the standard Carleman estimate for the solutions of the heat equation with homogeneous Neumann boundary conditions. (this estimate was already proved in [5]; however, in this paper, a careful study of the dependence of the constants on s, λ and T is needed).

2. Proof of theorem 1

The main arguments used below are similar to those in [6]. This is related to a general strategy which is used to relax the regularity assumptions on the various coefficients involved in the problem. Here, it will allow us to proceed without any kind of regularity on the coefficient $\beta = \beta(x,t)$.

Let us recall the definition of a weak solution: we say that φ is a *weak solution* to (1.4) if it satisfies

$$\begin{cases} \varphi \in L^2(0,T; H^1(\Omega)) \cap C^0([0,T]; L^2(\Omega)), \\ -\langle \varphi_t, v \rangle_{(H^1(\Omega))', H^1(\Omega)} + \displaystyle\int_\Omega \nabla\varphi \cdot \nabla v \, dx = \int_\Omega f_1(x,t)\, v \, dx \\ \qquad - \displaystyle\int_\Omega f_2(x,t) \cdot \nabla v \, dx + \int_{\partial\Omega} f_3(x,t)\, v \, d\sigma \\ \text{a.e. in } (0,T), \quad \forall v \in H^1(\Omega), \\ \varphi(x,T) = \varphi^0(x) \quad \text{in} \quad \Omega. \end{cases} \tag{1.12}$$

It is well known that, for $f_1 \in L^2(Q)$, $f_2 \in L^2(Q)^N$, $f_3 \in L^2(\Sigma)$ and $\varphi^0 \in L^2(\Omega)$, (1.4) possesses exactly one weak solution φ.

To prove the Carleman inequality (1.5), we will need two weight functions:

$$\xi(x,t) = \frac{e^{\lambda\eta^0(x)}}{t(T-t)}\,, \quad \alpha(x,t) = \frac{e^{2\lambda\|\eta^0\|_\infty} - e^{\lambda\eta^0(x)}}{t(T-t)}\,. \tag{1.13}$$

Here, $\lambda \geq 1$ is a parameter to be chosen below and $\eta^0 = \eta^0(x)$ is a function satisfying

$$\begin{gathered} \eta^0 \in C^2(\overline{\Omega}), \quad \eta^0(x) > 0 \text{ in } \Omega, \quad \eta^0(x) = 0 \text{ on } \partial\Omega, \\ |\nabla\eta^0(x)| > 0 \text{ in } \overline{\Omega \setminus \omega'}, \end{gathered} \tag{1.14}$$

where $\omega' \subset\subset \omega$ is a nonempty open set. The existence of η^0 satisfying (1.14) is proved in [5].

For the proof of theorem 1, we will need an auxiliary result: a Carleman inequality for the solutions to the heat equation with homogeneous Neumann boundary conditions. This is given in the following result:

Lemma 1 *Let $f \in L^2(Q)$ be given. There exist λ^*, σ^* and C only depending on Ω and ω such that, for any $\lambda \geq \lambda^*$, any $s \geq s^*(\lambda) = \sigma^*(e^{4\lambda\|\eta^0\|_\infty}T + T^2)$ and any $q^0 \in L^2(\Omega)$, the weak solution to*

$$\begin{cases} -q_t - \Delta q = f(x,t) & \text{in } Q, \\ \dfrac{\partial q}{\partial n} = 0 & \text{on } \Sigma, \\ q(x,T) = q^0(x) & \text{in } \Omega \end{cases}$$

satisfies

$$I_{s,\lambda}(q) \leq C\left(\iint_Q e^{-2s\alpha}\,|f|^2\,dx\,dt + s^3\lambda^4 \iint_{\omega\times(0,T)} e^{-2s\alpha}\,\xi^3\,|q|^2\,dx\,dt\right), \tag{1.15}$$

where we have used the notation

$$I_{s,\lambda}(q) = \iint_Q e^{-2s\alpha}\left((s\xi)^{-1}(|q_t|^2 + |\Delta q|^2) + s\lambda^2\xi\,|\nabla q|^2 + s^3\lambda^4\xi^3\,|q|^2\right)\,dx\,dt.$$

This result is a particular case of lemma 1.2 of Chapter I in [5]. For completeness and also in order to explain and justify the particular form of the constants λ^* and $s^*(\lambda)$, we give a complete proof in the Appendix, at the end of this paper.

Let us continue with the proof of theorem 1. We can view φ as a solution by transposition of (1.4). This means that φ is the unique function in $L^2(Q)$ satisfying

$$\begin{cases} \displaystyle\iint_Q \varphi\, h\,dx\,dt = \iint_Q f_1(x,t)\,z\,dx\,dt - \iint_Q f_2(x,t)\cdot\nabla z\,dx\,dt \\ \displaystyle\quad + \iint_\Sigma f_3(x,t)\,z\,d\sigma\,dt + \int_\Omega \varphi^0(x)\,z(x,T)\,dx \quad \forall h \in L^2(Q), \end{cases} \tag{1.16}$$

where we have denoted by z the (strong) solution of the following problem:

$$\begin{cases} z_t - \Delta z = h(x,t) & \text{in } Q, \\ \dfrac{\partial z}{\partial n} = 0 & \text{on } \Sigma, \\ z(x,0) = 0 & \text{in } \Omega. \end{cases}$$

We will argue as follows. Let us first estimate the second term in the left hand side of (1.5), i.e.

$$s^3 \lambda^4 \iint_Q e^{-2s\alpha}\, \xi^3\, |\varphi|^2 \, dx\, dt. \tag{1.17}$$

To this end, we will deal with techniques inspired by the arguments in [6].

Thus, let us see that the term in (1.17) can be bounded by the right hand side of (1.5), i.e.

$$\begin{aligned} s^3 \lambda^4 \iint_Q e^{-2s\alpha}\, \xi^3\, |\varphi|^2 \, dx\, dt \le C(\Omega,\omega) \Bigg(\iint_Q e^{-2s\alpha}\, |f_1|^2 \, dx\, dt \\ + s^2\lambda^2 \iint_Q e^{-2s\alpha}\, \xi^2\, |f_2|^2 \, dx\, dt + s\lambda \iint_\Sigma e^{-2s\alpha}\, \xi\, |f_3|^2 \, d\sigma\, dt \\ + s^3\lambda^4 \iint_{\omega\times(0,T)} e^{-2s\alpha}\, \xi^3\, |\varphi|^2 \, dx\, dt \Bigg) \end{aligned} \tag{1.18}$$

for a good choice of the parameters λ and s.

Let us consider the following constrained extremal problem:

$$\begin{cases} \text{Minimize } \dfrac{1}{2} \left(\displaystyle\iint_Q e^{2s\alpha}\, |z|^2 + s^{-3}\lambda^{-4} \iint_{\omega\times(0,T)} e^{2s\alpha}\, \xi^{-3}\, |v|^2 \, dx\, dt \right) \\ \text{subject to } v \in L^2(Q) \text{ and} \\ \quad \begin{cases} z_t - \Delta z = s^3\lambda^4 e^{-2s\alpha}\xi^3\varphi + v 1_\omega & \text{in } Q, \\ \dfrac{\partial z}{\partial n} = 0 & \text{on } \Sigma, \\ z(x,0) = 0, \quad z(x,T) = 0 & \text{in } \Omega. \end{cases} \end{cases} \tag{1.19}$$

Here, s and λ are chosen like in lemma 1.

By virtue of Lagrange's principle and arguing as in [6], we are led from (1.19) to the next optimality system, which is of fourth order in space and second order in time:

$$\begin{cases} \mathcal{L}(e^{-2s\alpha}\, \mathcal{L}^* p) + s^3\lambda^4 e^{-2s\alpha}\xi^3 p 1_\omega = s^3\lambda^4 e^{-2s\alpha}\xi^3\varphi & \text{in } Q, \\ \dfrac{\partial p}{\partial n} = 0, \quad \dfrac{\partial}{\partial n}(e^{-2s\alpha}\, \mathcal{L}^* p) = 0 & \text{on } \Sigma, \\ (e^{-2s\alpha}\mathcal{L}^* p)_{|t=0} = (e^{-2s\alpha}\mathcal{L}^* p)_{|t=T} = 0 & \text{in } \Omega. \end{cases} \tag{1.20}$$

Here, $\mathcal{L} = \partial_t - \Delta$ is the heat operator and $\mathcal{L}^* = -\partial_t - \Delta$ is its formal adjoint. If p is a solution to (1.20) (in an appropriate sense), then

$$\widehat{v} = -s^3\, \lambda^4\, e^{-2s\alpha}\, \xi^3\, p 1_\omega \quad \text{and} \quad \widehat{z} = e^{-2s\alpha}\mathcal{L}^* p \tag{1.21}$$

solve (1.19).

Let us show that (1.20) has a unique *weak* solution. To this end, we are going to rewrite this problem as a Lax-Milgram variational equation. Let us introduce the space

$$X_0 = \{\, z \in C^2(\overline{Q}) : \frac{\partial z}{\partial n} = 0 \text{ on } \Sigma \,\}$$

and the norm $\| \cdot \|_X$, with

$$\|q\|_X^2 = \iint_Q e^{-2s\alpha}\, |\mathcal{L}^* q|^2 \, dx\, dt + s^3 \lambda^4 \iint_{\omega\times(0,T)} e^{-2s\alpha}\, \xi^3\, |q|^2 \, dx\, dt$$

for all $q \in X_0$.

Due to lemma 1, $\| \cdot \|_X$ is indeed a norm in X_0 . Let X be the completion of X_0 for the norm $\| \cdot \|_X$. Then X is a Hilbert space for the scalar product $(\cdot\,,\cdot)_X$, with

$$(p, q)_X = \iint_Q e^{-2s\alpha}\, (\mathcal{L}^* p)(\mathcal{L}^* q) \, dx\, dt + s^3 \lambda^4 \iint_{\omega\times(0,T)} e^{-2s\alpha}\, \xi^3\, p\, q \, dx\, dt.$$

With this notation, system (1.20) is equivalent to find a function $p \in X$ such that

$$(p, q)_X = \ell(q) \quad \forall q \in X, \tag{1.22}$$

where

$$\ell(q) = s^3\, \lambda^4 \iint_Q e^{-2s\alpha}\, \xi^3\, \varphi\, q \, dx\, dt \quad \forall q \in X.$$

Of course, (1.22) is equivalent to another extremal problem

$$\begin{cases} \text{Minimize } \dfrac{1}{2}(q, q)_X - \ell(q) \\ \text{subject to } q \in X. \end{cases}$$

By virtue of lemma 1, one can easily check that $\ell \in X'$. Consequently, one can apply Lax-Milgram lemma and deduce that there exists a unique solution to (1.20).

Let us now take

$$h = s^3 \lambda^4 e^{-2s\alpha} \xi^3 \varphi + \widehat{v} 1_\omega$$

in (1.16). This gives

$$\begin{aligned} s^3 \lambda^4 \iint_Q e^{-2s\alpha}\, \xi^3\, |\varphi|^2 \; dx\, dt &= \iint_Q f_1\, \widehat{z} \, dx\, dt - \iint_Q f_2 \cdot \nabla \widehat{z} \, dx\, dt \\ &\quad + \iint_\Sigma f_3\, \widehat{z} \, d\sigma\, dt - \iint_{\omega\times(0,T)} \varphi\, \widehat{v} \, dx\, dt \end{aligned} \tag{1.23}$$

(recall that $\widehat{v}$ and $\widehat{z}$ are given by (1.21)). The idea of the proof of (1.18) is to bound $\widehat{z}$, $\nabla \widehat{z}$ and $\widehat{v}$ in Q and the trace of $\widehat{z}$ on Σ in terms of the left hand side of (1.23). For this purpose, we first multiply the equation in (1.20) by p and integrate in Q, which gives

$$\|p\|_X^2 \le \|\ell\|_{X'}\, \|p\|_X$$

and, consequently,

$$\begin{aligned}\|p\|_X^2 &= \iint_Q e^{2s\alpha}\,|\widehat{z}|^2\,dx\,dt + s^{-3}\lambda^{-4}\iint_{\omega\times(0,T)} e^{2s\alpha}\,\xi^{-3}\,|\widehat{v}|^2\,dx\,dt \\ &\le Cs^3\lambda^4\iint_Q e^{-2s\alpha}\,\xi^3\,|\varphi|^2\,dx\,dt,\end{aligned} \tag{1.24}$$

for $\lambda \ge \overline{\lambda}(\Omega,\omega)$, $s \ge \overline{\sigma}(\Omega,\omega)(e^{4\lambda\|\eta^0\|_\infty}T + T^2)$. This provides the desired bounds of $\widehat{z}$ and $\widehat{v}1_\omega$.

Let us now multiply the equation satisfied by $\widehat{z}$ by $s^{-2}\lambda^{-2}e^{2s\alpha}\xi^{-2}\widehat{z}$ and let us integrate in Q. After integration by parts, we obtain:

$$\begin{aligned}&\frac{1}{2}s^{-2}\lambda^{-2}\iint_Q e^{2s\alpha}\xi^{-2}\,\frac{\partial}{\partial t}|\widehat{z}|^2\,dx\,dt + s^{-2}\lambda^{-2}\iint_Q e^{2s\alpha}\xi^{-2}\,|\nabla\widehat{z}|^2\,dx\,dt \\ &-s^{-1}\lambda^{-1}\iint_Q e^{2s\alpha}\,\xi^{-1}\,\nabla\eta^0\cdot\nabla|\widehat{z}|^2\,dx\,dt \\ &-2s^{-2}\lambda^{-1}\iint_Q e^{2s\alpha}\,\xi^{-2}\,(\nabla\eta^0\cdot\nabla\widehat{z})\,\widehat{z}\,dx\,dt \\ &= s\lambda^2\iint_Q \xi\,\varphi\,\widehat{z}\,dx\,dt + s^{-2}\lambda^{-2}\iint_{\omega\times(0,T)} e^{2s\alpha}\,\xi^{-2}\,\widehat{v}\,\widehat{z}\,dx\,dt,\end{aligned}$$

whence

$$\begin{aligned}&s^{-2}\lambda^{-2}\iint_Q e^{2s\alpha}\,\xi^{-2}\,|\nabla\widehat{z}|^2\,dx\,dt - s^{-1}\lambda^{-1}\iint_\Sigma e^{2s\alpha}\xi^{-1}\,\frac{\partial\eta^0}{\partial n}\,|\widehat{z}|^2\,d\sigma\,dt \\ &= \frac{1}{2}s^{-2}\lambda^{-2}\iint_Q \frac{\partial}{\partial t}(e^{2s\alpha}\,\xi^{-2})\,|\widehat{z}|^2\,dx\,dt \\ &-s^{-1}\lambda^{-1}\iint_Q \nabla\cdot(e^{2s\alpha}\,\xi^{-1}\,\nabla\eta^0)\,|\widehat{z}|^2\,dx\,dt \\ &+2s^{-2}\lambda^{-1}\iint_Q e^{2s\alpha}\,\xi^{-2}\,\nabla\eta^0\cdot\nabla\widehat{z}\,\widehat{z}\,dx\,dt + s\lambda^2\iint_Q \xi\,\varphi\,\widehat{z}\,dx\,dt \\ &+s^{-2}\lambda^{-2}\iint_{\omega\times(0,T)} e^{2s\alpha}\,\xi^{-2}\,\widehat{v}\,\widehat{z}\,dx\,dt.\end{aligned} \tag{1.25}$$

We need now some estimates concerning the weight functions in order to preserve explicit bounds in s, λ and T. Notice that

$$\begin{aligned}\frac{\partial}{\partial t}(e^{2s\alpha}\,\xi^{-2}) &= -2(T-2t)\,e^{-\lambda\eta^0}\,e^{2s\alpha}(s\,e^{-\lambda\eta^0}(e^{2\lambda\|\eta^0\|_\infty} - e^{\lambda\eta^0}) - \xi^{-1}) \\ &\le C\,T\,e^{2s\alpha}(e^{2\lambda\|\eta^0\|_\infty}\,s + \xi^{-1}) \le C\,T\,s\,e^{2s\alpha}\,e^{2\lambda\|\eta^0\|_\infty},\end{aligned}$$

where we have taken $s \ge C\,T^2$. More generally, observe that, for any fixed m, one also has

$$|\nabla(e^{2s\alpha}\,\xi^m)| \le C_m(\Omega,\omega)\,s\,\lambda\,e^{2s\alpha}\,\xi^{m+1} \tag{1.26}$$

whenever $s \geq C\,T^2$. Indeed, we have

$$\nabla(e^{2s\alpha}\,\xi^m) = e^{2s\alpha}\,\lambda\,\nabla\eta^0\,\xi^m(2s\,\xi + m) \leq C(\Omega,\omega)\,e^{2s\alpha}\,\lambda\,\xi^m(s\,\xi+1)$$

and, taking into account that

$$C\,s\,\xi \geq 1 \quad \text{for } s \geq \frac{T^2}{4C}\,, \tag{1.27}$$

we directly get (1.26).

Turning back to (1.25), we obtain

$$\begin{aligned}
&s^{-2}\lambda^{-2}\iint_Q e^{2s\alpha}\,\xi^{-2}\,|\nabla\widehat{z}|^2\,dx\,dt - s^{-1}\lambda^{-1}\iint_\Sigma e^{2s\alpha}\xi^{-1}\,\frac{\partial\eta^0}{\partial n}\,|\widehat{z}|^2\,d\sigma\,dt\\
&\leq C(\Omega,\omega)\biggl(T\,s^{-1}\lambda^{-2}\,e^{2\lambda\|\eta^0\|_\infty}\iint_Q e^{2s\alpha}\,|\widehat{z}|^2\,dx\,dt\\
&+\iint_Q e^{2s\alpha}\,|\widehat{z}|^2\,dx\,dt + s^{-1}\lambda^{-1}\iint_Q e^{2s\alpha}\,\xi^{-1}\,|\widehat{z}|^2\,dx\,dt\\
&+s^{-2}\iint_Q e^{2s\alpha}\,\xi^{-2}\,|\widehat{z}|^2\,dx\,dt + s^2\lambda^4\iint_Q e^{-2s\alpha}\,\xi^2\,|\varphi|^2\,dx\,dt\\
&+s^{-4}\lambda^{-4}\iint_Q e^{2s\alpha}\,\xi^{-4}\,|\widehat{v}|^2\,dx\,dt\biggr) + \frac{1}{2}s^{-2}\lambda^{-2}\iint_Q e^{2s\alpha}\,\xi^{-2}\,|\nabla\widehat{z}|^2\,dx\,dt,
\end{aligned}$$

where we have taken $s \geq C\,T^2$. Now, we take into account (1.27) and we deduce that

$$\begin{aligned}
&s^{-2}\lambda^{-2}\iint_Q e^{2s\alpha}\,\xi^{-2}\,|\nabla\widehat{z}|^2\,dx\,dt - s^{-1}\lambda^{-1}\iint_\Sigma e^{2s\alpha}\xi^{-1}\,\frac{\partial\eta^0}{\partial n}\,|\widehat{z}|^2\,d\sigma\,dt\\
&\leq C(\Omega,\omega)\biggl(\iint_Q e^{2s\alpha}\,|\widehat{z}|^2\,dx\,dt + s^3\lambda^4\iint_Q e^{-2s\alpha}\,\xi^3\,|\varphi|^2\,dx\,dt\\
&\qquad s^{-3}\lambda^{-4}\iint_Q e^{2s\alpha}\,\xi^{-3}\,|\widehat{v}|^2\,dx\,dt\biggr)
\end{aligned}$$

for any $\lambda \geq C(\Omega,\omega)$ and any $s \geq C(\Omega,\omega)(e^{2\lambda\|\eta^0\|_\infty}\,T + T^2)$.

From (1.14), this gives an estimate of the gradient and the trace of $\widehat{z}$ in terms of $\widehat{z}$, $\widehat{v}1_\omega$ and φ. In view of (1.24), we now have

$$\begin{aligned}
&\iint_Q e^{2s\alpha}\,|\widehat{z}|^2\,dx\,dt + s^{-2}\lambda^{-2}\iint_Q e^{2s\alpha}\,\xi^{-2}\,|\nabla\widehat{z}|^2\,dx\,dt\\
&+s^{-1}\lambda^{-1}\iint_\Sigma e^{2s\alpha}\xi^{-1}\,|\widehat{z}|^2\,d\sigma\,dt + s^{-3}\lambda^{-4}\iint_{\omega\times(0,T)} e^{2s\alpha}\,\xi^{-3}\,|\widehat{v}|^2\,dx\,dt\\
&\qquad\leq C(\Omega,\omega)\,s^3\lambda^4\iint_Q e^{-2s\alpha}\,\xi^3\,|\varphi|^2\,dx\,dt
\end{aligned}$$

for $\lambda \geq C(\Omega,\omega)$, $s \geq C(\Omega,\omega)(e^{2\lambda\|\eta^0\|_\infty}\,T + T^2)$.

It suffices to combine this inequality and the identity (1.23) to deduce (1.18).

Let us now show that

$$\begin{aligned} s\lambda^2 \iint_Q e^{-2s\alpha}\,\xi\,|\nabla\varphi|^2\,dx\,dt \le C(\Omega,\omega)\Big(\iint_Q e^{-2s\alpha}\,|f_1|^2\,dx\,dt \\ +s^2\lambda^2 \iint_Q e^{-2s\alpha}\,\xi^2\,|f_2|^2\,dx\,dt + s\lambda \iint_\Sigma e^{-2s\alpha}\,\xi\,|f_3|^2\,d\sigma\,dt \\ +s^3\lambda^4 \iint_{\omega\times(0,T)} e^{-2s\alpha}\,\xi^3\,|\varphi|^2\,dx\,dt\Big). \end{aligned} \tag{1.28}$$

To this end, we now have to use not only that φ is a solution by transposition but a weak solution as well. More precisely, let us take

$$v = s\lambda^2 e^{-2s\alpha(\cdot,t)}\xi(\cdot,t)\varphi(\cdot,t)$$

in (1.12). Then, let us integrate in $(0,T)$ and let us perform integrations by parts similarly as we did before. We get:

$$\begin{aligned} &-\frac{1}{2}s\lambda^2 \iint_Q e^{-2s\alpha}\xi\,\frac{\partial}{\partial t}|\varphi|^2\,dx\,dt + s\lambda^2 \iint_Q e^{-2s\alpha}\xi\,|\nabla\varphi|^2\,dx\,dt \\ &+s\lambda^2 \iint_Q \nabla\varphi\cdot\nabla(e^{-2s\alpha}\,\xi)\,\varphi\,dx\,dt \\ &\qquad = s\lambda^2 \iint_Q e^{-2s\alpha}\,\xi\,f_1\,\varphi\,dx\,dt - s\lambda^2 \iint_Q f_2\cdot\nabla(e^{-2s\alpha}\,\xi\,\varphi)\,dx\,dt \\ &\qquad +s\lambda^2 \iint_\Sigma e^{-2s\alpha}\,\xi\,f_3\,\varphi\,d\sigma\,dt. \end{aligned}$$

We integrate by parts again and we obtain

$$\begin{aligned} &s\lambda^2 \iint_Q e^{-2s\alpha}\xi\,|\nabla\varphi|^2\,dx\,dt \\ &\quad = -\frac{1}{2}s\lambda^2 \iint_Q (e^{-2s\alpha}\xi)_t\,|\varphi|^2\,dx\,dt - s\lambda^2 \iint_Q \nabla\varphi\cdot\nabla(e^{-2s\alpha}\,\xi)\,\varphi\,dx\,dt \\ &\quad +s\lambda^2 \iint_Q e^{-2s\alpha}\,\xi\,f_1\,\varphi\,dx\,dt - s\lambda^2 \iint_Q f_2\cdot\nabla(e^{-2s\alpha}\,\xi)\,\varphi\,dx\,dt \\ &\quad -s\lambda^2 \iint_Q f_2\cdot\nabla\varphi\,e^{-2s\alpha}\,\xi\,dx\,dt + s\lambda^2 \iint_\Sigma e^{-2s\alpha}\,\xi\,f_3\,\varphi\,d\sigma\,dt. \end{aligned}$$

In view of (1.26), we find:

$$
\begin{aligned}
&s\lambda^2 \iint_Q e^{-2s\alpha}\xi\, |\nabla\varphi|^2\, dx\, dt \\
&\quad \le C(\Omega,\omega) \biggl(T\, s^2\, \lambda^2\, e^{2\lambda\|\eta^0\|_\infty} \iint_Q e^{-2s\alpha}\, \xi^3\, |\varphi|^2\, dx\, dt \\
&\quad + s^3\lambda^4 \iint_Q e^{-2s\alpha}\, \xi^3\, |\varphi|^2\, dx\, dt + \iint_Q e^{-2s\alpha}\, |f_1|^2\, dx\, dt \\
&\quad + s^2\lambda^4 \iint_Q e^{-2s\alpha}\, \xi^2\, |\varphi|^2\, dx\, dt + s\lambda^2 \iint_Q e^{-2s\alpha}\, \xi\, |f_2|^2\, dx\, dt \\
&\quad s\lambda \iint_\Sigma e^{-2s\alpha}\, \xi\, |f_3|^2\, d\sigma\, dt + s\lambda^3 \iint_\Sigma e^{-2s\alpha}\, \xi\, |\varphi|^2\, d\sigma\, dt \biggr) \\
&\quad + \frac{1}{2} s\lambda^2 \iint_Q e^{-2s\alpha}\xi\, |\nabla\varphi|^2\, dx\, dt,
\end{aligned}
$$

where we have taken $s \ge CT^2$ and $\lambda \ge C$. Making several simplifications, we easily see that

$$
\left\{
\begin{aligned}
&s\lambda^2 \iint_Q e^{-2s\alpha}\xi\, |\nabla\varphi|^2\, dx\, dt \le C \biggl(s^3\lambda^4 \iint_Q e^{-2s\alpha}\, \xi^3\, |\varphi|^2\, dx\, dt \\
&\quad + \iint_Q e^{-2s\alpha}\, |f_1|^2\, dx\, dt + s^2\lambda^2 \iint_Q e^{-2s\alpha}\, \xi^2\, |f_2|^2\, dx\, dt \\
&\quad + s\lambda \iint_\Sigma e^{-2s\alpha}\, \xi\, |f_3|^2\, d\sigma\, dt + s\lambda^3 \iint_\Sigma e^{-2s\alpha}\, \xi\, |\varphi|^2\, d\sigma\, dt \biggr),
\end{aligned}
\right. \tag{1.29}
$$

for $s \ge C(e^{2\lambda\|\eta^0\|_\infty}\, T + T^2)$ and $\lambda \ge C$, whence (1.28) follows easily.

Let us finally estimate the trace of φ in terms of φ and $\nabla\varphi$. Notice that

$$
\begin{aligned}
&-s^2\lambda^3 \iint_Q e^{-2s\alpha}\, \xi^2\, (\nabla\eta^0 \cdot \nabla\varphi)\, \varphi\, dx\, dt \\
&\qquad = -\frac{1}{2} s^2\lambda^3 \iint_\Sigma e^{-2s\alpha}\, \xi^2\, \frac{\partial\eta^0}{\partial n}\, |\varphi|^2\, d\sigma\, dt \\
&\qquad + \frac{1}{2} s^2\lambda^3 \iint_Q \nabla\cdot(e^{-2s\alpha}\, \xi^2\, \nabla\eta^0)\, |\varphi|^2\, dx\, dt.
\end{aligned}
$$

Taking into account (1.14), the following is found:

$$
\begin{aligned}
&s^2\lambda^3 \iint_\Sigma e^{-2s\alpha}\, \xi^2\, |\varphi|^2\, d\sigma\, dt \le C\, s^2\, \lambda^3 \iint_Q |\nabla\cdot(e^{-2s\alpha}\, \xi^2\, \nabla\eta^0)|\, |\varphi|^2\, dx\, dt \\
&\quad + C \left(s^3\lambda^4 \iint_Q e^{-2s\alpha}\, \xi^3\, |\varphi|^2\, dx\, dt + s\lambda^2 \iint_Q e^{-2s\alpha}\xi\, |\nabla\varphi|^2\, dx\, dt \right) \\
&\quad \le C \left(s^3\lambda^4 \iint_Q e^{-2s\alpha}\, \xi^3\, |\varphi|^2\, dx\, dt + s\lambda^2 \iint_Q e^{-2s\alpha}\xi\, |\nabla\varphi|^2\, dx\, dt \right),
\end{aligned}
$$

with $s \geq C(T+T^2)$ and $\lambda \geq C$.

This last inequality, together with (1.18) and (1.29), provides (1.5) and permits to achieve the proof of theorem 1.

3. Controllability of the linear system

This section is devoted to prove theorem 2. This will be a consequence of the Carleman inequality (1.5).

We will start with an explicit bound of the weak solution to the linear problem

$$\begin{cases} y_t - \Delta y + B(x,t)\cdot\nabla y + a(x,t)\,y = f(x,t) & \text{in } Q, \\ \dfrac{\partial y}{\partial n} + \beta(x,t) y = 0 & \text{on } \Sigma, \\ y(x,0) = y^0(x) & \text{in } \Omega, \end{cases} \tag{1.30}$$

where $f \in L^2(Q)$, $y^0 \in L^2(\Omega)$ and (1.2) is fulfilled. Then, we will use this result in combination with (1.5) to deduce the observability inequality (1.8) for the solutions to (1.6). Finally, we will end the proof of theorem 2 in a classical way, using this observability inequality.

Proposition 1 *Under the previous assumptions, the weak solution to* (1.30) *satisfies the estimate*

$$\|y\|_Y \leq e^{C\,T\,(1+\|a\|_\infty+\|B\|_\infty^2+\|\beta\|_\infty^2)}\,(\|f\|_{L^2(Q)} + \|y^0\|_{L^2(\Omega)}) \tag{1.31}$$

for some constant $C>0$. *Here,* Y *is the usual energy space:*

$$Y = L^2(0,T;H^1(\Omega)) \cap C^0([0,T];L^2(\Omega)).$$

Proof: The existence and uniqueness of a solution to (1.30) is well known. Furthermore, the following identity can be deduced for each $t \in (0,T)$ in a standard way:

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_\Omega |y(x,t)|^2\,dx + \int_\Omega |\nabla y(x,t)|^2\,dx + \int_{\partial\Omega} \beta(x,t)\,|y(x,t)|^2\,d\sigma \\ &\quad + \int_\Omega B(x,t)\cdot\nabla y(x,t)\,y(x,t)\,dx + \int_\Omega a(x,t)\,|y(x,t)|^2\,dx \\ &\quad = \int_\Omega f(x,t)\,y(x,t)\,dx. \end{aligned} \tag{1.32}$$

We will now use the following trace estimate for the functions in $H^1(\Omega)$:

$$\begin{cases} \displaystyle\int_{\partial\Omega} |u|^2\,d\sigma \leq C\left(\int_\Omega (|u|^2 + |\nabla u|^2)\,dx\right)^{1/2}\left(\int_\Omega |u|^2\,dx\right)^{1/2} \\ \forall u \in H^1(\Omega), \end{cases} \tag{1.33}$$

for some positive $C = C(\Omega)$. This inequality can be proved arguing first for regular functions in a dense subspace of $H^1(\Omega)$ and then passing to the limit. For a regular function u, (1.33) is very easy to establish when $\Omega = \mathbf{R}^N_+$. Then, a standard localization argument leads to the proof in the case of a general domain Ω.

In view of (1.32) and (1.33), we have:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega |y(x,t)|^2\,dx + \int_\Omega |\nabla y(x,t)|^2\,dx \\
&\quad\le -\int_\Omega B(x,t)\cdot\nabla y(x,t)\,y(x,t)\,dx - \int_\Omega a(x,t)\,|y(x,t)|^2\,dx \\
&\quad + \int_\Omega f(x,t)\,y(x,t)\,dx + C\,\|\beta\|_\infty\,\|y(\cdot,t)\|_{H^1(\Omega)}\,\|y(\cdot,t)\|_{L^2(\Omega)}.
\end{aligned}$$

Combining this and Young's inequality, we obtain:

$$\begin{aligned}
&\frac{d}{dt}\|y(\cdot,t)\|^2_{L^2(\Omega)} + \|y(\cdot,t)\|^2_{H^1(\Omega)} \\
&\quad\le C((1+\|a\|_\infty + \|B\|^2_\infty + \|\beta\|^2_\infty)\|y(\cdot,t)\|^2_{L^2(\Omega)} + \|f(\cdot,t)\|^2_{L^2(\Omega)})
\end{aligned}$$

for all $t\in(0,T)$. From these estimates, it is not difficult to obtain (1.31).

This ends the proof.

The announced observability estimate is proved in the following result:

Proposition 2 *For every $\varphi^0\in L^2(\Omega)$, the associated solution to (1.6) satisfies the observability inequality*

$$\iint_{\Omega\times(T/4,3T/4)} |\varphi|^2\,dx\,dt \le K\iint_{\omega\times(0,T)} |\varphi|^2\,dx\,dt. \tag{1.34}$$

for a constant K of the form

$$K = \exp\{C(1+\frac{1}{T} + \|a\|^{2/3}_\infty + \|B\|^2_\infty + \|\beta\|^2_\infty)\}. \tag{1.35}$$

Proof: Let $\varphi^0\in L^2(\Omega)$ be given. Notice that the corresponding φ solves (1.4) with

$$f_1 = -a\,\varphi\in L^2(Q),\quad f_2 = \varphi B\in L^2(0,T;L^2(\Omega)^N),\quad f_3 = -\beta\,\varphi\in L^2(\Sigma).$$

Thus, we can apply theorem 1 to φ and deduce that

$$\begin{aligned}
&\iint_Q e^{-2s\alpha}(s\,\lambda^2\,\xi\,|\nabla\varphi|^2 + s^3\,\lambda^4\,\xi^3\,|\varphi|^2)dx\,dt + s^2\lambda^3\iint_\Sigma e^{-2s\alpha}\,\xi^2\,|\varphi|^2\,d\sigma\,dt \\
&\le C(\Omega,\omega)\left(\|a\|^2_\infty\iint_Q e^{-2s\alpha}\,|\varphi|^2\,dx\,dt + s^2\lambda^2\|B\|^2_\infty\iint_Q e^{-2s\alpha}\,\xi^2\,|\varphi|^2\,dx\,dt\right. \\
&\quad\left. + s\,\lambda\,\|\beta\|^2_\infty\iint_\Sigma e^{-2s\alpha}\,\xi\,|\varphi|^2\,d\sigma\,dt + s^3\lambda^4\iint_{\omega\times(0,T)} e^{-2s\alpha}\,\xi^3\,|\varphi|^2\,dx\,dt\right)
\end{aligned}$$

for any $\lambda\ge\overline{\lambda}$ and any $s\ge\overline{\sigma}(e^{4\lambda\|\eta^0\|_\infty}T + T^2)$.

We will now try to eliminate the global terms in the right hand side of this inequality by making a convenient choice of the parameter s.

Taking $s \geq C\,T^2\,(\|a\|_\infty^{2/3} + \|B\|_\infty^2)$, we see that

$$C\left(s^2\lambda^2\|B\|_\infty^2 \iint_Q e^{-2s\alpha}\,\xi^2\,|\varphi|^2\,dx\,dt + \|a\|_\infty^2 \iint_Q e^{-2s\alpha}\,|\varphi|^2\,dx\,dt\right)$$
$$\leq \frac{1}{2}\,s^3\,\lambda^4 \iint_Q e^{-2s\alpha}\,\xi^3\,|\varphi|^2\,dx\,dt.$$

On the other hand, taking $s \geq C\,T^2\,\|\beta\|_\infty^2$, we find that

$$C\,s\,\lambda\,\|\beta\|_\infty^2 \iint_\Sigma e^{-2s\alpha}\,\xi\,|\varphi|^2\,d\sigma\,dt \leq \frac{1}{2}\,s^2\,\lambda^3 \iint_\Sigma e^{-2s\alpha}\,\xi^2\,|\varphi|^2\,d\sigma\,dt.$$

All this leads to the estimate

$$\iint_Q e^{-2s\alpha}\,\xi^3\,|\varphi|^2\,dx\,dt \leq C \iint_{\omega\times(0,T)} e^{-2s\alpha}\,\xi^3\,|\varphi|^2\,dx\,dt,$$

which holds for $\lambda \geq \overline{\lambda}$ and $s \geq \overline{\sigma}(e^{4\lambda\|\eta^0\|_\infty}\,T + T^2(1 + \|a\|_\infty^{2/3} + \|B\|_\infty^2 + \|\beta\|_\infty^2))$.

Taking into account the properties of the weight functions as well as the choice of s and λ we have made, it is not difficult to realize that the function

$$t \mapsto \exp\left(-2s\,\max_{x\in\overline{\Omega}}\alpha(t)\right)\,\min_{x\in\overline{\Omega}}\xi(t)^3$$

reaches its minimum in $(T/4, 3T/4)$ at $t = T/4$ and that the function

$$t \mapsto \exp\left(-2s\,\min_{x\in\overline{\Omega}}\alpha(t)\right)\,\max_{x\in\overline{\Omega}}\xi(t)^3$$

reaches its maximum in $(0,T)$ at $t = T/2$. With this, the previous Carleman inequality directly gives

$$\iint_{\Omega\times(T/4,3T/4)} |\varphi|^2\,dx\,dt \leq \exp\left\{-2s\left(\min_{x\in\overline{\Omega}}\alpha(x,\frac{T}{2}) - \max_{x\in\overline{\Omega}}\alpha(x,\frac{T}{4})\right)\right\}$$
$$\times \min_{x\in\overline{\Omega}}\xi(x,\frac{T}{4})^{-3}\,\max_{x\in\overline{\Omega}}\xi(x,\frac{T}{2})^3 \iint_{\omega\times(0,T)} |\varphi|^2\,dx\,dt,$$

for the same choice of the parameters s and λ.

Now, taking $\lambda = \overline{\lambda}$ and $s = \overline{s} = \overline{\sigma}(e^{4\overline{\lambda}\|\eta^0\|_\infty}\,T + T^2(1 + \|a\|_\infty^{2/3} + \|B\|_\infty^2 + \|\beta\|_\infty^2))$, we have

$$\iint_{\Omega\times(T/4,3T/4)} |\varphi|^2\,dx\,dt \leq C(\Omega,\omega)\,e^{C(\Omega,\omega)\,\overline{s}/T^2} \iint_{\omega\times(0,T)} |\varphi|^2\,dx\,dt,$$

which gives (1.35) and (1.34).

This ends the proof of proposition 2.

Let us now finish the proof of theorem 2. We will apply a well known argument that has already been used in several similar situations (see [3] and [5]).

Let us introduce a function $\eta \in C^\infty(0,T)$, with

$$\eta(t) = 1 \ \text{ for } t \in (0, T/4), \quad \eta(t) = 0 \ \text{ for } t \in (3T/4, T)$$

and

$$|\eta'(t)| \le C/T \text{ for } t \in (0,T).$$

Let χ be the weak solution of

$$\begin{cases} \chi_t - \Delta\chi + B(x,t)\cdot\nabla\chi + a(x,t)\,\chi = 0 & \text{in } Q, \\ \dfrac{\partial\chi}{\partial n} + \beta(x,t)\,\chi = 0 & \text{on } \Sigma, \\ \chi(x,0) = y^0(x) & \text{in } \Omega. \end{cases}$$

and let us put $y = w + \eta\chi$. If y is the state associated to v, i.e. the solution to (1.1), then w satisfies

$$\begin{cases} w_t - \Delta w + B(x,t)\cdot\nabla w + a(x,t)\,w = -\eta'(t)\,\chi + v1_\omega & \text{in } Q, \\ \dfrac{\partial w}{\partial n} + \beta(x,t)\,w = 0 & \text{on } \Sigma, \\ w(x,0) = 0 & \text{in } \Omega. \end{cases} \tag{1.36}$$

Our task is to find a control $v \in L^2(\omega \times (0,T))$ such that the associated solution to (1.36) satisfies

$$w(x,T) = 0 \quad \text{in} \quad \Omega. \tag{1.37}$$

After this, just taking $y = w + \eta\chi$ we will have proved our result with a control in $L^2(\omega \times (0,T))$.

For each $\varepsilon > 0$, let us consider the functional J_ε , with

$$\begin{cases} J_\varepsilon(\varphi^0) = \dfrac{1}{2}\displaystyle\iint_{\omega\times(0,T)} |\varphi|^2\,dx\,dt + \varepsilon\|\varphi^0\|_{L^2(\Omega)} - \iint_Q \eta'\,\chi\,\varphi\,dx\,dt \\ \forall\varphi^0 \in L^2(\Omega), \end{cases}$$

where, for each $\varphi^0 \in L^2(\Omega)$, φ is the solution to (1.6) associated to φ^0.

It is clear that

$$\varphi^0 \mapsto J_\varepsilon(\varphi^0)$$

is a continuous, strictly convex and (in view of (1.8)) coercive function on $L^2(\Omega)$. Consequently, it possesses exactly one minimizer φ^0_ε and it is not difficult to check that $\varphi^0_\varepsilon = 0$ if and only if the solution $\tilde{w}$ to (1.36) associated to $v = 0$ satisfies $\|\tilde{w}(\cdot,T)\|_{L^2(\Omega)} \le \varepsilon$.

Let us denote by φ_ε the solution to (1.6) associated to φ^0_ε , let us put

$$v_\varepsilon = \varphi_\varepsilon 1_\omega$$

and let us denote by w_ε the solution to (1.36) associated to the control v_ε. Then

$$\|w_\varepsilon(\cdot,T)\|_{L^2(\Omega)} \le \varepsilon. \tag{1.38}$$

Indeed, it is not restrictive to assume that $\varphi_\varepsilon^0 \neq 0$. Accordingly, J_ε is differentiable at φ_ε^0 and

$$(J'_\varepsilon(\varphi_\varepsilon^0), \varphi^0)_{L^2(\Omega)} = 0 \quad \forall \varphi^0 \in L^2(\Omega).$$

That is to say,

$$\begin{cases} \displaystyle\iint_{\omega\times(0,T)} \varphi_\varepsilon\, \varphi\, dx\, dt + (\varepsilon \frac{\varphi_\varepsilon^0}{\|\varphi_\varepsilon^0\|_{L^2}}, \varphi^0)_{L^2(\Omega)} - \iint_Q \eta'\, \chi\, \varphi\, dx\, dt = 0 \\ \forall \varphi^0 \in L^2(\Omega). \end{cases}$$

Since

$$\iint_{\omega\times(0,T)} \varphi_\varepsilon\, \varphi\, dx\, dt - \iint_Q \eta'\, \chi\, \varphi\, dx\, dt = (w_\varepsilon(\cdot, T), \varphi^0)_{L^2(\Omega)}\,,$$

we have

$$(w_\varepsilon(\cdot, T), \varphi^0)_{L^2(\Omega)} = -(\varepsilon \frac{\varphi_\varepsilon^0}{\|\varphi_\varepsilon^0\|_{L^2(\Omega)}}, \varphi^0)_{L^2(\Omega)} \quad \forall \varphi^0 \in L^2(\Omega),$$

which implies (1.38).

Since $J_\varepsilon(\varphi_\varepsilon^0) \leq J_\varepsilon(0) = 0$, we also have

$$\begin{aligned} &\|v_\varepsilon\|^2_{L^2(\omega\times(0,T))} \\ &\leq \left(\iint_{\Omega\times(T/4,3T/4)} |\varphi_\varepsilon|^2\, dx\, dt \right)^{1/2} \left(\iint_{\Omega\times(T/4,3T/4)} |\eta'\chi|^2\, dx\, dt \right)^{1/2}. \end{aligned}$$

From proposition 2 and the definition of v_ε, we deduce now that

$$\|v_\varepsilon\|^2_{L^2(\omega\times(0,T))} \leq \frac{C}{T} K^{1/2}\, \|v_\varepsilon\|_{L^2(Q)} \left(\iint_{\Omega\times(T/4,3T/4)} |\chi|^2\, dx\, dt \right)^{1/2}$$

and, using proposition 1, we have

$$\|v_\varepsilon\|_{L^2(\omega\times(0,T))} \leq C\, K^{1/2}\, \|\chi\|_Y \leq H\, \|y^0\|_{L^2(\Omega)}\,, \tag{1.39}$$

where the constant H is as in (1.11).

Consequently, $v_\varepsilon 1_\omega$ and w_ε are uniformly bounded in the spaces $L^2(\omega \times (0,T))$ and

$$Z = \{w \in L^2(0,T; H^1(\Omega)) : w_t \in L^2(0,T; H^{-1}(\Omega))\},$$

respectively. Obviously, we can extract sequences converging weakly to a control $v1_\omega$ and the associated solution w of (1.36), with

$$w(x,T) = 0 \quad \text{in} \quad \Omega.$$

We have thus proved the existence of a control $v \in L^2(Q)$ such that (1.10) and (1.37) are fulfilled.

This ends the proof of theorem 2.

Appendix: Proof of lemma 1

We divide the proof in three steps:
1 - First, we set $\psi = e^{-s\alpha}\, q$ and we prove the following inequality:

$$
\begin{aligned}
&\iint_Q (s^{-1}\,\xi^{-1}(|\psi_t|^2 + |\Delta\psi|^2) + s\,\lambda^2\,\xi\,|\nabla\psi|^2 + s^3\,\lambda^4\,\xi^3\,|\psi|^2)\,dx\,dt \\
&-2s^3\,\lambda^3 \iint_\Sigma |\nabla\eta^0|^2\,\xi^3\,\frac{\partial\eta^0}{\partial n}\,|\psi|^2\,d\sigma\,dt - 4s\,\lambda^2 \iint_\Sigma |\nabla\eta^0|^2\,\xi\,\frac{\partial\psi}{\partial n}\,\psi\,d\sigma\,dt \\
&-4s\,\lambda \iint_\Sigma \frac{\partial\eta^0}{\partial n}\,\xi\,\left|\frac{\partial\psi}{\partial n}\right|^2 d\sigma\,dt + 2s\,\lambda \iint_\Sigma \frac{\partial\eta^0}{\partial n}\,\xi\,|\nabla\psi|^2\,d\sigma\,dt \\
&+2 \iint_\Sigma \frac{\partial\psi}{\partial n}\,\psi_t\,d\sigma\,dt - 2s^2\,\lambda \iint_\Sigma \alpha_t\,\frac{\partial\eta^0}{\partial n}\,\xi\,|\psi|^2\,d\sigma\,dt \\
&\leq C\left(\iint_Q e^{-2s\alpha}\,|f|^2\,dx\,dt + s^3\,\lambda^4 \iint_{\omega\times(0,T)} \xi^3\,|\psi|^2\,dx\,dt\right)
\end{aligned} \tag{1.40}
$$

for $\lambda \geq C(\Omega,\omega)$ and $s \geq C(\Omega,\omega)(T\,e^{2\lambda\|\eta^0\|_\infty} + T^2)$.

2 - Then, we set $\tilde\psi = e^{-s\tilde\alpha}\, q$ and we prove that

$$
\begin{aligned}
&\iint_Q (s^{-1}\,\tilde\xi^{-1}(|\tilde\psi_t|^2 + |\Delta\tilde\psi|^2) + s\,\lambda^2\,\tilde\xi\,|\nabla\tilde\psi|^2 + s^3\,\lambda^4\,\tilde\xi^{\,3}\,|\tilde\psi|^2)dx\,dt \\
&+2s^3\,\lambda^3 \iint_\Sigma |\nabla\eta^0|^2\,\tilde\xi^{\,3}\,\frac{\partial\eta^0}{\partial n}\,|\tilde\psi|^2\,d\sigma\,dt - 4s\,\lambda^2 \iint_\Sigma |\nabla\eta^0|^2\,\tilde\xi\,\frac{\partial\tilde\psi}{\partial n}\,\tilde\psi\,d\sigma\,dt \\
&+4s\,\lambda \iint_\Sigma \frac{\partial\eta^0}{\partial n}\,\tilde\xi\,\left|\frac{\partial\tilde\psi}{\partial n}\right|^2 d\sigma\,dt - 2s\,\lambda \iint_\Sigma \frac{\partial\eta^0}{\partial n}\,\tilde\xi\,|\nabla\tilde\psi|^2\,d\sigma\,dt \\
&+2 \iint_\Sigma \frac{\partial\tilde\psi}{\partial n}\,\tilde\psi_t\,d\sigma\,dt + 2s^2\,\lambda \iint_\Sigma \tilde\alpha_t\,\frac{\partial\eta^0}{\partial n}\,\tilde\xi\,|\tilde\psi|^2\,d\sigma\,dt \\
&\leq C\left(\iint_Q e^{-2s\tilde\alpha}\,|f|^2\,dx\,dt + s^3\,\lambda^4 \iint_{\omega\times(0,T)} \tilde\xi^{\,3}\,|\tilde\psi|^2\,dx\,dt\right)
\end{aligned} \tag{1.41}
$$

for any $\lambda \geq C(\Omega,\omega)$ and $s \geq C(\Omega,\omega)(e^{4\lambda\|\eta^0\|_\infty}\,T + T^2)$. Here, $\tilde\xi$ and $\tilde\alpha$ stand for the functions

$$
\tilde\xi(x,t) = \frac{e^{-\lambda\eta^0(x)}}{t(T-t)}\ , \quad \tilde\alpha(x,t) = \frac{e^{2\lambda\|\eta^0\|_\infty} - e^{-\lambda\eta^0(x)}}{t(T-t)}\ .
$$

3 - Finally, we add the previous two inequalities and we come back to the original variable φ. This will give the desired inequality (1.15).

Step 1: Let us put $\psi = e^{-s\alpha} q$. Since $-q_t - \Delta q = f$, we also have

$$M_1\psi + M_2\psi = F, \tag{1.42}$$

where

$$M_1\psi = 2s\,\lambda^2\,\xi\,|\nabla\eta^0|^2\,\psi + 2s\,\lambda\,\xi\,\nabla\eta^0\cdot\nabla\psi - \psi_t\,,$$
$$M_2\psi = -s^2\,\lambda^2\,\xi^2\,|\nabla\eta^0|^2\,\psi - \Delta\psi - s\,\alpha_t\,\psi,$$
$$F = e^{-s\alpha}\,f - s\,\lambda\,\xi\,\Delta\eta^0\,\psi + s\,\lambda^2\,\xi\,|\nabla\eta^0|^2\,\psi.$$

From (1.42), we have that

$$\|M_1\psi\|^2_{L^2(Q)} + \|M_2\psi\|^2_{L^2(Q)} + 2(M_1\psi, M_2\psi)_{L^2(Q)} = \|F\|^2_{L^2(Q)}\,. \tag{1.43}$$

The main idea is to expand the term $2(M_1\psi, M_2\psi)_{L^2(Q)}$ and use the particular structure of α and the fact that s is large enough in order to obtain large positive terms in this scalar product. Denoting by $(M_i\psi)_j$ $(1\le i\le 2,\ 1\le j\le 3)$ the j-th term in the above expression of $M_i\psi$, we find that

$$(M_1\psi, M_2\psi)_{L^2(Q)} = \sum_{1\le i,j\le 3} ((M_1\psi)_i, (M_2\psi)_j)_{L^2(Q)}\,.$$

Let us compute each of these terms.

First, we have

$$((M_1\psi)_1, (M_2\psi)_1)_{L^2(Q)} = -2s^3\,\lambda^4 \iint_Q |\nabla\eta^0|^4\,\xi^3\,|\psi|^2\,dx\,dt = A.$$

Then,

$$\begin{aligned}
((M_1\psi)_2, (M_2\psi)_1)_{L^2(Q)} &= -2s^3\,\lambda^3 \iint_Q |\nabla\eta^0|^2\,\xi^3\,(\nabla\eta^0\cdot\nabla\psi)\,\psi\,dx\,dt \\
&= 3s^3\,\lambda^4 \iint_Q |\nabla\eta^0|^4\,\xi^3\,|\psi|^2\,dx\,dt + s^3\,\lambda^3 \iint_Q \Delta\eta^0\,|\nabla\eta^0|^2\,\xi^3\,|\psi|^2\,dx\,dt \\
&+2s^3\,\lambda^3 \iint_Q \partial_i\eta^0\,\partial_{ij}\eta^0\,\partial_j\eta^0\,\xi^3\,|\psi|^2\,dx\,dt \\
&-s^3\,\lambda^3 \iint_\Sigma |\nabla\eta^0|^2\,\xi^3\,\frac{\partial\eta^0}{\partial n}\,|\psi|^2\,d\sigma\,dt = B_1 + B_2 + B_3 + B_4\,.
\end{aligned}$$

We clearly have that $A + B_1$ is a positive term. As a consequence of the properties of η^0 (see (1.14)), we have

$$s^3\,\lambda^4 \iint_Q |\nabla\eta^0|^4\,\xi^3\,|\psi|^2\,dx\,dt \ge C\,s^3\,\lambda^4 \iint_Q \xi^3\,|\psi|^2\,dx\,dt$$
$$-C\,s^3\,\lambda^4 \iint_{\omega'\times(0,T)} \xi^3\,|\psi|^2\,dx\,dt$$

for some $C = C(\Omega, \omega)$. The first of these last two integrals will stay in the left hand side and the second one will go to the right.

The boundary term B_4 will also stay in the left hand side, while B_2 and B_3 will be absorbed by simply taking $\lambda \geq C(\Omega, \omega)$.

We also have

$$((M_1\psi)_3, (M_2\psi)_1)_{L^2(Q)} = s^2\,\lambda^2 \iint_Q |\nabla \eta^0|^2\, \xi^2\, \psi_t\, \psi\, dx\, dt$$
$$= -s^2\,\lambda^2 \iint_Q |\nabla \eta^0|^2\, \xi\, \xi_t\, |\psi|^2\, dx\, dt \leq C\, s^2\, \lambda^2\, T \iint_Q \xi^3\, |\psi|^2\, dx\, dt,$$

which is also absorbed by taking $\lambda \geq 1$ and $s \geq C(\Omega, \omega)\, T$.

Consequently, we obtain

$$\begin{aligned}(M_1\psi, (M_2\psi)_1)_{L^2(Q)} &= ((M_1\psi)_1 + (M_1\psi)_2 + (M_1\psi)_3, (M_2\psi)_1)_{L^2(Q)}\\ &\geq C\, s^3\,\lambda^4 \iint_Q \xi^3\, |\psi|^2\, dx\, dt - s^3\,\lambda^3 \iint_\Sigma |\nabla \eta^0|^2\, \xi^3\, \frac{\partial \eta^0}{\partial n}\, |\psi|^2\, d\sigma\, dt\\ &\quad -C\, s^3\,\lambda^4 \iint_{\omega' \times (0,T)} \xi^3\, |\psi|^2\, dx\, dt,\end{aligned} \tag{1.44}$$

for any $\lambda \geq C(\Omega, \omega)$ and $s \geq C(\Omega, \omega)\, T$.

On the other hand, we have

$$((M_1\psi)_1, (M_2\psi)_2)_{L^2(Q)} = -2s\,\lambda^2 \iint_Q |\nabla \eta^0|^2\, \xi\, \Delta\psi\, \psi\, dx\, dt$$
$$= -2s\,\lambda^2 \iint_\Sigma |\nabla \eta^0|^2\, \xi\, \frac{\partial \psi}{\partial n}\, \psi\, d\sigma\, dt + 2s\,\lambda^2 \iint_Q |\nabla \eta^0|^2\, \xi\, |\nabla \psi|^2\, dx\, dt$$
$$+4s\,\lambda^2 \iint_Q \partial_i \eta^0\, \partial_{ij}\eta^0\, \xi\, \partial_j \psi\, \psi\, dx\, dt + s\,\lambda^3 \iint_Q |\nabla \eta^0|^2\, \xi\, \nabla \eta^0 \cdot \nabla |\psi|^2\, dx\, dt$$
$$= C_1 + C_2 + C_3 + C_4\,.$$

We will keep C_1 and C_2 in the left hand side. For C_3 and C_4 , we have

$$C_3 \leq C\, s\,\lambda^4 \iint_Q \xi\, |\psi|^2\, dx\, dt + C\, s \iint_Q \xi\, |\nabla \psi|^2\, dx\, dt$$

and

$$C_4 \leq C\, s^2\,\lambda^4 \iint_Q \xi^2\, |\psi|^2\, dx\, dt + C\,\lambda^2 \iint_Q |\nabla \psi|^2\, dx\, dt.$$

Therefore, by taking $s \geq C\, T^2$, we find that

$$\begin{aligned}C_1 + C_2 + C_3 + C_4 &\geq -2s\,\lambda^2 \iint_\Sigma |\nabla \eta^0|^2\, \xi\, \frac{\partial \psi}{\partial n}\, \psi\, d\sigma\, dt\\ &\quad +2s\,\lambda^2 \iint_Q |\nabla \eta^0|^2\, \xi\, |\nabla \psi|^2\, dx\, dt - C\, s^2\,\lambda^4 \iint_Q \xi^2\, |\psi|^2\, dx\, dt\\ &\quad -C \iint_Q (s\,\xi + \lambda^2)\, |\nabla \psi|^2\, dx\, dt.\end{aligned} \tag{1.45}$$

We also have

$$
\begin{aligned}
((M_1\psi)_2,(M_2\psi)_2)_{L^2(Q)} &= -2s\,\lambda \iint_Q \xi\,(\nabla\eta^0\cdot\nabla\psi)\,\Delta\psi\,dx\,dt \\
&= -2s\,\lambda \iint_\Sigma \frac{\partial\eta^0}{\partial n}\,\xi\,\left|\frac{\partial\psi}{\partial n}\right|^2 d\sigma\,dt + 2s\,\lambda\iint_Q \partial_{ij}\eta^0\,\xi\,\partial_i\psi\,\partial_j\psi\,dx\,dt \\
&+2s\,\lambda^2 \iint_Q \xi\,|\nabla\eta^0\cdot\nabla\psi|^2\,dx\,dt + s\,\lambda\iint_Q \xi\,\nabla\eta^0\cdot\nabla|\nabla\psi|^2\,dx\,dt \\
&= D_1+D_2+D_3+D_4\,.
\end{aligned}
$$

As before, we will keep the boundary integral D_1 in the left hand side. Also,

$$
D_2 \le C\,s\,\lambda\iint_Q \xi\,|\nabla\psi|^2\,dx\,dt.
$$

Moreover, $D_3\ge 0$. After some additional computations, we also see that

$$
\begin{aligned}
D_4 = s\,\lambda\iint_Q \xi\,\nabla\eta^0\cdot\nabla|\nabla\psi|^2\,dx\,dt &= s\,\lambda\iint_\Sigma \xi\,\frac{\partial\eta^0}{\partial n}\,|\nabla\psi|^2\,d\sigma\,dt \\
-s\,\lambda^2\iint_Q |\nabla\eta^0|^2\,\xi\,|\nabla\psi|^2\,dx\,dt &- s\,\lambda\iint_Q \Delta\eta^0\,\xi\,|\nabla\psi|^2\,dx\,dt \\
&= D_{41}+D_{42}+D_{43}\,.
\end{aligned}
$$

Now, we keep once more D_{41} in the left and we notice that D_{43} can be bounded in the same form as D_2 .

Consequently,

$$
\begin{aligned}
D_1+D_2+D_3+D_4 &\ge -2s\,\lambda\iint_\Sigma \frac{\partial\eta^0}{\partial n}\,\xi\,\left|\frac{\partial\psi}{\partial n}\right|^2\,d\sigma\,dt \\
&+s\,\lambda\iint_\Sigma \frac{\partial\eta^0}{\partial n}\,\xi\,|\nabla\psi|^2\,d\sigma\,dt - s\,\lambda^2\iint_Q |\nabla\eta^0|^2\,\xi\,|\nabla\psi|^2\,dx\,dt \\
&-C\,s\,\lambda\iint_Q \xi\,|\nabla\psi|^2\,dx\,dt.
\end{aligned}
\tag{1.46}
$$

Additionally, we find that

$$
\begin{aligned}
((M_1\psi)_3,(M_2\psi)_2)_{L^2(Q)} &= \iint_Q \psi_t\,\Delta\psi\,dx\,dt \\
&= \iint_\Sigma \frac{\partial\psi}{\partial n}\,\psi_t\,d\sigma\,dt = E,
\end{aligned}
\tag{1.47}
$$

which will stay in the left hand side.

From (1.45)-(1.47), we deduce that

$$
\begin{aligned}
&(M_1\psi,(M_2\psi)_2)_{L^2(Q)} = ((M_1\psi)_1+(M_1\psi)_2+(M_1\psi)_3,(M_2\psi)_2)_{L^2(Q)}\\
&\geq s\,\lambda^2\iint_Q |\nabla\eta^0|^2\,\xi\,|\nabla\psi|^2\,dx\,dt - 2s\,\lambda^2\iint_\Sigma |\nabla\eta^0|^2\,\xi\,\frac{\partial\psi}{\partial n}\,\psi\,d\sigma\,dt\\
&-2s\,\lambda\iint_\Sigma \frac{\partial\eta^0}{\partial n}\,\xi\left|\frac{\partial\psi}{\partial n}\right|^2 d\sigma\,dt + s\,\lambda\iint_\Sigma \frac{\partial\eta^0}{\partial n}\,\xi\,|\nabla\psi|^2\,dx\,dt\\
&+\iint_\Sigma \frac{\partial\psi}{\partial n}\,\psi_t\,d\sigma\,dt - C\,s^2\,\lambda^4\iint_Q \xi^2\,|\psi|^2\,dx\,dt\\
&-C\iint_Q (s\,\lambda\,\xi+\lambda^2)\,|\nabla\psi|^2\,dx\,dt
\end{aligned}
$$

for any $\lambda \geq 1$. Hence, we have the following for any $\lambda \geq C(\Omega,\omega)$ and any $s \geq C(\Omega,\omega)\,T^2$:

$$
\begin{aligned}
&(M_1\psi,(M_2\psi)_2)_{L^2(Q)} \geq C\,s\,\lambda^2\iint_Q \xi\,|\nabla\psi|^2\,dx\,dt\\
&-2s\,\lambda^2\iint_\Sigma |\nabla\eta^0|^2\,\xi\,\frac{\partial\psi}{\partial n}\,\psi\,d\sigma\,dt - 2s\,\lambda\iint_\Sigma \frac{\partial\eta^0}{\partial n}\,\xi\left|\frac{\partial\psi}{\partial n}\right|^2 d\sigma\,dt\\
&+s\,\lambda\iint_\Sigma \frac{\partial\eta^0}{\partial n}\,\xi\,|\nabla\psi|^2\,dx\,dt + \iint_\Sigma \frac{\partial\psi}{\partial n}\,\psi_t\,d\sigma\,dt\\
&-C\,s^2\,\lambda^4\iint_Q \xi^2\,|\psi|^2\,dx\,dt - C\,s\,\lambda^2\iint_{\omega'\times(0,T)} \xi\,|\nabla\psi|^2\,dx\,dt.
\end{aligned}
\tag{1.48}
$$

Let us now consider the scalar product

$$
\begin{aligned}
((M_1\psi)_1,(M_2\psi)_3)_{L^2(Q)} &= -2s^2\,\lambda^2\iint_Q |\nabla\eta^0|^2\,\alpha_t\,\xi\,|\psi|^2\,dx\,dt\\
&\leq C(\Omega,\omega)\,e^{2\lambda\|\eta^0\|_\infty}\,s^2\,\lambda^2\,T\iint_Q \xi^3\,|\psi|^2\,dx\,dt,
\end{aligned}
\tag{1.49}
$$

Obviously, this will be absorbed by the term in $s^3\,\lambda^4$ in (1.44) if we take $\lambda \geq 1$ and $s \geq C(\Omega,\omega)\,e^{2\lambda\|\eta^0\|_\infty}\,T$.

Furthermore,

$$
\begin{aligned}
&((M_1\psi)_2,(M_2\psi)_3)_{L^2(Q)} = -2s^2\,\lambda\iint_Q \alpha_t\,\xi\,(\nabla\eta^0\cdot\nabla\psi)\,\psi\,dx\,dt\\
&= -s^2\,\lambda\iint_\Sigma \alpha_t\,\frac{\partial\eta^0}{\partial n}\,\xi\,|\psi|^2\,d\sigma\,dt + s^2\,\lambda^2\iint_Q \alpha_t\,|\nabla\eta^0|^2\,\xi\,|\psi|^2\,dx\,dt\\
&+s^2\,\lambda\iint_Q \nabla\alpha_t\cdot\nabla\eta^0\,\xi\,|\psi|^2\,dx\,dt + s^2\,\lambda\iint_Q \alpha_t\,\Delta\eta^0\,\xi\,|\psi|^2\,dx\,dt.
\end{aligned}
$$

With $\lambda \geq 1$, the last three terms in the left hand side can be bounded by

$$
C(\Omega,\omega)\,e^{2\lambda\|\eta^0\|_\infty}\,s^2\,\lambda^2\,T\iint_Q \xi^3\,|\psi|^2\,dx\,dt.
$$

Thus, we have

$$
\begin{aligned}
((M_1\psi)_2,(M_2\psi)_3)_{L^2(Q)} \geq -s^2\,\lambda \iint_\Sigma \alpha_t\,\frac{\partial \eta^0}{\partial n}\,\xi\,|\psi|^2\,dx\,dt \\
-C\,e^{2\lambda\|\eta^0\|_\infty}\,s^2\,\lambda^2\,T \iint_Q \xi^3\,|\psi|^2\,dx\,dt
\end{aligned}
\tag{1.50}
$$

Finally, we have

$$
\begin{aligned}
((M_1\psi)_3,(M_2\psi)_3)_{L^2(Q)} = s \iint_Q \alpha_t\,\psi_t\,\psi\,dx\,dt \\
\leq C\,e^{2\lambda\|\eta^0\|_\infty}\,s\,T^2 \iint_Q \xi^3\,|\psi|^2\,dx\,dt,
\end{aligned}
\tag{1.51}
$$

since

$$
\alpha_{tt} \leq C\,e^{2\lambda\|\eta^0\|_\infty}\,\xi^2(1+T^2\,\xi) \leq C\,e^{2\lambda\|\eta^0\|_\infty}\,T^2\,\xi^3.
$$

From (1.49)-(1.51), we deduce that, for $\lambda \geq C(\Omega,\omega)$ and $s \geq C(\Omega,\omega)\,e^{2\lambda\|\eta^0\|_\infty}\,T$, one has

$$
\begin{aligned}
(M_1\psi,(M_2\psi)_3)_{L^2(Q)} = ((M_1\psi)_1+(M_1\psi)_2+(M_1\psi)_3,(M_2\psi)_3)_{L^2(Q)} \\
\geq G - C\,s^3\,\lambda^2 \iint_Q \xi^3\,|\psi|^2\,dx\,dt,
\end{aligned}
\tag{1.52}
$$

where

$$
G = -s^2\,\lambda \iint_\Sigma \alpha_t\,\frac{\partial \eta^0}{\partial n}\,\xi\,|\psi|^2\,d\sigma\,dt.
$$

Taking into account (1.44), (1.48) and (1.52), we obtain

$$
\begin{aligned}
&(M_1\psi,M_2\psi)_{L^2(Q)} \geq C \iint_Q (s\,\lambda^2\,\xi\,|\nabla\psi|^2 + s^3\,\lambda^4\,\xi^3\,|\psi|^2)\,dx\,dt \\
&-s^3\,\lambda^3 \iint_\Sigma |\nabla\eta^0|^2\,\xi^3\,\frac{\partial \eta^0}{\partial n}\,|\psi|^2\,d\sigma\,dt - 2s\,\lambda^2 \iint_\Sigma |\nabla\eta^0|^2\,\xi\,\frac{\partial\psi}{\partial n}\,\psi\,d\sigma\,dt \\
&-2s\,\lambda \iint_\Sigma \frac{\partial \eta^0}{\partial n}\,\xi\,\left|\frac{\partial\psi}{\partial n}\right|^2\,d\sigma\,dt + s\,\lambda \iint_\Sigma \frac{\partial \eta^0}{\partial n}\,\xi\,|\nabla\psi|^2\,dx\,dt \\
&+\iint_\Sigma \frac{\partial\psi}{\partial n}\,\psi_t\,d\sigma\,dt - s^2\,\lambda \iint_\Sigma \alpha_t\,\frac{\partial \eta^0}{\partial n}\,\xi\,|\psi|^2\,d\sigma\,dt \\
&-C \iint_{\omega'\times(0,T)} (s\,\lambda^2\,\xi\,|\nabla\psi|^2 + s^3\,\lambda^4\,\xi^3\,|\psi|^2)\,dx\,dt
\end{aligned}
$$

for any $\lambda \geq C(\Omega,\omega)$ and $s \geq C(\Omega,\omega)(e^{2\lambda\|\eta^0\|_\infty} T + T^2)$. Using (1.43), this gives

$$
\begin{aligned}
&\|M_1\psi\|^2_{L^2(Q)} + \|M_2\psi\|^2_{L^2(Q)} + \iint_Q (s\,\lambda^2\,\xi\,|\nabla\psi|^2 + s^3\,\lambda^4\,\xi^3\,|\psi|^2)\,dx\,dt \\
&+2(B_4 + C_1 + D_1 + D_{41} + E + G) \leq C\Big(\|F\|^2_{L^2(Q)} \\
&+s\,\lambda^2 \iint_{\omega'\times(0,T)} \xi\,|\nabla\psi|^2\,dx\,dt + s^3\,\lambda^4 \iint_{\omega'\times(0,T)} \xi^3\,|\psi|^2)\,dx\,dt\Big) \\
&\leq C\Big(\iint_Q e^{-2s\alpha}\,|f|^2\,dx\,dt + s^2\,\lambda^4 \iint_Q \xi^2\,|\psi|^2\,dx\,dt \\
&+s\,\lambda^2 \iint_{\omega'\times(0,T)} \xi\,|\nabla\psi|^2\,dx\,dt + s^3\,\lambda^4 \iint_{\omega'\times(0,T)} \xi^3\,|\psi|^2)\,dx\,dt\Big).
\end{aligned}
$$

Thus, we also have

$$
\begin{aligned}
&\|M_1\psi\|^2_{L^2(Q)} + \|M_2\psi\|^2_{L^2(Q)} + \iint_Q (s\,\lambda^2\,\xi\,|\nabla\psi|^2 + s^3\,\lambda^4\,\xi^3\,|\psi|^2)\,dx\,dt \\
&\quad +2(B_4 + C_1 + D_1 + D_{41} + E + G) \leq C\Big(\iint_Q e^{-2s\alpha}\,|f|^2\,dx\,dt \\
&\quad +s\,\lambda^2 \iint_{\omega'\times(0,T)} \xi\,|\nabla\psi|^2\,dx\,dt + s^3\,\lambda^4 \iint_{\omega'\times(0,T)} \xi^3\,|\psi|^2)\,dx\,dt\Big)
\end{aligned}
\tag{1.53}
$$

for $\lambda \geq C(\Omega,\omega)$ and $s \geq C(\Omega,\omega)(e^{2\lambda\|\eta^0\|_\infty} T + T^2)$.

The next step is to try to add integrals of $|\Delta\psi|^2$ and $|\psi_t|^2$ to the left hand side of (1.53). This can be made using the expressions of $M_i\psi\,(i = 1, 2)$. Indeed, we have

$$
\begin{aligned}
s^{-1} \iint_Q \xi^{-1}\,|\psi_t|^2\,dx\,dt \leq C\Big(&s\,\lambda^2 \iint_Q \xi\,|\nabla\psi|^2\,dx\,dt \\
&+s\,\lambda^4 \iint_Q \xi\,|\psi|^2\,dx\,dt + \|M_1\psi\|^2_{L^2(Q)}\Big)
\end{aligned}
$$

and

$$
\begin{aligned}
s^{-1} \iint_Q \xi^{-1}\,|\Delta\psi|^2\,dx\,dt \leq C\Big(&s^3\,\lambda^4 \iint_Q \xi^3\,|\psi|^2\,dx\,dt \\
&+s\,T^2\,e^{4\lambda\|\eta^0\|_\infty} \iint_Q \xi^3\,|\psi|^2\,dx\,dt + \|M_2\psi\|^2_{L^2(Q)}\Big)
\end{aligned}
$$

for $s \geq C\,T^2$. Accordingly, we deduce from (1.53) that

$$\begin{aligned}
&\iint_Q \left((s\xi)^{-1}(|\psi_t|^2 + |\Delta\psi|^2) + s\,\lambda^2\,\xi\,|\nabla\psi|^2 + s^3\,\lambda^4\,\xi^3\,|\psi|^2\right)\,dx\,dt \\
&+2(B_4 + C_1 + D_1 + D_{41} + E + G) \leq C\left(\iint_Q e^{-2s\alpha}\,|f|^2\,dx\,dt\right. \\
&\left.+s\,\lambda^2 \iint_{\omega'\times(0,T)} \xi\,|\nabla\psi|^2\,dx\,dt + s^3\,\lambda^4 \iint_{\omega'\times(0,T)} \xi^3\,|\psi|^2)\,dx\,dt\right)
\end{aligned} \tag{1.54}$$

for $\lambda \geq C(\Omega,\omega)$ and $s \geq C(\Omega,\omega)(e^{2\lambda\|\eta^0\|_\infty}\,T + T^2)$.

We are now ready to eliminate the second integral in the right hand side. To this end, let us introduce a function $\theta = \theta(x)$, with

$$\theta \in C_c^2(\omega), \quad \theta \equiv 1 \text{ in } \omega', \quad 0 \leq \theta \leq 1$$

and let us make some computations:

$$\begin{aligned}
&s\,\lambda^2 \iint_{\omega'\times(0,T)} \xi\,|\nabla\psi|^2\,dx\,dt \leq\ s\,\lambda^2 \iint_{\omega\times(0,T)} \theta\,\xi\,|\nabla\psi|^2\,dx\,dt \\
&= -s\,\lambda^2 \iint_{\omega\times(0,T)} \theta\,\xi\,\Delta\psi\,\psi\,dx\,dt - s\,\lambda^2 \iint_{\omega\times(0,T)} \xi\,(\nabla\theta\cdot\nabla\psi)\,\psi\,dx\,dt \\
&-s\,\lambda^3 \iint_{\omega\times(0,T)} \theta\,\xi\,(\nabla\eta^0\cdot\nabla\psi)\,\psi\,dx\,dt \leq\ \varepsilon\,s^{-1} \iint_{\omega\times(0,T)} \xi^{-1}\,|\Delta\psi|^2\,dx\,dt \\
&+C\left(s^3\,\lambda^4 \iint_{\omega\times(0,T)} \xi^3\,|\psi|^2\,dx\,dt + s\,\lambda^4 \iint_{\omega\times(0,T)} \xi\,|\psi|^2\,dx\,dt\right),
\end{aligned}$$

where we have used that $\lambda \geq 1$. In view of this estimate, we deduce that the integral on $|\nabla\psi|^2$ of the right hand side of (1.54) can be suppressed if the last integral is performed in the slightly greater set $\omega \times (0,T)$. From (1.54) and this remark, we deduce (1.40).

Step 2: The proof of (1.41) is very similar to the proof of (1.40). We will only sketch the main points.

We start from the identity

$$M_1\tilde{\psi} + M_2\tilde{\psi} = \tilde{F},$$

where

$$\begin{aligned}
\tilde{M}_1\tilde{\psi} &= 2s\,\lambda^2\,|\nabla\eta^0|^2\,\tilde{\xi}\,\tilde{\psi} - 2s\,\lambda\,\tilde{\xi}\,\nabla\eta^0\cdot\nabla\tilde{\psi} - \tilde{\psi}_t\,, \\
\tilde{M}_2\tilde{\psi} &= -s^2\,\lambda^2\,|\nabla\eta^0|^2\,\tilde{\xi}^{\,2}\,\tilde{\psi} - \Delta\tilde{\psi} - s\,\tilde{\alpha}_t\,\tilde{\psi}, \\
\tilde{F} &= e^{-s\tilde{\alpha}}\,f + s\,\lambda\,\tilde{\xi}\,\Delta\eta^0\,\tilde{\psi} + s\,\lambda^2\,|\nabla\eta^0|^2\,\tilde{\xi}\,\tilde{\psi}.
\end{aligned}$$

We then have

$$\|\tilde{M}_1\tilde{\psi}\|^2_{L^2(Q)} + \|\tilde{M}_2\tilde{\psi}\|^2_{L^2(Q)} + 2(\tilde{M}_1\tilde{\psi}, \tilde{M}_2\tilde{\psi})_{L^2(Q)} = \|\tilde{F}\|^2_{L^2(Q)}\,. \tag{1.55}$$

After a lengthy computation, we find that

$$\begin{aligned}
&(\tilde{M}_1\psi, \tilde{M}_2\psi)_{L^2(Q)} \geq C \iint_Q (s\,\lambda^2\,\tilde{\xi}\,|\nabla\tilde{\psi}|^2 + s^3\,\lambda^4\,\tilde{\xi}^3\,|\tilde{\psi}|^2)\,dx\,dt \\
&+2(\tilde{B}_4 + \tilde{C}_1 + \tilde{D}_1 + \tilde{D}_{41} + \tilde{E} + \tilde{G}) \\
&-C \iint_{\omega'\times(0,T)} (s\,\lambda^2\,\tilde{\xi}\,|\nabla\tilde{\psi}|^2 + s^3\,\lambda^4\,\tilde{\xi}^3\,|\tilde{\psi}|^2)\,dx\,dt
\end{aligned}$$

for any $\lambda \geq C(\Omega,\omega)$ and $s \geq C(\Omega,\omega)(e^{2\lambda\|\eta^0\|_\infty}\,T + T^2)$, where

$$\tilde{B}_4 = s^3\,\lambda^3 \iint_\Sigma |\nabla\eta^0|^2\,\tilde{\xi}^3\,\frac{\partial\eta^0}{\partial n}\,|\tilde{\psi}|^2\,d\sigma\,dt,$$

$$\tilde{C}_1 = -2s\,\lambda^2 \iint_\Sigma |\nabla\eta^0|^2\,\tilde{\xi}\,\frac{\partial\tilde{\psi}}{\partial n}\,\tilde{\psi}\,d\sigma\,dt,$$

$$\tilde{D}_1 = 2s\,\lambda \iint_\Sigma \frac{\partial\eta^0}{\partial n}\,\tilde{\xi}\,\left|\frac{\partial\tilde{\psi}}{\partial n}\right|^2\,d\sigma\,dt, \quad \tilde{D}_{41} = -s\,\lambda \iint_\Sigma \frac{\partial\eta^0}{\partial n}\,\tilde{\xi}\,|\nabla\tilde{\psi}|^2\,dx\,dt,$$

$$\tilde{E} = \iint_\Sigma \frac{\partial\tilde{\psi}}{\partial n}\,\tilde{\psi}_t\,d\sigma\,dt, \quad \tilde{G} = s^2\,\lambda \iint_\Sigma \tilde{\alpha}_t\,\frac{\partial\eta^0}{\partial n}\,\tilde{\xi}\,|\tilde{\psi}|^2\,d\sigma\,dt.$$

This, together with (1.55), gives

$$\begin{aligned}
&\|M_1\tilde{\psi}\|^2_{L^2(Q)} + \|M_2\tilde{\psi}\|^2_{L^2(Q)} + \iint_Q (s\,\lambda^2\,\tilde{\xi}\,|\nabla\tilde{\psi}|^2 + s^3\,\lambda^4\,\tilde{\xi}^3\,|\tilde{\psi}|^2)\,dx\,dt \\
&+2(\tilde{B}_4 + \tilde{C}_1 + \tilde{D}_1 + \tilde{D}_{41} + \tilde{E} + \tilde{G}) \leq C \Big(\|\tilde{F}\|^2_{L^2(Q)} \\
&+s\,\lambda^2 \iint_{\omega'\times(0,T)} \tilde{\xi}\,|\nabla\tilde{\psi}|^2\,dx\,dt + s^3\,\lambda^4 \iint_{\omega'\times(0,T)} \tilde{\xi}^3\,|\tilde{\psi}|^2)\,dx\,dt \Big).
\end{aligned} \tag{1.56}$$

With similar arguments to those in the first step, we can now assume that, in (1.56), $\|\tilde{F}\|^2_{L^2(Q)}$ is replaced by

$$\iint_Q e^{-2s\tilde{\alpha}}\,|f|^2\,dx\,dt$$

and $\|\tilde{M}_1\tilde{\psi}\|^2_{L^2(Q)} + \|\tilde{M}_2\tilde{\psi}\|^2_{L^2(Q)}$ is replaced by

$$\iint_Q s^{-1}\,\xi^{-1}(|\tilde{\psi}_t|^2 + |\Delta\tilde{\psi}|^2)\,dx\,dt.$$

Finally, for $\lambda \geq C(\Omega,\omega)$ and $s \geq C(\Omega,\omega)\,T^2$ large enough, we can replace the integrals of $|\nabla\tilde{\psi}|^2$ and $|\tilde{\psi}|^2$ in the right hand side by

$$s^3\,\lambda^4 \iint_{\omega\times(0,T)} \tilde{\xi}^3\,|\tilde{\psi}|^2\,dx\,dt.$$

This yields the estimate (1.41).

Step 3: Now, let us add the inequalities (1.40) and (1.41) and let us check that all the integrals on Σ can be simplified, so that there will only remain integrals in Q.

Since $\eta^0 = 0$ on $\partial\Omega$, we have

$$\xi = \tilde{\xi}, \quad \alpha = \tilde{\alpha} \quad \text{and} \quad \psi = \tilde{\psi} \quad \text{on } \Sigma. \tag{1.57}$$

Consequently, $B_4 + \tilde{B}_4 = 0$ and $G + \tilde{G} = 0$.

Let us see that

$$\frac{\partial \tilde{\psi}}{\partial n} \equiv -\frac{\partial \psi}{\partial n} \text{ on } \Sigma. \tag{1.58}$$

From the definitions of ψ and $\tilde{\psi}$, we have

$$\partial_i \psi = e^{-s\alpha} \left(\partial_i q + s\,\lambda\, \partial_i \eta^0\, \xi\, q \right), \; \partial_i \tilde{\psi} = e^{-s\tilde{\alpha}} \left(\partial_i q - s\,\lambda\, \partial_i \eta^0\, \tilde{\xi}\, q \right), \tag{1.59}$$

whence

$$\frac{\partial \psi}{\partial n} = s\,\lambda\, \frac{\partial \eta^0}{\partial n}\, \xi\, e^{-s\alpha}\, q, \quad \frac{\partial \tilde{\psi}}{\partial n} = -s\,\lambda\, \frac{\partial \eta^0}{\partial n}\, \tilde{\xi}\, e^{-s\tilde{\alpha}}\, q \quad \text{on} \quad \Sigma$$

and we certainly have (1.58).

We deduce from (1.57) and (1.58) that $C_1 + \tilde{C}_1 = 0$, $D_1 + \tilde{D}_1 = 0$ and $E + \tilde{E} = 0$.

On the other hand, since φ satisfies a zero Neumann condition and $\eta^0 = 0$ on $\partial\Omega$, we also have

$$|\nabla \psi|^2 = |\nabla \tilde{\psi}|^2 \quad \text{on } \Sigma,$$

whence $D_{41} + \tilde{D}_{41} = 0$.

With all this, we obtain

$$\begin{aligned}
& s^{-1} \iint_Q (\xi^{-1}(|\psi_t|^2 + |\Delta\psi|^2) + \tilde{\xi}^{-1}(|\tilde{\psi}_t|^2 + |\Delta\tilde{\psi}|^2))\, dx\, dt \\
& + s\,\lambda^2 \iint_Q (\xi\, |\nabla\psi|^2 + \tilde{\xi}\, |\nabla\tilde{\psi}|^2)\, dx\, dt \\
& + s^3\,\lambda^4 \iint_Q (\xi^3\, |\psi|^2 + \tilde{\xi}^{\,3}\, |\tilde{\psi}|^2)\, dx\, dt \\
& \quad \le C \left(s^3\,\lambda^4 \iint_{\omega\times(0,T)} \left(\xi^3\, |\psi|^2 + \tilde{\xi}^{\,3}\, |\tilde{\psi}|^2 \right) dx\, dt \right. \\
& \quad \left. + \iint_Q \left(e^{-2s\alpha} + e^{-2s\tilde{\alpha}} \right) |f|^2\, dx\, dt \right),
\end{aligned} \tag{1.60}$$

for $\lambda \ge C(\Omega,\omega)$ and $s \ge C(\Omega,\omega)(e^{4\lambda\|\eta^0\|_\infty}\, T + T^2)$.

From the definitions of ξ, $\tilde{\xi}$, α and $\tilde{\alpha}$, we have

$$\tilde{\xi} \le \xi, \quad e^{-2s\tilde{\alpha}} \le e^{-2s\alpha} \quad \text{in } Q,$$

so (1.60) yields

$$\iint_Q \left((s\xi)^{-1}(|\psi_t|^2+|\Delta\psi|^2)+s\,\lambda^2\,\xi\,|\nabla\psi|^2+s^3\,\lambda^4\,\xi^3\,|\psi|^2\right)\,dx\,dt \\ \leq C\left(\iint_Q e^{-2s\alpha}\,|f|^2\,dx\,dt+s^3\,\lambda^4\iint_{\omega\times(0,T)}\xi^3\,|\psi|^2\,dx\,dt\right), \tag{1.61}$$

for any $\lambda \geq C(\Omega,\omega)$ and $s \geq C(\Omega,\omega)(e^{4\lambda\|\eta^0\|_\infty}\,T+T^2)$.

We finally turn back to φ. For the moment, we have

$$s^{-1}\iint_Q \xi^{-1}\,\left(|\psi_t|^2+|\Delta\psi|^2\right)\,dx\,dt \\ +s\,\lambda^2\iint_Q \xi\,|\nabla\psi|^2\,dx\,dt+s^3\,\lambda^4\iint_Q e^{-2s\alpha}\,\xi^3\,|q|^2\,dx\,dt \\ \leq C\left(\iint_Q e^{-2s\alpha}\,|f|^2\,dx\,dt+s^3\,\lambda^4\iint_{\omega\times(0,T)} e^{-2s\alpha}\,\xi^3\,|q|^2\,dx\,dt\right). \tag{1.62}$$

Using (1.59), we find that

$$s\,\lambda^2\iint_Q e^{-2s\alpha}\,\xi\,|\nabla q|^2\,dx\,dt \leq C\,s\,\lambda^2\iint_Q \xi\,|\nabla\psi|^2\,dx\,dt \\ +C\,s^3\,\lambda^4\iint_Q e^{-2s\alpha}\,\xi^3\,|q|^2\,dx\,dt.$$

Accordingly, the previous integral of $|\nabla q|^2$ can be added to the left hand side of (1.62):

$$s^{-1}\iint_Q \xi^{-1}\,|\psi_t|^2\,dx\,dt+s^{-1}\iint_Q \xi^{-1}\,|\Delta\psi|^2\,dx\,dt \\ +s\,\lambda^2\iint_Q \xi\,|\nabla q|^2\,dx\,dt+s^3\,\lambda^4\iint_Q e^{-2s\alpha}\,\xi^3\,|q|^2\,dx\,dt \\ \leq C\left(s^3\,\lambda^4\iint_{\omega\times(0,T)} e^{-2s\alpha}\,\xi^3\,|q|^2\,dx\,dt+\iint_Q e^{-2s\alpha}\,|f|^2\,dx\,dt\right).$$

For Δq, we use the identity

$$\Delta\psi = e^{-s\alpha}(\Delta q+s\,\lambda\,\Delta\eta^0\,\xi\,q+s\,\lambda^2\,|\nabla\eta^0|^2\,\xi\,q \\ +2s\,\lambda\,\xi\,\nabla\eta^0\cdot\nabla q+s^2\,\lambda^2\,|\nabla\eta^0|^2\,\xi^2\,q)$$

and we obtain

$$s^{-1}\iint_Q e^{-2s\alpha}\,\xi^{-1}\,|\Delta q|^2\,dx\,dt \leq C\left(s^{-1}\iint_Q \xi^{-1}\,|\Delta\psi|^2\,dx\,dt\right. \\ +s\,\lambda^2\iint_Q e^{-2s\alpha}\,\xi\,|q|^2\,dx\,dt+s\,\lambda^4\iint_Q e^{-2s\alpha}\,\xi\,|q|^2\,dx\,dt \\ \left.+s\,\lambda^2\iint_Q e^{-2s\alpha}\,\xi\,|\nabla q|^2\,dx\,dt+s^3\,\lambda^4\iint_Q e^{-2s\alpha}\,\xi^3\,|q|^2\,dx\,dt\right).$$

Finally, for q_t , we get

$$s^{-1} \iint_Q e^{-2s\alpha}\, \xi^{-1}\, |q_t|^2 \, dx \, dt \leq C(\Omega,\omega) \Bigg(s^{-1} \iint_Q \xi^{-1}\, |\psi_t|^2 \, dx \, dt$$

$$+ s\, e^{4\lambda \|\eta^0\|_{C(\overline{\Omega})}}\, T^2 \iint_Q e^{-2s\alpha}\, \xi^3\, |q|^2 \, dx \, dt \Bigg),$$

where we have used the identity

$$q_t = e^{s\alpha}(\psi_t + s\, \alpha_t\, \psi).$$

Thus, taking $\lambda \geq 1$ and $s \geq C(\Omega,\omega)(e^{2\lambda\|\eta^0\|_{C(\overline{\Omega})}}\, T + T^2)$, we are able to introduce all terms of $I_{s,\lambda}(q)$ in the left hand side of (1.62). This yields (1.15) and concludes the proof of lemma 1.

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Capítulo 2

Exact Controllability to the trajectories of the heat equation with Fourier boundary conditions: The semilinear case

Exact Controllability to the trajectories of the heat equation with Fourier boundary conditions: The semilinear case

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Abstract

This paper is concerned with the global exact controllability of the semilinear heat equation (with nonlinear terms involving the state and the gradient) completed with boundary conditions of the form $\frac{\partial y}{\partial n} + f(y) = 0$. We consider distributed controls, with support in a small set. The null controllability of similar linear systems has been analyzed in a previous first part of this work. In this second part we show that, when the nonlinear terms are locally Lipschitz-continuous and slightly superlinear, one has exact controllability to the trajectories.

1. Introduction

Let $\Omega \subset \mathbf{R}^N$ $(N \geq 1)$ be a bounded connected open set whose boundary $\partial\Omega$ is regular enough (for instance $\partial\Omega \in C^2$). Let $\omega \subset \Omega$ be a (small) nonempty open subset and let $T > 0$. We will use the notation $Q = \Omega \times (0,T)$ and $\Sigma = \partial\Omega \times (0,T)$ and we will denote by $n(x)$ the outward unit normal to Ω at the point $x \in \partial\Omega$.

We will consider the semilinear heat equation with nonlinear Fourier (or Robin) boundary conditions

$$\begin{cases} y_t - \Delta y + F(y, \nabla y) = v 1_\omega & \text{in } Q, \\ \dfrac{\partial y}{\partial n} + f(y) = 0 & \text{on } \Sigma, \\ y(x,0) = y^0(x) & \text{in } \Omega. \end{cases} \tag{2.1}$$

Here, we assume that $v \in L^2(\omega \times (0,T))$ (at least), 1_ω is the characteristic function of ω, $y^0 \in L^\infty(\Omega)$ and $F : \mathbf{R} \times \mathbf{R}^N \mapsto \mathbf{R}$ and $f : \mathbf{R} \mapsto \mathbf{R}$ are given functions. In (2.1), $y = y(x,t)$ is the state and $v = v(x,t)$ is the control; it is assumed that we can act on the system only through $\omega \times (0,T)$.

For the existence, uniqueness, regularity and general properties of the solutions to problems like (2.1), see for instance [1], [2] and [7]. An illustrative interpretation of the data and variables in (2.1) is the following. The function $y = y(x,t)$ can be viewed as the relative temperature of a medium (with respect to the exterior surrounding air) subject to transport and chemical reactions. The parabolic equation in (2.1) means, among other things, that a heat source $v1_\omega$ is applied on a part of the body. On the boundary, $-\frac{\partial y}{\partial n}$ can be viewed as the *normal heat flux*, inwards directed, up to a positive coefficient. Thus, the equality

$$-\frac{\partial y}{\partial n} = f(y)$$

means that this flux is a (nonlinear) function of the temperature. Accordingly, it is reasonable to assume that f is nondecreasing and $f(0) = 0$.

A simplified linear model which was considered in a previous paper [10] is the following:

$$\begin{cases} y_t - \Delta y + a(x,t)\, y + B(x,t) \cdot \nabla y = v 1_\omega & \text{in } Q, \\ \dfrac{\partial y}{\partial n} + \beta(x,t)\, y = 0 & \text{on } \Sigma, \\ y(x,0) = y^0(x) & \text{in } \Omega, \end{cases} \tag{2.2}$$

Here, it is assumed that the coefficients a, B and β satisfy

$$a \in L^\infty(Q), \quad B \in L^\infty(Q)^N, \quad \beta \in L^\infty(\Sigma) \tag{2.3}$$

and, for the reasons above, it is also natural to assume that $\beta \geq 0$ (although this assumption was not used in [10]).

The main goal of this paper is to analyze the controllability properties of the nonlinear system (2.1). More precisely, we will try to reach exactly uncontrolled solutions of (2.1), i.e. functions $\overline{y} = \overline{y}(x,t)$ satisfying

$$\begin{cases} \overline{y}_t - \Delta \overline{y} + F(\overline{y}, \nabla \overline{y}) = 0 & \text{in } Q, \\ \dfrac{\partial \overline{y}}{\partial n} + f(\overline{y}) = 0 & \text{on } \Sigma, \\ \overline{y}(x,0) = \overline{y}^0(x) & \text{in } \Omega. \end{cases} \tag{2.4}$$

It will be said that (2.1) is (globally) *exactly controllable to the trajectories* at time T if, for any solution of (2.4) with 'suitable' regularity and any $y^0 \in L^\infty(\Omega)$, there exist controls $v \in L^2(\omega \times (0,T))$ and associated solutions $y \in C^0([0,T]; L^2(\Omega))$ such that

$$y(x,T) = \overline{y}(x,T) \ \text{ in } \ \Omega. \tag{2.5}$$

Here, by suitable regularity we mean the following:

$$\overline{y} \in L^2(0,T; H^1(\Omega)) \cap C^0([0,T]; L^2(\Omega)) \cap L^\infty(Q), \quad \overline{y}^0 \in L^\infty(\Omega). \tag{2.6}$$

The controllability properties of semilinear time-dependent systems have been studied intensively these last years. See for instance [16], [13], [17], [5], [11] and [11], where nonlinearities of the form $f(y)$ are considered. See also the general treatise [14]. In particular, for parabolic systems completed with Dirichlet boundary conditions, nonlinear terms $f(y, \nabla y)$ depending on both the state and the gradient have been taken into account in [9] and [6]. For the similar linear system (2.2), the null controllability was analyzed more in detail in [10]. In the case of (2.1), some partial results have been given in [5].

Our main result concerns the global exact controllability to the trajectories of (2.1). It is the following:

Theorem 3 *Let us assume that F and f are locally Lipschitz-continuous and satisfy*

$$\lim_{|s|\to\infty} \frac{|F(s,p) - F(r,p)|}{|s-r| \log^{3/2}(1 + |s-r|)} = 0, \tag{2.7}$$

uniformly in $(r,p) \in [-K,K] \times \mathbf{R}^N \, \forall K > 0$*,*

$$\begin{cases} \forall L > 0,\, \exists M > 0 \ \text{ such that} \\ |F(s,p) - F(r,p)| \le M|s-r|, \quad |F(s,p) - F(s,q)| \le M|p-q| \\ \forall (s,r,p,q) \in [-L,L]^2 \times \mathbf{R}^N \times \mathbf{R}^N \end{cases} \tag{2.8}$$

and

$$\lim_{|s|\to\infty} \frac{|f(s) - f(r)|}{|s-r| \log^{1/2}(1+|s-r|)} = 0 \tag{2.9}$$

uniformly in $r \in [-K,K] \, \forall K > 0$*. Then, for each* $T > 0$*, the nonlinear system* (2.1) *is exactly controllable to the trajectories at time* T *with* L^∞ *controls.*

Remark 2 Conditions (2.7)–(2.9) are satisfied if F and f are globally Lipschitz continuous. Notice that (2.7) means that the function F can only be slightly superlinear in s, uniformly in p. In the similar case of Dirichlet boundary conditions, it is known that conditions like these are sharp. Indeed, for instance, when F does not depend on p and

$$|F(s) - F(r)| \sim |s-r| \log^{\beta}(1+|s-r|), \quad \beta > 2,$$

due to blow-up phenomena, the system fails to be controllable whenever $\omega \neq \Omega$ (see [11]). On the other hand, (2.9) is also a slightly superlinear growth assumption for f. It would be interesting to know whether a more superlinear f leading to blow up in the absence of control can also be an obstruction for the null controllability of (2.1). But this question does not seem obvious and remains open.

Remark 3 A result proved in [5] says that when $F \equiv 0$, f is smooth near zero and

$$f(s)\, s \ge 0 \quad \forall s \in \mathbf{R}, \tag{2.10}$$

the nonlinear system (2.1) is null controllable for large T. That is to say, under these assumptions, for each $y^0 \in L^2(\Omega)$ there exist $T(y^0) > 0$ and controls v in $L^\infty(\omega \times (0,T)$ such that the associated states y satisfy

$$y(x, T(y^0)) = 0 \ \text{ in } \ \Omega.$$

By inspection of the proof of theorem 3, we see that the same result holds for (2.1) with $F \equiv 0$ whenever f is locally Lipschitz-continuous and satisfies the *good sign condition* (2.10).

For the proof of theorem 3, we will first establish a null controllability result for (2.2) (see proposition 3 below). This will be used, together with an appropriate fixed point argument, to deduce the desired result.

This strategy was introduced in [16] in the framework of the exact controllability of the semilinear wave equation. See also [5] and [11] for similar results concerning the approximate and null controllability of the semilinear heat equation with Dirichlet or Neumann boundary conditions.

Our null controllability result for (2.2) is the following:

Proposition 3 *For every $T > 0$, system (2.2) is null controllable at time T, with controls in $L^\infty(\omega \times (0,T))$. More precisely, for each $y^0 \in L^2(\Omega)$, there exists $v \in L^\infty(\omega \times (0,T))$ such that the associated solution to (2.2) satisfies $y(x,T) = 0$ in Ω. Furthermore, the control v can be found satisfying*

$$\|v\|_{L^\infty(\omega\times(0,T))} \leq e^{C(\Omega,\omega)K(T,\|a\|_\infty,\|B\|_\infty,\|\beta\|_\infty)} \|y^0\|_{L^2(\Omega)}, \tag{2.11}$$

where

$$K = 1 + 1/T + \|a\|_\infty^{2/3} + \|B\|_\infty^2 + \|\beta\|_\infty^2 + T(1 + \|a\|_\infty + \|B\|_\infty^2 + \|\beta\|_\infty^2). \tag{2.12}$$

For the proof of proposition 3, we first introduce a control $L^2(\omega \times (0,T))$ which leads the solution of (2.2) to zero at time T. In a second step, arguing as in Section 2 in [4], a regularizing argument will lead to the desired L^∞ control.

The rest of this paper is organized as follows. In Section 2, we prove proposition 3. Section 3 is devoted to the proof of theorem 3. For completeness, we have also included an Appendix where the proof of a rather technical local regularity result is given in detail.

In the sequel, C denotes a generic positive constant only depending on Ω and ω.

2. A null controllability result for the linear system

In this Section we present the proof of proposition 3.

Let $y^0 \in L^2(\Omega)$ be given and let us introduce two open sets ω' and ω'', with $\omega'' \subset\subset \omega' \subset\subset \omega$. Then, we can use the main result in [10] (theorem 2) with control region $\omega'' \times (0,T)$ to deduce the existence of a control $\widetilde{v} \in L^2(\omega'' \times (0,T))$ such that the associated solution to (2.2) verifies $y(x,T) = 0$ in Ω and also the estimate

$$\|\widetilde{v}\|_{L^2(\omega''\times(0,T))} \leq e^{C(\Omega,\omega)\,K(T,\|a\|_\infty,\|B\|_\infty,\|\beta\|_\infty)} \|y^0\|_{L^2(\Omega)}, \tag{2.13}$$

where K is of the form (2.12).

Let us denote by $\widetilde{y}$ the state associated to $\widetilde{v}$. We now introduce a cut-off function $\eta = \eta(t)$ satisfying

$$\eta \in C^\infty([0,T]), \quad \eta(t) = 1 \text{ in } (0,T/4), \quad \eta(t) = 0 \text{ in } (3T/4,T)$$

and

$$0 \leq \eta(t) \leq 1,\ |\eta'(t)| \leq \frac{C}{t} \text{ in } (0,T)$$

and we denote by χ the solution to the system

$$\begin{cases} \chi_t - \Delta\chi + a(x,t)\,\chi + B(x,t)\cdot\nabla\chi = 0 & \text{in } Q, \\ \dfrac{\partial\chi}{\partial n} + \beta(x,t)\,\chi = 0 & \text{on } \Sigma, \\ \chi(x,0) = y^0(x) & \text{in } \Omega. \end{cases}$$

Then, the function $\widetilde{w} = \widetilde{y} - \eta\chi$ satisfies

$$\begin{cases} \widetilde{w}_t - \Delta\widetilde{w} + a(x,t)\,\widetilde{w} + B(x,t)\cdot\nabla\widetilde{w} = -\eta'(t)\,\chi + \widetilde{v}1_{\omega''} & \text{in } Q,\\ \dfrac{\partial\widetilde{w}}{\partial n} + \beta(x,t)\,\widetilde{w} = 0 & \text{on } \Sigma,\\ \widetilde{w}(x,0) = 0, \quad \widetilde{w}(x,T) = 0 & \text{in } \Omega. \end{cases}$$

Our aim is to construct a control $v \in L^\infty(\omega \times (0,T))$ which drives the solution of (2.2) to zero at time $t = T$. To this end, we will need a local regularity result for the solutions to linear heat equations with L^∞ coefficients a and B. This will be used below for the functions χ and $\widetilde{w}$ and reads as follows:

Lemma 2 *Let us denote by Y the space $L^2(0,T;H^1(\Omega)) \cap C^0([0,T];L^2(\Omega))$. Let $y \in Y$ be a solution to the equation*

$$y_t - \Delta y + a(x,t)\,y + B(x,t)\cdot\nabla y = f, \tag{2.14}$$

where $a \in L^\infty(Q)$, $B \in L^\infty(Q)^N$ and $f \in L^2(Q)$. Let $\mathcal{O} \subset \Omega$ be a nonempty open set and assume that f is L^∞ in the cylinder $\mathcal{O} \times (0,T)$. Then

$$y \in L^\infty(\delta, T; W^{1,\infty}(\mathcal{O}'))$$

for any $\delta \in (0,T)$ and any nonempty open set $\mathcal{O}' \subset\subset \mathcal{O}$. Furthermore, there exists a positive constant $C(\mathcal{O}')$ such that the following estimate holds:

$$\begin{aligned} \|y\|_{L^\infty(\delta,T;W^{1,\infty}(\mathcal{O}'))} \le C(\mathcal{O}')\,(T^{1/2} + T^{N/2})\times \\ \left(1 + \delta^{-1} + \|a\|_\infty + \|B\|_\infty\right)^{N+1}\left(\|y\|_Y + \|f\|_{L^\infty(\mathcal{O}\times(0,T))}\right). \end{aligned} \tag{2.15}$$

The previous regularity also holds with $\delta = 0$ if, besides (2.14)*, we have $y(x,0) = 0$ in Ω. In that case, one has an estimate similar to* (2.15) *without the term in δ.*

This lemma is implied by well known parabolic regularity theory. For completeness, its proof is given in an Appendix, at the end of this paper.

Let us now consider an open set ω_0 with $\omega' \subset\subset \omega_0 \subset\subset \omega$ and a cut-off function ξ, with

$$\xi \in C_0^2(\omega_0), \quad \xi \equiv 1 \text{ in } \omega'$$

and let us set $w = (1-\xi)\,\widetilde{w}$. Then we have:

$$\begin{cases} w_t - \Delta w + a(x,t)\,w + B(x,t)\cdot\nabla w = -\eta'(t)\,\chi + v1_\omega & \text{in } Q,\\ \dfrac{\partial w}{\partial n} + \beta(x,t)\,w = 0 & \text{on } \Sigma,\\ w(x,0) = 0, \quad w(x,T) = 0 & \text{in } \Omega, \end{cases}$$

with

$$v = \eta'\,\xi\,\chi + 2\nabla\xi\cdot\nabla\widetilde{w} + \Delta\xi\,\widetilde{w} - B\cdot\nabla\xi\,\widetilde{w}. \tag{2.16}$$

Let us remark that supp $v \subset \omega \times [0,T]$. Therefore, if we prove that $v \in L^\infty(\omega \times (0,T))$, we will have that the function $y = w + \eta\chi$ solves (together with v) the null controllability problem for (2.2).

Thus, let us check that $v \in L^\infty(\omega \times (0,T))$ and let us estimate its norm in this space:

• The regularity of the first term in the right hand side of (2.16) is implied by the interior regularity of χ not only in space but in time as well. From lemma 2 with $\mathcal{O} = \omega$, we deduce that $\chi \in L^\infty(\omega_0 \times (\delta,T))$ with supp $\xi \subset \omega_0 \subset\subset \omega$ (we even have $\chi \in L^\infty(\delta,T;W^{1,\infty}_{loc}(\omega))$) and

$$\|\chi\|_{L^\infty(\omega_0\times(\delta,T))} \leq C\,(T^{1/2}+T^{N/2})\,\big(1+\delta^{-1}+\|a\|_\infty+\|B\|_\infty\big)^{N+1}\,\|\chi\|_Y;$$

recall that $Y = L^2(0,T;H^1(\Omega)) \cap C^0([0,T];L^2(\Omega))$.

Consequently taking for instance $\delta = T/8$, since $\eta' \equiv 0$ in $(0,T/4)$, we get

$$\|\eta'\,\xi\,\chi\|_{L^\infty(\omega\times(0,T))} \leq C\,T^{-1}\,(T^{1/2}+T^{N/2})\times \\ \big(1+T^{-1}+\|a\|_\infty+\|B\|_\infty\big)^{N+1}\,\|\chi\|_Y.$$

• The regularity of the other three terms in the right hand side of (2.16) is related to the interior space regularity of $\widetilde{w}$. Thus, let us introduce ω_1 with $\omega_0 \subset\subset \omega_1 \subset\subset \omega$ and let us apply lemma 2 with $\mathcal{O} = \omega_1 \setminus \overline{\omega''}$. This gives $\widetilde{w} \in L^\infty(0,T;W^{1,\infty}(\omega_0 \setminus \overline{\omega'}))$ and the estimate

$$\|\widetilde{w}\|_{L^\infty(0,T;W^{1,\infty}(\omega_0\setminus\overline{\omega'}))} \leq C\,(T^{1/2}+T^{N/2})\times \\ (1+\|a\|_\infty+\|B\|_\infty)^{N+1}\,(\|\widetilde{w}\|_Y+\|\eta'\,\chi\|_{L^\infty(\omega_1\times(0,T))}),$$

whence

$$\left\{\begin{array}{l} \|2\nabla\xi\cdot\nabla\widetilde{w}+\Delta\xi\,\widetilde{w}-B\cdot\nabla\xi\,\widetilde{w}\|_{L^\infty(\omega\times(0,T))} \leq C(T^{1/2}+T^{N/2})\times \\ \qquad\qquad (1+\|a\|_\infty+\|B\|_\infty)^{N+2}\,(\|\widetilde{w}\|_Y+\|\eta'\,\chi\|_{L^\infty(\omega_1\times(0,T))}). \end{array}\right.$$

Putting the previous estimates together, we find that $v \in L^\infty(\omega \times (0,T))$ and

$$\|v\|_{L^\infty(\omega\times(0,T))} \leq C(1+T^{N-1})\times \\ \big(1+T^{-1}+\|a\|_\infty+\|B\|_\infty\big)^{2N+3}\,(\|\widetilde{w}\|_Y+\|\chi\|_Y). \tag{2.17}$$

At this point, notice that for any $f \in L^2(Q)$ and any $y^0 \in L^2(\Omega)$ the solution y to the linear system

$$\left\{\begin{array}{ll} y_t-\Delta y+a(x,t)\,y+B(x,t)\cdot\nabla y=f & \text{in } Q, \\ \dfrac{\partial y}{\partial n}+\beta(x,t)\,y=0 & \text{on } \Sigma, \\ y(x,0)=y^0(x) & \text{in } \Omega \end{array}\right. \tag{2.18}$$

satisfies

$$\|y\|_Y \leq e^{C\,T(1+\|a\|_\infty+\|B\|_\infty^2+\|\beta\|_\infty^2)}(\|f\|_{L^2(Q)}+\|y^0\|_{L^2(\Omega)}).$$

For a detailed proof, see for example proposition 1 in [10].

This can be used to estimate $\|\widetilde{w}\|_Y$ and $\|\chi\|_Y$ in terms of $\|\widetilde{v}\|_{L^2(\omega\times(0,T))}$ and $\|y^0\|_{L^2(\Omega)}$. In view of (2.17), we see that

$$\|v\|_{L^\infty(\omega\times(0,T))} \leq L\,(\|\widetilde{v}\|_{L^2(\omega''\times(0,T))} + \|y^0\|_{L^2(\Omega)}), \tag{2.19}$$

where

$$L = C\,T^{-1}(1+T^{N-1})\left(1+T^{-1}+\|a\|_\infty+\|B\|_\infty\right)^{2N+3}\times \exp\{C\,T(1+\|a\|_\infty+\|B\|_\infty^2+\|\beta\|_\infty^2)\}.$$

Combining this estimate and (2.13), we finally obtain that

$$\|v\|_{L^\infty(\omega\times(0,T))} \leq e^{C\,K(T,\|a\|_\infty,\|B\|_\infty,\|\beta\|_\infty)}\,\|y^0\|_{L^2(\Omega)}, \tag{2.20}$$

where K is given by (2.12).

This ends the proof of proposition 3.

3. Controllability of the nonlinear system

In this Section we will prove theorem 3. The following auxiliary result will be needed:

Proposition 4 *Let us assume that, in* (2.18)*, we have* $f \in L^\infty(Q)$ *and* $y^0 \in L^\infty(\Omega)$*. Let us also assume that the coefficients* a*,* B *and* β *satisfy* (2.3)*. Then* $y \in L^\infty(Q)$ *and*

$$\|y\|_\infty \leq e^{C\,T\left(1+\|a\|_\infty+\|B\|_\infty^2+\|\beta\|_\infty^2\right)}\left(\|y^0\|_\infty+\|f\|_\infty\right). \tag{2.21}$$

for some $C = C(\Omega)$*.*

Proof: We will consider two different situations:

Case 1 - We will first assume that $a \geq 1$ and $\beta \geq 0$ and we will establish (2.21) in this case. In fact, we will show that, under these assumptions,

$$\|y\|_\infty \leq \|y^0\|_\infty + \|f\|_\infty. \tag{2.22}$$

To this end, let us introduce the system

$$\begin{cases} z_t - \Delta z + a(x,t)\,z + B(x,t)\cdot\nabla z = h & \text{in } Q, \\ \dfrac{\partial z}{\partial n} + \beta(x,t)\,z = k & \text{on } \Sigma, \\ z(x,0) = z^0(x) & \text{in } \Omega, \end{cases}$$

where $h \in L^\infty(Q)$, $k \in L^\infty(\Sigma)$ and $z^0 \in L^\infty(\Omega)$ and let us show that, if h, z^0 and k are nonnegative, then this is also the case for z.

Indeed, by multiplying the equation satisfied by z by $z_-(\cdot,t)$ (the negative part of $z(\cdot,t)$) for each $t \in (0,T)$ and integrating in Ω, after several simplifications, we find:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega |z_-(x,t)|^2\,dx + \int_\Omega |\nabla z_-(x,t)|^2\,dx \\
&\quad + \int_{\partial\Omega} \beta(x,t)\,(z_-(x,t)+k(x,t))\,z_-(x,t)\,d\sigma + \int_\Omega a(x,t)\,|z_-(x,t)|^2\,dx \\
&\quad = -\int_\Omega h(x,t)\,z_-(x,t)\,dx - \int_\Omega B(x,t)\cdot\nabla z_-(x,t)\,z_-(x,t)\,dx.
\end{aligned}$$

From this identity, in view of the positiveness of a, h, β and k, we easily deduce that

$$\frac{d}{dt}\int_\Omega |z_-(x,t)|^2\,dx \le \|B\|_\infty^2 \int_\Omega |z_-(x,t)|^2\,dx,$$

whence $z \ge 0$ in Q.

Now, let $M > 0$ be a large constant (to be chosen below). The function $z = M - y$ satisfies

$$\begin{cases}
z_t - \Delta z + a(x,t)\,z + B(x,t)\cdot\nabla z = a(x,t)\,M - f & \text{in } Q,\\
\dfrac{\partial z}{\partial n} + \beta(x,t)\,z = \beta(x,t)\,M & \text{on } \Sigma,\\
z(x,0) = M - y^0(x) & \text{in } \Omega.
\end{cases}$$

Therefore, if we take

$$M \ge \text{máx}\{\,\|f\|_{L^\infty(Q)},\ \|y^0\|_{L^\infty(\Omega)}\},$$

we can apply the previous argument and deduce that $y \le M$. In a similar way, one can deduce that $y \ge -M$ and, consequently, $|y| \le M$. This proves that whenever $a \ge 1$ and $\beta \ge 0$, the estimate (2.22) holds.

CASE 2 - We will now prove (2.21) for general L^∞ coefficients a and β.

Let $\gamma \in C^2(\overline{\Omega})$ be a function satisfying

$$\begin{gathered}
\gamma \ge 0 \text{ in } \Omega, \quad \frac{\partial\gamma}{\partial n} \le -\|\beta\|_\infty \text{ on } \partial\Omega, \quad \|\gamma\|_\infty \le 1,\\
\|\nabla\gamma\|_\infty \le C\,\|\beta\|_\infty, \quad \|D^2\gamma\|_\infty \le C\,\|\beta\|_\infty^2.
\end{gathered} \tag{2.23}$$

We give here a sketch of the proof of the existence of such a function γ. To this end, let $\delta > 0$ be a parameter (depending on Ω) such that

$$x \in \Omega_\delta \mapsto \text{dist}(x,\partial\Omega)$$

is C^2, with $\Omega_\delta = \{x \in \Omega : \text{dist}(x,\partial\Omega) < \delta\}$. We distinguish two cases.

Let us first assume that $\|\beta\|_\infty \ge 1/\delta$. Then we take $\gamma(x) \equiv 1$ in $\Omega\backslash\Omega_\delta$, $\gamma(x) = \|\beta\|_\infty\,\text{dist}(x,\partial\Omega)$ in Ω_ε with $\varepsilon = 1/(2\|\beta\|_\infty)$ and a regularization of γ in $\Omega_\delta \setminus \Omega_\varepsilon$. This gives the desired properties for γ.

On the other hand, if $\|\beta\|_\infty < 1/\delta$, we take $\gamma(x) = \delta\,\|\beta\|_\infty$ in $\Omega\backslash\Omega_\delta$, $\gamma(x) = \|\beta\|_\infty \operatorname{dist}(x, \partial\Omega)$ in $\Omega_{\delta/2}$ and a regularization in $\Omega_\delta \setminus \Omega_{\delta/2}$. This also provides a desired function in this case.

Let us now set $\widehat{y} = e^{\gamma(x)}\, y$. Then $\widehat{y}$ satisfies

$$\begin{cases} \widehat{y}_t - \Delta\widehat{y} + \widehat{a}(x,t)\,\widehat{y} + \widehat{B}(x,t)\cdot\nabla\widehat{y} = e^{\gamma(x)} f & \text{in } Q, \\ \dfrac{\partial\widehat{y}}{\partial n} + \widehat{\beta}(x,t)\,\widehat{y} = 0 & \text{on } \Sigma, \\ \widehat{y}(x,0) = e^{\gamma(x)}\, y^0(x) & \text{in } \Omega, \end{cases} \tag{2.24}$$

where

$$\widehat{a} = a + \Delta\gamma - |\nabla\gamma|^2 - B\cdot\nabla\gamma,$$

$$\widehat{B} = B + 2\nabla\gamma, \quad \widehat{\beta} = \beta - \frac{\partial\gamma}{\partial n} \geq 0 \text{ on } \Sigma.$$

Notice that, from the inequalities (2.23) satisfied by γ, we know that

$$|a + \Delta\gamma - |\nabla\gamma|^2 - B\cdot\nabla\gamma| \leq C_1\,(\|a\|_\infty + \|B\|_\infty^2 + \|\beta\|_\infty^2) \text{ in } Q$$

for some $C_1 > 0$.

Now, let us set

$$\widetilde{y} = e^{-(C_1\,(\|a\|_\infty + \|B\|_\infty^2 + \|\beta\|_\infty^2)+1)t}\,\widehat{y}.$$

Then $\widetilde{y}$ satisfies

$$\begin{cases} \widetilde{y}_t - \Delta\widetilde{y} + \widetilde{a}(x,t)\,\widetilde{y} + (B(x,t) + 2\nabla\gamma(x))\cdot\nabla\widetilde{y} = \widetilde{f} & \text{in } Q, \\ \dfrac{\partial\widetilde{y}}{\partial n} + \widetilde{\beta}(x,t)\,\widetilde{y} = 0 & \text{on } \Sigma, \\ \widetilde{y}(x,0) = e^{\gamma(x)}\, y^0(x) & \text{in } \Omega, \end{cases} \tag{2.25}$$

where

$$\widetilde{a} = a + \Delta\gamma - |\nabla\gamma|^2 - B\cdot\nabla\gamma + C_1\,(\|a\|_\infty + \|B\|_\infty^2 + \|\beta\|_\infty^2) + 1,$$

$$\widetilde{f} = e^{-(C_1\,(\|a\|_\infty + \|B\|_\infty^2 + \|\beta\|_\infty^2)+1)t + \gamma(x)}\, f$$

and

$$\widetilde{\beta} = \widehat{\beta}.$$

Since $\widetilde{a} \geq 1$ and $\widetilde{\beta} \geq 0$, we can apply Case 1 to $\widetilde{y}$. This provides the estimates

$$\|y\|_\infty \leq \|\widehat{y}\|_\infty \leq e^{C\,T\left(1 + \|a\|_\infty + \|B\|_\infty^2 + \|\beta\|_\infty^2\right)}\,(\|y^0\|_\infty + \|f\|_\infty),$$

whence we deduce (2.21).

Let us now start with the proof of theorem 3. Let $y^0 \in L^\infty(\Omega)$ and $\overline{y}$ be given and assume that $\overline{y}$ satisfies (2.6) and (2.4) in the weak sense. Let us consider the nonlinear system

$$\begin{cases} w_t - \Delta w + F_1(w, \nabla w; x,t)w + F_2(\nabla w; x,t)\cdot \nabla w = v1_\omega & \text{in } Q, \\ \dfrac{\partial w}{\partial n} + F_3(w; x,t)w = 0 & \text{on } \Sigma, \\ w(x,0) = y^0(x) - \overline{y}(x,0) & \text{in } \Omega, \end{cases} \tag{2.26}$$

where we have used the notation

$$F_1(s,p;x,t) = \frac{F(\overline{y}(x,t)+s, \nabla\overline{y}(x,t)+p) - F(\overline{y}(x,t), \nabla\overline{y}(x,t)+p)}{s}, \tag{2.27}$$

$$F_2 = (F_{21}, \dots, F_{2N}), \quad F_{2j}(p;x,t) = \int_0^1 \frac{\partial F}{\partial p_j}(\overline{y}(x,t), \nabla\overline{y}(x,t) + \lambda p)\, d\lambda \tag{2.28}$$

and

$$F_3(s;x,t) = \frac{f(\overline{y}(x,t)+s) - f(\overline{y}(x,t))}{s} \tag{2.29}$$

for $s \in \mathbf{R}$ and $p \in \mathbf{R}^N$.

We will prove that there exist a control $v \in L^\infty(\omega \times (0,T))$ and an associated solution to (2.26) such that

$$w(x,T) = 0 \ \text{ in } \ \Omega. \tag{2.30}$$

With this control and the state $y = w + \overline{y}$, we will have solved the exact controllability problem for (2.1) and we will have thus proved theorem 3.

We will first assume that the functions F and f are continuously differentiable. Then, by a density argument, we will be able to prove the result in the general case.

3.1. The case in which F and f are C^1

The idea of the proof is well known: we introduce an appropriate (set-valued) fixed point mapping and we check that it possesses at least one fixed point; this will be a solution to the null controllability problem associated to (2.26).

Let $R > 0$ be given and let us introduce the following function:

$$M_R(s) = \begin{cases} -R & \text{if } s < -R, \\ s & \text{if } -R \le s \le R, \\ R & \text{if } s > R. \end{cases}$$

Let us denote by Z the Hilbert space $Z = L^2(0,T;H^1(\Omega))$ and let us set for each $R > 0$ and each $z \in Z$

$$a_{R,z}(x,t) = F_1(M_R(z(x,t)), \nabla z(x,t); x,t),$$

$$B_z(x,t) = F_2(\nabla z(x,t); x,t)$$

and

$$\beta_{R,z}(x,t) = F_3(M_R(z(x,t)); x,t).$$

Consider the linear null controllability problem

$$\begin{cases} w_t - \Delta w + a_{R,z}(x,t)\, w + B_z(x,t) \cdot \nabla w = v 1_\omega & \text{in } Q, \\ \dfrac{\partial w}{\partial n} + \beta_{R,z}(x,t)\, w = 0 & \text{on } \Sigma, \\ w(x,0) = y^0(x) - \overline{y}(x,0) & \text{in } \Omega, \end{cases} \tag{2.31}$$

together with (2.30).

From (2.6), (2.8) and the fact that $f \in C^1(\mathbf{R})$, we have

$$a_{R,z} \in L^\infty(Q), \quad B_z \in L^\infty(Q)^N, \quad \beta_{R,z} \in L^\infty(\Sigma).$$

Consequently, in view of proposition 3, (2.30)–(2.31) can be solved with controls in $L^\infty(\omega \times (0,T))$.

We are now going to select a particular solution to (2.30)–(2.31) constructed as in [11]. To do this, we first set $T_R = \text{mín}\{T, a_R^{-1/3}\} > 0$, where

$$a_R = \sup_{|s| \le R,\ p \in \mathbf{R}^N} \ \operatorname*{ess\,sup}_{(x,t) \in Q} |F_1(s,p;x,t)|.$$

We can follow the steps of Section 2 and construct a control $v_{R,z} \in L^\infty(\omega \times (0,T_R))$ such that the solution $w_{R,z}$ to (2.31) in $\Omega \times (0,T_R)$ verifies

$$w_{R,z}(x,T_R) = 0 \ \text{ in } \ \Omega.$$

The estimates we have been able to establish in propositions 3 and 4 written for $v_{R,z}$ and $w_{R,z}$ with final time T_R will now give

$$\|v_{R,z}\|_{L^\infty(\omega \times (0,T_R))} \le C_R \, \|w^0\|_{L^2(\Omega)}, \tag{2.32}$$

$$\|w_{R,z}\|_{L^2(0,T_R;H^1(\Omega))} \le C_R \, \|w^0\|_{L^2(\Omega)} \tag{2.33}$$

and

$$\|w_{R,z}\|_{L^\infty(\Omega \times (0,T_R))} \le C_R \, \|w^0\|_{L^\infty(\Omega)}, \tag{2.34}$$

where

$$C_R = \exp\left\{ C(\Omega,\omega,T) \left(1 + a_R^{2/3} + \overline{B}^2 + \beta_R^2 \right) \right\},$$

$$\overline{B} = \sup_{p \in \mathbf{R}^N} \ \operatorname*{ess\,sup}_{(x,t) \in Q} |F_2(p;x,t)|$$

and

$$\beta_R = \sup_{|s| \le R} \ \operatorname*{ess\,sup}_{(x,t) \in \Sigma} |F_3(s;x,t)|.$$

In fact, the estimates obtained in the previous section imply (2.32)–(2.34) with C_R replaced by $C_R(z)$, where

$$C_R(z) = \exp\Big\{ C(\Omega,\omega) \ \Big(1 + T_R^{-1} + T_R + \|a_{R,z}\|_\infty^{2/3} + \|B_z\|_\infty^2 + \|\beta_{R,z}\|_\infty^2 + T_R \left(\|a_{R,z}\|_\infty + \|B_z\|_\infty^2 + \|\beta_{R,z}\|_\infty^2 \right) \Big) \Big\};$$

but taking into account the definitions of T_R, a_R, $\overline{B}$ and β_R it is clear that $C_R(z) \leq C_R$ for all $z \in Z$.

At this moment, we can extend by zero the functions $v_{R,z}$ and $w_{R,z}$ for $t \in (T_R, T)$. In this way, we will have built a control $v_{R,z}$ and an associated state $w_{R,z}$ satisfying (2.30)–(2.31) and

$$\|v_{R,z}\|_{L^\infty(\omega\times(0,T))} \leq C_R \, \|w^0\|_{L^2(\Omega)}, \tag{2.35}$$

$$\|w_{R,z}\|_Z \leq C_R \, \|w^0\|_{L^2(\Omega)} \tag{2.36}$$

and

$$\|w_{R,z}\|_\infty \leq C_R \, \|w^0\|_{L^\infty(\Omega)}. \tag{2.37}$$

We will now introduce a set-valued mapping leading to the solution to our controllability problem.

We first consider the set of admissible controls $A_R(z)$. By definition, this is the set of controls $v_{R,z} \in L^\infty(\omega \times (0,T))$ which lead the solution to (2.31) to zero at time T and satisfy (2.35). Then, for each $z \in Z$, we denote by $\Lambda_R(z)$ the set of states $w_{R,z}$ associated to the controls $v_{R,z} \in A_R(z)$ furthermore satisfying (2.36) and (2.37). In view of the arguments above, $\Lambda_R(z)$ is a nonempty subset of Z.

The plan of the rest of the proof is the following:

- We will first see that, for each $R > 0$, Λ_R possesses a fixed point w_R. This will be implied by Kakutani's theorem.
- Then, we will find $R > 0$ (large enough) such that $M_R(w_R) = w_R$. At this level, the use of proposition 4 will be crucial.

As a consequence, for large R, the fixed point w_R of Λ_R will be, together with some $v_R \in L^\infty(\omega \times (0,T))$, a solution to (2.30)–(2.31).

Thus, let us recall Kakutani's fixed point theorem (see, for instance, [2]):

Theorem 4 *Let Z be a Banach space and let $\Lambda : Z \mapsto Z$ be a set-valued mapping satisfying the following assumptions:*

1. *$\Lambda(z)$ is a nonempty closed convex set of Z for every $z \in Z$.*
2. *There exists a nonempty convex compact set $K \subset Z$ such that $\Lambda(K) \subset K$.*
3. *Λ is upper-hemicontinuous in Z, i.e. for each $\sigma \in Z'$ the single-valued mapping*

$$z \mapsto \sup_{y\in\Lambda(z)} \langle \sigma, y\rangle_{Z',Z} \tag{2.38}$$

is upper-semicontinuous.

Then Λ possesses a fixed point in the set K, i.e. there exists $z \in K$ such that $z \in \Lambda(z)$.

Let us check that Kakutani's theorem can be applied to Λ_R.

That $\Lambda_R(z)$ is a nonempty closed convex set of Z for every $z \in Z$ is very easy to verify.

Let us prove that Λ_R maps a compact set into itself. In fact, let us see that Λ_R maps the whole space Z into a fixed convex compact set K_R.

Our argument will be the following: we choose an arbitrary sequence $\{z_n\}$ in Z and a sequence $\{w_n\}$ with $w_n \in \Lambda_R(z_n)$ for all n and we prove that $\{w_n\}$ possesses a strongly convergent subsequence.

Thus, let the sequences $\{z_n\}$ and $\{w_n\}$ be given. From (2.35)–(2.37), the equations satisfied by the functions w_n and the fact that $\|a_{R,z_n}\|_\infty \le a_R$, $\|B_{z_n}\|_\infty \le \overline{B}$ and $\|\beta_{R,z_n}\|_\infty \le \beta_R$ for all $n \ge 1$, we deduce the existence of subsequences $\{w_{n'}\}$ and $\{v_{n'}\}$ such that

$$w_{n'} \to w \quad \text{weakly in } Z,$$

$$w_{n',t} \to w_t \quad \text{weakly in } L^2(0,T;H^{-1}(\Omega))$$

and

$$v_{n'} \to v \quad \text{weakly-}* \text{ in } L^\infty(Q)$$

as $n' \to \infty$. We can also assume that the coefficients associated to $z_{n'}$ converge weakly-$*$ in $L^\infty(Q)$ and $L^\infty(\Sigma)$. Thus, we can pass to the limit in the weak formulations satisfied by $w_{n'}$ and deduce that w and v satisfy

$$\begin{cases} w_t - \Delta w + a(x,t)\, w + \theta(x,t) = v 1_\omega & \text{in } Q,\\ \dfrac{\partial w}{\partial n} + \beta(x,t)\, w = 0 & \text{on } \Sigma,\\ w(x,0) = w^0(x) & \text{in } \Omega \end{cases}$$

for some $a \in L^\infty(Q)$ and $\beta \in L^\infty(\Sigma)$, where $\theta \in L^2(Q)$ is the weak limit of $B_{z_{n'}} \cdot \nabla w_{n'}$ in $L^2(Q)$.

After substraction of the equations satisfied by the functions $w_{n'}$ and w, we find that

$$\begin{cases} (w_{n'} - w)_t - \Delta(w_{n'} - w) = a(x,t)\, w - a_{R,z_{n'}}(x,t)\, w_{n'} & \\ \qquad + \theta(x,t) - B_{z_{n'}}(x,t) \cdot \nabla w_{n'} + (v_{n'} - v)1_\omega & \text{in } Q,\\ \dfrac{\partial(w_{n'} - w)}{\partial n} + \beta_{R,z_{n'}}(x,t)\, w_{n'} - \beta(x,t)\, w = 0 & \text{on } \Sigma,\\ (w_{n'} - w)(x,0) = 0 & \text{in } \Omega. \end{cases}$$

Consequently,

$$
\begin{aligned}
\frac{1}{2}\int_\Omega |(w_{n'}-w)(x,T)|^2\,dx &+ \int_0^T\!\!\int_\Omega |\nabla(w_{n'}-w)(x,s)|^2\,dx\,ds \\
&= \int_0^T\!\!\int_{\partial\Omega} (\beta\, w - \beta_{R,z_{n'}}\, w_{n'})\,(w_{n'}-w)(x,s)\,d\sigma\,ds \\
&\quad + \int_0^T\!\!\int_\Omega (a\, w - a_{R,z_{n'}}\, w_{n'})\,(w_{n'}-w)(x,s)\,dx\,ds \\
&\quad + \int_0^T\!\!\int_\Omega (\theta - B_{z_{n'}}\cdot\nabla w_{n'})\,(w_{n'}-w)(x,s)\,dx\,ds \\
&\qquad + \int_0^T\!\!\int_\omega (v_{n'}-v)\,(w_{n'}-w)(x,s)\,dx\,ds.
\end{aligned} \tag{2.39}
$$

We are now going to check that all the terms in the right hand side of this last equality tends to zero. Among other things, this will imply that

$$w_{n'} \to w \quad \text{strongly in } Z. \tag{2.40}$$

• The first term in the right hand side converges to zero, since

$$w_{n'} \to w \quad \text{strongly in } L^2(\Sigma)$$

and consequently

$$\beta_{R,z_{n'}}\, w_{n'} \to \beta\, w \quad \text{weakly in } L^2(\Sigma).$$

Indeed, the strong convergence of $w_{n'}$ is an immediate consequence of the compact embedding of the space

$$\{\, z \in L^2(0,T;H^1(\Omega)) : z_t \in L^2(0,T;H^{-1}(\Omega))\,\}$$

in $L^2(0,T;H^s(\Omega))$ for all $s\in(1/2,1)$ and the fact that the *lateral trace* operator is well defined, linear and continuous from $L^2(0,T;H^s(\Omega))$ into $L^2(\Sigma)$.

• The convergence of the other three terms in the right hand side is a consequence of the weak convergence in $L^2(Q)$ of $a_{R,z_{n'}}\, w_{n'}$ and $B_{z_{n'}}\cdot\nabla w_{n'}$, the weak convergence in $L^2(\omega\times(0,T))$ of $v_{n'}$ and the *strong* convergence of $w_{n'}$ in $L^2(Q)$.

We have thus seen that $\{w_n\}$ possesses a strongly convergent subsequence and, consequently, Λ_R maps the space Z into a fixed compact set.

It remains to check that Λ_R is upper-hemicontinuous. Thus, assume that $\sigma\in Z'$ and let a sequence $\{z_n\}$ be given, with $z_n\to z$ strongly in Z. We must prove that

$$\limsup_{n\to+\infty}\ \sup_{w\in\Lambda_R(z_n)} \langle\sigma,w\rangle_{Z',Z} \le \sup_{w\in\Lambda_R(z)} \langle\sigma,w\rangle_{Z',Z}.$$

Let $\{z_{n'}\}$ be a subsequence of $\{z_n\}$ such that

$$\limsup_{n\to+\infty}\ \sup_{w\in\Lambda_R(z_n)} \langle\sigma,w\rangle_{Z',Z} = \lim_{n'\to+\infty}\ \sup_{w\in\Lambda_R(z_{n'})} \langle\sigma,w\rangle_{Z',Z}.$$

Since each $\Lambda_R(z_{n'})$ is a compact set of Z, for every n' we have

$$\sup_{w\in\Lambda_R(z_{n'})} \langle \sigma, w\rangle_{Z',Z} = \langle \sigma, w_{n'}\rangle_{Z',Z}$$

for some $w_{n'} \in \Lambda_R(z_{n'})$. On the other hand, since all the states $w_{n'}$ belong to the same compact set K_R, at least for a new subsequence (again indexed by n'), we must have (2.40). We will now prove that $w \in \Lambda_R(z)$. This will achieve the proof of the upper hemicontinuity of Λ_R.

Indeed, we can assume that the weak limits of the coefficients associated to $z_{n'}$ are $a_{R,z}$, B_z and $\beta_{R,z}$, since $z_{n'}$ converges strongly in Z towards z and therefore the coefficients $a_{R,z_{n'}}$, $B_{z_{n'}}$ and $\beta_{R,z_{n'}}$ converge almost everywhere (observe that we are using here the C^1 regularity of F and f).

On the other hand, it can be assumed that the controls $v_{n'}$ converge to a function v weakly-$*$ in $L^\infty(\omega\times(0,T))$. Then, w solves (2.31) and $w(T)=0$. Moreover, since inequality (2.35) is independent of n, v also satisfies (2.35). Therefore, $v \in A_R(z)$. Consequently, it is immediate that w is the solution to (2.31) associated to the control v.

This shows that $w \in \Lambda_R(z)$ and, therefore, Λ_R is upper hemicontinuous.

In view of these arguments, Kakutani's theorem can be applied and we deduce that, for each $R>0$, Λ_R possesses at least one fixed point w_R that belongs to Z and $L^\infty(Q)$.

Our aim is now to find $R>0$ such that

$$\|w_R\|_\infty \le R.$$

This will be a consequence of the estimates we know for w_R and the properties satisfied by the functions F_i.

From (2.37), we obtain

$$\|w_R\|_\infty \le e^{C(\Omega,\omega,T)(1+a_R^{2/3}+\overline{B}^2+\beta_R^2)}\,\|w^0\|_\infty. \tag{2.41}$$

On the other hand, from (2.8)–(2.9) it is also clear that, for every $\varepsilon>0$, there exists $C_\varepsilon>0$ such that

$$\left(\operatorname*{ess\,sup}_{(x,t)\in Q}|F_1(s,p;x,t)|\right)^{2/3} \le \varepsilon\,\log(1+|s|)+C_\varepsilon \quad \forall s\in\mathbf{R},\ \forall p\in\mathbf{R}^N,$$

$$\left(\operatorname*{ess\,sup}_{(x,t)\in Q}|F_2(p;x,t)|\right)^{2} \le C_\varepsilon \quad \forall p\in\mathbf{R}^N, \tag{2.42}$$

$$\left(\operatorname*{ess\,sup}_{(x,t)\in \Sigma}|F_1(s;x,t)|\right)^{2} \le \varepsilon\,\log(1+|s|)+C_\varepsilon \quad \forall s\in\mathbf{R}.$$

Consequently, it is also true that, for every $\varepsilon>0$, there exists $C_\varepsilon>0$ (independent of R) such that

$$a_R^{2/3}+\overline{B}^2+\beta_R^2 \le \varepsilon\,\log(1+R)+C_\varepsilon.$$

These estimates, together with (2.41) and the definitions of a_R, B and β_R, lead to the following inequality:

$$\|w_R\|_\infty \leq\, C(\Omega,\omega,T,\varepsilon)\,(1+R)^{C(\Omega,\omega,T)\,\varepsilon}\,\|w^0\|_\infty.$$

Accordingly, taking $\varepsilon > 0$ small enough to satisfy $C(\Omega,\omega,T)\,\varepsilon < 1$, we can ensure that, for $R > 0$ sufficiently large (depending on Ω, ω, T and $\|y^0 - \overline{y}^0\|_{L^\infty(\Omega)}$), one has

$$\|w_R\|_\infty \leq\, R.$$

This ends the proof of theorem 3 when F and f are C^1 functions.

3.2. The general case

We will now assume that f and F are locally Lipschitz-continuous functions satisfying (2.7)–(2.9).

Let us introduce the functions $\rho^1 \in C_c^\infty(\mathbf{R}\times\mathbf{R}^N)$, $\rho^2 \in C_c^\infty(\mathbf{R}^N)$ and $\rho^3 \in C_c^\infty(\mathbf{R})$, with $\rho^j \geq 0$, supp $\rho^1 \subset \overline{B}((0,0);1)$, supp $\rho^2 \subset \overline{B}(0;1)$, supp $\rho^3 \subset [-1,1]$ and

$$\iint_{\mathbf{R}\times\mathbf{R}^N} \rho^1(s,p)\,ds\,dp = \int_{\mathbf{R}^N} \rho^2(p)\,dp = \int_{\mathbf{R}} \rho^3(s)\,ds = 1.$$

Let us consider, for each $n \geq 1$, the associated *mollifiers*

$$\rho_n^1(s,p) = n^{N+1}\,\rho^1(ns,np), \quad \rho_n^2(p) = n^N\rho^2(np) \quad \forall (s,p)\in\mathbf{R}\times\mathbf{R}^N$$

and

$$\rho_n^3(s) = n\,\rho^3(ns) \quad \forall s\in\mathbf{R}$$

and the regularized functions

$$F_{i,n} = \rho_n^i * F_i \quad (i=1,2,3)$$

(the functions F_1, F_2 and F_3 were defined in (2.27)–(2.29)).

These functions satisfy the following:

- For each $n \geq 1$, $F_{1n} : \mathbf{R}\times\mathbf{R}^N\times Q \mapsto \mathbf{R}$, $F_{2n} : \mathbf{R}^N\times Q \mapsto \mathbf{R}^N$ and $F_{3n} : \mathbf{R}\times\Sigma\mapsto\mathbf{R}$ are Caratheodory functions (respectively continuous in (s,p), p and s and measurable in (x,t)).

- If we set

$$F_n(s,p;x,t) = F_{1,n}(s,p;x,t)s + F_{2,n}(p;x,t)\cdot p$$

and

$$f_n(s;x,t) = F_{3,n}(s;x,t)s,$$

then the asymptotic properties (2.8) and (2.42) remain true uniformly in n. In other words, for any $L > 0$, there exists $M > 0$ (independent of n) such that

$$\begin{cases} |F_n(s,p) - F_n(r,p)| \leq M|s-r|, \quad |F_n(s,p) - F_n(s,q)| \leq M|p-q| \\ \forall (s,r,p,q)\in[-L,L]^2\times\mathbf{R}^N\times\mathbf{R}^N. \end{cases}$$

Moreover, for each $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$\begin{cases} \left(\operatorname*{ess\,sup}_{(x,t)\in Q} |F_{1n}(s,p;x,t)| \right)^{2/3} \leq \varepsilon \log(1+|s|) + C_\varepsilon \quad \forall s \in \mathbf{R}, \forall p \in \mathbf{R}^N, \\ \left(\operatorname*{ess\,sup}_{(x,t)\in Q} |F_{2n}(p;x,t)| \right)^{2} \leq C_\varepsilon \quad \forall p \in \mathbf{R}^N, \\ \left(\operatorname*{ess\,sup}_{(x,t)\in \Sigma} |F_{3n}(s;x,t)| \right)^{2} \leq \varepsilon \log(1+|s|) + C_\varepsilon \quad \forall s \in \mathbf{R}, \end{cases}$$

for each $n \geq 1$.

- From the definitions of F_n and f_n, we also have that

$$F_n(z_n, \nabla z_n; \cdot) \to F_1(z, \nabla z; \cdot) z + F_2(\nabla z; \cdot) \cdot \nabla z \quad \text{weakly in } L^2(Q)$$

and

$$f_n(z_n; \cdot) \to F_3(z; \cdot) z \quad \text{weakly-}* \text{ in } L^\infty(\Sigma)$$

whenever

$$z_n \to z \quad \text{weakly-}* \text{ in } L^\infty(Q) \text{ and strongly in } L^2(0,T;H^1(\Omega)).$$

As a consequence, we can argue as in Subsection 3.1 and deduce that, for each n, there exists a control $v_n \in L^\infty(\omega \times (0,T))$ such that

$$\begin{cases} w_{n,t} - \Delta w_n + F_n(w_n, \nabla w_n; x, t) = v_n 1_\omega & \text{in } Q, \\ \dfrac{\partial w_n}{\partial n} + f_n(w_n; x, t) = 0 & \text{on } \Sigma, \\ w_n(x,0) = w^0(x) & \text{in } \Omega \end{cases} \tag{2.43}$$

and

$$w_n(x,T) = 0 \text{ in } \Omega. \tag{2.44}$$

In view of the properties satisfied by the functions F_{in}, the estimates we have established in Subsection 3.1 are independent of n. Accordingly, at least for a subsequence, we also have

$$v_n \to v \quad \text{weakly-}* \text{ in } L^\infty(\omega \times (0,T)),$$
$$w_{n,t} \to w_t \quad \text{weakly in } L^2(0,T;H^{-1}(\Omega))$$

and

$$w_n \to w \quad \text{weakly-}* \text{ in } L^\infty(Q) \text{ and strongly in } L^2(0,T;H^1(\Omega)).$$

Thus, we can pass to the limit in (2.43) and find a control $v \in L^\infty(\omega \times (0,T))$ such that the associated solution to (2.26) satisfies (2.30).

This ends the proof of theorem 3.

Remark 4 The proof of theorem 3 can also be achieved by applying another fixed point argument. More precisely, we can first introduce a small parameter $\varepsilon > 0$ and find a control v_ε such that the solution of (2.1) satisfies

$$\|y(\cdot, T) - \overline{y}(\cdot, T)\|_{L^2} \leq \varepsilon.$$

This can be made by previously solving an approximate controllability problem for the linear system (2.31) with the control of minimal norm in $L^p(\omega \times (0,T))$ for p large enough and, then, using Schauder's theorem. Since all the estimates we can establish are uniform in ε, we can pass to the limit as $\varepsilon \to 0$ and deduce the desired result.

Appendix: Proof of lemma 2

Let us introduce $N+1$ open subsets of $\mathcal{O}$ satisfying

$$\mathcal{O}_N = \mathcal{O}' \subset\subset \mathcal{O}_{N-1} \subset\subset ... \subset\subset \mathcal{O}_1 \subset\subset \mathcal{O}_0 \subset\subset \mathcal{O}.$$

We will also consider subintervals of $(0,T)$ of the form $(\delta/(i+1), T)$ for $0 \leq i \leq N$.

We will strict our considerations to the proof of lemma 2 in the case where no initial condition is imposed. The result concerning a vanishing initial condition will follow readily from the argument below.

We are first going to see that

$$y \in L^\infty(\delta/(N+1), T; H^1(\mathcal{O}_0)) \quad \text{and} \quad \Delta y \in L^2(\delta/(N+1), T; L^2(\mathcal{O}_0)),$$

with an estimate of the associated norms independent of T. To this end, let $\xi_0 \in C^2_c(\mathcal{O})$ and $\eta_0 \in C^1([0,T])$ be two functions satisfying

$$\xi_0(x) = 1 \text{ in } \mathcal{O}_0, \quad \eta_0(t) = 1 \text{ in } [\frac{\delta}{N+1}, T], \quad \eta_0(0) = 0, \quad |\eta_{0,t}(t)| \leq \frac{C}{\delta} \text{ in } (0,T)$$

(of course, C depends on N) and let us introduce the function $y_0 = \eta_0\, \xi_0\, y$. Then

$$\begin{cases} y_{0,t} - \Delta y_0 = f_0 & \text{in } Q, \\ y_0 = 0 & \text{on } \Sigma, \\ y_0(x,0) = 0 & \text{in } \Omega, \end{cases}$$

where

$$\begin{aligned} f_0 = \eta_0\, \xi_0\, f + \eta_{0,t}\, \xi_0\, y - 2\eta_0\, \nabla\xi_0 \cdot \nabla y - \eta_0\, \Delta\xi_0\, y \\ - a\, \eta_0\, \xi_0\, y - \eta_0\, \xi_0\, B \cdot \nabla y - \eta_0 (B \cdot \nabla \xi_0) y. \end{aligned}$$

We have $f_0 \in L^2(Q)$. Consequently, $\Delta y_0 \in L^2(Q)$, $y_0 \in C^0([0,T]; H^1_0(\Omega))$ and appropriate estimates are satisfied. Indeed, by multiplying the equation satisfied by y_0 by $-\Delta y_0$ and integrating with respect to x in Ω, we find

$$\frac{1}{2}\frac{d}{dt}\|\nabla y_0(\cdot,t)\|^2_{L^2} + \int_\Omega |\Delta y_0(x,t)|^2\, dx = -\int_\Omega f_0(x,t)\, \Delta y_0(x,t)\, dx. \tag{2.45}$$

Since

$$\|f_0\|_{L^2} \le C\left(\|f\|_{L^2} + (1+\delta^{-1} + \|a\|_\infty + \|B\|_\infty)\,\|y\|_Y\right),$$

we easily obtain from (2.45) that

$$\|y_0\|_{C^0([0,T];H_0^1(\Omega))} + \|\Delta y_0\|_{L^2(Q)} \le C\left(\|f\|_{L^2} + \left(1+\delta^{-1} + \|a\|_\infty + \|B\|_\infty\right)\,\|y\|_Y\right). \qquad (2.46)$$

Clearly, the same estimate holds for

$$\|y\|_{L^\infty(\delta/(N+1),T;H^1(\mathcal{O}_0))} + \|\Delta y\|_{L^2(\mathcal{O}_0\times(\delta/(N+1),T))}.$$

Let us now try to improve the local space regularity properties of y. To this end, we will use the following lemma:

Lemma 3 *Let us set $p_0 = 2$, let p_i be defined by*

$$\frac{1}{p_i} = \frac{1}{p_{i-1}} - \frac{1}{2N}$$

for $1 \le i \le N-1$ and let us set $p_N = +\infty$. Let us denote by X_i the space

$$X_i = L^\infty((i+1)\delta/(N+1),T;W^{1,p_i}(\mathcal{O}_i))$$

for $0 \le i \le N$ and suppose that $y \in X_{j-1}$ for some j. Then we also have $y \in X_j$ and

$$\|y\|_{X_j} \le C(\mathcal{O}')(T^{1/2}\,\|f\|_\infty + D(T,\delta,\|a\|_\infty,\|B\|_\infty)\,\|y\|_{X_{j-1}}),$$

where

$$D(T,\delta,\|a\|_\infty,\|B\|_\infty) = (T^{1/4}+T^{1/2})(1+\delta^{-1}+\|a\|_\infty+\|B\|_\infty). \qquad (2.47)$$

Proof of lemma 3: Let us introduce $\xi_j \in C_c^2(\mathcal{O}_{j-1})$ and $\eta_j \in C^1([0,T])$, with

$$\xi_j(x) = 1 \text{ in } \mathcal{O}_j\,, \quad \eta_j(t) = 1 \text{ in } [(j+1)\delta/(N+1),T],$$
$$\eta_j(t) = 0 \text{ in } [0,j\delta/(N+1)], \quad |\eta_{j,t}(t)| \le \frac{C}{\delta} \text{ in } (0,T)$$

and let us put $y_j = \eta_j\,\xi_j\,y$. Then y_j satisfies the following:

$$\begin{cases} y_{j,t} - \Delta y_j = f_j & \text{in } Q, \\ y_j = 0 & \text{on } \Sigma, \\ y_j(x,0) = 0 & \text{in } \Omega \end{cases} \qquad (2.48)$$

with

$$f_j = f_{j,1} + f_{j,2} + f_{j,3}\,,$$

where

$$f_{j,1} = \eta_j\,\xi_j\,f, \quad f_{j,2} = \eta_{j,t}\,\xi_j\,y - \eta_j\,\Delta\xi_j\,y - a\,\eta_j\,\xi_j\,y - \eta_j(B\cdot\nabla\xi_j)y,$$
$$f_{j,3} = -2\eta_j\,\nabla\xi_j\cdot\nabla y - \eta_j\,\xi_j\,B\cdot\nabla y.$$

From the fact that the system (2.48) is linear, we see that y_j can be written as the sum of three solutions to similar systems with right hand sides $f_{j,1}$, $f_{j,2}$ and $f_{j,3}$. Let us respectively denote them by $y_{j,1}$, $y_{j,2}$ and $y_{j,3}$. We are now going to deduce estimates of $y_{j,k}$ in X_j for $1 \le k \le 3$.

To this end, we will use the usual representation of $y_{j,k}$ provided by the semigroup $S(t)$ associated to the heat equation with homogeneous Dirichlet conditions, say

$$y_{j,k}(\cdot,t) = \int_0^t S(t-s)\, f_{j,k}(\cdot,s)\, ds$$

for all $t \in (0,T)$.

Since $f \in L^\infty(Q)$, we can write

$$\|y_{j,1}(\cdot,t)\|_{W^{1,p_j}(\Omega)} \le C \int_0^t (t-s)^{-1/2}\, \|f_{j,1}(\cdot,s)\|_{L^{p_j}(\Omega)}\, ds.$$

Therefore, from Young's inequality we find that $y_{j,1} \in L^\infty(0,T;W^{1,p_j}(\Omega))$ and

$$\begin{aligned} \|y_{j,1}\|_{L^\infty(0,T;W^{1,p_j}(\Omega))} &\le C\, T^{1/2}\, \|f_{j,1}\|_{L^\infty(0,T;L^{p_j}(\Omega))} \\ &\le C(\mathcal{O}')\, T^{1/2}\, \|f\|_{L^\infty(\mathcal{O}\times(0,T))} \,. \end{aligned}$$

Taking into account that $f_{j,2} \in L^\infty(0,T;L^{p^*_{j-1}}(\Omega))$ with

$$p^*_{j-1} = \begin{cases} \infty & \text{if } j > N-1, \\ p & \text{(arbitrary in } (1,+\infty)) \text{ if } j = N-1, \\ \dfrac{2N}{N-j-1} & \text{if } j < N-1, \end{cases}$$

we see that $f_{j,2}$ is not worse than $f_{j,1}$ and, again,

$$\|y_{j,2}(\cdot,t)\|_{W^{1,p_j}(\Omega)} \le C \int_0^t (t-s)^{-1/2}\, \|f_{j,2}(\cdot,s)\|_{L^{p_j}(\Omega)}\, ds$$

for all t. From Young's inequality and the assumption $y \in X_{j-1}$, we also get $y_{j,2} \in L^\infty(0,T;W^{1,p_j}(\Omega))$ and

$$\begin{aligned} \|y_{j,2}\|_{L^\infty(0,T;W^{1,p_j}(\Omega))} &\le C\, T^{1/2}\, \|f_{j,2}\|_{L^\infty(0,T;L^{p^*_{j-1}}(\Omega))} \\ &\le C(\mathcal{O}')\, T^{1/2}\, (1+\delta^{-1}+\|a\|_\infty)\, \|y\|_{X_{j-1}}. \end{aligned}$$

In the definition of $f_{j,3}$, we find ∇y.

Consequently, we can only ensure that $f_{j,3} \in L^\infty(0,T;L^{p_{j-1}}(\Omega))$. Since

$$-\frac{N}{2}\left(\frac{1}{p_{j-1}} - \frac{1}{p_j}\right) - \frac{1}{2} = -\frac{3}{4},$$

we have

$$\|y_{j,3}(\cdot,t)\|_{W^{1,p_j}(\Omega)} \le C \int_0^t (t-s)^{-3/4}\, \|f_{j,3}(\cdot,s)\|_{L^{p_{j-1}}(\Omega)}\, ds$$

and now Young's inequality gives $y_{j,3} \in L^\infty(0,T;W^{1,p_j}(\Omega))$ and

$$\|y_{j,3}\|_{L^\infty(0,T;W^{1,p_j}(\Omega))} \leq C\,T^{1/4}\,\|f_{j,3}\|_{L^\infty(0,T;L^{p_j-1}(\Omega))}$$
$$\leq C(\mathcal{O}')\,T^{1/4}\,(1+\|B\|_\infty)\,\|y\|_{X_{j-1}}.$$

Putting the estimates of $\|y_{j,k}\|_{L^\infty(0,T;W^{1,p_j}(\Omega))}$ together and taking into account the definitions of η_j and ξ_j , we obtain the desired inequality for $\|y\|_{X_j}$.

This concludes the proof of lemma 3. □

Since we already had $y \in X_0$, we deduce from lemma 3 that $y \in X_N$ and

$$\|y\|_{X_N} \leq C(\mathcal{O}')(T^{1/2}\,\|f\|_\infty + D(T,\delta,\|a\|_\infty,\|B\|_\infty)\,\|y\|_{X_{N-1}}),$$

where, D is given by (2.47).

We can apply lemma 3 subsequently for $j = N, N-1, \ldots, 1$. The estimates we find yield

$$\|y\|_{X_N} \leq C(T^{1/2}(1+D^{N-1})\|f\|_{L^\infty(\mathcal{O}\times(0,T))} + D^N\,\|y\|_{X_0}).$$

This, together with (2.46), yields

$$\|y\|_{X_N} \leq C(T^{1/2}+T^{N/2})D(T,\delta,\|a\|_\infty,\|B\|_\infty)^{N+1}(\|f\|_{L^\infty(\mathcal{O}\times(0,T))} + \|y\|_Y),$$

which is exactly (2.15).

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Capítulo 3

Some controllability results for the N-dimensional Navier-Stokes and Boussinesq systems with $N-1$ scalar controls

Some controllability results for the N-dimensional Navier-Stokes and Boussinesq systems with $N-1$ scalar controls

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Abstract

In this paper we deal with some controllability problems for systems of the Navier-Stokes and Boussinesq kind with distributed controls supported in small sets. Our main aim is to control N-dimensional systems ($N+1$ scalar unknowns in the case of the Navier-Stokes equations) with $N-1$ scalar control functions. In a first step, we present some global Carleman estimates for suitable adjoint problems of linearized Navier-Stokes and Boussinesq systems. In this way, we obtain null controllability properties for these systems. Then, we deduce results concerning the local exact controllability to the trajectories. We also present (global) null controllability results for some (truncated) approximations of the Navier-Stokes equations.

1. Introduction

Let $\Omega \subset \mathbf{R}^N$ ($N = 2$ or 3) be a bounded connected open set whose boundary $\partial\Omega$ is regular enough (for instance of class C^2). Let $\mathcal{O} \subset \Omega$ be a (small) nonempty open subset and let $T > 0$. We will use the notation $Q = \Omega \times (0,T)$ and $\Sigma = \partial\Omega \times (0,T)$ and we will denote by $n(x)$ the outward unit normal to Ω at the point $x \in \partial\Omega$.

On the other hand, we will denote by C, C_1, C_2, ... various positive constants (usually depending on Ω and $\mathcal{O}$).

We will be concerned with the following controlled Navier-Stokes and Boussinesq systems:

$$\begin{cases} y_t - \Delta y + (y,\nabla)y + \nabla p = v 1\!\!1_{\mathcal{O}}, \quad \nabla\cdot y = 0 & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = y^0 & \text{in } \Omega \end{cases} \tag{3.1}$$

and

$$\begin{cases} y_t - \Delta y + (y,\nabla)y + \nabla p = v 1\!\!1_{\mathcal{O}} + \theta\, e_N, \quad \nabla\cdot y = 0 & \text{in } Q, \\ \theta_t - \Delta\theta + y\cdot\nabla\theta = h 1\!\!1_{\mathcal{O}} & \text{in } Q, \\ y = 0, \quad \theta = 0 & \text{on } \Sigma, \\ y(0) = y^0, \quad \theta(0) = \theta^0 & \text{in } \Omega \end{cases} \tag{3.2}$$

(in both dimensions $N = 2$ and $N = 3$).

For $N = 2$, we will also consider the following approximation of the Navier-Stokes system with boundary conditions of the Navier kind:

$$\begin{cases} y_t - \Delta y + (y, \nabla)\mathbf{T}_M(y) + \nabla p = v\mathbb{1}_{\mathcal{O}}, \quad \nabla \cdot y = 0 & \text{in } Q, \\ y \cdot n = 0, \quad \nabla \times y = 0 & \text{on } \Sigma, \\ y(0) = y^0 & \text{in } \Omega, \end{cases} \tag{3.3}$$

where $M > 0$, $\mathbf{T}_M(y) = (T_M(y_1), T_M(y_2))$ and T_M is given by

$$T_M(s) = \begin{cases} -M & \text{if } s \le -M, \\ s & \text{if } -M \le s \le M, \\ M & \text{if } s \ge M. \end{cases}$$

In systems (3.1), (3.2) and (3.3), $v = v(x,t)$ and $h = h(x,t)$ stand for the control functions. They act during the whole time interval $(0,T)$ over the set $\mathcal{O}$. The symbol $\mathbb{1}_{\mathcal{O}}$ stands for the characteristic function of $\mathcal{O}$ and e_N is the N-th vector of the canonical basis of $\mathbf{R}^N$.

The controllability of Navier-Stokes systems has been the objective of considerable work along the last years. Until now, the best results have been given in [6], where a strategy based on the methods in [12] and [13] has been followed. Recently, the techniques in [6] have been adapted in [11] to cover Boussinesq systems (see also [3], [4], [7] and [9] for other results).

This paper can be viewed as a continuation of [6]. We will present some new results that indicate that the N-dimensional systems (3.1) and (3.2) can be controlled, at least under some geometrical assumptions, with only $N-1$ scalar controls in $L^2(\mathcal{O} \times (0,T))$. We will also prove that the two-dimensional system (3.3) can be controlled with controls of the form $v\mathbb{1}_{\mathcal{O}}$ where v is the curl of a function in $L^2(0,T;H^1(\mathcal{O}))$.

Along this paper, we will have to impose some regularity assumptions on the initial data. To this purpose, we introduce the spaces H, E and V, with

$$H = \{\, w \in L^2(\Omega)^N : \nabla \cdot w = 0 \text{ in } \Omega,\ w \cdot n = 0 \text{ on } \partial\Omega \,\}, \tag{3.4}$$

$$E = \begin{cases} H & \text{if } N = 2, \\ L^4(\Omega)^3 \cap H & \text{if } N = 3 \end{cases}$$

and

$$V = \{w \in H_0^1(\Omega)^N : \nabla \cdot w = 0 \text{ in } \Omega\}.$$

• For system (3.1), we will assume that the control region $\mathcal{O}$ is adjacent to the boundary $\partial\Omega$ (see assumption (3.10) below) and we will deal with the *local exact controllability to the trajectories*. More precisely, our task will be to prove that, for any bounded and sufficiently regular solution $(\overline{y}, \overline{p})$ of the uncontrolled Navier-Stokes equations

$$\begin{cases} \overline{y}_t - \Delta \overline{y} + (\overline{y}, \nabla)\overline{y} + \nabla \overline{p} = 0, \quad \nabla \cdot \overline{y} = 0 & \text{in } Q, \\ \overline{y} = 0 & \text{on } \Sigma, \\ \overline{y}(0) = \overline{y}^0 & \text{in } \Omega, \end{cases} \tag{3.5}$$

there exists $\delta > 0$ such that, whenever $y^0 \in E$ and

$$\|y^0 - \overline{y}^0\|_E \le \delta,$$

we can find L^2 controls v with $v_k \equiv 0$ for at least one k and associated states (y, p) satisfying

$$y(T) = \overline{y}(T) \ \text{ in } \Omega. \tag{3.6}$$

Notice that, under these circumstances, after time $t = T$ we can switch off the control and let the system follow the 'ideal' trajectory $(\overline{y}, \overline{p})$.

• For the Boussinesq system (3.2), we will assume that $\mathcal{O}$ is adjacent to $\partial\Omega$ near a point x^0 such that $n_k(x^0) \neq 0$ for some $k < N$. We will also be concerned with the local exact controllability to the trajectories. Now, a trajectory is a bounded and sufficiently regular solution $(\overline{y}, \overline{p}, \overline{\theta})$ of

$$\begin{cases} \overline{y}_t - \Delta\overline{y} + (\overline{y}, \nabla)\overline{y} + \nabla\overline{p} = \overline{\theta}\, e_N, \quad \nabla\cdot\overline{y} = 0 & \text{in } Q, \\ \overline{\theta}_t - \Delta\overline{\theta} + (\overline{y}\cdot\nabla\overline{\theta} = 0 & \text{in } Q, \\ \overline{y} = 0, \quad \overline{\theta} = 0 & \text{on } \Sigma, \\ \overline{y}(0) = \overline{y}^0, \quad \overline{\theta}(0) = \overline{\theta}^0 & \text{in } \Omega. \end{cases} \tag{3.7}$$

The goal will be to prove that there exists $\delta > 0$ such that, whenever $(y^0, \theta^0) \in E \times L^2(\Omega)$ and

$$\|(y^0, \theta^0) - (\overline{y}^0, \overline{\theta}^0)\|_{E\times L^2} \le \delta,$$

we can find L^2 controls v and h with $v_k \equiv v_N \equiv 0$ and associated states (y, p, θ) satisfying

$$y(T) = \overline{y}(T) \ \text{ and } \ \theta(T) = \overline{\theta}(T) \ \text{ in } \Omega. \tag{3.8}$$

In this context, the results established in [11] will be fundamental.

Notice that, in particular, when $N = 2$, we try to control the whole system (3.2) with just one scalar control h.

• As long as (3.3) is concerned, our goal will be to prove the *(global) null controllability*. That is to say, for each $y^0 \in H$, we will try to find controls of the form $v\mathbb{1}_{\mathcal{O}}$, where v belongs to the Hilbert space

$$W = \{\, \nabla\times z = (\partial_2 z, -\partial_1 z) : z \in L^2(0, T; H^1(\mathcal{O}))\,\},$$

such that the associated solutions (y, p) satisfy

$$y(T) = 0 \ \text{ in } \Omega. \tag{3.9}$$

Observe that in this system the boundary conditions are of the Navier kind (for their physical meaning, see for instance [10]). This and the fact that $N = 2$ will be essential in the arguments presented below.

Similarly to the previous situation, an extension by zero of the control after time $t = T$ will keep (y, p) at rest.

As mentioned above, some hypotheses will be imposed on the control domain and the trajectories. More precisely, we will frequently assume that

$$\exists x^0 \in \partial\Omega,\ \exists \varepsilon > 0 \ \text{ such that } \ \overline{\mathcal{O}} \cap \partial\Omega \supset B(x^0;\varepsilon) \cap \partial\Omega \tag{3.10}$$

($B(x^0;\varepsilon)$ is the ball centered at x^0 of radius ε),

$$\overline{y} \in L^\infty(Q)^N, \quad \overline{y}_t \in L^2(0,T;L^\sigma(\Omega)^N) \quad \begin{pmatrix} \sigma > 1 & \text{if } N = 2 \\ \sigma > 6/5 & \text{if } N = 3 \end{pmatrix} \tag{3.11}$$

and

$$\overline{\theta} \in L^\infty(Q), \quad \overline{\theta}_t \in L^2(0,T;L^\sigma(\Omega)) \quad \begin{pmatrix} \sigma > 1 & \text{if } N = 2 \\ \sigma > 6/5 & \text{if } N = 3 \end{pmatrix}. \tag{3.12}$$

Let us now present our main results in a precise form. The first one concerns the local exact controllability to the trajectories of system (3.1):

Theorem 5 *Assume that $\mathcal{O}$ satisfies* (3.10). *Then, for any $T > 0$,* (3.1) *is locally exactly controllable at time T to the trajectories $(\overline{y},\overline{p})$ satisfying* (3.11) *with controls $v \in L^2(\mathcal{O}\times(0,T))^N$ such that $v_k \equiv 0$ for at least one k.*

The second main result concerns the controllability of (3.2). It is the following:

Theorem 6 *Assume that $\mathcal{O}$ satisfies* (3.10) *with $n_k(x^0) \neq 0$ for some $k < N$. Then, for any $T > 0$,* (3.2) *is locally exactly controllable at time T to the trajectories $(\overline{y},\overline{p},\overline{\theta})$ satisfying* (3.11)–(3.12) *with L^2 controls v and h such that $v_k \equiv v_N \equiv 0$. In particular, if $N = 2$, we have local exact controllability to the trajectories with controls $v \equiv 0$ and $h \in L^2(\mathcal{O}\times(0,T))$.*

The last main result we present in this paper is the following:

Theorem 7 *Let $N = 2$. Then, for any $T > 0$ and any $M > 0$,* (3.3) *is null controllable at time T with controls of the form $v 1\!\!1_{\mathcal{O}}$, where $v \in W$.*

For the proofs of these results, following a standard approach, we will first deduce null controllability results for suitable linearized versions of (3.1), (3.2) and (3.3), namely:

$$\begin{cases} y_t - \Delta y + (\overline{y},\nabla)y + (y,\nabla)\overline{y} + \nabla p = f + v 1\!\!1_{\mathcal{O}}, \quad \nabla\cdot y = 0 & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = y^0 & \text{in } \Omega, \end{cases} \tag{3.13}$$

$$\begin{cases} y_t - \Delta y + (\overline{y},\nabla)y + (y,\nabla)\overline{y} + \nabla p = f + v 1\!\!1_{\mathcal{O}} + \theta\, e_N & \text{in } Q, \\ \nabla\cdot y = 0 & \text{in } Q, \\ \theta_t - \Delta\theta + \overline{y}\cdot\nabla\theta + y\cdot\nabla\overline{\theta} = k + h 1\!\!1_{\mathcal{O}} & \text{in } Q, \\ y = 0, \quad \theta = 0 & \text{on } \Sigma, \\ y(0) = y^0, \quad \theta(0) = \theta^0 & \text{in } \Omega \end{cases} \tag{3.14}$$

and

$$\begin{cases} y_t - \Delta y + (y,\nabla)\overline{y} + \nabla p = v\mathbb{1}_{\mathcal{O}}, \quad \nabla\cdot y = 0 & \text{in } Q,\\ y\cdot n = 0, \quad \nabla\times y = 0 & \text{on } \Sigma,\\ y(0) = y^0 & \text{in } \Omega. \end{cases} \tag{3.15}$$

Then, appropriate arguments will be used to deduce the controllability of the nonlinear systems (3.1)–(3.3).

Remark 5 When $N = 3$, it is very natural to ask whether a result similar to theorem 5 holds with controls having two zero components. In general, the answer is no. In fact, it seems difficult to identify the open sets Ω and $\mathcal{O}$ such that one has null controllability for all $T > 0$ with controls of this kind. This is unknown even for the classical Stokes equations for which, up to now, the unique known results concern *approximate controllability*; see [15].

Remark 6 Assume that $N = 2$. The arguments in [6] implicitly show that, under hypotheses (3.11), we can find controls $v\mathbb{1}_{\mathcal{O}}$ with $v \in W$ such that the associated solutions to (3.1) satisfy $y(T) = \overline{y}(T)$. Observe that the assumption (3.10) on the control domain is not necessary here.

This paper is organized as follows. We will first establish all the technical results needed in this work in Section 2. Section 3 will deal with null controllability results for the linear control systems (3.13)–(3.15). Finally, the proofs of theorems 5, 6 and 7 will be given in Section 4.

2. Some previous results

In this Section we will establish all the technical results needed in this paper. More precisely, we will present and prove the required *Carleman estimates* for the backward systems (3.18), (3.19) and (3.20), given below.

To do this, let us first introduce some weight functions:

$$\begin{gathered} \alpha(x,t) = \frac{e^{5/4\lambda m\|\eta^0\|_\infty} - e^{\lambda(m\|\eta^0\|_\infty+\eta^0(x))}}{t^4(T-t)^4},\\ \xi(x,t) = \frac{e^{\lambda(m\|\eta^0\|_\infty+\eta^0(x))}}{t^4(T-t)^4},\\ \widehat{\alpha}(t) = \min_{x\in\overline{\Omega}}\alpha(x,t) = \frac{e^{5/4\lambda m\|\eta^0\|_\infty} - e^{\lambda(m+1)\,\|\eta^0\|_\infty}}{t^4(T-t)^4},\\ \alpha^*(t) = \max_{x\in\overline{\Omega}}\alpha(x,t) = \frac{e^{5/4\lambda m\|\eta^0\|_\infty} - e^{\lambda m\|\eta^0\|_\infty}}{t^4(T-t)^4},\\ \widehat{\xi}(t) = \max_{x\in\overline{\Omega}}\xi(x,t) = \frac{e^{\lambda(m+1)\|\eta^0\|_\infty}}{t^4(T-t)^4}, \quad \xi^*(t) = \min_{x\in\overline{\Omega}}\xi(x,t) = \frac{e^{\lambda m\|\eta^0\|_\infty}}{t^4(T-t)^4}, \end{gathered} \tag{3.16}$$

where $m > 4$ is a fixed real number. Here, η^0 is a function verifying

$$\eta^0 \in C^2(\overline{\Omega}), \quad |\nabla\eta^0| > 0 \text{ in } \overline{\Omega\setminus\mathcal{O}_0}, \quad \eta^0 > 0 \text{ in } \Omega \text{ and } \eta^0 \equiv 0 \text{ on } \partial\Omega \tag{3.17}$$

with $\mathcal{O}_0$ a nonempty open subset of $\mathcal{O}$ that will be determined below. For any $\mathcal{O}_0$, the existence of such a function η^0 is proved in [8]. Remark that these weights have already been used in [6] and [11].

We will be dealing in this Section with the adjoint systems to (3.13), (3.14) and (3.15), that is to say

$$\begin{cases} -\varphi_t - \Delta\varphi - (D\varphi)\,\overline{y} + \nabla\pi = g, \quad \nabla\cdot\varphi = 0 & \text{in } Q, \\ \varphi = 0 & \text{on } \Sigma, \\ \varphi(T) = \varphi^0 & \text{in } \Omega, \end{cases} \tag{3.18}$$

$$\begin{cases} -\varphi_t - \Delta\varphi - (D\varphi)\,\overline{y} + \nabla\pi = g + \overline{\theta}\,\nabla\psi, \quad \nabla\cdot\varphi = 0 & \text{in } Q, \\ -\psi_t - \Delta\psi - \overline{y}\cdot\nabla\psi = q + \varphi_N & \text{in } Q, \\ \varphi = 0, \quad \psi = 0 & \text{on } \Sigma, \\ \varphi(T) = \varphi^0, \quad \psi(T) = \psi^0 & \text{in } \Omega \end{cases} \tag{3.19}$$

(where $D\varphi = \nabla\varphi + \nabla\varphi^t$) and

$$\begin{cases} -\rho_t - \Delta\rho - \nabla\times((\overline{y}\cdot\nabla\times)\nabla\gamma) = 0, \quad \Delta\gamma = \rho & \text{in } Q, \\ \gamma = 0, \quad \rho = 0 & \text{on } \Sigma, \\ \rho(T) = \rho^0 & \text{in } \Omega, \end{cases} \tag{3.20}$$

respectively. Here, $g \in L^2(Q)^N$, $q \in L^2(Q)$, $\varphi^0 \in H$, $\psi^0 \in L^2(\Omega)$ and $\rho^0 \in H^{-1}(\Omega)$ (of course, φ_N stands for the last component of the vector field φ).

2.1. New Carleman estimates for system (3.18)

In this paragraph, we will establish some new Carleman estimates for the solutions of (3.18). We will assume that $\mathcal{O}$ and $\overline{y}$ satisfy (3.10)–(3.11). To fix ideas, we will also assume for the moment that $N = 3$ and $n_1(x^0) \neq 0$ (x^0 appears in assumption (3.10)).

The desired Carleman inequalities will have the form

$$I(\varphi) \leq C\left(\iint_Q \rho_1^2\,|g|^2\,dx\,dt + \iint_{\mathcal{O}\times(0,T)} \rho_2^2\left(|\varphi_2|^2 + |\varphi_3|^2\right)\,dx\,dt\right),$$

where $I(\varphi)$ contains global weighted integrals of $|\varphi|^2$, $|\nabla\varphi|^2$, etc. and ρ_1 and ρ_2 are appropriate weights that vanish exponentially as $t \to T$. This will suffice to prove in Section 3 the null controllability of (3.13) with controls $v1\!\!1_{\mathcal{O}}$ satisfying $v_1 \equiv 0$.

Lemma 4 *Assume that* $N = 3$, $n_1(x^0) \neq 0$ *and* $\mathcal{O}$ *and* $\overline{y}$ *verify* (3.10)–(3.11). *Then there exists a positive constant* C *such that, for any* $g \in L^2(Q)^3$ *and any* $\varphi^0 \in H$, *the associated solution to*

(3.18) *satisfies:*

$$\begin{cases} I(\varphi) := \displaystyle\iint_Q e^{\frac{-2\overline{\alpha}}{t^4(T-t)^4}}\, t^{-12}(T-t)^{-12}\,|\varphi|^2\,dx\,dt \\ \quad + \displaystyle\iint_Q e^{\frac{-2\overline{\alpha}}{t^4(T-t)^4}}\, t^{-4}(T-t)^{-4}\,|\nabla\varphi|^2)\,dx\,dt \\ \quad + \displaystyle\iint_Q e^{\frac{-2\overline{\alpha}}{t^4(T-t)^4}}\, t^{4}(T-t)^{4}\,\left(|\Delta\varphi|^2+|\varphi_t|^2\right)\,dx\,dt \\ \quad \le C\left(\displaystyle\iint_Q e^{\frac{-4\widetilde{\alpha}+2\overline{\alpha}}{t^4(T-t)^4}} t^{-30}(T-t)^{-30}\,|g|^2\,dx\,dt \right.\\ \quad \left. + \displaystyle\iint_{\mathcal{O}\times(0,T)} e^{\frac{-16\widetilde{\alpha}+14\overline{\alpha}}{t^4(T-t)^4}}\, t^{-132}(T-t)^{-132}\,(|\varphi_2|^2+|\varphi_3|^2)\,dx\,dt\right). \end{cases} \tag{3.21}$$

Here, $\overline{\alpha}$ *and* $\widetilde{\alpha}$ *are constants only depending on* Ω, $\mathcal{O}$, T *and* $\overline{y}$ *satisfying* $0<\widetilde{\alpha}<\overline{\alpha}$ *and* $8\widetilde{\alpha}-7\overline{\alpha}>0$.

Proof: Let us first recall a Carleman inequality for the solutions of (3.18) which has been proved in [6] whenever (3.11) is fulfilled:

$$\begin{aligned} & s^3\lambda^4 \iint_Q e^{-2s\alpha}\xi^3|\varphi|^2\,dx\,dt + s\lambda^2\iint_Q e^{-2s\alpha}\xi|\nabla\varphi|^2\,dx\,dt \\ & \quad + s^{-1}\iint_Q e^{-2s\alpha}\xi^{-1}\,(|\varphi_t|^2+|\Delta\varphi|^2)\,dx\,dt \\ & \quad \le C_0(1+T^2)\left(s^{15/2}\lambda^{20}\iint_Q e^{-4s\widehat{\alpha}+2s\alpha^*}\widehat{\xi}^{15/2}|g|^2\,dx\,dt\right. \\ & \quad \left. + s^{16}\lambda^{40}\iint_{\mathcal{O}_0\times(0,T)} e^{-8s\widehat{\alpha}+6s\alpha^*}\widehat{\xi}^{16}|\varphi|^2\,dx\,dt\right). \end{aligned} \tag{3.22}$$

Here, $s\ge s_0$ and $\lambda\ge\lambda_0$ are arbitrarily large and C_0, s_0 and λ_0 are suitable constants depending on Ω, $\mathcal{O}_0$, T and $\overline{y}$; see theorem 1 in [6].

Recall that an inequality like (3.22) had already been proved in [12] using stronger properties on $\overline{y}$ than (3.11).

It is immediate from (3.22) that, for some C_1, $\overline{\alpha}$ and $\widetilde{\alpha}$ depending on Ω, $\mathcal{O}_0$, T and $\overline{y}$, we have:

$$\begin{cases} \displaystyle\iint_Q e^{\frac{-2\overline{\alpha}}{t^4(T-t)^4}}\left(t^{-12}(T-t)^{-12}\,|\varphi|^2 + t^{-4}(T-t)^{-4}\,|\nabla\varphi|^2\right)\,dx\,dt \\ \quad + \displaystyle\iint_Q e^{\frac{-2\overline{\alpha}}{t^4(T-t)^4}}\, t^{4}(T-t)^{4}\,\left(|\Delta\varphi|^2+|\varphi_t|^2\right)\,dx\,dt \\ \quad \le C_1\left(\displaystyle\iint_Q e^{\frac{-4\widetilde{\alpha}+2\overline{\alpha}}{t^4(T-t)^4}} t^{-30}(T-t)^{-30}\,|g|^2\,dx\,dt \right.\\ \quad \left. + \displaystyle\iint_{\mathcal{O}_0\times(0,T)} e^{\frac{-8\widetilde{\alpha}+6\overline{\alpha}}{t^4(T-t)^4}}\, t^{-64}(T-t)^{-64}\,|\varphi|^2\,dx\,dt\right). \end{cases} \tag{3.23}$$

Indeed, it suffices to choose

$$\begin{cases} \overline{\alpha} = s_0 \left(e^{5/4\lambda m \|\eta^0\|_\infty} - e^{\lambda m \|\eta^0\|_\infty} \right), \\ \widetilde{\alpha} = s_0 \left(e^{5/4\lambda m \|\eta^0\|_\infty} - e^{\lambda (m+1) \|\eta^0\|_\infty} \right) \end{cases} \tag{3.24}$$

and $C_1 = C_0(1+T^2)s_0^{16}\lambda^{40}e^{16\lambda(m+1)\|\eta^0\|_\infty}$ with $\lambda \geq \lambda_0$. Notice that $0 < \widetilde{\alpha} < \overline{\alpha}$. Moreover, it can be assumed that $8\widetilde{\alpha} - 7\overline{\alpha} > 0$ (it suffices to take λ large enough in (3.24)).

We will apply (3.23) for the open set $\mathcal{O}_0 \subset \mathcal{O}$ defined as follows. We choose $\kappa > 0$ such that

$$n_1(x) \neq 0 \quad \forall x \in B(x^0; \kappa) \cap \partial\mathcal{O} \cap \partial\Omega$$

and we denote this set by Γ_κ. Then, we define

$$\mathcal{O}_0 = \{ \, x \in \Omega : x = w + \tau \, e_1, \; w \in \Gamma_\kappa, \; |\tau| < \tau^0 \, \}, \tag{3.25}$$

with $\kappa, \tau^0 > 0$ small enough so that we still have

$$\mathcal{O}_0 \subset \mathcal{O} \quad \text{and} \quad d_0 := \operatorname{dist}(\overline{\mathcal{O}_0}, \partial\mathcal{O} \cap \Omega) > 0. \tag{3.26}$$

Observe that, with this choice, each $P \in \mathcal{O}_0$ verifies that one of the two points where the straight line $\{P + \mathbf{R}\, e_1\}$ intersects $\partial\Omega$ belongs to $\partial\mathcal{O}_0$.

Once $\mathcal{O}_0$ is defined, we apply inequality (3.23) in this open set and we try to bound the term

$$\iint_{\mathcal{O}_0 \times (0,T)} e^{\frac{-8\widetilde{\alpha}+6\overline{\alpha}}{t^4(T-t)^4}} \, t^{-64}(T-t)^{-64} \, |\varphi_1|^2 \, dx \, dt$$

in terms of local integrals of φ_2 and φ_3.

To this end, for each $(x,t) \in \mathcal{O}_0 \times (0,T)$ we denote by $l(x,t)$ (resp. $\widetilde{l}(x,t)$) the segment that starts from (x,t) with direction e_1 in the positive (resp. negative) sense and ends at $\partial\mathcal{O}_0$. Then, since φ is divergence-free, it is not difficult to see that

$$\varphi_1(x,t) = \int_{l(x,t)} (\partial_2\varphi_2 + \partial_3\varphi_3)(y_1, x_2, x_3, t) \, dy_1$$

for each $(x,t) \in \mathcal{O}_0 \times (0,T)$. For simplicity, let us introduce the notation

$$\beta(t) = e^{\frac{-8\widetilde{\alpha}+6\overline{\alpha}}{t^4(T-t)^4}} \, t^{-64}(T-t)^{-64} \quad \forall t \in (0,T).$$

Applying at this point Hölder's inequality and Fubini's formula, we obtain

$$\begin{aligned} &\iint_{\mathcal{O}_0 \times (0,T)} \beta(t) \, |\varphi_1|^2 \, dx \, dt \\ &\leq C_2 \iint_{\mathcal{O}_0 \times (0,T)} \beta(t) \left(\int_{l(x,t)} (|\partial_2\varphi_2|^2 + |\partial_3\varphi_3|^2) \, dy_1 \right) dx \, dt \\ &= C_2 \iint_{\mathcal{O}_0 \times (0,T)} (|\partial_2\varphi_2|^2 + |\partial_3\varphi_3|^2) \left(\int_{\widetilde{l}(y_1)} \beta(t) \, dx_1 \right) dy_1 \, dx_2 \, dx_3 \, dt \\ &\leq C_3 \iint_{\mathcal{O}_0 \times (0,T)} \beta(t)(|\partial_2\varphi_2|^2 + |\partial_3\varphi_3|^2) \, dx \, dt, \end{aligned} \tag{3.27}$$

where $\widetilde{l}(y_1)$ stands for the segment $\widetilde{l}(y_1, x_2, x_3, t)$. Then, we introduce a function $\zeta \in C^2(\overline{\mathcal{O}})$ such that

$$\zeta \equiv 1 \text{ in } \mathcal{O}_0, \quad 0 \le \zeta \le 1$$

and $\zeta(x) = 0$ at any point $x \in \mathcal{O}$ satisfying $\operatorname{dist}(x, \partial\mathcal{O} \cap \Omega) \le d_0/2$ (d_0 was defined in (3.26)). This and the fact that $\varphi|_\Sigma \equiv 0$ imply

$$\begin{aligned} &\iint_{\mathcal{O}_0 \times (0,T)} \beta(t)\, |\partial_i \varphi_i|^2 \, dx\, dt \le \iint_{\mathcal{O} \times (0,T)} \zeta\, \beta(t)\, |\partial_i \varphi_i|^2 \, dx\, dt \\ &= \frac{1}{2} \iint_{\mathcal{O} \times (0,T)} \partial_{ii}^2 \zeta\, \beta(t)\, |\varphi_i|^2 \, dx\, dt - \iint_{\mathcal{O} \times (0,T)} \zeta\, \beta(t)\, \partial_{ii}^2 \varphi_i\, \varphi_i \, dx\, dt \end{aligned}$$

for $i = 2, 3$. Finally, in view of Young's inequality and regularity estimates for φ_i in Ω ($\varphi_i \in H^2(\Omega)$ and $\|\varphi_i\|_{H^2} \le C \|\Delta \varphi_i\|_{L^2}$), we also have:

$$\begin{aligned} &\iint_{\mathcal{O}_0 \times (0,T)} \beta(t)\, |\partial_i \varphi_i|^2 \, dx\, dt \\ &\le C_4 \iint_{\mathcal{O} \times (0,T)} e^{\frac{-16\widetilde{\alpha} + 14\overline{\alpha}}{t^4 (T-t)^4}} t^{-132} (T-t)^{-132} \, |\varphi_i|^2 \, dx\, dt \\ &+ \frac{1}{2C_1 C_3} \iint_Q e^{\frac{-2\overline{\alpha}}{t^4 (T-t)^4}} t^4 (T-t)^4 \, |\Delta \varphi_i|^2 \, dx\, dt, \end{aligned}$$

which, combined with (3.23) and (3.27), yields (3.21).

Let us now present another Carleman inequality for (3.18) with weight functions not vanishing at time $t = 0$:

Lemma 5 *Assume that $N = 3$, $n_1(x^0) \neq 0$ and $\mathcal{O}$ and $\overline{y}$ verify* (3.10)–(3.11). *Then there exist positive constants C, $\overline{\alpha}$ and $\widetilde{\alpha}$ with $0 < \widetilde{\alpha} < \overline{\alpha}$ and $8\widetilde{\alpha} - 7\overline{\alpha} > 0$ depending on Ω, $\mathcal{O}$, T and $\overline{y}$ such that, for any $g \in L^2(Q)^3$ and any $\varphi^0 \in H$, the associated solution to* (3.18) *satisfies:*

$$\left\{ \begin{aligned} &\iint_Q e^{\frac{-2\overline{\alpha}}{\ell(t)^4}} \left(\ell(t)^{-12} \, |\varphi|^2 + \ell(t)^{-4} \, |\nabla \varphi|^2 \right) \, dx\, dt \\ &\le C \left(\iint_Q e^{\frac{-4\widetilde{\alpha} + 2\overline{\alpha}}{\ell(t)^4}} \ell(t)^{-30} \, |g|^2 \, dx\, dt \right. \\ &\left. + \iint_{\mathcal{O} \times (0,T)} e^{\frac{-16\widetilde{\alpha} + 14\overline{\alpha}}{\ell(t)^4}} \ell(t)^{-132} \left(|\varphi_2|^2 + |\varphi_3|^2 \right) dx\, dt \right), \end{aligned} \right. \tag{3.28}$$

where ℓ is the C^1 function given by

$$\ell(t) = \begin{cases} \dfrac{T^2}{4} & \text{for } 0 \le t \le T/2, \\ t(T-t) & \text{for } T/2 \le t \le T. \end{cases} \tag{3.29}$$

To prove (3.28), it suffices to use (3.21) and the classical parabolic estimates for the Stokes system satisfied by φ. The argument has already been used in [8], [12] and [6] in several similar situations, so we omit it for simplicity.

For completeness, let us state the similar result that can be established when $N = 2$. Here, we assume again that $n_1(x^0) \neq 0$.

Lemma 6 *Assume that $N = 2$, $n_1(x^0) \neq 0$ and $\mathcal{O}$ and $\overline{y}$ verify* (3.10)–(3.11). *Then there exist positive constants C, $\overline{\alpha}$ and $\widetilde{\alpha}$ with $0 < \widetilde{\alpha} < \overline{\alpha}$ and $8\widetilde{\alpha} - 7\overline{\alpha} > 0$ depending on Ω, $\mathcal{O}$, T and $\overline{y}$ such that, for any $g \in L^2(Q)^2$ and any $\varphi^0 \in H$, the associated solution to* (3.18) *satisfies:*

$$\left\{\begin{aligned} &\iint_Q e^{\frac{-2\overline{\alpha}}{\ell(t)^4}} \left(\ell(t)^{-12}|\varphi|^2 + \ell(t)^{-4}|\nabla\varphi|^2\right) dx\,dt \\ &\quad \leq C\left(\iint_Q e^{\frac{-4\widetilde{\alpha}+2\overline{\alpha}}{\ell(t)^4}} \ell(t)^{-30}|g|^2\,dx\,dt\right. \\ &\quad \left. + \iint_{\mathcal{O}\times(0,T)} e^{\frac{-16\widetilde{\alpha}+14\overline{\alpha}}{\ell(t)^4}} \ell(t)^{-132}|\varphi_2|^2\,dx\,dt\right), \end{aligned}\right. \tag{3.30}$$

where ℓ is the function given by (3.29).

2.2. New Carleman estimates for system (3.19)

In this paragraph, we will establish suitable Carleman inequalities for the solutions of (3.19). To this end, our approach will be similar to the one in subsection 2.1.

Thus, we will assume again that $N = 3$ and $n_1(x^0) \neq 0$ and we will prove an estimate of the form

$$K(\varphi,\psi) \leq C\left(\iint_Q \rho_3^2\,(|g|^2 + |q|^2)\,dx\,dt + \iint_{\mathcal{O}\times(0,T)} \rho_4^2\left(|\varphi_2|^2 + |\psi|^2\right) dx\,dt\right),$$

where $K(\varphi,\psi) = I(\varphi) + I(\psi)$ ($I(\varphi)$ has been given in (3.21)) and ρ_3 and ρ_4 are appropriate weights. This will be used in Section 3 to find controls $v\mathbb{1}_{\mathcal{O}}$ and $h\mathbb{1}_{\mathcal{O}}$ with $v_1 \equiv v_3 \equiv 0$ leading to the null controllability of (3.14).

Lemma 7 *Assume that $N = 3$, $n_1(x^0) \neq 0$ and $\mathcal{O}$ and $(\overline{y},\overline{\theta})$ satisfy* (3.10)–(3.12). *Then, there exist positive constants C, $\overline{\alpha}$ and $\widetilde{\alpha}$ depending on Ω, $\mathcal{O}$, T, $\overline{y}$ and $\overline{\theta}$ with $0 < \widetilde{\alpha} < \overline{\alpha}$ and $16\widetilde{\alpha} - 15\overline{\alpha} > 0$ such that, for any $g \in L^2(Q)^3$, $q \in L^2(Q)$, $\varphi^0 \in H$ and $\psi^0 \in L^2(\Omega)$, the* *associated solution to* (3.19) *satisfies:*

$$\begin{cases} I(\varphi)+I(\psi)\le C\left(\displaystyle\iint_Q e^{\frac{-4\tilde{\alpha}+2\overline{\alpha}}{t^4(T-t)^4}}t^{-30}(T-t)^{-30}\,|g|^2\,dx\,dt\right. \\ \qquad +\displaystyle\iint_Q e^{\frac{-32\tilde{\alpha}+30\overline{\alpha}}{t^4(T-t)^4}}\,t^{-252}(T-t)^{-252}\,|q|^2\,dx\,dt \\ \qquad +\displaystyle\iint_{\mathcal{O}\times(0,T)} e^{\frac{-16\tilde{\alpha}+14\overline{\alpha}}{t^4(T-t)^4}}\,t^{-132}(T-t)^{-132}\,|\varphi_2|^2\,dx\,dt \\ \qquad \left.+\displaystyle\iint_{\mathcal{O}\times(0,T)} e^{\frac{-32\tilde{\alpha}+30\overline{\alpha}}{t^4(T-t)^4}}\,t^{-268}(T-t)^{-268}\,|\psi|^2\,dx\,dt\right). \end{cases} \tag{3.31}$$

Proof: Let us first recall a Carleman inequality for the solutions of (3.19) which has recently been proved in [11] whenever (3.11)–(3.12) are fulfilled:

$$\begin{aligned} &s^3\lambda^4\iint_Q e^{-2s\alpha}\xi^3(|\varphi|^2+|\psi|^2)\,dx\,dt \\ &+s\lambda^2\iint_Q e^{-2s\alpha}\xi(|\nabla\varphi|^2+|\nabla\psi|^2)\,dx\,dt \\ &+s^{-1}\iint_Q e^{-2s\alpha}\xi^{-1}\left(|\varphi_t|^2+|\psi_t|^2+|\Delta\varphi|^2+|\Delta\psi|^2\right)dx\,dt \\ &\qquad\le C_5(1+T^2)\left(s^{15/2}\lambda^{28}\iint_Q e^{-4s\widehat{\alpha}+2s\alpha^*}\widehat{\xi}^{15/2}(|g|^2+|q|^2)\,dx\,dt\right. \\ &\qquad\left.+s^{16}\lambda^{56}\iint_{\mathcal{O}_0\times(0,T)} e^{-8s\widehat{\alpha}+6s\alpha^*}\widehat{\xi}^{16}(|\varphi|^2+|\psi|^2)\,dx\,dt\right). \end{aligned} \tag{3.32}$$

Here, $s\ge s_1$ and $\lambda\ge\lambda_1$ are arbitrarily large and C_5, s_1 and λ_1 are suitable constants depending on Ω, $\mathcal{O}_0$, T, $\overline{y}$ and $\overline{\theta}$; see proposition 1 in [11]. The proof of this inequality follows the same arguments employed in [6] to prove (3.22) and can be achieved without any further regularity on $\overline{y}$ or $\overline{\theta}$.

It is clear from (3.32) that, for some C_6, $\overline{\alpha}$ and $\widetilde{\alpha}$ depending on Ω, $\mathcal{O}_0$, T, $\overline{y}$ and $\overline{\theta}$, we have:

$$\begin{aligned} &\iint_Q e^{\frac{-2\overline{\alpha}}{t^4(T-t)^4}}t^{-12}(T-t)^{-12}\left(|\varphi|^2+|\psi|^2\right)dx\,dt \\ &+\iint_Q e^{\frac{-2\overline{\alpha}}{t^4(T-t)^4}}t^{-4}(T-t)^{-4}\left(|\nabla\varphi|^2+|\nabla\psi|^2\right)dx\,dt \\ &+\iint_Q e^{\frac{-2\overline{\alpha}}{t^4(T-t)^4}}\,t^4(T-t)^4\,\left(|\Delta\varphi|^2+|\Delta\psi|^2+|\varphi_t|^2+|\psi_t|^2\right)\,dx\,dt \\ &\le C_6\left(\iint_Q e^{\frac{-4\tilde{\alpha}+2\overline{\alpha}}{t^4(T-t)^4}}t^{-30}(T-t)^{-30}\left(|g|^2+|q|^2\right)dx\,dt\right. \\ &\quad\left.+\iint_{\mathcal{O}_0\times(0,T)} e^{\frac{-8\tilde{\alpha}+6\overline{\alpha}}{t^4(T-t)^4}}\,t^{-64}(T-t)^{-64}\left(|\varphi|^2+|\psi|^2\right)dx\,dt\right). \end{aligned} \tag{3.33}$$

Indeed, it suffices to take $\overline{\alpha}$ and $\widetilde{\alpha}$ as in (3.24) and

$$C_6 = C_5(1+T^2)s_1^{16}\lambda^{56}e^{16\lambda(m+1)\|\eta^0\|_\infty}$$

with $\lambda \geq \lambda_1$. We thus obtain $0 < \widetilde{\alpha} < \overline{\alpha}$ and, choosing λ large enough, $16\widetilde{\alpha} - 15\overline{\alpha} > 0$.

We apply (3.33) for the open set $\mathcal{O}_0$ defined in (3.25). Then we can argue as in paragraph 2.1 and deduce that

$$\begin{aligned}
&\iint_{\mathcal{O}_0\times(0,T)} e^{\frac{-8\widetilde{\alpha}+6\overline{\alpha}}{t^4(T-t)^4}}\, t^{-64}(T-t)^{-64}\,|\varphi_1|^2\,dx\,dt \\
&\qquad \leq C_7 \iint_{\mathcal{O}_1\times(0,T)} e^{\frac{-16\widetilde{\alpha}+14\overline{\alpha}}{t^4(T-t)^4}}\, t^{-132}(T-t)^{-132}\,(|\varphi_2|^2+|\varphi_3|^2)\,dx\,dt \\
&\qquad +\varepsilon \iint_Q e^{\frac{-2\overline{\alpha}}{t^4(T-t)^4}}\, t^4(T-t)^4\,(|\Delta\varphi_2|^2+|\Delta\varphi_3|^2)\,dx\,dt,
\end{aligned}$$

where $\mathcal{O}_1$ is an appropriate nonempty open set verifying

$$\mathcal{O}_0 \subset \mathcal{O}_1 \subset \mathcal{O}, \quad d_1 := \mathrm{dist}(\overline{\mathcal{O}_1}, \partial\mathcal{O}\cap\Omega) > 0.$$

This inequality combined with (3.33) yields:

$$\begin{aligned}
&\iint_Q e^{\frac{-2\overline{\alpha}}{t^4(T-t)^4}}\, t^{-12}(T-t)^{-12}\,(|\varphi|^2+|\psi|^2)\,dx\,dt \\
&+ \iint_Q e^{\frac{-2\overline{\alpha}}{t^4(T-t)^4}}\, t^{-4}(T-t)^{-4}\,(|\nabla\varphi|^2+|\nabla\psi|^2)\,dx\,dt \\
&+ \iint_Q e^{\frac{-2\overline{\alpha}}{t^4(T-t)^4}}\, t^{4}(T-t)^{4}\,\left(|\Delta\varphi|^2+|\Delta\psi|^2+|\varphi_t|^2+|\psi_t|^2\right)\,dx\,dt \\
&\qquad \leq C_8 \Bigg(\iint_Q e^{\frac{-4\widetilde{\alpha}+2\overline{\alpha}}{t^4(T-t)^4}}\, t^{-30}(T-t)^{-30}\,(|g|^2+|q|^2)\,dx\,dt \\
&\qquad + \iint_{\mathcal{O}_1\times(0,T)} e^{\frac{-16\widetilde{\alpha}+14\overline{\alpha}}{t^4(T-t)^4}}\, t^{-132}(T-t)^{-132}\,(|\varphi_2|^2+|\varphi_3|^2)\,dx\,dt \\
&\qquad + \iint_{\mathcal{O}\times(0,T)} e^{\frac{-8\widetilde{\alpha}+6\overline{\alpha}}{t^4(T-t)^4}}\, t^{-64}(T-t)^{-64}\,|\psi|^2\,dx\,dt \Bigg).
\end{aligned} \tag{3.34}$$

Our last task will be to estimate the integral

$$\iint_{\mathcal{O}_1\times(0,T)} e^{\frac{-16\widetilde{\alpha}+14\overline{\alpha}}{t^4(T-t)^4}}\, t^{-132}(T-t)^{-132}\,|\varphi_3|^2\,dx\,dt$$

in terms of $\varepsilon I(\varphi_3)$ and local integrals of ψ and q. To do this, we set

$$\beta_1(t) = e^{\frac{-16\widetilde{\alpha}+14\overline{\alpha}}{t^4(T-t)^4}}\, t^{-132}(T-t)^{-132}$$

and we introduce a function $\zeta \in C^2(\overline{\mathcal{O}})$ such that

$$\zeta \equiv 1 \text{ in } \mathcal{O}_1, \quad 0 \leq \zeta \leq 1$$

and $\zeta(x) = 0$ at any point $x \in \mathcal{O}$ satisfying $\mathrm{dist}(x, \partial\mathcal{O} \cap \Omega) \le d_1/2$. From the differential equation satisfied by ψ (see (3.19)), we have

$$\begin{aligned} \iint_{\mathcal{O}_1\times(0,T)} \beta_1(t)\,|\varphi_3|^2\,dx\,dt &\le \iint_{\mathcal{O}\times(0,T)} \beta_1(t)\,\zeta\,|\varphi_3|^2\,dx\,dt \\ &= \iint_{\mathcal{O}\times(0,T)} \beta_1(t)\,\zeta\,\varphi_3(-\psi_t - \Delta\psi - \overline{y}\cdot\nabla\psi - q)\,dx\,dt. \end{aligned} \tag{3.35}$$

To end the proof, we perform integrations by parts in the last integral and pass all the derivatives from ψ to φ_3:

• First, we integrate by parts in time taking into account that $\beta_1(0) = \beta_1(T) = 0$:

$$\begin{aligned} &-\iint_{\mathcal{O}\times(0,T)} \beta_1(t)\,\zeta\,\varphi_3\,\psi_t\,dx\,dt \\ &= \iint_{\mathcal{O}\times(0,T)} \beta_{1,t}(t)\,\zeta\,\varphi_3\,\psi\,dx\,dt + \iint_{\mathcal{O}\times(0,T)} \beta_1(t)\,\zeta\,\varphi_{3,t}\,\psi\,dx\,dt \\ &\le \varepsilon I(\varphi_3) + C_9(\varepsilon) \iint_{\mathcal{O}\times(0,T)} e^{\frac{-32\tilde{\alpha}+30\overline{\alpha}}{t^4(T-t)^4}}\,t^{-268}(T-t)^{-268}\,|\psi|^2\,dx\,dt. \end{aligned} \tag{3.36}$$

• Next, we integrate by parts twice in space. Here, we use the properties of the cut-off function ζ and the Dirichlet boundary conditions for φ_3 and ψ:

$$\begin{aligned} &-\iint_{\mathcal{O}\times(0,T)} \beta_1(t)\,\zeta\,\varphi_3\,\Delta\psi\,dx\,dt \\ &= \iint_{\mathcal{O}\times(0,T)} \beta_1(t)\,(-\Delta\zeta\,\varphi_3 - 2\nabla\zeta\cdot\nabla\varphi_3 - \zeta\,\Delta\varphi_3)\,\psi\,dx\,dt \\ &\le \varepsilon I(\varphi_3) + C_{10}(\varepsilon) \iint_{\mathcal{O}\times(0,T)} e^{\frac{-32\tilde{\alpha}+30\overline{\alpha}}{t^4(T-t)^4}}\,t^{-268}(T-t)^{-268}\,|\psi|^2\,dx\,dt. \end{aligned} \tag{3.37}$$

• We also integrate by parts in the third term with respect to x and we use the incompressibility condition on $\overline{y}$:

$$\begin{aligned} &-\iint_{\mathcal{O}\times(0,T)} \beta_1(t)\,\zeta\,\varphi_3\,\overline{y}\cdot\nabla\psi\,dx\,dt \\ &= \iint_{\mathcal{O}\times(0,T)} \beta_1(t)\,\overline{y}\cdot(\varphi_3\,\nabla\zeta + \zeta\,\nabla\varphi_3)\psi\,dx\,dt \\ &\le \varepsilon I(\varphi_3) + C_{11}(\varepsilon) \iint_{\mathcal{O}\times(0,T)} e^{\frac{-32\tilde{\alpha}+30\overline{\alpha}}{t^4(T-t)^4}}\,t^{-260}(T-t)^{-260}\,|\psi|^2\,dx\,dt. \end{aligned} \tag{3.38}$$

- We finally apply Young's inequality in the last term and we have:

$$
\begin{aligned}
&-\iint_{\mathcal{O}\times(0,T)} \beta_1(t)\,\zeta\,\varphi_3\,q\,dx\,dt \\
&\quad\leq \varepsilon I(\varphi_3) + C_{12}(\varepsilon)\iint_{\mathcal{O}\times(0,T)} e^{\frac{-32\widetilde{\alpha}+30\overline{\alpha}}{t^4(T-t)^4}}\,t^{-252}(T-t)^{-252}\,|q|^2\,dx\,dt.
\end{aligned}
\tag{3.39}
$$

From (3.34), (3.35) and (3.36)–(3.39), it is easy to deduce the desired inequality (3.31).

Arguing as in subsection 2.1, that is to say, combining the previous result and the classical energy estimates satisfied by φ and ψ, we can deduce the following Carleman inequality:

Lemma 8 *Assume that $N=3$, $n_1(x^0)\neq 0$ and $\mathcal{O}$ and $(\overline{y},\overline{\theta})$ satisfy (3.10)–(3.12). Then, there exist positive constants C, $\overline{\alpha}$ and $\widetilde{\alpha}$ depending on Ω, $\mathcal{O}$, T, $\overline{y}$ and $\overline{\theta}$ with $0<\widetilde{\alpha}<\overline{\alpha}$ and $16\widetilde{\alpha}-15\overline{\alpha}>0$ such that, for any $g\in L^2(Q)^3$, $q\in L^2(Q)$, $\varphi^0\in H$ and $\psi^0\in L^2(\Omega)$, the associated solution to (3.19) satisfies:*

$$
\left\{
\begin{aligned}
&\iint_Q e^{\frac{-2\overline{\alpha}}{\ell(t)^4}}\left(\ell(t)^{-12}\left(|\varphi|^2+|\psi|^2\right)+\ell(t)^{-4}\left(|\nabla\varphi|^2+|\nabla\psi|^2\right)\right)dx\,dt \\
&\quad\leq C\left(\iint_Q e^{\frac{-4\widetilde{\alpha}+2\overline{\alpha}}{\ell(t)^4}}\,\ell(t)^{-30}\,|g|^2\,dx\,dt\right. \\
&\quad+\iint_Q e^{\frac{-32\widetilde{\alpha}+30\overline{\alpha}}{\ell(t)^4}}\,\ell(t)^{-252}\,|q|^2\,dx\,dt \\
&\quad+\iint_{\mathcal{O}\times(0,T)} e^{\frac{-16\widetilde{\alpha}+14\overline{\alpha}}{\ell(t)^4}}\,\ell(t)^{-132}\,|\varphi_2|^2\,dx\,dt \\
&\quad\left.+\iint_{\mathcal{O}\times(0,T)} e^{\frac{-32\widetilde{\alpha}+30\overline{\alpha}}{\ell(t)^4}}\,\ell(t)^{-268}\,|\psi|^2\,dx\,dt\right),
\end{aligned}
\right.
\tag{3.40}
$$

where the function ℓ was defined in (3.29).

The similar result that can be established when $N=2$ is the following:

Lemma 9 *Assume that $N=2$, $n_1(x^0)\neq 0$ and $\mathcal{O}$ and $(\overline{y},\overline{\theta})$ satisfy (3.10)–(3.12). Then, there exist positive constants C, $\overline{\alpha}$ and $\widetilde{\alpha}$ depending on Ω, $\mathcal{O}$, T, $\overline{y}$ and $\overline{\theta}$ with $0<\widetilde{\alpha}<\overline{\alpha}$ and $16\widetilde{\alpha}-15\overline{\alpha}>0$ such that, for any $g\in L^2(Q)^2$, $q\in L^2(Q)$, $\varphi^0\in H$ and $\psi^0\in L^2(\Omega)$, the* *associated solution to* (3.19) *satisfies:*

$$\left\{\begin{array}{l}\displaystyle\iint_Q e^{\frac{-2\overline{\alpha}}{\ell(t)^4}} \left(\ell(t)^{-12}\left(|\varphi|^2+|\psi|^2\right)+\ell(t)^{-4}\left(|\nabla\varphi|^2+|\nabla\psi|^2\right)\right)\,dx\,dt\\ \qquad\displaystyle\leq C\left(\iint_Q e^{\frac{-4\widetilde{\alpha}+2\overline{\alpha}}{\ell(t)^4}}\ell(t)^{-30}\,|g|^2\,dx\,dt\right.\\ \qquad\displaystyle+\iint_Q e^{\frac{-32\widetilde{\alpha}+30\overline{\alpha}}{\ell(t)^4}}\ell(t)^{-252}\,|q|^2\,dx\,dt\\ \qquad\displaystyle\left.+\iint_{\mathcal{O}\times(0,T)} e^{\frac{-32\widetilde{\alpha}+30\overline{\alpha}}{\ell(t)^4}}\,\ell(t)^{-268}\,|\psi|^2\,dx\,dt\right).\end{array}\right. \tag{3.41}$$

2.3. An observability estimate for system (3.20)

In this paragraph, we will prove an observability estimate for the system

$$\left\{\begin{array}{ll}-\rho_t-\Delta\rho-\nabla\times((\overline{y}\cdot\nabla\times)\nabla\gamma)=0,\quad \Delta\gamma=\rho & \text{in } Q,\\ \gamma=0,\quad \rho=0 & \text{on } \Sigma,\\ \rho(T)=\rho^0 & \text{in } \Omega.\end{array}\right. \tag{3.42}$$

This estimate will be implied by a Carleman inequality of the form

$$S(\nabla\gamma)\leq C\iint_{\mathcal{O}\times(0,T)}|\nabla\gamma|^2\,dx\,dt,$$

where $S(\nabla\gamma)$ contains several global weighted integrals involving $\nabla\gamma$ (see (3.43)).

Lemma 10 *Assume that $N=2$ and $\overline{y}\in L^\infty(Q)^2$. There exist three positive constants C, $\overline{s}$ and $\overline{\lambda}$ depending on Ω, $\mathcal{O}$, T and $\overline{y}$ such that, for any $\rho^0\in H^{-1}(\Omega)$, the associated solution to* (3.42) *satisfies:*

$$\left\{\begin{array}{l}\displaystyle S(\nabla\gamma):=s^4\lambda^4\iint_Q e^{-2s\alpha}\,\xi^4|\nabla\gamma|^2\,dx\,dt\\ \qquad\displaystyle s\lambda^2\iint_Q e^{-2s\alpha}\,\xi|\nabla\rho|^2\,dx\,dt+s^3\lambda^4\iint_Q e^{-2s\alpha}\xi^3|\rho|^2\,dx\,dt\\ \qquad\displaystyle\leq C\,s^5\lambda^6\iint_{\mathcal{O}\times(0,T)} e^{-2s\alpha}\xi^5|\nabla\gamma|^2\,dx\,dt,\end{array}\right. \tag{3.43}$$

for any $s\geq\overline{s}$ and any $\lambda\geq\overline{\lambda}$. Recall that α and ξ were defined in (3.16).

Proof: Along the proof, s_j and λ_j ($j\geq 2$) will denote various positive constants that can eventually depend on Ω, $\mathcal{O}$, T and $\overline{y}$.

Let $\mathcal{O}_0$ be a nonempty open set satisfying $\mathcal{O}_0 \subset\subset \mathcal{O}$ and let us apply to ρ the Carleman inequality for parabolic systems with right hand sides in $L^2(0,T;H^{-1}(\Omega))$ proved in [14]:

$$
\begin{aligned}
s\lambda^2 \iint_Q e^{-2s\alpha}\,\xi|\nabla\rho|^2\,dx\,dt + s^3\lambda^4 \iint_Q e^{-2s\alpha}\xi^3|\rho|^2\,dx\,dt \\
\leq C_{13}\left(s^2\lambda^2\|\overline{y}\|_\infty^2 \iint_Q e^{-2s\alpha}\xi^2|\nabla(\nabla\times\gamma)|^2\,dx\,dt\right. \\
\left. + s^3\lambda^4 \iint_{\mathcal{O}_0\times(0,T)} e^{-2s\alpha}\xi^3|\rho|^2\,dx\,dt\right),
\end{aligned}
\tag{3.44}
$$

for any $s \geq s_2$ and $\lambda \geq \overline{\lambda}_2$.

Observe that, here, the assumption $\rho^0 \in H^{-1}(\Omega)$ may seem too weak to apply this result. Indeed, (3.44) can be proved as in [14] whenever $\rho \in C^1(\overline{Q})$ and, by a continuity argument, also for the solutions of problem (3.42) for which the left hand side of (3.44) is finite. This is our case, since one can ensure that $\rho \in L^2(Q)$ as soon as $\rho^0 \in H^{-1}(\Omega)$ (for instance, taking into account the definition of ρ as the solution by transposition of (3.42)).

Once (3.44) has been justified, let us first estimate the last integral in its right hand side. Thus, let $\zeta \in C^2(\overline{\mathcal{O}})$ be a cut-off function satisfying

$$
\zeta \equiv 1 \text{ in } \mathcal{O}_0, \quad 0 \leq \zeta \leq 1 \quad \text{and} \quad \zeta = 0 \text{ on } \partial\mathcal{O}.
$$

We have:

$$
\begin{aligned}
s^3\lambda^4 \iint_{\mathcal{O}_0\times(0,T)} e^{-2s\alpha}\xi^3|\Delta\gamma|^2\,dx\,dt &\leq s^3\lambda^4 \iint_{\mathcal{O}\times(0,T)} \zeta\, e^{-2s\alpha}\xi^3|\Delta\gamma|^2\,dx\,dt \\
&= -s^3\lambda^4 \iint_{\mathcal{O}\times(0,T)} e^{-2s\alpha}\xi^3(\nabla\zeta\cdot\nabla\gamma)\Delta\gamma\,dx\,dt \\
&\quad -3s^3\lambda^5 \iint_{\mathcal{O}\times(0,T)} \zeta\, e^{-2s\alpha}\xi^3(\nabla\eta^0\cdot\nabla\gamma)\Delta\gamma\,dx\,dt \\
&\quad +2s^4\lambda^5 \iint_{\mathcal{O}\times(0,T)} \zeta\, e^{-2s\alpha}\xi^4(\nabla\eta^0\cdot\nabla\gamma)\Delta\gamma\,dx\,dt \\
&\quad -s^3\lambda^4 \iint_{\mathcal{O}\times(0,T)} \zeta\, e^{-2s\alpha}\xi^3(\nabla\Delta\gamma\cdot\nabla\gamma)\,dx\,dt.
\end{aligned}
$$

Now, we apply Young's inequality several times and we obtain

$$
\begin{aligned}
& s^3\lambda^4 \iint_{\mathcal{O}_0\times(0,T)} e^{-2s\alpha}\xi^3|\Delta\gamma|^2\,dx\,dt \\
& \leq C_{14}(\varepsilon)\, s^5\lambda^6 \iint_{\mathcal{O}\times(0,T)} e^{-2s\alpha}\xi^5|\nabla\gamma|^2\,dx\,dt \\
& +\varepsilon\left(s^3\lambda^4 \iint_Q e^{-2s\alpha}\xi^3|\Delta\gamma|^2\,dx\,dt + s\lambda^2 \iint_Q e^{-2s\alpha}\xi|\nabla\Delta\gamma|^2\,dx\,dt\right),
\end{aligned}
$$

for $s \geq s_3$ and $\lambda \geq \lambda_3$ and for any small positive constant ε. Combining this, the fact that $\rho = \Delta\gamma$ and (3.44), we get

$$\begin{aligned} s\lambda^2 \iint_Q e^{-2s\alpha}\,\xi|\nabla\rho|^2\,dx\,dt + s^3\lambda^4 \iint_Q e^{-2s\alpha}\xi^3|\rho|^2\,dx\,dt \\ \leq C_{15}\left(s^2\lambda^2\|\overline{y}\|_\infty^2 \iint_Q e^{-2s\alpha}\xi^2|\nabla(\nabla\times\gamma)|^2\,dx\,dt \right. \\ \left. + s^5\lambda^6 \iint_{\mathcal{O}\times(0,T)} e^{-2s\alpha}\xi^5|\nabla\gamma|^2\,dx\,dt\right) \end{aligned} \tag{3.45}$$

for any $s \geq s_4$ and $\lambda \geq \lambda_4$.

Finally, we are going to estimate the first integral in the right hand side of (3.44). To this end, let us notice that, for $j = 1$ and 2 and almost every $t \in (0,T)$, the function $\partial_j\gamma(t)$ satisfies:

$$\Delta(\partial_j\gamma)(t) = \partial_j\rho(t) \text{ in } \Omega.$$

Let us apply the main result in [13] to $\partial_j\gamma$. This yields the existence of two numbers $\widetilde{\tau} > 1$ and $\widetilde{\lambda} > 1$ such that

$$\begin{aligned} \tau^4\lambda^4 \int_\Omega e^{2\tau\eta}\eta^4|\partial_j\gamma|^2(t)\,dx + \tau^2\lambda^2 \int_\Omega e^{2\tau\eta}\eta^2|\nabla(\partial_j\gamma)|^2(t)\,dx \\ \leq C_{16}\left(\tau \int_\Omega e^{2\tau\eta}\eta|\partial_j\rho|^2(t)\,dx + \tau^4\lambda^4 \int_{\mathcal{O}} e^{2\tau\eta}\eta^4|\partial_j\gamma|^2(t)\,dx \right. \\ \left. + \tau^{5/2}\lambda^2\, e^{2\tau}\|\partial_j\gamma(t)\|^2_{H^{1/2}(\partial\Omega)}\right) \end{aligned} \tag{3.46}$$

for $\tau \geq \widetilde{\tau}$ and $\lambda \geq \widetilde{\lambda}$. Here, we have introduced the function η, with

$$\eta(x) = e^{\lambda\eta^0(x)}.$$

In fact, the inequality one can find in [13] contains local integrals of $|\partial_j\gamma|^2$ and $|\nabla(\partial_j\gamma)|^2$ in the right hand side. But it can be written for a smaller set $\mathcal{O}' \subset\subset \mathcal{O}$. Using localizing arguments together with the fact that we actually have a global weighted integral of $|\Delta(\partial_j\gamma)|^2$ in the left hand side, (3.46) is easily found.

Following the same steps of [6], we set

$$\tau = \frac{s}{t^4(T-t)^4}\, e^{\lambda\, m\|\eta^0\|_\infty},$$

we multiply (3.46) by

$$\exp\left\{-2s\frac{e^{5/4\lambda m\|\eta^0\|_\infty}}{t^4(T-t)^4}\right\}$$

and we integrate in time over $(0,T)$. This gives

$$\begin{aligned}
&s^4\lambda^4 \iint_Q e^{-2s\alpha}\xi^4|\partial_j\gamma|^2\,dx\,dt + s^2\lambda^2 \iint_Q e^{-2s\alpha}\xi^2|\nabla(\partial_j\gamma)|^2\,dx\,dt \\
&\le C_{17}\left(s\iint_Q e^{-2s\alpha}\xi|\partial_j\rho|^2\,dx\,dt + s^4\lambda^4 \iint_{\mathcal{O}\times(0,T)} e^{-2s\alpha}\xi^4|\partial_j\gamma|^2\,dx\,dt \right.\\
&\left. + s^{5/2}\lambda^2 \int_0^T e^{-2s\alpha^*}(\xi^*)^{5/2}\|\partial_j\gamma\|^2_{H^{1/2}(\partial\Omega)} \right)
\end{aligned}$$

for $s \ge s_5$ and $\lambda \ge \widetilde{\lambda}$. Combining this estimate and (3.45), we have

$$\begin{aligned}
&s^4\lambda^4 \iint_Q e^{-2s\alpha}\xi^4|\partial_j\gamma|^2\,dx\,dt + s\lambda^2 \iint_Q e^{-2s\alpha}\,\xi|\nabla\rho|^2\,dx\,dt \\
&+ s^3\lambda^4 \iint_Q e^{-2s\alpha}\xi^3|\rho|^2\,dx\,dt \le C_{18}\left(s^{5/2}\lambda^2 \int_0^T e^{-2s\alpha^*}(\xi^*)^{5/2}\|\partial_j\gamma\|^2_{H^{1/2}(\partial\Omega)} \right.\\
&\left. + s^5\lambda^6 \iint_{\mathcal{O}\times(0,T)} e^{-2s\alpha}\xi^5|\nabla\gamma|^2\,dx\,dt \right)
\end{aligned}$$

for any $s \ge s_6$ and $\lambda \ge \lambda_5$. On the other hand, the boundary term can readily be bounded using the continuity of the trace operator:

$$\|\partial_j\gamma(t)\|^2_{H^{1/2}(\partial\Omega)} \le C_{19}(\|\partial_j\gamma(t)\|^2_{L^2} + \|\nabla(\partial_j\gamma)(t)\|^2_{L^2}).$$

Furthermore, since $\gamma|_\Sigma \equiv 0$, we know that there exists a positive constant C_{20} such that

$$\|\nabla(\partial_j\gamma)(t)\|_{L^2} \le C_{20}\|\Delta\gamma(t)\|_{L^2} \quad \text{a.e. in } (0,T) \text{ for } j=1,2.$$

Consequently,

$$\begin{aligned}
&s^4\lambda^4 \iint_Q e^{-2s\alpha}\xi^4|\partial_j\gamma|^2\,dx\,dt + s\lambda^2 \iint_Q e^{-2s\alpha}\,\xi|\nabla\rho|^2\,dx\,dt \\
&+ s^3\lambda^4 \iint_Q e^{-2s\alpha}\xi^3|\rho|^2\,dx\,dt \le C_{21}\, s^5\lambda^6 \iint_{\mathcal{O}\times(0,T)} e^{-2s\alpha}\xi^5|\nabla\gamma|^2\,dx\,dt
\end{aligned}$$

for $s \ge s_6$.

This implies (3.43) and ends the proof of lemma 12.

Remark 7 An almost immediate consequence of the Carleman estimate (3.43) is the following observability inequality:

$$\|(\nabla\gamma)(0)\|^2_{L^2} \le C \iint_{\mathcal{O}\times(0,T)} |\nabla\gamma|^2\,dx\,dt. \tag{3.47}$$

In fact, this is what will be used in Section 3 to prove the null controllability of system (3.15).

3. Null controllability of the linearized systems (3.13), (3.14) and (3.15)

3.1. Null controllability of (3.13)

We are dealing here with the following system:

$$\begin{cases} y_t - \Delta y + (\overline{y}, \nabla) y + (y, \nabla)\overline{y} + \nabla p = f + v 1\!\!1_{\mathcal{O}}, \quad \nabla \cdot y = 0 & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = y^0 & \text{in } \Omega, \end{cases} \tag{3.48}$$

where $\mathcal{O}$ satisfies (3.10) and $\overline{y}$ satisfies (3.11). Our goal will be to find a control v such that $y(T) = 0$ in Ω.

Let us introduce some weight functions:

$$\beta_2(t) = \exp\left\{\frac{\overline{\alpha}}{\ell(t)^4}\right\}\ell(t)^6, \quad \beta_3(t) = \exp\left\{\frac{2\widetilde{\alpha} - \overline{\alpha}}{\ell(t)^4}\right\}\ell(t)^{15}$$

and

$$\beta_4(t) = \exp\left\{\frac{8\widetilde{\alpha} - 7\overline{\alpha}}{\ell(t)^4}\right\}\ell(t)^{66}$$

(recall that ℓ was defined in (3.29)) where $\overline{\alpha}$ and $\widetilde{\alpha}$ are the constants provided by lemma 5 when $N = 3$ and lemma 6 when $N = 2$. Recall that, in particular, $0 < \widetilde{\alpha} < \overline{\alpha}$ and $8\widetilde{\alpha} - 7\overline{\alpha} > 0$.

Of course, we will need some specific conditions on f and y^0 to get the null controllability of (3.48). We will use the arguments in [6].

Thus, let us set

$$Ly = y_t - \Delta y + (\overline{y}, \nabla) y + (y, \nabla)\overline{y} \tag{3.49}$$

and let us introduce the spaces

$$E_2 = \{\, (y, p, v) : (y, v) \in E_0,\ \ell^{-4}\beta_2(Ly + \nabla p - v 1\!\!1_{\mathcal{O}}) \in L^2(0, T; H^{-1}(\Omega)^2) \,\}$$

when $N = 2$ and

$$\begin{aligned} E_3 = \{\, (y, p, v) : (y, v) \in E_0,\ &\ell^{-2}\beta_2^{1/2} y \in L^4(0, T; L^{12}(\Omega)^3), \\ &\ell^{-4}\beta_2(Ly + \nabla p - v 1\!\!1_{\mathcal{O}}) \in L^2(0, T; W^{-1,6}(\Omega)^3) \,\} \end{aligned}$$

when $N = 3$, where

$$\begin{aligned} E_0 = \{\, (y, v) : \beta_3\, y,\ \beta_4\, v 1\!\!1_{\mathcal{O}} \in L^2(Q)^N,\ v_1 \equiv 0, \\ \ell^{-2}\beta_2^{1/2} y \in L^2(0, T; V) \cap L^\infty(0, T; H) \,\}. \end{aligned}$$

It is clear that E_N is a Banach space for the norm $\|\cdot\|_{E_N}$, where

$$\begin{aligned} \|(y, p, v)\|_{E_2} = \Big(&\|\beta_3\, y\|_{L^2}^2 + \|\beta_4\, v 1\!\!1_{\mathcal{O}}\|_{L^2}^2 \\ &+ \|\ell^{-2}\beta_2^{1/2} y\|_{L^2(0,T;V)}^2 + \|\ell^{-2}\beta_2^{1/2} y\|_{L^\infty(0,T;H)}^2 \\ &+ \|\ell^{-4}\beta_2(Ly + \nabla p - v 1\!\!1_{\mathcal{O}})\|_{L^2(0,T;H^{-1})}^2\Big)^{1/2} \end{aligned}$$

and

$$\begin{aligned}\|(y,p,v)\|_{E_3} = \big(&\|\beta_3\, y\|^2_{L^2} + \|\beta_4\, v\mathbb{1}_{\mathcal{O}}\|^2_{L^2} \\ &+\|\ell^{-2}\beta_2^{1/2} y\|^2_{L^2(0,T;V)} + \|\ell^{-2}\beta_2^{1/2} y\|^2_{L^\infty(0,T;H)} \\ &+\|\ell^{-2}\beta_2^{1/2}\, y\|^2_{L^4(0,T;L^{12})} \\ &+\|\ell^{-4}\beta_2(Ly+\nabla p - v\mathbb{1}_{\mathcal{O}})\|^2_{L^2(0,T;W^{-1,6})}\big)^{1/2}.\end{aligned}$$

Proposition 5 *Assume that* $n_1(x^0) \neq 0$ *and* $\mathcal{O}$ *and* $\overline{y}$ *verify* (3.10)–(3.11). *Let* $y^0 \in E$ *and let us assume that*

$$\ell^{-4}\beta_1 f \in \begin{cases} L^2(0,T;H^{-1}(\Omega)^2) & \text{if } N=2, \\ L^2(0,T;W^{-1,6}(\Omega)^3) & \text{if } N=3. \end{cases}$$

Then, we can find a control v *such that the associated solution* (y,p) *to* (3.48) *satisfies* $(y,p,v) \in E_N$. *In particular,* $v_1 \equiv 0$ *and* $y(T)=0$.

Sketch of the proof: The proof of this proposition is very similar to the one of proposition 2 in [6], so we will just give the main ideas. For simplicity, we will only consider the case $N=3$. When $N=2$, the proof is even easier.

Following the arguments in [8] and [12], let us introduce the auxiliary optimal control problem

$$\begin{cases} \text{ínf } \dfrac{1}{2}\left(\displaystyle\iint_Q |\beta_3\, y|^2 dxdt + \iint_{\mathcal{O}\times(0,T)} |\beta_4\, v|^2 dxdt\right) \\ \text{subject to } v \in L^2(Q)^3,\ \operatorname{supp} v \subset \mathcal{O}\times(0,T),\ v_1 \equiv 0 \text{ and} \\ \qquad \begin{cases} Ly + \nabla p = f + v\mathbb{1}_{\mathcal{O}} & \text{in } Q, \\ \nabla\cdot y = 0 & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = y^0, \quad y(T) = 0 & \text{in } \Omega. \end{cases} \end{cases} \tag{3.50}$$

Notice that a solution $(\widehat{y},\widehat{p},\widehat{v})$ to (3.50) is a good candidate to satisfy $(\widehat{y},\widehat{p},\widehat{v}) \in E_3$.

For the moment, let us assume that (3.50) possesses a solution $(\widehat{y},\widehat{p},\widehat{v})$. Then, by virtue of Lagrange's principle, there must exist dual variables $\widehat{z}$ and $\widehat{q}$ such that

$$\begin{cases} \widehat{y} = \beta_3^{-2}(L^*\widehat{z} + \nabla\widehat{q}), \quad \nabla\cdot\widehat{z} = 0 & \text{in } Q, \\ \widehat{v}_1 \equiv 0, \quad \widehat{v}_i = -\beta_4^{-2}\,\widehat{z}_i \quad (i=2,3) & \text{in } \mathcal{O}\times(0,T), \\ \widehat{z} = 0 & \text{on } \Sigma, \end{cases} \tag{3.51}$$

where L^* is the adjoint operator of L, i.e.

$$L^*z = -z_t - \Delta z - (Dz)\,\overline{y}.$$

At least formally, the couple $(\widehat{z},\widehat{q})$ satisfies

$$a((\widehat{z},\widehat{q}),(w,h)) = \langle G,(w,h)\rangle \quad \forall (w,h) \in P_0, \tag{3.52}$$

where P_0 is the space

$$P_0 = \{\, (w,h) \in C^2(\overline{Q})^4 : \nabla\cdot w = 0,\ w = 0 \text{ on } \Sigma,\ \int_{\mathcal{O}} h(x,t)\,dx = 0 \,\}$$

and we have used the notation

$$a((\widehat{z},\widehat{q}),(w,h)) = \iint_Q \beta_3^{-2}\,(L^*\widehat{z} + \nabla\widehat{q})\cdot(L^*w + \nabla h)\,dx\,dt$$
$$+ \iint_{\mathcal{O}\times(0,T)} \beta_4^{-2}\,(\widehat{z}_2\,w_2 + \widehat{z}_3\,w_3)\,dx\,dt$$

and

$$\langle G,(w,h)\rangle = \int_0^T \langle f(t), w(t)\rangle_{H^{-1},H_0^1}\,dt + \int_\Omega y^0\cdot w(0)\,dx.$$

Conversely, if we are able to "solve" (3.52) and then use (3.51) to define $(\widehat{y},\widehat{p},\widehat{v})$, we will probably have found a solution to (3.50).

Thus, let us consider the linear space P_0. It is clear that $a(\cdot\,,\cdot) : P_0\times P_0 \mapsto \mathbf{R}$ is a symmetric, definite positive bilinear form on P_0. We will denote by P the completion of P_0 for the norm induced by $a(\cdot\,,\cdot)$. Then $a(\cdot\,,\cdot)$ is well-defined, continuous and again definite positive on P. Furthermore, in view of the Carleman estimate (3.28), the linear form $(w,h)\mapsto\langle G,(w,h)\rangle$ is well-defined and continuous on P. Hence, from Lax-Milgram's lemma, we deduce that the variational problem

$$\begin{cases} a((\widehat{z},\widehat{q}),(w,h)) = \langle G,(w,h)\rangle \\ \forall (w,h)\in P, \quad (\widehat{z},\widehat{q})\in P, \end{cases} \tag{3.53}$$

possesses exactly one solution $(\widehat{z},\widehat{q})$.

Let $\widehat{y}$ and $\widehat{v}$ be given by (3.51). Then, it is readily seen that they verify

$$\iint_Q \beta_3^2|\widehat{y}|^2 dx\,dt + \iint_{\mathcal{O}\times(0,T)} \beta_4^2|\widehat{v}|^2 dx\,dt < +\infty$$

and, also, that $\widehat{y}$ is, together with some pressure $\widehat{p}$, the weak solution (belonging to $L^2(0,T;V)\cap L^\infty(0,T;H)$) of the Stokes system in (3.50) for $v=\widehat{v}$.

In order to prove that $(\widehat{y},\widehat{p},\widehat{v})\in E_3$, it only remains to check that $\ell^{-2}\,\beta_2^{1/2}\,\widehat{y}$ is, together with $\ell^{-2}\,\beta_2^{1/2}\,\widehat{p}$, a weak solution of a Stokes problem of the kind (3.48) with a right hand side in $L^2(0,T;W^{-1,6}(\Omega)^3)$ that belongs to $L^4(0,T;L^{12}(\Omega)^3)$. To this end, we define the functions $y^* = \ell^{-2}\,\beta_2^{1/2}\,\widehat{y}$, $p^* = \ell^{-2}\,\beta_2^{1/2}\,\widehat{p}$ and $f^* = \ell^{-2}\,\beta_2^{1/2}(f + \widehat{v}1\!\!1_{\mathcal{O}})$. Then (y^*,p^*) satisfies

$$\begin{cases} Ly^* + \nabla p^* = f^* + (\ell^{-2}\beta_2^{-1/2})_t\,\widehat{y}, \quad \nabla\cdot y^* = 0 & \text{in } Q, \\ y^* = 0 & \text{on } \Sigma, \\ y^*(0) = \ell^{-2}(0)\beta_2^{1/2}(0)y^0 & \text{in } \Omega. \end{cases} \tag{3.54}$$

From the fact that $f^*\in L^2(0,T;H^{-1}(\Omega)^3)$ and $y^0\in H$, we have indeed

$$y^*\in L^2(0,T;V)\cap L^\infty(0,T;H).$$

Finally, we deduce that $y^*\in L^4(0,T;L^{12}(\Omega)^3)$ from lemma 2 in [6]. This ends the sketch of the proof of proposition 8.

3.2. Null controllability of system (3.14)

In this paragraph, we will establish the null controllability of the linear system

$$\begin{cases} y_t - \Delta y + (\overline{y}, \nabla)y + (y, \nabla)\overline{y} + \nabla p = f + v1\!\!1_{\mathcal{O}} + \theta\, e_N & \text{in } Q, \\ \nabla \cdot y = 0 & \text{in } Q, \\ \theta_t - \Delta\theta + \overline{y} \cdot \nabla\theta + y \cdot \nabla\overline{\theta} = k + h1\!\!1_{\mathcal{O}} & \text{in } Q, \\ y = 0, \quad \theta = 0 & \text{on } \Sigma, \\ y(0) = y^0, \quad \theta(0) = \theta^0 & \text{in } \Omega, \end{cases} \tag{3.55}$$

where $\mathcal{O}$ satisfies (3.10) and $\overline{y}$ and $\overline{\theta}$ satisfy (3.11) and (3.12), for suitable right hand sides f and k.

The arguments we present here are completely analogous to those in [11] and subsection 3.1 of this paper, so that we will only give a sketch. Thus, we restrict ourselves again to the three-dimensional case with $n_1(x^0) \neq 0$.

Let us introduce the weight functions

$$\beta_5(t) = \exp\left\{\frac{\overline{\alpha}}{\ell(t)^4}\right\}\ell(t)^6, \quad \beta_6(t) = \exp\left\{\frac{2\widetilde{\alpha} - \overline{\alpha}}{\ell(t)^4}\right\}\ell(t)^{15},$$

$$\beta_7(t) = \exp\left\{\frac{16\widetilde{\alpha} - 15\overline{\alpha}}{\ell(t)^4}\right\}\ell(t)^{126}, \quad \beta_8(t) = \exp\left\{\frac{8\widetilde{\alpha} - 7\overline{\alpha}}{\ell(t)^4}\right\}\ell(t)^{66}$$

and

$$\beta_9(t) = \exp\left\{\frac{16\widetilde{\alpha} - 15\overline{\alpha}}{\ell(t)^4}\right\}\ell(t)^{134},$$

where the constants $\overline{\alpha}$ and $\widetilde{\alpha}$ are furnished by lemma 8 when $N = 3$ and lemma 9 when $N = 2$ (and, in particular, $0 < \widetilde{\alpha} < \overline{\alpha}$ and $16\widetilde{\alpha} - 15\overline{\alpha} > 0$).

Let us set

$$P\theta = \theta_t - \Delta\theta + \overline{y} \cdot \nabla\theta \tag{3.56}$$

and let us introduce the spaces

$$\begin{aligned} \widetilde{E}_2 = \{\, & (y, p, \theta, v, h) : (y, \theta, v, h) \in \widetilde{E}_0, \\ & \ell^{-4}\beta_5(Ly + \nabla p - v1\!\!1_{\mathcal{O}}) \in L^2(0, T; H^{-1}(\Omega)^2), \\ & \ell^{-4}\beta_5(P\theta + y \cdot \nabla\overline{\theta} - h1\!\!1_{\mathcal{O}}) \in L^2(0, T; H^{-1}(\Omega)) \,\} \end{aligned}$$

when $N = 2$ and

$$\begin{aligned} \widetilde{E}_3 = \{\, & (y, p, \theta, v, h) : (y, \theta, v, h) \in \widetilde{E}_0, \\ & \ell^{-2}\beta_5^{1/2} y \in L^4(0, T; L^{12}(\Omega)^3), \\ & \ell^{-4}\beta_5(Ly + \nabla p - v1\!\!1_{\mathcal{O}}) \in L^2(0, T; W^{-1,6}(\Omega)^3), \\ & \ell^{-4}\beta_5(P\theta + y \cdot \nabla\overline{\theta} - h1\!\!1_{\mathcal{O}}) \in L^2(0, T; H^{-1}(\Omega)) \,\} \end{aligned}$$

when $N = 3$, where

$$\widetilde{E}_0 = \{ (y,\theta,v,h) : (\beta_6\, y)_i,\ \beta_7\,\theta,\ (\beta_8\, v 1\!\!1_{\mathcal{O}})_i,\ \beta_9\, h 1\!\!1_{\mathcal{O}} \in L^2(Q)\ (1 \le i \le N),$$
$$v_1 \equiv v_N \equiv 0,\ \ell^{-2}\beta_5^{1/2} y \in L^2(0,T;V) \cap L^\infty(0,T;H),$$
$$\ell^{-2}\beta_5^{1/2}\theta \in L^2(0,T;H_0^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega)) \}.$$

It can be readily seen now that $\widetilde{E}_0$, $\widetilde{E}_2$ and $\widetilde{E}_3$ are Banach spaces for the norms

$$\|(y,\theta,v,h)\|_{\widetilde{E}_0} = \big(\|\beta_6\, y\|_{L^2}^2 + \|\beta_7\,\theta\|_{L^2}^2 + \|\beta_8\, v\|_{L^2}^2$$
$$+\|\beta_9\, h\|_{L^2}^2 + \|\ell^{-2}\beta_5^{1/2} y\|_{L^2(0,T;V)}^2 + \|\ell^{-2}\beta_5^{1/2} y\|_{L^\infty(0,T;H)}^2$$
$$+\|\ell^{-2}\beta_5^{1/2}\theta\|_{L^2(0,T;H_0^1)}^2 + \|\ell^{-2}\beta_5^{1/2}\theta\|_{L^\infty(0,T;L^2)}^2\Big)^{1/2},$$

$$\|(y,p,\theta,v,h)\|_{\widetilde{E}_2} = \Big(\|(y,\theta,v,h)\|_{\widetilde{E}_0}^2$$
$$+\|\ell^{-4}\beta_5(Ly + \nabla p - v 1\!\!1_{\mathcal{O}})\|_{L^2(0,T;H^{-1})}^2$$
$$+\|\ell^{-4}\beta_5(P\theta + y\cdot\nabla\overline{\theta} - h 1\!\!1_{\mathcal{O}})\|_{L^2(0,T;H^{-1})}^2\Big)^{1/2}$$

and

$$\|(y,p,\theta,v,h)\|_{\widetilde{E}_3} = \Big(\|(y,\theta,v,h)\|_{\widetilde{E}_0}^2 + \|\ell^{-2}\beta_5^{1/2} y\|_{L^4(0,T;L^{12})}^2$$
$$+\|\ell^{-4}\beta_5(Ly + \nabla p - v 1\!\!1_{\mathcal{O}})\|_{L^2(0,T;W^{-1,6})}^2$$
$$+\|\ell^{-4}\beta_5(P\theta + y\cdot\nabla\overline{\theta} - h 1\!\!1_{\mathcal{O}})\|_{L^2(0,T;H^{-1})}^2\Big)^{1/2}$$

Proposition 6 *Assume that $n_1(x^0) \neq 0$ and $\mathcal{O}$ and $(\overline{y},\overline{\theta})$ satisfy* (3.10)–(3.12). *Let $y^0 \in E$, $\theta^0 \in L^2(\Omega)$ and let us assume that*

$$\ell^{-4}\beta_1(f,k) \in \begin{cases} L^2(0,T;H^{-1}(\Omega)^2) \times L^2(0,T;H^{-1}(\Omega)) & \text{if } N=2,\\ L^2(0,T;W^{-1,6}(\Omega)^3) \times L^2(0,T;H^{-1}(\Omega)) & \text{if } N=3. \end{cases}$$

Then, we can find controls v and h such that the associated solution to (3.55) *satisfies $(y,p,\theta,v,h) \in \widetilde{E}_N$. In particular, $v_1 \equiv v_N \equiv 0$ and $y(T) = \theta(T) = 0$.*

We omit the proof of this proposition, since it is essentially the same as the one of proposition 2 in [11] and follows the steps of proposition 5 above. As we have already indicated, the main ideas come from the paper [12].

3.3. Null controllability of system (3.15)

In this paragraph, we will prove the null controllability of the linear system

$$\begin{cases} y_t - \Delta y + (y,\nabla)\overline{y} + \nabla p = v 1\!\!1_{\mathcal{O}}, \quad \nabla\cdot y = 0 & \text{in } Q,\\ y\cdot n = 0, \quad \nabla\times y = 0 & \text{on } \Sigma,\\ y(0) = y^0 & \text{in } \Omega, \end{cases} \tag{3.57}$$

where $N = 2$ and $\overline{y} \in L^\infty(Q)^2$.

For this purpose, we first rewrite this system using the streamline-vorticity formulation. Thus, setting $\omega = \nabla \times y$, we have

$$\left\{ \begin{array}{ll} \omega_t - \Delta\omega + \nabla \times ((\nabla \times \psi, \nabla)\overline{y}) = \nabla \times (v 1\!\!1_{\mathcal{O}}), \quad \Delta\psi = \omega & \text{in } Q, \\ \psi = 0, \quad \omega = 0 & \text{on } \Sigma, \\ \omega(0) = \nabla \times y^0 & \text{in } \Omega. \end{array} \right. \tag{3.58}$$

Proposition 7 *Assume that $y^0 \in H$ and $\overline{y} \in L^\infty(Q)^2$. Then, there exist controls $v 1\!\!1_{\mathcal{O}}$ with $v \in W$ such that*

$$\|v\|_{L^2} \le C\|y^0\|_H \tag{3.59}$$

and the associated solutions of (3.57) *satisfy*

$$y \in L^2(0,T; H^1(\Omega)^2) \cap C^0([0,T]; L^2(\Omega)^2), \quad y_t \in L^2(0,T; H^{-1}(\Omega)^2), \tag{3.60}$$

and $y(T) = 0$, with

$$\|y\|_{L^2(0,T;H^1)} + \|y\|_{C^0([0,T];L^2)} + \|y_t\|_{L^2(0,T;H^{-1})} \le C\|y^0\|_H. \tag{3.61}$$

Proof: We first establish the null controllability property for y. This can be done in several ways. One of them is the following. We first define for each $\varepsilon > 0$ the functional

$$\left\{ \begin{array}{l} J_\varepsilon(\gamma^0) = \dfrac{1}{2} \displaystyle\iint_{\mathcal{O}\times(0,T)} |\nabla \times \gamma|^2 \, dx\, dt + \varepsilon \|\nabla\gamma^0\|_{L^2} + ((\nabla \times \gamma)(0), y^0)_{L^2} \\ \forall \gamma^0 \in H_0^1(\Omega), \end{array} \right.$$

where γ is given by (3.42) with $\rho^0 = \Delta\gamma^0 \in H^{-1}(\Omega)$.

It is not difficult to see from the observability inequality (3.47) that this functional possesses a unique minimizer $\gamma_\varepsilon^0 \in H_0^1(\Omega)$ (see, for instance, [5]). Now, from the necessary conditions for J_ε to reach a minimum, we have

$$\begin{array}{l} \displaystyle\iint_Q ((\nabla \times \gamma_\varepsilon) 1\!\!1_{\mathcal{O}}) \cdot (\nabla \times \gamma) \, dx\, dt + \varepsilon \Big(\frac{\nabla \times \gamma_\varepsilon^0}{\|\nabla \times \gamma_\varepsilon^0\|_{L^2}}, \nabla \times \gamma^0 \Big)_{L^2} \\ \quad + ((\nabla \times \gamma)(0), y^0)_{L^2} = 0 \quad \forall \gamma^0 \in H_0^1(\Omega). \end{array} \tag{3.62}$$

Thus, setting $v_\varepsilon = (\nabla \times \gamma_\varepsilon) 1\!\!1_{\mathcal{O}}$ and putting $\gamma^0 = \gamma_\varepsilon^0$, we find from (3.47) and (3.62) that (3.59) holds for v_ε for some C independent of ε:

$$\|\nabla \times \gamma_\varepsilon\|_{L^2(\mathcal{O}\times(0,T))^2} = \|v_\varepsilon\|_{L^2(\mathcal{O}\times(0,T))^2} \le C. \tag{3.63}$$

Let us denote by $(\omega_\varepsilon, \psi_\varepsilon)$ the solution to (3.58) for $v = v_\varepsilon$. Then, taking into account the systems satisfied by (ρ, γ) and $(\omega_\varepsilon, \psi_\varepsilon)$, we deduce that

$$\begin{array}{l} \displaystyle\iint_Q \nabla \times (v_\varepsilon 1_{\mathcal{O}}) \, \gamma \, dx\, dt + (\nabla \times \gamma^0, (\nabla \times \psi_\varepsilon)(T))_{L^2} \\ \quad - ((\nabla \times \gamma)(0), y^0)_{L^2} = 0 \quad \forall \gamma^0 \in H_0^1(\Omega). \end{array}$$

Combining this and (3.62), we obtain

$$\|(\nabla \times \psi_\varepsilon)(T)\|_{L^2} \leq \varepsilon. \tag{3.64}$$

From (3.63) and (3.64) written for each $\varepsilon > 0$, we deduce that, at least for a subsequence, $v_\varepsilon \to v$ weakly in $L^2(\mathcal{O}\times(0,T))^2$, where the control $v1\!\!1_{\mathcal{O}}$ is such that the corresponding solution (ω, ψ) to (3.58) satisfies

$$(\nabla \times \psi)(T) = y(T) = 0 \quad \text{in } \Omega.$$

Since $v \in L^2(\mathcal{O} \times (0,T))^2$ and $\nabla \cdot v = 0$ in $\mathcal{O} \times (0,T)$, we necessarily have $v \in W$ (from De Rham's lemma applied to $(v_2, -v_1)$).

In order to obtain the desired regularity for y, we will consider again the equations satisfied by ψ and ω and we will check that

$$\psi \in L^2(0,T;H^2(\Omega)) \cap C^0([0,T];H^1_0(\Omega)) \ \text{ and } \ \psi_t \in L^2(Q), \tag{3.65}$$

with appropriate estimates.

For simplicity, we will only present the estimates. The rigorous argument relies on introducing a standard Galerkin approximation of (3.58) with a 'special' basis of $H^1_0(\Omega)$ (more precisely, the basis formed by the eigenfunctions of the Laplacian-Dirichlet operator in Ω) and deducing for the associated approximate solutions the estimates below.

Thus, let us multiply the first equation in (3.58) by ψ and let us integrate by parts. We find that

$$\frac{1}{2}\int_\Omega |\nabla\psi(t)|^2\,dx + \int_0^t\int_\Omega |\Delta\psi|^2\,dx\,d\tau = \int_0^t\int_{\mathcal{O}} v\cdot(\nabla\times\psi)\,dx\,d\tau$$
$$- \int_0^t\int_\Omega (((\nabla\times\psi),\nabla)\,\overline{y})\cdot(\nabla\times\psi)\,dx\,d\tau + \frac{1}{2}\|(\nabla\psi)(0)\|^2_{L^2}$$

for all $t \in (0,T)$. If we integrate by parts in the last integral, we also have

$$- \int_0^t\int_\Omega (((\nabla\times\psi)\cdot\nabla)\,\overline{y})\cdot(\nabla\times\psi)\,dx\,d\tau$$
$$= \int_0^t\int_\Omega ((\nabla\times\psi)\cdot\nabla)(\nabla\times\psi)\cdot\overline{y}\,dx\,d\tau.$$

Since $\psi|_\Sigma \equiv 0$, we deduce that

$$\psi \in L^2(0,T;H^2(\Omega)) \cap L^\infty(0,T;H^1_0(\Omega)) \tag{3.66}$$

and

$$\|\psi\|_{L^2(0,T;H^2)} + \|\psi\|_{L^\infty(0,T;H^1_0)} \leq C\|y^0\|_{L^2}. \tag{3.67}$$

Now, let us introduce for each t the function $\psi^*(t) = \Delta^{-1}\psi_t(t)$, i.e. the solution to

$$\begin{cases} -\Delta\psi^*(t) = \psi_t(t) & \text{in } \Omega \\ \psi^*(t) = 0 & \text{on } \partial\Omega. \end{cases}$$

Observe that, whenever $\psi_t(t) \in L^2(\Omega)$, this function satisfies $\psi^*(t) \in H^2(\Omega) \cap H_0^1(\Omega)$ and

$$\|\psi^*(t)\|_{H^2} \leq C\|\psi_t(t)\|_{L^2}. \tag{3.68}$$

Then, we multiply the first equation of (3.58) by ψ^* and we integrate by parts. This gives

$$\iint_Q |\psi_t|^2 \, dx \, dt = \iint_Q (\Delta\psi) \, \psi_t \, dx \, dt - \iint_Q ((\nabla \times \psi) \cdot \nabla)(\nabla \times \psi^*) \cdot \overline{y} \, dx \, dt$$
$$+ \iint_{\mathcal{O}\times(0,T)} v \cdot (\nabla \times \psi^*) \, dx \, dt.$$

Using that $v \in L^2(Q)^2$ and we already have (3.67) and (3.68), we conclude that $\psi_t \in L^2(Q)$ and

$$\|\psi_t\|_{L^2} \leq C\|y^0\|_{L^2}. \tag{3.69}$$

From (3.67) and (3.69), we immediately obtain (3.65), (3.60) and (3.61).

This ends the proof of proposition 9.

4. Proofs of the controllability results for the nonlinear systems

In this last Section, we will give the proofs of theorems 5, 6 and 7. For the proofs of theorems 5 and 6 we employ an inverse mapping theorem, while a fixed point argument is used for theorem 7.

4.1. Proof of Theorem 5

We also follow here the steps in [6].

Thus, we set $y = \overline{y} + z$ and $p = \overline{p} + \chi$ and we use these identities in (3.1). Taking into account that $(\overline{y}, \overline{p})$ solves (3.5), we find:

$$\begin{cases} Lz + (z, \nabla)z + \nabla\chi = v1\!\!1_{\mathcal{O}}, \quad \nabla \cdot z = 0 & \text{in } Q, \\ z = 0 & \text{on } \Sigma, \\ z(0) = y^0 - \overline{y}^0 & \text{in } \Omega \end{cases} \tag{3.70}$$

(recall that L was defined in (3.49)).

This way, we have reduced our problem to a local null controllability result for the solution (z, χ) to the *nonlinear* problem (3.70).

We will use the following inverse mapping theorem (see [1]):

Theorem 8 *Let B_1 and B_2 be two Banach spaces and let $\mathcal{A} : B_1 \mapsto B_2$ satisfy $\mathcal{A} \in C^1(B_1; B_2)$. Assume that $b_0 \in B_1$, $\mathcal{A}(b_0) = d_0$ and also that $\mathcal{A}'(b_0) : B_1 \mapsto B_2$ is surjective. Then there exists $\delta > 0$ such that, for every $d \in B_2$ satisfying $\|d - d_0\|_{B_2} < \delta$, there exists a solution of the equation*

$$\mathcal{A}(b) = d, \quad b \in B_1.$$

We will apply this result with $B_1 = E_N$,

$$B_2 = \begin{cases} L^2(\ell^{-4}\beta_2; 0,T; H^{-1}(\Omega)^2) \times H & \text{if } N = 2, \\ L^2(\ell^{-4}\beta_2; 0,T; W^{-1,6}(\Omega)^3) \times (H \cap L^4(\Omega)^3) & \text{if } N = 3 \end{cases}$$

and

$$\mathcal{A}(z,\chi,v) = (Lz + (z,\nabla)z + \nabla\chi - v\mathbb{1}_{\mathcal{O}}, z(0)) \quad \forall (z,\chi,v) \in E_N.$$

From the facts that $\ell^{-2}\beta_2^{1/2} y \in L^4(0,T; L^{12}(\Omega)^3)$ and $\mathcal{A}$ is bilinear, it is not difficult to check that $\mathcal{A} \in C^1(B_1; B_2)$; more details can be found in [12] or [6].

Let b_0 be the origin in B_1. Notice that $\mathcal{A}'(0,0,0) : B_1 \mapsto B_2$ is given by

$$\mathcal{A}'(0,0,0)(z,\chi,v) = (Lz + \nabla\chi - v\mathbb{1}_{\mathcal{O}}, z(0)) \quad \forall (z,\chi,v) \in E_N$$

and is surjective, in view of the null controllability result for (3.13) given in proposition 8.

Consequently, we can indeed apply theorem 8 with these data and there exists $\delta > 0$ such that, if $\|z(0)\|_E \le \delta$, then we find a control v satisfying $v_1 \equiv 0$ such that the associated solution to (3.70) verifies $z(T) = 0$ in Ω.

This concludes the proof of theorem 5.

4.2. Proof of theorem 6

Again, we follow here the ideas of [11].

Therefore, we set $y = \overline{y} + z$, $p = \overline{p} + \chi$ and $\theta = \overline{\theta} + \rho$, so from (3.2) and (3.7), we find:

$$\begin{cases} Lz + (z,\nabla)z + \nabla\chi = v\mathbb{1}_{\mathcal{O}} + \rho\, e_N, \quad \nabla\cdot z = 0 & \text{in } Q, \\ P\rho + (z,\nabla)\rho + z\cdot\nabla\overline{\theta} = h\mathbb{1}_{\mathcal{O}} & \text{in } Q, \\ z = 0, \quad \rho = 0 & \text{on } \Sigma, \\ z(0) = y^0 - \overline{y}^0, \quad \rho(0) = \theta^0 - \overline{\theta}(0) & \text{in } \Omega \end{cases} \tag{3.71}$$

(L and P were respectively defined in (3.49) and (3.56)).

We are thus led to prove the local null controllability of (3.71). To this end, we will use again theorem 8, which was presented in subsection 4.1. Using the same notation as there, we set $B_1 = \widetilde{E}_N$,

$$B_2 = L^2(\ell^{-4}\beta_5; 0,T; H^{-1}(\Omega)^3) \times H \times L^2(\Omega)$$

if $N = 2$ and

$$B_2 = L^2(\ell^{-4}\beta_5; 0,T; W^{-1,6}(\Omega)^3 \times H^{-1}(\Omega)) \times (L^4(\Omega)^3 \cap H) \times L^2(\Omega).$$

if $N = 3$.

Let us introduce $\mathcal{A}$, with

$$\mathcal{A}(z,\chi,\rho,v,h) = (\mathcal{A}_1(z,\chi,\rho,v), \mathcal{A}_2(z,\rho,h), z(0), \rho(0)),$$

$$\mathcal{A}_1(z,\chi,\rho,v) = Lz + (z,\nabla)z + \nabla\chi - v\mathbb{1}_{\mathcal{O}} - \rho e_N$$

and

$$\mathcal{A}_2(z,\rho,h) = P\rho + (z,\nabla)\rho + z\cdot\nabla\overline{\theta} - h\mathbb{1}_{\mathcal{O}}$$

for every $(z,\chi,\rho,v,h)\in\widetilde{E}_N$.

Using the fact that $\ell^{-2}\beta_5^{1/2}z\in L^4(0,T;L^{12}(\Omega)^3)$, it can be checked that $\mathcal{A}_1$ is C^1. Then, since $\ell^{-2}\beta_5^{1/2}\rho\in L^2(0,T;H^1(\Omega))\cap L^\infty(0,T;L^2(\Omega))$ and this space is continuously embedded in $L^4(0,T;L^3(\Omega))$, we deduce that

$$\ell^{-4}\beta_5(z,\nabla)\rho=\nabla\cdot(z\,\rho)\in L^2(0,T;W^{-1,12/5}(\Omega))\subset L^2(0,T;H^{-1}(\Omega))$$

and, consequently, $\mathcal{A}$ is well-defined and satisfies $\mathcal{A}\in C^1(B_1;B_2)$.

The fact that $\mathcal{A}'(0,0,0,0,0):B_1\longmapsto B_2$ is onto is an immediate consequence of the result given in proposition 6.

As a conclusion, we can apply theorem 8 and the null controllability for system (3.71) holds.

4.3. Proof of theorem 7

Let us recall the nonlinear system we are dealing with:

$$\begin{cases} y_t-\Delta y+(y,\nabla)\mathbf{T}_M(y)+\nabla p=v1\!\!1_{\mathcal{O}} & \text{in } Q,\\ \nabla\cdot y=0 & \text{in } Q,\\ y\cdot n=0,\quad \nabla\times y=0 & \text{on } \Sigma,\\ y(0)=y^0 & \text{in } \Omega.\end{cases}$$

In this case, we are going to apply *Kakutani's fixed point* theorem (see for instance [2]):

Theorem 9 *Let Z be a Banach space and let $\Lambda:Z\mapsto Z$ be a set-valued mapping satisfying the following assumptions:*

- *$\Lambda(z)$ is a nonempty closed convex set of Z for every $z\in Z$.*
- *There exists a convex compact set $K\subset Z$ such that $\Lambda(K)\subset K$.*
- *Λ is upper-hemicontinuous in Z, i.e. for each $\sigma\in Z'$ the single-valued mapping*

$$z\mapsto\sup_{y\in\Lambda(z)}\langle\sigma,y\rangle_{Z',Z}\tag{3.72}$$

is upper-semicontinuous.

Then Λ possesses a fixed point in the set K, i.e. there exists $z\in K$ such that $z\in\Lambda(z)$.

In order to apply this result, we set $Z=L^2(Q)^2$ and, for each $z\in Z$, we consider the following system:

$$\begin{cases} y_t-\Delta y+(y,\nabla)\mathbf{T}_M(z)+\nabla p=v1\!\!1_{\mathcal{O}} & \text{in } Q,\\ \nabla\cdot y=0 & \text{in } Q,\\ y\cdot n=0,\quad \nabla\times y=0 & \text{on } \Sigma,\\ y(0)=y^0 & \text{in } \Omega.\end{cases}\tag{3.73}$$

Then, for each $z\in Z$, we denote by $A(z)$ the set of controls $v1\!\!1_{\mathcal{O}}$ with $v\in W$ that drive system (3.73) to zero and satisfy (3.59). Finally, our set-valued mapping is given as follows: for each $z \in Z$, $\Lambda(z)$ is the set of functions y that solve, together with some p, the linear system (3.73) corresponding to a control $v \in A(z)$.

Let us check that the assumptions of theorem 9 are satisfied in this setting. The first one holds easily, so we omit the proof. Next, the estimates (3.60) and (3.61) tell us that the whole space Z is actually mapped into a compact set.

Let us finally see that Λ is upper-hemicontinuous in Z. Assume that $\sigma \in Z'$ and let $\{z_n\}$ be a sequence in Z such that $z_n \to z$ in Z. We have to prove that

$$\limsup_{n\to\infty} \sup_{y\in\Lambda(z_n)} \langle \sigma, y\rangle_{Z',Z} \leq \sup_{y\in\Lambda(z)} \langle \sigma, y\rangle_{Z',Z}. \tag{3.74}$$

Let us choose a subsequence $\{z_{n'}\}$ such that

$$\limsup_{n\to\infty} \sup_{y\in\Lambda(z_n)} \langle \sigma, y\rangle_{Z',Z} = \lim_{n'\to\infty} \sup_{y\in\Lambda(z_{n'})} \langle \sigma, y\rangle_{Z',Z}. \tag{3.75}$$

From the fact that $\Lambda(z_{n'})$ is a compact set of Z, for each n' we have

$$\sup_{y\in\Lambda(z_{n'})} \langle \sigma, y\rangle_{Z',Z} = \langle \sigma, y_{n'}\rangle_{Z',Z}$$

for some $y_{n'} \in \Lambda(z_{n'})$. Obviously, it can be assumed that

$$z_{n'}(x,t) \to z(x,t) \text{ a.e. } (x,t) \in Q \tag{3.76}$$

and

$$v_{n'} \rightharpoonup v \text{ weakly in } L^2(Q)^2 \tag{3.77}$$

with $v \in A(z)$. Furthermore, since all the $y_{n'}$ belong to a fixed compact set, we can also assume that

$$y_{n'} \to y \;\text{ in } Z.$$

This, together with (3.75)–(3.77) implies that $y \in \Lambda(z)$. As a conclusion, (3.74) holds and the proof of theorem 7 is achieved.

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Capítulo 4

Remarks on the controllability of the anisotropic Lamé system

Remarks on the controllability of the anisotropic Lamé system

S. Guerrero y O. Yu. Imanuvilov

Abstract

In this paper we establish a Carleman estimate for the elasticity system with residual stresses. As an application of this estimate we obtain exact controllability results for the same system with locally distributed control.

1. Introduction

Let us denote $x = (x_0, x')$, where x_0 (resp. x') stands for the time (resp. spatial) variable. This paper is concerned with global Carleman estimates for the Lamé system

$$\rho\partial_{x_0}^2 u_i - \sum_{j=1}^{3} \partial_{x_i}(\sigma_{ij}) = f_i \quad \text{in } Q,\ 1 \le i \le 3, \tag{4.1}$$

where Ω is a bounded domain with boundary $\partial\Omega \in C^3$, $Q = (0,T) \times \Omega$, $\mathbf{u}(x) = (u_1, u_2, u_3)$ is the displacement, $\mathbf{f} = (f_1, f_2, f_3)$ is the density of external forces and σ_{ij} is the stress tensor:

$$\sigma_{ij} = a_{ijhk}(x)\partial_{x_h} u_k.$$

On the boundary, we equip the Lamé system with zero Dirichlet boundary conditions:

$$\mathbf{u} = 0 \quad \text{on } \Sigma,$$

where we have denoted $\Sigma = (0,T) \times \partial\Omega$.

We introduce the following standard assumptions on the coefficients a_{ijhk}

$$\begin{cases} a_{ijhk} = a_{jikh} = a_{hkij}, \\ a_{ijhk} X_{ij} X_{kh} \ge \alpha X_{ij} X_{ij} \quad \forall X \in \mathbb{R}^9 \text{ with } X_{ij} = X_{ji}, \end{cases} \tag{4.2}$$

where α is some positive number.

In this paper we will strict to the case of the anisotropic Lamé system with residual free stresses R:

$$\begin{aligned} \sigma = {} & R + (\nabla u)R + \lambda(tr\epsilon)I + 2\mu\epsilon + \beta_1(tr\epsilon)(trR)I + \beta_2(trR)\epsilon \\ & + \beta_3((tr\epsilon)R + tr(\epsilon R)I) + \beta_4(\epsilon R + R\epsilon), \end{aligned} \tag{4.3}$$

where

$$\epsilon = \frac{1}{2}(\nabla u + (\nabla u)^t)$$

and

$$\nabla \cdot R = 0 \quad \text{in } Q. \tag{4.4}$$

We will assume for simplicity that $\beta_1 = \beta_2 = \beta_3 = \beta_4 = 0$ and therefore

$$\sigma = R + (\nabla u)R + \lambda(tr\epsilon)I + 2\mu\epsilon, \tag{4.5}$$

for some λ, μ and R. We will impose the following regularity and positivity assumptions on the Lamé coefficients

$$\rho,\ \lambda,\ \mu,\ R_{ij} \in C^2(\overline{\Omega}), \quad \rho > 0,\ \ \mu - R_{33} > 0, \ \ \text{and} \ \ \lambda + 2\mu - R_{33} > 0 \ \text{ in } \Omega, \tag{4.6}$$

for $i, j = 1, 2, 3$.

The first goal of this paper is to establish appropriate global Carleman estimates for the Lamé system. For displacements $\mathbf{u}$ with compact support, such estimates were obtained in the previous works [27], [25], [18]. More results are available for the isotropic Lamé system. Thus, in the stationary case we refer to [8] and [31] for displacements with compact support and [17] for displacements satisfying Dirichlet boundary conditions. For the nonstationary isotropic Lamé system, see [10] for displacements with compact support and [14]–[16] in the other case. In this paper we have extended the techniques in [15] to consider the anisotropic Lamé system (4.1) with σ given by (4.5). Our Carleman estimates will hold for displacements u satisfying zero Dirichlet coditions on Σ.

The last section of this paper is devoted to the exact controllability of the Lamé system. To our best knowledge, the first observability result for the Lamé system was proved in [24] using multipliers of the form $(x_i - x_i^0)\frac{\partial \mathbf{u}}{\partial x_i}$, which led to the observability inequality

$$E(t) = \int_\Omega (\mu|\nabla \mathbf{u}|^2 + (\lambda + \mu)|div\, \mathbf{u}|^2)dx \le C \int_{(0,T)\times\Gamma_0} \left|\frac{\partial \mathbf{u}}{\partial n}\right|^2 d\sigma$$

when the Lamé coefficients μ and λ are constants. Here, $E(t)$ is the energy. The control is exerted at $(0,T) \times \Gamma_0$ where $\Gamma_0 \subset \partial\Omega$ and homogeneous Dirichlet boundary conditions are assumed for u. Further results have been deduced in [1] for the anisotropic Lamé system by essentially the same multipliers method. Several questions concerning the approximate controllability and the uniqueness of the Lamé system were studied in [10] by means of Carleman estimates. They obtained approximate controllability with a control distributed over any open subset of the boundary for a sufficiently large time. A series of important results have been obtained quite recently in [5]–[7]. In particular, a "logarithmic type" energy decay estimate has been proved in the case where the geometrical control condition of Bardos, Lebeau, Rauch is not fulfilled. An interesting result was also proved in [32], for the isotropic Lamé system with the locally distributed control $\mathbf{v}$ of the form $\chi_\omega \mathbf{v}$, where $\mathbf{v} = (v_1, \ldots, v_n)$ and $v_n \equiv 0$. Under some geometric assumptions on the domain D he established the approximate controllability for the isotropic Lamé system.

Several works are devoted to the construction of dissipative "feedback" boundary conditions for the Lamé system. In [2], dissipative boundary conditions of the form $\sigma(u)n + Au + B\partial_t u = 0$ were introduced on the controlled part of the boundary Γ_0. Under some geometric conditions on Γ_0, they established the exponential decay of the energy

$$E(t) = \frac{1}{2}\int_\Omega (|\partial_t \mathbf{u}|^2 + \sigma_{ij}\epsilon_{ij}(\mathbf{u}))dx + \frac{1}{2}\int_{\Gamma_0} A|\mathbf{u}|^2 dS \le e^{-\omega t}$$

for the anisotropic Lamé system. In the isotropic case, stabilizing boundary feedbacks were also constructed in [13] and [26] but under less restrictive geometrical assumptions.

A closely related question is the control and stabilization of layered plate models. Concerning the control of thermoelastic systems, the uniform stabilization of thermoelastic Reissner plates were proved in [21] using thermal and mechanical boundary feedbacks. In [20], the same author proved the exact controllability of the mechanical component of a thermoelastic Kirchhoff plate using mechanical boundary controls. In one-dimensional cases, this was improved to exact null controllability (of thermal and mechanical components) by the mechanical variable on the boundary in [12]. In [22], this result was extended to the case of a three-dimensional thermoelastic Lamé system with a mechanical distributed control on a neighborhood of the boundary. In [4], a related result for the boundary control of a thermoelastic Kirchhoff plate was established, but with no restriction on the size of the coupling constant.

In the last part of this paper, we will present a controllability result for the anisotropic Lamé system with distributed controls supported by $Q_\omega = \omega \times (0, T)$, where $\omega \subset \Omega$ is a nonempty open set.

Notation: Let $x = (x_0, x')$, where $x_0 = t$ stands for the time variable. We denote by $H^{2,s}(Q)$ and $H^{1,s}(Q)$ the Hilbert spaces $H^2(Q)$ and $H^1(Q)$ with the norms

$$\|u\|^2_{H^{2,s}(Q)} = \sum_{|\alpha|=0}^{2} s^{4-2|\alpha|}\|D^\alpha u\|^2_{L^2(Q)}$$

and

$$\|u\|^2_{H^{1,s}(Q)} = s^2\int_Q |u|^2\,dx + \int_Q |\nabla u|^2\,dx,$$

respectively.

We also introduce the following norms:

$$\|\mathbf{u}\|^2_{\mathcal{B}_\phi(Q)} = \int_Q e^{2s\phi}\left(\textstyle\sum_{|\alpha|=0}^2 s^{4-2|\alpha|}|D^\alpha \mathbf{u}|^2 + s|\nabla(\nabla\times\mathbf{u})|^2 \right. \\ \left. + s^3|\nabla\times\mathbf{u}|^2 + s|\nabla(\nabla\cdot\mathbf{u})|^2 + s^3|\nabla\cdot\mathbf{u}|^2\right)dx \tag{4.7}$$

and

$$\|\mathbf{u}\|^2_{\mathcal{Y}_\phi(Q)} = \|\mathbf{u}\|^2_{\mathcal{B}_\phi(Q)} + s\left\|e^{s\phi}\frac{\partial\mathbf{u}}{\partial n}\right\|^2_{H^{1,s}(\Sigma)} + s\left\|e^{s\phi}\frac{\partial^2\mathbf{u}}{\partial n^2}\right\|^2_{L^2(\Sigma)}, \tag{4.8}$$

where ϕ is a given function.

In the sequel, we will denote by $\varepsilon(\delta)$ a positive function such that

$$\lim_{\delta\to 0^+}\varepsilon(\delta) = 0.$$

2. A Carleman estimate for the anisotropic Lamé system

Let us consider the following anisotropic Lamé system completed with Dirichlet boundary conditions:

$$\begin{cases} P(x,D)\mathbf{u} \equiv \rho(x')\partial_{x_0}^2\mathbf{u} - L(x,D)\mathbf{u} = \mathbf{f} & \text{in } Q, \\ \mathbf{u} = 0 & \text{on } \Sigma, \\ \mathbf{u}(\cdot,T) = \partial_{x_0}\mathbf{u}(\cdot,T) = \mathbf{u}(\cdot,0) = \partial_{x_0}\mathbf{u}(\cdot,0) = 0 & \text{in } \Omega, \end{cases} \tag{4.9}$$

where

$$L(x,D)\mathbf{u} = \mu\Delta\mathbf{u} + (\lambda+\mu)\nabla(\nabla\cdot\mathbf{u}) - [(R,\nabla)\nabla^t]\mathbf{u}.$$

Here, $\mathbf{f}$ and $\mathbf{u}$ are three-dimensional vector fields functions and R is a symmetric matrix-valued function, whose components will be denoted by

$$\mathbf{u} = (u_j)_{j=1}^3, \quad \mathbf{f} = (f_j)_{j=1}^3 \quad \text{and} \quad R = (R_{jk})_{j,k=1}^3.$$

Let $\xi = (\xi_0,\xi_1,\xi_2,\xi_3) = (\xi_0,\xi')$. We introduce the symbols

$$\begin{cases} p_1(x,\xi) = \rho(x')\xi_0^2 - \mu(x')(\xi_1^2+\xi_2^2+\xi_3^2) + (R(x')\xi)\xi, \\ p_2(x,\xi) = \rho(x')\xi_0^2 - (\lambda(x')+2\mu(x'))(\xi_1^2+\xi_2^2+\xi_3^2) + (R(x')\xi)\xi. \end{cases}$$

For any two smooth functions $w(x,\xi)$ and $z(x,\xi)$, let us introduce the formula for the Poisson bracket

$$\{w,z\} = \sum_{j=0}^{3}(\partial_{\xi_j}w\,\partial_{x_j}z - \partial_{\xi_j}z\,\partial_{x_j}w).$$

Through this paper we will assume the existence of a function ψ satisfying the following condition:

Condition A *There exists a function* $\psi \in C^3(\overline{Q})$ *such that*
- $|\nabla_{x'}\psi|_{\overline{Q\setminus Q_\omega}} \neq 0$
- $\{p_j,\{p_j,\psi\}\}(x,\xi) > 0$ *for all* $(x,\xi) \in (\overline{Q\setminus Q_\omega}) \times \mathbf{R}^4$ *with* $\xi \neq 0$ *such that*

$$p_j(x,\xi) = \langle\nabla_\xi p_j, \nabla\psi\rangle = 0 \quad \textit{for } j = 1,2$$

and
- $\{p_j(x,\xi - is\nabla\psi(x)), p_j(x,\xi + is\nabla\psi(x))\}/(2is) > 0$ *for all* $\xi \in \mathbf{R}^4\setminus\{0\}$ *and* $s \in \mathbf{R}\setminus\{0\}$ *satisfying*

$$p_j(x,\xi + is\nabla\psi(x)) = \langle\nabla_\xi p_j(x,\xi+is\nabla\psi(x)), \nabla\psi(x)\rangle = 0,\ x \in \overline{Q\setminus Q_\omega},\ j = 1,2.$$

On the boundary, the following is required:

$$p_1(x,\nabla\psi(x)) < 0 \quad \forall x \in \overline{\partial\Omega\times(0,T)}$$

and

$$\begin{cases} (\lambda+2\mu)\dfrac{\partial\psi}{\partial n} - \displaystyle\sum_{j=1}^{3} R_{i3}\dfrac{\partial\psi}{\partial x_i} < 0 \quad \textit{and} \quad \mu\dfrac{\partial\psi}{\partial n} - \displaystyle\sum_{j=1}^{3} R_{i3}\dfrac{\partial\psi}{\partial x_i} < 0 \\ \forall x \in (\overline{\partial\Omega \setminus \partial\omega}) \times (0,T). \end{cases}$$

Let thus fix a function ψ satisfying the previous conditions and let us set

$$\phi(x) = e^{\tau\psi(x)} \quad \forall x \in \overline{Q}$$

for some positive parameter τ which will be chosen later on.

Theorem 10 *Let $f \in H^1(Q)^3$ and let (4.2), (4.4) and (4.6) hold. Suppose there exists a function ψ satisfying condition A and such that its normal derivative is large enough. Then, there exists $\tau^* > 0$ such that, for any $\tau > \tau^*$, there exists $s^* > 0$ such that*

$$\|\mathbf{u}\|_{\mathcal{Y}_\phi(Q)} \le C(\|e^{s\phi}\mathbf{f}\|_{H^{1,s}(Q)^3} + \|\mathbf{u}\|_{\mathcal{B}_\phi(Q_\omega)}) \quad \forall s > s^* \tag{4.10}$$

for some $C > 0$ independent of s and for any solution $u \in L^2(0,T;H^2(\Omega)^3) \cap H^1(Q)^3$ of system (4.9)*.*

Proof: Let us first write down the equations verified by $\nabla\times\mathbf{u}$ and $\nabla\cdot\mathbf{u}$:

$$\begin{aligned} P_\mu(x,D)(\nabla\times\mathbf{u}) &\equiv \partial_{x_0}^2(\nabla\times\mathbf{u}) - \mu\Delta(\nabla\times\mathbf{u}) + (R,\nabla)\nabla^T(\nabla\times\mathbf{u}) \\ &= \nabla\times\mathbf{f} + \tilde{P}_1(x,D)\mathbf{u} \end{aligned} \tag{4.11}$$

and

$$\begin{aligned} P_{\lambda+2\mu}(x,D)(\nabla\cdot\mathbf{u}) &\equiv \partial_{x_0}^2(\nabla\cdot\mathbf{u}) - (\lambda+2\mu)\Delta(\nabla\cdot\mathbf{u}) + (R,\nabla)\nabla^T(\nabla\cdot\mathbf{u}) \\ &= \nabla\cdot\mathbf{f} + \tilde{P}_2(x,D)\mathbf{u}, \end{aligned} \tag{4.12}$$

where the $\{\tilde{P}_j\}$ are second order differential operators.

In this situation, we are able to apply the Carleman estimates obtained in [28] and [9] to the equations (4.11) and (4.12). Combining them, we obtain

$$\begin{aligned} &s^3\tau^3(\|\phi^{3/2}e^{s\phi}(\nabla\times\mathbf{u})\|^2_{L^2(Q)^3} + \|\phi^{3/2}e^{s\phi}(\nabla\cdot\mathbf{u})\|^2_{L^2(Q)^3}) \\ &+s\tau(\|\phi^{1/2}e^{s\phi}(\nabla\cdot\mathbf{u})\|^2_{H^1(Q)^3} + \|\phi^{1/2}e^{s\phi}(\nabla\times\mathbf{u})\|^2_{H^1(Q)^3}) \\ &\quad\le C\Bigg(\|e^{s\phi}\mathbf{f}\|^2_{H^1(Q)^3} + \sum_{|\alpha|\le 2}\|e^{s\phi}D^\alpha\mathbf{u}\|^2_{L^2(Q)^3} + s\tau\left\|\phi^{1/2}e^{s\phi}\frac{\partial\mathbf{u}}{\partial n}\right\|^2_{L^2(\Sigma)^3} \\ &\quad+s^3\tau^3\left\|\phi^{3/2}e^{s\phi}\frac{\partial(\partial_{tg}\mathbf{u})}{\partial n}\right\|^2_{L^2(\Sigma)^3} + s\tau\|\phi^{1/2}e^{s\phi}\nabla(\nabla\times\mathbf{u})\|^2_{L^2(Q_\omega)^9} \\ &\quad+s\tau\|\phi^{1/2}e^{s\phi}\nabla(\nabla\cdot\mathbf{u})\|^2_{L^2(Q_\omega)^3}) + s^3\tau^3\|\phi^{3/2}e^{s\phi}(\nabla\times\mathbf{u},\nabla\cdot\mathbf{u})\|^2_{L^2(Q_\omega)^4} \\ &\quad+\sum_{|\alpha|=0}^{2}(s\tau)^{4-2|\alpha|}\|\phi^{2-|\alpha|}e^{s\alpha}D^\alpha\mathbf{u}\|^2_{L^2(Q_\omega)^3}\Bigg) \quad \forall\tau\ge\tau_0,\ \forall s\ge s_0(\tau), \end{aligned} \tag{4.13}$$

where C is independent of s and τ. We recall that the definition of $\|\cdot\|_{\mathcal{B}_\phi(Q)}$ was given in (4.7).

Now, from the well known identity

$$\Delta \mathbf{u} = -\nabla \times (\nabla \times \mathbf{u}) + \nabla(\nabla \cdot \mathbf{u}),$$

we can use the Carleman inequality proved in [11] for the case of a (simpler) elliptic equation with homogeneous Dirichlet conditions and combine this with (4.13), so we deduce in a standard way that

$$\begin{aligned} &s^4\tau^5\|\phi^2 e^{s\phi} u\|_{L^2(Q)^3} + s^2\tau^3\|\phi e^{s\phi}\nabla u\|^2_{L^2(Q)^9} + \tau\|e^{s\phi}D^2 u\|^2_{L^2(Q)^3} \\ &\quad \le C\Bigg(\|e^{s\phi}\mathbf{f}\|^2_{H^1(Q)^3} + \sum_{|\alpha|\le 2}\|e^{s\phi}D^\alpha \mathbf{u}\|^2_{L^2(Q)^3} + s\tau\left\|\phi^{1/2}e^{s\phi}\frac{\partial \mathbf{u}}{\partial n}\right\|^2_{L^2(\Sigma)^3} \\ &\quad +s^3\tau^3\left\|\phi^{3/2}e^{s\phi}\frac{\partial(\partial_{tg}\mathbf{u})}{\partial n}\right\|^2_{L^2(\Sigma)^3} + s\tau\|\phi^{1/2}e^{s\phi}\nabla(\nabla\times\mathbf{u})\|^2_{L^2(Q_\omega)^9} \\ &\quad +s\tau\|\phi^{1/2}e^{s\phi}\nabla(\nabla\cdot\mathbf{u})\|^2_{L^2(Q_\omega)^3}) + s^3\tau^3\|\phi^{3/2}e^{s\phi}(\nabla\times\mathbf{u},\nabla\cdot\mathbf{u})\|^2_{L^2(Q_\omega)^4} \\ &\quad + \sum_{|\alpha|=0}^{2}(s\tau)^{4-2|\alpha|}\|\phi^{2-|\alpha|}e^{s\alpha}D^\alpha\mathbf{u}\|^2_{L^2(Q_\omega)^3}\Bigg) \quad \forall \tau\ge\tau_1,\ s\ge s_1(\tau). \end{aligned} \tag{4.14}$$

Taking τ large enough the global term in the right hand side concerning $\mathbf{u}$ can be absorbed.

On the other hand, from this point of the proof we forget about the dependence on τ and possible powers of ϕ (which is a regular function) in our inequalities, since it will not be crucial. Consequently, for the moment, we have

$$\begin{aligned} \|\mathbf{u}\|_{\mathcal{Y}_\phi(Q)} \le C\Big(&\|e^{s\phi}\mathbf{f}\|^2_{H^{1,s}(Q)^3} + s\left\|e^{s\phi}\frac{\partial\mathbf{u}}{\partial n}\right\|^2_{H^{1,s}(\Sigma)^3} + s\left\|e^{s\phi}\frac{\partial^2\mathbf{u}}{\partial n^2}\right\|^2_{L^2(\Sigma)^3} \\ &+\|\mathbf{u}\|^2_{\mathcal{B}_\phi(Q_\omega)}\Big) \quad \forall\tau\ge\tau_1,\ s\ge s_1(\tau). \end{aligned} \tag{4.15}$$

The norm $\|\cdot\|_{\mathcal{Y}_\phi(Q)}$ was introduced in (4.8).

The goal will be now to estimate the previous boundary terms. For this purpose, we consider another weight function φ such that $\varphi = \phi$ on Σ. For instance, let us take

$$\varphi = e^{\tau\tilde{\psi}} \quad \text{with} \quad \tilde{\psi} = \psi - \frac{1}{Z^2}\ell_1 + Z\ell_1^2,$$

where Z is a large positive number and ℓ_1 is a regular function verifying

$$\ell_1 = 0 \text{ on } \partial\Omega, \quad \ell_1 > 0 \text{ in } \Omega \quad \text{and} \quad \nabla\ell_1 \ne 0 \text{ on } \partial\Omega.$$

Moreover, if we set

$$\Omega_{1/Z^2} = \{x' = (x_1,x_2,x_3)\in\Omega : dist(x',\partial\Omega) < 1/Z^2\},$$

we can suppose that the function ℓ_1 is chosen such that

$$\varphi(x) < \phi(x) \quad \forall x \in \Omega_{1/Z^2} \times (0,T)$$

provided Z is large enough.

For this new weight function, we will be able to prove the following lemma:

Lemma 11 *Under the previous conditions, the following inequality holds*

$$\|\mathbf{u}\|_{\mathcal{Y}_\varphi(Q)} \leq C\left(\|e^{s\varphi}\mathbf{f}\|_{H^{1,s}(Q)^3} + \|\mathbf{u}\|_{\mathcal{B}_\varphi(Q_\omega)}\right) \quad \forall s \geq s_0, \tag{4.16}$$

for functions $\mathbf{u}$ *verifying*

$$supp\, \mathbf{u} \subset \overline{\Omega}_{1/Z^2} \times [0,T].$$

Let us suppose that lemma 11 holds and let us deduce theorem 10 from it. Suppose that Z is already fixed such that (4.16) holds and take $\varepsilon \in (0, 1/Z^2)$. Then we have

$$\varphi(x) < \phi(x) \quad \forall x \in (\overline{\Omega_\varepsilon \setminus \Omega_{\varepsilon/2}}) \times [0,T]. \tag{4.17}$$

Let us now introduce a cut-off function $\theta \in C_c^2(\Omega_\varepsilon)$ such that $\theta \equiv 1$ in $\Omega_{\varepsilon/2}$. Then, it readily follows that the function $\theta\mathbf{u}$ fulfills the following system

$$\begin{cases} P(x,D)(\theta\mathbf{u}) \equiv \theta\mathbf{f} + [P,\theta](x,D)\mathbf{u} & \text{in } Q, \\ \theta\mathbf{u} = 0 & \text{on } \Sigma, \\ (\theta\mathbf{u})(\cdot,T) = \partial_{x_0}(\theta\mathbf{u})(\cdot,T) = (\theta\mathbf{u})(\cdot,0) = \partial_{x_0}(\theta\mathbf{u})(\cdot,0) = 0 & \text{in } \Omega. \end{cases}$$

Consequently, we can apply estimate (4.16) to $\theta\mathbf{u}$ and deduce that

$$\begin{aligned} s\left\|e^{s\phi}\frac{\partial\mathbf{u}}{\partial n}\right\|^2_{H^{1,s}(\partial\Omega\times(0,T))^3} &+ s\left\|e^{s\phi}\frac{\partial^2\mathbf{u}}{\partial n^2}\right\|^2_{L^2(\partial\Omega\times(0,T))^3} \\ &\leq C\left(\|e^{s\varphi}\mathbf{f}\|^2_{H^{1,s}(Q)^3} + \|e^{s\varphi}[P,\theta]\mathbf{u}\|^2_{H^{1,s}(Q)^3} + \|\mathbf{u}\|^2_{\mathcal{B}_\varphi(Q_\omega)}\right) \quad \forall s \geq s_0, \end{aligned} \tag{4.18}$$

since $\varphi \equiv \phi$ on the boundary. Next, we observe that the support of the function $[P,\theta]\mathbf{u}$ is contained in $\overline{\Omega_\varepsilon \setminus \Omega_{\varepsilon/2}} \times [0,T]$, while $\varphi < \phi$ in that set (see (4.17)). Thus,

$$\|e^{s\varphi}[P,\theta]\mathbf{u}\|_{H^{1,s}(Q)^3} \leq C\sum_{|\alpha|=0}^{2}\|e^{s\phi}D^\alpha\mathbf{u}\|_{L^2(Q)^3} \quad \forall s > 0.$$

Finally, we put this together with (4.18) and (4.14) and we obtain the desired inequality (4.10).

This ends the proof of theorem 10.

Proof of Lemma 11: Assume that $supp\,\mathbf{u} \subset B_\delta \cap ([0,T] \times \overline{\Omega}_{1/Z^2})$ and that the small part of the boundary where we are working on is given by the equation

$$x_3 = \ell(x_1, x_2).$$

Without lack of generality, the function $\ell \in C^3$ can be taken to satisfy

$$\nabla' \ell(0,0) = (\ell_{x_1}, \ell_{x_2})(0,0) = 0.$$

In order to work in an appropriate frame, we perform the change of variables

$$\begin{cases} y_1 = x_1, \\ y_2 = x_2, \\ y_3 = x_3 - \ell(x_1, x_2) \end{cases}$$

and we set $y^* = (y_0, 0, 0, 0)$. Let us denote by $\mathbf{w} = (\mathbf{w}', w_4)$ the functions $\nabla \times \mathbf{u}$ and $\nabla \cdot \mathbf{u}$ in the new coordinates. Then, some simple computations show that, in the new variables, the main symbols of equations (4.11) and (4.12) are

$$\begin{aligned} p_\beta(y, \xi) = -\xi_0^2 + (\beta - R_{11})\xi_1^2 + (\beta - R_{22})\xi_2^2 + \{[(\beta E_3 - R)G^t]G\}\xi_3^2 \\ -2R_{12}\xi_1\xi_2 - 2\textstyle\sum_{j=1}^2 (R_{j3} + \ell_{y_j}(\beta - R_{jj}) - \ell_{y_{3-j}} R_{12})\xi_j\xi_3, \end{aligned}$$

where

$$\beta = \begin{cases} \mu & \text{for } \nabla \times u, \\ \lambda + 2\mu & \text{for } \nabla \cdot u \end{cases}$$

and

$$G = (-\ell_{y_1}, -\ell_{y_2}, 1)^t. \tag{4.19}$$

Let us set some other standard ingredients in the microlocal analysis frame. Thus, we consider the unit sphere in $\mathbf{R}^4$, say

$$S^3 = \{\zeta = (s, \xi') : s^2 + \xi_0^2 + \xi_1^2 + \xi_2^2 = 1\}$$

and the following associated finite covering:

$$\{\zeta \in S^3 : |\zeta - \zeta_\nu^*| < \delta_1\}_{1 \le \nu \le M(\delta_1)},$$

with $\zeta_\nu^* \in S^3$. To this covering we associate the partition of unity $\{\chi_\nu\}_{1 \le \nu \le M}$, extending χ_ν out of S^3 like a homogenous function of order 0 with support contained in the conic neighborhood

$$\mathcal{O}(\delta_1) = \left\{ \zeta : \left| \frac{\zeta}{|\zeta|} - \zeta_\nu^* \right| < \delta_1 \right\}.$$

In order to finish the proof, we need another lemma:

Lemma 12 *Let $\gamma^* = (y^*, \zeta^*) \in \partial\mathcal{G} \times S^3$ be fixed and supp $\chi_\nu \subset \mathcal{O}(\delta_1)$. Then, for sufficiently small δ and δ_1, we have*

$$s\|\mathbf{z}_\nu\|^2_{H^{1,s}(\mathcal{G})^4} + s(\|\mathbf{z}_\nu\|^2_{H^{1,s}(\partial\mathcal{G})^4} + \|\partial_{y_3}\mathbf{z}_\nu\|^2_{L^2(\partial\mathcal{G})^4}) \le C(\|e^{s\varphi}\mathbf{f}\|^2_{H^{1,s}(\mathcal{G})^3} + \|\mathbf{z}\|^2_{H^{1,s}(\mathcal{G})^4}). \tag{4.20}$$

Here, we have denoted $\mathcal{G} = \mathbf{R}^3 \times [0, 1/Z^2]$ and $\mathbf{z}_\nu = \chi_\nu(s, D')\mathbf{z}$, with $\mathbf{z} = e^{s\varphi}\mathbf{w}$.

Let us suppose that lemma 12 holds. Then we have

$$\begin{aligned} & s\|\mathbf{z}\|^2_{H^{1,s}(\mathcal{G})^4} + s(\|\mathbf{z}\|^2_{H^{1,s}(\partial\mathcal{G})^4} + \|\partial_{y_3}\mathbf{z}\|^2_{L^2(\partial\mathcal{G})^4}) \\ & \quad \leq Cs\sum_{\nu=1}^{M}(\|\mathbf{z}_\nu\|^2_{H^{1,s}(\mathcal{G})^4} + \|\mathbf{z}_\nu\|^2_{H^{1,s}(\partial\mathcal{G})^4} + \|\partial_{y_3}\mathbf{z}_\nu\|^2_{L^2(\partial\mathcal{G})^4}) \\ & \quad \leq C(\|e^{s\varphi}\mathbf{f}\|^2_{H^{1,s}(\mathcal{G})^3} + \|e^{s\varphi}\mathbf{u}\|^2_{H^{2,s}(\mathcal{G})^3}). \end{aligned} \tag{4.21}$$

Let us now use the result stated in proposition 4.2 of [14]:

$$Z\sum_{|\alpha|=0}^{2} s^{4-2|\alpha|}\|e^{s\varphi}D^\alpha_{y'}\mathbf{u}\|^2_{L^2(\mathcal{G})^3} \leq C(\|e^{s\varphi}\mathbf{f}\|^2_{H^{1,s}(\mathcal{G})^3} + \|e^{s\varphi}\mathbf{u}\|^2_{H^{2,s}(\mathcal{G})^3}),$$

where C is independent of s and Z.

From the differential equation in (4.9), we deduce that $\|e^{s\varphi}\partial^2_{y_0}\mathbf{u}\|^2_{L^2(\mathcal{G})^3}$ can be added to the left hand side of the previous inequality. The same can be said of $\|e^{s\varphi}\partial^2_{y_0y_1}\mathbf{u}\|^2_{L^2(\mathcal{G})^3}$ in view of well known interpolation arguments. Consequently, we have

$$Z\sum_{|\alpha|=0}^{2} s^{4-2|\alpha|}\|e^{s\varphi}D^\alpha\mathbf{u}\|^2_{L^2(\mathcal{G})^3} \leq C\|e^{s\varphi}\mathbf{f}\|^2_{H^{1,s}(\mathcal{G})^3}.$$

Combining this with (4.21), we obtain:

$$\sum_{|\alpha|=0}^{2} s^{4-2|\alpha|}\|e^{s\varphi}D^\alpha\mathbf{u}\|^2_{L^2(\mathcal{G})^3} + s(\|\mathbf{z}\|^2_{H^{1,s}(\partial\mathcal{G})^4} + \|\partial_{y_3}\mathbf{z}\|^2_{L^2(\partial\mathcal{G})^4}) \leq C\|e^{s\varphi}\mathbf{f}\|^2_{H^{1,s}(\mathcal{G})^3}. \tag{4.22}$$

Let us finally see that we can deduce inequality (4.16) from (4.22). To do this, we just have to provide 'good' estimates for the terms

$$s\,\|e^{s\varphi}\partial_{y_3}\mathbf{u}\|^2_{H^{1,s}(\partial\mathcal{G})^3} \quad \text{and} \quad s\,\left\|e^{s\varphi}\partial^2_{y_3}\mathbf{u}\right\|^2_{L^2(\partial\mathcal{G})^3}.$$

From the definition of the y variables in terms of x and the Dirichlet boundary condition on u, we see that the following estimates hold:

$$|\partial_{y_3}u_j| \leq |(\nabla\times u)_{3-j}| + \varepsilon(\delta)|\partial_{y_3}\mathbf{u}| \quad \text{on } \partial\mathcal{G} \quad \text{for } j=1,2,$$

$$|\partial_{y_3}u_3| \leq |\nabla\cdot\mathbf{u}| + \varepsilon(\delta)|\partial_{y_3}\mathbf{u}| \quad \text{on } \partial\mathcal{G}.$$

These two estimates tell that

$$|e^{s\varphi}\partial_{y_3}\mathbf{u}| \leq |z_1| + |z_2| + |z_4| \quad \text{on } \partial\mathcal{G}. \tag{4.23}$$

Additionally,

$$|\partial^2_{y_jy_3}u_k| \leq |\partial_{y_j}(\nabla\times\mathbf{u})_{3-k}| + \varepsilon(\delta)|\partial_{y_jy_3}\mathbf{u}| \quad \text{on } \partial\mathcal{G} \quad \text{for } j=0,1,2,\ k=1,2$$

and

$$|\partial^2_{y_j y_3} u_3| \le |\partial_{y_j}(\nabla\cdot\mathbf{u})| + \varepsilon(\delta)|\partial_{y_j y_3}\mathbf{u}| \quad \text{on } \partial\mathcal{G} \quad \text{for } j=0,1,2,$$

whence we deduce that

$$|e^{s\varphi}\nabla^{tg}_y \partial_{y_3}\mathbf{u}| \le |\nabla^{tg}_y z_1| + |\nabla^{tg}_y z_2| + |\nabla^{tg}_y z_4| \quad \text{on } \partial\mathcal{G}. \tag{4.24}$$

Finally, we have

$$|\partial^2_{y_3} u_j| \le |\partial_{y_3}(\nabla\times\mathbf{u})_{3-j}| + |\partial_{y_j}(\nabla\cdot\mathbf{u})| + \varepsilon(\delta)|\partial_{y_3}\nabla_y\mathbf{u}| \quad \text{on } \partial\mathcal{G} \quad \text{for } j=1,2$$

and

$$|\partial^2_{y_3} u_3| \le |\partial_{y_1}(\nabla\times\mathbf{u})_2| + |\partial_{y_2}(\nabla\times\mathbf{u})_1| + |\partial_{y_3}(\nabla\cdot\mathbf{u})| + \varepsilon(\delta)|\partial_{y_3}\nabla_y\mathbf{u}| \quad \text{on } \partial\mathcal{G},$$

which lead to the estimate

$$|e^{s\varphi}\partial^2_{y_3}\mathbf{u}| \le |\nabla_y z_1| + |\nabla_y z_2| + |\nabla_y z_4| + \varepsilon(\delta)|e^{s\varphi}\partial_{y_3}\nabla^{tg}_y\mathbf{u}| \quad \text{on } \partial\mathcal{G}. \tag{4.25}$$

One can readily see that (4.23)–(4.25) imply that

$$\|e^{s\varphi}\partial_{y_3}\mathbf{u}\|^2_{H^{1,s}(\partial\mathcal{G})^3} + \left\|e^{s\varphi}\partial^2_{y_3}\mathbf{u}\right\|^2_{L^2(\partial\mathcal{G})} \le C(\|\mathbf{z}\|^2_{H^{1,s}(\partial\mathcal{G})^4} + \|\partial_{y_3}\mathbf{z}\|^2_{L^2(\partial\mathcal{G})^4}),$$

as we wanted to prove.

As a conclusion it suffices to prove lemma 12, so the rest of this section will be dedicated to it.

Proof of lemma 12: Let us introduce the notation

$$\mathbb{D} = D + is\nabla\varphi.$$

Then the main symbol of the differential operator $P_\beta(y,\mathbb{D})$ is

$$\begin{aligned} p_{\beta,s}(y,s,\xi) = &-(\xi_0+is\varphi_{y_0})^2 + (\beta-R_{11})(\xi_1+is\varphi_{y_1})^2 \\ &+(\beta-R_{22})(\xi_2+is\varphi_{y_2})^2 + \{[(\beta E_3 - R)G^t]G\}(\xi_3+is\varphi_{y_3})^2 \\ &-2R_{12}(\xi_1+is\varphi_{y_1})(\xi_2+is\varphi_{y_2}) \\ &-2\textstyle\sum_{j=1}^2 (R_{j3}+\ell_{y_j}(\beta-R_{jj})-\ell_{y_{3-j}}R_{12})(\xi_j+is\varphi_{y_j})(\xi_3+is\varphi_{y_3}). \end{aligned} \tag{4.26}$$

We recall that $G=(-\ell_{y_1},-\ell_{y_2},1)^t$. The roots of this polynomial with respect to the ξ_3 variable are

$$\Gamma^\pm_\beta(y,s,\xi') = -is\varphi_{y_3} + \alpha^\pm_\beta(y,s,\xi'), \tag{4.27}$$

where

$$\alpha^\pm_\beta = \frac{\sum_{j=1}^2 (R_{j3}+\ell_{y_j}(\beta-R_{jj})-\ell_{y_{3-j}}R_{12})(\xi_j+is\varphi_{y_j}) \pm \sqrt{r_\beta(y,s,\xi')}}{[(\beta E_3-R)G^t]G}$$

and

$$\begin{aligned} r_\beta(y,s,\xi') = & \Big(\textstyle\sum_{j=1}^2 (R_{j3} + \ell_{y_j}(\beta - R_{jj}) - \ell_{y_{3-j}} R_{12})(\xi_j + is\varphi_{y_j})\Big)^2 \\ & -[(\beta E_3 - R)G^t]G(-(\xi_0 + is\varphi_{y_0})^2 + (\beta - R_{11})(\xi_1 + is\varphi_{y_1})^2 \\ & +(\beta - R_{22})(\xi_2 + is\varphi_{y_2})^2 - 2R_{12}(\xi_1 + is\varphi_{y_1})(\xi_2 + is\varphi_{y_2})). \end{aligned} \tag{4.28}$$

It will be useful for the sequel to factorize $P_{\beta,s}$ as the product of two first order operators. This is made in the following proposition:

Proposition 8 *Let $\beta \in \{\mu, \lambda + 2\mu\}$ and $|r_\beta(\gamma)| \geq C > 0$ for all $\gamma \in (B_\delta \cap \mathcal{G}) \times \mathcal{O}(2\delta_1)$. Then, for any function v such that supp $v \subset B_\delta \cap \mathcal{G}$, we have*

$$P_{\beta,s}(y,D)v_\nu = [(\beta E_3 - R)G^t]G(D_{y_3} - \Gamma_\beta^-(y,s,D'))(D_{y_3} - \Gamma_\beta^+(y,s,D'))v_\nu + T_{\beta,s}v_\nu,$$

where $T_{\beta,s}$ is a continuous operator

$$T_{\beta,s} : H^{1,s}(\mathcal{G})^3 \longmapsto L^2(\mathcal{G})^3.$$

Once this decomposition can be done, one would desire to obtain appropriate estimates of some norms of v_ν. More precisely, let $\tilde{v}$ satisfy

$$(D_{y_3} - \Gamma_\beta^-(y,s,D'))\tilde{v}_\nu = q, \quad \tilde{v}|_{y_3 = 1/Z^2} = 0, \quad \operatorname{supp} \tilde{v} \subset B_\delta \cap \mathcal{G}.$$

We can then prove the following:

Proposition 9 *Let $\beta \in \{\mu, \lambda + 2\mu\}$ and $|r_\beta(\gamma)| \geq C > 0$ for all $\gamma \in B_\delta \times \mathcal{O}(2\delta_1)$. Then,*

$$s\|\tilde{v}_\nu\|^2_{L^2(\partial\mathcal{G})} \leq C\|q\|^2_{L^2(\partial\mathcal{G})},$$

for some positive constant C independent of s and Z.

Proposition 8 and proposition 9 can both be proved as in [14].

The next step will be to obtain a Carleman inequality for a function satisfying our (second order hyperbolic) differential equation but with no imposed boundary conditions. This will be fundamental to obtain the desired estimate (4.20). Indeed, let w satisfy

$$P_\beta(y, I\!\!D)w = g \text{ in } \mathcal{G}, \quad \operatorname{supp} w \subset B_\delta \times [0, 1/Z^2).$$

Let us denote by $P_\beta^*(\cdot, I\!\!D)$ the adjoint operator of $P_\beta(\cdot, I\!\!D)$ ($\beta \in \{\mu, \lambda + 2\mu\}$) and let us set

$$L_{+,\beta}(\cdot, I\!\!D) = \frac{P_\beta(\cdot, I\!\!D) + P_\beta^*(\cdot, I\!\!D)}{2}, \quad L_{-,\beta}(\cdot, I\!\!D) = \frac{P_\beta(\cdot, I\!\!D) - P_\beta^*(\cdot, I\!\!D)}{2}.$$

We have

$$L_{+,\beta}(\cdot, I\!\!D)w + L_{-,\beta}(\cdot, I\!\!D)w = g.$$

After several computations involving integration by parts, we get

$$\|L_{+,\beta}(\cdot, I\!\!D)w\|^2_{L^2(\mathcal{G})} + \|L_{-,\beta}(\cdot, I\!\!D)w\|^2_{L^2(\mathcal{G})} + \operatorname{Re} \int_{\mathcal{G}} [L_{+,\beta}, L_{-,\beta}]w \, dy + \Sigma_\beta(w) = \|g\|^2_{L^2(\mathcal{G})}, \tag{4.29}$$

with $\Sigma_\beta(w) = (\Sigma_\beta^1 + \Sigma_\beta^2)(w)$, where Σ_β^1 can be written as $\Sigma_\beta^{1,1} + \Sigma_\beta^{1,2} + \Sigma_\beta^{1,3}$ with

$$\Sigma_\beta^{1,1}(w) = s \int_{\partial\mathcal{G}} (\beta - R_{33})^2(y^*)\varphi_{\tilde{y}_3}(y^*)(|\partial_{\tilde{y}_3} w|^2 + s^2\varphi_{\tilde{y}_3}^2 |w|^2)\, dy', \tag{4.30}$$

$$\begin{aligned}\Sigma_\beta^{1,2}(w) = s \int_{\partial\mathcal{G}} &(\beta - R_{33})(y^*)\varphi_{\tilde{y}_3}(y^*)\, \Big[|\partial_{y_0} w|^2 - s^2\varphi_{y_0}^2(y^*)|w|^2 \\ &- \left(\beta - R_{11} - \frac{R_{13}^2}{\beta - R_{33}}\right)(y^*)(|\partial_{y_1} w|^2 - s^2\varphi_{y_1}^2(y^*)|w|^2) \\ &- \left(\beta - R_{22} - \frac{R_{23}^2}{\beta - R_{33}}\right)(y^*)(|\partial_{y_2} w|^2 - s^2\varphi_{y_2}^2(y^*)|w|^2) \\ &+2\left(R_{12} + \frac{R_{13}R_{23}}{\beta - R_{33}}\right)(y^*)(\partial_{y_1} w\, \partial_{y_2} w - s^2\varphi_{y_1}(y^*)\varphi_{y_2}(y^*)|w|^2)\Big]\, dy'\end{aligned} \tag{4.31}$$

and

$$\begin{aligned}\Sigma_\beta^{1,3}(w) = -2s\,\mathrm{Re} \int_{\partial\mathcal{G}} &((\beta - R_{33})(y^*)\partial_{y_3} w - R_{13}\partial_{y_1} w - R_{23}\partial_{y_2} w) \\ &\Bigg(\overline{\varphi_{y_0}(y^*)\partial_{y_0} w - \left(\beta - R_{11} - \frac{R_{13}^2}{\beta - R_{33}}\right)(y^*)\varphi_{y_1}(y^*)\partial_{y_1} w} \\ &\overline{- \left(\beta - R_{22} - \frac{R_{23}^2}{\beta - R_{33}}\right)(y^*)\varphi_{y_2}(y^*)\partial_{y_2} w} \\ &\overline{+ \left(R_{12} + \frac{R_{13}R_{23}}{\beta - R_{33}}\right)(y^*)(\varphi_{y_2}(y^*)\partial_{y_1} w + \varphi_{y_1}(y^*)\partial_{y_2} w)}\Bigg)\, dy'\end{aligned} \tag{4.32}$$

and

$$\Sigma_\beta^2(w) \le \varepsilon(\delta) s \left(\|w\|^2_{H^{1,s}(\partial\mathcal{G})} + \|\partial_{y_3} w\|^2_{L^2(\partial\mathcal{G})}\right), \tag{4.33}$$

with $\varepsilon(\delta) \to 0$ when $\delta \to 0^+$. In (4.30) and (4.31), we have used the notation

$$\partial_{\tilde{y}_3} = -\frac{R_{13}}{\beta - R_{33}}(y^*)\partial_{y_1} - \frac{R_{23}}{\beta - R_{33}}(y^*)\partial_{y_2} + \partial_{y_3}. \tag{4.34}$$

In fact, the expressions of $L_{+,\beta}(\cdot, \mathbb{D})w$ and $L_{-,\beta}(\cdot, \mathbb{D})w$ are

$$\begin{aligned}L_{+,\beta}(y, \mathbb{D})w = \partial_{y_0}^2 w + s^2\varphi_{y_0}^2 w &- (\beta - R_{11})(\partial_{y_1}^2 w + s^2\varphi_{y_1}^2 w) \\ &- (\beta - R_{22})(\partial_{y_2}^2 w + s^2\varphi_{y_2}^2 w) - (\beta - R_{33})(\partial_{y_3}^2 w + s^2\varphi_{y_3}^2 w) \\ &+ 2R_{12}(\partial_{y_1y_2}^2 w + s^2\varphi_{y_1}\varphi_{y_2} w) + 2R_{13}(\partial_{y_1y_3}^2 w + s^2\varphi_{y_1}\varphi_{y_3} w) \\ &+ 2R_{23}(\partial_{y_2y_3}^2 w + s^2\varphi_{y_2}\varphi_{y_3} w)\end{aligned}$$

and

$$L_{-,\beta}(y,\mathbb{D})w = s\,(-\partial_{y_0}(\varphi_{y_0}w) - \varphi_{y_0}\partial_{y_0}w + (\beta - R_{11})(\partial_{y_1}(\varphi_{y_1}w) + \varphi_{y_1}\partial_{y_1}w)$$
$$+ (\beta - R_{22})(\partial_{y_2}(\varphi_{y_2}w) + \varphi_{y_2}\partial_{y_2}w) + (\beta - R_{33})(\partial_{y_3}(\varphi_{y_3}w) + \varphi_{y_3}\partial_{y_3}w)$$
$$- R_{12}(\partial_{y_2}(\varphi_{y_1}w) + \varphi_{y_1}\partial_{y_2}w + \partial_{y_1}(\varphi_{y_2}w) + \varphi_{y_2}\partial_{y_1}w)$$
$$- R_{13}(\partial_{y_3}(\varphi_{y_1}w) + \varphi_{y_1}\partial_{y_3}w + \partial_{y_1}(\varphi_{y_3}w) + \varphi_{y_3}\partial_{y_1}w)$$
$$- R_{23}(\partial_{y_2}(\varphi_{y_3}w) + \varphi_{y_3}\partial_{y_2}w + \partial_{y_3}(\varphi_{y_2}w) + \varphi_{y_2}\partial_{y_3}w))\,.$$

One just has to integrate by parts, keeping the boundary terms in order to conclude that

$$\Sigma_\beta(w) = \Sigma^1_\beta(w) + \Sigma^2_\beta(w),$$

where $\Sigma^1_\beta(w)$ is given by (4.30)–(4.32) and Σ^2_β verifies the estimate (4.33).

Using (4.29), one can prove in the same way as in Appendix II of [14] that the following inequality holds:

$$s\|w\|_{H^{1,s}(\mathcal{G})} \leq \|L_{+,\beta}(\cdot,\mathbb{D})w\|^2_{L^2(\mathcal{G})} + \|L_{-,\beta}(\cdot,\mathbb{D})w\|^2_{L^2(\mathcal{G})} + \operatorname{Re}\left([L_{+,\beta}, L_{-,\beta}]w, w\right)_{L^2(\mathcal{G})}$$
$$+ sC\|w\|_{L^2(\partial\mathcal{G})}\|\partial_{y_3}w\|_{L^2(\partial\mathcal{G})} \quad \forall s \geq s_0.$$

Combining this with (4.29), we get

$$C_1 s\|w\|^2_{H^{1,s}(\mathcal{G})} + \Sigma_\beta(w) \leq C_2(\|g\|^2_{L^2(\mathcal{G})} + s\|w\|_{L^2(\partial\mathcal{G})}\|\partial_{y_3}w\|_{L^2(\partial\mathcal{G})}) \quad \forall s \geq s_0. \tag{4.35}$$

Having this inequality in mind, we will prove lemma 12 distinguishing several cases according to the values of $r_\beta(\gamma^*)$ (recall that $\gamma^* = (y^*, \zeta^*)$):

- **First Case:** $r_\mu(\gamma^*) = 0$, $r_{\lambda+2\mu}(\gamma^*) \neq 0$.
- **Second case:** $r_{\lambda+2\mu}(\gamma^*) = 0$, $r_\mu(\gamma^*) \neq 0$.
- **Third Case:** $r_\mu(\gamma^*) \neq 0$, $r_{\lambda+2\mu}(\gamma^*) \neq 0$ or $r_\mu(\gamma^*) = r_{\lambda+2\mu}(\gamma^*) = 0$.

2.1. First Case: $r_\mu(\gamma^*) = 0$, $r_{\lambda+2\mu}(\gamma^*) \neq 0$.

In this situation, taking δ and δ_1 small enough, one can suppose that

$$|r_{\lambda+2\mu}(\gamma)| \geq C > 0 \quad \forall \gamma = (y,\zeta) \in B_\delta \times (\mathcal{O}(\delta_1) \cap \{|\zeta| \geq 1\}). \tag{4.36}$$

Let us start applying estimate (4.35) to $\mathbf{z}'_\nu = \chi_\nu(s, D')e^{s\varphi}(\nabla\times\mathbf{u})$. Recall that $\mathbf{w} = (\mathbf{w}', w_4) = (\nabla\times\mathbf{u}, \nabla\cdot\mathbf{u})$, where the differential operators are taken in the y variables. This yields

$$s\|\mathbf{z}'_\nu\|^2_{H^{1,s}(\mathcal{G})^3} + \Sigma^1_\mu(\mathbf{z}'_\nu) \leq \varepsilon(\delta)s(\|\mathbf{z}'_\nu\|^2_{H^{1,s}(\partial\mathcal{G})^3} + \|\partial_{y_3}\mathbf{z}'_\nu\|^2_{L^2(\partial\mathcal{G})^3})$$
$$+ C(\|e^{s\varphi}\mathbf{f}\|^2_{H^{1,s}(\mathcal{G})^3} + \|\mathbf{z}\|^2_{H^{1,s}(\mathcal{G})^4}), \tag{4.37}$$

with $\Sigma^1_\mu(\mathbf{z}'_\nu)$ given by (4.30)–(4.32). Let us rewrite the boundary terms in the form

$$\Sigma^1_\mu(\mathbf{z}'_\nu) = \Sigma^{1,1}_\mu(\mathbf{z}'_\nu) + \Sigma^{1,2}_\mu(\mathbf{z}'_\nu) + \Sigma^{1,3}_\mu(\mathbf{z}'_\nu), \tag{4.38}$$

with

$$\Sigma_\mu^{1,1}(\mathbf{z}'_\nu) = s\int_{\partial\mathcal{G}} (\mu - R_{33})^2(y^*)\varphi_{\tilde{y}_3}(y^*)(|\partial_{\tilde{y}_3}\mathbf{z}'_\nu|^2 + s^2\varphi^2_{\tilde{y}_3}|\mathbf{z}'_\nu|^2)\,dy',$$

$$\begin{aligned}\Sigma_\mu^{1,2}(\mathbf{z}'_\nu) = s\int_{\partial\mathcal{G}} &(\mu - R_{33})(y^*)\varphi_{\tilde{y}_3}(y^*)\left[|\partial_{y_0}\mathbf{z}'_\nu|^2 - s^2\varphi^2_{y_0}(y^*)|\mathbf{z}'_\nu|^2\right.\\ &-\left(\mu - R_{11} - \frac{R_{13}^2}{\mu - R_{33}}\right)(y^*)(|\partial_{y_1}\mathbf{z}'_\nu|^2 - s^2\varphi^2_{y_1}(y^*)|\mathbf{z}'_\nu|^2)\\ &-\left(\mu - R_{22} - \frac{R_{23}^2}{\mu - R_{33}}\right)(y^*)(|\partial_{y_2}\mathbf{z}'_\nu|^2 - s^2\varphi^2_{y_2}(y^*)|\mathbf{z}'_\nu|^2)\\ &\left.+2\left(R_{12} + \frac{R_{13}R_{23}}{\mu - R_{33}}\right)(y^*)(\partial_{y_1}\mathbf{z}'_\nu\,\overline{\partial_{y_2}\mathbf{z}'_\nu} - s^2\varphi_{y_1}(y^*)\varphi_{y_2}(y^*))|\mathbf{z}'_\nu|^2\right]dy'\end{aligned}$$

and

$$\begin{aligned}\Sigma_\mu^{1,3}(\mathbf{z}'_\nu) = -2s\,\mathrm{Re}\int_{\partial\mathcal{G}} &\big((\mu - R_{33})(y^*)\partial_{y_3}\mathbf{z}'_\nu - R_{13}(y^*)\partial_{y_1}\mathbf{z}'_\nu - R_{23}(y^*)\partial_{y_2}\mathbf{z}'_\nu\big)\\ &\times\left(\overline{\varphi_{y_0}(y^*)\partial_{y_0}\mathbf{z}'_\nu - \left(\mu - R_{11} - \frac{R_{13}^2}{\mu - R_{33}}\right)(y^*)\varphi_{y_1}(y^*)\partial_{y_1}\mathbf{z}'_\nu}\right.\\ &\overline{-\left(\mu - R_{22} - \frac{R_{23}^2}{\mu - R_{33}}\right)(y^*)\varphi_{y_2}(y^*)\partial_{y_2}\mathbf{z}'_\nu}\\ &\left.\overline{+\left(R_{12} + \frac{R_{13}R_{23}}{\mu - R_{33}}\right)(y^*)(\varphi_{y_2}(y^*)\partial_{y_1}\mathbf{z}'_\nu + \varphi_{y_1}(y^*)\partial_{y_2}\mathbf{z}'_\nu)}\right)dy'.\end{aligned} \tag{4.39}$$

Taking into account (4.28), we observe that

$$\begin{aligned}&0 = \mathrm{Re}\, r_\mu(\gamma^*) = R_{13}^2[(\xi_1^*)^2 - (s^*\varphi_{y_1}(y^*))^2] + R_{23}^2[(\xi_2^*)^2 - (s^*\varphi_{y_2}(y^*))^2]\\ &+2R_{13}(y^*)R_{23}(y^*)[\xi_1^*\xi_2^* - (s^*)^2\varphi_{y_1}(y^*)\varphi_{y_2}(y^*)]\\ &-(\mu - R_{33})(y^*)\{[-(\xi_0^*)^2 + (s^*\varphi_{y_0}(y^*))^2] + (\mu - R_{11})(y^*)[(\xi_1^*)^2 - (s^*\varphi_{y_1}(y^*))^2]\\ &+(\mu - R_{22})(y^*)[(\xi_2^*)^2 - (s^*\varphi_{y_2}(y^*))^2] - 2R_{12}(y^*)[\xi_1^*\,\xi_2^* - (s^*)^2\varphi_{y_1}(y^*)\varphi_{y_2}(y^*)]\}.\end{aligned}$$

Consequently,

$$|\mathrm{Re}\, r_\mu(\gamma)| \le \varepsilon(\delta_1)(s^2 + |\xi_0|^2 + |\xi_1|^2 + |\xi_2|^2) \quad \forall\gamma = (y^*, s, \xi') \in \mathcal{O}(\delta_1).$$

Since

$$\Sigma_\mu^{1,2}(z'_\nu) = s\int_{\mathbf{R}^3} (\mu - R_{33})(y^*)\varphi_{\tilde{y}_3}(y^*)\,\mathrm{Re}\, r_\mu(y^*, s, \xi')|\widehat{z'_\nu}|^2\,d\xi',$$

we readily deduce that

$$\Sigma_\mu^{1,2}(\mathbf{z}'_\mu) \le \varepsilon(\delta_1)s\|\mathbf{z}'_\nu\|^2_{H^{1,s}(\partial\mathcal{G})^3}. \tag{4.40}$$

Also, from $r_\mu(\gamma^*) = 0$ we find that

$$|\xi_0|^2 \le C(s^2 + |\xi_1|^2 + |\xi_2|^2) \quad \forall(s, \xi_0, \xi_1, \xi_2) \in \mathcal{O}(\delta_1) \tag{4.41}$$

for a positive constant C, provided δ_1 is small enough.

In order to estimate $\Sigma_\mu^{1,3}(\mathbf{z}'_\nu)$, we will have to distinguish again whether s^* is equal to zero or not.

Taking into account (4.36), an application of proposition 8 provides the identity

$$P_{\lambda+2\mu,s}(y,D)z_{4,\nu} = [(\lambda+2\mu)E_3 - R)G^t]G(D_{y_3} - \Gamma^-_{\lambda+2\mu}(y,s,D'))z^+_{4,\nu} + Tz_{4,\nu},$$

for some $T \in \mathcal{L}(H^{1,s}(\mathcal{G}); L^2(\mathcal{G}))$, where we have put

$$z^+_{4,\nu} = (D_{y_3} - \Gamma^+_{\lambda+2\mu}(y,s,D'))z_{4,\nu}.$$

Then, proposition 9 applied to $z^+_{4,\nu}$ yields

$$s\|(I\!\!D_{y_3} - \alpha^+_{\lambda+2\mu})z_{4,\nu}\|^2_{L^2(\partial\mathcal{G})} \le (\|e^{s\varphi}f\|^2_{H^{1,s}(\mathcal{G})^3} + \|z\|^2_{H^{1,s}(\mathcal{G})^4}). \tag{4.42}$$

Case 2.1.1: $s^* \neq 0$

We recall here the expression of $\operatorname{Im} r_\mu(\gamma^*)$:

$$\begin{aligned} 0 = \operatorname{Im} r_\mu(\gamma^*) = 2s^*\{ & R^2_{13}(y^*)\varphi_{y_1}(y^*)\xi_1^* + R^2_{23}(y^*)\varphi_{y_2}(y^*)\xi_2^* \\ & + R_{13}(y^*)R_{23}(y^*)(\varphi_{y_2}(y^*)\xi_1^* + \varphi_{y_1}(y^*)\xi_2^*) \\ & - (\mu - R_{33})(y^*)[-\varphi_{y_0}(y^*)\xi_0^* + (\mu - R_{11})(y^*)(\varphi_{y_1}(y^*)\xi_1^*) \\ & + (\mu - R_{22})(y^*)(\varphi_{y_2}(y^*)\xi_2^*) - R_{12}(y^*)(\varphi_{y_2}(y^*)\xi_1^* + \varphi_{y_1}(y^*)\xi_2^*)]\}, \end{aligned}$$

so we have

$$|\operatorname{Im} r_\mu(\gamma)| \le \varepsilon(\delta_1)(s^2 + |\xi_0|^2 + |\xi_1|^2 + |\xi_2|^2) \quad \forall \gamma = (y^*, s, \xi') \in \mathcal{O}(\delta_1).$$

Then, a similar argument to the one above leads us to estimate the term $\Sigma_\mu^{1,3}(\mathbf{z}'_\nu)$:

$$\Sigma_\mu^{1,3}(\mathbf{z}'_\nu) \le \varepsilon(\delta_1)s(\|\mathbf{z}'_\nu\|^2_{H^{1,s}(\partial\mathcal{G})^3} + \|\partial_{y_3}\mathbf{z}'_\nu\|^2_{L^2(\partial\mathcal{G})^3}). \tag{4.43}$$

Therefore, from the expression of $\Sigma^1_\mu(\mathbf{z}'_\nu)$ (see (4.30)–(4.32)), (4.40), (4.43) and the positiveness of $\Sigma_\mu^{1,1}(\mathbf{z}'_\nu)$, we deduce that

$$\Sigma^1_\nu(\mathbf{z}'_\mu) \ge Cs(\|\mathbf{z}'_\nu\|^2_{H^{1,s}(\partial\mathcal{G})^3} + \|\partial_{y_3}\mathbf{z}'_\nu\|^2_{L^2(\partial\mathcal{G})^3})$$

for some positive constant C. Here, we have used Condition A (at the beginning of section 2) on ψ, $\mu - R_{33} > 0$ (see (4.6)) and the fact that $s^2 \ge C(\xi_0^2 + \xi_1^2 + \xi_2^2)$ in $\mathcal{O}(\delta_1)$. Combining this and (4.37), we get

$$s(\|\mathbf{z}'_\nu\|^2_{H^{1,s}(\mathcal{G})^3} + \|\mathbf{z}'_\nu\|^2_{H^{1,s}(\partial\mathcal{G})^3} + \|\partial_{y_3}\mathbf{z}'_\nu\|^2_{L^2(\partial\mathcal{G})^3}) \le C(\|e^{s\varphi}\mathbf{f}\|^2_{H^{1,s}(\mathcal{G})^3} + \|\mathbf{z}\|^2_{H^{1,s}(\mathcal{G})^4}), \tag{4.44}$$

In order to estimate the boundary norms of $z_{4,\nu}$, we will use the boundary Dirichlet conditions and the equations of the Lamé system written on $\partial\mathcal{G}$. Indeed, from the two first ones, it is not difficult to deduce that

$$\begin{cases} |I\!\!D_{y_j} z_{4,\nu}| \le C(|e^{s\varphi}\mathbf{f}| + s|\mathbf{z}'_\nu| + |\nabla_y \mathbf{z}'_\nu|) + \varepsilon(\delta)(s|\mathbf{z}_\nu| + |\nabla_y^{tg}\mathbf{z}_\nu| + |\partial_{y_3}\mathbf{z}_\nu|) \\ \text{on } \partial\mathcal{G}, \quad \text{for } j = 1,2 \end{cases}$$

and from the third one

$$|\mathbb{D}_{y_3} z_{4,\nu}| \leq C(|e^{s\varphi}\mathbf{f}| + s|\mathbf{z}'_\nu| + |\nabla_y^{tg}\mathbf{z}'_\nu| + |\mathbb{D}_{y_1} z_{4,\nu}| + |\mathbb{D}_{y_2} z_{4,\nu}|)$$
$$+\varepsilon(\delta)(s|\mathbf{z}_\nu| + |\nabla_y^{tg}\mathbf{z}_\nu| + |\partial_{y_3}\mathbf{z}_\nu|) \quad \text{on } \partial\mathcal{G}.$$

Now, using that $\lambda + 2\mu - R_{33} > 0$ along with the Dirichlet boundary conditions, we have

$$\|\mathbb{D}_{y_3} z_{4,\nu}\|^2_{L^2(\partial\mathcal{G})} \leq C(\|e^{s\varphi}\mathbf{f}\|^2_{L^2(\partial\mathcal{G})^3} + \|\mathbf{z}'_\nu\|^2_{H^{1,s}(\partial\mathcal{G})^3} + \|\partial_{y_3}\mathbf{z}'_\nu\|^2_{L^2(\partial\mathcal{G})^4})$$
$$+\varepsilon(\delta)(\|\mathbf{z}_\nu\|^2_{H^{1,s}(\partial\mathcal{G})^4} + \|\partial_{y_3}\mathbf{z}_\nu\|^2_{L^2(\partial\mathcal{G})^4}). \tag{4.45}$$

In addition, combining (4.45) and (4.42), we find an estimate of the L^2 norm of z_4 and its tangential derivatives on $\partial\mathcal{G}$:

$$s\|z_{4,\nu}\|^2_{H^{1,s}(\partial\mathcal{G})} \leq C(\|e^{s\varphi}\mathbf{f}\|^2_{H^{1,s}(\mathcal{G})^3} + s\|\mathbf{z}\|^2_{H^{1,s}(\mathcal{G})^4} + s\|\mathbf{z}'_\nu\|^2_{H^{1,s}(\partial\mathcal{G})^3} + s\|\partial_{y_3}\mathbf{z}'_\nu\|^2_{L^2(\partial\mathcal{G})^4})$$
$$+\varepsilon(\delta)s(\|\mathbf{z}_\nu\|^2_{H^{1,s}(\partial\mathcal{G})^4} + \|\partial_{y_3}\mathbf{z}_\nu\|^2_{L^2(\partial\mathcal{G})^4}). \tag{4.46}$$

Indeed, in view of (4.36), we can apply Gärding's inequality after (4.42) and obtain (4.46).

Finally, (4.46) and (4.45) give an estimate of the normal derivative of $z_{4,\nu}$ in the L^2 norm:

$$s\|z_{4,\nu}\|^2_{H^{1,s}(\partial\mathcal{G})} + s\|\partial_{y_3} z_{4,\nu}\|^2_{L^2(\partial\mathcal{G})} \leq C(\|e^{s\varphi}\mathbf{f}\|^2_{H^{1,s}(\mathcal{G})^3} + s\|\mathbf{z}\|^2_{H^{1,s}(\mathcal{G})^4} + s\|\mathbf{z}'_\nu\|^2_{H^{1,s}(\partial\mathcal{G})^3}$$
$$+s\|\partial_{y_3}\mathbf{z}'_\nu\|^2_{L^2(\partial\mathcal{G})^4}) + \varepsilon(\delta)s(\|\mathbf{z}_\nu\|^2_{H^{1,s}(\partial\mathcal{G})^4} + \|\partial_{y_3}\mathbf{z}_\nu\|^2_{L^2(\partial\mathcal{G})^4}). \tag{4.47}$$

This and (4.44) provide the desired inequality (4.20). This ends the proof of lemma 12 in this case.

Case 2.1.2: $s^* = 0$

We first remark that, thanks to the Dirichlet boundary conditions, we have

$$s\|z_{3,\nu}\|^2_{H^{1,s}(\partial\mathcal{G})} \leq \varepsilon(\delta)s(\|\mathbf{z}_\nu\|^2_{H^{1,s}(\partial\mathcal{G})^4} + \|\partial_{y_3}\mathbf{z}_\nu\|^2_{L^2(\partial\mathcal{G})^4}). \tag{4.48}$$

Once the tangential derivatives of $z_{3,\nu}$ are bounded, an application of (4.35) for $w = z_{3,\nu}$ also gives an estimate for its normal derivative (see the expression of $\Sigma^1_\mu(z_{3,\nu})$ in (4.30)–(4.32), (4.34) and (4.33)):

$$s(\|z_{3,\nu}\|^2_{H^{1,s}(\partial\mathcal{G})} + \|\partial_{y_3} z_{3,\nu}\|^2_{L^2(\partial\mathcal{G})}) \leq \varepsilon(\delta)s(\|\mathbf{z}_\nu\|^2_{H^{1,s}(\partial\mathcal{G})^4} + \|\partial_{y_3}\mathbf{z}_\nu\|^2_{L^2(\partial\mathcal{G})^4})$$
$$+ C(\|e^{s\varphi}\mathbf{f}\|^2_{H^{1,s}(\mathcal{G})} + \|\mathbf{z}\|^2_{H^{1,s}(\mathcal{G})^4}). \tag{4.49}$$

We next deal with the terms $\Sigma^1_\mu(z_{1,\nu})$ and $\Sigma^1_\mu(z_{2,\nu})$. In fact, (4.40) together with (4.41) and the fact that $\varphi_{\tilde{y}_3}$ is large enough yields

$$\Sigma^1_\mu(z_{j,\nu}) \geq s(\|(\mu - R_{33})\partial_{y_3} z_{j,\nu} - (R_{13}\partial_{y_1} z_{j,\nu} + R_{23}\partial_{y_2} z_{j,\nu})\|^2_{L^2(\partial\mathcal{G})} + s^2\|z_{j,\nu}\|^2_{L^2(\partial\mathcal{G})})$$
$$- C(\|e^{s\varphi}\mathbf{f}\|^2_{H^{1,s}(\mathcal{G})^3} + \|\mathbf{z}\|^2_{H^{1,s}(\mathcal{G})^4}) - \varepsilon(\delta)s\|\nabla'_y z_{j,\nu}\|^2_{L^2(\partial\mathcal{G})}$$

for $j = 1, 2$.

An application of (4.37) taking into account (4.49), leads to the inequality

$$\begin{aligned} &s\sum_{j=1}^{2}(\|(\mu-R_{33})\partial_{y_3}z_{j,\nu}-(R_{13}\partial_{y_1}z_{j,\nu}+R_{23}\partial_{y_2}z_{j,\nu})\|^2_{L^2(\partial\mathcal{G})}+s^2\|z_{j,\nu}\|^2_{L^2(\partial\mathcal{G})})\\ &+s\|z'_\nu\|^2_{L^2(\mathcal{G})^3}+s(\|z_{3,\nu}\|^2_{H^{1,s}(\partial\mathcal{G})}+\|\partial_{y_3}z_{3,\nu}\|^2_{L^2(\partial\mathcal{G})})\\ &\qquad\le C(\|e^{s\varphi}\mathbf{f}\|^2_{H^{1,s}(\mathcal{G})^3}+\|\mathbf{z}\|^2_{H^{1,s}(\mathcal{G})^4})+\varepsilon(\delta)s\sum_{j=1}^{2}\|\nabla'_y z_{j,\nu}\|^2_{L^2(\partial\mathcal{G})}. \end{aligned} \tag{4.50}$$

At this point of the proof, we divide it into two subcases:

• **Case 2.1.2:** A) $R_{13}(y^*)\xi_1^*+R_{23}(y^*)\xi_2^*=0$

In this situation, inequality (4.50) readily provides an estimate for the normal derivatives of $z_{1,\nu}$ and $z_{2,\nu}$ in $L^2(\mathcal{G})$, so we deduce that

$$\begin{aligned} &s\sum_{j=1}^{2}(\|\partial_{y_3}z_{j,\nu}\|^2_{L^2(\partial\mathcal{G})}+s^2\|z_{j,\nu}\|^2_{L^2(\partial\mathcal{G})})+s\|\mathbf{z}'_\nu\|^2_{L^2(\mathcal{G})^3}\\ &+s(\|z_{3,\nu}\|^2_{H^{1,s}(\partial\mathcal{G})}+\|\partial_{y_3}z_{3,\nu}\|^2_{L^2(\partial\mathcal{G})})\\ &\qquad\le C(\|e^{s\varphi}\mathbf{f}\|^2_{H^{1,s}(\mathcal{G})^3}+\|\mathbf{z}\|^2_{H^{1,s}(\mathcal{G})^4})+\varepsilon(\delta)s\sum_{j=1}^{2}\|\nabla'_y z_{j,\nu}\|^2_{L^2(\partial\mathcal{G})}. \end{aligned} \tag{4.51}$$

On the other hand, the two first equations of our Lamé system provide

$$\begin{cases} -(\mu-R_{33})\partial_{y_3}z_{2,\nu}+2R_{13}\partial_{y_1}z_{2,\nu}+2R_{23}\partial_{y_2}z_{2,\nu}\\ \qquad\qquad =F_1+(\lambda+2\mu-R_{33})\partial_{y_1}z_{4,\nu},\\ (\mu-R_{33})\partial_{y_3}z_{1,\nu}-2R_{13}\partial_{y_1}z_{1,\nu}-2R_{23}\partial_{y_2}z_{1,\nu}\\ \qquad\qquad =F_2+(\lambda+2\mu-R_{33})\partial_{y_2}z_{4,\nu}, \end{cases} \tag{4.52}$$

with F_1 and F_2 satisfying estimate

$$\begin{aligned} s\|F_j\|^2_{L^2(\partial\mathcal{G})}&\le C(\|e^{s\varphi}\mathbf{f}\|^2_{H^{1,s}(\mathcal{G})^3}+\|\mathbf{z}\|^2_{H^{1,s}(\mathcal{G})^4})\\ &\quad+\varepsilon(\delta)s(\|\mathbf{z}_\nu\|^2_{H^{1,s}(\partial\mathcal{G})^4}+\|\partial_{y_3}\mathbf{z}_\nu\|^2_{L^2(\partial\mathcal{G})^4})\quad j=1,2. \end{aligned} \tag{4.53}$$

Consequently, this combined with (4.41) show that the term

$$s\|z_{4,\nu}\|^2_{H^{1,s}(\partial\mathcal{G})}$$

can be added to the left hand side of (4.51). The same happens for the normal derivative of $z_{4,\nu}$, after an application of (4.42).

In this way, we have

$$
\begin{aligned}
& s\sum_{j=1}^{2}(\|\partial_{y_3} z_{j,\nu}\|^2_{L^2(\partial\mathcal{G})} + s^2\|z_{j,\nu}\|^2_{L^2(\partial\mathcal{G})}) + s\|\mathbf{z}'_\nu\|^2_{L^2(\mathcal{G})^3} \\
& + s\sum_{j=3}^{4}(\|z_{j,\nu}\|^2_{H^{1,s}(\partial\mathcal{G})} + \|\partial_{y_3} z_{j,\nu}\|^2_{L^2(\partial\mathcal{G})}) \\
& \qquad \leq C(\|e^{s\varphi}\mathbf{f}\|^2_{H^{1,s}(\mathcal{G})^3} + \|\mathbf{z}\|^2_{H^{1,s}(\mathcal{G})^4}) + \varepsilon(\delta) s\sum_{j=1}^{2}\|\nabla'_y z_{j,\nu}\|^2_{L^2(\partial\mathcal{G})}.
\end{aligned}
\tag{4.54}
$$

Next, in order to estimate the tangential derivatives of $z_{1,\nu}$ and $z_{2,\nu}$, we use (4.41) together with the fact that estimate (4.49) also holds for

$$\partial_{y_1} z_{1,\nu} + \partial_{y_2} z_{2,\nu} \quad \text{and} \quad \partial_{y_2} z_{1,\nu} - \partial_{y_1} z_{2,\nu}. \tag{4.55}$$

Let us briefly say why (4.49) holds for $\partial_{y_1} z_{1,\nu} + \partial_{y_2} z_{2,\nu}$ and $\partial_{y_2} z_{1,\nu} - \partial_{y_1} z_{2,\nu}$. For the first one, it suffices to realize that the definition of $\mathbf{z} = \nabla \times \mathbf{u}$ implies that $\nabla \cdot \mathbf{z} = 0$ while for the second one we use the fact that we already have estimates for all the norms of $z_{4,\nu}$ and the following expression for the third equation of the Lamé system:

$$
\begin{aligned}
(\lambda + 2\mu - R_{33})\partial_{y_3} z_{4,\nu} - 2R_{13}\partial_{y_1} z_{4,\nu} - 2R_{23}\partial_{y_2} z_{4,\nu} \\
= F_3 + (\mu - R_{33})(\partial_{y_1} z_{2,\nu} - \partial_{y_2} z_{1,\nu}),
\end{aligned}
\tag{4.56}
$$

where F_3 satisfies estimate (4.53).

Finally, since the determinant of

$$\begin{pmatrix} \xi_1^* & \xi_2^* \\ \xi_1^* & -\xi_2^* \end{pmatrix}$$

is nonzero (observe that the situation $\xi_1^* = \xi_2^* = 0$ is not possible in case 2.1.2), we can bound the tangential derivatives of $z_{1,\nu}$ and $z_{2,\nu}$ as in (4.54). As a conclusion, we deduce inequality (4.20) and we conclude the proof of lemma 12 also in this case.

• **Case 2.1.2:** B) $R_{13}(y^*)\xi_1^* + R_{23}(y^*)\xi_2^* \neq 0$

In this second subcase we will first obtain estimates for the norms of $z_{4,\nu}$ (getting profit of the estimates of $\partial_{\tilde{y}_3}\mathbf{z}'_\nu$ given by (4.50)) and then for those of $\mathbf{z}'_\nu$.

Applying (4.42), we have that

$$\begin{aligned} s\|(\partial_{\tilde{y}_3} - \sqrt{r_{\lambda+2\mu}}^+(y,s,D'))z_{4,\nu}\|^2_{L^2(\partial\mathcal{G})} \leq (\|e^{s\varphi}\mathbf{f}\|^2_{H^{1,s}(\mathcal{G})^3} + \|\mathbf{z}\|^2_{H^{1,s}(\mathcal{G})^4}) \\ +\varepsilon(\delta)s(\|\mathbf{z}_\nu\|^2_{H^{1,s}(\partial\mathcal{G})^4} + \|\partial_{y_3}\mathbf{z}_\nu\|^2_{L^2(\partial\mathcal{G})^4}). \end{aligned} \tag{4.57}$$

Then, the third equation of the Lamé system (see (4.56)) implies that

$$(\sqrt{r_{\lambda+2\mu}}^+(y,s,D') - R_{13}\partial_{y_1} - R_{23}\partial_{y_2})z_{4,\nu} = (\mu - R_{33})(\partial_{y_1}z_{2,\nu} - \partial_{y_2}z_{1,\nu}) + F_4, \tag{4.58}$$

with F_4 satisfying estimate (4.53).

We are now going to express $z_{1,\nu}$ and $z_{2,\nu}$ in terms of $z_{4,\nu}$, using the two first equations of the Lamé systems, written in the form (4.52):

$$\begin{cases} R_{13}\partial_{y_1}z_{2,\nu} + R_{23}\partial_{y_2}z_{2,\nu} = F_5 + (\lambda + 2\mu - R_{33})\partial_{y_1}z_{4,\nu}, \\ R_{13}\partial_{y_1}z_{1,\nu} + R_{23}\partial_{y_2}z_{1,\nu} = F_6 - (\lambda + 2\mu - R_{33})\partial_{y_2}z_{4,\nu}. \end{cases} \tag{4.59}$$

Here, F_5 and F_6 satisfy estimate (4.53) and so we have taken into account (4.50) in order to estimate $\partial_{\tilde{y}_3}z_{1,\nu}$ and $\partial_{\tilde{y}_3}z_{2,\nu}$. Since $R_{13}(y^*)\xi_1^* + R_{23}(y^*)\xi_2^* \neq 0$, we have

$$\widehat{z}_{k,\nu} = (-1)^k \frac{(\lambda + 2\mu - R_{33})(y^*)\xi_{3-k}}{R_{13}(y^*)\xi_1 + R_{23}(y^*)\xi_2}\widehat{z}_{4,\nu} + \widehat{F}_7.$$

Plugging this into (4.58), we obtain

$$\left(\sqrt{r_{\lambda+2\mu}}^+(s^*,\xi') - R_{13}\xi_1 - R_{23}\xi_2 - \frac{(\lambda+2\mu - R_{33})(\mu - R_{33})(\xi_1^2 + \xi_2^2)}{R_{13}\xi_1 + R_{23}\xi_2}\right)\widehat{z}_{4,\nu} = \widehat{F}_8.$$

This directly gives an estimate of the the L^2 norm of $z_{4,\nu}$ and its tangential derivatives in the following way:

$$s\|z_{4,\nu}\|^2_{H^{1,s}(\partial\mathcal{G})} \leq (\|e^{s\varphi}\mathbf{f}\|^2_{H^{1,s}(\mathcal{G})^3} + \|\mathbf{z}\|^2_{H^{1,s}(\mathcal{G})^4}) + \varepsilon(\delta)s(\|\mathbf{z}_\nu\|^2_{H^{1,s}(\partial\mathcal{G})^4} + \|\partial_{y_3}\mathbf{z}_\nu\|^2_{L^2(\partial\mathcal{G})^4}).$$

Now, (4.57) also provides an estimate for the normal derivative of $z_{4,\nu}$, so for the moment we have:

$$\begin{aligned} &s\sum_{j=1}^{2}(\|\partial_{\tilde{y}_3}z_{j,\nu}\|^2_{L^2(\partial\mathcal{G})} + s^2\|z_{j,\nu}\|^2_{L^2(\partial\mathcal{G})}) + s\|\mathbf{z}'_\nu\|^2_{L^2(\mathcal{G})^3} \\ &+s\sum_{j=3}^{4}(\|z_{j,\nu}\|^2_{H^{1,s}(\partial\mathcal{G})} + \|\partial_{y_3}\mathbf{z}_{j,\nu}\|^2_{L^2(\partial\mathcal{G})}) \\ &\qquad \leq C(\|e^{s\varphi}\mathbf{f}\|^2_{H^{1,s}(\mathcal{G})^3} + \|\mathbf{z}\|^2_{H^{1,s}(\mathcal{G})^4}) + \varepsilon(\delta)s(\|\mathbf{z}_\nu\|^2_{H^{1,s}(\partial\mathcal{G})^4} + \|\partial_{y_3}\mathbf{z}_\nu\|^2_{L^2(\partial\mathcal{G})^4}). \end{aligned} \tag{4.60}$$

Finally, we use the same argument as in (4.55) in order to deduce that the tangential derivatives of $z_{1,\nu}$ and $z_{2,\nu}$ are also bounded. Since $\partial_{\tilde{y}_3}z_{1,\nu}$ and $\partial_{\tilde{y}_3}z_{2,\nu}$ are already bounded, the normal derivatives of $z_{1,\nu}$ and $z_{2,\nu}$ are too.

As a conclusion, we also obtain (4.20) in this situation.

2.2. Second Case: $r_\mu(\gamma^*) \neq 0$, $r_{\lambda+2\mu}(\gamma^*) = 0$

In a way similar to above, one can take δ and δ_1 to be small enough so that

$$|r_\mu(\gamma)| \geq C > 0 \quad \forall \gamma = (y, \zeta) \in B_\delta \times \mathcal{O}(\delta_1). \tag{4.61}$$

In this situation, condition $r_{\lambda+2\mu}(\gamma^*) = 0$ also provides the estimate (4.41).

We distinguish now two different situations depending on s^*.

Case 2.2.1: $s^* = 0$

First, we can suppose that

$$\varphi_{\tilde{y}_3}(y^*) > \mathrm{Im}(r_\mu(\gamma^*)/s)/(2\sqrt{\mathrm{Re}(r_\mu(\gamma^*))}),$$

since $\varphi_{\tilde{y}_3}$ is large enough.

Under these hypotheses, we can take δ and δ_1 small enough and suppose that

$$-\mathrm{Im}\, \Gamma_\mu^\pm(y, \zeta) \geq Cs \quad \forall \gamma = (y, \zeta) \in B_\delta \times \mathcal{O}(\delta_1),$$

since $\varphi_{\tilde{y}_3} > 0$ (see the expression of $\Gamma_\mu^\pm$ in (4.27)).

Consequently, we can apply proposition 8 in two different ways and deduce that

$$\begin{aligned} P_{\mu,s}(y, D)\mathbf{z}'_\nu &= [(\mu E_3 - R)G^t]G(D_{y_3} - \Gamma_\mu^-(y, s, D'))\mathbf{w}'^+_\nu + T^+_{\mu,s}\mathbf{w}'^+_\nu \\ &= [(\mu E_3 - R)G^t]G(D_{y_3} - \Gamma_\mu^+(y, s, D'))\mathbf{w}'^-_\nu + T^-_{\mu,s}\mathbf{w}'^-_\nu, \end{aligned}$$

with $T^\pm_{\mu,s} \in \mathcal{L}(H^{1,s}(\mathcal{G})^3, L^2(\mathcal{G})^3)$. Here, we have set

$$\mathbf{w}'^\pm_\nu = (D_{y_3} - \Gamma_\mu^\pm(y, s, D'))\mathbf{z}'_\nu.$$

Next, we apply proposition 9 and we get the following estimates:

$$s\|\mathbf{w}'^\pm_\nu\|^2_{L^2(\partial\mathcal{G})} \leq C(\|e^{s\varphi}\mathbf{f}\|^2_{H^{1,s}(\mathcal{G})^3} + \|\mathbf{z}\|^2_{H^{1,s}(\mathcal{G})^4}). \tag{4.62}$$

On the other hand, since

$$\mathbf{w}'^-_\nu - \mathbf{w}'^+_\nu = 2\sqrt{r_\mu}(y, s, D')\mathbf{z}'_\nu \quad \text{on } \partial\mathcal{G}$$

and $r_\mu(\gamma^*) \neq 0$, we can apply Gärding's inequality and obtain from (4.62) that

$$s\|\mathbf{z}'_\nu\|^2_{H^{1,s}(\partial\mathcal{G})^3} \leq C(\|e^{s\varphi}\mathbf{f}\|^2_{H^{1,s}(\mathcal{G})^3} + \|\mathbf{z}\|^2_{H^{1,s}(\mathcal{G})^4}).$$

Again, (4.62) indicates that the normal derivative of $\mathbf{z}'_\nu$ is also bounded:

$$s(\|\mathbf{z}'_\nu\|^2_{H^{1,s}(\partial\mathcal{G})^3} + \|\partial_{y_3}\mathbf{z}'_\nu\|^2_{L^2(\partial\mathcal{G})^3}) \leq C(\|e^{s\varphi}\mathbf{f}\|^2_{H^{1,s}(\mathcal{G})^3} + \|\mathbf{z}\|^2_{H^{1,s}(\mathcal{G})^4}). \tag{4.63}$$

Next, we can estimate $\partial_{y_1} z_{4,\nu}$ and $\partial_{y_2} z_{4,\nu}$ by means of the two first equations of the Lamé system (see (4.52)). This, together with (4.41), provides the estimate

$$\begin{aligned} s\|z_{4,\nu}\|^2_{H^{1,s}(\partial\mathcal{G})} &\leq C(\|e^{s\varphi}\mathbf{f}\|^2_{H^{1,s}(\mathcal{G})^3} + \|\mathbf{z}\|^2_{H^{1,s}(\mathcal{G})^4}) \\ &\quad + \varepsilon(\delta)s(\|\mathbf{z}_\nu\|^2_{H^{1,s}(\partial\mathcal{G})^4} + \|\partial_{y_3}\mathbf{z}_\nu\|^2_{L^2(\partial\mathcal{G})^4}). \end{aligned} \tag{4.64}$$

Finally, the third equation of the Lamé system (see (4.56)) gives an estimate of the normal derivative of $z_{4,\nu}$. Combining this, (4.64) and (4.63), we deduce the desired inequality (4.20).

Case 2.2.2: $s^* \neq 0$

We first apply estimate (4.37) to $z_{4,\nu}$ and get

$$\begin{aligned} s\|z_{4,\nu}\|^2_{H^{1,s}(\mathcal{G})} + \Sigma^1_{\lambda+2\mu}(z_{4,\nu}) \leq C(\|e^{s\varphi}\mathbf{f}\|^2_{H^{1,s}(\mathcal{G})^3} + \|\mathbf{z}\|^2_{H^{1,s}(\mathcal{G})^4}) \\ + \varepsilon(\delta)s(\|z_{4,\nu}\|^2_{H^{1,s}(\partial\mathcal{G})} + \|\partial_{y_3}z_{4,\nu}\|^2_{L^2(\partial\mathcal{G})}). \end{aligned} \tag{4.65}$$

We recall that we may write

$$\Sigma^1_{\lambda+2\mu}(z_{4,\nu}) = \Sigma^{1,1}_{\lambda+2\mu}(z_{4,\nu}) + \Sigma^{1,2}_{\lambda+2\mu}(z_{4,\nu}) + \Sigma^{1,3}_{\lambda+2\mu}(z_{4,\nu}),$$

where $\Sigma^{1,i}_{\lambda+2\mu}(z_{4,\nu})$ $(1 \leq i \leq 3)$ corresponds to the expressions given just after (4.38) with $\lambda+2\mu$ instead of μ and $z_{4,\nu}$ instead of $\mathbf{z}'_\nu$.

In order to estimate $\Sigma^{1,2}_{\lambda+2\mu}(z_{4,\nu})$, we use that $r_{\lambda+2\mu}(\gamma^*) = 0$ and we obtain

$$|\Sigma^{1,2}_{\lambda+2\mu}(z_{4,\nu})| \leq \varepsilon(\delta_1)s\|z_{4,\nu}\|^2_{H^{1,s}(\partial\mathcal{G})}.$$

Also, from $r_{\lambda+2\mu}(\gamma^*) = 0$ and (4.41), we obtain for small δ_1 that

$$\begin{aligned} s\Bigg|\varphi_{y_0}(y^*)\xi_0 - \left(\lambda+2\mu - R_{11} - \frac{R_{13}^2}{\lambda+2\mu-R_{33}}\right)(y^*)\varphi_{y_1}(y^*)\xi_1 \\ - \left(\lambda+2\mu - R_{22} - \frac{R_{23}^2}{\lambda+2\mu-R_{33}}\right)\varphi_{y_2}(y^*)\xi_2 \\ + \left(R_{12} + \frac{R_{13}R_{23}}{\lambda+2\mu-R_{33}}\right)(y^*)\left(\varphi_{y_2}(y^*)\xi_1 + \varphi_{y_1}(y^*)\xi_2\right)\Bigg| \\ \leq \varepsilon(\delta_1)(s^2+\xi_1^2+\xi_2^2) \quad \forall \zeta \in \mathcal{O}(\delta_1), \end{aligned}$$

which yields

$$|\Sigma^{1,3}_{\lambda+2\mu}(z_{4,\nu})| \leq \varepsilon(\delta_1)s(\|z_{4,\nu}\|^2_{H^{1,s}(\partial\mathcal{G})} + \|\partial_{y_3}z_{4,\nu}\|^2_{L^2(\partial\mathcal{G})}).$$

Consequently, we have

$$\begin{aligned} \Sigma^1_{\lambda+2\mu}(z_{4,\nu}) \geq -\varepsilon(\delta_1)s(\|z_{4,\nu}\|^2_{H^{1,s}(\partial\mathcal{G})} + \|\partial_{y_3}z_{4,\nu}\|^2_{L^2(\partial\mathcal{G})}) \\ + s(s^2\|z_{4,\nu}\|^2_{L^2(\partial\mathcal{G})} + \|\partial_{\tilde{y}_3}z_{4,\nu}\|^2_{L^2(\partial\mathcal{G})}). \end{aligned}$$

From the fact that $s^* \neq 0$, we also deduce that

$$\Sigma^1_{\lambda+2\mu}(z_{4,\nu}) \geq s(\|z_{4,\nu}\|^2_{H^{1,s}(\partial\mathcal{G})} + \|\partial_{y_3}z_{4,\nu}\|^2_{L^2(\partial\mathcal{G})}),$$

which, together with (4.65), gives

$$\begin{aligned} s\|z_{4,\nu}\|^2_{H^{1,s}(\mathcal{G})} + s(\|z_{4,\nu}\|^2_{H^{1,s}(\partial\mathcal{G})} + \|\partial_{y_3}z_{4,\nu}\|^2_{L^2(\partial\mathcal{G})}) \\ \leq C(\|e^{s\varphi}\mathbf{f}\|^2_{H^{1,s}(\mathcal{G})^3} + \|\mathbf{z}\|^2_{H^{1,s}(\mathcal{G})^4}). \end{aligned} \tag{4.66}$$

In this situation, estimate (4.48) also holds. Furthermore, we can apply proposition 9 to $z_{3,\nu}$, so we deduce

$$\begin{aligned} s(\|z_{3,\nu}\|^2_{H^{1,s}(\partial\mathcal{G})} + \|\partial_{y_3} z_{3,\nu}\|^2_{L^2(\partial\mathcal{G})}) \leq C(\|e^{s\varphi}\mathbf{f}\|^2_{H^{1,s}(\mathcal{G})^3} + \|\mathbf{z}\|^2_{H^{1,s}(\mathcal{G})^4}) \\ +\varepsilon(\delta)s(\|\mathbf{z}_\nu\|^2_{H^{1,s}(\partial\mathcal{G})^4} + \|\partial_{y_3}\mathbf{z}_\nu\|^2_{L^2(\partial\mathcal{G})^4}). \end{aligned} \tag{4.67}$$

The next step will be to estimate the functions $z_{1,\nu}$ and $z_{2,\nu}$. To this end, we first observe that the case where

$$\sqrt{r_\mu}(\gamma^*) - R_{13}(\xi_1^* + is^*\varphi_{y_1}) - R_{23}(\xi_2^* + is^*\varphi_{y_2}) \neq 0 \tag{4.68}$$

directly provides estimate (4.20); indeed, the two first equations in (4.9) can be rewritten as

$$(\sqrt{r_\mu}(y,s,D') - R_{13}(D_{y_1} + is\varphi_{y_1}) - R_{23}(D_{y_2} + is\varphi_{y_2}))z_{2,\nu} = F_9$$

and

$$(\sqrt{r_\mu}(y,s,D') - R_{13}(D_{y_1} + is\varphi_{y_1}) - R_{23}(D_{y_2} + is\varphi_{y_2}))z_{1,\nu} = F_{10}$$

respectively, with F_9 and F_{10} satisfying estimate (4.53).

To prove this, it suffices to take into account (4.66) and proposition 9 applied to $z_{1,\nu}$ and $z_{2,\nu}$. Now, from (4.68), we can apply Gärding's inequality to the two previous equations and we find out estimates of the $H^{1,s}$ norm of $z_{1,\nu}$ and $z_{2,\nu}$ on $\partial\mathcal{G}$. Finally, again an application of proposition 9 also provides the estimate of the normal derivative of $z_{1,\nu}$ and $z_{2,\nu}$.

Outside the particular situation (4.68), we distinguish another two cases:

- If $(\xi_1^* + is\varphi_{y_1}(y^*))^2 + (\xi_2^* + is\varphi_{y_2}(y^*))^2 \neq 0$

In this case, we are going to represent $(z_{1,\nu}, z_{2,\nu})$ in terms of $z_{4,\nu}$ and we will then use estimate (4.66).

In order to do this, let us introduce the following differential operator:

$$M(y,s,D')(z_{1,\nu}, z_{2,\nu}) = (\mathbb{D}_{y_1} z_{1,\nu} + \mathbb{D}_{y_2} z_{2,\nu}, (\mu - R_{33})(\mathbb{D}_{y_1} z_{2,\nu} - \mathbb{D}_{y_2} z_{1,\nu})) \tag{4.69}$$

From the third equation of our Lamé system, we have

$$\begin{aligned} &(\mu - R_{33})(\mathbb{D}_{y_1} z_{2,\nu} - \mathbb{D}_{y_2} z_{1,\nu}) \\ &\quad = (\lambda + 2\mu - R_{33})\mathbb{D}_{y_3} z_{4,\nu} - 2R_{13}\mathbb{D}_{y_1} z_{4,\nu} - 2R_{23}\mathbb{D}_{y_2} z_{4,\nu} + F_{11} \end{aligned} \tag{4.70}$$

where F_{11} satisfies estimate (4.53).

Then, since the divergence of a curl is identically zero and we have estimate (4.49) for $\partial_{y_3} z_{3,\nu}$, we find that

$$\begin{aligned} s\|\mathbb{D}_{y_1} z_{1,\nu} + \mathbb{D}_{y_2} z_{2,\nu}\|^2_{L^2(\partial\mathcal{G})} \leq \varepsilon(\delta)s(\|z_\nu\|^2_{H^{1,s}(\partial\mathcal{G})^4} + \|\partial_{y_3} z_\nu\|^2_{L^2(\partial\mathcal{G})^4}) \\ + C(\|e^{s\varphi}\mathbf{f}\|^2_{H^{1,s}(\mathcal{G})} + \|\mathbf{z}\|^2_{H^{1,s}(\mathcal{G})^4}). \end{aligned} \tag{4.71}$$

The principal symbol of the operator M is

$$\begin{pmatrix} \xi_1 + is\varphi_{y_1} & \xi_2 + is\varphi_{y_2} \\ -(\mu - R_{33})(\xi_2 + is\varphi_{y_2}) & (\mu - R_{33})(\xi_1 + is\varphi_{y_1}) \end{pmatrix},$$

which has a nonzero determinant at the point γ^* because of the hypothesis made in this case. Therefore, we deduce that there exists a parametrix of this operator such that

$$(z_{1,\nu}, z_{2,\nu}) = M^{-1}(y, s, D')(\mathbb{D}_{y_1} z_{1,\nu} + \mathbb{D}_{y_2} z_{2,\nu}, (\mu - R_{33})(\mathbb{D}_{y_1} z_{2,\nu} - \mathbb{D}_{y_2} z_{1,\nu})) + T(z_{1,\nu}, z_{2,\nu}),$$

with $T \in S^{-1}$.

Consequently, if we use the identity (4.70) and we combine estimates (4.71) with (4.66), we find

$$s \sum_{j=1}^{2} \|z_{j,\nu}\|^2_{H^{1,s}(\partial\mathcal{G})} \leq C(\|e^{s\varphi}\mathbf{f}\|^2_{H^{1,s}(\mathcal{G})^3} + \|\mathbf{z}\|^2_{H^{1,s}(\mathcal{G})^4}) + \varepsilon(\delta)s(\|\mathbf{z}_\nu\|^2_{H^{1,s}(\partial\mathcal{G})^4} + \|\partial_{y_3}\mathbf{z}_\nu\|^2_{L^2(\partial\mathcal{G})^4}).$$

Finally, the estimate of $\partial_{y_3} z_{j,\nu}$ $(j = 1, 2)$ comes from proposition 9 applied to $z_{1,\nu}$ and $z_{2,\nu}$. As a conclusion, we have

$$\begin{aligned} s \sum_{j=1}^{2} (\|z_{j,\nu}\|^2_{H^{1,s}(\partial\mathcal{G})} + \|\partial_{y_3} z_{j,\nu}\|^2_{L^2(\partial\mathcal{G})}) &\leq C(\|e^{s\varphi}\mathbf{f}\|^2_{H^{1,s}(\mathcal{G})^3} + \|\mathbf{z}\|^2_{H^{1,s}(\mathcal{G})^4}) \\ &\quad + \varepsilon(\delta)s(\|\mathbf{z}_\nu\|^2_{H^{1,s}(\partial\mathcal{G})^4} + \|\partial_{y_3}\mathbf{z}_\nu\|^2_{L^2(\partial\mathcal{G})^4}) \end{aligned}$$

which, combined with (4.66) and (4.67), gives (4.20).

- If $(\xi_1^* + is^*\varphi_{y_1}(y^*))^2 + (\xi_2^* + is^*\varphi_{y_2}(y^*))^2 = 0$

After an application of estimate (4.37) to $z_{1,\nu}$ and $z_{2,\nu}$, we obtain that

$$\begin{aligned} s\|z_{j,\nu}\|^2_{H^{1,s}(\mathcal{G})} + \Sigma^1_\mu(z_{j,\nu}) &\leq \varepsilon(\delta)s(\|z_{j,\nu}\|^2_{H^{1,s}(\partial\mathcal{G})} + \|\partial_{y_3} z_{j,\nu}\|^2_{L^2(\partial\mathcal{G})}) \\ &\quad + C(\|e^{s\varphi}\mathbf{f}\|^2_{H^{1,s}(\mathcal{G})^3} + \|\mathbf{z}\|^2_{H^{1,s}(\mathcal{G})^4}) \quad \text{for } j = 1, 2, \end{aligned} \tag{4.72}$$

where $\Sigma^1_\mu(z_{j,\nu})$ are given just after (4.38).

Let us estimate $\Sigma^{1,2}_\mu(z_{j,\nu})$. First, we rewrite it in the form

$$\begin{aligned} \Sigma^{1,2}_\mu(z_{j,\nu}) &= s \int_{\partial\mathcal{G}} (\mu - R_{33})(y^*)\varphi_{\tilde{y}_3}(y^*) \left[|\partial_{y_0} z_{j,\nu}|^2 - s^2\varphi^2_{y_0}(y^*)|z_{j,\nu}|^2 \right. \\ &\quad - \left(\lambda + 2\mu - R_{11} - \frac{R^2_{13}}{\lambda + 2\mu - R_{33}} \right)(y^*)(|\partial_{y_1} z_{j,\nu}|^2 - s^2\varphi^2_{y_1}(y^*)|z_{j,\nu}|^2) \\ &\quad - \left(\lambda + 2\mu - R_{22} - \frac{R^2_{23}}{\lambda + 2\mu - R_{33}} \right)(y^*)(|\partial_{y_2} z_{j,\nu}|^2 - s^2\varphi^2_{y_2}(y^*)|z_{j,\nu}|^2) \\ &\quad \left. +2\left(R_{12} + \frac{R_{13}R_{23}}{\lambda + 2\mu - R_{33}} \right)(y^*)(\partial_{y_1} z_{j,\nu}\, \partial_{y_2} z_{j,\nu} - s^2\varphi_{y_1}(y^*)\varphi_{y_2}(y^*))|z_{j,\nu}|^2 \right] dy' \\ &\quad +s \int_{\partial\mathcal{G}} (\mu - R_{33})(y^*)\varphi_{\tilde{y}_3} \left[(\lambda + \mu)\left(|\partial_{y_1} z_{j,\nu}|^2 + |\partial_{y_2} z_{j,\nu}|^2 - s^2\varphi^2_{y_1}(y^*)|z_{j,\nu}|^2 \right. \right. \\ &\quad \left. -s^2\varphi^2_{y_2}(y^*)|z_{j,\nu}|^2 \right) - (\lambda + 2\mu - R_{33})^{-1}(y^*)\left(R^2_{13}(y^*)\left(|\partial_{y_1} z_{j,\nu}|^2 \right. \right. \\ &\quad \left. -s^2\varphi^2_{y_1}(y^*)|z_{j,\nu}|^2 \right) + R^2_{23}(y^*)\left(|\partial_{y_2} z_{j,\nu}|^2 - s^2\varphi^2_{y_2}(y^*)|z_{j,\nu}|^2 \right) \\ &\quad \left. \left. +2R_{13}(y^*)R_{23}(y^*)\left(\partial_{y_1} z_{j,\nu}\partial_{y_2} z_{j,\nu} - s^2\varphi_{y_1}\varphi_{y_2}|z_{j,\nu}|^2 \right) \right) \right] dy' \\ &= J_4(z_{j,\nu}) + J_5(z_{j,\nu}), \end{aligned}$$

where

$$\begin{aligned}J_4(z_{j,\nu}) = s\int_{\partial\mathcal{G}} &(\mu - R_{33})(y^*)\varphi_{\tilde{y}_3}(y^*)\left[|\partial_{y_0} z_{j,\nu}|^2 - s^2\varphi_{y_0}^2(y^*)|z_{j,\nu}|^2\right.\\ &-\left(\lambda+2\mu-R_{11}-\frac{R_{13}^2}{\lambda+2\mu-R_{33}}\right)(y^*)(|\partial_{y_1} z_{j,\nu}|^2 - s^2\varphi_{y_1}^2(y^*)|z_{j,\nu}|^2)\\ &-\left(\lambda+2\mu-R_{22}-\frac{R_{23}^2}{\lambda+2\mu-R_{33}}\right)(y^*)(|\partial_{y_2} z_{j,\nu}|^2 - s^2\varphi_{y_2}^2(y^*)|z_{j,\nu}|^2)\\ &\left.+2\left(R_{12}+\frac{R_{13}R_{23}}{\lambda+2\mu-R_{33}}\right)(y^*)(\partial_{y_1} z_{j,\nu}\,\partial_{y_2} z_{j,\nu} - s^2\varphi_{y_1}(y^*)\varphi_{y_2}(y^*))|z_{j,\nu}|^2\right] dy'\end{aligned}$$

and

$$J_5(z_{j,\nu}) = \Sigma_\mu^{1,2}(z_{j,\nu}) - J_4(z_{j,\nu}).$$

From $\mathrm{Re}(r_{\lambda+2\mu}(\gamma^*)) = 0$, we deduce that

$$|J_4(z_{j,\nu})| \le \varepsilon(\delta_1)s\|z_{j,\nu}\|^2_{H^{1,s}(\partial\mathcal{G})}. \tag{4.73}$$

Let us now estimate $J_5(z_{j,\nu})$; we observe that

$$\begin{aligned}J_5(z_{j,\nu}) = s\int_{\mathbf{R}^2} &(\mu - R_{33})(y^*)\varphi_{\tilde{y}_3}(y^*)\left[(\lambda+\mu)\left(\xi_1^2+\xi_2^2-s^2\varphi_{y_1}^2(y^*)\right.\right.\\ &\left.-s^2\varphi_{y_2}^2(y^*)\right) - (\lambda+2\mu-R_{33})^{-1}(y^*)\,\mathrm{Re}\,(R_{13}(y^*)(\xi_1+is\varphi_{y_1}(y^*))\\ &\left.+R_{23}(y^*)(\xi_2+is\varphi_{y_2}(y^*)))^2\right]|\widehat{z}_{j,\nu}|^2\,d\xi_1\,d\xi_2.\end{aligned}$$

We first realize that, since $r_{\lambda+2\mu}(\gamma^*) = 0$ and

$$\sqrt{r_\mu}(\gamma^*) - R_{13}(\xi_1^* + is^*\varphi_{y_1}) - R_{23}(\xi_2^* + is^*\varphi_{y_2}) = 0,$$

we have

$$\begin{aligned}&\mathrm{Re}\,(R_{13}(y^*)(\xi_1^*+is\varphi_{y_1}(y^*)) + R_{23}(y^*)(\xi_2^*+is\varphi_{y_2}(y^*)))^2\\ &= (\lambda+2\mu-R_{33})(\lambda+\mu)(y^*)\,\mathrm{Re}\left((\xi_1^*+is\varphi_{y_1}(y^*))^2+(\xi_2^*+is\varphi_{y_2}(y^*))^2\right).\end{aligned}$$

Then, it is readily seen that $J_5(z_{j,\nu})$ can be bounded as in (4.73). Consequently,

$$|\Sigma_\mu^{1,2}(z_{j,\nu})| \le \varepsilon(\delta_1)s\|z_{j,\nu}\|^2_{H^{1,s}(\partial\mathcal{G})}. \tag{4.74}$$

On the other hand, the term $\Sigma_\mu^{1,1}(z_{j,\nu})$ provides estimates for

$$C^*s(\|\partial_{\tilde{y}_3} z_{j,\nu}\|^2_{L^2(\partial\mathcal{G})} + s^2\|z_{j,\nu}\|^2_{L^2(\partial\mathcal{G})}),$$

with C^* large (since $\varphi_{\tilde{y}_3}$ is too). Then, since $s^* \ne 0$, we also have estimates for the tangential derivatives of $z_{j,\nu}$ (that is to say, for $\|z_{j,\nu}\|^2_{H^{1,s}(\partial\mathcal{G})}, j = 1, 2$) and hence for the normal derivative too.

Finally, the fact that C^* is large is used for estimating $\Sigma_\mu^{1,3}(z_{j,\nu})$.

As a conclusion, from (4.66)–(4.67) and (4.72)–(4.74), we deduce the desired inequality (4.20).

2.3. Third Case: $r_\mu(\gamma^*) \neq 0$ and $r_{\lambda+2\mu}(\gamma^*) \neq 0$ or $r_\mu(\gamma^*) = r_{\lambda+2\mu}(\gamma^*) = 0$.

It is in this paragraph where we will use Calderon's method. The ideas we develop here are similar to those in [15].

Let us introduce the variables $\mathbf{U} = (U_j)_{j=1}^6$, given by

$$(U_1, U_2, U_3) = \Lambda(s, D')(e^{s\varphi}\mathbf{u}), \quad (U_4, U_5, U_6) = I\!D_{y_3}(e^{s\varphi}\mathbf{u}),$$

where Λ is the pseudodifferential operator with symbol $(1 + s^2 + |\xi'|^2)$. With these notations, we can rewrite the Lamé system (4.9) as

$$\begin{cases} D_{y_3}\mathbf{U} = K(y, s, D')\mathbf{U} + F & \text{in } \mathbf{R}^3 \times [0, 1/Z^2], \\ (U_1, U_2, U_3) = 0 & \text{on } \{y_3 \equiv 0\}, \\ \mathbf{U} = 0 & \text{on } \{y_3 \equiv 1/Z^2\}, \end{cases} \tag{4.75}$$

where $F = (0, e^{s\varphi}\mathbf{f})$ and $K(y, s, D')$ is a pseudodifferential operator whose principal symbol is

$$K_1(\gamma) = \begin{pmatrix} 0 & \Lambda_1 E_3 \\ A^{-1}K_{11}\Lambda^{-1} & A^{-1}K_{12} \end{pmatrix} - is\varphi_{y_3}E_6.$$

Here, we have set

$$\begin{aligned} A &= (\lambda + \mu)GG^t + [((\mu E_3 - R)G^t)G]E_3, \\ K_{11} &= -(\lambda + \mu)\theta\theta^t + [(\xi_0 + is\varphi_{y_0})^2 - ((\mu E_3 - R)\theta^t)\theta]E_3, \\ K_{12} &= -(\lambda + \mu)(\theta G^t + G\theta^t) - 2((\mu E_3 - R)\theta^t)GE_3, \end{aligned} \tag{4.76}$$

with $\theta = (\xi_1 + is\varphi_{y_1}, \xi_2 + is\varphi_{y_2}, 0)^t$. Recall that G was defined (also as a column vector field) in (4.19).

In this context, one can check that the eigenvalues of the matrix K_1 coincide with $\Gamma_\beta^\pm$ (given in (4.27)).

Let us consider again several situations:

Case 2.3.1: $r_\mu(\gamma^*) = 0 = r_{\lambda+2\mu}(\gamma^*)$

Then, the expression of r_β indicates directly that $\mathrm{Im}(\Gamma_\beta^\pm((\gamma^*)) < 0$ for $\beta \in \{\mu, \lambda + 2\mu\}$; this comes from the fact that $\varphi_{\tilde{y}_3} > 0$. Consequently, the eigenvalues of the matrix K_1 have negative imaginary parts and we can suppose that

$$\mathrm{Im}(\Gamma_\beta^\pm(\gamma)) < -C|\zeta| \quad \forall \gamma \in B_\delta \times \mathcal{O}(\delta_1).$$

Now, using standard arguments (see, for instance, Chapter 7 in [19]), we obtain that

$$\|\chi_\nu \mathbf{U}\|^2_{H^{2,s}(\mathcal{G})^6} \leq C(\|e^{s\varphi}\mathbf{f}\|^2_{H^{1,s}(\mathcal{G})^3} + \|e^{s\varphi}\mathbf{u}\|^2_{H^{1,s}(\mathcal{G})^3}).$$

From this inequality, estimate (4.20) is readily deduced.

Case 2.3.2: $r_\mu(\gamma^*) \neq 0$, $r_{\lambda+2\mu}(\gamma^*) \neq 0$ and $r_\mu(\gamma^*) \neq r_{\lambda+2\mu}(\gamma^*)$

In this situation, the matrix K_1 has four eigenvalues which are given by (4.27) and the corresponding eigenvectors:

$$\begin{aligned} v_1^{\pm} &= ((\theta^t + \alpha^{\pm}_{\lambda+2\mu} G^t)\Lambda_1^{-1}, \alpha^{\pm}_{\lambda+2\mu}(\theta^t + \alpha^{\pm}_{\lambda+2\mu} G^t)\Lambda_1^{-2})^t, \\ v_2^{\pm} &= (v_{21}^{\pm}, \alpha^{\pm}_{\mu} v_{21}^{\pm}\Lambda_1^{-1})^t, \quad v_3^{\pm} = (v_{31}^{\pm}, \alpha^{\pm}_{\mu} v_{31}^{\pm}\Lambda_1^{-1})^t, \end{aligned} \tag{4.77}$$

with

$$v_{21}^{\pm} = (\ell_{y_2}\alpha^{\pm}_{\mu} - \theta_2, -\ell_{y_1}\alpha^{\pm}_{\mu} + \theta_1, 0)\Lambda_1^{-1}$$

and

$$v_{31}^{\pm} = (\alpha^{\pm}_{\mu}(\theta_1 - \ell_{y_1}\alpha^{\pm}_{\mu}), \alpha^{\pm}_{\mu}(\theta_2 - \ell_{y_2}\alpha^{\pm}_{\mu}), -\sum_{j=1}^{2}(\theta_j - \ell_{y_j}\alpha^{\pm}_{\mu})^2)\Lambda_1^{-2}.$$

Observe that $\{v_2^{\pm}, v_3^{\pm}\}$ is a basis of the orthogonal space to the vector $\theta + \alpha^{\pm}_{\mu} G$.

Let us take the symbol $S(\gamma)$ in the form $S = \{v_1^{+}, v_2^{+}, v_3^{+}, v_1^{-}, v_2^{-}, v_3^{-}\}$ and let us extend it as an homogeneous function of order 0 and of class C^3 in the ζ variables. Now, $\mathbf{U}$ is determined by (4.75). Let us introduce $\mathbf{W} = S^{-1}(y, s, D')U$. Then, system (4.75) takes the form

$$D_{y_3}\mathbf{W} = \widetilde{K}(y, s, D')\mathbf{W} + T(y, s, D')\mathbf{W} + \widetilde{\mathbf{F}},$$

where $\widetilde{K}$ is a diagonal matrix and $T \in L^{\infty}(0, 1; \mathcal{L}(H^{1,s}(\mathbf{R}^3)^3); H^{1,s}(\mathbf{R}^3)^3)$. A standard argument for pseudodifferential systems allows to estimate the last three components of $\mathbf{W}$ as follows (see for instance [19]):

$$s\|(W_4, W_5, W_6)\|^2_{H^{1,s}(\partial\mathcal{G})^3} \le C(\|e^{s\varphi}\mathbf{f}\|^2_{H^{1,s}(\mathcal{G})^3} + \|e^{s\varphi}\mathbf{u}\|^2_{H^{2,s}(\mathcal{G})^3}). \tag{4.78}$$

Finally, the first three components can be bounded in terms of the last three by means of the first-squared 3×3 matrix inside S, which is

$$\Lambda_1^{-1}\begin{pmatrix} \theta_1 - \alpha^{+}_{\lambda+2\mu}\ell_{y_1} & -\theta_2 + \ell_{y_2}\alpha^{+}_{\mu} & \alpha^{+}_{\mu}(\theta_1 - \ell_{y_1}\alpha^{+}_{\mu})\Lambda_1^{-1} \\ \theta_2 - \alpha^{+}_{\lambda+2\mu}\ell_{y_2} & \theta_1 - \ell_{y_1}\alpha^{+}_{\mu} & \alpha^{+}_{\mu}(\theta_2 - \ell_{y_2}\alpha^{+}_{\mu})\Lambda_1^{-1} \\ \alpha^{+}_{\lambda+2\mu} & 0 & -\sum_{j=1}^{2}(\theta_j - \ell_{y_j}\alpha^{+}_{\mu})^2\Lambda_1^{-1} \end{pmatrix}$$

This yields

$$s\|(W_1, W_2, W_3)\|^2_{H^{1,s}(\partial\mathcal{G})^3} \le C(\|e^{s\varphi}\mathbf{f}\|^2_{H^{1,s}(\mathcal{G})^3} + \|e^{s\varphi}\mathbf{u}\|^2_{H^{2,s}(\mathcal{G})^3}),$$

which, in combination with (4.78) and (4.35), provides (4.20).

Case 2.3.3: $r_{\mu}(\gamma^*) = r_{\lambda+2\mu}(\gamma^*) \neq 0$

The argument needed to prove estimate (4.20) in this case coincides exactly with the one developed in [15].

This ends the proof of Lemma 12.

3. Controllability of the Lamé system.

In this section we obtain some exact controllability results for the Lamé system with controls locally distributed over the cylinder $Q_\omega = \omega \times (0,T)$.

First, we need a Carleman estimate similar to (4.10) but with the right hand side in the spaces $L^2(Q)$ and $H^{-1}(Q)$. In order to prove such an estimate, we need a pseudoconvex function ψ which satisfies the additional condition:

$$\partial_{x_0}\psi(\cdot,0) > 0, \quad \partial_{x_0}\psi(\cdot,T) < 0 \quad \forall x \in \overline{\Omega}. \tag{4.79}$$

We have the following result:

Theorem 11 *Let* $\mathbf{f} \in L^2(Q)^3$ *and assume that* (4.2), (4.4), (4.6) *and* (4.79) *hold true. Suppose there exists a function* ψ *which satisfies condition A. Then, there exists* $\tau^* > 0$ *such that, for any* $\tau > \tau^*$, *there exists* $s^* > 0$ *such that*

$$\|\mathbf{u}\|_{H^{1,s}(Q)^3} \le C(\|e^{s\phi}\mathbf{f}\|_{L^2(Q)^3} + \|\mathbf{u}\|_{H^{1,s}(Q_\omega)^3}) \quad \forall s > s^* \tag{4.80}$$

for some positive constant C *independent of* s *and for any solution* $\mathbf{u} \in H^1(Q)^3$ *of* (4.9).

Next, we consider a situation with the function f in the space $H^{-1}(Q)$.

Theorem 12 *Let* $\mathbf{f} = \mathbf{f_{-1}} + \sum_{k=0}^{3} \partial_{x_k}\mathbf{f_k}$, *where* $\mathbf{f_0}, \mathbf{f_1}, \mathbf{f_2}, \mathbf{f_3} \in L^2(Q)^3$ *and assume that* (4.2), (4.4), (4.6) *and* (4.79) *hold true. Suppose there exists a function* ψ *which satisfies condition A. Then, there exists* $\tau^* > 0$ *such that, for any* $\tau > \tau^*$, *there exists* $s^* > 0$ *such that*

$$\|\mathbf{u}\|_{L^2(Q)^3} \le C(\|e^{s\phi}\mathbf{f_{-1}}\|_{H^{-1}(Q)^3} + \sum_{k=0}^{3} \|e^{s\phi}\mathbf{f_k}\|_{L^2(Q)^3} + \|\mathbf{u}\|_{L^2(Q_\omega)^3}) \quad \forall s > s^* \tag{4.81}$$

for some positive constant C *independent of* s *and for any solution* $\mathbf{u} \in L^2(Q)^3$ *of* (4.9).

The proofs of theorems 11 and 12 rely on theorem 10 and are similar to the proofs of the similar results presented in [14]).

Next, we consider the controllability problem

$$\begin{cases} P\mathbf{u} = \mathbf{f} + \chi_\omega \mathbf{v} \quad \text{in } Q, \quad \mathbf{u} = 0 \quad \text{on } \Sigma, \\ \mathbf{u}(\cdot,0) = \mathbf{u_0}, \quad \partial_{x_0}\mathbf{u}(\cdot,0) = \mathbf{u_1} \quad \text{in } \Omega. \end{cases} \tag{4.82}$$

Here $\mathbf{f}, \mathbf{u_0}, \mathbf{u_1}$ are given functions and $\mathbf{v}$ is a control. Recall that the operator P was defined in (4.9). Suppose that two target functions $\mathbf{u_2}$ and $\mathbf{u_3}$ are given. We have to find a control $\mathbf{v}$ such that

$$\mathbf{u}(\cdot,T) = \mathbf{u_2}, \quad \partial_{x_0}\mathbf{u}(\cdot,T) = \mathbf{u_3} \quad \text{in } \Omega. \tag{4.83}$$

Condition B. Suppose that there exists $t^* \in (0,T)$ such that

$$\underline{\text{mín}}_{x' \in \Omega\setminus\omega} \psi(x',t^*) > \text{máx}\{\underline{\text{máx}}_{x' \in \Omega\setminus\omega} \psi(x',0), \underline{\text{máx}}_{x' \in \Omega\setminus\omega} \psi(x',T)\}.$$

Before presenting the main result of this section, let us remind a well known result on the solvability of the Lamé system

$$\begin{cases} P\mathbf{p} = \mathbf{q} \quad \text{in } Q, \quad \mathbf{p} = 0 \quad \text{on } \Sigma, \\ \mathbf{p}(\cdot, 0) = \mathbf{p_0}, \quad \partial_{x_0}\mathbf{p}(\cdot, 0) = \mathbf{p_1} \quad \text{in } \Omega. \end{cases} \tag{4.84}$$

We have the following:

Theorem 13 *Assume that* (4.2) *and* (4.6) *hold. Then, if* $\mathbf{q} \in L^2(Q)^3$, $\mathbf{p_0} \in H_0^1(\Omega)^3$ *and* $\mathbf{p_1} \in L^2(\Omega)^3$, *problem* (4.84) *possesses exactly one solution* $\mathbf{p} \in H^1(Q)^3$.

Furthermore, if $\mathbf{q} \in H^{-1}(Q)^3$, $\mathbf{p_0} \in L^2(\Omega)^3$ *and* $\mathbf{p_1} \in H^{-1}(\Omega)^3$, *problem* (4.84) *possesses exactly one solution* $\mathbf{p} \in L^2(Q)^3$.

Now, we state the main result of this paper:

Theorem 14 *Assume that* (4.2), (4.4) *and* (4.6) *hold. Suppose that there exists a function* ψ *which satisfies condition A, condition B and* (4.79). *Then if* $\mathbf{f} \in L^2(Q)^3$, $\mathbf{u_0} \in H_0^1(\Omega)^3$ *and* $\mathbf{u_1} \in L^2(\Omega)^3$, *there exists a solution* $(\mathbf{u}, \mathbf{v}) \in H^1(Q)^3 \times L^2(Q_\omega)^3$ *to the controllability problem* (4.82)–(4.83).

Furthermore, if $\mathbf{f} \in H^{-1}(Q)^3$, $\mathbf{u_0} \in L^2(\Omega)^3$ *and* $\mathbf{u_1} \in H^{-1}(\Omega)^3$, *there exists a solution* $(\mathbf{u}, \mathbf{v}) \in L^2(Q)^3 \times H^{-1}(Q_\omega)^3$ *to* (4.82)–(4.83).

Proof. Let ϵ be a sufficiently small positive number such that $t^* \in (\epsilon, T-\epsilon)$ and set

$$M = \min_{x' \in \overline{\Omega \setminus \omega}} \psi(x', t^*) > \max\{ \max_{x' \in \overline{\Omega \setminus \omega}} \psi(x', \epsilon), \max_{x' \in \overline{\Omega \setminus \omega}} \psi(x', T-\epsilon)\}$$

We introduce a cut-off function $\chi(x_0) \in C_0^\infty([0, T])$ such that $\chi|_{[\epsilon, T-\epsilon]} = 1$. Let $\mathbf{p}$ be a solution to the system

$$\begin{cases} P\mathbf{p}(x) \equiv \rho(x')\partial_{x_0}^2\mathbf{p} - L\mathbf{p} = \mathbf{q} & \text{in } Q, \\ \mathbf{p} = 0 & \text{on } \Sigma. \end{cases} \tag{4.85}$$

Then the function $\tilde{\mathbf{p}} = \chi(x_0)\mathbf{p}$ satisfies the equation

$$\begin{cases} P\tilde{\mathbf{p}}(x) = \chi\mathbf{q} - [\chi, \partial_{x_0}^2]\mathbf{p} & \text{in } Q, \\ \mathbf{p} = 0 & \text{on } \Sigma. \end{cases} \tag{4.86}$$

Note that $[\chi, \partial_{x_0}^2]$ is a first order operator with coefficients having support in $[0, \epsilon] \cup [T-\epsilon, T] \times \Omega$. Therefore, we have the following a priori estimates:

$$\|[\chi, \partial_{x_0}^2]\mathbf{p}e^{s\phi}\|_{L^2(Q)^3} \leq e^{\delta M}\|\mathbf{p}\|_{H^1(Q)^3}, \quad \|[\chi, \partial_{x_0}^2]\mathbf{p}e^{s\phi}\|_{H^{-1}(Q)^3} \leq e^{\delta M}\|\mathbf{p}\|_{L^2(Q)^3}.$$

Additionally, we apply Carleman estimate (4.10) to the equation (4.86) and we obtain that

$$\|\tilde{\mathbf{p}}\|_{H^{k,s}(Q)^3} \leq C(\|e^{s\phi}\mathbf{q}\|_{H^{k-1}(Q)^3} + \|\tilde{\mathbf{p}}\|_{H^{k,s}(Q_\omega)^3} + e^{s\delta M}\|\mathbf{p}\|_{H^{1-k}(Q)^3}).$$

Note that, for any sufficiently small $\epsilon > 0$ there exists a $\delta_1 > \delta$ such that

$$\begin{aligned} &e^{s\delta_1 M}\|(\mathbf{p}(\cdot, t^*), \partial_{x_0}\mathbf{p}(\cdot, t^*))\|_{H^k(\Omega)^3 \times H^{k-1}(\Omega)^3} \\ &\qquad \leq C(\|\tilde{\mathbf{p}}\|_{H^{k,s}(Q)^3} + \|e^{s\phi}\mathbf{q}\|_{H^{k-1}(Q)^3} + \|\tilde{\mathbf{p}}\|_{H^{k,s}(Q_\omega)^3} + e^{\delta M}\|\mathbf{p}\|_{H^{1-k}(Q)^3}). \end{aligned}$$

Since $\delta_1 > \delta$, taking the parameter s sufficiently large in the previous inequality thanks to the theorem 13, we arrive at the observability estimate

$$\|\mathbf{p}\|_{H^k(Q)^3} \leq C(\|\mathbf{q}\|_{H^{k-1}(Q)^3} + \|\mathbf{p}\|_{H^k(Q_\omega)^3}).$$

This observability estimate can be readily converted into the controllability result stated in our theorem by the well known HUM method (see [24]).

This ends the proof of Theorem 14.

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Apéndice: desigualdad de Carleman para el sistema de Stokes

Desigualdad de Carleman para el sistema de Stokes

En esta sección presentamos un resumen de la demostración de la desigualdad de Carleman para el problema de Stokes con condiciones de Dirichlet que ha sido probada con detalle en [2]. Hemos visto que es esencial para el desarrollo de la tercera parte de esta memoria, correspondiente al trabajo [3].

Para $a \in L^\infty(Q)^N$, introducimos el siguiente sistema retrógrado de ecuaciones de tipo Stokes:

$$\begin{cases} -\varphi_t - \Delta\varphi - D\varphi\, a + \nabla\pi = g & \text{en } Q \\ \nabla\cdot\varphi = 0 & \text{en } Q, \\ \varphi = 0 & \text{sobre } \Sigma, \\ \varphi(T) = \varphi^0 & \text{en } \Omega. \end{cases} \tag{A.1}$$

Aquí hemos denotado

$$D\varphi = \nabla\varphi + \nabla\varphi^t.$$

Las siguientes hipótesis de regularidad tendrán que ser impuestas sobre a:

$$a \in L^\infty(Q)^N, \quad a_t \in L^2(0,T; L^\sigma(\Omega)^N) \quad \begin{pmatrix} \sigma > 1 & \text{si } N = 2 \\ \sigma > 6/5 & \text{si } N = 3 \end{pmatrix}. \tag{A.2}$$

Probaremos una desigualdad del tipo

$$\iint_Q \rho_1^2(x,t)\,|\varphi|^2\,dx\,dt \le C \iint_{\omega\times(0,T)} \rho_2^2(t)\,|\varphi|^2\,dx\,dt$$

para toda $\varphi^0 \in H$, donde $C > 0$ es una constante que sólo depende de Ω, ω, T y a.

Para probar esto, en primer lugar definimos las funciones peso que utilizaremos. La función peso básica es $\eta^0 \in C^2(\overline{\Omega})$, que verifica

$$\eta^0 > 0 \text{ en } \Omega, \quad \eta^0 = 0 \text{ sobre } \partial\Omega, \quad |\nabla\eta^0| > 0 \text{ en } \overline{\Omega\setminus\omega_1},$$

con $\omega_1 \subset\subset \omega$ un abierto no vacío. La existencia de una tal η^0 está probada en [4]. Ahora, para ciertos números positivos s y λ, definimos

$$\begin{aligned}
\alpha(x,t) &= \frac{e^{5/4\lambda m\|\eta^0\|_\infty} - e^{\lambda(m\|\eta^0\|_\infty + \eta^0(x))}}{t^4(T-t)^4}, \\
\xi(x,t) &= \frac{e^{\lambda(m\|\eta^0\|_\infty + \eta^0(x))}}{t^4(T-t)^4}, \\
\widehat{\alpha}(t) = \min_{x\in\overline{\Omega}} \alpha(x,t) &= \frac{e^{5/4\lambda m\|\eta^0\|_\infty} - e^{\lambda(m+1)\,\|\eta^0\|_\infty}}{t^4(T-t)^4}, \\
\alpha^*(t) = \max_{x\in\overline{\Omega}} \alpha(x,t) &= \frac{e^{5/4\lambda m\|\eta^0\|_\infty} - e^{\lambda m\|\eta^0\|_\infty}}{t^4(T-t)^4}, \\
\widehat{\xi}(t) = \max_{x\in\overline{\Omega}} \xi(x,t) = \frac{e^{\lambda(m+1)\|\eta^0\|_\infty}}{t^4(T-t)^4}, &\quad \xi^*(t) = \min_{x\in\overline{\Omega}} \xi(x,t) = \frac{e^{\lambda m\|\eta^0\|_\infty}}{t^4(T-t)^4}, \\
\widehat{\theta}(t) = s\lambda e^{-s\widehat{\alpha}}\widehat{\xi}, &\quad \theta(t) = s^{15/4}e^{-2s\widehat{\alpha}+s\alpha^*}\widehat{\xi}^{15/4},
\end{aligned} \tag{A.3}$$

donde $m > 4$ es un número fijo.

Por comodidad, durante la prueba usaremos la notación

$$I(s,\lambda;\varphi) = s^{-1}\iint_Q e^{-2s\alpha}\xi^{-1}(|\varphi_t|^2 + |\Delta\varphi|^2)\,dx\,dt$$
$$+s\lambda^2\iint_Q e^{-2s\alpha}\xi|\nabla\varphi|^2\,dx\,dt + s^3\lambda^4\iint_Q e^{-2s\alpha}\xi^3|\varphi|^2\,dx\,dt.$$

El resultado principal de esta sección es el siguiente:

Teorema 1 *Supongamos que a verifica* (A.2). *Entonces existen tres constantes positivas C, $\overline{s}_0$ y $\overline{\lambda}_0$ tales que para todo $\varphi^0 \in H$ y toda $g \in L^2(Q)^N$, la solución de* (A.1) *verifica*

$$\begin{aligned}
I(s,\lambda;\varphi) \le C\Bigg(& s^{15/2}\lambda^{20}\iint_Q e^{-4s\widehat{\alpha}+2s\alpha^*}\widehat{\xi}^{15/2}|g|^2\,dx\,dt \\
& + s^{16}\lambda^{40}\iint_{\omega\times(0,T)} e^{-8s\widehat{\alpha}+6s\alpha^*}\widehat{\xi}^{16}|\varphi|^2\,dx\,dt\Bigg),
\end{aligned} \tag{A.4}$$

para todo $\lambda \ge \overline{\lambda}_0(\Omega,\omega,T,a)$ y todo $s \ge \overline{s}_0(\Omega,\omega,T)$.

Observación 1 En el teorema anterior es posible conocer de qué forma dependen $\overline{\lambda}_0$ y $\overline{s}_0$ de T y de a. En concreto, se prueba en [2] que

$$\overline{\lambda}_0 = \overline{\lambda}(1 + \|a\|_\infty + \|a_t\|^2_{L^2(0,T;L^\sigma(\Omega)^N)} + e^{\overline{\lambda}T\|a\|^2_\infty})$$

y

$$\overline{s}_0 = \overline{s}(T^4 + T^8),$$

donde $\overline{\lambda}$ y $\overline{s}$ son constantes positivas que dependen de Ω y de ω.

La estimación (A.4) es una nueva desigualdad global de Carleman que ha sido demostrada en [2] con un método novedoso, si bien la prueba obedece al esquema marcado por O. Yu. Imanuvilov en [6]:

PRIMERA PARTE: Consideramos (A.1) como un sistema de N ecuaciones parabólicas y utilizamos la desigualdad de Carleman para la ecuación del calor con condiciones de Dirichlet. Una prueba de ésta se puede encontrar en [4]. Esto proporciona una estimación de la velocidad en función de la presión:

$$I(s,\lambda;\varphi) \leq C \left(\iint_{\omega\times(0,T)} \rho_3^2(x,t)|\varphi|^2\,dx\,dt + \iint_Q \rho_4^2(x,t)\,(|\pi|^2+|g|^2)\,dx\,dt \right)$$

(véase (A.7) más abajo).

SEGUNDA PARTE: Tomando el operador divergencia en la ecuación satisfecha por φ, obtenemos la ecuación elíptica verificada por la presión π. Usando la desigualdad de Carleman para las soluciones débiles de problemas elípticos probada en [7], deducimos una estimación de la presión en función de su traza sobre la frontera y de la velocidad:

$$\iint_Q \rho_4^2(x,t)\,|\pi|^2\,dx\,dt \leq C \left(\iint_{\omega\times(0,T)} \rho_5^2(x,t)|\pi|^2\,dx\,dt \right.$$
$$\left. \iint_Q \rho_6^2(x,t)(|\nabla\varphi|^2+|g|^2)\,dx\,dt + \int_0^T \rho_7^2(t)\|\pi(t)\|^2_{H^{1/2}(\partial\Omega)}\,dt \right)$$

(véase (A.11)). Seguimos ahora las ideas de [6] y usamos resultados de regularidad para el sistema de Stokes para acotar la traza de la presión. Combinado con lo anterior, esto nos dará

$$I(s,\lambda;\varphi) \leq C \left(\iint_{\omega\times(0,T)} \rho_8^2(x,t)(|\varphi|^2+|\pi|^2)\,dx\,dt + \iint_Q \rho_9^2(x,t)\,|g|^2\,dx\,dt \right)$$

(desigualdad (A.15)).

En lo que queda de prueba, nos centraremos en estimar el término local de la presión. En virtud de la ecuación que verifica φ, la tarea consiste en conseguir estimaciones adecuadas de integrales locales de $\Delta\varphi$ y φ_t.

TERCERA PARTE: En un primer apartado, indicaremos cómo se acota localmente $\Delta\varphi$. Esta es una desigualdad clásica y no conlleva argumentos de dificultad importantes, de ahí que no hagamos la prueba detallada.

CUARTA PARTE: A continuación, demostraremos la estimación local de φ_t. En ella, descompondremos nuestra solución φ (salvo un peso dependiente de t) en otras dos soluciones de problemas de tipo Stokes. En ambas podremos razonar para realizar acotaciones locales de sus derivadas en tiempo, valiéndonos exclusivamente de resultados de regularidad de las soluciones del problema de Stokes.

Antes de comenzar la prueba del teorema 1, haremos un comentario que nos será de utilidad.

Observación 2 Sean $\varphi^0 \in H$ y $g \in L^2(Q)^N$ dados y sea (φ, π) la correspondiente solución de (A.1). Sabemos que $\varphi \in L^2(0,T;V) \cap C^0([0,T];H)$. Sea $\gamma \in C^1([0,T])$ una función tal que $\gamma(T) = 0$. Entonces $(\widetilde{\varphi}, \widetilde{\pi}) := (\gamma\varphi, \gamma\pi)$ resuelve el sistema

$$\begin{cases} -\widetilde{\varphi}_t - \Delta\widetilde{\varphi} - D\widetilde{\varphi}\,a + \nabla\widetilde{\pi} = \gamma g - \gamma'\varphi & \text{en } Q, \\ \nabla \cdot \widetilde{\varphi} = 0 & \text{en } Q, \\ \widetilde{\varphi} = 0 & \text{sobre } \Sigma, \\ \widetilde{\varphi}(T) = 0 & \text{en } \Omega. \end{cases} \tag{A.5}$$

Por tanto, podemos decir que $(\widetilde{\varphi}, \widetilde{\pi})$ es una solución fuerte de (A.5). En particular,

$$\pi(t) \in H^1(\Omega), \quad \varphi(t) \in H^2(\Omega)^N, \quad \text{y} \quad \varphi_t(t) \in L^2(\Omega)^N$$

para casi todo $t \in (0,T)$.

Estimación de Carleman para la ecuación del calor

Aplicamos la desigualdad de Carleman habitual para la ecuación del calor con segundo miembro en $L^2(Q)$ a la ecuación satisfecha por φ_i, que tiene como segundo miembro

$$G_i = g_i + (D\varphi\,a)_i - \partial_i\pi.$$

La consecuencia es que existen tres constantes positivas $C_1(\Omega,\omega)$, $\lambda_0(\Omega,\omega) \geq 1$ y $s_0(\Omega,\omega) > 0$ tales que

$$\begin{aligned} I(s,\lambda;\varphi) \leq C_1 \bigg(& \iint_Q e^{-2s\alpha}(|g|^2 + |D\varphi\,a|^2 + |\nabla\pi|^2)\,dx\,dt \\ & + s^3\lambda^4 \iint_{\omega_1\times(0,T)} e^{-2s\alpha}\xi^3|\varphi|^2\,dx\,dt \bigg) \end{aligned} \tag{A.6}$$

para todo $\lambda \geq \lambda_0$ y todo $s \geq s_0(T^7 + T^8)$. De hecho, para probar (A.6) basta con seguir las ideas de [4] teniendo en cuenta que

$$|\xi^{-1}| \leq C\,T^8 \quad \text{and} \quad |\alpha_t| \leq C\,T\,\xi^{5/4}$$

para cierta $C > 0$ independiente de λ y después razonar como en [1].

Ahora, eliminamos el término de $D\varphi\,a$ en la derecha de (A.6), usando que

$$C_1\,|D\varphi\,a|^2 \leq Cs\|a\|_\infty^2\xi|\nabla\varphi|^2 \leq \frac{1}{2}s\lambda^2\xi|\nabla\varphi|^2,$$

para $\lambda \geq \lambda_1(\Omega,\omega)\|a\|_\infty$ y $s \geq s_1(\Omega,\omega)T^8$. Deducimos que

$$\begin{aligned} I(s,\lambda;\varphi) \leq C_2 \bigg(& \iint_Q e^{-2s\alpha}(|g|^2 + |\nabla\pi|^2)\,dx\,dt \\ & + s^3\lambda^4 \iint_{\omega_1\times(0,T)} e^{-2s\alpha}\xi^3|\varphi|^2\,dx\,dt \bigg) \end{aligned} \tag{A.7}$$

para todo $\lambda \geq \lambda_2(\Omega,\omega)(1+\|a\|_\infty)$ y todo $s \geq s_2(\Omega,\omega)(T^7+T^8)$.

Localización de la presión

En este apartado acotaremos la integral de $|\nabla\pi|^2$ a la derecha de (A.7) en función de un término local de $|\pi|^2$, la norma en $H^{1/2}$ de la traza de π y otros dos términos globales de $|g|^2$ y $|\nabla\varphi|^2$; éste último será absorbido más adelante con $I(s,\lambda;\varphi)$.

Esto será conseguido con la ayuda de una desigualdad de Carleman aplicada a la ecuación elíptica que verifica la presión; esta desigualdad se probó en [7].

Concretamente, aplicando el operador divergencia a la primera ecuación de (A.1), tenemos

$$\Delta\pi(t) = \nabla\cdot(D\varphi\, a + g)(t) \quad \text{en } \Omega, \tag{A.8}$$

para casi todo $t \in (0,T)$. Obsérvese que el segundo miembro de (A.8) pertenece a $H^{-1}(\Omega)$. Usando la Observación 2, también sabemos que $\pi(t) \in H^1(\Omega)$. Aplicamos entonces el resultado principal contenido en [7] (véase la desigualdad (2.10) en dicha referencia), el cual nos dice que existe una constante $\overline{C}_1(\Omega,\omega) > 0$ y dos números $\overline{\lambda} > 1$, $\overline{\tau} > 1$ tal que

$$\begin{aligned}\int_\Omega e^{2\tau\eta}|\nabla\pi(t)|^2\,dx \leq \overline{C}_1\Bigg(\tau\int_\Omega e^{2\tau\eta}\eta(|D\varphi\, a|^2+|g|^2)(t)\,dx \\ +\tau^{1/2}e^{2\tau}\|\pi(t)\|^2_{H^{1/2}(\partial\Omega)} + \int_{\omega_1} e^{2\tau\eta}(|\nabla\pi|^2+\tau^2\lambda^2\eta^2|\pi|^2)(t)\,dx\Bigg)\end{aligned} \tag{A.9}$$

para todo $\lambda \geq \overline{\lambda}$ y todo $\tau \geq \overline{\tau}$, donde η está dada por

$$\eta(x) = e^{\lambda\eta^0(x)} \quad \forall x \in \Omega.$$

La integral local de $|\nabla\pi|^2$ en (A.9) puede ser acotada a costa de tener $|\pi|^2$ en un abierto ω_2 más grande: $\omega_1 \subset\subset \omega_2 \subset\subset \omega$. Para ello, consideremos una función $\zeta \in C^2_c(\omega_2)$ tal que

$$\zeta(x) = 1 \text{ en } \omega_1, \quad 0 \leq \zeta \leq 1$$

e integremos por partes varias veces:

$$\begin{aligned}\int_{\omega_1} e^{2\tau\eta}|\nabla\pi(t)|^2\,dx &\leq \int_{\omega_2} e^{2\tau\eta}\zeta\nabla\pi(t)\cdot\nabla\pi(t)\,dx \\ &= -\frac{1}{2}\int_{\omega_2}\nabla(e^{2\tau\eta}\zeta)\cdot\nabla|\pi(t)|^2\,dx - \langle e^{2\tau\eta}\Delta\pi(t),\zeta\pi(t)\rangle_{H^{-1}(\omega_2),H^1_0(\omega_2)} \\ &= \frac{1}{2}\int_{\omega_2}\Delta(e^{2\tau\eta}\zeta)|\pi(t)|^2\,dx \\ &\quad -\langle e^{2\tau\eta}\nabla\cdot(D\varphi\, a+g)(t),\zeta\pi(t)\rangle_{H^{-1}(\omega_2),H^1_0(\omega_2)}.\end{aligned} \tag{A.10}$$

Como

$$|\Delta(e^{2\tau\eta}\zeta)| \leq 2\overline{C}_2\tau^2\lambda^2\eta^2e^{2\tau\eta} \quad \text{en } \omega_2$$

para $\lambda \geq \overline{\lambda}_0(\Omega,\omega)$ y cierta constante $\overline{C}_2(\Omega,\omega) > 0$, podemos acotar el primer término de la derecha de (A.10) por

$$\overline{C}_2\tau^2\lambda^2\int_{\omega_2} e^{2\tau\eta}\eta^2|\pi(t)|^2\,dx.$$

Integramos de nuevo por partes en el otro término. Obtenemos que

$$
\begin{aligned}
&-\langle e^{2\tau\eta}\nabla\cdot(D\varphi\, a+g)(t),\zeta\pi(t)\rangle_{H^{-1}(\omega_2),H_0^1(\omega_2)}\\
&=\int_{\omega_2}\nabla(e^{2\tau\eta}\zeta)\cdot(D\varphi\, a+g)(t)\pi(t)\,dx\\
&+\int_{\omega_2}e^{2\tau\eta}\zeta(D\varphi\, a+g)(t)\cdot\nabla\pi(t)\,dx\\
&\leq\overline{C}_4\left(\tau^2\lambda^2\int_{\omega_2}e^{2\tau\eta}\eta^2|\pi(t)|^2\,dx+\int_{\omega_2}e^{2\tau\eta}(|D\varphi\, a|^2+|g|^2)(t)\,dx\right)\\
&+\frac{1}{2}\int_{\omega_2}e^{2\tau\eta}\zeta|\nabla\pi(t)|^2\,dx
\end{aligned}
$$

para cierta constante $\overline{C}_4(\Omega,\omega)>0$, pues

$$|\nabla(e^{2\tau\eta}\zeta)|\leq\overline{C}_3(\Omega,\omega)\tau\lambda\eta e^{2\tau\eta}\qquad\text{en }\omega_2$$

para $\lambda\geq\overline{\lambda}_1(\Omega,\omega)$. A partir de (A.10), tenemos ahora que

$$
\begin{aligned}
\int_{\omega_1}e^{2\tau\eta}|\nabla\pi(t)|^2\,dx\leq\overline{C}_5\bigg(&\tau^2\lambda^2\int_{\omega_2}e^{2\tau\eta}\eta^2|\pi(t)|^2\,dx\\
&+\int_{\omega_2}e^{2\tau\eta}(|D\varphi\, a|^2+|g|^2)(t)\,dx\bigg)
\end{aligned}
$$

para $\lambda\geq\overline{\lambda}_2(\Omega,\omega)$, lo cual, junto con (A.9), nos da

$$
\begin{aligned}
\int_{\Omega}e^{2\tau\eta}|\nabla\pi(t)|^2\,dx\leq\overline{C}_6\bigg(&\tau\int_{\Omega}e^{2\tau\eta}\eta(|D\varphi\,\overline{y}|^2+|g|^2)(t)\,dx\\
&+\tau^{1/2}e^{2\tau}\|\pi(t)\|^2_{H^{1/2}(\partial\Omega)}+\tau^2\lambda^2\int_{\omega_2}e^{2\tau\eta}\eta^2|\pi(t)|^2\,dx\bigg),
\end{aligned}
$$

para $\lambda\geq\overline{\lambda}_2$ y $\tau\geq\overline{\tau}$.

Para conectar esta estimación elíptica con (A.7), tomamos

$$\tau=\frac{s}{t^4(T-t)^4}\,e^{\lambda m\|\eta^0\|_\infty},$$

multiplicamos la expresión por

$$\exp\left\{-2s\frac{e^{5/4\lambda m\|\eta^0\|_\infty}}{t^4(T-t)^4}\right\}$$

e integramos con respecto a t en $(0,T)$. Obsérvese que, para que esta elección de τ sea mayor que $\overline{\tau}$, bastará tomar $s \geq (\overline{\tau}/2^8)\, T^8$ y entonces obtenemos

$$
\begin{aligned}
&\iint_Q e^{-2s\alpha} |\nabla \pi|^2 \, dx \, dt \leq \overline{C}_7 \Bigg(s \iint_Q e^{-2s\alpha} \xi |g|^2 \, dx \, dt \\
&+ s \iint_Q e^{-2s\alpha} \xi |D\varphi \, a|^2 \, dx \, dt + s^{1/2} \int_0^T e^{-2s\alpha^*} (\xi^*)^{1/2} \|\pi(t)\|^2_{H^{1/2}(\partial\Omega)} \, dt \\
&+ s^2 \lambda^2 \iint_{\omega_2 \times (0,T)} e^{-2s\alpha} \xi^2 |\pi|^2 \, dx \, dt \Bigg)
\end{aligned}
\tag{A.11}
$$

para todo $\lambda \geq \overline{\lambda}_2$ y todo $s \geq \overline{s}_0 T^8$.

Tal y como comentamos al comienzo, seguimos ahora las ideas de [6] para acotar la traza de π. En concreto, usaremos estimaciones fuertes para las soluciones de sistemas de Stokes.

Definamos las funciones

$$
\varphi^* = s^{1/4} e^{-s\alpha^*} (\xi^*)^{1/4} \varphi, \quad \pi^* = s^{1/4} e^{-s\alpha^*} (\xi^*)^{1/4} \pi,
$$

que satisfacen el sistema

$$
\begin{cases}
-\varphi^*_t - \Delta \varphi^* + \nabla \pi^* = g^* & \text{en } Q, \\
\nabla \cdot \varphi^* = 0 & \text{en } Q, \\
\varphi^* = 0 & \text{sobre } \Sigma, \\
\varphi^*(T) = 0 & \text{en } \Omega,
\end{cases}
$$

con

$$
g^* = s^{1/4} e^{-s\alpha^*} (\xi^*)^{1/4} g + s^{1/4} e^{-s\alpha^*} (\xi^*)^{1/4} D\varphi \, a - s^{1/4} (e^{-s\alpha^*} (\xi^*)^{1/4})_t \varphi.
$$

Usando propiedades de regularidad bien conocidas para el sistema de Stokes (véase por ejemplo [8]), deducimos que $\varphi^* \in L^2(0,T; H^2(\Omega)^N \cap V) \cap L^\infty(0,T;V)$, $\varphi^*_t \in L^2(0,T;H)$, $\pi^* \in L^2(0,T; H^1(\Omega))$ y, también, que estas funciones están acotadas en esos espacios por la norma L^2 del segundo miembro. En particular,

$$
\iint_Q (|\pi^*|^2 + |\nabla \pi^*|^2) \, dx \, dt \leq \overline{C}_8 \iint_Q |g^*|^2 \, dx \, dt
$$

y, en consecuencia, tenemos que

$$
\begin{aligned}
&\int_0^T \|\pi^*(t)\|^2_{H^{1/2}(\partial\Omega)} \, dt \\
&\quad \leq \overline{C}_9 \Bigg(s^{1/2} \iint_Q e^{-2s\alpha^*} (\xi^*)^{1/2} |g|^2 \, dx \, dt \\
&\quad + s^{1/2} \|a\|^2_\infty \iint_Q e^{-2s\alpha^*} (\xi^*)^{1/2} |\nabla \varphi|^2 \, dx \, dt \\
&\quad + s^{1/2} \iint_Q |(e^{-s\alpha^*} (\xi^*)^{1/4})_t|^2 |\varphi|^2 \, dx \, dt \Bigg).
\end{aligned}
\tag{A.12}
$$

Teniendo en cuenta la definiciíon de α^* y de ξ^* (véase (A.3)), vemos que las dos primeras integrales de la derecha de (A.12) pueden acotarse por

$$s \iint_Q e^{-2s\alpha}\xi|g|^2\,dx\,dt + s\|a\|_\infty^2 \iint_Q e^{-2s\alpha}\xi|\nabla\varphi|^2\,dx\,dt,$$

si $s \geq \overline{s}_1 T^8$.

Finalmente, obtenemos una estimación de la derivada en tiempo del peso $e^{-s\alpha^*}(\xi^*)^{1/4}$:

$$\begin{aligned}(e^{-s\alpha^*}(\xi^*)^{1/4})_t &= e^{-s\alpha^*}(-s\alpha_t^*(\xi^*)^{1/4} + 1/4(\xi^*)^{-3/4}\xi_t^*)\\ &\leq \overline{C}_{10}e^{-s\alpha^*}(sT(\xi^*)^{3/2} + T(\xi^*)^{1/2}) \leq \overline{C}_{11}e^{-s\alpha^*}sT(\xi^*)^{3/2},\end{aligned}$$

para cierta constante $\overline{C}_{11} > 0$ independiente de λ; aquí se ha elegido $s \geq \overline{s}_2 T^8$. Con esto, podemos acotar el último término de (A.12). Teniendo en cuenta (A.11), obtenemos:

$$\begin{aligned}\iint_Q e^{-2s\alpha}|\nabla\pi|^2\,dx\,dt \leq \overline{C}_{12}\Bigg(s\iint_Q e^{-2s\alpha}\xi|g|^2\,dx\,dt\\ +s\|a\|_\infty^2 \iint_Q e^{-2s\alpha}\xi|\nabla\varphi|^2\,dx\,dt + s^{5/2}T^2\iint_Q e^{-2s\alpha}\xi^3|\varphi|^2\,dx\,dt\\ +s^2\lambda^2\iint_{\omega_2\times(0,T)} e^{-2s\alpha}\xi^2|\pi|^2\,dx\,dt\Bigg)\end{aligned} \tag{A.13}$$

para todo $\lambda \geq \overline{\lambda}_0$ y todo $s \geq \overline{s}_3 T^8$.

Ahora, combinamos esta desigualdad con (A.7) y llegamos a que

$$\begin{aligned}I(s,\lambda;\varphi) \leq C_3\Bigg(s^3\lambda^4\iint_{\omega_1\times(0,T)} e^{-2s\alpha}\xi^3|\varphi|^2\,dx\,dt\\ +s^2\lambda^2\iint_{\omega_2\times(0,T)} e^{-2s\alpha}\xi^2|\pi|^2\,dx\,dt + s\iint_Q e^{-2s\alpha}\xi|g|^2\,dx\,dt\ ,\\ +s^{5/2}T^2\iint_Q e^{-2s\alpha}\xi^3|\varphi|^2\,dx\,dt + s\|a\|_\infty^2\iint_Q e^{-2s\alpha}\xi|\nabla\varphi|^2\,dx\,dt\Bigg)\end{aligned} \tag{A.14}$$

para $\lambda \geq \lambda_3(1+\|a\|_\infty)$ y $s \geq s_3(T^7 + T^8)$.

A continuación, absorbemos los dos últimos términos de (A.14) tomando $s \geq s_4 T^4$ y $\lambda \geq \lambda_4\|a\|_\infty$ de modo que

$$C_3 s^{5/2}T^2 \leq \frac{1}{2}s^3, \quad C_3\|a\|_\infty^2 \leq \frac{1}{2}\lambda^2.$$

Por tanto, obtenemos la desigualdad

$$\begin{aligned}I(s,\lambda;\varphi) \leq C_4\Bigg(s^3\lambda^4\iint_{\omega_1\times(0,T)} e^{-2s\alpha}\xi^3|\varphi|^2\,dx\,dt\\ +s^2\lambda^2\iint_{\omega_2\times(0,T)} e^{-2s\alpha}\xi^2|\pi|^2\,dx\,dt + s\iint_Q e^{-2s\alpha}\xi|g|^2\,dx\,dt\Bigg)\end{aligned} \tag{A.15}$$

para $\lambda \geq \lambda_5(1 + \|a\|_\infty)$ y $s \geq s_5(T^4 + T^8)$.

Como anunciamos al comienzo, el resto de la demostración tiene como objetivo la eliminación del término local de la presión que aparece a la derecha de (A.15). Podemos mencionar dos grandes dificultades para llevar a cabo esta tarea: el hecho de estimar localmente la presión de un problema de Stokes como (A.1) en función de términos locales que sólo dependen del campo de velocidades es bastante delicado; también lo es que tengamos que tratar con funciones peso que dependen de la variable x.

Por consiguiente, en primer lugar reemplazaremos la función peso por otra que sólo dependa de t. Esto nos permitirá reducirnos a estimar una integral de $|\nabla\pi|^2$ en vez de $|\pi|^2$. Entonces, teniendo en cuenta la ecuación que verifican φ y π, el objetivo será acotar integrales locales de $|\Delta\varphi|^2$ y $|\varphi_t|^2$.

De hecho, las definiciones de $\widehat{\alpha}$, $\widehat{\xi}$ y $\widehat{\theta}$ (véase (A.3)) dan directamente

$$s^2\lambda^2 \iint_{\omega_2\times(0,T)} e^{-2s\alpha}\xi^2|\pi|^2\,dx\,dt \leq \iint_{\omega_2\times(0,T)} |\widehat{\theta}|^2|\pi|^2\,dx\,dt.$$

Podemos tomar $\pi(t)$ cumpliendo

$$\int_{\omega_2} \pi(t)\,dx = 0$$

para cada $t \in (0,T)$. Así, usando la desigualdad de Poincaré-Wirtinger, sabemos que existe $C_5 > 0$ tal que

$$\iint_{\omega_2\times(0,T)} |\widehat{\theta}|^2|\pi|^2\,dx\,dt \leq C_5 \iint_{\omega_2\times(0,T)} |\widehat{\theta}|^2|\nabla\pi|^2\,dx\,dt.$$

De la primera ecuación de (A.1), deducimos por tanto que

$$\begin{aligned} s^2\lambda^2 \iint_{\omega_2\times(0,T)} e^{-2s\alpha}\xi^2|\pi|^2\,dx\,dt \leq C_6 \Bigg(&\iint_{\omega_2\times(0,T)} |\widehat{\theta}|^2|g|^2\,dx\,dt \\ + \iint_{\omega_2\times(0,T)} |\widehat{\theta}|^2|\Delta\varphi|^2\,dx\,dt &+ \iint_{\omega_2\times(0,T)} |\widehat{\theta}|^2|\varphi_t|^2\,dx\,dt \\ +\|a\|_\infty^2 &\iint_{\omega_2\times(0,T)} |\widehat{\theta}|^2|\nabla\varphi|^2\,dx\,dt \Bigg). \end{aligned} \tag{A.16}$$

Estimación local de $|\Delta\varphi|^2$

En este apartado vamos a indicar cómo se puede acotar el segundo término a la derecha de (A.16). Sea ω_3 un abierto no vacío tal que

$$\omega_2 \subset\subset \omega_3 \subset\subset \omega$$

y sea $\rho \in \mathcal{D}(\omega_3)$ una función verificando $\rho \equiv 1$ en ω_2.

Pongamos

$$u(x,t) = \widehat{\theta}(t)\rho(x)\Delta\varphi(x,t) \quad \text{en } \mathbf{R}^N \times (0,T).$$

Aquí, hemos supuesto que u se extiende por cero fuera de ω_3. El objetivo será acotar la integral

$$\iint_{\omega_2\times(0,T)} |u|^2\,dx\,dt.$$

Veamos qué ecuación del calor verifica u. Para ello, aplicamos el operador de Laplace a la ecuación satisfecha por φ y tenemos en cuenta (A.8); obtenemos

$$(\Delta\varphi)_t - \Delta(\Delta\varphi) = f \quad \text{en } Q, \tag{A.17}$$

donde

$$f = \Delta(D\varphi\, a) + \Delta g - \nabla(\nabla\cdot(D\varphi\, a)) - \nabla(\nabla\cdot g).$$

A partir de (A.17), deducimos

$$\begin{cases} u_t - \Delta u = F & \text{en } \mathbf{R}^N\times(0,T), \\ u(0) = 0 & \text{en } \mathbf{R}^N, \end{cases} \tag{A.18}$$

con

$$F = \widehat{\theta}\rho f + \widehat{\theta}'\rho\Delta\varphi - 2\widehat{\theta}\nabla\rho\cdot\nabla\Delta\varphi - \widehat{\theta}\Delta\rho\Delta\varphi.$$

Obsérvese que $F\in L^2(0,T;H^{-2}(\mathbf{R}^N)^N)$ y que *a priori* sabemos que $u\in L^2(\mathbf{R}^N\times(0,T))^N$ (Observación 2). De la propia ecuación de (A.18), deducimos que $u_t\in L^2(0,T;H^{-2}(\mathbf{R}^N)^N)$, luego $u(0)$ tiene sentido. Además, es posible probar que el sistema (A.18) posee una única solución en esta clase.

Ahora, reescribimos F como la suma de derivadas segundas de g, $D\varphi\,\overline{y}$ y φ más el resto de términos. En concreto, escribimos $F = F_1 + F_2$, con

$$F_1 = \widehat{\theta}\Delta(\rho(D\varphi\, a)) + \widehat{\theta}\Delta(\rho g) - \widehat{\theta}\,\nabla(\nabla\cdot(\rho(D\varphi\, a)))$$
$$-\widehat{\theta}\nabla(\nabla\cdot(\rho\, g)) + \widehat{\theta}'\Delta(\rho\varphi)$$

y

$$F_2 = -2\widehat{\theta}\nabla\rho\cdot\nabla(D\varphi\, a) - \widehat{\theta}\Delta\rho(D\varphi\, a) - 2\widehat{\theta}\nabla\rho\cdot\nabla g$$
$$-\widehat{\theta}\Delta\rho g + \widehat{\theta}\nabla(\nabla\rho\cdot(D\varphi\, a)) + \widehat{\theta}\nabla\rho(\nabla\cdot(D\varphi\, a))$$
$$\widehat{\theta}\nabla(\nabla\rho\cdot g) + \widehat{\theta}\nabla\rho(\nabla\cdot g) - 2\widehat{\theta}'\nabla\rho\cdot\nabla\varphi$$
$$-\widehat{\theta}'\Delta\rho\varphi - 2\widehat{\theta}\nabla\rho\cdot\nabla\Delta\varphi - \widehat{\theta}\Delta\rho\Delta\varphi.$$

Nótese que $F_1\in L^2(0,T;H^{-2}(\mathbf{R}^N)^N)$, mientras que $F_2\in L^2(0,T;H^{-1}(\mathbf{R}^N)^N)$. A continuación, definiremos y estimaremos dos funciones u^1 y u^2 en $L^2(\mathbf{R}^N\times(0,T))^N$ cumpliendo

$$\begin{cases} u^i_t - \Delta u^i = F_i & \text{en } \mathbf{R}^N\times(0,T), \\ u^i(0) = 0 & \text{en } \mathbf{R}^N \end{cases} \tag{A.19}$$

para $i = 1,2$. Por tanto, bastará con estimar las integrales

$$\iint_{\omega_2\times(0,T)} |u^i|^2\,dx\,dt.$$

Para la primera de ellas, definimos $u_1 \in L^2(\mathbf{R}^N \times (0,T))$ como la solución por transposición de (A.19) para $i = 1$. En otras palabras, u_1 es la única función de $L^2(\mathbf{R}^N \times (0,T))$ que verifica para cada $h \in L^2(\mathbf{R}^N \times (0,T))$ la igualdad

$$\begin{aligned} \iint_{\mathbf{R}^N \times (0,T)} & u^1 \cdot h \, dx \, dt \\ & = \iint_{\mathbf{R}^N \times (0,T)} (\widehat{\theta}\rho\,(g + D\varphi\, a)) \cdot \Delta z) \, dx \, dt \\ & - \iint_{\mathbf{R}^N \times (0,T)} \widehat{\theta}\rho(D\varphi\, a) \cdot \nabla(\nabla \cdot z) \, dx \, dt \\ & - \iint_{\mathbf{R}^N \times (0,T)} \widehat{\theta}\rho g \cdot \nabla(\nabla \cdot z) \, dx \, dt \\ & + \iint_{\mathbf{R}^N \times (0,T)} \widehat{\theta}'\rho\varphi \cdot \Delta z \, dx \, dt, \end{aligned} \tag{A.20}$$

donde z es la solución de

$$\begin{cases} z_t - \Delta z = h & \text{en } \mathbf{R}^N \times (0,T), \\ z(0) = 0 & \text{en } \mathbf{R}^N. \end{cases} \tag{A.21}$$

A partir de la definición, no es difícil obtener la estimación L^2 de u_1 (para más detalles, véase [2]):

$$\begin{aligned} \iint_{\omega_2 \times (0,T)} |u^1|^2 \, dx \, dt & \leq \iint_{\mathbf{R}^N \times (0,T)} |u^1|^2 \, dx \, dt \\ & \leq \widehat{C}_1 \left(\iint_{\omega_3 \times (0,T)} |\widehat{\theta} g|^2 \, dx \, dt + \iint_{\omega_3 \times (0,T)} |\widehat{\theta}'\varphi|^2 \, dx \, dt \right. \\ & + \left. \iint_{\omega_3 \times (0,T)} |\widehat{\theta} D\varphi\, a|^2 \, dx \, dt \right), \end{aligned} \tag{A.22}$$

con $\widehat{C}_1(\omega) > 0$.

Para realizar la estimación de u_2, basta definirla como la solución débil de (A.19) para $i = 2$ (puesto que $F_2 \in L^2(0,T; H^{-1}(\mathbf{R}^N))$). Directamente, deducimos

$$\begin{aligned} \iint_{\omega_2 \times (0,T)} (|u^2|^2 + |\nabla u^2|^2) \, dx \, dt \leq \widehat{C}_2 & \left(\iint_{\omega_3 \times (0,T)} |\widehat{\theta}'\varphi|^2 \, dx \, dt \right. \\ & + \left. \iint_{\omega_3 \times (0,T)} |\widehat{\theta}|^2 (|g|^2 + |D\varphi\, a|^2 + |\varphi|^2) \, dx \, dt \right). \end{aligned} \tag{A.23}$$

Naturalmente, esto también es consecuencia de que u_2 es la solución de (A.19) en el sentido del semigrupo asociado a la ecuación del calor en $\mathbf{R}^N$; para más detalles, véase [2].

En definitiva, combinando esto con (A.22), llegamos a que

$$\begin{aligned}\iint_{\omega_2\times(0,T)} |\widehat{\theta}|^2|\Delta\varphi|^2\,dx\,dt \leq \widehat{C}_3 \Bigg(\iint_{\omega_3\times(0,T)} |\widehat{\theta}\,'|^2|\varphi|^2\,dx\,dt \\ + \iint_{\omega_3\times(0,T)} |\widehat{\theta}|^2(|g|^2 + |D\varphi\,a|^2 + |\varphi|^2)\,dx\,dt\Bigg).\end{aligned} \tag{A.24}$$

Estimación local de $|\varphi_t|^2$

Este parágrafo contiene la parte más complicada de la prueba. Resumiremos el argumento que está desarrollado en su completitud en [2].

Sean (ψ_1, q_1) y (ψ_2, q_2) las soluciones respectivas de

$$\begin{cases} -\psi_{1,t} - \Delta\psi_1 - D\psi_1\,a + \nabla q_1 = \theta g & \text{en } Q, \\ \nabla\cdot\psi_1 = 0 & \text{en } Q, \\ \psi_1 = 0 & \text{sobre } \Sigma, \\ \psi_1(T) = 0 & \text{en } \Omega \end{cases} \tag{A.25}$$

y

$$\begin{cases} -\psi_{2,t} - \Delta\psi_2 - D\psi_2\,a + \nabla q_2 = -\theta'\varphi & \text{en } Q, \\ \nabla\cdot\psi_2 = 0 & \text{en } Q, \\ \psi_2 = 0 & \text{sobre } \Sigma, \\ \psi_2(T) = 0 & \text{en } \Omega \end{cases} \tag{A.26}$$

(θ se definió en (A.3)). Por la unicidad del sistema de Stokes, tenemos que

$$\theta\varphi = \psi_1 + \psi_2 \quad \text{y} \quad \theta\pi = q_1 + q_2.$$

Luego debemos acotar el término

$$\begin{aligned}\iint_{\omega_2\times(0,T)} |\widehat{\theta}|^2|\varphi_t|^2\,dx\,dt &= \iint_{\omega_2\times(0,T)} \theta^{-2}|\widehat{\theta}|^2|\theta\,\varphi_t|^2\,dx\,dt \\ &= s^{-11/2}\lambda^2 \iint_{\omega_2\times(0,T)} e^{-2s\alpha^*+2s\widehat{\alpha}}\widehat{\xi}^{-11/2}|\psi_{1,t} + \psi_{2,t} - \theta'\varphi|^2\,dx\,dt.\end{aligned} \tag{A.27}$$

A continuación, acotamos las integrales locales asociadas a $\psi_{1,t}$ y $\psi_{2,t}$.

La estimación del término $e^{-2s\alpha^*+2s\widehat{\alpha}}\widehat{\xi}^{-11/2}|\psi_{1,t}|^2$ se hará en realidad en $\Omega\times(0,T)$ y sin la ayuda de la función peso. En concreto, es fácil ver que la función peso

$$e^{-2s\alpha^*+2s\widehat{\alpha}}\widehat{\xi}^{-11/2}$$

está uniformemente acotada en $(0,T)$. Ahora, aplicamos estimaciones de regularidad para el sistema de Stokes (véase, por ejemplo, [8]) a (A.25) y deducimos que

$$\psi_1 \in L^2(0,T;H^2(\Omega)^N) \cap H^1(0,T;L^2(\Omega)^N)$$

y

$$\|\psi_{1,t}\|^2_{L^2(Q)^N} + \|\psi_1\|^2_{L^2(0,T;H^2(\Omega)^N} \leq C_1^*(1+\|a\|^2_\infty e^{C_2^* T\|a\|^2_\infty})\|\theta g\|^2_{L^2(Q)^N}.$$

Luego, tenemos

$$\begin{aligned} s^{-11/2}\lambda^2 & \iint_{\omega_2\times(0,T)} e^{-2s\alpha^*+2s\widehat{\alpha}}\widehat{\xi}^{-11/2}|\psi_{1,t}|^2\,dx\,dt \\ & \leq C_3^*(1+\|a\|^2_\infty e^{C_2^* T\|a\|^2_\infty}) \iint_Q |\theta|^2|g|^2\,dx\,dt. \end{aligned} \tag{A.28}$$

En lo que se refiere a la estimación de $\psi_{2,t}$, indiquemos que en el transcurso de la prueba necesitaremos acotar la segunda derivada en tiempo de ψ_2. Esto será posible bajo las hipótesis de regularidad sobre a impuestas en (A.2).

En concreto, integrando por partes dos veces con respecto a t, obtenemos

$$\begin{aligned} & s^{-11/2}\lambda^2 \iint_{\omega_2\times(0,T)} e^{-2s\alpha^*+2s\widehat{\alpha}}\widehat{\xi}^{-11/2}|\psi_{2,t}|^2\,dx\,dt \\ & = \frac{1}{2}s^{-11/2}\lambda^2 \iint_{\omega_2\times(0,T)} (e^{-2s\alpha^*+2s\widehat{\alpha}}\widehat{\xi}^{-11/2})_{tt}|\psi_2|^2\,dx\,dt \\ & -s^{-11/2}\lambda^2 \iint_{\omega_2\times(0,T)} e^{-2s\alpha^*+2s\widehat{\alpha}}\widehat{\xi}^{-11/2}\psi_{2,tt}\cdot\psi_2\,dx\,dt. \end{aligned} \tag{A.29}$$

De nuevo, no es difícil ver que la función peso

$$(s^{-11/2}\lambda^2 e^{-2s\alpha^*+2s\widehat{\alpha}}\widehat{\xi}^{-11/2})_{tt}$$

está acotada uniformemente.

Introduzcamos la función

$$\theta^* = s^{-11/2}\lambda^{-4}e^{-2s\alpha^*+2s\widehat{\alpha}}\widehat{\xi}^{-11/2}$$

y acotemos el segundo término de la derecha de (A.29). Usando la desigualdad de Hölder, deducimos que

$$\begin{aligned} -\lambda^6 & \iint_{\omega_2\times(0,T)} \theta^*\psi_{2,tt}\psi_2\,dx\,dt \\ & \leq \lambda^6\|\theta^*\psi_{2,tt}\|_{L^2(0,T;L^r(\omega_2)^N)}\|\psi_2\|_{L^2(0,T;L^{r'}(\omega_2)^N)} \\ & \leq \frac{1}{2}\|\theta^*\psi_{2,tt}\|^2_{L^2(0,T;L^r(\omega_2)^N)} + \frac{1}{2}\lambda^{12}\|\psi_2\|^2_{L^2(0,T;L^{r'}(\omega_2)^N)}, \end{aligned} \tag{A.30}$$

donde $6/5 < r < \sigma$ si $N=3$ y $1 < r < \sigma$ si $N=2$ (σ se definió en (A.2)).

Acotemos en primer lugar el último término de (A.30). Sea $\zeta \in C^2(\omega_3)$ tal que

$$\operatorname{supp}\zeta \subset \omega_3 \quad \text{y} \quad \zeta = 1 \text{ en } \omega_2.$$

Entonces,

$$\begin{aligned}\|\psi_2\|^2_{L^2(0,T;L^{r'}(\omega_2)^N)} &\le C_4^*\|\Delta(\zeta\psi_2)\|^2_{L^2(0,T;L^2(\omega_3)^N)}\\ &= C_4^*\|\psi_2\Delta\zeta + 2\nabla\zeta\cdot\nabla\psi_2 + \zeta\Delta\psi_2\|^2_{L^2(0,T;L^2(\omega_3)^N)},\end{aligned}$$

pues $H^2(\omega_3)^N \cap H_0^1(\omega_3)^N$ se inyecta continuamente en $L^{r'}(\omega_3)^N$, para todo $r' < \infty$.

En este punto, podemos aplicar la estimación local obtenida en el parágrafo anterior a $\|\zeta\Delta\psi_2\|^2_{L^2(0,T;L^2(\omega_3)^N)}$. De hecho, como $\psi_2(T) = 0$, basta aplicar la desigualdad (A.24) con $\varphi = \psi_2$, $\widehat{\theta} = 1$ y $g = -\theta'\varphi$. Esto da

$$\begin{aligned}&\|\psi_2\|^2_{L^2(0,T;L^{r'}(\omega_2)^N)}\\ &\qquad \le C_5^*(1+\|a\|^2_\infty)\iint_{\omega_3\times(0,T)}(|\psi_2|^2 + |\nabla\psi_2|^2 + |\theta'\varphi|^2)\,dx\,dt.\end{aligned}$$

De las definiciones de ψ_1 y ψ_2, también deducimos que

$$|\psi_2|^2 + |\nabla\psi_2|^2 \le 2(|\psi_1|^2 + |\nabla\psi_1|^2 + |\theta|^2(|\varphi|^2 + |\nabla\varphi|^2)).$$

Como consecuencia, si vemos ψ_1 como la solución débil de (A.25) y usamos estimaciones globales, llegamos a

$$\begin{aligned}\|\psi_2\|^2_{L^2(0,T;L^{r'}(\omega_2)^N)} &\le C_6^*(1+\|a\|^2_\infty)\Bigg(e^{C_7^*T\|a\|^2_\infty}\|\theta g\|^2_{L^2(Q)^N}\\ &+\iint_{\omega_3\times(0,T)}(|\theta|^2+|\theta'|^2)|\varphi|^2\,dx\,dt + \iint_{\omega_3\times(0,T)}|\theta|^2|\nabla\varphi|^2\,dx\,dt\Bigg).\end{aligned}\tag{A.31}$$

Por último, digamos cómo se puede obtener una cota del término de $\psi_{2,tt}$ en (A.30).

Consideramos el par $(\psi, q) := (\theta^*\psi_{2,t}, \theta^* q_{2,t})$. Se puede probar (por ejemplo, usando un argumento de densidad sobre a) que este par verifica

$$\begin{cases} -\psi_t - \Delta\psi - D\psi\, a + \nabla q = G & \text{en } Q,\\ \nabla\cdot\psi = 0 & \text{en } Q,\\ \psi = 0 & \text{sobre } \Sigma,\\ \psi(T) = 0 & \text{en } \Omega,\end{cases}\tag{A.32}$$

donde

$$G = -\theta^*\theta''\varphi - \theta^*\theta'\varphi_t + \theta^* D\psi_2\, a_t - (\theta^*)'\psi_{2,t}.$$

Con el objetivo de estimar ψ_t en $L^2(0,T;L^r(\Omega)^N)$, primero acotamos el término de transporte $(D\psi\, a)$ en el mismo espacio. De hecho, si miramos ψ como la solución débil de (A.32), deducimos que $\psi \in L^2(0,T;V)$ y

$$\|\psi\|_{L^2(0,T;V)} \le C_8^* e^{C_9^*T\|a\|^2_\infty}\|G\|_{L^2(0,T;H^{-1}(\Omega)^N)},\tag{A.33}$$

luego tenemos la misma cota para $\|D\psi\|_{L^2(Q)^N}$.

Tras diversas estimaciones de cierta complejidad, se puede demostrar que

$$\theta^* D\psi_2\, a_t \in L^2(0,T;L^r(\Omega)^N)$$

y que

$$\begin{aligned}\|\theta^* D\psi_2\, a_t\|_{L^2(0,T;L^r(\Omega)^N)} \\ \leq C_{10}^* \|a_t\|_{L^2(0,T;L^\sigma(\Omega)^N)}\, T^{-1/2+1/k_3'}(1+\|a\|_\infty^2)(\|\theta^*\Delta\psi_2\|_{L^2(Q)^N} \\ +\|(\theta^*)'\Delta\psi_2\|_{L^2(Q)^N} + \|(\theta^*\psi_2)_t\|_{L^2(Q)^N} + \|((\theta^*)'\psi_2)_t\|_{L^2(Q)^N} \\ +\|\theta^*\theta'\Delta\varphi\|_{L^2(Q)^N} + \|(\theta^*\theta'\varphi)_t\|_{L^2(Q)^N}).\end{aligned} \tag{A.34}$$

La prueba detallada de esto se encuentra en la etapa 6 de la prueba del teorema 1 de [2].

Bajo estas condiciones, podemos aplicar el teorema 2.8 de [5] para deducir que $\psi_t \in L^2(0,T;L^r(\Omega)^N)$, junto con la estimación

$$\|\psi_t\|_{L^2(0,T;L^r(\Omega)^N)} \leq C_{11}^* \|G\|_{L^2(0,T;L^r(\Omega)^N)} \tag{A.35}$$

para cierta constante $C_{11}^* > 0$ que depende de Ω pero no de T. Como $L^r(\Omega)$ se inyecta continuamente en $H^{-1}(\Omega)$, de (A.33) y (A.35) deducimos que

$$\begin{aligned}\|\theta^*\psi_{2,tt}\|_{L^2(0,T;L^r(\Omega)^N)} \leq C_{12}^*(1+\|a\|_\infty)e^{C_{13}^* T\|a\|_\infty^2}(\|\theta^*\theta''\varphi\|_{L^2(Q)^N} \\ +\|\theta^*\theta'\varphi_t\|_{L^2(Q)^N} + \|(\theta^*)'\psi_{2,t}\|_{L^2(Q)^N} + \|\theta^* D\psi_2\, a_t\|_{L^2(0,T;L^r(\Omega)^N)}).\end{aligned}$$

Usando (A.29)–(A.31) y esta última desigualdad, tenemos

$$\begin{aligned}s^{-11/2}\lambda^2 \iint_{\omega_2\times(0,T)} e^{-2s\alpha^*+2s\widehat{\alpha}}\widehat{\xi}^{-11/2}|\psi_{2,t}|^2\,dx\,dt \\ \leq C_{14}^*(1+\|a\|_\infty^2)e^{C_{15}^* T\|a\|_\infty^2}\Big(\lambda^{12}(\|\theta g\|_{L^2(Q)^N}^2 \\ +\|\theta\varphi\|_{L^2(0,T;L^2(\omega_5)^N)}^2 + \|\theta'\varphi\|_{L^2(0,T;L^2(\omega_5)^N)}^2 + \|\theta\nabla\varphi\|_{L^2(0,T;L^2(\omega_5)^N)}^2) \\ +\|\theta^*\theta''\varphi\|_{L^2(Q)^N}^2 + \|\theta^*\theta'\varphi_t\|_{L^2(Q)^N}^2 + \|(\theta^*)'\psi_{2,t}\|_{L^2(Q)^N}^2 \\ +\|\theta^* D\psi_2\, a_t\|_{L^2(0,T;L^r(\Omega)^N)}^2\Big).\end{aligned} \tag{A.36}$$

Para obtener la estimación de la derivada en tiempo de φ, combinamos (A.27), (A.28) y (A.36) y obtenemos que

$$\begin{aligned}\iint_{\omega_2\times(0,T)} |\widehat{\theta}|^2|\varphi_t|^2\,dx\,dt \\ \leq C_{16}^*(1+\|a\|_\infty^2)e^{C_{17}^* T\|a\|_\infty^2}\Big(\lambda^{12}(\|\theta g\|_{L^2(Q)^N}^2 \\ +\|\theta\varphi\|_{L^2(0,T;L^2(\omega_5)^N)}^2 + \|\theta'\varphi\|_{L^2(0,T;L^2(\omega_5)^N)}^2 + \|\theta\nabla\varphi\|_{L^2(0,T;L^2(\omega_5)^N)}^2) \\ +\|\theta^*\theta''\varphi\|_{L^2(Q)^N}^2 + \|\theta^*\theta'\varphi_t\|_{L^2(Q)^N}^2 + \|(\theta^*)'\psi_{2,t}\|_{L^2(Q)^N}^2 \\ +\|\theta^* D\psi_2\, a_t\|_{L^2(0,T;L^r(\Omega)^N)}^2\Big).\end{aligned} \tag{A.37}$$

Para acabar, en primer lugar usamos la expresión de $\psi_2 = -\psi_1 + \theta\varphi$ junto con la estimación (A.28) para ψ_1. Esto nos lleva a la desigualdad

$$\begin{aligned}
&\iint_{\omega_2\times(0,T)} |\widehat{\theta}|^2|\varphi_t|^2\,dx\,dt \\
&\quad \le C_{18}^*(1+\|a\|_\infty^6)\|a_t\|^2_{L^2(0,T;L^\sigma(\Omega)^N)} e^{C_{19}^* T\|a\|_\infty^2} \\
&\quad \Big(\lambda^{12}(\|\theta\, g\|^2_{L^2(Q)^N} + \|\theta\varphi\|^2_{L^2(0,T;L^2(\omega_5)^N)} \\
&\quad +\|\theta'\varphi\|^2_{L^2(0,T;L^2(\omega_5)^N)} + \|\theta\nabla\varphi\|^2_{L^2(0,T;L^2(\omega_5)^N)}) \\
&\quad +(1+T^{1/2})(\|\theta^*\theta'\varphi\|^2_{L^2(Q)^N} + \|(\theta^*)'\theta'\varphi\|^2_{L^2(Q)^N} \\
&\quad +\|(\theta^*)''\theta\varphi\|^2_{L^2(Q)^N} + \|\theta^*\theta''\varphi\|^2_{L^2(Q)^N} + \|\theta^*\theta\Delta\varphi\|^2_{L^2(Q)^N} \\
&\quad +\|\theta^*\theta'\Delta\varphi\|_{L^2(Q)^N} + \|(\theta^*)'\theta\Delta\varphi\|^2_{L^2(Q)^N} + \|\theta^*\theta\varphi_t\|^2_{L^2(Q)^N} \\
&\quad +\|\theta^*\theta'\varphi_t\|^2_{L^2(Q)^N} + \|(\theta^*)'\theta\varphi_t\|^2_{L^2(Q)^N})\Big)\,.
\end{aligned} \tag{A.38}$$

Teniendo en cuenta la definición de las funciones peso (véase A.3) y haciendo una buena elección de los parámetros s y λ, obtenemos

$$\begin{aligned}
&C_4C_6\iint_{\omega_2\times(0,T)} |\widehat{\theta}|^2|\varphi_t|^2\,dx\,dt \\
&\le C_{20}^*\lambda^{20}(\|\theta g\|^2_{L^2(Q)^N} + \|\theta\varphi\|^2_{L^2(0,T;L^2(\omega_5)^N)} + \|\theta\varphi\|^2_{L^2(0,T;L^2(\omega_5)^N)} \\
&+\|\theta\nabla\varphi\|^2_{L^2(0,T;L^2(\omega_5)^N)}) + \frac{1}{2}I(s,\lambda;\varphi).
\end{aligned}$$

Finalmente, combinamos esta desigualdad con (A.15), (A.16) y (A.24) y llegamos a que

$$\begin{aligned}
I(s,\lambda;\varphi) \le C_7\lambda^{20}\Big(&\|\theta g\|^2_{L^2(Q)^N} + \|\theta\varphi\|^2_{L^2(0,T;L^2(\omega_5)^N)} \\
&+\|\theta\varphi\|^2_{L^2(0,T;L^2(\omega_5)^N)} + \|\theta\nabla\varphi\|^2_{L^2(0,T;L^2(\omega_5)^N)}\Big)\,,
\end{aligned} \tag{A.39}$$

de donde deducimos (A.4) casi directamente.

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