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Résultats sur la contrôlabilité exacte aux trajectoires de quelques systèmes paraboliques nonlinéaires.

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Introduction

Dans ce travail on présente des résultats sur la contrôlabilité de quelques systèmes d'équations aux derivées partielles paraboliques, notamment le système de réaction-diffusion et le système de Navier-Stokes. Tout au long de ce mémoire, $\Omega \subset \mathbf{R}^N$ dénotera un ouvert borné, $\omega \subset \Omega$ le domaine de contrôle, T un nombre strictement positif qui représente le temps final de l'évolution et $\mathbb{1}_\omega$ la fonction caractéristique de l'ouvert ω. On rappelle les définitions de contrôlabilité qui seront utilisées dans la suite :

• On dit qu'un problème parabolique général de contrôle est *contrôlable à zéro* au temps $T > 0$ si, pour toute condition initiale y^0, on peut trouver une fonction contrôle v telle que l'état associé est amené à zéro à l'instant T.

• Pour des systèmes nonlinéaires il ne sera pas toujours possible de déduire des résultats de contrôlabilité (globale) à zéro. Il est donc habituel d'introduire le concept de *contrôlabilité locale à zéro* au temps T qui consiste, sous l'hypothèse supplémentaire que y^0 soit suffisament petit, à amener le système à zéro à l'instant T.

• Une question aussi très intéressante est la *contrôlabilité exacte aux trajectoires* des mêmes systèmes ; c'est à dire, étant donnée une solution non contrôlée de notre système, est-il possible d'amener, à l'aide d'un contrôle, la solution du système contrôlé à cette trajectoire à l'instant T ? De plus, cette dernière question semble être le problème le plus naturel à considérer.

Dans ce mémoire, on montrera de nouveaux résultats de contrôlabilité exacte aux trajectoires de systèmes nonlinéaires, après avoir établi des résultats de contrôlabilité à zéro appropriés pour des systèmes linéarisés associés.

Beaucoup de progrès ont été faits ces dernières années dans le cadre de la contrôlabilité des équations paraboliques. Dans un premier temps, tant que les conditions aux limites sont de type Dirichlet, la contrôlabilité à zéro du système

$$\begin{cases} y_t - \Delta y = v\mathbb{1}_\omega & \text{dans } Q = \Omega \times (0,T), \\ y = 0 & \text{sur } \Sigma = \partial\Omega \times (0,T), \\ y(0) = y^0 & \text{dans } \Omega \end{cases} \tag{1}$$

a été démontrée, d'une part par G. Lebeau et L. Robbiano dans [20] et d'autre part par A. V. Fursikov et O. Yu. Imanuvilov dans [15] en utilisant une méthode qui peut s'adapter à systèmes assez generaux reposant sur des *inégalités de Carleman globales*. Ce type d'inégalités constitue une estimation très importante du point de vue de la contrôlabilité. Plus précisément, les inégalités globales de Carleman impliquent en particulier l'observabilité pour le problème adjoint associé au problème de contrôle linéaire et il est bien connu que l'observabilité du problème adjoint entraîne la contrôlabilité à zéro.

Dans [23], l'auteur résout pour la première fois la contrôlabilité d'un système nonlinéaire, à l'aide d'un argument de point fixe. Depuis, des avancées fructueuses ont été faites dans ce domaine. Un exemple est le travail [10], qui démontre la *contrôlabilité approchée* du système de réaction-diffusion nonlinéaire (1).

Dans [14], E. Fernández-Cara et E. Zuazua ont étudié la contrôlabilité à zéro de systèmes nonlinéaires avec nonlinéarités "explosives". En fait, pour des f satisfaisant

$$\lim_{|s|\to\infty} \frac{f(s)}{s\log^{3/2}(1+|s|)} = 0, \tag{2}$$

ils démontrent que le système

$$\begin{cases} y_t - \Delta y + f(y) = v\mathbb{1}_\omega & \text{dans } Q, \\ y = 0 & \text{sur } \Sigma, \\ y(0) = y^0 & \text{dans } \Omega \end{cases} \tag{3}$$

est contrôlable à zéro. Très récemment et comme application du travail [19], A. Doubova et al. ont étendu dans [7] ce résultat à des nonlinéarités qui dépendent de l'état et du gradient de l'état. Ils ont déduit la contrôlabilité à zéro du système

$$\begin{cases} y_t - \Delta y + F(y, \nabla y) = v\mathbb{1}_\omega & \text{dans } Q, \\ y = 0 & \text{sur } \Sigma, \\ y(0) = y^0 & \text{dans } \Omega, \end{cases} \tag{4}$$

pourvu que la dérivée de $F(s,p)$ par rapport à la première variable (resp. aux N dernières variables) soit strictement majoré à l'infini par $\log^{3/2}(1+|s|+|p|)$ (resp. $\log^{1/2}(1+|s|+|p|)$).

Ensuite, la contrôlabilité de l'équation de la chaleur avec conditions aux limites de type Fourier (ou Robin) a été étudiée pour la première fois dans [15] où les auteurs ont résolu la contrôlabilité à zéro du système

$$\begin{cases} y_t - \Delta y = v\mathbb{1}_\omega & \text{dans } Q, \\ \dfrac{\partial y}{\partial n} + \beta\, y = 0 & \text{sur } \Sigma, \\ y(0) = y^0 & \text{dans } \Omega, \end{cases} \tag{5}$$

pour des coefficients $\beta \in C^1(\overline{\Sigma})$. En utilisant ce résultat, A. Doubova et al. [6] ont montré la contrôlabilité locale à zéro pour tout $T > 0$ et la contrôlabilité à zéro pour T assez grand (qui dépend de y^0) du système

$$\begin{cases} y_t - \Delta y + G(y) = v\mathbb{1}_\omega & \text{dans } Q, \\ \dfrac{\partial y}{\partial n} + g(y) = 0 & \text{sur } \Sigma, \\ y(0) = y^0 & \text{dans } \Omega, \end{cases} \tag{6}$$

où G vérifie (2) et $g \in C^4(\mathbf{R})$ a une dérivée première positive.

Les deux premières parties de ce mémoire seront consacrées au système de réaction-diffusion avec conditions aux limites de type Fourier linéaires et nonlinéaires. Dans une première partie, on établit un résultat de contrôlabilité à zéro pour le système

$$\begin{cases} y_t - \Delta y + a\, y + B \cdot \nabla y = v \mathbb{1}_\omega & \text{dans } Q, \\ \dfrac{\partial y}{\partial n} + \beta\, y = 0 & \text{sur } \Sigma, \\ y(0) = y^0 & \text{dans } \Omega, \end{cases} \tag{7}$$

avec des coefficients a, B et β appartenant à l'espace L^∞ (cette partie correspond à l'article [11]). Ceci permet ensuite d'obtenir un résultat de contrôlabilité globale pour le système nonlinéaire

$$\begin{cases} y_t - \Delta y + G(y, \nabla y) = v \mathbb{1}_\omega & \text{dans } Q, \\ \dfrac{\partial y}{\partial n} + g(y) = 0 & \text{sur } \Sigma, \\ y(0) = y^0 & \text{dans } \Omega, \end{cases} \tag{8}$$

qui est l'object de l'article [12]. Pour déduire un résultat de caractère global, on obtient la contrôlabilité nulle du système (7) seulement avec les hypothéses L^∞ sur les coefficients et puis on applique les idées développées dans [14]. Plus de détails seront donnés plus bas.

La contrôlabilité de systèmes de type Navier-Stokes a été aussi très étudiée ces derniers temps. Quelques résultats concernant la *contrôlabilité approchée* ont été démontrés par J.-M. Coron [5], C. Fabre [9] et J.-L. Lions et E. Zuazua [21]. On définit quelques espaces classiques dans le cadre des fluides incompressibles :

$$H = \{u \in L^2(\Omega)^N : \nabla \cdot u = 0 \text{ dans } \Omega,\ u \cdot n = 0 \text{ sur } \partial\Omega\} \tag{9}$$

et

$$V = \{u \in H_0^1(\Omega)^N : \nabla \cdot u = 0 \text{ dans } \Omega\}. \tag{10}$$

En ce qui concerne la contrôlabilité exacte pour les équations de Navier-Stokes avec conditions au bord naturelles de type Dirichlet, le seul résultat positif était dû à O. Yu. Imanuvilov. Dans [17], il montre la contrôlabilité exacte locale aux trajectoires du système de Navier-Stokes avec conditions aux limites de type Dirichlet :

$$\begin{cases} y_t - \Delta y + (y, \nabla) y + \nabla p = v \mathbb{1}_\omega & \text{dans } Q, \\ \nabla \cdot y = 0 & \text{dans } Q, \\ y = 0 & \text{sur } \Sigma, \\ y(0) = y^0 & \text{dans } \Omega. \end{cases} \tag{11}$$

Plus précisément, pour les solutions du problème

$$\begin{cases} \overline{y}_t - \Delta \overline{y} + (\overline{y}, \nabla) \overline{y} + \nabla \overline{p} = 0 & \text{dans } Q, \\ \nabla \cdot \overline{y} = 0 & \text{dans } Q, \\ \overline{y} = 0 & \text{on } \Sigma \end{cases} \tag{12}$$

appartenant à l'espace

$$W^{1,\infty}(0,T;L^\infty(\Omega)^N)\cap L^\infty(0,T;W^{1,\infty}(\Omega)^N\cap V),$$

il démontre qu'il existe un $\delta>0$ tel que, pour des données initiales y^0 vérifiant $\|y^0-\overline{y}(0)\|_V\leq\delta$, on peut trouver des contrôles v qui conduisent la solution de (11) à la solution de (12) en un temps T.

Dans la troisième partie de ce manuscript on améliorera ce résultat, dans le sens où des trajectoires moins régulières du système (11) seront approchées. Cette partie correspond à l'article [13].

Enfin, dans la quatrième et dernière partie, on s'intéresse à la contrôlabilité du système de Navier-Stokes avec conditions aux limites nonlinéaires de type Navier :

$$\begin{cases} y_t-\Delta y+(y,\nabla)y+\nabla p=v\mathbb{1}_\omega & \text{dans } Q,\\ \nabla\cdot y=0 & \text{dans } Q,\\ y\cdot n=0,\ (\sigma(y,p)\,n)_{tg}+(f(y))_{tg}=0 & \text{sur } \Sigma,\\ y(0)=y^0 & \text{dans } \Omega, \end{cases}\tag{13}$$

où on a noté

$$w_{tg}=w-(w\cdot n)n$$

et où le tenseur σ est donné par

$$\sigma(y,p)=-p\,Id+(\nabla y+\nabla^t y).$$

Auparavant, le seul résultat de contrôlabilité d'un système similaire avait été démontré par J.-M. Coron dans [5], où un résultat de type contrôlabilité approchée est établi dans l'espace $W^{-1,\infty}(\Omega)$ pour le système

$$\begin{cases} y_t-\Delta y+(y,\nabla)y+\nabla p=v_1\mathbb{1}_\omega & \text{dans } Q,\\ \nabla\cdot y=0 & \text{dans } Q,\\ (y\cdot n, s(\sigma(y,p)\,n)_{tg}+(1-s)(y\cdot\tau))=v_2\mathbb{1}_\gamma & \text{sur } \Sigma,\\ y(0)=y^0 & \text{dans } \Omega, \end{cases}\tag{14}$$

avec $N=2$, $\gamma\subset\partial\Omega$ et $0<s\leq 1$.

On va montrer un résultat de contrôlabilité locale aux trajectoires du système (13). Pour des fonctions $(\overline{y},\overline{p})$ avec une certaine régularité et satisfaisant

$$\begin{cases} \overline{y}_t-\Delta\overline{y}+(\overline{y},\nabla)\overline{y}+\nabla\overline{p}=0 & \text{dans } Q,\\ \nabla\cdot\overline{y}=0 & \text{dans } Q,\\ \overline{y}\cdot n=0,\ (\sigma(\overline{y},\overline{p})\,n)_{tg}+(f(\overline{y}))_{tg}=0 & \text{sur } \Sigma, \end{cases}\tag{15}$$

on trouve des contrôles qui permettent d'atteindre $\overline{y}$ à l'instant T, pourvu que y et $\overline{y}$ soient suffisamment proches à l'instant initial. Cette partie a été l'object du travail [16].

On va maintenant développer avec plus de détails les différentes parties de ce mémoire, composé de quatre travaux de recherche en langue anglaise.

Contrôlabilité à zéro du système de réaction-diffusion linéaire avec conditions au bord de type Fourier

Dans la première partie de cette thèse on travaille sur le système

$$\begin{cases} y_t - \Delta y + B \cdot \nabla y + a\, y = v \mathbb{1}_\omega & \text{dans } Q, \\ \dfrac{\partial y}{\partial n} + \beta\, y = 0 & \text{sur } \Sigma, \\ y(0) = y^0 & \text{dans } \Omega \end{cases} \tag{16}$$

et on démontre qu'il est contrôlable à zéro au temps T pour des coefficients a, B et β appartenants à L^∞.

Avant de donner une idée de la preuve, une interprétation possible des données et des variables de (16) est la suivante : on peut voir l'état y comme la température relative d'un corps N-dimensionel par rapport à l'air environnant. L'équation différentielle de (16) nous dit que la source de chaleur $v\mathbb{1}_\omega$ peut seulement agir sur une partie du corps. Sur le bord, $-\frac{\partial y}{\partial n}$ représente le flux normal donc, l'identité

$$-\frac{\partial y}{\partial n} = \beta\, y$$

traduit le fait que ce flux est une fonction linéaire de la température (par conséquent, il serait raisonnable de supposer $\beta \geq 0$, mais cela ne sera pas nécéssaire).

Le premier résultat important que l'on montre est une inégalité globale de Carleman pour les solutions faibles de problèmes rétrogrades qui s'écrivent sous la forme

$$\begin{cases} -\varphi_t - \Delta\varphi = f_1 + \nabla \cdot f_2 & \text{dans } Q, \\ (\nabla\varphi + f_2) \cdot n = f_3 & \text{sur } \Sigma, \\ \varphi(T) = \varphi^0 & \text{dans } \Omega, \end{cases} \tag{17}$$

avec $f_1 \in L^2(Q)$, $f_2 \in L^2(Q)^N$ et $f_3 \in L^2(\Sigma)$. On peut réaliser que, pour des solutions $\varphi \in L^2(Q)$, la condition au bord a *'a priori'* un sens puisque $\nabla\varphi + f_2 \in L^2(Q)^N$ et $\nabla \cdot (\nabla\varphi + f_2) \in H^{-1}(0, T; L^2(\Omega))$.

Cette inégalité est la suivante :

Théorème 1 *Si f_1, f_2 et f_3 vérifient les hypothèses ci-dessus, il existe quatre constantes $\overline{\lambda}$, σ_1, σ_2 et C qui dépendent seulement de Ω et ω telles que, pour tout $\lambda \geq \overline{\lambda}$, tout $s \geq \overline{s} = \sigma_1(e^{\sigma_2\lambda}T + T^2)$*

et tout $\varphi^0 \in L^2(\Omega)$*, la solution faible de* (17) *satisfait*

$$\begin{aligned}
&\iint_Q e^{-2s\alpha}(s\,\lambda^2\xi\,|\nabla\varphi|^2 + s^3\,\lambda^4\,\xi^3\,|\varphi|^2)dx\,dt \\
&\qquad\qquad +s^2\lambda^3 \iint_\Sigma e^{-2s\alpha}\,\xi^2\,|\varphi|^2\,d\sigma\,dt \\
&\le C\left(\iint_Q e^{-2s\alpha}(|f_1|^2 + \ s^2\,\lambda^2\,\xi^2\,|f_2|^2)dx\,dt\right. \\
&\left. +s\lambda \iint_\Sigma e^{-2s\alpha}\,\xi\,|f_3|^2\,d\sigma\,dt + s^3\lambda^4 \iint_{\omega\times(0,T)} e^{-2s\alpha}\,\xi^3\,|\varphi|^2\,dx\,dt\right).
\end{aligned} \tag{18}$$

Les fonctions poids $\xi = \xi(x,t)$ et $\alpha = \alpha(x,t)$ sont données par

$$\xi(x,t) = \frac{e^{\lambda\eta^0(x)}}{t(T-t)}\ , \quad \alpha(x,t) = \frac{e^{2\lambda\|\eta^0\|_\infty} - e^{\lambda\eta^0(x)}}{t(T-t)}, \tag{19}$$

où $\eta^0 = \eta^0(x)$ est une fonction satisfaisant

$$\begin{gathered}
\eta^0 \in C^2(\overline{\Omega}), \quad \eta^0(x) > 0 \text{ dans } \Omega, \quad \eta^0(x) = 0 \text{ sur } \partial\Omega, \\
|\nabla\eta^0(x)| > 0 \text{ dans } \overline{\Omega\setminus\omega'},
\end{gathered} \tag{20}$$

avec $\omega' \subset\subset \omega$ un ouvert non vide.

La preuve de ce résultat utilise les idées du travail [19]. Plus précisément, on suit une stratégie de dualité qui permet de relaxer les hypothèses de régularité que l'on impose sur les données. Ceci va nous permettre de n'imposer aucune régularité plus forte que L^∞ sur $\beta = \beta(x,t)$.

Pour utiliser cette technique, on doit se baser sur une inégalité de Carleman pour un système *homogène* associé à (17). Précisément, les solutions du problème

$$\begin{cases}
-q_t - \Delta q = f & \text{dans } Q, \\
\dfrac{\partial q}{\partial n} = 0 & \text{sur } \Sigma, \\
q(T) = q^0 & \text{dans } \Omega
\end{cases}$$

avec $f \in L^2(Q)$ et $q^0 \in L^2(\Omega)$, satisfont

$$I_{s,\lambda}(q) \le C\left(\iint_Q e^{-2s\alpha}\,|f|^2\,dx\,dt + s^3\lambda^4 \iint_{\omega\times(0,T)} e^{-2s\alpha}\,\xi^3\,|q|^2\,dx\,dt\right), \tag{21}$$

pour des s et λ appropriés. Le terme $I_{s,\lambda}(q)$ est donné par

$$I_{s,\lambda}(q) = \iint_Q e^{-2s\alpha}\left((s\xi)^{-1}(|q_t|^2 + |\Delta q|^2) + s\lambda^2\xi\,|\nabla q|^2 + s^3\lambda^4\xi^3\,|q|^2\right)\,dx\,dt.$$

On regarde φ comme la solution par transposition de (17), c'est à dire, l'unique fonction de $L^2(Q)$ qui vérifie

$$\left\{ \begin{array}{l} \displaystyle\iint_Q \varphi\, h\, dx\, dt = \iint_Q f_1(x,t)\, z\, dx\, dt - \iint_Q f_2(x,t)\cdot\nabla z\, dx\, dt \\ \displaystyle\qquad + \iint_\Sigma f_3(x,t)\, z\, d\sigma\, dt + \int_\Omega \varphi^0(x)\, z(x,T)\, dx \quad \forall h \in L^2(Q), \end{array} \right. \tag{22}$$

où z est, pour chaque $h \in L^2(Q)$, la solution du problème

$$\left\{ \begin{array}{ll} z_t - \Delta z = h & \text{dans } Q, \\ \dfrac{\partial z}{\partial n} = 0 & \text{sur } \Sigma, \\ z(0) = 0 & \text{dans } \Omega. \end{array} \right.$$

Ensuite, on considère le problème suivant :

$$\left\{ \begin{array}{l} \text{Minimiser } \dfrac{1}{2}\left(\displaystyle\iint_Q e^{2s\alpha}\,|z|^2 + s^{-3}\lambda^{-4} \iint_{\omega\times(0,T)} e^{2s\alpha}\,\xi^{-3}\,|v|^2\, dx\, dt \right) \\ \text{restreint à } v \in L^2(Q) \text{ et} \\ \qquad \left\{ \begin{array}{ll} z_t - \Delta z = s^3\lambda^4 e^{-2s\alpha}\xi^3\varphi + v 1\!\!1_\omega & \text{dans } Q, \\ \dfrac{\partial z}{\partial n} = 0 & \text{sur } \Sigma, \\ z(0) = 0, \quad z(T) = 0 & \text{dans } \Omega. \end{array} \right. \end{array} \right. \tag{23}$$

Si l'on suit les idées de [19], on est amené au système d'optimalité

$$\left\{ \begin{array}{ll} \mathcal{L}(e^{-2s\alpha}\,\mathcal{L}^* p) + s^3\lambda^4 e^{-2s\alpha}\xi^3 p 1_\omega = s^3\lambda^4 e^{-2s\alpha}\xi^3\varphi & \text{dans } Q, \\ \dfrac{\partial p}{\partial n} = 0, \quad \dfrac{\partial}{\partial n}(e^{-2s\alpha}\,\mathcal{L}^* p) = 0 & \text{sur } \Sigma, \\ (e^{-2s\alpha}\mathcal{L}^* p)_{|t=0} = (e^{-2s\alpha}\mathcal{L}^* p)_{|t=T} = 0 & \text{dans } \Omega, \end{array} \right. \tag{24}$$

avec $\mathcal{L} = \partial_t - \Delta$ et $\mathcal{L}^* = -\partial_t - \Delta$. Alors, on peut montrer que (24) (donc (23) aussi) admet une seule solution p et que

$$\widehat{v} = -s^3\,\lambda^4\, e^{-2s\alpha}\,\xi^3\, p 1\!\!1_\omega \quad \text{et} \quad \widehat{z} = e^{-2s\alpha}\mathcal{L}^* p$$

est la solution de (23). On montre que cette solution satisfait l'inégalité

$$\begin{array}{l} \displaystyle\iint_Q e^{2s\alpha}\,|\widehat{z}|^2\, dx\, dt + s^{-2}\lambda^{-2} \iint_Q e^{2s\alpha}\,\xi^{-2}\,|\nabla\widehat{z}|^2\, dx\, dt \\ \displaystyle + s^{-1}\lambda^{-1} \iint_\Sigma e^{2s\alpha}\xi^{-1}\,|\widehat{z}|^2\, d\sigma\, dt + s^{-3}\lambda^{-4} \iint_{\omega\times(0,T)} e^{2s\alpha}\,\xi^{-3}\,|\widehat{v}|^2\, dx\, dt \\ \displaystyle\qquad \le C(\Omega,\omega)\, s^3\lambda^4 \iint_Q e^{-2s\alpha}\,\xi^3\,|\varphi|^2\, dx\, dt, \end{array} \tag{25}$$

pour des s et λ convenables.

En choisissant

$$h = s^3\lambda^4 e^{-2s\alpha}\xi^3\varphi + \widehat{v}\mathbb{1}_\omega$$

dans (22) et en utilisant (25), on arrive à

$$\begin{array}{c} s^3\lambda^4 \displaystyle\iint_Q e^{-2s\alpha}\,\xi^3\,|\varphi|^2\,dx\,dt \le C(\Omega,\omega)\Bigg(\iint_Q e^{-2s\alpha}\,|f_1|^2\,dx\,dt \\ +s^2\lambda^2 \displaystyle\iint_Q e^{-2s\alpha}\,\xi^2\,|f_2|^2\,dx\,dt + s\lambda \iint_\Sigma e^{-2s\alpha}\,\xi\,|f_3|^2\,d\sigma\,dt \\ +s^3\lambda^4 \displaystyle\iint_{\omega\times(0,T)} e^{-2s\alpha}\,\xi^3\,|\varphi|^2\,dx\,dt\Bigg) \end{array} \tag{26}$$

pour des s et λ appropriés. Ensuite, on déduit (18) à partir de (26) de façon directe.

Comme conséquence du théorème 1, on peut déduire directement une *inégalité d'observabilité* pour le système adjoint associé à (16) :

$$\begin{cases} -\varphi_t - \Delta\varphi - \nabla\cdot(\varphi B) + a\,\varphi = 0 & \text{dans } Q, \\ (\nabla\varphi + \varphi B)\cdot n + \beta\varphi = 0 & \text{sur } \Sigma, \\ \varphi(T) = \varphi^0 & \text{dans } \Omega. \end{cases} \tag{27}$$

En fait, les solutions de (27) satisfont

$$\iint_{\Omega\times(T/4,3T/4)} |\varphi|^2\,dx\,dt \le K \iint_{\omega\times(0,T)} |\varphi|^2\,dx\,dt, \tag{28}$$

avec une constante K de la forme

$$K = \exp\{C(1 + \frac{1}{T} + \|a\|_\infty^{2/3} + \|B\|_\infty^2 + \|\beta\|_\infty^2)\}. \tag{29}$$

Le deuxième résultat important de ce travail établit la contrôlabilité nulle du système (16) :

Théorème 2 *Supposons que $a \in L^\infty(Q)$, $B \in L^\infty(Q)^N$ et $\beta \in L^\infty(\Sigma)$. Alors, (16) est contrôlable à zéro pour tout $T > 0$ avec des contrôles v tels que*

$$\|v\|_{L^2(\omega\times(0,T))} \le \widetilde{K}\,\|y^0\|_{L^2(\Omega)}\,, \tag{30}$$

avec une constante $\widetilde{K}$ de la forme

$$\widetilde{K} = \exp\{C(1 + \frac{1}{T} + \|a\|_\infty^{2/3} + \|B\|_\infty^2 + \|\beta\|_\infty^2 + T(\|a\|_\infty + \|B\|_\infty^2 + \|\beta\|_\infty^2))\} \tag{31}$$

pour une certaine $C = C(\Omega,\omega)$.

La preuve du théorème 2 à partir de l'inégalité d'observabilité (28) est classique.

Dans la deuxième partie, on présente un autre travail concernant l'équation de réaction-diffusion avec conditions aux limites de type Fourier mais nonlinéaires. Celui-ci représente une continuation naturelle du premier travail.

En fait, dans son analyse, les résultats présentés dans les théorèmes 1 et 2 seront cruciaux.

Contrôlabilité exacte aux trajectoires du système de réaction-diffusion nonlinéaire avec conditions au bord de type Fourier

A l'aide des résultats établis dans la partie précédente, on va démontrer la contrôlabilité exacte (globale) aux trajectoires du système

$$\begin{cases} y_t - \Delta y + G(y, \nabla y) = v 1\!\!1_\omega & \text{dans } Q, \\ \dfrac{\partial y}{\partial n} + g(y) = 0 & \text{sur } \Sigma, \\ y(0) = y^0 & \text{dans } \Omega. \end{cases} \tag{32}$$

Dans ce cas, on considèrera $y^0 \in L^\infty(\Omega)$ et deux fonctions $G : \mathbf{R} \times \mathbf{R}^N \mapsto \mathbf{R}$ et $g : \mathbf{R} \mapsto \mathbf{R}$. Quelques résultats d'existence, d'unicité, de régularité et d'autres proprietés des solutions de problèmes du type (32) se trouvent par exemple dans [1], [2] et [8]. Par le même principe énoncé dans le paragraphe précédent, il est raisonnable de supposer que g ne décroît pas et que $g(0) = 0$.

On fixe une trajectoire non contrôlée du système (32) :

$$\begin{cases} \overline{y}_t - \Delta \overline{y} + G(\overline{y}, \nabla \overline{y}) = 0 & \text{dans } Q, \\ \dfrac{\partial \overline{y}}{\partial n} + g(\overline{y}) = 0 & \text{sur } \Sigma, \\ \overline{y}(0) = \overline{y}^0 & \text{dans } \Omega. \end{cases} \tag{33}$$

Les hypothèses que l'on impose à cette trajectoire sont :

$$\overline{y} \in L^2(0,T; H^1(\Omega)) \cap C^0([0,T]; L^2(\Omega)) \cap L^\infty(Q) \quad \text{et} \quad \overline{y}^0 \in L^\infty(\Omega). \tag{34}$$

Le résultat principal de ce travail est le suivant :

Théorème 3 *Supposons que G et g sont deux fonctions localement lipschitziennes qui vérifient*

$$\lim_{|s|\to\infty} \frac{|G(s,p) - G(r,p)|}{|s-r| \log^{3/2}(1 + |s-r|)} = 0, \tag{35}$$

uniformément en $(r,p) \in [-K,K] \times \mathbf{R}^N$ pour tout $K > 0$,

$$\begin{cases} \forall L > 0,\ \exists M > 0 \ \textit{ tel que} \\ |G(s,p) - G(r,p)| \le M|s-r|, \quad |G(s,p) - G(s,q)| \le M|p-q| \\ \forall (s,r,p,q) \in [-L,L]^2 \times \mathbf{R}^N \times \mathbf{R}^N \end{cases} \tag{36}$$

et

$$\lim_{|s|\to\infty} \frac{|g(s)-g(r)|}{|s-r|\log^{1/2}(1+|s-r|)} = 0 \tag{37}$$

uniformément en $r \in [-K,K]$ *pour tout* $K>0$*. Alors, pour chaque* $T>0$*, on a la contrôlabilité exacte aux trajectoires* $\overline{y}$ *qui vérifient* (34) *avec des contrôles* $v \in L^\infty(\omega\times(0,T))$*.*

Avant de faire un résumé de la preuve du théorème 3, on a besoin d'un résultat de contrôlabilité nulle pour le système (16) avec des contrôles dans $L^\infty(\omega\times(0,T))$:

Proposition 1 *Pour tout* $T>0$*, le système* (16) *est contrôlable à zéro avec des contrôles dans* $L^\infty(\omega\times(0,T))$*. De plus, on peut trouver les contrôles* v *satisfaisant*

$$\|v\|_{L^\infty(\omega\times(0,T))} \le e^{C(\Omega,\omega)K(T,\|a\|_\infty,\|B\|_\infty,\|\beta\|_\infty)}\,\|y^0\|_{L^2(\Omega)}\,, \tag{38}$$

où

$$K = 1+1/T+\|a\|_\infty^{2/3}+\|B\|_\infty^2+\|\beta\|_\infty^2+T(1+\|a\|_\infty+\|B\|_\infty^2+\|\beta\|_\infty^2). \tag{39}$$

Soit $y^0\in L^2(\Omega)$ et soient ω' et ω'' deux ouverts avec $\omega''\subset\subset\omega'\subset\subset\omega$. Alors, si l'on utilise le théorème 2 avec domaine de contrôle ω'', on déduit qu'il existe un contrôle $\widetilde{v}\in L^2(\omega''\times(0,T))$ tel que la solution $\widetilde{y}$ de (16) vérifie $\widetilde{y}(T)=0$ et on a l'estimation

$$\|\widetilde{v}\|_{L^2(\omega''\times(0,T))} \le e^{C(\Omega,\omega)\,K(T,\|a\|_\infty,\|B\|_\infty,\|\beta\|_\infty)}\,\|y^0\|_{L^2(\Omega)}, \tag{40}$$

avec K de la forme (39).

On introduit maintenant une fonction "cut-off" $\eta\in C^\infty([0,T])$ satisfaisant

$$\eta(t)=1 \text{ dans } (0,T/4), \quad \eta(t)=0 \text{ dans } (3T/4,T)$$

et

$$0\le\eta(t)\le 1 \text{ dans } (0,T)$$

et on considère la solution χ du système

$$\begin{cases} \chi_t-\Delta\chi+a\,\chi+B\cdot\nabla\chi=0 & \text{dans } Q,\\ \dfrac{\partial\chi}{\partial n}+\beta\,\chi=0 & \text{sur } \Sigma,\\ \chi(0)=y^0 & \text{dans } \Omega. \end{cases}$$

Alors, $\widetilde{w}=\widetilde{y}-\eta\chi$ satisfait

$$\begin{cases} \widetilde{w}_t-\Delta\widetilde{w}+a\,\widetilde{w}+B\cdot\nabla\widetilde{w}=-\eta'(t)\,\chi+\widetilde{v}\mathbb{1}_{\omega''} & \text{dans } Q,\\ \dfrac{\partial\widetilde{w}}{\partial n}+\beta\,\widetilde{w}=0 & \text{sur } \Sigma,\\ \widetilde{w}(0)=0, \quad \widetilde{w}(T)=0 & \text{dans } \Omega. \end{cases} \tag{41}$$

On considére ensuite un ouvert ω_0 avec $\omega'\subset\subset\omega_0\subset\subset\omega$ et encore une fonction "cut-off" ξ qui vérifie

$$\xi\in C_0^2(\omega_0) \quad \text{et} \quad \xi\equiv 1 \text{ dans } \omega'.$$

Alors, si on pose $w = (1-\xi)\,\widetilde{w}$ on a :

$$\begin{cases} w_t - \Delta w + a\,w + B\cdot\nabla w = -\eta'(t)\,\chi + v\mathbb{1}_\omega & \text{dans } Q, \\ \dfrac{\partial w}{\partial n} + \beta\, w = 0 & \text{sur } \Sigma, \\ w(0) = 0, \quad w(T) = 0 & \text{dans } \Omega, \end{cases}$$

avec

$$v = \eta'\,\xi\,\chi + 2\nabla\xi\cdot\nabla\widetilde{w} + \Delta\xi\,\widetilde{w} - B\cdot\nabla\xi\,\widetilde{w}. \tag{42}$$

D'après quelques résultats de régularité locale pour les solutions du système (41), il n'est pas trop difficile de déduire que $v \in L^\infty(\omega\times(0,T))$ et que la fonction $y = w + \eta\chi$ résout (avec ce v) la contrôlabilité à zéro de (16).

Un autre résultat que l'on utilisera pour démontrer le théorème 2 est une estimation L^∞ des solutions du système (16).

Proposition 2 *Supposons que $v \in L^\infty(Q)$, $y^0 \in L^\infty(\Omega)$ et que les coefficients a, B, β sont dans L^∞. Alors $y \in L^\infty(Q)$ et on a*

$$\|y\|_\infty \le e^{C\,T\left(1+\|a\|_\infty+\|B\|_\infty^2+\|\beta\|_\infty^2\right)}\left(\|y^0\|_\infty + \|f\|_\infty\right). \tag{43}$$

avec une constante $C>0$ ne dépendant que de Ω.

On revient à la démonstration du théorème 2 et on considère le système

$$\begin{cases} w_t - \Delta w + F_1(w,\nabla w;x,t)w + F_2(\nabla w;x,t)\cdot\nabla w = v\mathbb{1}_\omega & \text{dans } Q, \\ \dfrac{\partial w}{\partial n} + F_3(w;x,t)w = 0 & \text{sur } \Sigma, \\ w(0) = y^0 - \overline{y}(0) & \text{dans } \Omega, \end{cases} \tag{44}$$

où

$$F_1(s,p;x,t) = \frac{F(\overline{y}(x,t)+s,\nabla\overline{y}(x,t)+p) - F(\overline{y}(x,t),\nabla\overline{y}(x,t)+p)}{s}, \tag{45}$$

$$F_2 = (F_{21},\dots,F_{2N}), \quad F_{2j}(p;x,t) = \int_0^1 \frac{\partial F}{\partial p_j}(\overline{y}(x,t),\nabla\overline{y}(x,t)+\lambda p)\,d\lambda \tag{46}$$

et

$$F_3(s;x,t) = \frac{f(\overline{y}(x,t)+s) - f(\overline{y}(x,t))}{s} \tag{47}$$

pour $s \in \mathbf{R}$ et $p \in \mathbf{R}^N$.

Avec $y = w + \overline{y}$, on est ramené à démontrer la contrôlabilité nulle du système (44) et pour cela, on se restreint (par densité) au cas où les fonctions G et g ont une dérivée continue.

L'idée de la preuve est à peu près classique et elle est basée sur un argument de point fixe dans l'espace $Z = L^2(0,T;H^1(\Omega))$. Pour chaque $R>0$, on considère la fonction

$$M_R(s) = \begin{cases} -R & \text{if } s < -R, \\ s & \text{si } -R \le s \le R, \\ R & \text{si } s > R \end{cases}$$

et les coefficients

$$a_{R,z}(x,t) = F_1(M_R(z(x,t)), \nabla z(x,t); x,t),$$

$$B_z(x,t) = F_2(\nabla z(x,t); x,t)$$

et

$$\beta_{R,z}(x,t) = F_3(M_R(z(x,t)); x,t).$$

Alors, on sait déjà que le problème

$$\begin{cases} w_t - \Delta w + a_{R,z}\, w + B_z \cdot \nabla w = v 1\!\!1_\omega & \text{dans } Q, \\ \dfrac{\partial w}{\partial n} + \beta_{R,z}\, w = 0 & \text{sur } \Sigma, \\ w(0) = y^0 - \overline{y}(0), \quad w(T) = 0 & \text{dans } \Omega, \end{cases} \tag{48}$$

a une solution, en vertu de la proposition 1.

Ensuite, on choisit une solution particulière du problème (48), en procédant comme dans [14], qui nous donne des bonnes estimations pour le contrôle et l'état, c'est-à-dire,

$$\|v_{R,z}\|_{L^\infty(\omega\times(0,T))} \le C_R\, \|w^0\|_{L^2(\Omega)}, \tag{49}$$

$$\|w_{R,z}\|_Z \le C_R\, \|w^0\|_{L^2(\Omega)}, \tag{50}$$

et

$$\|w_{R,z}\|_\infty \le C_R\, \|w^0\|_{L^\infty(\Omega)} \tag{51}$$

avec

$$C_R = \exp\left\{ C(\Omega, \omega, T)\left(1 + a_R^{2/3} + \overline{B}^2 + \beta_R^2\right)\right\},$$

où

$$a_R = \sup_{|s|\le R,\ p\in\mathbf{R}^N}\ \operatorname*{ess\,sup}_{(x,t)\in Q} |F_1(s,p;x,t)|,$$

$$\overline{B} = \sup_{p\in\mathbf{R}^N}\ \operatorname*{ess\,sup}_{(x,t)\in Q} |F_2(p;x,t)|$$

et

$$\beta_R = \sup_{|s|\le R}\ \operatorname*{ess\,sup}_{(x,t)\in\Sigma} |F_3(s;x,t)|.$$

Finalement, on introduit l'application Λ_R. A chaque $z \in Z$, Λ_R associe l'ensemble des états $w_{R,z}$ qui satisfont (50) et (51) correspondants aux contrôles $v_{R,z} \in L^\infty(\omega \times (0,T))$ tels que $w_{R,z}(T) = 0$ et tels que $v_{R,z}$ satisfait (49). On peut diviser la preuve en deux étapes :

- On montre d'abord que, pour chaque $R > 0$, Λ_R a un point fixe w_R. Pour cela, on utilise le théorème de Kakutani (voir [3]). La partie la plus délicate de l'application de ce résultat est l'appartenance de $\Lambda_R(z)$ à un compact fixe (qui dépend de R), pour tout $z \in Z$.
- Enfin, on montre qu'il existe $R > 0$ suffisamment grand tel que $M_R(w_R) = w_R$.

Contrôlabilité exacte aux trajectoires du système de Navier-Stokes

Dans ce paragraphe, on s'intéresse au système contrôlé de Navier-Stokes

$$\begin{cases} y_t - \Delta y + \nabla \cdot (y \otimes y) + \nabla p = v 1\!\!1_\omega & \text{dans } Q, \\ \nabla \cdot y = 0 & \text{dans } Q, \\ y = 0 & \text{sur } \Sigma, \\ y(0) = y^0 & \text{dans } \Omega, \end{cases} \tag{52}$$

où

$$(\nabla \cdot (y^1 \otimes y^2))_i = \sum_{j=1}^{N} \partial_j (y_i^1 y_j^2) \quad i = 1, \ldots, N.$$

En outre, soit $\overline{y}$ une solution du système non contrôlé de Navier-Stokes

$$\begin{cases} \overline{y}_t - \Delta \overline{y} + \nabla \cdot (\overline{y} \otimes \overline{y}) + \nabla \overline{p} = 0 & \text{dans } Q, \\ \nabla \cdot \overline{y} = 0 & \text{dans } Q, \\ \overline{y} = 0 & \text{sur } \Sigma, \\ \overline{y}(0) = \overline{y}^0 & \text{dans } \Omega. \end{cases} \tag{53}$$

Comme précédemment annoncé, on montrera que si y^0 est suffisamment près de $\overline{y}^0$, on peut trouver des contrôles v et des états associés (y, p) tels que

$$y(T) = \overline{y}(T) \ \text{ dans } \Omega.$$

Les hypothèses que l'on va imposer aux trajectoires $\overline{y}$ sont :

$$\overline{y} \in L^\infty(Q)^N, \quad \overline{y}_t \in L^2(0, T; L^\sigma(\Omega))^N \quad \left(\begin{array}{ll} \sigma > 1 & \text{si } N = 2 \\ \sigma > 6/5 & \text{si } N = 3 \end{array} \right). \tag{54}$$

Le résultat principal de ce travail est présenté dans le théorème suivant :

Théorème 4 *Soit $\overline{y}$ une solution de* (53) *avec $\overline{y}^0 \in L^{2N-2}(\Omega)^N \cap H$ qui satisfait* (54)*. Alors il existe $\delta > 0$ tel que, pour chaque $y^0 \in L^{2N-2}(\Omega)^N \cap H$ satisfaisant $\|y^0 - \overline{y}^0\|_{L^{2N-2}(\Omega)^N} \leq \delta$, il existe un contrôle $v \in L^2(\omega \times (0, T))^N$ et une solution (y, p) de* (52) *avec*

$$y(T) = \overline{y}(T) \ \textit{ dans } \Omega.$$

La preuve de ce théorème utilise les idées de [17] et repose sur un théorème d'inversion locale. Par l'application de celui-ci, on est amené à démontrer un résultat de contrôlabilité nulle pour un système linéarisé autour de $\overline{y}$ et avec un second membre non nul (voir proposition 3 plus loin) :

$$\begin{cases} Ly + \nabla p = f + v 1\!\!1_\omega & \text{dans } Q, \\ \nabla \cdot y = 0 & \text{dans } Q, \\ y = 0 & \text{sur } \Sigma, \\ y(0) = y^0 & \text{dans } \Omega, \end{cases} \tag{55}$$

où

$$Ly = y_t - \Delta y + \nabla \cdot (y \otimes \overline{y}) + \nabla \cdot (\overline{y} \otimes y). \tag{56}$$

Supposons que la contrôlabilité nulle de (55) soit démontrée pour un second membre $f \in X$ (X etant un espace de Banach approprié). Alors, pour pouvoir déduire ce qu'on veut pour le problème nonlinéaire (théorème 4) ce résultat ne suffira pas, en effet, on devra aussi trouver une solution y telle que $\nabla \cdot (y \otimes y) \in X$.

Cela nous conduit à l'argument suivant :

• Dans un premier temps, on montre une inégalité globale de type Carleman pour les solutions du problème adjoint :

$$\begin{cases} -\varphi_t - \Delta\varphi - D\varphi\,\overline{y} + \nabla\pi = g & \text{dans } Q, \\ \nabla \cdot \varphi = 0 & \text{dans } Q, \\ \varphi = 0 & \text{sur } \Sigma, \\ \varphi(T) = \varphi^0 & \text{dans } \Omega, \end{cases} \tag{57}$$

avec

$$D\varphi = \nabla\varphi + \nabla\varphi^t.$$

On la présente dans le théorème suivant :

Théorème 5 *Supposons que $\overline{y}$ vérifie (54). Alors, il existe trois constantes positives $\overline{s}$, $\overline{\lambda}$, C qui dépendent de Ω et ω telles que, pour chaque $\varphi^0 \in H$ et $g \in L^2(Q)^N$, la solution de (57) vérifie*

$$\begin{aligned} & s^3\lambda^4 \iint_Q e^{-2s\alpha}\xi^3|\varphi|^2\,dx\,dt + s\lambda^2 \iint_Q e^{-2s\alpha}\xi|\nabla\varphi|^2\,dx\,dt \\ & + s^{-1} \iint_Q e^{-2s\alpha}\xi^{-1}\left(|\varphi_t|^2 + |\Delta\varphi|^2\right)dx\,dt \\ & \qquad \leq C(1+T^2)\left(s^{15/2}\lambda^{20} \iint_Q e^{-4s\widehat{\alpha}+2s\alpha^*}\widehat{\xi}^{15/2}|g|^2\,dx\,dt\right. \\ & \qquad \left. + s^{16}\lambda^{40} \iint_{\omega\times(0,T)} e^{-8s\widehat{\alpha}+6s\alpha^*}\widehat{\xi}^{16}|\varphi|^2\,dx\,dt\right), \end{aligned} \tag{58}$$

pour tout $\lambda \geq \overline{\lambda}(1 + \|\overline{y}\|_\infty + \|\overline{y}_t\|^2_{L^2(0,T;L^\sigma(\Omega)^N)} + e^{\overline{\lambda}T\|\overline{y}\|^2_\infty})$ et tout $s \geq \overline{s}(T^4 + T^8)$.

Les fonctions poids utilisés sont similaires à ceux donnés par (19)-(20). Ils seront définies dans la troisième partie.

La méthode de la preuve de cette inégalité est originale même si elle suit les étapes générales du travail [17] :

Première étape : On regarde (57) comme N équations paraboliques et on utilise l'inégalité de Carleman pour ces équations. Une preuve de ce résultat peut être trouvée, par exemple, dans [15]. Cela donne une estimation de la vitesse en termes d'une intégrale globale la pression.

Deuxième étape : En prenant l'opérateur divergence dans la première équation de (57), on a

$$-\Delta\pi(t) = -\nabla\cdot(g + D\varphi\,\overline{y})(t) \ \text{ dans } \Omega$$

pour presque tout $t \in (0,T)$. Ici, on peut appliquer l'estimation de Carleman déduite dans [18], pour les solutions faibles d'équations elliptiques. Cela donne une estimation de la pression en termes de sa trace et de deux intégrales locales de la vitesse et de la pression même.

Troisième étape : Les résultats de régularité pour le système de Stokes donnent une estimation de la trace de la pression par rapport à des termes qui peuvent être bien bornés. Cette idée a déjà été utilisée dans [17].

A ce point de la preuve, on a une inégalité de Carleman avec quelques intégrales globales à gauche et deux termes locaux de pression et de vitesse à droite :

$$\begin{aligned} I(s,\lambda;\varphi) \le C_4 \Bigg(s^3\lambda^4 \iint_{\omega_1\times(0,T)} & e^{-2s\alpha}\xi^3|\varphi|^2\,dx\,dt \\ +s^2\lambda^2 \iint_{\omega_2\times(0,T)} e^{-2s\alpha}\xi^2|\pi|^2\,dx\,dt &+ s\iint_Q e^{-2s\alpha}\xi|g|^2\,dx\,dt \Bigg), \end{aligned} \tag{59}$$

où $I(s,\lambda;\varphi)$ est le terme à gauche en (58) et ω_1 et ω_2 sont deux ouverts qui vérifient $\omega_1 \subset \omega_2 \subset \omega$.

Cette estimation est particulièrement importante et a intérêt par elle même. Précisément, elle nous permet de traîter des problèmes de contrôlabilité où le contrôle agit non seulement comme un second membre de l'équation de mouvement, mais aussi sur la condition de la divergence de la vitesse. En faisant une extension de notre ouvert Ω, on peut donc considérer le problème d'un contrôle agissant sur une (petite) partie de la frontière.

Quatrième étape : Le reste de la preuve est consacré à l'estimation de l'intégrale locale de la pression. En utilisant la première équation du système (57), l'idée est d'estimer le Laplacien et la derivée en temps de la vitesse au lieu de la pression. Cette partie est la plus délicate et la principale contribution originale.

L'estimation du Laplacien de la vitesse n'est pas très difficile, tandis que celle de la dérivée en temps complique considérablement la démonstration. Pour faire cela, notre intention est toujours d'intégrer par parties par rapport au temps mais ceci ne peut pas être fait directement, puisque g_t n'a pas de sens. On est donc obligé de décomposer (globalement) notre solution en une somme de deux fonctions ayant des propriétés de régularité différentes.

• Enfin, on établit la contrôlabilité nulle du système (55) avec un second membre qui décroît exponentiellement quand $t \to T^-$, sans oublier que l'on doit aussi obtenir cette décroissance pour le terme $\nabla\cdot(y\otimes y)$.

Avant de commencer ceci, on note que d'après l'inégalité (58), il est toujours possible de trouver une autre inégalité avec des fonctions poids qui ne sont pas dégénérées en $t=0$:

$$\begin{aligned}\iint_Q e^{-2s\beta}\gamma^3|\varphi|^2\,dx\,dt + \iint_Q e^{-2s\beta}\gamma|\nabla\varphi|^2\,dx\,dt + \|\varphi(0)\|^2_{L^2(\Omega)^N} \\ \le C\left(\iint_Q e^{-4s\widehat{\beta}+2s\beta^*}\widehat{\gamma}^{15/2}\,|g|^2\,dx\,dt\right. \\ \left. + \iint_{\omega\times(0,T)} e^{-8s\widehat{\beta}+6s\beta^*}\widehat{\gamma}^{16}|\varphi|^2\,dx\,dt\right),\end{aligned} \tag{60}$$

avec un choix des paramètres s et λ comme dans le théorème 5. Les nouvelles fonctions poids seront définies au début de la section 3 du travail correspondant.

Une fois que l'on a obtenu (60), on définit les espaces (de Banach) où on va résoudre la contrôlabilité à zéro du système (55) :

$$\begin{aligned}E_2 = \{(y,v)\in E_0 : \exists p \ \text{ tel que} \\ e^{s\beta^*}(\gamma^*)^{-1/2}(Ly+\nabla p - v1_\omega)\in L^2(0,T;H^{-1}(\Omega)^2)\}\end{aligned}$$

en dimension $N=2$ et

$$\begin{aligned}E_3 = \{(y,v)\in E_0 : e^{s\beta^*/2}(\gamma^*)^{-1/4}y\in L^4(0,T;L^{12}(\Omega)^3), \\ \exists p : e^{s\beta^*}(\gamma^*)^{-1/2}(Ly+\nabla p - v1_\omega)\in L^2(0,T;W^{-1,6}(\Omega)^3)\},\end{aligned}$$

en dimension $N=3$, avec

$$\begin{aligned}E_0 = \{(y,v) : e^{2s\widehat{\beta}-s\beta^*}\widehat{\gamma}^{-15/4}y,\ e^{4s\widehat{\beta}-3s\beta^*}\widehat{\gamma}^{-8}v1_\omega \in L^2(Q)^N, \\ e^{s\beta^*/2}(\gamma^*)^{-1/4}y\in L^2(0,T;V)\cap L^\infty(0,T;H)\}.\end{aligned}$$

Remarque Si $(y,v)\in E_N$ $(N=2,3)$ alors $y(T)=0$, donc y et v résolvent, avec une certaine pression p, la contrôlabilité à zéro du système (55) avec un second membre f adéquat.

Avec ces notations, on a le résultat suivant :

Proposition 3 *Supposons que $\overline{y}$ satisfait* (54) *et que l'on a :*

- *Si $N=2$: $y^0\in H$, $e^{s\beta^*}(\gamma^*)^{-1/2}f\in L^2(0,T;H^{-1}(\Omega)^2)$.*
- *Si $N=3$: $y^0\in H\cap L^4(\Omega)^3$, $e^{s\beta^*}(\gamma^*)^{-1/2}f\in L^2(0,T;W^{-1,6}(\Omega)^3)$.*

Alors, il existe un contrôle $v\in L^2(\omega\times(0,T))^N$ tel que, si y est la solution de (55), *on a $(y,v)\in E_N$. En particulier, $y(T)=0$.*

Dans la preuve on utilise les arguments de [17], donc on introduit le problème de minimisation :

$$\left\{\begin{array}{l}\inf \dfrac{1}{2}\left(\displaystyle\iint_Q e^{4s\widehat{\beta}-2s\beta^*}\widehat{\gamma}^{-15/2}|y|^2dxdt + \int_0^T\!\!\int_\omega e^{8s\widehat{\beta}-6s\beta^*}\widehat{\gamma}^{-16}|v|^2dxdt\right) \\ \text{restreint à } v\in L^2(Q)^N,\ \text{supp}\, v\subset\omega\times(0,T) \text{ et} \\ \qquad\qquad \left\{\begin{array}{ll} Ly+\nabla p = f + v\mathbb{1}_\omega & \text{dans } Q, \\ \nabla\cdot y = 0 & \text{dans } Q, \\ y = 0 & \text{sur } \Sigma, \\ y(0)=y^0, \quad y(T)=0 & \text{dans } \Omega.\end{array}\right.\end{array}\right. \tag{61}$$

Maintenant, il n'est pas très difficile de démontrer que le problème (61) posséde une seule solution $(\widehat{y}, \widehat{v})$, qui vérifie donc

$$\iint_Q e^{4s\widehat{\beta}-2s\beta^*}\widehat{\gamma}^{-15/2}|\widehat{y}|^2 dx\, dt + \int_0^T\int_\omega e^{8s\widehat{\beta}-6s\beta^*}\widehat{\gamma}^{-16}|\widehat{v}|^2 dx\, dt < +\infty.$$

Pour cela, il faut utiliser d'une façon essentielle l'inégalité de Carleman (60).

En outre, il est possible démontrer que $\widehat{y}$ est en fait (pour une certaine pression $\widehat{p}$) la solution faible du système de Stokes

$$\begin{cases} L\widehat{y} + \nabla\widehat{p} = f + \widehat{v}\mathbb{1}_\omega & \text{dans } Q, \\ \nabla\cdot\widehat{y} = 0 & \text{dans } Q, \\ \widehat{y} = 0 & \text{sur } \Sigma, \\ \widehat{y}(0) = y^0 & \text{dans } \Omega. \end{cases}$$

Finalement, non seulement $(\widehat{y}, \widehat{v})$ vérifie

$$e^{2s\widehat{\beta}-s\beta^*}\widehat{\gamma}^{-15/4}\widehat{y},\ e^{4s\widehat{\beta}-3s\beta^*}\widehat{\gamma}^{-8}\widehat{v}1_\omega \in L^2(Q)^N,$$

$$e^{s\beta^*}(\gamma^*)^{-1/2}(L\widehat{y} + \nabla\widehat{p} - \widehat{v}\mathbb{1}_\omega) \in L^2(0,T;H^{-1}(\Omega)^2) \quad \text{si } N = 2$$

et

$$e^{s\beta^*}(\gamma^*)^{-1/2}(L\widehat{y} + \nabla\widehat{p} - \widehat{v}\mathbb{1}_\omega) \in L^2(0,T;W^{-1,6}(\Omega)^3) \quad \text{si } N = 3$$

(direct, d'après (61) et les hypothèses sur f) mais on arrive aussi à montrer que

$$e^{s\beta^*/2}(\gamma^*)^{-1/4}\widehat{y} \in L^2(0,T;V) \cap L^\infty(0,T;H)$$

et $e^{s\beta^*/2}(\gamma^*)^{-1/4}\widehat{y} \in L^4(0,T;L^{12}(\Omega)^3)$ en dimension $N = 3$. Là, on utilise essentiellement les résultats d'interpolation inclus dans [22].

Avec cette dernière propriété de régularité, il est clair que $\widehat{y}$ vérifie également

$$e^{s\beta^*}(\gamma^*)^{-1/2}\nabla\cdot(\widehat{y}\otimes\widehat{y}) \in \begin{cases} L^2(0,T;H^{-1}(\Omega)^2) & \text{si } N = 2, \\ L^2(0,T;W^{-1,6}(\Omega)^3) & \text{si } N = 3. \end{cases}$$

Contrôlabilité exacte aux trajectoires du système de Navier-Stokes avec conditions de glissement du type Navier

On travaille ici sur un système contrôlé de Navier-Stokes avec conditions aux limites de glissement de type Navier. Etant donnée une fonction *régulière* $f : \mathbf{R}^N \mapsto \mathbf{R}^N$ et un état initial y^0, on considère le système

$$\begin{cases} y_t - \Delta y + (y,\nabla)y + \nabla p = v1_\omega & \text{dans } Q, \\ \nabla\cdot y = 0 & \text{dans } Q, \\ y\cdot n = 0, \quad (\sigma(y,p)\cdot n)_{tg} + f(y)_{tg} = 0 & \text{sur } \Sigma, \\ y(0) = y^0 & \text{dans } \Omega. \end{cases} \tag{62}$$

Ici, $n = n(x)$ est, pour chaque $x \in \partial\Omega$, le vecteur normal unitaire en x dirigé vers l'extérieure de Ω et $\sigma(y,p)$ est donné par

$$\sigma(y,p) = -pId + Dy, \quad Dy = \nabla y + \nabla y^t.$$

La première condition aux limites est la *condition de glissement* qui traduit le fait que les particules du fluide ne traversent pas la frontière. Par ailleurs, $\sigma(y,p)\cdot n$ représente la force exercée par la paroi sur le fluide. Par conséquent, la condition

$$(\sigma(y,p)\cdot n)_{tg} = -ky_{tg}, \quad k \text{ étant une constante},$$

est très naturelle et signifie que la composante tangentielle de cette force est proportionnelle à la vitesse sur le bord. Mais k peut dépendre de $|y|$ d'une façon nonlinéaire, donc

$$(\sigma(y,p)\cdot n)_{tg} = -(f(y))_{tg}$$

est une condition plus générale. Une autre interprétation de cette deuxième condition aux limites est présentée dans la quatrième partie.

Les trajectoires que l'on fixe vérifient :

$$\begin{cases} \overline{y}_t - \Delta\overline{y} + (\overline{y},\nabla)\overline{y} + \nabla\overline{p} = 0 & \text{dans } Q, \\ \nabla\cdot\overline{y} = 0 & \text{dans } Q, \\ \overline{y}\cdot n = 0,\ (\sigma(\overline{y},\overline{p})\cdot n)_{tg} + (f(\overline{y}))_{tg} = 0 & \text{sur } \Sigma. \end{cases} \tag{63}$$

Quelques hypothèses de régularité seront imposées sur les données de notre problème. Soit $\ell > 0$ arbitrairement proche de $1/2$. Pour certains vecteurs $d(x,t)$, on supposera :

$$d \in L^\infty(Q)^N, \quad d_t \in L^2(0,T;L^r(\Omega)^N) \quad \begin{pmatrix} r=4 & \text{si } N=2 \\ r=6 & \text{if } N=3 \end{pmatrix}, \tag{64}$$

tandis que pour une fonction matricielle A on imposera :

$$A \in L^\infty(\Sigma)^{N\times N}, \tag{65}$$

$$A \in H^{1-\ell}(0,T;W^{\nu_1,\nu_1+1}(\partial\Omega)^{N\times N}), \tag{66}$$

$$A \in H^{(3-\ell)/2}(0,T;H^{\nu_2}(\partial\Omega)^{N\times N}), \tag{67}$$

avec $\nu_1 > 1$ (arbitrairement petit) en dimension 3 et $\nu_1 = 1$ en dimension 2 et $\nu_2 = 1/2(3-N) + (1-\ell)(N-2)$.

La stratégie que l'on suit repose sur un argument de point fixe. On commence donc par établir la contrôlabilité à zéro du système linéaire

$$\begin{cases} w_t - \Delta w + (a+b,\nabla)w + (w,\nabla)b + \nabla q = v\mathbb{1}_\omega & \text{dans } Q, \\ \nabla\cdot w = 0 & \text{dans } Q, \\ w\cdot n = 0, \quad (\sigma(w,q)\cdot n)_{tg} + (Aw)_{tg} = 0 & \text{sur } \Sigma, \\ w(0) = w^0 & \text{dans } \Omega. \end{cases} \tag{68}$$

Théorème 6 *Soit $w^0 \in H$ et supposons que A satisfait* (65)-(67) *et a, b sont deux fonctions vectorielles à divergence nulle qui satisfont* (64)*. Alors, il existe des contrôles $v \in H^1(0,T;L^2(\omega)^N) \cap C^0([0,T];H^1(\omega)^N)$ tels que la solution de* (68) *vérifie $w(T)=0$.*

De plus, il existe une constante C qui dépend de Ω, ω, T, $\|a\|_\infty$, $\|b\|_\infty$, $\|a_t\|_{L^2(L^r)}$, $\|b_t\|_{L^2(L^r)}$, $\|A\|_{H^{1-\ell}(W^{\nu_1,\nu_1+1})}$ et $\|A\|_{H^{(3-\ell)/2}(H^{\nu_2})}$, telle que

$$\|v\|_{H^1(L^2)} + \|v\|_{L^\infty(H^1)} \leq C\|w^0\|_H.$$

La preuve du théorème 6 repose sur une inégalité d'observabilité pour le problème adjoint

$$\begin{cases} -\varphi_t - \Delta\varphi - (a,\nabla)\varphi - D\varphi\, b + \nabla\pi = 0 & \text{dans } Q, \\ \nabla\cdot\varphi = 0 & \text{dans } Q, \\ \varphi\cdot n = 0, \quad (\sigma(\varphi,\pi)\cdot n)_{tg} + (A^t\varphi)_{tg} = 0 & \text{sur } \Sigma, \\ \varphi(T) = \varphi^0 & \text{dans } \Omega. \end{cases} \tag{69}$$

Comme d'habitude, pour avoir l'observabilité de (69) on montrera une inégalité de Carleman. La preuve contient deux étapes :

• Dans un premier temps, on obtient une inégalité de Carleman pour l'opérateur (vectoriel) de la chaleur avec ces conditions aux limites. Plus précisément, on considère une fonction φ qui vérifie

$$\begin{cases} -\varphi_t - \Delta\varphi = G \in L^2(Q)^N, \quad \nabla\cdot\varphi = 0 & \text{dans } Q, \\ \varphi\cdot n = 0, \quad (D\varphi\cdot n)_{tg} + (A\varphi)_{tg} = 0 & \text{sur } \Sigma \end{cases}$$

et, tant que A vérifie (65)–(66), on montre l'estimation

$$I(s,\lambda;\varphi) \leq C\left(s^3\lambda^4 \iint_{\omega_0\times(0,T)} e^{-2s\alpha}\xi^3|\varphi|^2\,dx\,dt + \lambda\iint_Q e^{-2s\alpha}|G|^2\,dx\,dt\right)$$

pour un choix convenable des paramètres s et λ, où $\omega_0 \subset \omega$ est un ouvert.

L'idée générale utilisée est celle du livre [15], mais la tâche est beaucoup plus difficile ici à cause des conditions aux limites. Plus précisément, en utilisant les mêmes fonctions poids que dans [15] (et que l'on définira dans la quatrième partie), on pose

$$\psi = e^{-s\alpha}\varphi, \quad \tilde\psi = e^{-s\tilde\alpha}\varphi$$

et on trouve deux estimations : l'une pour ψ et l'autre pour $\tilde\psi$ mais avec des intégrales sur Σ sans signe. Enfin, les définitions des poids α et $\tilde\alpha$ font disparaître les termes du bord.

La nouveauté principale repose sur la stratégie de preuve des inegalités pour ψ et $\tilde\psi$, où on a réussi à profiter de tous les ingrédients que l'on a à gauche de nos inégalités depuis le début. Tous les détails peuvent être trouvés dans le paragraphe 2.1 du dernier travail.

• Enfin, les mêmes arguments du troisième travail et ceux de [17] peuvent être utilisés pour trouver une inégalité de Carleman pour le système (69) de la forme

$$I(s,\lambda;\varphi) \leq Cs^{15/2}\lambda^8 \iint_{\omega\times(0,T)} e^{-4s\widehat{\alpha}+2s\alpha^*}\widehat{\xi}^{15/2}|\varphi|^2\,dx\,dt. \tag{70}$$

Pour faire cela, il faut imposer sur A les hypothèses (65)–(67).

A partir de cette inégalité, le résultat énoncé dans le théorème 6 avec contrôles dans $L^2(\omega \times (0,T))^N$ est classique, mais ici on doit régulariser ces contrôles pour pouvoir exécuter un point fixe sur le problème (68).

Le procédé de régularisation que l'on utilise a été introduit dans [4]. Il consiste à profiter au maximum des termes qui apparaissent à gauche de notre inégalité de Carleman (70) pour déterminer la régularité du contrôle. Plus précisément, il est bien connu que les contrôles qui donnent la contrôlabilité approchée d'un système linéaire sont en fait les états de problèmes adjoints appropriés. Il est donc raisonnable espérer que la régularité des contrôles qui vont amener la solution du système (68) à zéro soit déterminée par la régularité de φ donnée par (70).

Finalement, on présente le résultat principal de ce travail concernant la contrôlabilité locale aux trajectoires du système (62). Ces trajectoires devront vérifier quelques hypothèses de régularité :

$$\begin{gathered}\overline{y} \in L^\infty(Q)^N, \quad \overline{y} \in H^{1-\ell}(0,T;W^{\nu_1,\nu_1+1}(\partial\Omega)^N),\\ \overline{y} \in H^{(3-\ell)/2}(0,T;H^{\nu_2}(\partial\Omega)^N),\end{gathered} \tag{71}$$

$$\begin{gathered}\overline{y}(0) \in H^3(\Omega)^N \cap W,\\ (D\overline{y}(0)\cdot n)_{tg} + (f(\overline{y}(0)))_{tg} = 0 \text{ sur } \partial\Omega\end{gathered} \tag{72}$$

(ℓ, ν_1 et ν_2 ont été définis dans (65)–(67)), avec

$$W = \{y \in H^1(\Omega)^N : \nabla\cdot y = 0 \text{ dans } \Omega,\ y\cdot n = 0 \text{ sur } \partial\Omega\}.$$

On a le résultat suivant :

Théorème 7 *Supposons que $f \in C^3(\mathbf{R}^N \times \mathbf{R}^N)$ et $y^0 \in H^3(\Omega)^N \cap W$ vérifie la condition de compatibilité*

$$(Dy^0\cdot n)_{tg} + (f(y^0))_{tg} = 0 \text{ sur } \partial\Omega. \tag{73}$$

Alors, on a la contrôlabilité exacte locale aux trajectoires du système (62) satisfaisant (71)–(72), c'est-à-dire, il existe $\delta > 0$ tel que si $\|y^0 - \overline{y}(0)\|_{H^3\cap W} \le \delta$, il existe des contrôles v et des solutions associées (y,p) de (62) satisfaisant

$$y(T) = \overline{y}(T) \text{ dans } \Omega.$$

En plus, ces contrôles appartiennent à

$$H^1(0,T;L^2(\omega)^N) \cap C([0,T];H^1(\omega)^N).$$

On va donner ici les principaux arguments de la démonstration. D'abord, on soustrait le système (63) vérifié par $\overline{y}$ de (62). Alors, $w = y - \overline{y}$ vérifie

$$\begin{cases} w_t - \Delta w + (w,\nabla)w + (w,\nabla)\overline{y} + (\overline{y},\nabla)w + \nabla q = v 1\!\!1_\omega & \text{dans } Q,\\ \nabla\cdot w = 0 & \text{dans } Q,\\ w\cdot n = 0,\ (\sigma(w,q)\cdot n)_{tg} + (F(\overline{y};w)w)_{tg} = 0, & \text{sur } \Sigma,\\ w(0) = y^0 - \overline{y}^0 = w^0 & \text{dans } \Omega,\end{cases} \tag{74}$$

où

$$F(\overline{y};w) = \int_0^1 \nabla f(\overline{y} + lw)\, dl \in \mathbf{R}^N \times \mathbf{R}^N.$$

On est donc ramené à démontrer par un argument de point fixe la contrôlabilité nulle locale du système (74).

On introduit l'espace de Banach

$$Z = \{z \in H^{(3-\ell)/2}(0,T;H^{\nu_2+1/2}(\Omega)^N \cap W) \cap H^{1-\ell}(0,T;W^{\nu_1+1/2,\nu_1+1}(\Omega)^N \cap W)\}$$

et un sous-espace affine fermé

$$Z_0 = \{z \in Z : z(\cdot,0) = w^0 \text{ dans } \Omega\}.$$

Alors, pour chaque $z \in Z_0$, le théorème 6 donne l'existence d'un contrôle $v_z \in H^1(0,T;L^2(\omega)^N) \cap C^0([0,T];H^1(\omega)^N)$ tel que la solution w_z de

$$\begin{cases} w_t + \widetilde{L}w + \nabla p = \zeta v_z & \text{dans } Q, \\ \nabla \cdot w = 0 & \text{dans } Q, \\ w \cdot n = 0,\ (Dw \cdot n)_{tg} + (F(\overline{y};z)w)_{tg} = 0 & \text{sur } \Sigma, \\ w(0) = w^0 & \text{dans } \Omega \end{cases} \tag{75}$$

soit nulle à l'instant T. Dans ce système, on a noté

$$\widetilde{L}w = -\Delta w + (z,\nabla)w + (w,\nabla)\overline{y} + (\overline{y},\nabla)w$$

et ζ est une fonction "cut-off" dans ω.

De plus, on peut construire v_z tel que

$$\|v_z\|_{H^1(L^2)} + \|v_z\|_{L^\infty(L^2)} \le C(\Omega,\omega,T,\|z\|_Z,\|F(\overline{y};z)\|_Z)\|w^0\|_H. \tag{76}$$

Ensuite, on définit $\Lambda(z)$ comme l'ensemble de contrôles

$$v_z \in H^1(0,T;L^2(\omega)^N) \cap C^0([0,T];H^1(\omega)^N)$$

qui conduisent la solution w_z de (75) à zéro au temps T et tels que (76) est vérifié. On introduit enfin l'application multivoque A, avec

$$A(z) = \{w_z \text{ solution de (75)} : v_z \in \Lambda(z)\} \quad \forall z \in Z.$$

Tout ceci permet d'appliquer le théorème de Kakutani (voir [3]) et on en déduit que l'application A posséde un point fixe.

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Chapitre 1

Null controllability of the heat equation with Fourier boundary conditions : The linear case

Null controllability of the heat equation with Fourier boundary conditions : The linear case

E. Fernández-Cara, M. González-Burgos, S. Guerrero et J.-P. Puel

Abstract

In this paper, we prove the global null controllability of the linear heat equation completed with linear Fourier boundary conditions of the form $\frac{\partial y}{\partial n} + \beta\, y = 0$. We consider distributed controls with support in a small set and nonregular coefficients $\beta = \beta(x,t)$. For the proof of null controllability, a crucial tool will be a new Carleman estimate for the weak solutions of the classical heat equation with nonhomogeneous Neumann boundary conditions.

1 Introduction

Let $\Omega \subset \mathbf{R}^N$ be a bounded connected open set whose boundary $\partial\Omega$ is regular enough ($N \geq 1$). Let $\omega \subset \Omega$ be a (small) nonempty open subset and let $T > 0$. We will use the notation $Q = \Omega \times (0,T)$ and $\Sigma = \partial\Omega \times (0,T)$ and we will denote by $n(x)$ the outward unit normal to Ω at $x \in \partial\Omega$. On the other hand, we will denote by C, C_1, C_2, ... generic positive constants (usually depending on Ω and ω).

We will consider the linear heat equation with linear Fourier (or Robin) conditions

$$\begin{cases} y_t - \Delta y + B(x,t)\cdot\nabla y + a(x,t)\, y = v(x,t)1_\omega & \text{in } Q, \\ \dfrac{\partial y}{\partial n} + \beta(x,t)\, y = 0 & \text{on } \Sigma, \\ y(x,0) = y^0(x) & \text{in } \Omega. \end{cases} \tag{1.1}$$

Here, it will be assumed that the coefficients a, B and β satisfy

$$a \in L^\infty(Q), \quad B \in L^\infty(Q)^N, \quad \beta \in L^\infty(\Sigma). \tag{1.2}$$

On the other hand, we suppose that $v \in L^2(\omega \times (0,T))$, 1_ω is the characteristic function of ω and $y^0 \in L^2(\Omega)$. In (1.1), $y = y(x,t)$ is the state and $v = v(x,t)$ is the control. It is assumed that we can act on the system only through $\omega \times (0,T)$.

An illustrative interpretation of the data and variables in (1.1) is the following. The function y can be viewed as the relative temperature of a body (with respect to the exterior surrounding air). The parabolic equation in (1.1) means, among other things, that a heat source $v1_\omega$ acts on a part of the body. On the boundary, $-\frac{\partial y}{\partial n}$ must be viewed as the *normal heat flux*, directed inwards, up to a positive coefficient. Thus, the equality

$$-\frac{\partial y}{\partial n} = \beta\, y$$

means that this flux is a linear function of the temperature. Thus, it is reasonable to suppose that $\beta \geq 0$ (although this assumption will not be imposed in this paper).

The main goal of this paper is to analyze the controllability properties of (1.1). It will be said that this system is *null controllable* at time T if, for each $y^0 \in L^2(\Omega)$, there exists $v \in L^2(\omega \times (0,T))$ such that the associated solution satisfies

$$y(x,T) = 0 \quad \text{in} \quad \Omega. \tag{1.3}$$

The null controllability of linear parabolic equations has been intensively studied these last years ; see for instance [8], [7], [12], [4] and [2].

In this paper, we will be concerned with (1.1), where the main difficulties arise from the particular form of the boundary condition. Indeed, it has been shown in [12] and [5] that this is more difficult to analyze than the case of Dirichlet boundary conditions, considered in [7], [12] and [4].

More precisely, what has been proved until now is that (1.1) is null controllable with $B \equiv 0$ under the assumptions (1.2) whenever $\beta_t \in L^\infty(\Sigma)$. This was shown in [12]. However, it would be important to prove the null controllability of (1.1) without this regularity hypothesis on β_t in view of applications to control systems with *nonlinear* boundary conditions.

The first main result in this paper concerns a Carleman inequality for a general (adjoint) system of the form

$$\begin{cases} -\varphi_t - \Delta\varphi = f_1(x,t) + \nabla\cdot f_2(x,t) & \text{in } Q, \\ (\nabla\varphi + f_2(x,t))\cdot n = f_3(x,t) & \text{on } \Sigma, \\ \varphi(x,T) = \varphi^0(x) & \text{in } \Omega, \end{cases} \tag{1.4}$$

where $f_1 \in L^2(Q)$, $f_2 \in L^2(Q)^N$ and $f_3 \in L^2(\Sigma)$. Observe that, as long as $\varphi \in L^2(Q)$, $\nabla\varphi + f_2 \in L^2(Q)^N$ and $\nabla\cdot(\nabla\varphi + f_2) \in H^{-1}(0,T;L^2(\Omega))$, we can give a sense to the boundary condition in the space $H^{-1}(0,T;H^{-1/2}(\partial\Omega))$.

We present now this result :

Theorem 1 *Under the previous assumptions on f_1 , f_2 and f_3 , there exist $\overline{\lambda}$, σ_1, σ_2 and C, only depending on Ω and ω, such that, for any $\lambda \geq \overline{\lambda}$, any $s \geq \overline{s} = \sigma_1(e^{\sigma_2\lambda}T + T^2)$ and any*

$\varphi^0 \in L^2(\Omega)$, *the weak solution to* (1.4) *satisfies*

$$\begin{aligned}&\iint_Q e^{-2s\alpha}(s\,\lambda^2\xi\,|\nabla\varphi|^2 + s^3\,\lambda^4\,\xi^3\,|\varphi|^2)dx\,dt\\&+s^2\lambda^3\iint_\Sigma e^{-2s\alpha}\,\xi^2\,|\varphi|^2\,d\sigma\,dt\\&\leq C\left(\iint_Q e^{-2s\alpha}(|f_1|^2 + \ s^2\,\lambda^2\,\xi^2\,|f_2|^2)dx\,dt\right.\\&\left.+s\lambda\iint_\Sigma e^{-2s\alpha}\,\xi\,|f_3|^2\,d\sigma\,dt + s^3\lambda^4\iint_{\omega\times(0,T)} e^{-2s\alpha}\,\xi^3\,|\varphi|^2\,dx\,dt\right).\end{aligned} \tag{1.5}$$

Here, $\alpha = \alpha(x,t)$ and $\xi = \xi(x,t)$ are appropriate positive functions, again only depending on Ω and ω. They are given below ; see (1.13)–(1.14).

As a consequence of theorem 1, we can deduce an *observability inequality* for the adjoint system associated to (1.1). More precisely, let us consider the backward in time system

$$\begin{cases} -\varphi_t - \Delta\varphi - \nabla\cdot(\varphi B(x,t)) + a(x,t)\,\varphi = 0 & \text{in } Q,\\ (\nabla\varphi + \varphi B(x,t))\cdot n + \beta(x,t)\varphi = 0 & \text{on } \Sigma,\\ \varphi(x,T) = \varphi^0(x) & \text{in } \Omega,\end{cases} \tag{1.6}$$

where $\varphi^0 \in L^2(\Omega)$. It will be seen that, for some K of the form

$$K = e^{C(1+\frac{1}{T}+\|a\|_\infty^{2/3}+\|B\|_\infty^2+\|\beta\|_\infty^2)}, \tag{1.7}$$

the solutions of (1.6) satisfy

$$\iint_{\Omega\times(T/4,3T/4)} |\varphi|^2\,dx\,dt \leq K\iint_{\omega\times(0,T)} |\varphi|^2\,dx\,dt. \tag{1.8}$$

Remark 1 In fact, (1.8) is not the unique way of saying that (1.6) is observable. It is indeed more frequent to use other inequalities of the form

$$\|\varphi(\cdot,0)\|^2_{L^2(\Omega)} \leq C\iint_{\omega\times(0,T)} |\varphi|^2\,dx\,dt \tag{1.9}$$

for some C. The estimates (1.9) can be easily deduced from (1.8) and the energy inequalities satisfied by φ.

The second main result in this paper concerns the *null controllability* of (1.1). It is the following :

Theorem 2 *Let us assume that* (1.2) *is satisfied. Then, for each* $T > 0$, (1.1) *is null controllable at time* T *with controls* $v \in L^2(\omega\times(0,T))$. *Moreover, one can find* v *such that*

$$\|v\|_{L^2(\omega\times(0,T))} \leq H\,\|y^0\|_{L^2}\,, \tag{1.10}$$

with a constant H of the form

$$H = e^{C(1+\frac{1}{T}+\|a\|_\infty^{2/3}+\|B\|_\infty^2+\|\beta\|_\infty^2+T(\|a\|_\infty+\|B\|_\infty^2+\|\beta\|_\infty^2))} \tag{1.11}$$

for some $C = C(\Omega, \omega)$.

In the proof of theorem 2, the main tool is the estimate (1.8). This arises from a general principle that asserts that the null controllability of (1.1) with controls in $L^2(\omega \times (0,T))$ (depending continuously on the data) is equivalent to the observability of (1.6). More details will be given below.

In a second part of this work, which will appear in a forthcoming paper, we will consider controllability questions for semilinear heat equations completed with *nonlinear* Fourier boundary conditions of the form

$$\frac{\partial y}{\partial n} + f(y) = 0 \quad \text{on} \quad \Sigma,$$

where $f : \mathbf{R} \mapsto \mathbf{R}$ is locally Lipschitz-continuous. For the analysis of these systems, theorems 1 and 2 of the present paper will be crucial.

The rest of this paper is organized as follows. Section 2 is devoted to the proof of theorem 1. In section 3, we deduce the observability inequality (1.8) and we prove theorem 2. For completeness, we have included an Appendix, where we give a detailed proof of the standard Carleman estimate for the solutions of the heat equation with homogeneous Neumann boundary conditions. (this estimate was already proved in [12] ; however, in this paper, a careful study of the dependence of the constants on s, λ and T is needed).

2 Proof of theorem 1

The main arguments used below are similar to those in [6]. This is related to a general strategy which is used to relax the regularity assumptions on the various coefficients involved in the problem. Here, it will allow us to proceed without any kind of regularity on the coefficient $\beta = \beta(x,t)$.

Let us recall the definition of a weak solution : we say that φ is a *weak solution* to (1.4) if it satisfies

$$\begin{cases} \varphi \in L^2(0,T;H^1(\Omega)) \cap C^0([0,T];L^2(\Omega)), \\ -\langle \varphi_t, v\rangle_{(H^1(\Omega))',H^1(\Omega)} + \displaystyle\int_\Omega \nabla\varphi\cdot\nabla v\,dx = \int_\Omega f_1(x,t)\,v\,dx \\ \qquad -\displaystyle\int_\Omega f_2(x,t)\cdot\nabla v\,dx + \int_{\partial\Omega} f_3(x,t)\,v\,d\sigma \\ \text{a.e. in } (0,T), \quad \forall v \in H^1(\Omega), \\ \varphi(x,T) = \varphi^0(x) \quad \text{in} \quad \Omega. \end{cases} \tag{1.12}$$

It is well known that, for $f_1 \in L^2(Q)$, $f_2 \in L^2(Q)^N$, $f_3 \in L^2(\Sigma)$ and $\varphi^0 \in L^2(\Omega)$, (1.4) possesses exactly one weak solution φ.

To prove the Carleman inequality (1.5), we will need two weight functions :

$$\xi(x,t) = \frac{e^{\lambda\eta^0(x)}}{t(T-t)}\ , \quad \alpha(x,t) = \frac{e^{2\lambda\|\eta^0\|_\infty} - e^{\lambda\eta^0(x)}}{t(T-t)}\ . \tag{1.13}$$

Here, $\lambda \geq 1$ is a parameter to be chosen below and $\eta^0 = \eta^0(x)$ is a function satisfying

$$\begin{gathered} \eta^0 \in C^2(\overline{\Omega}), \quad \eta^0(x) > 0 \text{ in } \Omega, \quad \eta^0(x) = 0 \text{ on } \partial\Omega, \\ |\nabla\eta^0(x)| > 0 \text{ in } \overline{\Omega \setminus \omega'}, \end{gathered} \tag{1.14}$$

where $\omega' \subset\subset \omega$ is a nonempty open set. The existence of η^0 satisfying (1.14) is proved in [12].

For the proof of theorem 1, we will need an auxiliary result : a Carleman inequality for the solutions to the heat equation with homogeneous Neumann boundary conditions. This is given in the following result :

Lemma 1 *Let $f \in L^2(Q)$ be given. There exist λ^*, σ^* and C only depending on Ω and ω such that, for any $\lambda \geq \lambda^*$, any $s \geq s^*(\lambda) = \sigma^*(e^{4\lambda\|\eta^0\|_\infty}T + T^2)$ and any $q^0 \in L^2(\Omega)$, the weak solution to*

$$\begin{cases} -q_t - \Delta q = f(x,t) & \text{in } Q, \\ \dfrac{\partial q}{\partial n} = 0 & \text{on } \Sigma, \\ q(x,T) = q^0(x) & \text{in } \Omega \end{cases}$$

satisfies

$$I_{s,\lambda}(q) \leq C\left(\iint_Q e^{-2s\alpha}\,|f|^2\,dx\,dt + s^3\lambda^4 \iint_{\omega\times(0,T)} e^{-2s\alpha}\,\xi^3\,|q|^2\,dx\,dt\right), \tag{1.15}$$

where we have used the notation

$$I_{s,\lambda}(q) = \iint_Q e^{-2s\alpha}\left((s\xi)^{-1}(|q_t|^2 + |\Delta q|^2) + s\lambda^2\xi\,|\nabla q|^2 + s^3\lambda^4\xi^3\,|q|^2\right)\,dx\,dt.$$

This result is a particular case of lemma 1.2 of Chapter I in [12]. For completeness and also in order to explain and justify the particular form of the constants λ^* and $s^*(\lambda)$, we give a complete proof in the Appendix, at the end of this paper.

Let us continue with the proof of theorem 1. We can view φ as a solution by transposition of (1.4). This means that φ is the unique function in $L^2(Q)$ satisfying

$$\begin{cases} \displaystyle\iint_Q \varphi\,h\,dx\,dt = \iint_Q f_1(x,t)\,z\,dx\,dt - \iint_Q f_2(x,t)\cdot\nabla z\,dx\,dt \\ \displaystyle\quad + \iint_\Sigma f_3(x,t)\,z\,d\sigma\,dt + \int_\Omega \varphi^0(x)\,z(x,T)\,dx \quad \forall h \in L^2(Q), \end{cases} \tag{1.16}$$

where we have denoted by z the (strong) solution of the following problem :

$$\begin{cases} z_t - \Delta z = h(x,t) & \text{in } Q, \\ \dfrac{\partial z}{\partial n} = 0 & \text{on } \Sigma, \\ z(x,0) = 0 & \text{in } \Omega. \end{cases}$$

We will argue as follows. Let us first estimate the second term in the left hand side of (1.5), i.e.

$$s^3\lambda^4 \iint_Q e^{-2s\alpha}\,\xi^3\,|\varphi|^2\,dx\,dt. \tag{1.17}$$

To this end, we will deal with techniques inspired by the arguments in [6].

Thus, let us see that the term in (1.17) can be bounded by the right hand side of (1.5), i.e.

$$\begin{aligned} s^3\lambda^4 \iint_Q e^{-2s\alpha}\,\xi^3\,|\varphi|^2\,dx\,dt &\le C(\Omega,\omega)\Bigg(\iint_Q e^{-2s\alpha}\,|f_1|^2\,dx\,dt \\ &+ s^2\lambda^2 \iint_Q e^{-2s\alpha}\,\xi^2\,|f_2|^2\,dx\,dt + s\lambda \iint_\Sigma e^{-2s\alpha}\,\xi\,|f_3|^2\,d\sigma\,dt \\ &+ s^3\lambda^4 \iint_{\omega\times(0,T)} e^{-2s\alpha}\,\xi^3\,|\varphi|^2\,dx\,dt\Bigg) \end{aligned} \tag{1.18}$$

for a good choice of the parameters λ and s.

Let us consider the following constrained extremal problem :

$$\begin{cases} \text{Minimize } \dfrac{1}{2}\left(\displaystyle\iint_Q e^{2s\alpha}\,|z|^2 + s^{-3}\lambda^{-4}\iint_{\omega\times(0,T)} e^{2s\alpha}\,\xi^{-3}\,|v|^2\,dx\,dt\right) \\ \text{subject to } v \in L^2(Q) \text{ and} \\ \quad \begin{cases} z_t - \Delta z = s^3\lambda^4 e^{-2s\alpha}\xi^3\varphi + v1_\omega & \text{in } Q, \\ \dfrac{\partial z}{\partial n} = 0 & \text{on } \Sigma, \\ z(x,0) = 0, \quad z(x,T) = 0 & \text{in } \Omega. \end{cases} \end{cases} \tag{1.19}$$

Here, s and λ are chosen like in lemma 1.

By virtue of Lagrange's principle and arguing as in [6], we are led from (1.19) to the next optimality system, which is of fourth order in space and second order in time :

$$\begin{cases} \mathcal{L}(e^{-2s\alpha}\,\mathcal{L}^*p) + s^3\lambda^4 e^{-2s\alpha}\xi^3 p1_\omega = s^3\lambda^4 e^{-2s\alpha}\xi^3\varphi & \text{in } Q, \\ \dfrac{\partial p}{\partial n} = 0, \quad \dfrac{\partial}{\partial n}(e^{-2s\alpha}\,\mathcal{L}^*p) = 0 & \text{on } \Sigma, \\ (e^{-2s\alpha}\mathcal{L}^*p)_{|t=0} = (e^{-2s\alpha}\mathcal{L}^*p)_{|t=T} = 0 & \text{in } \Omega. \end{cases} \tag{1.20}$$

Here, $\mathcal{L} = \partial_t - \Delta$ is the heat operator and $\mathcal{L}^* = -\partial_t - \Delta$ is its formal adjoint. If p is a solution to (1.20) (in an appropriate sense), then

$$\widehat{v} = -s^3\,\lambda^4\,e^{-2s\alpha}\,\xi^3\,p1_\omega \quad \text{and} \quad \widehat{z} = e^{-2s\alpha}\mathcal{L}^*p \tag{1.21}$$

solve (1.19).

Let us show that (1.20) has a unique *weak* solution. To this end, we are going to rewrite this problem as a Lax-Milgram variational equation. Let us introduce the space

$$X_0 = \{\, z \in C^2(\overline{Q}) : \frac{\partial z}{\partial n} = 0 \text{ on } \Sigma \,\}$$

and the norm $\| \cdot \|_X$, with

$$\|q\|_X^2 = \iint_Q e^{-2s\alpha}\, |\mathcal{L}^* q|^2 \, dx\, dt + s^3 \lambda^4 \iint_{\omega\times(0,T)} e^{-2s\alpha}\, \xi^3\, |q|^2 \, dx\, dt$$

for all $q \in X_0$.

Due to lemma 1, $\| \cdot \|_X$ is indeed a norm in X_0 . Let X be the completion of X_0 for the norm $\| \cdot \|_X$. Then X is a Hilbert space for the scalar product $(\cdot\,,\cdot)_X$, with

$$(p, q)_X = \iint_Q e^{-2s\alpha}\, (\mathcal{L}^* p)(\mathcal{L}^* q) \, dx\, dt + s^3 \lambda^4 \iint_{\omega\times(0,T)} e^{-2s\alpha}\, \xi^3\, p\, q \, dx\, dt.$$

With this notation, system (1.20) is equivalent to find a function $p \in X$ such that

$$(p, q)_X = \ell(q) \quad \forall q \in X, \tag{1.22}$$

where

$$\ell(q) = s^3\, \lambda^4 \iint_Q e^{-2s\alpha}\, \xi^3\, \varphi\, q \, dx\, dt \quad \forall q \in X.$$

Of course, (1.22) is equivalent to another extremal problem

$$\begin{cases} \text{Minimize } \dfrac{1}{2}(q, q)_X - \ell(q) \\ \text{subject to } q \in X. \end{cases}$$

By virtue of lemma 1, one can easily check that $\ell \in X'$. Consequently, one can apply Lax-Milgram lemma and deduce that there exists a unique solution to (1.20).

Let us now take

$$h = s^3 \lambda^4 e^{-2s\alpha} \xi^3 \varphi + \widehat{v} 1_\omega$$

in (1.16). This gives

$$\begin{aligned} s^3 \lambda^4 \iint_Q e^{-2s\alpha}\, \xi^3\, |\varphi|^2 \, dx\, dt &= \iint_Q f_1\, \widehat{z}\, dx\, dt - \iint_Q f_2 \cdot \nabla \widehat{z}\, dx\, dt \\ &+ \iint_\Sigma f_3\, \widehat{z}\, d\sigma\, dt - \iint_{\omega\times(0,T)} \varphi\, \widehat{v}\, dx\, dt \end{aligned} \tag{1.23}$$

(recall that $\widehat{v}$ and $\widehat{z}$ are given by (1.21)). The idea of the proof of (1.18) is to bound $\widehat{z}$, $\nabla \widehat{z}$ and $\widehat{v}$ in Q and the trace of $\widehat{z}$ on Σ in terms of the left hand side of (1.23). For this purpose, we first multiply the equation in (1.20) by p and integrate in Q, which gives

$$\|p\|_X^2 \le \|\ell\|_{X'}\, \|p\|_X$$

and, consequently,

$$\begin{aligned}\|p\|_X^2 &= \iint_Q e^{2s\alpha}\,|\widehat{z}|^2\,dx\,dt + s^{-3}\lambda^{-4}\iint_{\omega\times(0,T)} e^{2s\alpha}\,\xi^{-3}\,|\widehat{v}|^2\,dx\,dt\\ &\le Cs^3\lambda^4\iint_Q e^{-2s\alpha}\,\xi^3\,|\varphi|^2\,dx\,dt,\end{aligned}\tag{1.24}$$

for $\lambda \ge \overline{\lambda}(\Omega,\omega)$, $s \ge \overline{\sigma}(\Omega,\omega)(e^{4\lambda\|\eta^0\|_\infty}T + T^2)$. This provides the desired bounds of $\widehat{z}$ and $\widehat{v}1_\omega$.

Let us now multiply the equation satisfied by $\widehat{z}$ by $s^{-2}\lambda^{-2}e^{2s\alpha}\xi^{-2}\widehat{z}$ and let us integrate in Q. After integration by parts, we obtain :

$$\begin{aligned}&\frac{1}{2}s^{-2}\lambda^{-2}\iint_Q e^{2s\alpha}\xi^{-2}\,\frac{\partial}{\partial t}|\widehat{z}|^2\,dx\,dt + s^{-2}\lambda^{-2}\iint_Q e^{2s\alpha}\xi^{-2}\,|\nabla\widehat{z}|^2\,dx\,dt\\ &\quad -s^{-1}\lambda^{-1}\iint_Q e^{2s\alpha}\,\xi^{-1}\,\nabla\eta^0\cdot\nabla|\widehat{z}|^2\,dx\,dt\\ &\quad -2s^{-2}\lambda^{-1}\iint_Q e^{2s\alpha}\,\xi^{-2}\,(\nabla\eta^0\cdot\nabla\widehat{z})\,\widehat{z}\,dx\,dt\\ &\quad = s\lambda^2\iint_Q \xi\,\varphi\,\widehat{z}\,dx\,dt + s^{-2}\lambda^{-2}\iint_{\omega\times(0,T)} e^{2s\alpha}\,\xi^{-2}\,\widehat{v}\,\widehat{z}\,dx\,dt,\end{aligned}$$

whence

$$\begin{aligned}&s^{-2}\lambda^{-2}\iint_Q e^{2s\alpha}\,\xi^{-2}\,|\nabla\widehat{z}|^2\,dx\,dt - s^{-1}\lambda^{-1}\iint_\Sigma e^{2s\alpha}\xi^{-1}\,\frac{\partial\eta^0}{\partial n}\,|\widehat{z}|^2\,d\sigma\,dt\\ &\quad = \frac{1}{2}s^{-2}\lambda^{-2}\iint_Q \frac{\partial}{\partial t}(e^{2s\alpha}\,\xi^{-2})\,|\widehat{z}|^2\,dx\,dt\\ &\quad -s^{-1}\lambda^{-1}\iint_Q \nabla\cdot(e^{2s\alpha}\,\xi^{-1}\,\nabla\eta^0)\,|\widehat{z}|^2\,dx\,dt\\ &\quad +2s^{-2}\lambda^{-1}\iint_Q e^{2s\alpha}\,\xi^{-2}\,\nabla\eta^0\cdot\nabla\widehat{z}\,\widehat{z}\,dx\,dt + s\lambda^2\iint_Q \xi\,\varphi\,\widehat{z}\,dx\,dt\\ &\quad +s^{-2}\lambda^{-2}\iint_{\omega\times(0,T)} e^{2s\alpha}\,\xi^{-2}\,\widehat{v}\,\widehat{z}\,dx\,dt.\end{aligned}\tag{1.25}$$

We need now some estimates concerning the weight functions in order to preserve explicit bounds in s, λ and T. Notice that

$$\begin{aligned}\frac{\partial}{\partial t}(e^{2s\alpha}\,\xi^{-2}) &= -2(T-2t)\,e^{-\lambda\eta^0}\,e^{2s\alpha}(s\,e^{-\lambda\eta^0}(e^{2\lambda\|\eta^0\|_\infty} - e^{\lambda\eta^0}) - \xi^{-1})\\ &\le C\,T\,e^{2s\alpha}(e^{2\lambda\|\eta^0\|_\infty}\,s + \xi^{-1}) \le C\,T\,s\,e^{2s\alpha}\,e^{2\lambda\|\eta^0\|_\infty},\end{aligned}$$

where we have taken $s \ge C\,T^2$. More generally, observe that, for any fixed m, one also has

$$|\nabla(e^{2s\alpha}\,\xi^m)| \le C_m(\Omega,\omega)\,s\,\lambda\,e^{2s\alpha}\,\xi^{m+1}\tag{1.26}$$

whenever $s \geq C\,T^2$. Indeed, we have

$$\nabla(e^{2s\alpha}\,\xi^m) = e^{2s\alpha}\,\lambda\,\nabla\eta^0\,\xi^m(2s\,\xi+m) \leq C(\Omega,\omega)\,e^{2s\alpha}\,\lambda\,\xi^m(s\,\xi+1)$$

and, taking into account that

$$C\,s\,\xi \geq 1 \quad \text{for } s \geq \frac{T^2}{4C}\,, \tag{1.27}$$

we directly get (1.26).

Turning back to (1.25), we obtain

$$\begin{aligned}
&s^{-2}\lambda^{-2}\iint_Q e^{2s\alpha}\,\xi^{-2}\,|\nabla\widehat{z}|^2\,dx\,dt - s^{-1}\lambda^{-1}\iint_\Sigma e^{2s\alpha}\xi^{-1}\,\frac{\partial\eta^0}{\partial n}\,|\widehat{z}|^2\,d\sigma\,dt \\
&\leq C(\Omega,\omega)\Big(T\,s^{-1}\lambda^{-2}\,e^{2\lambda\|\eta^0\|_\infty}\iint_Q e^{2s\alpha}\,|\widehat{z}|^2\,dx\,dt \\
&+\iint_Q e^{2s\alpha}\,|\widehat{z}|^2\,dx\,dt + s^{-1}\lambda^{-1}\iint_Q e^{2s\alpha}\,\xi^{-1}\,|\widehat{z}|^2\,dx\,dt \\
&+s^{-2}\iint_Q e^{2s\alpha}\,\xi^{-2}\,|\widehat{z}|^2\,dx\,dt + s^2\lambda^4\iint_Q e^{-2s\alpha}\,\xi^2\,|\varphi|^2\,dx\,dt \\
&+s^{-4}\lambda^{-4}\iint_Q e^{2s\alpha}\,\xi^{-4}\,|\widehat{v}|^2\,dx\,dt\Big) + \frac{1}{2}s^{-2}\lambda^{-2}\iint_Q e^{2s\alpha}\,\xi^{-2}\,|\nabla\widehat{z}|^2\,dx\,dt,
\end{aligned}$$

where we have taken $s \geq C\,T^2$. Now, we take into account (1.27) and we deduce that

$$\begin{aligned}
&s^{-2}\lambda^{-2}\iint_Q e^{2s\alpha}\,\xi^{-2}\,|\nabla\widehat{z}|^2\,dx\,dt - s^{-1}\lambda^{-1}\iint_\Sigma e^{2s\alpha}\xi^{-1}\,\frac{\partial\eta^0}{\partial n}\,|\widehat{z}|^2\,d\sigma\,dt \\
&\leq C(\Omega,\omega)\Big(\iint_Q e^{2s\alpha}\,|\widehat{z}|^2\,dx\,dt + s^3\lambda^4\iint_Q e^{-2s\alpha}\,\xi^3\,|\varphi|^2\,dx\,dt \\
&\qquad s^{-3}\lambda^{-4}\iint_Q e^{2s\alpha}\,\xi^{-3}\,|\widehat{v}|^2\,dx\,dt\Big)
\end{aligned}$$

for any $\lambda \geq C(\Omega,\omega)$ and any $s \geq C(\Omega,\omega)(e^{2\lambda\|\eta^0\|_\infty}\,T + T^2)$.

From (1.14), this gives an estimate of the gradient and the trace of $\widehat{z}$ in terms of $\widehat{z}$, $\widehat{v}1_\omega$ and φ. In view of (1.24), we now have

$$\begin{aligned}
&\iint_Q e^{2s\alpha}\,|\widehat{z}|^2\,dx\,dt + s^{-2}\lambda^{-2}\iint_Q e^{2s\alpha}\,\xi^{-2}\,|\nabla\widehat{z}|^2\,dx\,dt \\
&+s^{-1}\lambda^{-1}\iint_\Sigma e^{2s\alpha}\xi^{-1}\,|\widehat{z}|^2\,d\sigma\,dt + s^{-3}\lambda^{-4}\iint_{\omega\times(0,T)} e^{2s\alpha}\,\xi^{-3}\,|\widehat{v}|^2\,dx\,dt \\
&\qquad \leq C(\Omega,\omega)\,s^3\lambda^4\iint_Q e^{-2s\alpha}\,\xi^3\,|\varphi|^2\,dx\,dt
\end{aligned}$$

for $\lambda \geq C(\Omega,\omega)$, $s \geq C(\Omega,\omega)(e^{2\lambda\|\eta^0\|_\infty}\,T + T^2)$.

It suffices to combine this inequality and the identity (1.23) to deduce (1.18).

Let us now show that

$$\begin{aligned} s\lambda^2 \iint_Q e^{-2s\alpha}\,\xi\,|\nabla\varphi|^2\,dx\,dt \le C(\Omega,\omega)\Bigg(\iint_Q e^{-2s\alpha}\,|f_1|^2\,dx\,dt \\ +s^2\lambda^2 \iint_Q e^{-2s\alpha}\,\xi^2\,|f_2|^2\,dx\,dt + s\lambda \iint_\Sigma e^{-2s\alpha}\,\xi\,|f_3|^2\,d\sigma\,dt \\ +s^3\lambda^4 \iint_{\omega\times(0,T)} e^{-2s\alpha}\,\xi^3\,|\varphi|^2\,dx\,dt\Bigg). \end{aligned} \tag{1.28}$$

To this end, we now have to use not only that φ is a solution by transposition but a weak solution as well. More precisely, let us take

$$v = s\lambda^2 e^{-2s\alpha(\cdot,t)}\xi(\cdot,t)\varphi(\cdot,t)$$

in (1.12). Then, let us integrate in $(0,T)$ and let us perform integrations by parts similarly as we did before. We get :

$$\begin{aligned} &-\frac{1}{2}s\lambda^2 \iint_Q e^{-2s\alpha}\xi\,\frac{\partial}{\partial t}|\varphi|^2\,dx\,dt + s\lambda^2 \iint_Q e^{-2s\alpha}\xi\,|\nabla\varphi|^2\,dx\,dt \\ &+s\lambda^2 \iint_Q \nabla\varphi\cdot\nabla(e^{-2s\alpha}\,\xi)\,\varphi\,dx\,dt \\ &\qquad = s\lambda^2 \iint_Q e^{-2s\alpha}\,\xi\,f_1\,\varphi\,dx\,dt - s\lambda^2 \iint_Q f_2\cdot\nabla(e^{-2s\alpha}\,\xi\,\varphi)\,dx\,dt \\ &\qquad +s\lambda^2 \iint_\Sigma e^{-2s\alpha}\,\xi\,f_3\,\varphi\,d\sigma\,dt. \end{aligned}$$

We integrate by parts again and we obtain

$$\begin{aligned} &s\lambda^2 \iint_Q e^{-2s\alpha}\xi\,|\nabla\varphi|^2\,dx\,dt \\ &\quad = -\frac{1}{2}s\lambda^2 \iint_Q (e^{-2s\alpha}\xi)_t\,|\varphi|^2\,dx\,dt - s\lambda^2 \iint_Q \nabla\varphi\cdot\nabla(e^{-2s\alpha}\,\xi)\,\varphi\,dx\,dt \\ &\quad +s\lambda^2 \iint_Q e^{-2s\alpha}\,\xi\,f_1\,\varphi\,dx\,dt - s\lambda^2 \iint_Q f_2\cdot\nabla(e^{-2s\alpha}\,\xi)\,\varphi\,dx\,dt \\ &\quad -s\lambda^2 \iint_Q f_2\cdot\nabla\varphi\,e^{-2s\alpha}\,\xi\,dx\,dt + s\lambda^2 \iint_\Sigma e^{-2s\alpha}\,\xi\,f_3\,\varphi\,d\sigma\,dt. \end{aligned}$$

In view of (1.26), we find :

$$
\begin{aligned}
& s\lambda^2 \iint_Q e^{-2s\alpha}\xi\, |\nabla\varphi|^2\, dx\, dt \\
& \quad \leq C(\Omega,\omega) \Big(T\, s^2\, \lambda^2\, e^{2\lambda\|\eta^0\|_\infty} \iint_Q e^{-2s\alpha}\, \xi^3\, |\varphi|^2\, dx\, dt \\
& \quad + s^3\lambda^4 \iint_Q e^{-2s\alpha}\, \xi^3\, |\varphi|^2\, dx\, dt + \iint_Q e^{-2s\alpha}\, |f_1|^2\, dx\, dt \\
& \quad + s^2\lambda^4 \iint_Q e^{-2s\alpha}\, \xi^2\, |\varphi|^2\, dx\, dt + s\lambda^2 \iint_Q e^{-2s\alpha}\, \xi\, |f_2|^2\, dx\, dt \\
& \quad s\lambda \iint_\Sigma e^{-2s\alpha}\, \xi\, |f_3|^2\, d\sigma\, dt + s\lambda^3 \iint_\Sigma e^{-2s\alpha}\, \xi\, |\varphi|^2\, d\sigma\ dt \Big) \\
& \quad + \frac{1}{2} s\lambda^2 \iint_Q e^{-2s\alpha}\xi\, |\nabla\varphi|^2\, dx\, dt,
\end{aligned}
$$

where we have taken $s \geq CT^2$ and $\lambda \geq C$. Making several simplifications, we easily see that

$$
\left\{
\begin{aligned}
& s\lambda^2 \iint_Q e^{-2s\alpha}\xi\, |\nabla\varphi|^2\, dx\, dt \leq C \Big(s^3\lambda^4 \iint_Q e^{-2s\alpha}\, \xi^3\, |\varphi|^2\, dx\, dt \\
& \quad + \iint_Q e^{-2s\alpha}\, |f_1|^2\, dx\, dt + s^2\lambda^2 \iint_Q e^{-2s\alpha}\, \xi^2\, |f_2|^2\, dx\, dt \\
& \quad + s\lambda \iint_\Sigma e^{-2s\alpha}\, \xi\, |f_3|^2\, d\sigma\, dt + s\lambda^3 \iint_\Sigma e^{-2s\alpha}\, \xi\, |\varphi|^2\, d\sigma\ dt \Big),
\end{aligned}
\right.
\tag{1.29}
$$

for $s \geq C(e^{2\lambda\|\eta^0\|_\infty}\, T + T^2)$ and $\lambda \geq C$, whence (1.28) follows easily.

Let us finally estimate the trace of φ in terms of φ and $\nabla\varphi$. Notice that

$$
\begin{aligned}
& -s^2\lambda^3 \iint_Q e^{-2s\alpha}\, \xi^2\, (\nabla\eta^0 \cdot \nabla\varphi)\, \varphi\, dx\, dt \\
& \qquad = -\frac{1}{2} s^2\lambda^3 \iint_\Sigma e^{-2s\alpha}\, \xi^2\, \frac{\partial\eta^0}{\partial n}\, |\varphi|^2\, d\sigma\, dt \\
& \qquad + \frac{1}{2} s^2\lambda^3 \iint_Q \nabla\cdot(e^{-2s\alpha}\, \xi^2\, \nabla\eta^0)\, |\varphi|^2\, dx\, dt.
\end{aligned}
$$

Taking into account (1.14), the following is found :

$$
\begin{aligned}
& s^2\lambda^3 \iint_\Sigma e^{-2s\alpha}\, \xi^2\, |\varphi|^2\, d\sigma\, dt \leq C\, s^2\, \lambda^3 \iint_Q |\nabla\cdot(e^{-2s\alpha}\, \xi^2\, \nabla\eta^0)|\, |\varphi|^2\, dx\, dt \\
& \quad + C\Big(s^3\lambda^4 \iint_Q e^{-2s\alpha}\, \xi^3\, |\varphi|^2\, dx\, dt + s\lambda^2 \iint_Q e^{-2s\alpha}\xi\, |\nabla\varphi|^2\, dx\, dt \Big) \\
& \quad \leq C\Big(s^3\lambda^4 \iint_Q e^{-2s\alpha}\, \xi^3\, |\varphi|^2\, dx\, dt + s\lambda^2 \iint_Q e^{-2s\alpha}\xi\, |\nabla\varphi|^2\, dx\, dt \Big),
\end{aligned}
$$

with $s \geq C(T+T^2)$ and $\lambda \geq C$.

This last inequality, together with (1.18) and (1.29), provides (1.5) and permits to achieve the proof of theorem 1.

3 Controllability of the linear system

This section is devoted to prove theorem 2. This will be a consequence of the Carleman inequality (1.5).

We will start with an explicit bound of the weak solution to the linear problem

$$\begin{cases} y_t - \Delta y + B(x,t)\cdot\nabla y + a(x,t)\, y = f(x,t) & \text{in } Q, \\ \dfrac{\partial y}{\partial n} + \beta(x,t) y = 0 & \text{on } \Sigma, \\ y(x,0) = y^0(x) & \text{in } \Omega, \end{cases} \tag{1.30}$$

where $f \in L^2(Q)$, $y^0 \in L^2(\Omega)$ and (1.2) is fulfilled. Then, we will use this result in combination with (1.5) to deduce the observability inequality (1.8) for the solutions to (1.6). Finally, we will end the proof of theorem 2 in a classical way, using this observability inequality.

Proposition 4 *Under the previous assumptions, the weak solution to* (1.30) *satisfies the estimate*

$$\|y\|_Y \leq e^{CT\,(1+\|a\|_\infty+\|B\|_\infty^2+\|\beta\|_\infty^2)}\,(\|f\|_{L^2(Q)} + \|y^0\|_{L^2(\Omega)}) \tag{1.31}$$

for some constant $C>0$. Here, Y is the usual energy space :

$$Y = L^2(0,T;H^1(\Omega)) \cap C^0([0,T];L^2(\Omega)).$$

Proof : The existence and uniqueness of a solution to (1.30) is well known. Furthermore, the following identity can be deduced for each $t \in (0,T)$ in a standard way :

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_\Omega |y(x,t)|^2\,dx + \int_\Omega |\nabla y(x,t)|^2\,dx + \int_{\partial\Omega} \beta(x,t)\,|y(x,t)|^2\,d\sigma \\ &\quad + \int_\Omega B(x,t)\cdot\nabla y(x,t)\,y(x,t)\,dx + \int_\Omega a(x,t)\,|y(x,t)|^2\,dx \\ &\quad = \int_\Omega f(x,t)\,y(x,t)\,dx. \end{aligned} \tag{1.32}$$

We will now use the following trace estimate for the functions in $H^1(\Omega)$:

$$\begin{cases} \displaystyle\int_{\partial\Omega} |u|^2\,d\sigma \leq C\left(\int_\Omega (|u|^2 + |\nabla u|^2)\,dx\right)^{1/2}\left(\int_\Omega |u|^2\,dx\right)^{1/2} \\ \forall u \in H^1(\Omega), \end{cases} \tag{1.33}$$

for some positive $C = C(\Omega)$. This inequality can be proved arguing first for regular functions in a dense subspace of $H^1(\Omega)$ and then passing to the limit. For a regular function u, (1.33) is very easy to establish when $\Omega = \mathbf{R}^N_+$. Then, a standard localization argument leads to the proof in the case of a general domain Ω.

In view of (1.32) and (1.33), we have :

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega |y(x,t)|^2\,dx + \int_\Omega |\nabla y(x,t)|^2\,dx \\ \leq -\int_\Omega B(x,t)\cdot\nabla y(x,t)\,y(x,t)\,dx - \int_\Omega a(x,t)\,|y(x,t)|^2\,dx \\ + \int_\Omega f(x,t)\,y(x,t)\,dx + C\,\|\beta\|_\infty\,\|y(\cdot,t)\|_{H^1(\Omega)}\,\|y(\cdot,t)\|_{L^2(\Omega)}.\end{aligned}$$

Combining this and Young's inequality, we obtain :

$$\begin{aligned}\frac{d}{dt}\|y(\cdot,t)\|^2_{L^2(\Omega)} + \|y(\cdot,t)\|^2_{H^1(\Omega)} \\ \leq C((1+\|a\|_\infty + \|B\|^2_\infty + \|\beta\|^2_\infty)\|y(\cdot,t)\|^2_{L^2(\Omega)} + \|f(\cdot,t)\|^2_{L^2(\Omega)})\end{aligned}$$

for all $t \in (0,T)$. From these estimates, it is not difficult to obtain (1.31).

This ends the proof.

The announced observability estimate is proved in the following result :

Proposition 5 *For every $\varphi^0 \in L^2(\Omega)$, the associated solution to* (1.6) *satisfies the observability inequality*

$$\iint_{\Omega\times(T/4,3T/4)} |\varphi|^2\,dx\,dt \leq K\iint_{\omega\times(0,T)} |\varphi|^2\,dx\,dt. \tag{1.34}$$

for a constant K of the form

$$K = \exp\{C(1+\frac{1}{T} + \|a\|^{2/3}_\infty + \|B\|^2_\infty + \|\beta\|^2_\infty)\}. \tag{1.35}$$

Proof : Let $\varphi^0 \in L^2(\Omega)$ be given. Notice that the corresponding φ solves (1.4) with

$$f_1 = -a\,\varphi \in L^2(Q), \quad f_2 = \varphi B \in L^2(0,T;L^2(\Omega)^N), \quad f_3 = -\beta\,\varphi \in L^2(\Sigma).$$

Thus, we can apply theorem 1 to φ and deduce that

$$\begin{aligned}\iint_Q e^{-2s\alpha}(s\,\lambda^2\,\xi\,|\nabla\varphi|^2 + s^3\,\lambda^4\,\xi^3\,|\varphi|^2)dx\,dt + s^2\lambda^3\iint_\Sigma e^{-2s\alpha}\,\xi^2\,|\varphi|^2\,d\sigma\,dt \\ \leq C(\Omega,\omega)\left(\|a\|^2_\infty\iint_Q e^{-2s\alpha}\,|\varphi|^2\,dx\,dt + s^2\lambda^2\|B\|^2_\infty\iint_Q e^{-2s\alpha}\,\xi^2\,|\varphi|^2\,dx\,dt\right. \\ \left. + s\,\lambda\,\|\beta\|^2_\infty\iint_\Sigma e^{-2s\alpha}\,\xi\,|\varphi|^2\,d\sigma\,dt + s^3\lambda^4\iint_{\omega\times(0,T)} e^{-2s\alpha}\,\xi^3\,|\varphi|^2\,dx\,dt\right)\end{aligned}$$

for any $\lambda \geq \overline{\lambda}$ and any $s \geq \overline{\sigma}(e^{4\lambda\|\eta^0\|_\infty}T + T^2)$.

We will now try to eliminate the global terms in the right hand side of this inequality by making a convenient choice of the parameter s.

Taking $s \geq C\,T^2\,(\|a\|_\infty^{2/3} + \|B\|_\infty^2)$, we see that

$$C\left(s^2\lambda^2\|B\|_\infty^2 \iint_Q e^{-2s\alpha}\,\xi^2\,|\varphi|^2\,dx\,dt + \|a\|_\infty^2 \iint_Q e^{-2s\alpha}\,|\varphi|^2\,dx\,dt\right) \\ \leq \frac{1}{2}\,s^3\,\lambda^4 \iint_Q e^{-2s\alpha}\,\xi^3\,|\varphi|^2\,dx\,dt.$$

On the other hand, taking $s \geq C\,T^2\,\|\beta\|_\infty^2$, we find that

$$C\,s\,\lambda\,\|\beta\|_\infty^2 \iint_\Sigma e^{-2s\alpha}\,\xi\,|\varphi|^2\,d\sigma\,dt \leq \frac{1}{2}\,s^2\,\lambda^3 \iint_\Sigma e^{-2s\alpha}\,\xi^2\,|\varphi|^2\,d\sigma\,dt.$$

All this leads to the estimate

$$\iint_Q e^{-2s\alpha}\,\xi^3\,|\varphi|^2\,dx\,dt \leq C \iint_{\omega\times(0,T)} e^{-2s\alpha}\,\xi^3\,|\varphi|^2\,dx\,dt,$$

which holds for $\lambda \geq \overline{\lambda}$ and $s \geq \overline{\sigma}(e^{4\lambda\|\eta^0\|_\infty}\,T + T^2(1 + \|a\|_\infty^{2/3} + \|B\|_\infty^2 + \|\beta\|_\infty^2))$.

Taking into account the properties of the weight functions as well as the choice of s and λ we have made, it is not difficult to realize that the function

$$t \mapsto \exp\left(-2s\max_{x\in\overline{\Omega}}\alpha(t)\right)\,\min_{x\in\overline{\Omega}}\xi(t)^3$$

reaches its minimum in $(T/4, 3T/4)$ at $t = T/4$ and that the function

$$t \mapsto \exp\left(-2s\min_{x\in\overline{\Omega}}\alpha(t)\right)\,\max_{x\in\overline{\Omega}}\xi(t)^3$$

reaches its maximum in $(0,T)$ at $t = T/2$. With this, the previous Carleman inequality directly gives

$$\iint_{\Omega\times(T/4,3T/4)} |\varphi|^2\,dx\,dt \leq \exp\left\{-2s\left(\min_{x\in\overline{\Omega}}\alpha(x,\frac{T}{2}) - \max_{x\in\overline{\Omega}}\alpha(x,\frac{T}{4})\right)\right\} \\ \times \min_{x\in\overline{\Omega}}\xi(x,\frac{T}{4})^{-3}\,\max_{x\in\overline{\Omega}}\xi(x,\frac{T}{2})^3 \iint_{\omega\times(0,T)} |\varphi|^2\,dx\,dt,$$

for the same choice of the parameters s and λ.

Now, taking $\lambda = \overline{\lambda}$ and $s = \overline{s} = \overline{\sigma}(e^{4\overline{\lambda}\|\eta^0\|_\infty}\,T + T^2(1 + \|a\|_\infty^{2/3} + \|B\|_\infty^2 + \|\beta\|_\infty^2))$, we have

$$\iint_{\Omega\times(T/4,3T/4)} |\varphi|^2\,dx\,dt \leq C(\Omega,\omega)\,e^{C(\Omega,\omega)\,\overline{s}/T^2} \iint_{\omega\times(0,T)} |\varphi|^2\,dx\,dt,$$

which gives (1.35) and (1.34).

This ends the proof of proposition 5.

Let us now finish the proof of theorem 2. We will apply a well known argument that has already been used in several similar situations (see [6] and [12]).

Let us introduce a function $\eta \in C^\infty(0,T)$, with

$$\eta(t) = 1 \ \text{ for } t \in (0, T/4), \quad \eta(t) = 0 \ \text{ for } t \in (3T/4, T)$$

and

$$|\eta'(t)| \le C/T \text{ for } t \in (0,T).$$

Let χ be the weak solution of

$$\begin{cases} \chi_t - \Delta\chi + B(x,t)\cdot\nabla\chi + a(x,t)\,\chi = 0 & \text{in } Q, \\ \dfrac{\partial\chi}{\partial n} + \beta(x,t)\,\chi = 0 & \text{on } \Sigma, \\ \chi(x,0) = y^0(x) & \text{in } \Omega. \end{cases}$$

and let us put $y = w + \eta\chi$. If y is the state associated to v, i.e. the solution to (1.1), then w satisfies

$$\begin{cases} w_t - \Delta w + B(x,t)\cdot\nabla w + a(x,t)\,w = -\eta'(t)\,\chi + v1_\omega & \text{in } Q, \\ \dfrac{\partial w}{\partial n} + \beta(x,t)\,w = 0 & \text{on } \Sigma, \\ w(x,0) = 0 & \text{in } \Omega. \end{cases} \tag{1.36}$$

Our task is to find a control $v \in L^2(\omega\times(0,T))$ such that the associated solution to (1.36) satisfies

$$w(x,T) = 0 \quad \text{in} \quad \Omega. \tag{1.37}$$

After this, just taking $y = w + \eta\chi$ we will have proved our result with a control in $L^2(\omega\times(0,T))$.

For each $\varepsilon > 0$, let us consider the functional J_ε, with

$$\begin{cases} J_\varepsilon(\varphi^0) = \dfrac{1}{2}\displaystyle\iint_{\omega\times(0,T)} |\varphi|^2\,dx\,dt + \varepsilon\|\varphi^0\|_{L^2(\Omega)} - \iint_Q \eta'\,\chi\,\varphi\,dx\,dt \\ \forall\varphi^0 \in L^2(\Omega), \end{cases}$$

where, for each $\varphi^0 \in L^2(\Omega)$, φ is the solution to (1.6) associated to φ^0.

It is clear that

$$\varphi^0 \mapsto J_\varepsilon(\varphi^0)$$

is a continuous, strictly convex and (in view of (1.8)) coercive function on $L^2(\Omega)$. Consequently, it possesses exactly one minimizer φ^0_ε and it is not difficult to check that $\varphi^0_\varepsilon = 0$ if and only if the solution $\tilde{w}$ to (1.36) associated to $v = 0$ satisfies $\|\tilde{w}(\cdot,T)\|_{L^2(\Omega)} \le \varepsilon$.

Let us denote by φ_ε the solution to (1.6) associated to φ^0_ε, let us put

$$v_\varepsilon = \varphi_\varepsilon 1_\omega$$

and let us denote by w_ε the solution to (1.36) associated to the control v_ε. Then

$$\|w_\varepsilon(\cdot,T)\|_{L^2(\Omega)} \le \varepsilon. \tag{1.38}$$

Indeed, it is not restrictive to assume that $\varphi_\varepsilon^0 \neq 0$. Accordingly, J_ε is differentiable at φ_ε^0 and

$$(J_\varepsilon'(\varphi_\varepsilon^0), \varphi^0)_{L^2(\Omega)} = 0 \quad \forall \varphi^0 \in L^2(\Omega).$$

That is to say,

$$\begin{cases} \displaystyle\iint_{\omega\times(0,T)} \varphi_\varepsilon \, \varphi \, dx\, dt + (\varepsilon \frac{\varphi_\varepsilon^0}{\|\varphi_\varepsilon^0\|_{L^2}}, \varphi^0)_{L^2(\Omega)} - \iint_Q \eta' \, \chi \, \varphi \, dx \, dt = 0 \\ \forall \varphi^0 \in L^2(\Omega). \end{cases}$$

Since

$$\iint_{\omega\times(0,T)} \varphi_\varepsilon \, \varphi \, dx\, dt - \iint_Q \eta' \, \chi \, \varphi \, dx \, dt = (w_\varepsilon(\cdot, T), \varphi^0)_{L^2(\Omega)} \, ,$$

we have

$$(w_\varepsilon(\cdot, T), \varphi^0)_{L^2(\Omega)} = -(\varepsilon \frac{\varphi_\varepsilon^0}{\|\varphi_\varepsilon^0\|_{L^2(\Omega)}}, \varphi^0)_{L^2(\Omega)} \quad \forall \varphi^0 \in L^2(\Omega),$$

which implies (1.38).

Since $J_\varepsilon(\varphi_\varepsilon^0) \leq J_\varepsilon(0) = 0$, we also have

$$\begin{aligned} &\|v_\varepsilon\|^2_{L^2(\omega\times(0,T))} \\ &\leq \left(\iint_{\Omega\times(T/4,3T/4)} |\varphi_\varepsilon|^2 \, dx \, dt \right)^{1/2} \left(\iint_{\Omega\times(T/4,3T/4)} |\eta'\chi|^2 \, dx \, dt \right)^{1/2} . \end{aligned}$$

From proposition 5 and the definition of v_ε, we deduce now that

$$\|v_\varepsilon\|^2_{L^2(\omega\times(0,T))} \leq \frac{C}{T} K^{1/2} \, \|v_\varepsilon\|_{L^2(Q)} \left(\iint_{\Omega\times(T/4,3T/4)} |\chi|^2 \, dx \, dt \right)^{1/2}$$

and, using proposition 4, we have

$$\|v_\varepsilon\|_{L^2(\omega\times(0,T))} \leq C \, K^{1/2} \, \|\chi\|_Y \leq H \, \|y^0\|_{L^2(\Omega)} \, , \tag{1.39}$$

where the constant H is as in (1.11).

Consequently, $v_\varepsilon 1_\omega$ and w_ε are uniformly bounded in the spaces $L^2(\omega \times (0,T))$ and

$$Z = \{ w \in L^2(0,T; H^1(\Omega)) : w_t \in L^2(0,T; H^{-1}(\Omega)) \},$$

respectively. Obviously, we can extract sequences converging weakly to a control $v1_\omega$ and the associated solution w of (1.36), with

$$w(x,T) = 0 \quad \text{in} \quad \Omega.$$

We have thus proved the existence of a control $v \in L^2(Q)$ such that (1.10) and (1.37) are fulfilled.

This ends the proof of theorem 2.

Appendix : Proof of lemma 1

We divide the proof in three steps :
1 - First, we set $\psi = e^{-s\alpha}\, q$ and we prove the following inequality :

$$\begin{aligned}
&\iint_Q (s^{-1}\,\xi^{-1}(|\psi_t|^2 + |\Delta\psi|^2) + s\,\lambda^2\,\xi\,|\nabla\psi|^2 + s^3\,\lambda^4\,\xi^3\,|\psi|^2)\,dx\,dt \\
&-2s^3\,\lambda^3 \iint_\Sigma |\nabla\eta^0|^2\,\xi^3\,\frac{\partial\eta^0}{\partial n}\,|\psi|^2\,d\sigma\,dt - 4s\,\lambda^2 \iint_\Sigma |\nabla\eta^0|^2\,\xi\,\frac{\partial\psi}{\partial n}\,\psi\,d\sigma\,dt \\
&-4s\,\lambda \iint_\Sigma \frac{\partial\eta^0}{\partial n}\,\xi\,\left|\frac{\partial\psi}{\partial n}\right|^2 d\sigma\,dt + 2s\,\lambda \iint_\Sigma \frac{\partial\eta^0}{\partial n}\,\xi\,|\nabla\psi|^2\,d\sigma\,dt \\
&+2\iint_\Sigma \frac{\partial\psi}{\partial n}\,\psi_t\,d\sigma\,dt - 2s^2\,\lambda \iint_\Sigma \alpha_t\,\frac{\partial\eta^0}{\partial n}\,\xi\,|\psi|^2\,d\sigma\,dt \\
&\le C\left(\iint_Q e^{-2s\alpha}\,|f|^2\,dx\,dt + s^3\,\lambda^4 \iint_{\omega\times(0,T)} \xi^3\,|\psi|^2\,dx\,dt\right)
\end{aligned} \tag{1.40}$$

for $\lambda \ge C(\Omega,\omega)$ and $s \ge C(\Omega,\omega)(T\,e^{2\lambda\|\eta^0\|_\infty} + T^2)$.

2 - Then, we set $\tilde\psi = e^{-s\tilde\alpha}\, q$ and we prove that

$$\begin{aligned}
&\iint_Q (s^{-1}\,\tilde\xi^{-1}(|\tilde\psi_t|^2 + |\Delta\tilde\psi|^2) + s\,\lambda^2\,\tilde\xi\,|\nabla\tilde\psi|^2 + s^3\,\lambda^4\,\tilde\xi^{\,3}\,|\tilde\psi|^2)dx\,dt \\
&+2s^3\,\lambda^3 \iint_\Sigma |\nabla\eta^0|^2\,\tilde\xi^{\,3}\,\frac{\partial\eta^0}{\partial n}\,|\tilde\psi|^2\,d\sigma\,dt - 4s\,\lambda^2 \iint_\Sigma |\nabla\eta^0|^2\,\tilde\xi\,\frac{\partial\tilde\psi}{\partial n}\,\tilde\psi\,d\sigma\,dt \\
&+4s\,\lambda \iint_\Sigma \frac{\partial\eta^0}{\partial n}\,\tilde\xi\,\left|\frac{\partial\tilde\psi}{\partial n}\right|^2 d\sigma\,dt - 2s\,\lambda \iint_\Sigma \frac{\partial\eta^0}{\partial n}\,\tilde\xi\,|\nabla\tilde\psi|^2\,d\sigma\,dt \\
&+2\iint_\Sigma \frac{\partial\tilde\psi}{\partial n}\,\tilde\psi_t\,d\sigma\,dt + 2s^2\,\lambda \iint_\Sigma \tilde\alpha_t\,\frac{\partial\eta^0}{\partial n}\,\tilde\xi\,|\tilde\psi|^2\,d\sigma\,dt \\
&\le C\left(\iint_Q e^{-2s\tilde\alpha}\,|f|^2\,dx\,dt + s^3\,\lambda^4 \iint_{\omega\times(0,T)} \tilde\xi^{\,3}\,|\tilde\psi|^2\,dx\,dt\right)
\end{aligned} \tag{1.41}$$

for any $\lambda \ge C(\Omega,\omega)$ and $s \ge C(\Omega,\omega)(e^{4\lambda\|\eta^0\|_\infty}\,T + T^2)$. Here, $\tilde\xi$ and $\tilde\alpha$ stand for the functions

$$\tilde\xi(x,t) = \frac{e^{-\lambda\eta^0(x)}}{t(T-t)}\ , \quad \tilde\alpha(x,t) = \frac{e^{2\lambda\|\eta^0\|_\infty} - e^{-\lambda\eta^0(x)}}{t(T-t)}\ .$$

3 - Finally, we add the previous two inequalities and we come back to the original variable φ. This will give the desired inequality (1.15).

STEP 1 : Let us put $\psi = e^{-s\alpha}\, q$. Since $-q_t - \Delta q = f$, we also have

$$M_1\psi + M_2\psi = F, \tag{1.42}$$

where

$$M_1\psi = 2s\,\lambda^2\,\xi\,|\nabla\eta^0|^2\,\psi + 2s\,\lambda\,\xi\,\nabla\eta^0\cdot\nabla\psi - \psi_t\,,$$
$$M_2\psi = -s^2\,\lambda^2\,\xi^2\,|\nabla\eta^0|^2\,\psi - \Delta\psi - s\,\alpha_t\,\psi,$$
$$F = e^{-s\alpha}\,f - s\,\lambda\,\xi\,\Delta\eta^0\,\psi + s\,\lambda^2\,\xi\,|\nabla\eta^0|^2\,\psi.$$

From (1.42), we have that

$$\|M_1\psi\|^2_{L^2(Q)} + \|M_2\psi\|^2_{L^2(Q)} + 2(M_1\psi, M_2\psi)_{L^2(Q)} = \|F\|^2_{L^2(Q)}\,. \tag{1.43}$$

The main idea is to expand the term $2(M_1\psi, M_2\psi)_{L^2(Q)}$ and use the particular structure of α and the fact that s is large enough in order to obtain large positive terms in this scalar product. Denoting by $(M_i\psi)_j$ $(1 \le i \le 2,\ 1 \le j \le 3)$ the j-th term in the above expression of $M_i\psi$, we find that

$$(M_1\psi, M_2\psi)_{L^2(Q)} = \sum_{1\le i,j\le 3} ((M_1\psi)_i, (M_2\psi)_j)_{L^2(Q)}\,.$$

Let us compute each of these terms.

First, we have

$$((M_1\psi)_1, (M_2\psi)_1)_{L^2(Q)} = -2s^3\,\lambda^4 \iint_Q |\nabla\eta^0|^4\,\xi^3\,|\psi|^2\,dx\,dt = A.$$

Then,

$$((M_1\psi)_2, (M_2\psi)_1)_{L^2(Q)} = -2s^3\,\lambda^3 \iint_Q |\nabla\eta^0|^2\,\xi^3\,(\nabla\eta^0\cdot\nabla\psi)\,\psi\,dx\,dt$$
$$= 3s^3\,\lambda^4 \iint_Q |\nabla\eta^0|^4\,\xi^3\,|\psi|^2\,dx\,dt + s^3\,\lambda^3 \iint_Q \Delta\eta^0\,|\nabla\eta^0|^2\,\xi^3\,|\psi|^2\,dx\,dt$$
$$+2s^3\,\lambda^3 \iint_Q \partial_i\eta^0\,\partial_{ij}\eta^0\,\partial_j\eta^0\,\xi^3\,|\psi|^2\,dx\,dt$$
$$-s^3\,\lambda^3 \iint_\Sigma |\nabla\eta^0|^2\,\xi^3\,\frac{\partial\eta^0}{\partial n}\,|\psi|^2\,d\sigma\,dt = B_1 + B_2 + B_3 + B_4\,.$$

We clearly have that $A + B_1$ is a positive term. As a consequence of the properties of η^0 (see (1.14)), we have

$$s^3\,\lambda^4 \iint_Q |\nabla\eta^0|^4\,\xi^3\,|\psi|^2\,dx\,dt \ge C\,s^3\,\lambda^4 \iint_Q \xi^3\,|\psi|^2\,dx\,dt$$
$$-C\,s^3\,\lambda^4 \iint_{\omega'\times(0,T)} \xi^3\,|\psi|^2\,dx\,dt$$

for some $C = C(\Omega, \omega)$. The first of these last two integrals will stay in the left hand side and the second one will go to the right.

The boundary term B_4 will also stay in the left hand side, while B_2 and B_3 will be absorbed by simply taking $\lambda \geq C(\Omega, \omega)$.

We also have

$$((M_1\psi)_3, (M_2\psi)_1)_{L^2(Q)} = s^2\,\lambda^2 \iint_Q |\nabla\eta^0|^2\,\xi^2\,\psi_t\,\psi\,dx\,dt$$

$$= -s^2\,\lambda^2 \iint_Q |\nabla\eta^0|^2\,\xi\,\xi_t\,|\psi|^2\,dx\,dt \leq C\,s^2\,\lambda^2\,T \iint_Q \xi^3\,|\psi|^2\,dx\,dt,$$

which is also absorbed by taking $\lambda \geq 1$ and $s \geq C(\Omega, \omega)\,T$.

Consequently, we obtain

$$\begin{gathered}(M_1\psi, (M_2\psi)_1)_{L^2(Q)} = ((M_1\psi)_1 + (M_1\psi)_2 + (M_1\psi)_3, (M_2\psi)_1)_{L^2(Q)} \\ \geq C\,s^3\,\lambda^4 \iint_Q \xi^3\,|\psi|^2\,dx\,dt - s^3\,\lambda^3 \iint_\Sigma |\nabla\eta^0|^2\,\xi^3\,\frac{\partial\eta^0}{\partial n}\,|\psi|^2\,d\sigma\,dt \\ -C\,s^3\,\lambda^4 \iint_{\omega'\times(0,T)} \xi^3\,|\psi|^2\,dx\,dt,\end{gathered} \tag{1.44}$$

for any $\lambda \geq C(\Omega, \omega)$ and $s \geq C(\Omega, \omega)\,T$.

On the other hand, we have

$$\begin{aligned}&((M_1\psi)_1, (M_2\psi)_2)_{L^2(Q)} = -2s\,\lambda^2 \iint_Q |\nabla\eta^0|^2\,\xi\,\Delta\psi\,\psi\,dx\,dt \\ &= -2s\,\lambda^2 \iint_\Sigma |\nabla\eta^0|^2\,\xi\,\frac{\partial\psi}{\partial n}\,\psi\,d\sigma\,dt + 2s\,\lambda^2 \iint_Q |\nabla\eta^0|^2\,\xi\,|\nabla\psi|^2\,dx\,dt \\ &+4s\,\lambda^2 \iint_Q \partial_i\eta^0\,\partial_{ij}\eta^0\,\xi\,\partial_j\psi\,\psi\,dx\,dt + s\,\lambda^3 \iint_Q |\nabla\eta^0|^2\,\xi\,\nabla\eta^0\cdot\nabla|\psi|^2\,dx\,dt \\ &= C_1 + C_2 + C_3 + C_4\,.\end{aligned}$$

We will keep C_1 and C_2 in the left hand side. For C_3 and C_4 , we have

$$C_3 \leq C\,s\,\lambda^4 \iint_Q \xi\,|\psi|^2\,dx\,dt + C\,s \iint_Q \xi\,|\nabla\psi|^2\,dx\,dt$$

and

$$C_4 \leq C\,s^2\,\lambda^4 \iint_Q \xi^2\,|\psi|^2\,dx\,dt + C\,\lambda^2 \iint_Q |\nabla\psi|^2\,dx\,dt.$$

Therefore, by taking $s \geq C\,T^2$, we find that

$$\begin{aligned}&C_1 + C_2 + C_3 + C_4 \geq -2s\,\lambda^2 \iint_\Sigma |\nabla\eta^0|^2\,\xi\,\frac{\partial\psi}{\partial n}\,\psi\,d\sigma\,dt \\ &+2s\,\lambda^2 \iint_Q |\nabla\eta^0|^2\,\xi\,|\nabla\psi|^2\,dx\,dt - C\,s^2\,\lambda^4 \iint_Q \xi^2\,|\psi|^2\,dx\,dt \\ &-C \iint_Q (s\,\xi + \lambda^2)\,|\nabla\psi|^2\,dx\,dt.\end{aligned} \tag{1.45}$$

We also have

$$
\begin{aligned}
((M_1\psi)_2,(M_2\psi)_2)_{L^2(Q)} &= -2s\,\lambda \iint_Q \xi\,(\nabla\eta^0\cdot\nabla\psi)\,\Delta\psi\,dx\,dt\\
&= -2s\,\lambda \iint_\Sigma \frac{\partial\eta^0}{\partial n}\,\xi\,\left|\frac{\partial\psi}{\partial n}\right|^2 d\sigma\,dt + 2s\,\lambda\iint_Q \partial_{ij}\eta^0\,\xi\,\partial_i\psi\,\partial_j\psi\,dx\,dt\\
&\quad +2s\,\lambda^2\iint_Q \xi\,|\nabla\eta^0\cdot\nabla\psi|^2\,dx\,dt + s\,\lambda\iint_Q \xi\,\nabla\eta^0\cdot\nabla|\nabla\psi|^2\,dx\,dt\\
&= D_1+D_2+D_3+D_4\,.
\end{aligned}
$$

As before, we will keep the boundary integral D_1 in the left hand side. Also,

$$
D_2 \le C\,s\,\lambda\iint_Q \xi\,|\nabla\psi|^2\,dx\,dt.
$$

Moreover, $D_3 \ge 0$. After some additional computations, we also see that

$$
\begin{aligned}
D_4 = s\,\lambda\iint_Q \xi\,\nabla\eta^0\cdot\nabla|\nabla\psi|^2\,dx\,dt &= s\,\lambda\iint_\Sigma \xi\,\frac{\partial\eta^0}{\partial n}\,|\nabla\psi|^2\,d\sigma\,dt\\
-s\,\lambda^2\iint_Q |\nabla\eta^0|^2\,\xi\,|\nabla\psi|^2\,dx\,dt &- s\,\lambda\iint_Q \Delta\eta^0\,\xi\,|\nabla\psi|^2\,dx\,dt\\
&= D_{41}+D_{42}+D_{43}\,.
\end{aligned}
$$

Now, we keep once more D_{41} in the left and we notice that D_{43} can be bounded in the same form as D_2 .

Consequently,

$$
\begin{aligned}
D_1+D_2+D_3+D_4 &\ge -2s\,\lambda\iint_\Sigma \frac{\partial\eta^0}{\partial n}\,\xi\,\left|\frac{\partial\psi}{\partial n}\right|^2 d\sigma\,dt\\
&\quad +s\,\lambda\iint_\Sigma \frac{\partial\eta^0}{\partial n}\,\xi\,|\nabla\psi|^2\,d\sigma\,dt - s\,\lambda^2\iint_Q |\nabla\eta^0|^2\,\xi\,|\nabla\psi|^2\,dx\,dt\\
&\quad -C\,s\,\lambda\iint_Q \xi\,|\nabla\psi|^2\,dx\,dt.
\end{aligned} \tag{1.46}
$$

Additionally, we find that

$$
\begin{aligned}
((M_1\psi)_3,(M_2\psi)_2)_{L^2(Q)} &= \iint_Q \psi_t\,\Delta\psi\,dx\,dt\\
&= \iint_\Sigma \frac{\partial\psi}{\partial n}\,\psi_t\,d\sigma\,dt = E,
\end{aligned} \tag{1.47}
$$

which will stay in the left hand side.

From (1.45)-(1.47), we deduce that

$$(M_1\psi,(M_2\psi)_2)_{L^2(Q)} = ((M_1\psi)_1 + (M_1\psi)_2 + (M_1\psi)_3,(M_2\psi)_2)_{L^2(Q)}$$
$$\geq s\,\lambda^2 \iint_Q |\nabla\eta^0|^2\,\xi\,|\nabla\psi|^2\,dx\,dt - 2s\,\lambda^2 \iint_\Sigma |\nabla\eta^0|^2\,\xi\,\frac{\partial\psi}{\partial n}\,\psi\,d\sigma\,dt$$
$$-2s\,\lambda \iint_\Sigma \frac{\partial\eta^0}{\partial n}\,\xi\,\left|\frac{\partial\psi}{\partial n}\right|^2\,d\sigma\,dt + s\,\lambda \iint_\Sigma \frac{\partial\eta^0}{\partial n}\,\xi\,|\nabla\psi|^2\,dx\,dt$$
$$+ \iint_\Sigma \frac{\partial\psi}{\partial n}\,\psi_t\,d\sigma\,dt - C\,s^2\,\lambda^4 \iint_Q \xi^2\,|\psi|^2\,dx\,dt$$
$$-C \iint_Q (s\,\lambda\,\xi + \lambda^2)\,|\nabla\psi|^2\,dx\,dt$$

for any $\lambda \geq 1$. Hence, we have the following for any $\lambda \geq C(\Omega,\omega)$ and any $s \geq C(\Omega,\omega)\,T^2$:

$$\begin{aligned}
&(M_1\psi,(M_2\psi)_2)_{L^2(Q)} \geq C\,s\,\lambda^2 \iint_Q \xi\,|\nabla\psi|^2\,dx\,dt \\
&-2s\,\lambda^2 \iint_\Sigma |\nabla\eta^0|^2\,\xi\,\frac{\partial\psi}{\partial n}\,\psi\,d\sigma\,dt - 2s\,\lambda \iint_\Sigma \frac{\partial\eta^0}{\partial n}\,\xi\,\left|\frac{\partial\psi}{\partial n}\right|^2\,d\sigma\,dt \\
&+s\,\lambda \iint_\Sigma \frac{\partial\eta^0}{\partial n}\,\xi\,|\nabla\psi|^2\,dx\,dt + \iint_\Sigma \frac{\partial\psi}{\partial n}\,\psi_t\,d\sigma\,dt \\
&-C\,s^2\,\lambda^4 \iint_Q \xi^2\,|\psi|^2\,dx\,dt - C\,s\,\lambda^2 \iint_{\omega'\times(0,T)} \xi\,|\nabla\psi|^2\,dx\,dt.
\end{aligned} \tag{1.48}$$

Let us now consider the scalar product

$$\begin{aligned}
((M_1\psi)_1,(M_2\psi)_3)_{L^2(Q)} &= -2s^2\,\lambda^2 \iint_Q |\nabla\eta^0|^2\,\alpha_t\,\xi\,|\psi|^2\,dx\,dt \\
&\leq C(\Omega,\omega)\,e^{2\lambda\|\eta^0\|_\infty}\,s^2\,\lambda^2\,T \iint_Q \xi^3\,|\psi|^2\,dx\,dt,
\end{aligned} \tag{1.49}$$

Obviously, this will be absorbed by the term in $s^3\,\lambda^4$ in (1.44) if we take $\lambda \geq 1$ and $s \geq C(\Omega,\omega)\,e^{2\lambda\|\eta^0\|_\infty}\,T$.

Furthermore,

$$((M_1\psi)_2,(M_2\psi)_3)_{L^2(Q)} = -2s^2\,\lambda \iint_Q \alpha_t\,\xi\,(\nabla\eta^0\cdot\nabla\psi)\,\psi\,dx\,dt$$
$$= -s^2\,\lambda \iint_\Sigma \alpha_t\,\frac{\partial\eta^0}{\partial n}\,\xi\,|\psi|^2\,d\sigma\,dt + s^2\,\lambda^2 \iint_Q \alpha_t\,|\nabla\eta^0|^2\,\xi\,|\psi|^2\,dx\,dt$$
$$+ s^2\,\lambda \iint_Q \nabla\alpha_t\cdot\nabla\eta^0\,\xi\,|\psi|^2\,dx\,dt + s^2\,\lambda \iint_Q \alpha_t\,\Delta\eta^0\,\xi\,|\psi|^2\,dx\,dt.$$

With $\lambda \geq 1$, the last three terms in the left hand side can be bounded by

$$C(\Omega,\omega)\,e^{2\lambda\|\eta^0\|_\infty}\,s^2\,\lambda^2\,T \iint_Q \xi^3\,|\psi|^2\,dx\,dt.$$

Thus, we have

$$\begin{aligned}((M_1\psi)_2,(M_2\psi)_3)_{L^2(Q)} \geq & -s^2\,\lambda \iint_\Sigma \alpha_t\,\frac{\partial \eta^0}{\partial n}\,\xi\,|\psi|^2\,dx\,dt \\ & -C\,e^{2\lambda\|\eta^0\|_\infty}\,s^2\,\lambda^2\,T \iint_Q \xi^3\,|\psi|^2\,dx\,dt\end{aligned} \tag{1.50}$$

Finally, we have

$$\begin{aligned}((M_1\psi)_3,(M_2\psi)_3)_{L^2(Q)} &= s \iint_Q \alpha_t\,\psi_t\,\psi\,dx\,dt \\ &\leq C\,e^{2\lambda\|\eta^0\|_\infty}\,s\,T^2 \iint_Q \xi^3\,|\psi|^2\,dx\,dt,\end{aligned} \tag{1.51}$$

since

$$\alpha_{tt} \leq C\,e^{2\lambda\|\eta^0\|_\infty}\,\xi^2(1+T^2\,\xi) \leq C\,e^{2\lambda\|\eta^0\|_\infty}\,T^2\,\xi^3.$$

From (1.49)-(1.51), we deduce that, for $\lambda \geq C(\Omega,\omega)$ and $s \geq C(\Omega,\omega)\,e^{2\lambda\|\eta^0\|_\infty}\,T$, one has

$$\begin{aligned}(M_1\psi,(M_2\psi)_3)_{L^2(Q)} &= ((M_1\psi)_1+(M_1\psi)_2+(M_1\psi)_3,(M_2\psi)_3)_{L^2(Q)} \\ &\geq G - C\,s^3\,\lambda^2 \iint_Q \xi^3\,|\psi|^2\,dx\,dt,\end{aligned} \tag{1.52}$$

where

$$G = -s^2\,\lambda \iint_\Sigma \alpha_t\,\frac{\partial \eta^0}{\partial n}\,\xi\,|\psi|^2\,d\sigma\,dt.$$

Taking into account (1.44), (1.48) and (1.52), we obtain

$$\begin{aligned}&(M_1\psi,M_2\psi)_{L^2(Q)} \geq C \iint_Q (s\,\lambda^2\,\xi\,|\nabla\psi|^2 + s^3\,\lambda^4\,\xi^3\,|\psi|^2)\,dx\,dt \\ &-s^3\,\lambda^3 \iint_\Sigma |\nabla\eta^0|^2\,\xi^3\,\frac{\partial \eta^0}{\partial n}\,|\psi|^2\,d\sigma\,dt - 2s\,\lambda^2 \iint_\Sigma |\nabla\eta^0|^2\,\xi\,\frac{\partial \psi}{\partial n}\,\psi\,d\sigma\,dt \\ &-2s\,\lambda \iint_\Sigma \frac{\partial \eta^0}{\partial n}\,\xi\,\left|\frac{\partial \psi}{\partial n}\right|^2\,d\sigma\,dt + s\,\lambda \iint_\Sigma \frac{\partial \eta^0}{\partial n}\,\xi\,|\nabla\psi|^2\,dx\,dt \\ &+\iint_\Sigma \frac{\partial \psi}{\partial n}\,\psi_t\,d\sigma\,dt - s^2\,\lambda \iint_\Sigma \alpha_t\,\frac{\partial \eta^0}{\partial n}\,\xi\,|\psi|^2\,d\sigma\,dt \\ &-C \iint_{\omega'\times(0,T)} (s\,\lambda^2\,\xi\,|\nabla\psi|^2 + s^3\,\lambda^4\,\xi^3\,|\psi|^2)\,dx\,dt\end{aligned}$$

for any $\lambda \geq C(\Omega,\omega)$ and $s \geq C(\Omega,\omega)(e^{2\lambda\|\eta^0\|_\infty} T + T^2)$. Using (1.43), this gives

$$
\begin{aligned}
&\|M_1\psi\|^2_{L^2(Q)} + \|M_2\psi\|^2_{L^2(Q)} + \iint_Q (s\,\lambda^2\,\xi\,|\nabla\psi|^2 + s^3\,\lambda^4\,\xi^3\,|\psi|^2)\,dx\,dt \\
&+2(B_4 + C_1 + D_1 + D_{41} + E + G) \leq C\Big(\|F\|^2_{L^2(Q)} \\
&+s\,\lambda^2 \iint_{\omega'\times(0,T)} \xi\,|\nabla\psi|^2\,dx\,dt + s^3\,\lambda^4 \iint_{\omega'\times(0,T)} \xi^3\,|\psi|^2)\,dx\,dt\Big) \\
&\leq C\Big(\iint_Q e^{-2s\alpha}\,|f|^2\,dx\,dt + s^2\,\lambda^4 \iint_Q \xi^2\,|\psi|^2\,dx\,dt \\
&+s\,\lambda^2 \iint_{\omega'\times(0,T)} \xi\,|\nabla\psi|^2\,dx\,dt + s^3\,\lambda^4 \iint_{\omega'\times(0,T)} \xi^3\,|\psi|^2)\,dx\,dt\Big).
\end{aligned}
$$

Thus, we also have

$$
\begin{aligned}
&\|M_1\psi\|^2_{L^2(Q)} + \|M_2\psi\|^2_{L^2(Q)} + \iint_Q (s\,\lambda^2\,\xi\,|\nabla\psi|^2 + s^3\,\lambda^4\,\xi^3\,|\psi|^2)\,dx\,dt \\
&\quad +2(B_4 + C_1 + D_1 + D_{41} + E + G) \leq C\Big(\iint_Q e^{-2s\alpha}\,|f|^2\,dx\,dt \\
&\quad +s\,\lambda^2 \iint_{\omega'\times(0,T)} \xi\,|\nabla\psi|^2\,dx\,dt + s^3\,\lambda^4 \iint_{\omega'\times(0,T)} \xi^3\,|\psi|^2)\,dx\,dt\Big)
\end{aligned}
\tag{1.53}
$$

for $\lambda \geq C(\Omega,\omega)$ and $s \geq C(\Omega,\omega)(e^{2\lambda\|\eta^0\|_\infty} T + T^2)$.

The next step is to try to add integrals of $|\Delta\psi|^2$ and $|\psi_t|^2$ to the left hand side of (1.53). This can be made using the expressions of $M_i\psi\,(i = 1, 2)$. Indeed, we have

$$
\begin{aligned}
s^{-1} \iint_Q \xi^{-1}\,|\psi_t|^2\,dx\,dt \leq C\Big(&s\,\lambda^2 \iint_Q \xi\,|\nabla\psi|^2\,dx\,dt \\
&+s\,\lambda^4 \iint_Q \xi\,|\psi|^2\,dx\,dt + \|M_1\psi\|^2_{L^2(Q)}\Big)
\end{aligned}
$$

and

$$
\begin{aligned}
s^{-1} \iint_Q \xi^{-1}\,|\Delta\psi|^2\,dx\,dt \leq C\Big(&s^3\,\lambda^4 \iint_Q \xi^3\,|\psi|^2\,dx\,dt \\
&+s\,T^2\,e^{4\lambda\|\eta^0\|_\infty} \iint_Q \xi^3\,|\psi|^2\,dx\,dt + \|M_2\psi\|^2_{L^2(Q)}\Big)
\end{aligned}
$$

for $s \geq C\,T^2$. Accordingly, we deduce from (1.53) that

$$\iint_Q \left((s\xi)^{-1}(|\psi_t|^2 + |\Delta\psi|^2) + s\,\lambda^2\,\xi\,|\nabla\psi|^2 + s^3\,\lambda^4\,\xi^3\,|\psi|^2\right)\,dx\,dt$$
$$+2(B_4 + C_1 + D_1 + D_{41} + E + G) \leq C\left(\iint_Q e^{-2s\alpha}\,|f|^2\,dx\,dt\right. \tag{1.54}$$
$$\left.+s\,\lambda^2 \iint_{\omega'\times(0,T)} \xi\,|\nabla\psi|^2\,dx\,dt + s^3\,\lambda^4 \iint_{\omega'\times(0,T)} \xi^3\,|\psi|^2)\,dx\,dt\right)$$

for $\lambda \geq C(\Omega,\omega)$ and $s \geq C(\Omega,\omega)(e^{2\lambda\|\eta^0\|_\infty}\,T + T^2)$.

We are now ready to eliminate the second integral in the right hand side. To this end, let us introduce a function $\theta = \theta(x)$, with

$$\theta \in C^2_c(\omega), \quad \theta \equiv 1 \text{ in } \omega', \quad 0 \leq \theta \leq 1$$

and let us make some computations :

$$s\,\lambda^2 \iint_{\omega'\times(0,T)} \xi\,|\nabla\psi|^2\,dx\,dt \leq\ s\,\lambda^2 \iint_{\omega\times(0,T)} \theta\,\xi\,|\nabla\psi|^2\,dx\,dt$$
$$= -s\,\lambda^2 \iint_{\omega\times(0,T)} \theta\,\xi\,\Delta\psi\,\psi\,dx\,dt - s\,\lambda^2 \iint_{\omega\times(0,T)} \xi\,(\nabla\theta\cdot\nabla\psi)\,\psi\,dx\,dt$$
$$-s\,\lambda^3 \iint_{\omega\times(0,T)} \theta\,\xi\,(\nabla\eta^0\cdot\nabla\psi)\,\psi\,dx\,dt \leq\ \varepsilon\,s^{-1} \iint_{\omega\times(0,T)} \xi^{-1}\,|\Delta\psi|^2\,dx\,dt$$
$$+C\left(s^3\,\lambda^4 \iint_{\omega\times(0,T)} \xi^3\,|\psi|^2\,dx\,dt + s\,\lambda^4 \iint_{\omega\times(0,T)} \xi\,|\psi|^2\,dx\,dt\right),$$

where we have used that $\lambda \geq 1$. In view of this estimate, we deduce that the integral on $|\nabla\psi|^2$ of the right hand side of (1.54) can be suppressed if the last integral is performed in the slightly greater set $\omega \times (0,T)$. From (1.54) and this remark, we deduce (1.40).

Step 2 : The proof of (1.41) is very similar to the proof of (1.40). We will only sketch the main points.

We start from the identity

$$M_1\tilde\psi + M_2\tilde\psi = \tilde F,$$

where

$$\tilde M_1\tilde\psi = 2s\,\lambda^2\,|\nabla\eta^0|^2\,\tilde\xi\,\tilde\psi - 2s\,\lambda\,\tilde\xi\,\nabla\eta^0\cdot\nabla\tilde\psi - \tilde\psi_t\,,$$
$$\tilde M_2\tilde\psi = -s^2\,\lambda^2\,|\nabla\eta^0|^2\,\tilde\xi^{\,2}\,\tilde\psi - \Delta\tilde\psi - s\,\tilde\alpha_t\,\tilde\psi,$$
$$\tilde F = e^{-s\tilde\alpha}\,f + s\,\lambda\,\tilde\xi\,\Delta\eta^0\,\tilde\psi + s\,\lambda^2\,|\nabla\eta^0|^2\,\tilde\xi\,\tilde\psi.$$

We then have

$$\|\tilde M_1\tilde\psi\|^2_{L^2(Q)} + \|\tilde M_2\tilde\psi\|^2_{L^2(Q)} + 2(\tilde M_1\tilde\psi, \tilde M_2\tilde\psi)_{L^2(Q)} = \|\tilde F\|^2_{L^2(Q)}\,. \tag{1.55}$$

After a lengthy computation, we find that

$$(\tilde{M}_1\psi, \tilde{M}_2\psi)_{L^2(Q)} \geq C \iint_Q (s\,\lambda^2\,\tilde{\xi}\,|\nabla\tilde{\psi}|^2 + s^3\,\lambda^4\,\tilde{\xi}^3\,|\tilde{\psi}|^2)\,dx\,dt$$
$$+2(\tilde{B}_4 + \tilde{C}_1 + \tilde{D}_1 + \tilde{D}_{41} + \tilde{E} + \tilde{G})$$
$$-C \iint_{\omega'\times(0,T)} (s\,\lambda^2\,\tilde{\xi}\,|\nabla\tilde{\psi}|^2 + s^3\,\lambda^4\,\tilde{\xi}^3\,|\tilde{\psi}|^2)\,dx\,dt$$

for any $\lambda \geq C(\Omega,\omega)$ and $s \geq C(\Omega,\omega)(e^{2\lambda\|\eta^0\|_\infty}\,T + T^2)$, where

$$\tilde{B}_4 = s^3\,\lambda^3 \iint_\Sigma |\nabla\eta^0|^2\,\tilde{\xi}^3\,\frac{\partial\eta^0}{\partial n}\,|\tilde{\psi}|^2\,d\sigma\,dt,$$

$$\tilde{C}_1 = -2s\,\lambda^2 \iint_\Sigma |\nabla\eta^0|^2\,\tilde{\xi}\,\frac{\partial\tilde{\psi}}{\partial n}\,\tilde{\psi}\,d\sigma\,dt,$$

$$\tilde{D}_1 = 2s\,\lambda \iint_\Sigma \frac{\partial\eta^0}{\partial n}\,\tilde{\xi}\,\left|\frac{\partial\tilde{\psi}}{\partial n}\right|^2 d\sigma\,dt, \quad \tilde{D}_{41} = -s\,\lambda \iint_\Sigma \frac{\partial\eta^0}{\partial n}\,\tilde{\xi}\,|\nabla\tilde{\psi}|^2\,dx\,dt,$$

$$\tilde{E} = \iint_\Sigma \frac{\partial\tilde{\psi}}{\partial n}\,\tilde{\psi}_t\,d\sigma\,dt, \quad \tilde{G} = s^2\,\lambda \iint_\Sigma \tilde{\alpha}_t\,\frac{\partial\eta^0}{\partial n}\,\tilde{\xi}\,|\tilde{\psi}|^2\,d\sigma\,dt.$$

This, together with (1.55), gives

$$\begin{aligned}&\|M_1\tilde{\psi}\|^2_{L^2(Q)} + \|M_2\tilde{\psi}\|^2_{L^2(Q)} + \iint_Q (s\,\lambda^2\,\tilde{\xi}\,|\nabla\tilde{\psi}|^2 + s^3\,\lambda^4\,\tilde{\xi}^3\,|\tilde{\psi}|^2)\,dx\,dt\\ &+2(\tilde{B}_4 + \tilde{C}_1 + \tilde{D}_1 + \tilde{D}_{41} + \tilde{E} + \tilde{G}) \leq C\Big(\|\tilde{F}\|^2_{L^2(Q)}\\ &+s\,\lambda^2 \iint_{\omega'\times(0,T)} \tilde{\xi}\,|\nabla\tilde{\psi}|^2\,dx\,dt + s^3\,\lambda^4 \iint_{\omega'\times(0,T)} \tilde{\xi}^3\,|\tilde{\psi}|^2)\,dx\,dt\Big).\end{aligned} \tag{1.56}$$

With similar arguments to those in the first step, we can now assume that, in (1.56), $\|\tilde{F}\|^2_{L^2(Q)}$ is replaced by

$$\iint_Q e^{-2s\tilde{\alpha}}\,|f|^2\,dx\,dt$$

and $\|\tilde{M}_1\tilde{\psi}\|^2_{L^2(Q)} + \|\tilde{M}_2\tilde{\psi}\|^2_{L^2(Q)}$ is replaced by

$$\iint_Q s^{-1}\,\xi^{-1}(|\tilde{\psi}_t|^2 + |\Delta\tilde{\psi}|^2)\,dx\,dt.$$

Finally, for $\lambda \geq C(\Omega,\omega)$ and $s \geq C(\Omega,\omega)\,T^2$ large enough, we can replace the integrals of $|\nabla\tilde{\psi}|^2$ and $|\tilde{\psi}|^2$ in the right hand side by

$$s^3\,\lambda^4 \iint_{\omega\times(0,T)} \tilde{\xi}^3\,|\tilde{\psi}|^2\,dx\,dt.$$

This yields the estimate (1.41).

STEP 3 : Now, let us add the inequalities (1.40) and (1.41) and let us check that all the integrals on Σ can be simplified, so that there will only remain integrals in Q.

Since $\eta^0 = 0$ on $\partial\Omega$, we have

$$\xi = \tilde{\xi}, \quad \alpha = \tilde{\alpha} \quad \text{and} \quad \psi = \tilde{\psi} \quad \text{on } \Sigma. \tag{1.57}$$

Consequently, $B_4 + \tilde{B}_4 = 0$ and $G + \tilde{G} = 0$.

Let us see that

$$\frac{\partial\tilde{\psi}}{\partial n} \equiv -\frac{\partial\psi}{\partial n} \text{ on } \Sigma. \tag{1.58}$$

From the definitions of ψ and $\tilde{\psi}$, we have

$$\partial_i\psi = e^{-s\alpha}\left(\partial_i q + s\,\lambda\,\partial_i\eta^0\,\xi\,q\right),\ \partial_i\tilde{\psi} = e^{-s\tilde{\alpha}}\left(\partial_i q - s\,\lambda\,\partial_i\eta^0\,\tilde{\xi}\,q\right), \tag{1.59}$$

whence

$$\frac{\partial\psi}{\partial n} = s\,\lambda\,\frac{\partial\eta^0}{\partial n}\,\xi\,e^{-s\alpha}\,q, \quad \frac{\partial\tilde{\psi}}{\partial n} = -s\,\lambda\,\frac{\partial\eta^0}{\partial n}\,\tilde{\xi}\,e^{-s\tilde{\alpha}}\,q \quad \text{on} \quad \Sigma$$

and we certainly have (1.58).

We deduce from (1.57) and (1.58) that $C_1 + \tilde{C}_1 = 0$, $D_1 + \tilde{D}_1 = 0$ and $E + \tilde{E} = 0$.

On the other hand, since φ satisfies a zero Neumann condition and $\eta^0 = 0$ on $\partial\Omega$, we also have

$$|\nabla\psi|^2 = |\nabla\tilde{\psi}|^2 \quad \text{on } \Sigma,$$

whence $D_{41} + \tilde{D}_{41} = 0$.

With all this, we obtain

$$\begin{aligned}
&s^{-1}\iint_Q (\xi^{-1}(|\psi_t|^2 + |\Delta\psi|^2) + \tilde{\xi}^{-1}(|\tilde{\psi}_t|^2 + |\Delta\tilde{\psi}|^2))\,dx\,dt \\
&+ s\,\lambda^2\iint_Q (\xi\,|\nabla\psi|^2 + \tilde{\xi}\,|\nabla\tilde{\psi}|^2)\,dx\,dt \\
&+ s^3\,\lambda^4\iint_Q (\xi^3\,|\psi|^2 + \tilde{\xi}^{\,3}\,|\tilde{\psi}|^2)\,dx\,dt \\
&\qquad \le C\left(s^3\,\lambda^4\iint_{\omega\times(0,T)}\left(\xi^3\,|\psi|^2 + \tilde{\xi}^{\,3}\,|\tilde{\psi}|^2\right)dx\,dt\right. \\
&\qquad \left. + \iint_Q \left(e^{-2s\alpha} + e^{-2s\tilde{\alpha}}\right)|f|^2\,dx\,dt\right),
\end{aligned} \tag{1.60}$$

for $\lambda \ge C(\Omega,\omega)$ and $s \ge C(\Omega,\omega)(e^{4\lambda\|\eta^0\|_\infty}\,T + T^2)$.

From the definitions of ξ, $\tilde{\xi}$, α and $\tilde{\alpha}$, we have

$$\tilde{\xi} \le \xi, \quad e^{-2s\tilde{\alpha}} \le e^{-2s\alpha} \quad \text{in } Q,$$

so (1.60) yields

$$\iint_Q \left((s\xi)^{-1}(|\psi_t|^2 + |\Delta\psi|^2) + s\,\lambda^2\,\xi\,|\nabla\psi|^2 + s^3\,\lambda^4\,\xi^3\,|\psi|^2\right)\,dx\,dt$$
$$\leq C\left(\iint_Q e^{-2s\alpha}\,|f|^2\,dx\,dt + s^3\,\lambda^4\iint_{\omega\times(0,T)}\xi^3\,|\psi|^2\,dx\,dt\right), \tag{1.61}$$

for any $\lambda \geq C(\Omega,\omega)$ and $s \geq C(\Omega,\omega)(e^{4\lambda\|\eta^0\|_\infty}\,T + T^2)$.

We finally turn back to φ. For the moment, we have

$$s^{-1}\iint_Q \xi^{-1}\,\left(|\psi_t|^2 + |\Delta\psi|^2\right)\,dx\,dt$$
$$+s\,\lambda^2\iint_Q \xi\,|\nabla\psi|^2\,dx\,dt + s^3\,\lambda^4\iint_Q e^{-2s\alpha}\,\xi^3\,|q|^2\,dx\,dt \tag{1.62}$$
$$\leq C\left(\iint_Q e^{-2s\alpha}\,|f|^2\,dx\,dt + s^3\,\lambda^4\iint_{\omega\times(0,T)} e^{-2s\alpha}\,\xi^3\,|q|^2\,dx\,dt\right).$$

Using (1.59), we find that

$$s\,\lambda^2\iint_Q e^{-2s\alpha}\,\xi\,|\nabla q|^2\,dx\,dt \leq C\,s\,\lambda^2\iint_Q \xi\,|\nabla\psi|^2\,dx\,dt$$
$$+C\,s^3\,\lambda^4\iint_Q e^{-2s\alpha}\,\xi^3\,|q|^2\,dx\,dt.$$

Accordingly, the previous integral of $|\nabla q|^2$ can be added to the left hand side of (1.62) :

$$s^{-1}\iint_Q \xi^{-1}\,|\psi_t|^2\,dx\,dt + s^{-1}\iint_Q \xi^{-1}\,|\Delta\psi|^2\,dx\,dt$$
$$+s\,\lambda^2\iint_Q \xi\,|\nabla q|^2\,dx\,dt + s^3\,\lambda^4\iint_Q e^{-2s\alpha}\,\xi^3\,|q|^2\,dx\,dt$$
$$\leq C\left(s^3\,\lambda^4\iint_{\omega\times(0,T)} e^{-2s\alpha}\,\xi^3\,|q|^2\,dx\,dt + \iint_Q e^{-2s\alpha}\,|f|^2\,dx\,dt\right).$$

For Δq, we use the identity

$$\Delta\psi = e^{-s\alpha}(\Delta q + s\,\lambda\,\Delta\eta^0\,\xi\,q + s\,\lambda^2\,|\nabla\eta^0|^2\,\xi\,q$$
$$+2s\,\lambda\,\xi\,\nabla\eta^0\cdot\nabla q + s^2\,\lambda^2\,|\nabla\eta^0|^2\,\xi^2\,q)$$

and we obtain

$$s^{-1}\iint_Q e^{-2s\alpha}\,\xi^{-1}\,|\Delta q|^2\,dx\,dt \leq C\left(s^{-1}\iint_Q \xi^{-1}\,|\Delta\psi|^2\,dx\,dt\right.$$
$$+s\,\lambda^2\iint_Q e^{-2s\alpha}\,\xi\,|q|^2\,dx\,dt + s\,\lambda^4\iint_Q e^{-2s\alpha}\,\xi\,|q|^2\,dx\,dt$$
$$\left.+s\,\lambda^2\iint_Q e^{-2s\alpha}\,\xi\,|\nabla q|^2\,dx\,dt + s^3\,\lambda^4\iint_Q e^{-2s\alpha}\,\xi^3\,|q|^2\,dx\,dt\right).$$

Finally, for q_t , we get

$$s^{-1} \iint_Q e^{-2s\alpha}\, \xi^{-1}\, |q_t|^2\, dx\, dt \le C(\Omega, \omega) \left(s^{-1} \iint_Q \xi^{-1}\, |\psi_t|^2\, dx\, dt \right.$$
$$\left. + s\, e^{4\lambda \|\eta^0\|_{C(\overline{\Omega})}}\, T^2 \iint_Q e^{-2s\alpha}\, \xi^3\, |q|^2\, dx\, dt \right),$$

where we have used the identity

$$q_t = e^{s\alpha}(\psi_t + s\, \alpha_t\, \psi).$$

Thus, taking $\lambda \ge 1$ and $s \ge C(\Omega, \omega)(e^{2\lambda \|\eta^0\|_{C(\overline{\Omega})}}\, T + T^2)$, we are able to introduce all terms of $I_{s,\lambda}(q)$ in the left hand side of (1.62). This yields (1.15) and concludes the proof of lemma 1.

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Chapitre 2

Exact Controllability to the trajectories of the heat equation with Fourier boundary conditions : The semilinear case

Exact Controllability to the trajectories of the heat equation with Fourier boundary conditions : The semilinear case

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Abstract

This paper is concerned with the global exact controllability of the semilinear heat equation (with nonlinear terms involving the state and the gradient) completed with boundary conditions of the form $\frac{\partial y}{\partial n} + f(y) = 0$. We consider distributed controls, with support in a small set. The null controllability of similar linear systems has been analyzed in a previous first part of this work. In this second part we show that, when the nonlinear terms are locally Lipschitz-continuous and slightly superlinear, one has exact controllability to the trajectories.

1 Introduction

Let $\Omega \subset \mathbf{R}^N$ $(N \geq 1)$ be a bounded connected open set whose boundary $\partial\Omega$ is regular enough (for instance $\partial\Omega \in C^2$). Let $\omega \subset \Omega$ be a (small) nonempty open subset and let $T > 0$. We will use the notation $Q = \Omega \times (0,T)$ and $\Sigma = \partial\Omega \times (0,T)$ and we will denote by $n(x)$ the outward unit normal to Ω at the point $x \in \partial\Omega$.

We will consider the semilinear heat equation with nonlinear Fourier (or Robin) boundary conditions

$$\begin{cases} y_t - \Delta y + F(y, \nabla y) = v1_\omega & \text{in } Q, \\ \dfrac{\partial y}{\partial n} + f(y) = 0 & \text{on } \Sigma, \\ y(x,0) = y^0(x) & \text{in } \Omega. \end{cases} \tag{2.1}$$

Here, we assume that $v \in L^2(\omega \times (0,T))$ (at least), 1_ω is the characteristic function of ω, $y^0 \in L^\infty(\Omega)$ and $F : \mathbf{R} \times \mathbf{R}^N \mapsto \mathbf{R}$ and $f : \mathbf{R} \mapsto \mathbf{R}$ are given functions. In (2.1), $y = y(x,t)$ is the state and $v = v(x,t)$ is the control; it is assumed that we can act on the system only through $\omega \times (0,T)$.

For the existence, uniqueness, regularity and general properties of the solutions to problems like (2.1), see for instance [1], [2] and [7]. An illustrative interpretation of the data and variables in (2.1) is the following. The function $y = y(x,t)$ can be viewed as the relative temperature of a medium (with respect to the exterior surrounding air) subject to transport and chemical reactions. The parabolic equation in (2.1) means, among other things, that a heat source $v1_\omega$ is applied on a part of the body. On the boundary, $-\frac{\partial y}{\partial n}$ can be viewed as the *normal heat flux*, inwards directed, up to a positive coefficient. Thus, the equality

$$-\frac{\partial y}{\partial n} = f(y)$$

means that this flux is a (nonlinear) function of the temperature. Accordingly, it is reasonable to assume that f is nondecreasing and $f(0) = 0$.

A simplified linear model which was considered in a previous paper [10] is the following :

$$\begin{cases} y_t - \Delta y + a(x,t)\, y + B(x,t) \cdot \nabla y = v 1_\omega & \text{in } Q, \\ \dfrac{\partial y}{\partial n} + \beta(x,t)\, y = 0 & \text{on } \Sigma, \\ y(x,0) = y^0(x) & \text{in } \Omega, \end{cases} \tag{2.2}$$

Here, it is assumed that the coefficients a, B and β satisfy

$$a \in L^\infty(Q), \quad B \in L^\infty(Q)^N, \quad \beta \in L^\infty(\Sigma) \tag{2.3}$$

and, for the reasons above, it is also natural to assume that $\beta \geq 0$ (although this assumption was not used in [10]).

The main goal of this paper is to analyze the controllability properties of the nonlinear system (2.1). More precisely, we will try to reach exactly uncontrolled solutions of (2.1), i.e. functions $\overline{y} = \overline{y}(x,t)$ satisfying

$$\begin{cases} \overline{y}_t - \Delta \overline{y} + F(\overline{y}, \nabla \overline{y}) = 0 & \text{in } Q, \\ \dfrac{\partial \overline{y}}{\partial n} + f(\overline{y}) = 0 & \text{on } \Sigma, \\ \overline{y}(x,0) = \overline{y}^0(x) & \text{in } \Omega. \end{cases} \tag{2.4}$$

It will be said that (2.1) is (globally) *exactly controllable to the trajectories* at time T if, for any solution of (2.4) with 'suitable' regularity and any $y^0 \in L^\infty(\Omega)$, there exist controls $v \in L^2(\omega \times (0,T))$ and associated solutions $y \in C^0([0,T]; L^2(\Omega))$ such that

$$y(x,T) = \overline{y}(x,T) \ \text{ in } \ \Omega. \tag{2.5}$$

Here, by suitable regularity we mean the following :

$$\overline{y} \in L^2(0,T; H^1(\Omega)) \cap C^0([0,T]; L^2(\Omega)) \cap L^\infty(Q), \quad \overline{y}^0 \in L^\infty(\Omega). \tag{2.6}$$

The controllability properties of semilinear time-dependent systems have been studied intensively these last years. See for instance [16], [13], [17], [6], [12] and [11], where nonlinearities of the form $f(y)$ are considered. See also the general treatise [14]. In particular, for parabolic systems completed with Dirichlet boundary conditions, nonlinear terms $f(y, \nabla y)$ depending on both the state and the gradient have been taken into account in [9] and [6]. For the similar linear system (2.2), the null controllability was analyzed more in detail in [10]. In the case of (2.1), some partial results have been given in [5].

Our main result concerns the global exact controllability to the trajectories of (2.1). It is the following :

Theorem 3 *Let us assume that F and f are locally Lipschitz-continuous and satisfy*

$$\lim_{|s| \to \infty} \frac{|F(s,p) - F(r,p)|}{|s-r| \log^{3/2}(1 + |s-r|)} = 0, \tag{2.7}$$

uniformly in $(r,p) \in [-K,K] \times \mathbf{R}^N \, \forall K > 0,$

$$\begin{cases} \forall L > 0, \, \exists M > 0 \;\; \textit{such that} \\ |F(s,p) - F(r,p)| \le M|s-r|, \quad |F(s,p) - F(s,q)| \le M|p-q| \\ \forall (s,r,p,q) \in [-L,L]^2 \times \mathbf{R}^N \times \mathbf{R}^N \end{cases} \tag{2.8}$$

and

$$\lim_{|s|\to\infty} \frac{|f(s) - f(r)|}{|s-r| \log^{1/2}(1+|s-r|)} = 0 \tag{2.9}$$

uniformly in $r \in [-K,K] \, \forall K > 0$*. Then, for each* $T > 0$*, the nonlinear system* (2.1) *is exactly controllable to the trajectories at time* T *with* L^∞ *controls.*

Remark 2 Conditions (2.7)–(2.9) are satisfied if F and f are globally Lipschitz continuous. Notice that (2.7) means that the function F can only be slightly superlinear in s, uniformly in p. In the similar case of Dirichlet boundary conditions, it is known that conditions like these are sharp. Indeed, for instance, when F does not depend on p and

$$|F(s) - F(r)| \sim |s-r| \log^\beta (1 + |s-r|), \quad \beta > 2,$$

due to blow-up phenomena, the system fails to be controllable whenever $\omega \neq \Omega$ (see [11]). On the other hand, (2.9) is also a slightly superlinear growth assumption for f. It would be interesting to know whether a more superlinear f leading to blow up in the absence of control can also be an obstruction for the null controllability of (2.1). But this question does not seem obvious and remains open.

Remark 3 A result proved in [5] says that when $F \equiv 0$, f is smooth near zero and

$$f(s)\, s \ge 0 \quad \forall s \in \mathbf{R}, \tag{2.10}$$

the nonlinear system (2.1) is null controllable for large T. That is to say, under these assumptions, for each $y^0 \in L^2(\Omega)$ there exist $T(y^0) > 0$ and controls v in $L^\infty(\omega \times (0,T))$ such that the associated states y satisfy

$$y(x, T(y^0)) = 0 \;\; \text{in} \;\; \Omega.$$

By inspection of the proof of theorem 3, we see that the same result holds for (2.1) with $F \equiv 0$ whenever f is locally Lipschitz-continuous and satisfies the *good sign condition* (2.10).

For the proof of theorem 3, we will first establish a null controllability result for (2.2) (see proposition 6 below). This will be used, together with an appropriate fixed point argument, to deduce the desired result.

This strategy was introduced in [16] in the framework of the exact controllability of the semilinear wave equation. See also [6] and [12] for similar results concerning the approximate and null controllability of the semilinear heat equation with Dirichlet or Neumann boundary conditions.

Our null controllability result for (2.2) is the following :

Proposition 6 *For every $T > 0$, system* (2.2) *is null controllable at time T, with controls in $L^\infty(\omega \times (0,T))$. More precisely, for each $y^0 \in L^2(\Omega)$, there exists $v \in L^\infty(\omega \times (0,T))$ such that the associated solution to* (2.2) *satisfies $y(x,T) = 0$ in Ω. Furthermore, the control v can be found satisfying*

$$\|v\|_{L^\infty(\omega\times(0,T))} \le e^{C(\Omega,\omega)K(T,\|a\|_\infty,\|B\|_\infty,\|\beta\|_\infty)} \|y^0\|_{L^2(\Omega)}\,, \tag{2.11}$$

where

$$K = 1 + 1/T + \|a\|_\infty^{2/3} + \|B\|_\infty^2 + \|\beta\|_\infty^2 + T(1 + \|a\|_\infty + \|B\|_\infty^2 + \|\beta\|_\infty^2). \tag{2.12}$$

For the proof of proposition 6, we first introduce a control $L^2(\omega \times (0,T))$ which leads the solution of (2.2) to zero at time T. In a second step, arguing as in Section 2 in [4], a regularizing argument will lead to the desired L^∞ control.

The rest of this paper is organized as follows. In Section 2, we prove proposition 6. Section 3 is devoted to the proof of theorem 3. For completeness, we have also included an Appendix where the proof of a rather technical local regularity result is given in detail.

In the sequel, C denotes a generic positive constant only depending on Ω and ω.

2 A null controllability result for the linear system

In this Section we present the proof of proposition 6.

Let $y^0 \in L^2(\Omega)$ be given and let us introduce two open sets ω' and ω'', with $\omega'' \subset\subset \omega' \subset\subset \omega$. Then, we can use the main result in [10] (theorem 2) with control region $\omega'' \times (0,T)$ to deduce the existence of a control $\widetilde{v} \in L^2(\omega'' \times (0,T))$ such that the associated solution to (2.2) verifies $y(x,T) = 0$ in Ω and also the estimate

$$\|\widetilde{v}\|_{L^2(\omega''\times(0,T))} \le e^{C(\Omega,\omega)\,K(T,\|a\|_\infty,\|B\|_\infty,\|\beta\|_\infty)} \|y^0\|_{L^2(\Omega)}, \tag{2.13}$$

where K is of the form (2.12).

Let us denote by $\widetilde{y}$ the state associated to $\widetilde{v}$. We now introduce a cut-off function $\eta = \eta(t)$ satisfying

$$\eta \in C^\infty([0,T]), \quad \eta(t) = 1 \text{ in } (0,T/4), \quad \eta(t) = 0 \text{ in } (3T/4,T)$$

and

$$0 \le \eta(t) \le 1,\ |\eta'(t)| \le \frac{C}{t} \text{ in } (0,T)$$

and we denote by χ the solution to the system

$$\begin{cases} \chi_t - \Delta\chi + a(x,t)\,\chi + B(x,t)\cdot\nabla\chi = 0 & \text{in } Q, \\ \dfrac{\partial\chi}{\partial n} + \beta(x,t)\,\chi = 0 & \text{on } \Sigma, \\ \chi(x,0) = y^0(x) & \text{in } \Omega. \end{cases}$$

Then, the function $\widetilde{w} = \widetilde{y} - \eta\chi$ satisfies

$$\begin{cases} \widetilde{w}_t - \Delta\widetilde{w} + a(x,t)\,\widetilde{w} + B(x,t)\cdot\nabla\widetilde{w} = -\eta'(t)\,\chi + \widetilde{v}1_{\omega''} & \text{in } Q, \\ \dfrac{\partial\widetilde{w}}{\partial n} + \beta(x,t)\,\widetilde{w} = 0 & \text{on } \Sigma, \\ \widetilde{w}(x,0) = 0, \quad \widetilde{w}(x,T) = 0 & \text{in } \Omega. \end{cases}$$

Our aim is to construct a control $v \in L^\infty(\omega \times (0,T))$ which drives the solution of (2.2) to zero at time $t = T$. To this end, we will need a local regularity result for the solutions to linear heat equations with L^∞ coefficients a and B. This will be used below for the functions χ and $\widetilde{w}$ and reads as follows :

Lemma 2 *Let us denote by Y the space $L^2(0,T;H^1(\Omega)) \cap C^0([0,T];L^2(\Omega))$. Let $y \in Y$ be a solution to the equation*

$$y_t - \Delta y + a(x,t)\,y + B(x,t)\cdot\nabla y = f, \tag{2.14}$$

where $a \in L^\infty(Q)$, $B \in L^\infty(Q)^N$ and $f \in L^2(Q)$. Let $\mathcal{O} \subset \Omega$ be a nonempty open set and assume that f is L^∞ in the cylinder $\mathcal{O} \times (0,T)$. Then

$$y \in L^\infty(\delta,T;W^{1,\infty}(\mathcal{O}'))$$

for any $\delta \in (0,T)$ and any nonempty open set $\mathcal{O}' \subset\subset \mathcal{O}$. Furthermore, there exists a positive constant $C(\mathcal{O}')$ such that the following estimate holds :

$$\begin{aligned} \|y\|_{L^\infty(\delta,T;W^{1,\infty}(\mathcal{O}'))} \le C(\mathcal{O}')\,(T^{1/2} + T^{N/2})\times \\ \left(1 + \delta^{-1} + \|a\|_\infty + \|B\|_\infty\right)^{N+1}\left(\|y\|_Y + \|f\|_{L^\infty(\mathcal{O}\times(0,T))}\right). \end{aligned} \tag{2.15}$$

The previous regularity also holds with $\delta = 0$ if, besides (2.14), we have $y(x,0) = 0$ in Ω. In that case, one has an estimate similar to (2.15) without the term in δ.

This lemma is implied by well known parabolic regularity theory. For completeness, its proof is given in an Appendix, at the end of this paper.

Let us now consider an open set ω_0 with $\omega' \subset\subset \omega_0 \subset\subset \omega$ and a cut-off function ξ, with

$$\xi \in C_0^2(\omega_0), \quad \xi \equiv 1 \text{ in } \omega'$$

and let us set $w = (1-\xi)\,\widetilde{w}$. Then we have :

$$\begin{cases} w_t - \Delta w + a(x,t)\,w + B(x,t)\cdot\nabla w = -\eta'(t)\,\chi + v1_\omega & \text{in } Q, \\ \dfrac{\partial w}{\partial n} + \beta(x,t)\,w = 0 & \text{on } \Sigma, \\ w(x,0) = 0, \quad w(x,T) = 0 & \text{in } \Omega, \end{cases}$$

with

$$v = \eta'\,\xi\,\chi + 2\nabla\xi\cdot\nabla\widetilde{w} + \Delta\xi\,\widetilde{w} - B\cdot\nabla\xi\,\widetilde{w}. \tag{2.16}$$

Let us remark that supp $v \subset \omega \times [0,T]$. Therefore, if we prove that $v \in L^\infty(\omega \times (0,T))$, we will have that the function $y = w + \eta\chi$ solves (together with v) the null controllability problem for (2.2).

Thus, let us check that $v \in L^\infty(\omega \times (0,T))$ and let us estimate its norm in this space :

• The regularity of the first term in the right hand side of (2.16) is implied by the interior regularity of χ not only in space but in time as well. From lemma 2 with $\mathcal{O} = \omega$, we deduce that $\chi \in L^\infty(\omega_0 \times (\delta, T))$ with supp $\xi \subset \omega_0 \subset\subset \omega$ (we even have $\chi \in L^\infty(\delta, T; W^{1,\infty}_{loc}(\omega))$) and

$$\|\chi\|_{L^\infty(\omega_0\times(\delta,T))} \leq C\,(T^{1/2} + T^{N/2})\,\big(1 + \delta^{-1} + \|a\|_\infty + \|B\|_\infty\big)^{N+1}\,\|\chi\|_Y;$$

recall that $Y = L^2(0,T;H^1(\Omega)) \cap C^0([0,T];L^2(\Omega))$.

Consequently taking for instance $\delta = T/8$, since $\eta' \equiv 0$ in $(0,T/4)$, we get

$$\begin{aligned}\|\eta'\,\xi\,\chi\|_{L^\infty(\omega\times(0,T))} \leq C\,T^{-1}\,(T^{1/2} + T^{N/2})\times& \\ \big(1 + T^{-1} + \|a\|_\infty + \|B\|_\infty\big)^{N+1}&\,\|\chi\|_Y.\end{aligned}$$

• The regularity of the other three terms in the right hand side of (2.16) is related to the interior space regularity of $\widetilde{w}$. Thus, let us introduce ω_1 with $\omega_0 \subset\subset \omega_1 \subset\subset \omega$ and let us apply lemma 2 with $\mathcal{O} = \omega_1 \setminus \overline{\omega''}$. This gives $\widetilde{w} \in L^\infty(0,T;W^{1,\infty}(\omega_0 \setminus \overline{\omega'}))$ and the estimate

$$\begin{aligned}\|\widetilde{w}\|_{L^\infty(0,T;W^{1,\infty}(\omega_0\setminus\overline{\omega'}))} \leq C\,(T^{1/2} + T^{N/2})\times& \\ (1 + \|a\|_\infty + \|B\|_\infty)^{N+1}\,(\|\widetilde{w}\|_Y + \|\eta'\,\chi\|_{L^\infty(\omega_1\times(0,T))}&),\end{aligned}$$

whence

$$\left\{\begin{aligned}\|2\nabla\xi\cdot\nabla\widetilde{w} + \Delta\xi\,\widetilde{w} - B\cdot\nabla\xi\,\widetilde{w}\|_{L^\infty(\omega\times(0,T))} \leq C(T^{1/2} + T^{N/2})\times& \\ (1 + \|a\|_\infty + \|B\|_\infty)^{N+2}\,(\|\widetilde{w}\|_Y + \|\eta'\,\chi\|_{L^\infty(\omega_1\times(0,T))}&).\end{aligned}\right.$$

Putting the previous estimates together, we find that $v \in L^\infty(\omega \times (0,T))$ and

$$\begin{aligned}\|v\|_{L^\infty(\omega\times(0,T))} \leq C(1 + T^{N-1})\times& \\ \big(1 + T^{-1} + \|a\|_\infty + \|B\|_\infty\big)^{2N+3}\,(\|\widetilde{w}\|_Y + \|\chi\|_Y&).\end{aligned} \tag{2.17}$$

At this point, notice that for any $f \in L^2(Q)$ and any $y^0 \in L^2(\Omega)$ the solution y to the linear system

$$\left\{\begin{array}{ll} y_t - \Delta y + a(x,t)\,y + B(x,t)\cdot\nabla y = f & \text{in } Q, \\ \dfrac{\partial y}{\partial n} + \beta(x,t)\,y = 0 & \text{on } \Sigma, \\ y(x,0) = y^0(x) & \text{in } \Omega \end{array}\right. \tag{2.18}$$

satisfies

$$\|y\|_Y \leq e^{C\,T(1+\|a\|_\infty + \|B\|_\infty^2 + \|\beta\|_\infty^2)}(\|f\|_{L^2(Q)} + \|y^0\|_{L^2(\Omega)}).$$

For a detailed proof, see for example proposition 1 in [10].

This can be used to estimate $\|\widetilde{w}\|_Y$ and $\|\chi\|_Y$ in terms of $\|\widetilde{v}\|_{L^2(\omega\times(0,T))}$ and $\|y^0\|_{L^2(\Omega)}$. In view of (2.17), we see that

$$\|v\|_{L^\infty(\omega\times(0,T))} \le L\,(\|\widetilde{v}\|_{L^2(\omega''\times(0,T))} + \|y^0\|_{L^2(\Omega)}), \tag{2.19}$$

where

$$L = C\,T^{-1}(1+T^{N-1})\left(1+T^{-1}+\|a\|_\infty+\|B\|_\infty\right)^{2N+3}\times \\ \exp\{C\,T(1+\|a\|_\infty+\|B\|_\infty^2+\|\beta\|_\infty^2)\}.$$

Combining this estimate and (2.13), we finally obtain that

$$\|v\|_{L^\infty(\omega\times(0,T))} \le e^{C\,K(T,\|a\|_\infty,\|B\|_\infty,\|\beta\|_\infty)}\,\|y^0\|_{L^2(\Omega)}, \tag{2.20}$$

where K is given by (2.12).

This ends the proof of proposition 6.

3 Controllability of the nonlinear system

In this Section we will prove theorem 3. The following auxiliary result will be needed :

Proposition 7 *Let us assume that, in* (2.18)*, we have $f \in L^\infty(Q)$ and $y^0 \in L^\infty(\Omega)$. Let us also assume that the coefficients a, B and β satisfy* (2.3)*. Then $y \in L^\infty(Q)$ and*

$$\|y\|_\infty \le e^{C\,T\left(1+\|a\|_\infty+\|B\|_\infty^2+\|\beta\|_\infty^2\right)}\left(\|y^0\|_\infty+\|f\|_\infty\right). \tag{2.21}$$

for some $C = C(\Omega)$.

Proof : We will consider two different situations :

Case 1 - We will first assume that $a \ge 1$ and $\beta \ge 0$ and we will establish (2.21) in this case. In fact, we will show that, under these assumptions,

$$\|y\|_\infty \le \|y^0\|_\infty + \|f\|_\infty. \tag{2.22}$$

To this end, let us introduce the system

$$\begin{cases} z_t - \Delta z + a(x,t)\,z + B(x,t)\cdot\nabla z = h & \text{in } Q, \\ \dfrac{\partial z}{\partial n} + \beta(x,t)\,z = k & \text{on } \Sigma, \\ z(x,0) = z^0(x) & \text{in } \Omega, \end{cases}$$

where $h \in L^\infty(Q)$, $k \in L^\infty(\Sigma)$ and $z^0 \in L^\infty(\Omega)$ and let us show that, if h, z^0 and k are nonnegative, then this is also the case for z.

Indeed, by multiplying the equation satisfied by z by $z_-(\cdot,t)$ (the negative part of $z(\cdot,t)$) for each $t \in (0,T)$ and integrating in Ω, after several simplifications, we find :

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega |z_-(x,t)|^2\,dx + \int_\Omega |\nabla z_-(x,t)|^2\,dx\\&\quad+\int_{\partial\Omega}\beta(x,t)\,(z_-(x,t)+k(x,t))\,z_-(x,t)\,d\sigma + \int_\Omega a(x,t)\,|z_-(x,t)|^2\,dx\\&\quad= -\int_\Omega h(x,t)\,z_-(x,t)\,dx - \int_\Omega B(x,t)\cdot\nabla z_-(x,t)\,z_-(x,t)\,dx.\end{aligned}$$

From this identity, in view of the positiveness of a, h, β and k, we easily deduce that

$$\frac{d}{dt}\int_\Omega |z_-(x,t)|^2\,dx \le \|B\|_\infty^2 \int_\Omega |z_-(x,t)|^2\,dx,$$

whence $z \ge 0$ in Q.

Now, let $M > 0$ be a large constant (to be chosen below). The function $z = M - y$ satisfies

$$\begin{cases} z_t - \Delta z + a(x,t)\,z + B(x,t)\cdot\nabla z = a(x,t)\,M - f & \text{in } Q,\\ \dfrac{\partial z}{\partial n} + \beta(x,t)\,z = \beta(x,t)\,M & \text{on } \Sigma,\\ z(x,0) = M - y^0(x) & \text{in } \Omega.\end{cases}$$

Therefore, if we take

$$M \ge \max\{\,\|f\|_{L^\infty(Q)},\ \|y^0\|_{L^\infty(\Omega)}\},$$

we can apply the previous argument and deduce that $y \le M$. In a similar way, one can deduce that $y \ge -M$ and, consequently, $|y| \le M$. This proves that whenever $a \ge 1$ and $\beta \ge 0$, the estimate (2.22) holds.

Case 2 - We will now prove (2.21) for general L^∞ coefficients a and β.

Let $\gamma \in C^2(\overline{\Omega})$ be a function satisfying

$$\begin{aligned}\gamma \ge 0 \text{ in } \Omega,\quad \frac{\partial\gamma}{\partial n} \le -\|\beta\|_\infty \text{ on } \partial\Omega,\quad \|\gamma\|_\infty \le 1,\\ \|\nabla\gamma\|_\infty \le C\,\|\beta\|_\infty,\quad \|D^2\gamma\|_\infty \le C\,\|\beta\|_\infty^2.\end{aligned} \tag{2.23}$$

We give here a sketch of the proof of the existence of such a function γ. To this end, let $\delta > 0$ be a parameter (depending on Ω) such that

$$x \in \Omega_\delta \mapsto \mathrm{dist}(x,\partial\Omega)$$

is C^2, with $\Omega_\delta = \{x \in \Omega : \mathrm{dist}(x,\partial\Omega) < \delta\}$. We distinguish two cases.

Let us first assume that $\|\beta\|_\infty \ge 1/\delta$. Then we take $\gamma(x) \equiv 1$ in $\Omega\backslash\Omega_\delta$, $\gamma(x) = \|\beta\|_\infty\,\mathrm{dist}(x,\partial\Omega)$ in Ω_ε with $\varepsilon = 1/(2\|\beta\|_\infty)$ and a regularization of γ in $\Omega_\delta \setminus \Omega_\varepsilon$. This gives the desired properties for γ.

On the other hand, if $\|\beta\|_\infty < 1/\delta$, we take $\gamma(x) = \delta\,\|\beta\|_\infty$ in $\Omega\backslash\Omega_\delta$, $\gamma(x) = \|\beta\|_\infty\,\mathrm{dist}(x,\partial\Omega)$ in $\Omega_{\delta/2}$ and a regularization in $\Omega_\delta \setminus \Omega_{\delta/2}$. This also provides a desired function in this case.

Let us now set $\widehat{y} = e^{\gamma(x)}\,y$. Then $\widehat{y}$ satisfies

$$\begin{cases} \widehat{y}_t - \Delta\widehat{y} + \widehat{a}(x,t)\,\widehat{y} + \widehat{B}(x,t)\cdot\nabla\widehat{y} = e^{\gamma(x)}f & \text{in } Q,\\ \dfrac{\partial\widehat{y}}{\partial n} + \widehat{\beta}(x,t)\,\widehat{y} = 0 & \text{on } \Sigma,\\ \widehat{y}(x,0) = e^{\gamma(x)}\,y^0(x) & \text{in } \Omega, \end{cases} \tag{2.24}$$

where

$$\widehat{a} = a + \Delta\gamma - |\nabla\gamma|^2 - B\cdot\nabla\gamma,$$

$$\widehat{B} = B + 2\nabla\gamma, \quad \widehat{\beta} = \beta - \frac{\partial\gamma}{\partial n} \geq 0 \text{ on } \Sigma.$$

Notice that, from the inequalities (2.23) satisfied by γ, we know that

$$|a + \Delta\gamma - |\nabla\gamma|^2 - B\cdot\nabla\gamma| \leq C_1\,(\|a\|_\infty + \|B\|_\infty^2 + \|\beta\|_\infty^2) \text{ in } Q$$

for some $C_1 > 0$.

Now, let us set

$$\widetilde{y} = e^{-(C_1\,(\|a\|_\infty + \|B\|_\infty^2 + \|\beta\|_\infty^2)+1)t}\,\widehat{y}.$$

Then $\widetilde{y}$ satisfies

$$\begin{cases} \widetilde{y}_t - \Delta\widetilde{y} + \widetilde{a}(x,t)\,\widetilde{y} + (B(x,t) + 2\nabla\gamma(x))\cdot\nabla\widetilde{y} = \widetilde{f} & \text{in } Q,\\ \dfrac{\partial\widetilde{y}}{\partial n} + \widetilde{\beta}(x,t)\,\widetilde{y} = 0 & \text{on } \Sigma,\\ \widetilde{y}(x,0) = e^{\gamma(x)}\,y^0(x) & \text{in } \Omega, \end{cases} \tag{2.25}$$

where

$$\widetilde{a} = a + \Delta\gamma - |\nabla\gamma|^2 - B\cdot\nabla\gamma + C_1\,(\|a\|_\infty + \|B\|_\infty^2 + \|\beta\|_\infty^2) + 1,$$

$$\widetilde{f} = e^{-(C_1\,(\|a\|_\infty + \|B\|_\infty^2 + \|\beta\|_\infty^2)+1)t+\gamma(x)}\,f$$

and

$$\widetilde{\beta} = \widehat{\beta}.$$

Since $\widetilde{a} \geq 1$ and $\widetilde{\beta} \geq 0$, we can apply Case 1 to $\widetilde{y}$. This provides the estimates

$$\|y\|_\infty \leq \|\widehat{y}\|_\infty \leq e^{C\,T\left(1+\|a\|_\infty + \|B\|_\infty^2 + \|\beta\|_\infty^2\right)}\,(\|y^0\|_\infty + \|f\|_\infty),$$

whence we deduce (2.21).

Let us now start with the proof of theorem 3. Let $y^0 \in L^\infty(\Omega)$ and $\overline{y}$ be given and assume that $\overline{y}$ satisfies (2.6) and (2.4) in the weak sense. Let us consider the nonlinear system

$$\begin{cases} w_t - \Delta w + F_1(w, \nabla w; x,t)w + F_2(\nabla w; x,t) \cdot \nabla w = v1_\omega & \text{in } Q, \\ \dfrac{\partial w}{\partial n} + F_3(w; x,t)w = 0 & \text{on } \Sigma, \\ w(x,0) = y^0(x) - \overline{y}(x,0) & \text{in } \Omega, \end{cases} \tag{2.26}$$

where we have used the notation

$$F_1(s,p;x,t) = \frac{F(\overline{y}(x,t)+s, \nabla\overline{y}(x,t)+p) - F(\overline{y}(x,t), \nabla\overline{y}(x,t)+p)}{s}, \tag{2.27}$$

$$F_2 = (F_{21}, \dots, F_{2N}), \quad F_{2j}(p;x,t) = \int_0^1 \frac{\partial F}{\partial p_j}(\overline{y}(x,t), \nabla\overline{y}(x,t) + \lambda p)\, d\lambda \tag{2.28}$$

and

$$F_3(s;x,t) = \frac{f(\overline{y}(x,t)+s) - f(\overline{y}(x,t))}{s} \tag{2.29}$$

for $s \in \mathbf{R}$ and $p \in \mathbf{R}^N$.

We will prove that there exist a control $v \in L^\infty(\omega \times (0,T))$ and an associated solution to (2.26) such that

$$w(x,T) = 0 \ \text{ in } \ \Omega. \tag{2.30}$$

With this control and the state $y = w + \overline{y}$, we will have solved the exact controllability problem for (2.1) and we will have thus proved theorem 3.

We will first assume that the functions F and f are continuously differentiable. Then, by a density argument, we will be able to prove the result in the general case.

3.1 The case in which F and f are C^1

The idea of the proof is well known : we introduce an appropriate (set-valued) fixed point mapping and we check that it possesses at least one fixed point ; this will be a solution to the null controllability problem associated to (2.26).

Let $R > 0$ be given and let us introduce the following function :

$$M_R(s) = \begin{cases} -R & \text{if } s < -R, \\ s & \text{if } -R \le s \le R, \\ R & \text{if } s > R. \end{cases}$$

Let us denote by Z the Hilbert space $Z = L^2(0,T;H^1(\Omega))$ and let us set for each $R > 0$ and each $z \in Z$

$$a_{R,z}(x,t) = F_1(M_R(z(x,t)), \nabla z(x,t); x,t),$$

$$B_z(x,t) = F_2(\nabla z(x,t); x,t)$$

and

$$\beta_{R,z}(x,t) = F_3(M_R(z(x,t)); x,t).$$

Consider the linear null controllability problem

$$\begin{cases} w_t - \Delta w + a_{R,z}(x,t)\, w + B_z(x,t)\cdot\nabla w = v1_\omega & \text{in } Q,\\ \dfrac{\partial w}{\partial n} + \beta_{R,z}(x,t)\, w = 0 & \text{on } \Sigma,\\ w(x,0) = y^0(x) - \overline{y}(x,0) & \text{in } \Omega, \end{cases} \tag{2.31}$$

together with (2.30).

From (2.6), (2.8) and the fact that $f \in C^1(\mathbf{R})$, we have

$$a_{R,z} \in L^\infty(Q), \quad B_z \in L^\infty(Q)^N, \quad \beta_{R,z} \in L^\infty(\Sigma).$$

Consequently, in view of proposition 6, (2.30)–(2.31) can be solved with controls in $L^\infty(\omega \times (0,T))$.

We are now going to select a particular solution to (2.30)–(2.31) constructed as in [11]. To do this, we first set $T_R = \min\{T, a_R^{-1/3}\} > 0$, where

$$a_R = \sup_{|s|\le R,\ p\in\mathbf{R}^N} \ \operatorname*{ess\,sup}_{(x,t)\in Q} |F_1(s,p;x,t)|.$$

We can follow the steps of Section 2 and construct a control $v_{R,z} \in L^\infty(\omega\times(0,T_R))$ such that the solution $w_{R,z}$ to (2.31) in $\Omega\times(0,T_R)$ verifies

$$w_{R,z}(x,T_R) = 0 \ \text{ in } \ \Omega.$$

The estimates we have been able to establish in propositions 6 and 7 written for $v_{R,z}$ and $w_{R,z}$ with final time T_R will now give

$$\|v_{R,z}\|_{L^\infty(\omega\times(0,T_R))} \le C_R\, \|w^0\|_{L^2(\Omega)}, \tag{2.32}$$

$$\|w_{R,z}\|_{L^2(0,T_R;H^1(\Omega))} \le C_R\, \|w^0\|_{L^2(\Omega)} \tag{2.33}$$

and

$$\|w_{R,z}\|_{L^\infty(\Omega\times(0,T_R))} \le C_R\, \|w^0\|_{L^\infty(\Omega)}, \tag{2.34}$$

where

$$C_R = \exp\left\{ C(\Omega,\omega,T)\left(1 + a_R^{2/3} + \overline{B}^2 + \beta_R^2\right)\right\},$$

$$\overline{B} = \sup_{p\in\mathbf{R}^N} \ \operatorname*{ess\,sup}_{(x,t)\in Q} |F_2(p;x,t)|$$

and

$$\beta_R = \sup_{|s|\le R} \ \operatorname*{ess\,sup}_{(x,t)\in\Sigma} |F_3(s;x,t)|.$$

In fact, the estimates obtained in the previous section imply (2.32)–(2.34) with C_R replaced by $C_R(z)$, where

$$C_R(z) = \exp\Big\{ C(\Omega,\omega)\ \Big(1 + T_R^{-1} + T_R + \|a_{R,z}\|_\infty^{2/3} + \|B_z\|_\infty^2 + \|\beta_{R,z}\|_\infty^2 + T_R\left(\|a_{R,z}\|_\infty + \|B_z\|_\infty^2 + \|\beta_{R,z}\|_\infty^2\right)\Big)\Big\};$$

but taking into account the definitions of T_R, a_R, $\overline{B}$ and β_R it is clear that $C_R(z) \leq C_R$ for all $z \in Z$.

At this moment, we can extend by zero the functions $v_{R,z}$ and $w_{R,z}$ for $t \in (T_R, T)$. In this way, we will have built a control $v_{R,z}$ and an associated state $w_{R,z}$ satisfying (2.30)–(2.31) and

$$\|v_{R,z}\|_{L^\infty(\omega\times(0,T))} \leq C_R \, \|w^0\|_{L^2(\Omega)}, \tag{2.35}$$

$$\|w_{R,z}\|_Z \leq C_R \, \|w^0\|_{L^2(\Omega)} \tag{2.36}$$

and

$$\|w_{R,z}\|_\infty \leq C_R \, \|w^0\|_{L^\infty(\Omega)}. \tag{2.37}$$

We will now introduce a set-valued mapping leading to the solution to our controllability problem.

We first consider the set of admissible controls $A_R(z)$. By definition, this is the set of controls $v_{R,z} \in L^\infty(\omega \times (0,T))$ which lead the solution to (2.31) to zero at time T and satisfy (2.35). Then, for each $z \in Z$, we denote by $\Lambda_R(z)$ the set of states $w_{R,z}$ associated to the controls $v_{R,z} \in A_R(z)$ furthermore satisfying (2.36) and (2.37). In view of the arguments above, $\Lambda_R(z)$ is a nonempty subset of Z.

The plan of the rest of the proof is the following :

- We will first see that, for each $R > 0$, Λ_R possesses a fixed point w_R. This will be implied by Kakutani's theorem.
- Then, we will find $R > 0$ (large enough) such that $M_R(w_R) = w_R$. At this level, the use of proposition 7 will be crucial.

As a consequence, for large R, the fixed point w_R of Λ_R will be, together with some $v_R \in L^\infty(\omega \times (0,T))$, a solution to (2.30)–(2.31).

Thus, let us recall Kakutani's fixed point theorem (see, for instance, [1]) :

Theorem 4 *Let Z be a Banach space and let $\Lambda : Z \mapsto Z$ be a set-valued mapping satisfying the following assumptions :*

(i) $\Lambda(z)$ is a nonempty closed convex set of Z for every $z \in Z$.

(ii) There exists a nonempty convex compact set $K \subset Z$ such that $\Lambda(K) \subset K$.

(iii) Λ is upper-hemicontinuous in Z, i.e. for each $\sigma \in Z'$ the single-valued mapping

$$z \mapsto \sup_{y\in\Lambda(z)} \langle \sigma, y \rangle_{Z',Z} \tag{2.38}$$

is upper-semicontinuous.

Then Λ possesses a fixed point in the set K, i.e. there exists $z \in K$ such that $z \in \Lambda(z)$.

Let us check that Kakutani's theorem can be applied to Λ_R.

That $\Lambda_R(z)$ is a nonempty closed convex set of Z for every $z \in Z$ is very easy to verify.

Let us prove that Λ_R maps a compact set into itself. In fact, let us see that Λ_R maps the whole space Z into a fixed convex compact set K_R.

Our argument will be the following : we choose an arbitrary sequence $\{z_n\}$ in Z and a sequence $\{w_n\}$ with $w_n \in \Lambda_R(z_n)$ for all n and we prove that $\{w_n\}$ possesses a strongly convergent subsequence.

Thus, let the sequences $\{z_n\}$ and $\{w_n\}$ be given. From (2.35)–(2.37), the equations satisfied by the functions w_n and the fact that $\|a_{R,z_n}\|_\infty \leq a_R$, $\|B_{z_n}\|_\infty \leq \overline{B}$ and $\|\beta_{R,z_n}\|_\infty \leq \beta_R$ for all $n \geq 1$, we deduce the existence of subsequences $\{w_{n'}\}$ and $\{v_{n'}\}$ such that

$$w_{n'} \to w \quad \text{weakly in } Z,$$
$$w_{n',t} \to w_t \quad \text{weakly in } L^2(0,T;H^{-1}(\Omega))$$

and

$$v_{n'} \to v \quad \text{weakly-}* \text{ in } L^\infty(Q)$$

as $n' \to \infty$. We can also assume that the coefficients associated to $z_{n'}$ converge weakly-$*$ in $L^\infty(Q)$ and $L^\infty(\Sigma)$. Thus, we can pass to the limit in the weak formulations satisfied by $w_{n'}$ and deduce that w and v satisfy

$$\begin{cases} w_t - \Delta w + a(x,t)\, w + \theta(x,t) = v 1_\omega & \text{in } Q, \\ \dfrac{\partial w}{\partial n} + \beta(x,t)\, w = 0 & \text{on } \Sigma, \\ w(x,0) = w^0(x) & \text{in } \Omega \end{cases}$$

for some $a \in L^\infty(Q)$ and $\beta \in L^\infty(\Sigma)$, where $\theta \in L^2(Q)$ is the weak limit of $B_{z_{n'}} \cdot \nabla w_{n'}$ in $L^2(Q)$.

After substraction of the equations satisfied by the functions $w_{n'}$ and w, we find that

$$\begin{cases} (w_{n'} - w)_t - \Delta(w_{n'} - w) = a(x,t)\, w - a_{R,z_{n'}}(x,t)\, w_{n'} & \\ \qquad\qquad + \theta(x,t) - B_{z_{n'}}(x,t) \cdot \nabla w_{n'} + (v_{n'} - v) 1_\omega & \text{in } Q, \\ \dfrac{\partial (w_{n'} - w)}{\partial n} + \beta_{R,z_{n'}}(x,t)\, w_{n'} - \beta(x,t)\, w = 0 & \text{on } \Sigma, \\ (w_{n'} - w)(x,0) = 0 & \text{in } \Omega. \end{cases}$$

Consequently,

$$\begin{aligned} &\frac{1}{2} \int_\Omega |(w_{n'} - w)(x,T)|^2 \, dx + \int_0^T \int_\Omega |\nabla (w_{n'} - w)(x,s)|^2 \, dx \, ds \\ &\quad = \int_0^T \int_{\partial\Omega} (\beta\, w - \beta_{R,z_{n'}}\, w_{n'})\, (w_{n'} - w)(x,s) \, d\sigma \, ds \\ &\quad + \int_0^T \int_\Omega (a\, w - a_{R,z_{n'}}\, w_{n'})\, (w_{n'} - w)(x,s) \, dx \, ds \\ &\quad + \int_0^T \int_\Omega (\theta - B_{z_{n'}} \cdot \nabla w_{n'})\, (w_{n'} - w)(x,s) \, dx \, ds \\ &\qquad + \int_0^T \int_\omega (v_{n'} - v)\, (w_{n'} - w)(x,s) \, dx \, ds. \end{aligned} \tag{2.39}$$

We are now going to check that all the terms in the right hand side of this last equality tends to zero. Among other things, this will imply that

$$w_{n'} \to w \quad \text{strongly in } Z. \tag{2.40}$$

• The first term in the right hand side converges to zero, since

$$w_{n'} \to w \quad \text{strongly in } L^2(\Sigma)$$

and consequently

$$\beta_{R,z_{n'}} w_{n'} \to \beta\, w \quad \text{weakly in } L^2(\Sigma).$$

Indeed, the strong convergence of $w_{n'}$ is an immediate consequence of the compact embedding of the space

$$\{\, z \in L^2(0,T;H^1(\Omega)) : z_t \in L^2(0,T;H^{-1}(\Omega)) \,\}$$

in $L^2(0,T;H^s(\Omega))$ for all $s \in (1/2,1)$ and the fact that the *lateral trace* operator is well defined, linear and continuous from $L^2(0,T;H^s(\Omega))$ into $L^2(\Sigma)$.

• The convergence of the other three terms in the right hand side is a consequence of the weak convergence in $L^2(Q)$ of $a_{R,z_{n'}} w_{n'}$ and $B_{z_{n'}} \cdot \nabla w_{n'}$, the weak convergence in $L^2(\omega \times (0,T))$ of $v_{n'}$ and the *strong* convergence of $w_{n'}$ in $L^2(Q)$.

We have thus seen that $\{w_n\}$ possesses a strongly convergent subsequence and, consequently, Λ_R maps the space Z into a fixed compact set.

It remains to check that Λ_R is upper-hemicontinuous. Thus, assume that $\sigma \in Z'$ and let a sequence $\{z_n\}$ be given, with $z_n \to z$ strongly in Z. We must prove that

$$\limsup_{n\to+\infty} \sup_{w\in\Lambda_R(z_n)} \langle \sigma, w\rangle_{Z',Z} \le \sup_{w\in\Lambda_R(z)} \langle \sigma, w\rangle_{Z',Z}.$$

Let $\{z_{n'}\}$ be a subsequence of $\{z_n\}$ such that

$$\limsup_{n\to+\infty} \sup_{w\in\Lambda_R(z_n)} \langle \sigma, w\rangle_{Z',Z} = \lim_{n'\to+\infty} \sup_{w\in\Lambda_R(z_{n'})} \langle \sigma, w\rangle_{Z',Z}.$$

Since each $\Lambda_R(z_{n'})$ is a compact set of Z, for every n' we have

$$\sup_{w\in\Lambda_R(z_{n'})} \langle \sigma, w\rangle_{Z',Z} = \langle \sigma, w_{n'}\rangle_{Z',Z}$$

for some $w_{n'} \in \Lambda_R(z_{n'})$. On the other hand, since all the states $w_{n'}$ belong to the same compact set K_R, at least for a new subsequence (again indexed by n'), we must have (2.40). We will now prove that $w \in \Lambda_R(z)$. This will achieve the proof of the upper hemicontinuity of Λ_R.

Indeed, we can assume that the weak limits of the coefficients associated to $z_{n'}$ are $a_{R,z}$, B_z and $\beta_{R,z}$, since $z_{n'}$ converges strongly in Z towards z and therefore the coefficients $a_{R,z_{n'}}$, $B_{z_{n'}}$ and $\beta_{R,z_{n'}}$ converge almost everywhere (observe that we are using here the C^1 regularity of F and f).

On the other hand, it can be assumed that the controls $v_{n'}$ converge to a function v weakly-$*$ in $L^\infty(\omega \times (0,T))$. Then, w solves (2.31) and $w(T) = 0$. Moreover, since inequality (2.35) is independent of n, v also satisfies (2.35). Therefore, $v \in A_R(z)$. Consequently, it is immediate that w is the solution to (2.31) associated to the control v.

This shows that $w \in \Lambda_R(z)$ and, therefore, Λ_R is upper hemicontinuous.

In view of these arguments, Kakutani's theorem can be applied and we deduce that, for each $R > 0$, Λ_R possesses at least one fixed point w_R that belongs to Z and $L^\infty(Q)$.

Our aim is now to find $R > 0$ such that

$$\|w_R\|_\infty \le R.$$

This will be a consequence of the estimates we know for w_R and the properties satisfied by the functions F_i.

From (2.37), we obtain

$$\|w_R\|_\infty \le e^{C(\Omega,\omega,T)(1+a_R^{2/3}+\overline{B}^2+\beta_R^2)}\,\|w^0\|_\infty. \tag{2.41}$$

On the other hand, from (2.8)–(2.9) it is also clear that, for every $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$\begin{aligned}
&\left(\operatorname*{ess\,sup}_{(x,t)\in Q}|F_1(s,p;x,t)|\right)^{2/3} \le \varepsilon\,\log(1+|s|)+C_\varepsilon \quad \forall s\in\mathbf{R},\ \forall p\in\mathbf{R}^N,\\
&\left(\operatorname*{ess\,sup}_{(x,t)\in Q}|F_2(p;x,t)|\right)^{2} \le C_\varepsilon \quad \forall p\in\mathbf{R}^N,\\
&\left(\operatorname*{ess\,sup}_{(x,t)\in \Sigma}|F_1(s;x,t)|\right)^{2} \le \varepsilon\,\log(1+|s|)+C_\varepsilon \quad \forall s\in\mathbf{R}.
\end{aligned} \tag{2.42}$$

Consequently, it is also true that, for every $\varepsilon > 0$, there exists $C_\varepsilon > 0$ (independent of R) such that

$$a_R^{2/3}+\overline{B}^2+\beta_R^2 \le \varepsilon\,\log(1+R)+C_\varepsilon.$$

These estimates, together with (2.41) and the definitions of a_R, B and β_R, lead to the following inequality :

$$\|w_R\|_\infty \le\ C(\Omega,\omega,T,\varepsilon)\,(1+R)^{C(\Omega,\omega,T)\,\varepsilon}\,\|w^0\|_\infty.$$

Accordingly, taking $\varepsilon > 0$ small enough to satisfy $C(\Omega,\omega,T)\,\varepsilon < 1$, we can ensure that, for $R > 0$ sufficiently large (depending on Ω, ω, T and $\|y^0-\overline{y}^0\|_{L^\infty(\Omega)}$), one has

$$\|w_R\|_\infty \le\ R.$$

This ends the proof of theorem 3 when F and f are C^1 functions.

3.2 The general case

We will now assume that f and F are locally Lipschitz-continuous functions satisfying (2.7)–(2.9).

Let us introduce the functions $\rho^1 \in C_c^\infty(\mathbf{R} \times \mathbf{R}^N)$, $\rho^2 \in C_c^\infty(\mathbf{R}^N)$ and $\rho^3 \in C_c^\infty(\mathbf{R})$, with $\rho^j \geq 0$, supp $\rho^1 \subset \overline{B}((0,0);1)$, supp $\rho^2 \subset \overline{B}(0;1)$, supp $\rho^3 \subset [-1,1]$ and

$$\iint_{\mathbf{R}\times\mathbf{R}^N} \rho^1(s,p)\, ds\, dp = \int_{\mathbf{R}^N} \rho^2(p)\, dp = \int_{\mathbf{R}} \rho^3(s)\, ds = 1.$$

Let us consider, for each $n \geq 1$, the associated *mollifiers*

$$\rho_n^1(s,p) = n^{N+1} \rho^1(ns, np), \quad \rho_n^2(p) = n^N \rho^2(np) \quad \forall (s,p) \in \mathbf{R} \times \mathbf{R}^N$$

and

$$\rho_n^3(s) = n\, \rho^3(ns) \quad \forall s \in \mathbf{R}$$

and the regularized functions

$$F_{i,n} = \rho_n^i * F_i \quad (i = 1,2,3)$$

(the functions F_1, F_2 and F_3 were defined in (2.27)–(2.29)).

These functions satisfy the following :

- For each $n \geq 1$, $F_{1n} : \mathbf{R} \times \mathbf{R}^N \times Q \mapsto \mathbf{R}$, $F_{2n} : \mathbf{R}^N \times Q \mapsto \mathbf{R}^N$ and $F_{3n} : \mathbf{R} \times \Sigma \mapsto \mathbf{R}$ are Caratheodory functions (respectively continuous in (s,p), p and s and measurable in (x,t)).

- If we set

$$F_n(s,p;x,t) = F_{1,n}(s,p;x,t)s + F_{2,n}(p;x,t) \cdot p$$

and

$$f_n(s;x,t) = F_{3,n}(s;x,t)s,$$

then the asymptotic properties (2.8) and (2.42) remain true uniformly in n. In other words, for any $L > 0$, there exists $M > 0$ (independent of n) such that

$$\begin{cases} |F_n(s,p) - F_n(r,p)| \leq M|s-r|, \quad |F_n(s,p) - F_n(s,q)| \leq M|p-q| \\ \forall (s,r,p,q) \in [-L,L]^2 \times \mathbf{R}^N \times \mathbf{R}^N. \end{cases}$$

Moreover, for each $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$\begin{cases} \left(\operatorname*{ess\,sup}_{(x,t)\in Q} |F_{1n}(s,p;x,t)| \right)^{2/3} \leq \varepsilon \log(1+|s|) + C_\varepsilon \quad \forall s \in \mathbf{R}, \forall p \in \mathbf{R}^N, \\ \left(\operatorname*{ess\,sup}_{(x,t)\in Q} |F_{2n}(p;x,t)| \right)^2 \leq C_\varepsilon \quad \forall p \in \mathbf{R}^N, \\ \left(\operatorname*{ess\,sup}_{(x,t)\in \Sigma} |F_{3n}(s;x,t)| \right)^2 \leq \varepsilon \log(1+|s|) + C_\varepsilon \quad \forall s \in \mathbf{R}, \end{cases}$$

for each $n \geq 1$.

- From the definitions of F_n and f_n, we also have that

$$F_n(z_n, \nabla z_n; \cdot) \to F_1(z, \nabla z; \cdot)z + F_2(\nabla z; \cdot) \cdot \nabla z \quad \text{weakly in } L^2(Q)$$

and

$$f_n(z_n; \cdot) \to F_3(z; \cdot)z \quad \text{weakly-}* \text{ in } L^\infty(\Sigma)$$

whenever

$$z_n \to z \quad \text{weakly-}* \text{ in } L^\infty(Q) \text{ and strongly in } L^2(0,T; H^1(\Omega)).$$

As a consequence, we can argue as in Subsection 3.1 and deduce that, for each n, there exists a control $v_n \in L^\infty(\omega \times (0,T))$ such that

$$\begin{cases} w_{n,t} - \Delta w_n + F_n(w_n, \nabla w_n; x, t) = v_n 1_\omega & \text{in } Q, \\ \dfrac{\partial w_n}{\partial n} + f_n(w_n; x, t) = 0 & \text{on } \Sigma, \\ w_n(x, 0) = w^0(x) & \text{in } \Omega \end{cases} \tag{2.43}$$

and

$$w_n(x, T) = 0 \quad \text{in} \quad \Omega. \tag{2.44}$$

In view of the properties satisfied by the functions F_{in}, the estimates we have established in Subsection 3.1 are independent of n. Accordingly, at least for a subsequence, we also have

$$v_n \to v \quad \text{weakly-}* \text{ in } L^\infty(\omega \times (0,T)),$$
$$w_{n,t} \to w_t \quad \text{weakly in } L^2(0,T; H^{-1}(\Omega))$$

and

$$w_n \to w \quad \text{weakly-}* \text{ in } L^\infty(Q) \text{ and strongly in } L^2(0,T; H^1(\Omega)).$$

Thus, we can pass to the limit in (2.43) and find a control $v \in L^\infty(\omega \times (0,T))$ such that the associated solution to (2.26) satisfies (2.30).

This ends the proof of theorem 3.

Remark 4 The proof of theorem 3 can also be achieved by applying another fixed point argument. More precisely, we can first introduce a small parameter $\varepsilon > 0$ and find a control v_ε such that the solution of (2.1) satisfies

$$\|y(\cdot, T) - \overline{y}(\cdot, T)\|_{L^2} \leq \varepsilon.$$

This can be made by previously solving an approximate controllability problem for the linear system (2.31) with the control of minimal norm in $L^p(\omega \times (0,T))$ for p large enough and, then, using Schauder's theorem. Since all the estimates we can establish are uniform in ε, we can pass to the limit as $\varepsilon \to 0$ and deduce the desired result.

Appendix : Proof of lemma 2

Let us introduce $N+1$ open subsets of $\mathcal{O}$ satisfying

$$\mathcal{O}_N = \mathcal{O}' \subset\subset \mathcal{O}_{N-1} \subset\subset ... \subset\subset \mathcal{O}_1 \subset\subset \mathcal{O}_0 \subset\subset \mathcal{O}.$$

We will also consider subintervals of $(0,T)$ of the form $(\delta/(i+1),T)$ for $0 \le i \le N$.

We will strict our considerations to the proof of lemma 2 in the case where no initial condition is imposed. The result concerning a vanishing initial condition will follow readily from the argument below.

We are first going to see that

$$y \in L^\infty(\delta/(N+1),T;H^1(\mathcal{O}_0)) \quad \text{and} \quad \Delta y \in L^2(\delta/(N+1),T;L^2(\mathcal{O}_0)),$$

with an estimate of the associated norms independent of T. To this end, let $\xi_0 \in C^2_c(\mathcal{O})$ and $\eta_0 \in C^1([0,T])$ be two functions satisfying

$$\xi_0(x) = 1 \text{ in } \mathcal{O}_0, \quad \eta_0(t) = 1 \text{ in } [\frac{\delta}{N+1},T], \quad \eta_0(0) = 0, \quad |\eta_{0,t}(t)| \le \frac{C}{\delta} \text{ in } (0,T)$$

(of course, C depends on N) and let us introduce the function $y_0 = \eta_0\,\xi_0\,y$. Then

$$\begin{cases} y_{0,t} - \Delta y_0 = f_0 & \text{in } Q, \\ y_0 = 0 & \text{on } \Sigma, \\ y_0(x,0) = 0 & \text{in } \Omega, \end{cases}$$

where

$$\begin{aligned} f_0 &= \eta_0\,\xi_0\,f + \eta_{0,t}\,\xi_0\,y - 2\eta_0\,\nabla\xi_0\cdot\nabla y - \eta_0\,\Delta\xi_0\,y \\ &\quad - a\,\eta_0\,\xi_0\,y - \eta_0\,\xi_0\,B\cdot\nabla y - \eta_0(B\cdot\nabla\xi_0)y. \end{aligned}$$

We have $f_0 \in L^2(Q)$. Consequently, $\Delta y_0 \in L^2(Q)$, $y_0 \in C^0([0,T];H^1_0(\Omega))$ and appropriate estimates are satisfied. Indeed, by multiplying the equation satisfied by y_0 by $-\Delta y_0$ and integrating with respect to x in Ω, we find

$$\frac{1}{2}\frac{d}{dt}\|\nabla y_0(\cdot,t)\|^2_{L^2} + \int_\Omega |\Delta y_0(x,t)|^2\,dx = -\int_\Omega f_0(x,t)\,\Delta y_0(x,t)\,dx. \tag{2.45}$$

Since

$$\|f_0\|_{L^2} \le C\left(\|f\|_{L^2} + (1+\delta^{-1}+\|a\|_\infty+\|B\|_\infty)\,\|y\|_Y\right),$$

we easily obtain from (2.45) that

$$\|y_0\|_{C^0([0,T];H^1_0(\Omega))} + \|\Delta y_0\|_{L^2(Q)} \le C\left(\|f\|_{L^2} + \left(1+\delta^{-1}+\|a\|_\infty+\|B\|_\infty\right)\,\|y\|_Y\right). \tag{2.46}$$

Clearly, the same estimate holds for

$$\|y\|_{L^\infty(\delta/(N+1),T;H^1(\mathcal{O}_0))} + \|\Delta y\|_{L^2(\mathcal{O}_0\times(\delta/(N+1),T))}.$$

Let us now try to improve the local space regularity properties of y. To this end, we will use the following lemma :

Lemma 3 *Let us set $p_0 = 2$, let p_i be defined by*

$$\frac{1}{p_i} = \frac{1}{p_{i-1}} - \frac{1}{2N}$$

for $1 \leq i \leq N-1$ and let us set $p_N = +\infty$. Let us denote by X_i the space

$$X_i = L^\infty((i+1)\delta/(N+1), T; W^{1,p_i}(\mathcal{O}_i))$$

for $0 \leq i \leq N$ and suppose that $y \in X_{j-1}$ for some j. Then we also have $y \in X_j$ and

$$\|y\|_{X_j} \leq C(\mathcal{O}')(T^{1/2}\|f\|_\infty + D(T, \delta, \|a\|_\infty, \|B\|_\infty)\|y\|_{X_{j-1}}),$$

where

$$D(T, \delta, \|a\|_\infty, \|B\|_\infty) = (T^{1/4} + T^{1/2})(1 + \delta^{-1} + \|a\|_\infty + \|B\|_\infty). \tag{2.47}$$

Proof of lemma 3 : Let us introduce $\xi_j \in C_c^2(\mathcal{O}_{j-1})$ and $\eta_j \in C^1([0,T])$, with

$$\xi_j(x) = 1 \text{ in } \mathcal{O}_j\,, \quad \eta_j(t) = 1 \text{ in } [(j+1)\delta/(N+1), T],$$
$$\eta_j(t) = 0 \text{ in } [0, j\delta/(N+1)], \quad |\eta_{j,t}(t)| \leq \frac{C}{\delta} \text{ in } (0,T)$$

and let us put $y_j = \eta_j\, \xi_j\, y$. Then y_j satisfies the following :

$$\begin{cases} y_{j,t} - \Delta y_j = f_j & \text{in } Q, \\ y_j = 0 & \text{on } \Sigma, \\ y_j(x,0) = 0 & \text{in } \Omega \end{cases} \tag{2.48}$$

with

$$f_j = f_{j,1} + f_{j,2} + f_{j,3}\,,$$

where

$$f_{j,1} = \eta_j\, \xi_j\, f, \quad f_{j,2} = \eta_{j,t}\, \xi_j\, y - \eta_j\, \Delta\xi_j\, y - a\, \eta_j\, \xi_j\, y - \eta_j(B \cdot \nabla\xi_j)y,$$
$$f_{j,3} = -2\eta_j\, \nabla\xi_j \cdot \nabla y - \eta_j\, \xi_j\, B \cdot \nabla y.$$

From the fact that the system (2.48) is linear, we see that y_j can be written as the sum of three solutions to similar systems with right hand sides $f_{j,1}$, $f_{j,2}$ and $f_{j,3}$. Let us respectively denote them by $y_{j,1}$, $y_{j,2}$ and $y_{j,3}$. We are now going to deduce estimates of $y_{j,k}$ in X_j for $1 \leq k \leq 3$.

To this end, we will use the usual representation of $y_{j,k}$ provided by the semigroup $S(t)$ associated to the heat equation with homogeneous Dirichlet conditions, say

$$y_{j,k}(\cdot, t) = \int_0^t S(t-s)\, f_{j,k}(\cdot, s)\, ds$$

for all $t \in (0,T)$.

Since $f \in L^\infty(Q)$, we can write

$$\|y_{j,1}(\cdot,t)\|_{W^{1,p_j}(\Omega)} \leq C \int_0^t (t-s)^{-1/2}\, \|f_{j,1}(\cdot,s)\|_{L^{p_j}(\Omega)}\, ds.$$

Therefore, from Young's inequality we find that $y_{j,1} \in L^\infty(0,T;W^{1,p_j}(\Omega))$ and

$$\begin{aligned}\|y_{j,1}\|_{L^\infty(0,T;W^{1,p_j}(\Omega))} &\leq C\,T^{1/2}\,\|f_{j,1}\|_{L^\infty(0,T;L^{p_j}(\Omega))}\\ &\leq C(\mathcal{O}')\,T^{1/2}\,\|f\|_{L^\infty(\mathcal{O}\times(0,T))}\,.\end{aligned}$$

Taking into account that $f_{j,2} \in L^\infty(0,T;L^{p^*_{j-1}}(\Omega))$ with

$$p^*_{j-1} = \begin{cases} \infty & \text{if } j > N-1,\\ p & \text{(arbitrary in } (1,+\infty)) \text{ if } j = N-1,\\ \dfrac{2N}{N-j-1} & \text{if } j < N-1,\end{cases}$$

we see that $f_{j,2}$ is not worse than $f_{j,1}$ and, again,

$$\|y_{j,2}(\cdot,t)\|_{W^{1,p_j}(\Omega)} \leq C\int_0^t (t-s)^{-1/2}\,\|f_{j,2}(\cdot,s)\|_{L^{p_j}(\Omega)}\,ds$$

for all t. From Young's inequality and the assumption $y \in X_{j-1}$, we also get $y_{j,2} \in L^\infty(0,T;W^{1,p_j}(\Omega))$ and

$$\begin{aligned}\|y_{j,2}\|_{L^\infty(0,T;W^{1,p_j}(\Omega))} &\leq C\,T^{1/2}\,\|f_{j,2}\|_{L^\infty(0,T;L^{p^*_{j-1}}(\Omega))}\\ &\leq C(\mathcal{O}')\,T^{1/2}\,(1+\delta^{-1}+\|a\|_\infty)\,\|y\|_{X_{j-1}}.\end{aligned}$$

In the definition of $f_{j,3}$, we find ∇y.

Consequently, we can only ensure that $f_{j,3} \in L^\infty(0,T;L^{p_{j-1}}(\Omega))$. Since

$$-\frac{N}{2}\left(\frac{1}{p_{j-1}}-\frac{1}{p_j}\right)-\frac{1}{2} = -\frac{3}{4},$$

we have

$$\|y_{j,3}(\cdot,t)\|_{W^{1,p_j}(\Omega)} \leq C\int_0^t (t-s)^{-3/4}\,\|f_{j,3}(\cdot,s)\|_{L^{p_{j-1}}(\Omega)}\,ds$$

and now Young's inequality gives $y_{j,3} \in L^\infty(0,T;W^{1,p_j}(\Omega))$ and

$$\begin{aligned}\|y_{j,3}\|_{L^\infty(0,T;W^{1,p_j}(\Omega))} &\leq C\,T^{1/4}\,\|f_{j,3}\|_{L^\infty(0,T;L^{p_{j-1}}(\Omega))}\\ &\leq C(\mathcal{O}')\,T^{1/4}\,(1+\|B\|_\infty)\,\|y\|_{X_{j-1}}.\end{aligned}$$

Putting the estimates of $\|y_{j,k}\|_{L^\infty(0,T;W^{1,p_j}(\Omega))}$ together and taking into account the definitions of η_j and ξ_j , we obtain the desired inequality for $\|y\|_{X_j}$.

This concludes the proof of lemma 3. □

Since we already had $y \in X_0$, we deduce from lemma 3 that $y \in X_N$ and

$$\|y\|_{X_N} \leq C(\mathcal{O}')(T^{1/2}\,\|f\|_\infty + D(T,\delta,\|a\|_\infty,\|B\|_\infty)\,\|y\|_{X_{N-1}}),$$

where, D is given by (2.47).

We can apply lemma 3 subsequently for $j = N, N-1, \ldots, 1$. The estimates we find yield

$$\|y\|_{X_N} \leq C(T^{1/2}(1 + D^{N-1})\|f\|_{L^\infty(\mathcal{O}\times(0,T))} + D^N \|y\|_{X_0}).$$

This, together with (2.46), yields

$$\|y\|_{X_N} \leq C(T^{1/2} + T^{N/2})D(T, \delta, \|a\|_\infty, \|B\|_\infty)^{N+1}(\|f\|_{L^\infty(\mathcal{O}\times(0,T))} + \|y\|_Y),$$

which is exactly (2.15).

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Chapitre 3

Local exact controllability of the Navier-Stokes system

Local exact controllability of the Navier-Stokes system

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Abstract

In this paper we deal with the local exact controllability of the Navier-Stokes system with distributed controls supported in small sets. In a first step, we present a new Carleman inequality for the linearized Navier-Stokes system, which leads to null controllability at any time $T > 0$. Then, we deduce a local result concerning the exact controllability to the trajectories of the Navier-Stokes system.

1 Introduction

Let $\Omega \subset \mathbf{R}^N$ ($N = 2$ or 3) be a bounded connected open set whose boundary $\partial\Omega$ is regular (for instance of class C^2). Let $\omega \subset \Omega$ be a (small) nonempty open subset and let $T > 0$. We will use the notation $Q = \Omega \times (0, T)$ and $\Sigma = \partial\Omega \times (0, T)$ and we will denote by $n(x)$ the outward unit normal to Ω at the point $x \in \partial\Omega$.

On the other hand, we will denote by C, C_1, C_2, ... various positive constants (usually depending on Ω and ω).

In this paper, we will be concerned with the controlled Navier-Stokes system

$$\begin{cases} y_t - \Delta y + \nabla \cdot (y \otimes y) + \nabla p = v 1_\omega & \text{in } Q, \\ \nabla \cdot y = 0 & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = y^0 & \text{in } \Omega. \end{cases} \tag{3.1}$$

Here, v stands for the control function which acts over the set ω during the time interval $(0, T)$ and we have denoted

$$(\nabla \cdot (y^1 \otimes y^2))_i = \sum_{j=1}^{N} \partial_j (y^1_i y^2_j) \quad i = 1, \dots, N.$$

Let us recall the definition of some usual spaces in the context of Navier-Stokes equations :

$$V = \{ y \in H^1_0(\Omega)^N : \nabla \cdot y = 0 \text{ in } \Omega \}$$

and

$$H = \{ y \in L^2(\Omega)^N : \nabla \cdot y = 0 \text{ in } \Omega, \; y \cdot n = 0 \text{ on } \partial\Omega \}.$$

It is clear that we cannot expect exact controllability for the Navier-Stokes equations with an arbitrary target function, in particular because of the dissipative and non reversible properties of the system.

On the other hand, approximate controllability is still an open question for this system. Some results in this direction have been obtained in [5] for different boundary conditions (Navier slip boundary conditions) and in [3] for a different nonlinearity. However, the notion of approximate controllability does not appear to be really meaningful. Indeed, even if we could reach an arbitrary neighborhood of a given target y^1 at time T by the action of a control, the question of what to do after time T to stay in the same neighbourhood would remain open.

Let us now introduce the concept of *exact controllability to the trajectories* for the Navier-Stokes system. Even if we cannot reach every element of the state space, the goal is here to reach (in finite time T) any point on any trajectory of the same operator.

Thus, let $\overline{y}$ be a solution of the uncontrolled Navier-Stokes system

$$\begin{cases} \overline{y}_t - \Delta \overline{y} + \nabla \cdot (\overline{y} \otimes \overline{y}) + \nabla \overline{p} = 0 & \text{in } Q, \\ \nabla \cdot \overline{y} = 0 & \text{in } Q, \\ \overline{y} = 0 & \text{on } \Sigma, \\ \overline{y}(0) = \overline{y}^0 & \text{in } \Omega. \end{cases} \tag{3.2}$$

The problem is to look for a control v such that at least one solution of (3.1) verifies

$$y(T) = \overline{y}(T) \text{ in } \Omega.$$

If we can find such a control, then after time T we can switch off the control and the system will follow the 'ideal' trajectory $\overline{y}$. Another point that makes this concept meaningful is that it is natural to drive our evolution system to another evolution system which obeys to the same conservation laws.

At present, we do not know any global result concerning exact controllability to the trajectories for (3.1). In this work, we give a result of *local exact controllability to the trajectories* for the Navier-Stokes equations. That is to say, for a fixed trajectory $\overline{y}$, a solution of (3.2) satisfying suitable regularity properties, we will show the existence of $\delta > 0$ such that, for every $y^0 \in X$ satisfying

$$\|\overline{y}^0 - y^0\|_X \leq \delta$$

(X is an appropriate Banach space), there exists a control v such that the corresponding solution to (3.1) verifies

$$y(T) = \overline{y}(T) \text{ in } \Omega.$$

This problem has already been treated in [11]. Local exact controllability was proved there if $\overline{y} \in W^{1,\infty}(0,T;W^{1,\infty}(\Omega)^N \cap V)$ and $\overline{y}^0 \in V$. The result we present here improves the one obtained in that article. More precisely, we suppose less regularity on the trajectory $\overline{y}$ and our assumptions are more accessible for the solutions of the Navier-Stokes systems. They are the following :

$$\overline{y} \in L^\infty(Q)^N, \quad \overline{y}_t \in L^2(0,T;L^\sigma(\Omega))^N \quad \begin{pmatrix} \sigma > 6/5 & \text{if } N=3 \\ \sigma > 1 & \text{if } N=2 \end{pmatrix}. \tag{3.3}$$

The same question was addressed in [7] for the Navier-Stokes system with boundary conditions imposed on the *curl* of the solution. The fact that we consider Dirichlet boundary conditions here, which are natural for these equations, increases a lot the mathematical difficulty of the control problem.

A relevant related control system is the linearization of (3.1) around $\overline{y}$, namely :

$$\begin{cases} y_t - \Delta y + \nabla \cdot (\overline{y} \otimes y + y \otimes \overline{y}) + \nabla p = f + v 1_\omega & \text{in } Q, \\ \nabla \cdot y = 0 & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = y^0 & \text{in } \Omega. \end{cases} \tag{3.4}$$

Our strategy is as follows :

- We first prove a global Carleman inequality for the adjoint system associated to (3.4), say :

$$\begin{cases} -\varphi_t - \Delta \varphi - D\varphi\, \overline{y} + \nabla \pi = g & \text{in } Q, \\ \nabla \cdot \varphi = 0 & \text{in } Q, \\ \varphi = 0 & \text{on } \Sigma, \\ \varphi(T) = \varphi^0 & \text{in } \Omega, \end{cases} \tag{3.5}$$

where

$$D\varphi = \nabla \varphi + \nabla \varphi^t.$$

This inequality constitutes the first important result of the present paper and is given in the following theorem :

Theorem 5 *Let us suppose that* (3.3) *holds. Then, there exist three positive constants* $\overline{s}$, $\overline{\lambda}$, C *depending on* Ω *and* ω *such that, for every* $\varphi^0 \in H$ *and* $g \in L^2(Q)^N$*, the corresponding solution to* (3.5) *verifies*

$$\begin{aligned} & s^3\lambda^4 \iint_Q e^{-2s\alpha}\xi^3|\varphi|^2\,dx\,dt + s\lambda^2 \iint_Q e^{-2s\alpha}\xi|\nabla\varphi|^2\,dx\,dt \\ & + s^{-1} \iint_Q e^{-2s\alpha}\xi^{-1}\left(|\varphi_t|^2 + |\Delta\varphi|^2\right)dx\,dt \\ & \qquad \le C(1+T^2)\left(s^{15/2}\lambda^{20} \iint_Q e^{-4s\widehat{\alpha}+2s\alpha^*}\widehat{\xi}^{15/2}|g|^2\,dx\,dt \right. \\ & \qquad \left. + s^{16}\lambda^{40} \iint_{\omega\times(0,T)} e^{-8s\widehat{\alpha}+6s\alpha^*}\widehat{\xi}^{16}|\varphi|^2\,dx\,dt \right) \end{aligned} \tag{3.6}$$

for all $\lambda \ge \overline{\lambda}(1 + \|\overline{y}\|_\infty + \|\overline{y}_t\|^2_{L^2(0,T;L^\sigma(\Omega)^N)} + e^{\overline{\lambda}T\|\overline{y}\|^2_\infty})$ *and* $s \ge \overline{s}(T^4 + T^8)$ *and appropriate positive weight functions* α, ξ, $\widehat{\alpha}$, α^*, $\widehat{\xi}$ *which will be defined in* (3.8).

Concerning the proof of theorem 5, we present now a useful remark :

Remark 5 Let $\varphi^0 \in H$ and $g \in L^2(Q)^N$ be given and let (φ, π) be the corresponding solution to (3.5). We know that $\varphi \in L^2(0,T;V) \cap C^0([0,T];H)$. Now, let $\gamma \in C^1([0,T])$ verify $\gamma(T) = 0$. Then $(\widetilde{\varphi}, \widetilde{\pi}) := (\gamma\varphi, \gamma\pi)$ solves the system

$$\begin{cases} -\widetilde{\varphi}_t - \Delta\widetilde{\varphi} - D\widetilde{\varphi}\,\overline{y} + \nabla\widetilde{\pi} = \gamma g - \gamma'\varphi & \text{in } Q, \\ \nabla \cdot \widetilde{\varphi} = 0 & \text{in } Q, \\ \widetilde{\varphi} = 0 & \text{on } \Sigma, \\ \widetilde{\varphi}(T) = 0 & \text{in } \Omega. \end{cases} \tag{3.7}$$

Therefore, $(\widetilde{\varphi}, \widetilde{\pi})$ is a strong solution of (3.7). In particular, we have

$$\pi(t) \in H^1(\Omega),\ \varphi(t) \in H^2(\Omega)^N,\ \varphi_t(t) \in L^2(\Omega)^N$$

for almost every $t \in (0,T)$.

The estimate (3.6) is a new global Carleman inequality, proved by an original method. However, the plan of the proof follows the one in [11] and contains four parts.

First part : We look at (3.5) as a system of N heat equations and we use a Carleman estimate for the heat equation, which is 'classic' by now. For a proof, see for instance [9]. This provides an estimate of the velocity vector field in terms of the pressure.

Second part : From the elliptic equation satisfied by the pressure, we can deduce an estimate of the pressure in terms of its trace on the boundary and the velocity vector field. This estimate was obtained in [12].

Third part : Following the ideas of [11] and using regularity results for the Stokes system, we can also obtain an estimate of the trace of the pressure.

At this point of the proof, we will have a Carleman inequality with 'good' terms in the left hand side and local in space integrals of the pressure and the velocity vector field in the right hand side. This estimate has its own interest and allows us to deal with controllability problems where the control function acts not only as a right hand side in the momentum equation but on the divergence of the velocity as well. By an extension of our open set Ω, we can in this way solve the case where the control function acts over a (small) part of the boundary.

Fourth part : The rest of the proof is devoted to estimate of the local term of the pressure. The idea is to work with the Laplacian and the time derivative of the velocity instead of the pressure term. It is not possible to obtain a local estimate of the L^2 norm of the pressure just in terms of a local L^2 norm of the velocity vector field. In addition, we have not been able to estimate the local term involving the time derivative of the velocity in terms of local norms of the velocity. This way, we are led to consider global systems, see (3.30) and (3.31). Moreover, as far as the local term involving the time derivative is concerned, some technical considerations and complicated computations lead us to impose the hypotheses on $\overline{y}$ given in (3.3) to be able to conclude the proof.

Finally, this Carleman estimate allows us to deduce a null controllability result for system (3.4) with a right hand side satisfying suitable decreasing properties near $t = T$ (see proposition 9 below).

• The second main result of the present paper concerns the local exact controllability to the trajectories for the Navier-Stokes system. It is given in the following theorem :

Theorem 6 *Let $\overline{y}$ be a solution of* (3.2) *with* $\overline{y}^0 \in L^{2N-2}(\Omega)^N \cap H$ *and assume that* (3.3) *holds. Then there exists* $\delta > 0$ *such that, for any* $y^0 \in L^{2N-2}(\Omega)^N \cap H$ *satisfying* $\|y^0 - \overline{y}^0\|_{L^{2N-2}(\Omega)^N} \leq \delta$, *we can find a control* $v \in L^2(\omega \times (0,T))^N$ *and an associated solution* (y,p) *to* (3.1) *such that*

$$y(T) = \overline{y}(T) \text{ in } \Omega.$$

That is to say, the local exact controllability to the trajectories holds in the space $X = L^{2N-2}(\Omega)^N \cap H$.

The proof of this theorem follows the ideas of [11] and is based on the application of an inverse mapping theorem.

Let us remark that the 'ideal' hypotheses on $\overline{y}$ are still far to be reached. In particular, it would be very interesting to know whether a similar result holds without imposing the L^∞ property on $\overline{y}$.

The paper is organized as follows : we prove theorem 5 in section 2. Section 3 deals with the null controllability result for the linear control system with a right hand side. Finally, the proof of theorem 6 is given in the fourth section.

2 A new Carleman inequality

In this section we will deduce the Carleman inequality given in theorem 5. To this end, we first define several weight functions which will be useful in the sequel.

The basic weight will be a function $\eta^0 \in C^2(\overline{\Omega})$ verifying

$$\eta^0 > 0 \text{ in } \Omega, \quad \eta^0 = 0 \text{ on } \partial\Omega, \quad |\nabla \eta^0| > 0 \text{ in } \overline{\Omega \setminus \omega_1},$$

where $\omega_1 \subset\subset \omega$ is a nonempty open set. The existence of such a function η^0 is proved in [9]. Then, for some positive real numbers s and λ, we introduce

$$\begin{aligned}
\alpha(x,t) &= \frac{e^{5/4\lambda m\|\eta^0\|_\infty} - e^{\lambda(m\|\eta^0\|_\infty + \eta^0(x))}}{t^4(T-t)^4}, \\
\xi(x,t) &= \frac{e^{\lambda(m\|\eta^0\|_\infty + \eta^0(x))}}{t^4(T-t)^4}, \\
\widehat{\alpha}(t) = \min_{x\in\overline{\Omega}} \alpha(x,t) &= \frac{e^{5/4\lambda m\|\eta^0\|_\infty} - e^{\lambda(m+1)\,\|\eta^0\|_\infty}}{t^4(T-t)^4}, \\
\alpha^*(t) = \max_{x\in\overline{\Omega}} \alpha(x,t) &= \frac{e^{5/4\lambda m\|\eta^0\|_\infty} - e^{\lambda m\|\eta^0\|_\infty}}{t^4(T-t)^4}, \\
\widehat{\xi}(t) = \max_{x\in\overline{\Omega}} \xi(x,t) = \frac{e^{\lambda(m+1)\|\eta^0\|_\infty}}{t^4(T-t)^4}, &\quad \xi^*(t) = \min_{x\in\overline{\Omega}} \xi(x,t) = \frac{e^{\lambda m\|\eta^0\|_\infty}}{t^4(T-t)^4}, \\
\widehat{\theta}(t) = s\lambda e^{-s\widehat{\alpha}}\widehat{\xi}, &\quad \theta(t) = s^{15/4} e^{-2s\widehat{\alpha} + s\alpha^*} \widehat{\xi}^{15/4},
\end{aligned} \tag{3.8}$$

where $m > 4$ is a fixed real number.

Along the proof, we will also use the notation

$$I(s,\lambda;\varphi) = s^{-1}\iint_Q e^{-2s\alpha}\xi^{-1}(|\varphi_t|^2 + |\Delta\varphi|^2)\,dx\,dt$$
$$+s\lambda^2\iint_Q e^{-2s\alpha}\xi|\nabla\varphi|^2\,dx\,dt + s^3\lambda^4\iint_Q e^{-2s\alpha}\xi^3|\varphi|^2\,dx\,dt.$$

For an easier comprehension, we divide the proof in several steps :

STEP 1. *Carleman estimate for the heat equation.*

We will first apply the usual Carleman inequality for the heat equation with right hand side in $L^2(Q)$ to the equation satisfied by φ_i, for which the right hand side is

$$G_i = g_i + (D\varphi\,\overline{y})_i - \partial_i\pi.$$

This can be done since $G_i \in L^2(e^{-2s\widehat{\alpha}}(0,T);L^2(\Omega))$ (from Remark 1) and $e^{-2s\alpha} \le e^{-2s\widehat{\alpha}}$. This Carleman estimate can be found in [9] (for the explicit dependence with respect to λ, s and T, see for instance [5]).

Consequently, there exist a positive constant $C_1(\Omega,\omega)$ and two numbers $\lambda_0(\Omega,\omega) \ge 1$, $s_0(\Omega,\omega) > 0$ such that

$$\begin{aligned} I(s,\lambda;\varphi) \le C_1 \Bigg(&\iint_Q e^{-2s\alpha}(|g|^2 + |D\varphi\,\overline{y}|^2 + |\nabla\pi|^2)\,dx\,dt \\ &+s^3\lambda^4 \iint_{\omega_1\times(0,T)} e^{-2s\alpha}\xi^3|\varphi|^2\,dx\,dt \Bigg), \end{aligned} \tag{3.9}$$

for all $\lambda \ge \lambda_0$ and $s \ge s_0(T^7 + T^8)$ (a proof of (3.9) can be achieved taking into account that

$$|\xi^{-1}| \le C\,T^8 \quad \text{and} \quad |\alpha_t| \le C\,T\,\xi^{5/4}$$

for some $C > 0$ independent of λ and, then, arguing as in [5]).

Now, we eliminate the term involving $D\varphi\,\overline{y}$ in the right hand side of (3.9), taking into account that

$$C_1\,|D\varphi\,\overline{y}|^2 \le Cs\|\overline{y}\|_\infty^2\xi|\nabla\varphi|^2 \le \frac{1}{2}s\lambda^2\xi|\nabla\varphi|^2,$$

for $\lambda \ge \lambda_1(\Omega,\omega)\|\overline{y}\|_\infty$ and $s \ge s_1(\Omega,\omega)T^8$. We deduce that

$$\begin{aligned} I(s,\lambda;\varphi) \le C_2 \Bigg(&\iint_Q e^{-2s\alpha}(|g|^2 + |\nabla\pi|^2)\,dx\,dt \\ &+s^3\lambda^4 \iint_{\omega_1\times(0,T)} e^{-2s\alpha}\xi^3|\varphi|^2\,dx\,dt \Bigg), \end{aligned} \tag{3.10}$$

for all $\lambda \ge \lambda_2(\Omega,\omega)(1 + \|\overline{y}\|_\infty)$ and $s \ge s_2(\Omega,\omega)(T^7 + T^8)$.

STEP 2. *Estimate of the pressure.*

In this step we bound the integral of $|\nabla\pi|^2$ in the right hand side of (3.10) in terms of a local integral of $|\pi|^2$, a term concerning the trace of π and other two global terms involving $|g|^2$ and $|\nabla\varphi|^2$; the last one will be absorbed later on by the corresponding integral appearing in $I(s,\lambda;\varphi)$.

This estimate will be made with the help of an elliptic Carleman inequality applied to the (weak) differential equation satisfied by the pressure ; this inequality was proved in [12].

Indeed, applying the divergence operator to the first equation in (3.5)), we see that

$$\Delta\pi(t) = \nabla\cdot(D\varphi\,\overline{y} + g)(t) \quad \text{in } \Omega, \tag{3.11}$$

for almost every $t\in(0,T)$. Notice that the right hand side of (3.11) belongs to $H^{-1}(\Omega)$. From Remark 5, we also know that $\pi(t)\in H^1(\Omega)$. We can apply here the main result in [12] (see inequality (2.10) in this reference), which tells that there exist a constant $\overline{C}_1(\Omega,\omega)>0$ and two numbers $\overline{\lambda}>1$, $\overline{\tau}>1$ such that

$$\begin{aligned}\int_\Omega e^{2\tau\eta}|\nabla\pi(t)|^2\,dx \le \overline{C}_1\Big(&\tau\int_\Omega e^{2\tau\eta}\eta(|D\varphi\,\overline{y}|^2+|g|^2)(t)\,dx\\ &+\tau^{1/2}e^{2\tau}\|\pi(t)\|^2_{H^{1/2}(\partial\Omega)} + \int_{\omega_1} e^{2\tau\eta}(|\nabla\pi|^2+\tau^2\lambda^2\eta^2|\pi|^2)(t)\,dx\Big),\end{aligned} \tag{3.12}$$

for all $\lambda\ge\overline{\lambda}$ and $\tau\ge\overline{\tau}$, where η is given by

$$\eta(x) = e^{\lambda\eta^0(x)} \quad \forall x\in\Omega.$$

We can eliminate the local integral of $|\nabla\pi|^2$ in (3.12) at the price of integrating $|\pi|^2$ in a open set ω_2 satisfying $\omega_1\subset\subset\omega_2\subset\subset\omega$. To this end, let us introduce a function $\zeta\in C^2_c(\omega_2)$ such that

$$\zeta\equiv 1 \text{ in } \omega_1, \quad 0\le\zeta\le 1$$

and let us integrate by parts several times :

$$\begin{aligned}&\int_{\omega_1} e^{2\tau\eta}|\nabla\pi(t)|^2\,dx \le \int_{\omega_2} e^{2\tau\eta}\zeta\nabla\pi(t)\cdot\nabla\pi(t)\,dx\\ &= -\frac12\int_{\omega_2}\nabla(e^{2\tau\eta}\zeta)\cdot\nabla|\pi(t)|^2\,dx - \langle e^{2\tau\eta}\Delta\pi(t),\zeta\pi(t)\rangle_{H^{-1}(\omega_2),H^1_0(\omega_2)}\\ &= \frac12\int_{\omega_2}\Delta(e^{2\tau\eta}\zeta)|\pi(t)|^2\,dx\\ &-\langle e^{2\tau\eta}\nabla\cdot(D\varphi\,\overline{y}+g)(t),\zeta\pi(t)\rangle_{H^{-1}(\omega_2),H^1_0(\omega_2)}.\end{aligned} \tag{3.13}$$

Since

$$|\Delta(e^{2\tau\eta}\zeta)| \le 2\overline{C}_2\tau^2\lambda^2\eta^2e^{2\tau\eta} \quad \text{in } \omega_2$$

for $\lambda\ge\overline{\lambda}_0(\Omega,\omega)$ and for some constant $\overline{C}_2(\Omega,\omega)>0$, the first term in the right hand side can be estimated by

$$\overline{C}_2\tau^2\lambda^2\int_{\omega_2} e^{2\tau\eta}\eta^2|\pi(t)|^2\,dx.$$

We integrate by parts again in the other term and we obtain

$$
\begin{aligned}
&-\langle e^{2\tau\eta}\nabla\cdot(D\varphi\,\overline{y}+g)(t),\zeta\pi(t)\rangle_{H^{-1}(\omega_2),H_0^1(\omega_2)}\\
&=\int_{\omega_2}\nabla(e^{2\tau\eta}\zeta)\cdot(D\varphi\,\overline{y}+g)(t)\pi(t)\,dx\\
&+\int_{\omega_2}e^{2\tau\eta}\zeta(D\varphi\,\overline{y}+g)(t)\cdot\nabla\pi(t)\,dx\\
&\leq\overline{C}_4\left(\tau^2\lambda^2\int_{\omega_2}e^{2\tau\eta}\eta^2|\pi(t)|^2\,dx+\int_{\omega_2}e^{2\tau\eta}(|D\varphi\,\overline{y}|^2+|g|^2)(t)\,dx\right)\\
&+\frac{1}{2}\int_{\omega_2}e^{2\tau\eta}\zeta|\nabla\pi(t)|^2\,dx,
\end{aligned}
$$

for a constant $\overline{C}_4(\Omega,\omega)>0$, since

$$
|\nabla(e^{2\tau\eta}\zeta)|\leq\overline{C}_3(\Omega,\omega)\tau\lambda\eta e^{2\tau\eta}\quad\text{ in }\omega_2
$$

for $\lambda\geq\overline{\lambda}_1(\Omega,\omega)$. From (3.13), we have

$$
\begin{aligned}
\int_{\omega_1}e^{2\tau\eta}|\nabla\pi(t)|^2\,dx\leq\overline{C}_5\Bigg(&\tau^2\lambda^2\int_{\omega_2}e^{2\tau\eta}\eta^2|\pi(t)|^2\,dx\\
&+\int_{\omega_2}e^{2\tau\eta}(|D\varphi\,\overline{y}|^2+|g|^2)(t)\,dx\Bigg)
\end{aligned}
$$

for $\lambda\geq\overline{\lambda}_2(\Omega,\omega)$ which, together with (3.12), gives

$$
\begin{aligned}
\int_{\Omega}e^{2\tau\eta}|\nabla\pi(t)|^2\,dx\leq\overline{C}_6\Bigg(&\tau\int_{\Omega}e^{2\tau\eta}\eta(|D\varphi\,\overline{y}|^2+|g|^2)(t)\,dx\\
&+\tau^{1/2}e^{2\tau}\|\pi(t)\|^2_{H^{1/2}(\partial\Omega)}+\tau^2\lambda^2\int_{\omega_2}e^{2\tau\eta}\eta^2|\pi(t)|^2\,dx\Bigg),
\end{aligned}
$$

for $\lambda\geq\overline{\lambda}_2$ and $\tau\geq\overline{\tau}$.

To connect this elliptic estimate with (3.10), we put

$$
\tau=\frac{s}{t^4(T-t)^4}\,e^{\lambda m\|\eta^0\|_\infty},
$$

we multiply by

$$
\exp\left\{-2s\frac{e^{5/4\lambda m\|\eta^0\|_\infty}}{t^4(T-t)^4}\right\}
$$

and we integrate with respect to t in $(0,T)$. Let us remark that the last choice of τ will be greater than $\overline{\tau}$ if we take $s \geq (\overline{\tau}/2^8)\, T^8$, so we get

$$\begin{aligned}&\iint_Q e^{-2s\alpha}|\nabla\pi|^2\,dx\,dt \leq \overline{C}_7 \Bigg(s\iint_Q e^{-2s\alpha}\xi|g|^2\,dx\,dt \\ &+s\iint_Q e^{-2s\alpha}\xi|D\varphi\,\overline{y}|^2\,dx\,dt + s^{1/2}\int_0^T e^{-2s\alpha^*}(\xi^*)^{1/2}\|\pi(t)\|^2_{H^{1/2}(\partial\Omega)}\,dt \\ &+s^2\lambda^2\iint_{\omega_2\times(0,T)} e^{-2s\alpha}\xi^2|\pi|^2\,dx\,dt\Bigg),\end{aligned} \tag{3.14}$$

for all $\lambda \geq \overline{\lambda}_2$ and $s \geq \overline{s}_0 T^8$.

STEP 3. *Estimate of the trace of* π.

Following [11], we will use classical estimates for the Stokes system. This will provide an estimate of the third term in the right hand side of (3.14) by global integrals involving $|g|^2$, $|\varphi|^2$ and $|\nabla\varphi|^2$ with suitable powers of the parameters s and λ. Combining this estimate and the inequalities (3.10) and (3.14), we will then be able to absorb the integrals of $|\varphi|^2$ and $|\nabla\varphi|^2$ with $I(s,\lambda;\varphi)$. In this way, we will find an evidence of the relevance of the terms $t^4(T-t)^4$ in the definitions of α and ξ.

Let us introduce the functions

$$\varphi^* = s^{1/4}e^{-s\alpha^*}(\xi^*)^{1/4}\varphi, \quad \pi^* = s^{1/4}e^{-s\alpha^*}(\xi^*)^{1/4}\pi,$$

which fulfill the system

$$\begin{cases} -\varphi^*_t - \Delta\varphi^* + \nabla\pi^* = g^* & \text{in } Q, \\ \nabla\cdot\varphi^* = 0 & \text{in } Q, \\ \varphi^* = 0 & \text{on } \Sigma, \\ \varphi^*(T) = 0 & \text{in } \Omega, \end{cases}$$

with

$$g^* = s^{1/4}e^{-s\alpha^*}(\xi^*)^{1/4}g + s^{1/4}e^{-s\alpha^*}(\xi^*)^{1/4}D\varphi\,\overline{y} - s^{1/4}(e^{-s\alpha^*}(\xi^*)^{1/4})_t\varphi.$$

Using well known regularity properties of the evolution Stokes equation (see, for instance, [19]), we deduce that $\varphi^* \in L^2(0,T;H^2(\Omega)^N)\cap L^\infty(0,T;V)$, $\varphi^*_t \in L^2(0,T;H)$, $\pi^* \in L^2(0,T;H^1(\Omega))$ and, also, that these functions are bounded in these spaces by the L^2-norm of the right hand side. In particular,

$$\iint_Q (|\pi^*|^2 + |\nabla\pi^*|^2)\,dx\,dt \leq \overline{C}_8 \iint_Q |g^*|^2\,dx\,dt$$

and, consequently, the following holds :

$$
\begin{aligned}
&\int_0^T \|\pi^*(t)\|^2_{H^{1/2}(\partial\Omega)}\,dt \\
&\qquad \leq \overline{C}_9 \left(s^{1/2} \iint_Q e^{-2s\alpha^*}(\xi^*)^{1/2}|g|^2\,dx\,dt \right. \\
&\qquad + s^{1/2}\|\overline{y}\|^2_\infty \iint_Q e^{-2s\alpha^*}(\xi^*)^{1/2}|\nabla\varphi|^2\,dx\,dt \\
&\qquad \left. + s^{1/2} \iint_Q |(e^{-s\alpha^*}(\xi^*)^{1/4})_t|^2|\varphi|^2\,dx\,dt \right).
\end{aligned}
\tag{3.15}
$$

Taking into account the definitions of α^* and ξ^* (see (3.8)), we see that the first two integrals in the right hand side of (3.15) can be estimated by

$$
s\|\overline{y}\|^2_\infty \iint_Q e^{-2s\alpha}\xi|\nabla\varphi|^2\,dx\,dt + s\iint_Q e^{-2s\alpha}\xi|g|^2\,dx\,dt,
$$

for $s \geq \overline{s}_1 T^8$.

Finally, we obtain an estimate of the time derivative of the weight function $e^{-s\alpha^*}(\xi^*)^{1/4}$:

$$
\begin{aligned}
&(e^{-s\alpha^*}(\xi^*)^{1/4})_t = e^{-s\alpha^*}(-s\alpha^*_t(\xi^*)^{1/4} + 1/4(\xi^*)^{-3/4}\xi^*_t) \\
&\leq \overline{C}_{10}e^{-s\alpha^*}(sT(\xi^*)^{3/2} + T(\xi^*)^{1/2}) \leq \overline{C}_{11}e^{-s\alpha^*}sT(\xi^*)^{3/2},
\end{aligned}
$$

for a constant $\overline{C}_{11} > 0$ independent of λ, where we have taken $s \geq \overline{s}_2 T^8$. With this, we can estimate the last term in (3.15). In view of (3.14), we obtain :

$$
\begin{aligned}
&\iint_Q e^{-2s\alpha}|\nabla\pi|^2\,dx\,dt \leq \overline{C}_{12}\left(s\|\overline{y}\|^2_\infty \iint_Q e^{-2s\alpha}\xi|\nabla\varphi|^2\,dx\,dt \right. \\
&\quad + s\iint_Q e^{-2s\alpha}\xi|g|^2\,dx\,dt + s^{5/2}T^2 \iint_Q e^{-2s\alpha}\xi^3|\varphi|^2\,dx\,dt \\
&\quad \left. + s^2\lambda^2 \iint_{\omega_2\times(0,T)} e^{-2s\alpha}\xi^2|\pi|^2\,dx\,dt \right),
\end{aligned}
\tag{3.16}
$$

for all $\lambda \geq \overline{\lambda}_0$ and $s \geq \overline{s}_3 T^8$.

Now we plug this inequality into (3.10) and we get

$$
\begin{aligned}
&I(s,\lambda;\varphi) \leq C_3 \left(s^3\lambda^4 \iint_{\omega_1\times(0,T)} e^{-2s\alpha}\xi^3|\varphi|^2\,dx\,dt \right. \\
&+ s^2\lambda^2 \iint_{\omega_2\times(0,T)} e^{-2s\alpha}\xi^2|\pi|^2\,dx\,dt + s\iint_Q e^{-2s\alpha}\xi|g|^2\,dx\,dt\ , \\
&\left. + s^{5/2}T^2 \iint_Q e^{-2s\alpha}\xi^3|\varphi|^2\,dx\,dt + s\|\overline{y}\|^2_\infty \iint_Q e^{-2s\alpha}\xi|\nabla\varphi|^2\,dx\,dt \right),
\end{aligned}
\tag{3.17}
$$

for $\lambda \geq \lambda_3(1+\|\overline{y}\|_\infty)$ and $s \geq s_3(T^7+T^8)$.

As mentioned above, we can now absorb the last two terms in (3.17) just taking $s \geq s_4 T^4$ and $\lambda \geq \lambda_4 \|\overline{y}\|_\infty$ in such a way that

$$C_3 s^{5/2} T^2 \leq \frac{1}{2} s^3, \quad C_3 \|\overline{y}\|_\infty^2 \leq \frac{1}{2}\lambda^2.$$

Therefore, we get the inequality

$$\begin{aligned} I(s,\lambda;\varphi) \leq C_4 \Bigg(& s^3\lambda^4 \iint_{\omega_1\times(0,T)} e^{-2s\alpha}\xi^3|\varphi|^2\,dx\,dt \\ & + s^2\lambda^2 \iint_{\omega_2\times(0,T)} e^{-2s\alpha}\xi^2|\pi|^2\,dx\,dt + s\iint_Q e^{-2s\alpha}\xi|g|^2\,dx\,dt \Bigg), \end{aligned} \tag{3.18}$$

for $\lambda \geq \lambda_5(1+\|\overline{y}\|_\infty)$ and $s \geq s_5(T^4+T^8)$.

The rest of the proof is intended to eliminate the local term of the pressure appearing in the right hand side of (3.18). Two main difficulties appear : to obtain a local estimate of the pressure in terms of a local term of the velocity vector field is not an easy task in Stokes systems like (3.5) and the fact that the weight function multiplying the pressure depends on x complicates the argument a lot.

Accordingly, our strategy will be the following. We first replace the weight in the local integral of the pressure by another one that does not depend on x, but just on t. This will allow us to reduce our problem to an estimate of an integral of $|\nabla\pi|^2$ instead of $|\pi|^2$. Then, using the equation verified by φ and π, the goal will be to estimate two local integrals involving $|\Delta\varphi|^2$ and $|\varphi_t|^2$. The integral of $|\Delta\varphi|^2$ will be treated in the fourth step and we will deal with the integral of $|\varphi_t|^2$ in step 5.

Indeed, the definitions of $\widehat{\alpha}$, $\widehat{\xi}$ and $\widehat{\theta}$ (see (3.8)) readily give

$$s^2\lambda^2 \iint_{\omega_2\times(0,T)} e^{-2s\alpha}\xi^2|\pi|^2\,dx\,dt \leq \iint_{\omega_2\times(0,T)} |\widehat{\theta}|^2|\pi|^2\,dx\,dt.$$

We can take $\pi(t)$ to satisfy

$$\int_{\omega_2} \pi(t)\,dx = 0$$

for each $t \in (0,T)$. So, using Poincare-Wirtinger's inequality, there exists $C_5 > 0$ such that

$$\iint_{\omega_2\times(0,T)} |\widehat{\theta}|^2|\pi|^2\,dx\,dt \leq C_5 \iint_{\omega_2\times(0,T)} |\widehat{\theta}|^2|\nabla\pi|^2\,dx\,dt.$$

Then, from the equation in (3.5), we have

$$\begin{aligned} s^2\lambda^2 \iint_{\omega_2\times(0,T)} e^{-2s\alpha}\xi^2|\pi|^2\,dx\,dt \le C_6 \Bigg(\iint_{\omega_2\times(0,T)} |\widehat{\theta}|^2|g|^2\,dx\,dt \\ + \iint_{\omega_2\times(0,T)} |\widehat{\theta}|^2|\Delta\varphi|^2\,dx\,dt + \iint_{\omega_2\times(0,T)} |\widehat{\theta}|^2|\varphi_t|^2\,dx\,dt \\ + \|\overline{y}\|_\infty^2 \iint_{\omega_2\times(0,T)} |\widehat{\theta}|^2|\nabla\varphi|^2\,dx\,dt \Bigg). \end{aligned} \tag{3.19}$$

STEP 4. *Estimate of the local integral of* $|\Delta\varphi|^2$.

In this step we will bound the second term in the right hand side of (3.19). Let us first introduce two open sets ω_3 and ω_4 such that

$$\omega_2 \subset\subset \omega_3 \subset\subset \omega_4 \subset\subset \omega$$

and a function $\rho \in \mathcal{D}(\omega_4)$ with $\rho \equiv 1$ in ω_3.

Secondly, let us put

$$u(x,t) = \widehat{\theta}(t)\rho(x)\Delta\varphi(x,T-t) \quad \text{in } \mathbf{R}^N \times (0,T).$$

Let us remark that, here, u is extended by zero outside ω_4. The goal of this step is to estimate

$$\iint_{\omega_2\times(0,T)} |u|^2\,dx\,dt.$$

Let us first see which is the heat equation satisfied by u. Thus, applying Laplace's operator to the equation verified by φ and keeping in mind (3.11), we get

$$(\Delta\varphi(T-t))_t - \Delta(\Delta\varphi(T-t)) = f \quad \text{in } Q, \tag{3.20}$$

where

$$f = \Delta(D\varphi\,\overline{y})(T-t) + \Delta g(T-t) - \nabla(\nabla\cdot(D\varphi\,\overline{y})(T-t)) - \nabla(\nabla\cdot g(T-t)).$$

From (3.20), we deduce that

$$\begin{cases} u_t - \Delta u = F & \text{in } \mathbf{R}^N \times (0,T), \\ u(0) = 0 & \text{in } \mathbf{R}^N, \end{cases} \tag{3.21}$$

where

$$F = \widehat{\theta}\rho f + \widehat{\theta}'\rho\Delta\varphi(T-t) - 2\widehat{\theta}\nabla\rho\cdot\nabla\Delta\varphi(T-t) - \widehat{\theta}\Delta\rho\Delta\varphi(T-t).$$

Notice that $F \in L^2(0,T;H^{-2}(\mathbf{R}^N)^N)$ and we a priori know that $u \in L^2(\mathbf{R}^N \times (0,T))^N$ (Remark 5). From the equation in (3.21), we have that $u_t \in L^2(0,T;H^{-2}(\mathbf{R}^N)^N)$, so that $u(0)$

makes sense. On the other hand, it is possible to show that (3.21) possesses exactly one solution in this class.

Now, we rewrite F in a more appropriate way, so that it is given by the sum of two functions : in the first one, we collect all the terms where we find second partial derivatives of g, $D\varphi\,\overline{y}$ and φ ; in the second one, we include all the other terms. Notice that this second function has a support contained in $\omega_4 \setminus \overline{\omega}_3$ (because derivatives of ρ appear everywhere).

More precisely, we put $F = F_1 + F_2$, with

$$F_1 = \widehat{\theta}\Delta(\rho(D\varphi\,\overline{y})(T-t)) + \widehat{\theta}\Delta(\rho g(T-t)) - \widehat{\theta}\,\nabla(\nabla\cdot(\rho(D\varphi\,\overline{y})(T-t))) \\ -\widehat{\theta}\nabla(\nabla\cdot(\rho\, g(T-t))) + \widehat{\theta}'\Delta(\rho\varphi(T-t))$$

and

$$F_2 = -2\widehat{\theta}\nabla\rho\cdot\nabla(D\varphi\,\overline{y})(T-t) - \widehat{\theta}\Delta\rho(D\varphi\,\overline{y})(T-t) - 2\widehat{\theta}\nabla\rho\cdot\nabla g(T-t) \\ -\widehat{\theta}\Delta\rho g(T-t) + \widehat{\theta}\nabla(\nabla\rho\cdot(D\varphi\,\overline{y})(T-t)) + \widehat{\theta}\nabla\rho(\nabla\cdot(D\varphi\,\overline{y})(T-t)) \\ \widehat{\theta}\nabla(\nabla\rho\cdot g(T-t)) + \widehat{\theta}\nabla\rho(\nabla\cdot g(T-t)) - 2\widehat{\theta}'\nabla\rho\cdot\nabla\varphi(T-t) \\ -\widehat{\theta}'\Delta\rho\varphi(T-t) - 2\widehat{\theta}\nabla\rho\cdot\nabla\Delta\varphi(T-t) - \widehat{\theta}\Delta\rho\Delta\varphi(T-t).$$

Notice that $F, F_1 \in L^2(0,T;H^{-2}(\mathbf{R}^N)^N)$, while $F_2 \in L^2(0,T;H^{-1}(\mathbf{R}^N)^N)$. If we were able to find two functions u^1 and u^2 in $L^2(\mathbf{R}^N\times(0,T))^N$ satisfying

$$\begin{cases} u^i_t - \Delta u^i = F_i & \text{in } \mathbf{R}^N\times(0,T), \\ u^i(0) = 0 & \text{in } \mathbf{R}^N \end{cases} \tag{3.22}$$

for $i = 1,2$, then we would have $u = u^1 + u^2$, and it would suffice to estimate the integrals

$$\iint_{\omega_2\times(0,T)} |u^i|^2\,dx\,dt.$$

STEP 4.1. *Definition and estimate of u^1.*

By definition, we will say that u^1 is the solution by transposition of the Cauchy problem for the heat equation (3.22) for $i = 1$. This means that u^1 is the unique function in $L^2(\mathbf{R}^N\times(0,T))^N$ that, for each $h \in L^2(\mathbf{R}^N\times(0,T))^N$, one has

$$\begin{aligned} &\iint_{\mathbf{R}^N\times(0,T)} u^1\cdot h\,dx\,dt \\ &\quad= \iint_{\mathbf{R}^N\times(0,T)} (\widehat{\theta}\rho\,(g + D\varphi\,\overline{y})(T-t))\cdot\Delta z)\,dx\,dt \\ &\quad- \iint_{\mathbf{R}^N\times(0,T)} \widehat{\theta}\rho(D\varphi\,\overline{y})(T-t)\cdot\nabla(\nabla\cdot z)\,dx\,dt \\ &\quad- \iint_{\mathbf{R}^N\times(0,T)} \widehat{\theta}\rho g(T-t)\cdot\nabla(\nabla\cdot z)\,dx\,dt \\ &\quad+ \iint_{\mathbf{R}^N\times(0,T)} \widehat{\theta}'\rho\varphi(T-t)\cdot\Delta z\,dx\,dt, \end{aligned} \tag{3.23}$$

where z is the solution of

$$\begin{cases} -z_t - \Delta z = h & \text{in } \mathbf{R}^N \times (0,T), \\ z(T) = 0 & \text{in } \mathbf{R}^N. \end{cases} \tag{3.24}$$

Remark that, for every $h \in L^2(\mathbf{R}^N \times (0,T))^N$, (3.24) possesses exactly one solution $z \in L^2(0,T;H^2(\mathbf{R}^N)^N)$ that depends continuously on h. Therefore, u_1 is well defined and

$$\|u^1\|_{L^2(\mathbf{R}^N\times(0,T))^N} \le \widehat{C}_1 \|hF_1\|_{L^2(0,T;H^{-2}(\mathbf{R}^N)^N)}, \tag{3.25}$$

for a positive constant $\widehat{C}_1$. Furthermore, it is not difficult to show that $u^1 \in C^0([0,T];H^{-2}(\mathbf{R}^N)^N)$ and solves (3.22) for $i = 1$ in the distributional sense.

We can easily deduce from (3.25) that

$$\iint_{\mathbf{R}^N\times(0,T)} |u^1|^2 \,dx\,dt \le \widehat{C}_2 \left(\iint_{\mathbf{R}^N\times(0,T)} |\widehat{\theta}\rho g|^2 \,dx\,dt \right.$$
$$\left. + \iint_{\mathbf{R}^N\times(0,T)} |\widehat{\theta}\rho D\varphi\,\overline{y}|^2 \,dx\,dt + \iint_{\mathbf{R}^N\times(0,T)} |\widehat{\theta}'\rho\varphi|^2 \,dx\,dt \right),$$

for a constant $\widehat{C}_2 > 0$. Here, we have used the fact that

$$\widehat{\theta}(T-t) = \widehat{\theta}(t) \quad \forall t \in (0,T).$$

Thanks to the properties of ρ, we finally get

$$\begin{aligned} \iint_{\omega_2\times(0,T)} |u^1|^2 \,dx\,dt &\le \iint_{\mathbf{R}^N\times(0,T)} |u^1|^2 \,dx\,dt \\ &\le \widehat{C}_3 \left(\iint_{\omega_4\times(0,T)} |\widehat{\theta} g|^2 \,dx\,dt + \iint_{\omega_4\times(0,T)} |\widehat{\theta}'\varphi|^2 \,dx\,dt \right. \\ &\quad \left. + \iint_{\omega_4\times(0,T)} |\widehat{\theta} D\varphi\,\overline{y}|^2 \,dx\,dt \right), \end{aligned} \tag{3.26}$$

with $\widehat{C}_3(\omega) > 0$.

STEP 4.2. *Definition and estimate of* u^2.

Now, we deal with the Cauchy problem (3.22) for $i = 2$, where the right hand side is in $L^2(0,T;H^{-1}(\mathbf{R}^N)^N)$. The existence and uniqueness of a solution $u^2 \in L^2(0,T;H^1(\mathbf{R}^N)^N)$ is classical. Recall that $F_1(t)$ has support in $\omega_4 \setminus \overline{\omega}_3$ for t a.e., while we would like to estimate the L^2-norm of the solution in ω_2 and ω_2 is disjoint of $\omega_4 \setminus \overline{\omega}_3$. This fact will play a very important role in the sequel.

We will start by writing u^2 in terms of the fundamental solution $G = G(x,t)$ of the heat equation. To do this, we first notice that F_2 can be written in the form

$$F_2 = F_{21} + \nabla \cdot F_{22},$$

where F_{21} and F_{22} are L^2 functions supported by $(\omega_4 \setminus \overline{\omega}_3) \times [0,T]$ which can be written as sums of derivatives up to the second order of products $\widehat{\theta} D^\beta \rho g$, $\widehat{\theta} D^\beta \rho\varphi$, $\widehat{\theta} D^\beta \rho D\varphi\, \overline{y}$ and $\widehat{\theta}' D^\beta \rho\varphi$ with $1 \le |\beta| \le 4$.

Observe that, for any $y \in \omega_4 \setminus \overline{\omega}_3$ and any $x \in \omega_2$, one has $|x-y| \ge \operatorname{dist}(\partial\omega_3, \partial\omega_4) = d > 0$. Then, we have :

$$\begin{aligned} u^2(x,t) &= \int_0^t \int_{\omega_4 \setminus \overline{\omega}_3} G(x-y, t-s) F_{21}(y,s)\, dy\, ds \\ &- \int_0^t \int_{\omega_4 \setminus \overline{\omega}_3} \nabla_y G(x-y, t-s) \cdot F_{22}(y,s)\, dy\, ds \end{aligned} \tag{3.27}$$

for all $(x,t) \in \omega_2 \times (0,T)$, where

$$G(x,t) = \frac{1}{(4\pi t)^{N/2}} e^{-|x|^2/2t} \quad \forall x \in \mathbf{R}^N, \quad \forall t > 0.$$

Now we integrate by parts with respect to y in (3.27), passing all the derivatives from F_{21} and F_{22} to G and $\nabla_y G$. This is possible because we are integrating in a region where G is of class C^∞. This yields an expression for u^2 of the form

$$u^2(x,t) = \iint_{(\omega_4 \setminus \overline{\omega}_3) \times (0,t)} \sum_{\alpha \in I,\, \beta \in J} D_y^\alpha G(x-y, t-s) D_y^\beta \rho(y) z_{\alpha,\beta}(y,s)\, dy\, ds,$$

where all $\alpha \in I$ satisfy $|\alpha| \le 3$, all $\beta \in J$ satisfy $1 \le |\beta| \le 4$ and

$$\begin{aligned} z_{\alpha,\beta}(y,s) &= \widehat{\theta}(s)(C_{\alpha,\beta} g(y,s) + D_{\alpha,\beta}\varphi(y,s) + E_{\alpha,\beta}(D\varphi\, \overline{y})(y,s)) \\ &+ U_{\alpha,\beta} \widehat{\theta}'(s)\varphi(y,s), \qquad C_{\alpha,\beta},\ D_{\alpha,\beta},\ E_{\alpha,\beta},\ U_{\alpha,\beta} \in \mathbf{R}. \end{aligned}$$

From the previous considerations, we readily have

$$|u^2(x,t)| \le \iint_{(\omega_4 \setminus \overline{\omega}_3) \times (0,t)} \sum_{\alpha \in I} |D_y^\alpha G(x-y, t-s)| |z(y,s)|\, dy\, ds$$

for all $(x,t) \in \omega_2 \times (0,T)$, where

$$z(y,s) = \widehat{\theta}(s)(\widehat{C}_4 g(y,s) + \widehat{C}_5 \varphi(y,s) + \widehat{C}_6 (D\varphi\, \overline{y})(y,s)) + \widehat{C}_7 \widehat{\theta}'(s)\varphi(y,s).$$

Obviously, for every $0 < \delta < d$ there exists a positive constant $\widehat{C}_8(\delta,\omega)$ such that

$$|D^\alpha G(x-y, t-s)| \le \widehat{C}_8 \exp\left(\frac{-\delta^2}{2(t-s)} \right)$$

for all $(x,t) \in \omega_2 \times (0,T)$, all $(y,s) \in (\omega_4 \setminus \overline{\omega_3}) \times (0,t)$ and any $\alpha \in I$, so that

$$|u^2(x,t)| \le \widehat{C}_9 \iint_{(\omega_2 \setminus \overline{\omega}_3) \times (0,t)} \exp\left(\frac{-\delta^2}{2(t-s)} \right) |z(y,s)|\, dy\, ds$$

with $\widehat{C}_9 = \widehat{C}_9(\omega) > 0$.

At this moment, we integrate in $\omega_2 \times (0,T)$ and we obtain :

$$\iint_{\omega_2\times(0,T)} |u^2|^2\,dx\,dt \le \widehat{C}_{10} \int_0^T \left(\int_0^t \int_{\omega_4\setminus\overline{\omega}_3} \exp\left(\frac{-\delta^2}{2(t-s)}\right) |z(y,s)|\,dy\,ds \right)^2 dt$$
$$\le \widehat{C}_{11}\, T \int_0^T \left(\int_0^t \exp\left(\frac{-\delta^2}{2(t-s)}\right) \|z(s)\|^2_{L^2(\omega_4)}\,ds \right) dt$$

for some $\widehat{C}_{11}(\omega) > 0$.

Finally, we write the last term as a convolution, say

$$\int_0^T (f_1 * f_2)(t)\,dt,$$

where

$$f_1(t) = e^{-\delta^2/t} 1_{(0,+\infty)}(t) \in L^1(\mathbf{R}), \quad f_2(t) = \|z(t)\|^2_{L^2(\omega_4)} 1_{[0,T]}(t) \in L^1(\mathbf{R})$$

and we use Young's inequality. This provides

$$\iint_{\omega_2\times(0,T)} |u^2|^2\,dx\,dt \le \widehat{C}_{12} T \iint_{\omega_4\times(0,T)} |z|^2\,dx\,dt.$$

Taking into account the expression of z, we get

$$\begin{aligned}\iint_{\omega_2\times(0,T)} |u^2|^2\,dx\,dt \le \widehat{C}_{13} T \Bigg(&\iint_{\omega_4\times(0,T)} |\widehat{\theta}'\varphi|^2\,dx\,dt \\ &+ \iint_{\omega_4\times(0,T)} |\widehat{\theta}|^2 (|g|^2 + |D\varphi\,\overline{y}|^2 + |\varphi|^2)\,dx\,dt \Bigg).\end{aligned} \tag{3.28}$$

Putting this together with (3.26), we arrive at the estimate searched in the fourth step :

$$\begin{aligned}\iint_{\omega_2\times(0,T)} |\widehat{\theta}|^2 |\Delta\varphi|^2\,dx\,dt \le \widehat{C}_{14}(1+T) \Bigg(&\iint_{\omega_4\times(0,T)} |\widehat{\theta}'|^2 |\varphi|^2\,dx\,dt \\ &+ \iint_{\omega_4\times(0,T)} |\widehat{\theta}|^2 (|g|^2 + |D\varphi\,\overline{y}|^2 + |\varphi|^2)\,dx\,dt \Bigg).\end{aligned} \tag{3.29}$$

STEP 5. *Estimate of the local integral of* $|\varphi_t|^2$.

In this step we are going to bound the third term in the right hand side of (3.19). To this end, we will decompose our solution (up to a weight function depending on t) as the sum of other two solutions of Stokes systems, ψ_1 and ψ_2, with different properties. The first one will receive a global treatment and only energy estimates of the Stokes system will be employed there. On the other hand, we will deal with local terms of ψ_2 but with the advantage that $\psi_{2,tt}$ will make sense.

Thus, let (ψ_1, q_1) and (ψ_2, q_2) be the solutions to the following systems :

$$\begin{cases} -\psi_{1,t} - \Delta\psi_1 - D\psi_1\,\overline{y} + \nabla q_1 = \theta g & \text{in } Q, \\ \nabla\cdot\psi_1 = 0 & \text{in } Q, \\ \psi_1 = 0 & \text{on } \Sigma, \\ \psi_1(T) = 0 & \text{in } \Omega \end{cases} \tag{3.30}$$

and

$$\begin{cases} -\psi_{2,t} - \Delta\psi_2 - D\psi_2\,\overline{y} + \nabla q_2 = -\theta'\varphi & \text{in } Q, \\ \nabla\cdot\psi_2 = 0 & \text{in } Q, \\ \psi_2 = 0 & \text{on } \Sigma, \\ \psi_2(T) = 0 & \text{in } \Omega \end{cases} \tag{3.31}$$

(recall that θ was defined in (3.8)). Adding (3.30) and (3.31), we see that $(\psi_1+\psi_2, q_1+q_2)$ solves the same system as $(\theta\varphi, \theta\pi)$, where (φ, π) is the solution to (3.5). By uniqueness of the Stokes system, we thus have

$$\theta\varphi = \psi_1 + \psi_2 \quad \text{and} \quad \theta\pi = q_1 + q_2.$$

The term to be bounded is

$$\begin{aligned} \iint_{\omega_2\times(0,T)} |\widehat{\theta}|^2|\varphi_t|^2\,dx\,dt &= \iint_{\omega_2\times(0,T)} \theta^{-2}|\widehat{\theta}|^2|\theta\,\varphi_t|^2\,dx\,dt \\ = s^{-11/2}\lambda^2 \iint_{\omega_2\times(0,T)} & e^{-2s\alpha^*+2s\widehat{\alpha}}\widehat{\xi}^{-11/2}|\psi_{1,t}+\psi_{2,t}-\theta'\varphi|^2\,dx\,dt, \end{aligned} \tag{3.32}$$

so we will focus our attention on estimating the time derivatives of ψ_1 and ψ_2. One must realize that the weight function

$$e^{-2s\alpha^*+2s\widehat{\alpha}}$$

is "small".

STEP 5.1. *Estimate of* $\psi_{1,t}$.

In this step, we will bound the integral of $e^{-2s\alpha^*+2s\widehat{\alpha}}\widehat{\xi}^{-11/2}|\psi_{1,t}|^2$ in $\omega_2\times(0,T)$. At present, we do not know how to obtain local weighted estimates for $\psi_{1,t}$ depending just on the data. Therefore, we will bound $\psi_{1,t}$ globally in $\Omega\times(0,T)$ using well known estimates for the Stokes system and without the help of the weight function (in other words, we will forget the 'smallness' of the weight in the estimates).

Taking s and λ such that $s \geq s_1^* T^8$ and $\lambda \geq \lambda_1^*$, we get

$$e^{-2s\alpha^*+2s\widehat{\alpha}} \leq e^{-C_2^* s T^{-8} e^{\lambda m\|\eta^0\|_\infty}} \leq e^{-C_1^* e^{\lambda m\|\eta^0\|_\infty}}$$

and

$$s^{-11/2}\lambda^2\widehat{\xi}^{-11/2}e^{-2s\alpha^*+2s\widehat{\alpha}} \leq C_3^*\lambda^2 e^{-C_1^* e^{\lambda m\|\eta^0\|_\infty}},$$

which is bounded uniformly in λ for $\lambda \geq \lambda_1^*$.

Now, we apply regularity estimates for the Stokes system (see, for instance, [19]) to (3.30) and we deduce (among other things) that $\psi_{1,t} \in L^2(Q)^N$ and

$$\|\psi_{1,t}\|^2_{L^2(Q)^N} + \|\psi_1\|^2_{L^2(0,T;H^2(\Omega)^N} \leq C_4^*(1 + \|\overline{y}\|^2_\infty e^{C_5^* T\|\overline{y}\|^2_\infty})\|\theta g\|^2_{L^2(Q)^N}.$$

Hence, the estimate we obtain is

$$\begin{aligned} s^{-11/2}\lambda^2 \iint_{\omega_2\times(0,T)} & e^{-2s\alpha^*+2s\widehat{\alpha}}\widehat{\xi}^{-11/2}|\psi_{1,t}|^2\,dx\,dt \\ & \leq C_6^*(1+\|\overline{y}\|^2_\infty e^{C_5^* T\|\overline{y}\|^2_\infty}) \iint_Q |\theta|^2|g|^2\,dx\,dt. \end{aligned} \tag{3.33}$$

STEP 5.2. *Estimate of* $\psi_{2,t}$.

Now, we will be concerned with the integral of $e^{-2s\alpha^*+2s\widehat{\alpha}}\widehat{\xi}^{-11/2}|\psi_{2,t}|^2$. In this step, we will need an estimate of $\psi_{2,tt}$ in terms of other integrals that will be absorbed later on (see the terms in the right hand side of (3.40)). This will be possible by imposing the regularity hypotheses in (3.3) on $\overline{y}$. The tools we use here are classical a priori estimates for the Stokes and heat equations.

More precisely, by integrating by parts twice with respect to t, we obtain

$$\begin{aligned} & s^{-11/2}\lambda^2 \iint_{\omega_2\times(0,T)} e^{-2s\alpha^*+2s\widehat{\alpha}}\widehat{\xi}^{-11/2}|\psi_{2,t}|^2\,dx\,dt \\ & = \frac{1}{2}s^{-11/2}\lambda^2 \iint_{\omega_2\times(0,T)} (e^{-2s\alpha^*+2s\widehat{\alpha}}\widehat{\xi}^{-11/2})_{tt}|\psi_2|^2\,dx\,dt \\ & -s^{-11/2}\lambda^2 \iint_{\omega_2\times(0,T)} e^{-2s\alpha^*+2s\widehat{\alpha}}\widehat{\xi}^{-11/2}\psi_{2,tt}\cdot\psi_2\,dx\,dt. \end{aligned} \tag{3.34}$$

First, observe that we can make very similar computations to those made in the previous step to deduce that the weight function

$$(s^{-11/2}\lambda^2 e^{-2s\alpha^*+2s\widehat{\alpha}}\widehat{\xi}^{-11/2})_{tt}$$

is bounded from above for $s \geq s_2^*(T^4+T^8)$ and $\lambda \geq \lambda_2^*$ uniformly.

Let us now introduce the function

$$\theta^* = s^{-11/2}\lambda^{-4}e^{-2s\alpha^*+2s\widehat{\alpha}}\widehat{\xi}^{-11/2}$$

and let us concentrate in the estimate of the second integral in the right hand side of (3.34). Using Hölder's inequality, we deduce that

$$\begin{aligned} -\lambda^6 & \iint_{\omega_2\times(0,T)} \theta^*\psi_{2,tt}\psi_2\,dx\,dt \\ & \leq \lambda^6\|\theta^*\psi_{2,tt}\|_{L^2(0,T;L^r(\omega_2)^N)}\|\psi_2\|_{L^2(0,T;L^{r'}(\omega_2)^N)} \\ & \leq \frac{1}{2}\|\theta^*\psi_{2,tt}\|^2_{L^2(0,T;L^r(\omega_2)^N)} + \frac{1}{2}\lambda^{12}\|\psi_2\|^2_{L^2(0,T;L^{r'}(\omega_2)^N)}, \end{aligned} \tag{3.35}$$

where $6/5 < r < \sigma$ if $N = 3$ and $1 < r < \sigma$ if $N = 2$ (σ was introduced in (3.3)).

A key point of the proof will turn out to be the way we have applied Hölder's inequality here. Notice that the whole weight function is now accompanying $\psi_{2,tt}$, while there is no weight function with ψ_2. This will give desirable consequences, since we will be able to make a local treatment of the term on ψ_2, whereas just a global argument will be possible for the term on $\psi_{2,tt}$.

Let us deal with the last term in the right hand side of (3.35). Thus, let us introduce a cut-off function $\zeta \in C^2(\omega_3)$ such that

$$\operatorname{supp} \zeta \subset \omega_3 \quad \text{and} \quad \zeta = 1 \text{ in } \omega_2.$$

Then,

$$\begin{aligned}\|\psi_2\|^2_{L^2(0,T;L^{r'}(\omega_2)^N)} &\leq C_7^* \|\Delta(\zeta\psi_2)\|^2_{L^2(0,T;L^2(\omega_3)^N)} \\ &= C_7^* \|\psi_2 \Delta\zeta + 2\nabla\zeta \cdot \nabla\psi_2 + \zeta\Delta\psi_2\|^2_{L^2(0,T;L^2(\omega_3)^N)},\end{aligned}$$

since $H^2(\omega_3)^N \cap H_0^1(\omega_3)^N$ is continuously imbedded in $L^{r'}(\omega_3)^N$, for every $r' < \infty$.

Now, arguing as in step 4, we can get a bound of $\|\zeta\Delta\psi_2\|^2_{L^2(0,T;L^2(\omega_3)^N)}$. In fact, since $\psi_2(T) = 0$, it suffices to apply the inequality (3.29) with $\varphi = \psi_2$, $\widehat{\theta} = 1$, $g = -\theta'\varphi$, $\omega_2 = \omega_3$ and $\omega_4 = \omega_5$, where ω_5 is a new open set verifying

$$\omega_4 \subset\subset \omega_5 \subset\subset \omega.$$

This gives

$$\begin{aligned}&\|\psi_2\|^2_{L^2(0,T;L^{r'}(\omega_2)^N)} \\ &\qquad \leq C_8^*(1 + \|\overline{y}\|^2_\infty)(1+T) \iint_{\omega_5\times(0,T)} (|\psi_2|^2 + |\nabla\psi_2|^2 + |\theta'\varphi|^2)\,dx\,dt.\end{aligned}$$

From the definitions of ψ_1 and ψ_2, we also find that

$$|\psi_2|^2 + |\nabla\psi_2|^2 \leq 2(|\psi_1|^2 + |\nabla\psi_1|^2 + |\theta|^2(|\varphi|^2 + |\nabla\varphi|^2)).$$

Consequently, viewing ψ_1 as the weak solution of (3.30) and using again global estimates, we see that

$$\begin{aligned}\|\psi_2\|^2_{L^2(0,T;L^{r'}(\omega_2)^N)} &\leq C_9^*(1 + \|\overline{y}\|^2_\infty)(1+T) \Bigg(e^{C_{10}^* T \|\overline{y}\|^2_\infty} \|\theta g\|^2_{L^2(Q)} \\ &+ \iint_{\omega_5\times(0,T)} (|\theta|^2 + |\theta'|^2)|\varphi|^2\,dx\,dt + \iint_{\omega_5\times(0,T)} |\theta|^2|\nabla\varphi|^2\,dx\,dt \Bigg).\end{aligned} \tag{3.36}$$

We will keep the first two terms in the right hand side of (3.36) in our final inequality, while the third one will be estimated later on.

Let us now consider the norm involving $\psi_{2,tt}$ in (3.35).

The couple $(\psi, q) := (\theta^*\psi_{2,t}, \theta^* q_{2,t})$ satisfies

$$\begin{cases} -\psi_t - \Delta\psi - D\psi\,\overline{y} + \nabla q = G & \text{in } Q, \\ \nabla \cdot \psi = 0 & \text{in } Q, \\ \psi = 0 & \text{on } \Sigma, \\ \psi(T) = 0 & \text{in } \Omega, \end{cases} \tag{3.37}$$

where

$$G = -\theta^*\theta''\varphi - \theta^*\theta'\varphi_t + \theta^* D\psi_2\overline{y}_t - (\theta^*)'\psi_{2,t}.$$

This can be proved by first considering a sequence $\{\overline{y}^n\}$ of regular functions satisfying

$$\overline{y}^n \to \overline{y} \text{ weakly star in } L^\infty(Q)^N \quad \text{and} \quad \overline{y}^n_t \to \overline{y}_t \text{ weakly in } L^2(0,T;L^\sigma(\Omega)^N)$$

and setting

$$G^n = -\theta^*\theta''\varphi - \theta^*\theta'\varphi_t + \theta^* D\psi_2\overline{y}^n_t - (\theta^*)'\psi_{2,t}.$$

for all $n \geq 1$. For each n, one can easily establish the existence and uniqueness of a solution (ψ^n, q^n) to (3.37) with G replaced by G^n. Then, one can pass to the limit and deduce that (ψ, q) is actually the solution of (3.37).

In order to obtain an estimate of ψ_t in $L^2(0,T;L^r(\Omega)^N)$, we will first deduce an estimate of the transport term $(D\psi\,\overline{y})$ in the same space. In fact, if we look at ψ as the weak solution to (3.37), we have that $\psi \in L^2(0,T;V)$ and

$$\|\psi\|_{L^2(0,T;V)} \leq C^*_{11} e^{C^*_{12}T\|\overline{y}\|^2_\infty} \|G\|_{L^2(0,T;H^{-1}(\Omega)^N)}, \tag{3.38}$$

so the same bound holds for $\|D\psi\|_{L^2(Q)^N}$.

For the moment, let us assume that $\theta^* D\psi_2\,\overline{y}_t \in L^2(0,T;L^r(\Omega)^N)$; this will be proved in the sixth step. Now, we decompose the terms in the equation satisfied by ψ that are not divergence free. More precisely, in view of the Helmholtz's decomposition, there exist four functions g_1, g_2, g_3 and g_4 with g_1, $\nabla g_2 \in L^2(Q)^N$, $\nabla \cdot g_1 = 0$, g_3, $\nabla g_4 \in L^2(0,T;L^r(\Omega)^N)$ and $\nabla \cdot g_3 = 0$ such that

$$D\psi\,\overline{y} = g_1 + \nabla g_2 \quad \text{and} \quad \theta^* D\psi_2\,\overline{y}_t = g_3 + \nabla g_4,$$

with g_1, ∇g_2 and g_3, ∇g_4 depending continuously on $D\psi\,\overline{y}$ and $\theta^* D\psi_2\,\overline{y}_t$ in the spaces $L^2(Q)^N$ and $L^2(0,T;L^r(\Omega)^N)$, respectively.

This way, the equation verified by ψ can be written in the form

$$-\psi_t - \Delta\psi + \nabla\tilde{q} = J,$$

where

$$\tilde{q} = q - g_2 - g_4 \quad \text{and} \quad J = -\theta^*\theta''\varphi - \theta^*\theta'\varphi_t + g_3 - (\theta^*)'\psi_{2,t} + g_1.$$

Observe that J is divergence free. Under these conditions, we can apply theorem 2.8 in [8] and deduce that, among other properties, $\psi_t \in L^2(0,T;L^r(\Omega)^N)$ and

$$\|\psi_t\|_{L^2(0,T;L^r(\Omega)^N)} \leq C^*_{13}\|J\|_{L^2(0,T;L^r(\Omega)^N)} \tag{3.39}$$

for a positive constant C_{13}^* depending on Ω but not on T. Since $L^r(\Omega)$ is continuously imbedded in $H^{-1}(\Omega)$, (3.38) and (3.39) yield

$$\|\theta^*\psi_{2,tt}\|_{L^2(0,T;L^r(\Omega)^N)} \le C_{14}^*(1+\|\overline{y}\|_\infty)e^{C_{15}^*T\|\overline{y}\|_\infty^2}(\|\theta^*\theta''\varphi\|_{L^2(Q)^N} \\ +\|\theta^*\theta'\varphi_t\|_{L^2(Q)^N} + \|(\theta^*)'\psi_{2,t}\|_{L^2(Q)^N} + \|\theta^* D\psi_2\,\overline{y}_t\|_{L^2(0,T;L^r(\Omega)^N)}).$$

From (3.34)–(3.36) and this last inequality, we have

$$\begin{aligned} &s^{-11/2}\lambda^2\iint_{\omega_2\times(0,T)} e^{-2s\alpha^*+2s\widehat{\alpha}}\widehat{\xi}^{-11/2}|\psi_{2,t}|^2\,dx\,dt \\ &\le C_{16}^*(1+\|\overline{y}\|_\infty^2)e^{C_{17}^*T\|\overline{y}\|_\infty^2}\Big(\lambda^{12}(1+T)(\|\theta g\|_{L^2(Q)^N}^2 \\ &+\|\theta\varphi\|_{L^2(0,T;L^2(\omega_5)^N)}^2 + \|\theta'\varphi\|_{L^2(0,T;L^2(\omega_5)^N)}^2 + \|\theta\nabla\varphi\|_{L^2(0,T;L^2(\omega_5)^N)}^2) \\ &+\|\theta^*\theta''\varphi\|_{L^2(Q)^N}^2 + \|\theta^*\theta'\varphi_t\|_{L^2(Q)^N}^2 + \|(\theta^*)'\psi_{2,t}\|_{L^2(Q)^N}^2 \\ &+\|\theta^* D\psi_2\,\overline{y}_t\|_{L^2(0,T;L^r(\Omega)^N)}^2\Big)\,. \end{aligned} \tag{3.40}$$

To finish this step, we combine (3.32), (3.33) and (3.40) and we obtain the following estimate of φ_t :

$$\begin{aligned} &\iint_{\omega_2\times(0,T)} |\widehat{\theta}|^2|\varphi_t|^2\,dx\,dt \\ &\le C_{18}^*(1+\|\overline{y}\|_\infty^2)e^{C_{19}^*T\|\overline{y}\|_\infty^2}\Big(\lambda^{12}(1+T)(\|\theta g\|_{L^2(Q)^N}^2 \\ &+\|\theta\varphi\|_{L^2(0,T;L^2(\omega_5)^N)}^2 + \|\theta'\varphi\|_{L^2(0,T;L^2(\omega_5)^N)}^2 + \|\theta\nabla\varphi\|_{L^2(0,T;L^2(\omega_5)^N)}^2) \\ &+\|\theta^*\theta''\varphi\|_{L^2(Q)^N}^2 + \|\theta^*\theta'\varphi_t\|_{L^2(Q)^N}^2 + \|(\theta^*)'\psi_{2,t}\|_{L^2(Q)^N}^2 \\ &+\|\theta^* D\psi_2\,\overline{y}_t\|_{L^2(0,T;L^r(\Omega)^N)}^2\Big)\,. \end{aligned} \tag{3.41}$$

STEP 6. *Estimate of* $\theta^* D\psi_2\,\overline{y}_t$ *in* $L^2(0,T;L^r(\Omega)^N)$.

The strategy we follow in this step is to deduce an estimate of $\theta^*\psi_2$ in $L^\infty(0,T;W^{1,l}(\Omega)^N)$ for every $l<\infty$, with explicit dependence with respect to all the data. We will only use that $\overline{y}\in L^\infty(Q)^N$.

Observe that, once this is achieved, from (3.3) and the choice we have made of r, it is easy to obtain an estimate of $\theta^* D\psi_2\,\overline{y}_t$ in $L^2(0,T;L^r(\Omega)^N)$.

The function $(\theta^*\psi_2,\theta^* q_2)$ solves the following system :

$$\begin{cases} -(\theta^*\psi_2)_t - \Delta(\theta^*\psi_2) - D(\theta^*\psi_2)\,\overline{y} + \nabla(\theta^* q_2) \\ \qquad\qquad = -(\theta^*)'\psi_2 - \theta^*\theta'\varphi & \text{in } Q, \\ \nabla\cdot(\theta^*\psi_2) = 0 & \text{in } Q, \\ \theta^*\psi_2 = 0 & \text{on } \Sigma, \\ (\theta^*\psi_2)(T) = 0 & \text{in } \Omega. \end{cases} \tag{3.42}$$

From well known interpolation inequalities, we easily deduce that, for $N \leq 3$,

$$L^2(0,T;H^2(\Omega)^N) \cap L^\infty(0,T;V) \subset L^{k_1}(0,T;L^{k_2}(\Omega)^N)$$

with

$$\frac{2}{k_1} + \frac{6}{k_2} = 1 \tag{3.43}$$

and

$$L^2(0,T;L^6(\Omega)^N) \cap L^\infty(0,T;L^2(\Omega)^N) \subset L^{k_3}(0,T;L^{k_4}(\Omega)^N)$$

with

$$\frac{4/3}{k_3} + \frac{2}{k_4} = 1. \tag{3.44}$$

The desired regularity property of $\theta^*\psi_2$ will be the consequence of a *bootstrap* argument with only two steps.

• First, since $\theta^*\psi_2$ is the strong solution of (3.42),

$$D(\theta^*\psi_2)\overline{y} \in L^2(0,T;L^6(\Omega)^N) \cap L^\infty(0,T;L^2(\Omega)^N)$$

and we can put

$$D(\theta^*\psi_2)\,\overline{y} = g_5 + \nabla g_6 \in L^{k_3}(0,T;L^{k_4}(\Omega)^N), \quad \nabla \cdot g_5 = 0,$$

for some g_5 and g_6, with continuous dependence of g_5 and ∇g_6 in the space $L^{k_3}(0,T;L^{k_4}(\Omega)^N)$. Now, we can look at equation (3.42) with a pressure $\theta^* q_2 - g_6$ and a right hand side $-(\theta^*)'\psi_2 - \theta^*\theta'\varphi + g_5$, which is divergence free. Using the same regularity result as before, we deduce that $\theta^*\psi_2 \in L^{k_3}(0,T;W^{2,k_4}(\Omega)^N)$ and

$$\|\theta^*\psi_2\|_{L^{k_3}(0,T;W^{2,k_4}(\Omega)^N)} \leq C_7\| -(\theta^*)'\psi_2 - \theta^*\theta'\varphi + g_5\|_{L^{k_3}(0,T;L^{k_4}(\Omega)^N)}. \tag{3.45}$$

Accordingly, $\theta^* D\psi_2 \in L^{k_3}(0,T;W^{1,k_4}(\Omega)^N)$.

In this argument, k_4 can be any number satisfying $3 \leq k_4 < 6$. This gives $\theta^* D\psi_2 \in L^{k_1}(0,T;L^{k_2}(\Omega)^N)$, with $k_1 = k_3$ given by (3.44) and $k_2 = l > 6$.

• Secondly, we make another Helmholtz's decomposition of $\theta^* D\psi_2\,\overline{y}$ and we put

$$\theta^* D\psi_2\,\overline{y} = g_7 + \nabla g_8, \quad \nabla \cdot g_7 = 0,$$

but this time in the space $L^{k_3}(0,T;L^l(\Omega)^N)$. Hence, we obtain (for instance) $\nabla(\theta^* q_2 - g_8) \in L^{k_3}(0,T;L^l(\Omega)^N)$ and

$$\|\nabla(\theta^* q_2 - g_8)\|_{L^{k_3}(0,T;L^l(\Omega)^N)} \leq C_8\| -(\theta^*)'\psi_2 - \theta^*\theta'\varphi + g_7\|_{L^{k_3}(0,T;L^l(\Omega)^N)}.$$

Then, we take into account (3.45) and the continuous dependence of g_7 and g_5 with respect to $D(\theta^*\psi_2)\,\overline{y}$ and we get

$$\begin{aligned}\|\nabla(\theta^* q_2 - g_8)\|_{L^{k_3}(0,T;L^l(\Omega)^N)} &\leq C_9(1 + \|\overline{y}\|_\infty^2)(\|\theta^*\Delta\psi_2\|_{L^2(Q)^N} \\ &+\|(\theta^*)'\Delta\psi_2\|_{L^2(Q)^N} + \|(\theta^*\psi_2)_t\|_{L^2(Q)^N} + \|((\theta^*)'\psi_2)_t\|_{L^2(Q)^N} \\ &+\|\theta^*\theta'\Delta\varphi\|_{L^2(Q)^N} + \|(\theta^*\theta'\varphi)_t\|_{L^2(Q)^N}).\end{aligned} \tag{3.46}$$

Let us see that this suffices to ensure that $\theta^*\psi_2 \in L^\infty(0,T;W^{1,l}(\Omega))^N$ with explicit estimates. To this end, we look at (3.42) as N heat systems with right hand sides $B_i := -(\theta^*)'\psi_2^i - \theta^*\theta'\varphi^i - \partial_i(\theta^* q_2 - g_8) + g_7^i$ and we represent the solution in terms of the semigroup of the heat operator. Then, we have

$$\|\theta^*\psi_2(t)\|_{W^{1,l}(\Omega)^N} \le C_{10}\int_0^t (t-s)^{-1/2}\|B(s)\|_{L^l(\Omega)^N} \quad \forall t\in(0,T)$$

(see [11] for more details). Since $\|B(\cdot)\|_{L^l(\Omega)} \in L^{k_3}(0,T)$ with $k_3 > 2$, Young's inequality implies $\|\theta^*\psi_2(\cdot)\|_{W^{1,l}(\Omega)} \in L^\infty(0,T)$ and

$$\|\theta^*\psi_2\|_{L^\infty(0,T;W^{1,l}(\Omega)^N)} \le C_{10}(1-k_3'/2)^{-1/k_3'}\,T^{-1/2+1/k_3'}\|B\|_{L^{k_3}(0,T;L^l(\Omega)^N)}.$$

From (3.46), we obtain the desired regularity of $\theta^*\psi_2$:

$$\begin{aligned}\|\theta^*\psi_2\|_{L^\infty(0,T;W^{1,l}(\Omega)^N)} &\le C_{11}T^{-1/2+1/k_3'}(1+\|\overline{y}\|_\infty^2)(\|\theta^*\Delta\psi_2\|_{L^2(Q)^N}\\ &+\|(\theta^*)'\Delta\psi_2\|_{L^2(Q)^N} + \|(\theta^*\psi_2)_t\|_{L^2(Q)^N} + \|((\theta^*)'\psi_2)_t\|_{L^2(Q)^N}\\ &+\|\theta^*\theta'\Delta\varphi\|_{L^2(Q)^N} + \|(\theta^*\theta'\varphi)_t\|_{L^2(Q)^N}).\end{aligned} \tag{3.47}$$

As mentioned above, combining (3.47) and (3.3), we find that $\theta^* D\psi_2\,\overline{y}_t$ in $L^2(0,T;L^r(\Omega)^N)$ and

$$\begin{aligned}&\|\theta^* D\psi_2\,\overline{y}_t\|_{L^2(0,T;L^r(\Omega)^N)} \le \|\theta^* D\psi_2\|_{L^\infty(0,T;L^l(\Omega)^N)}\|\overline{y}_t\|_{L^2(0,T;L^\sigma(\Omega)^N)}\\ &\le C_{12}\,\|\overline{y}_t\|_{L^2(0,T;L^\sigma(\Omega)^N)}\,T^{-1/2+1/k_3'}(1+\|\overline{y}\|_\infty^2)(\|\theta^*\Delta\psi_2\|_{L^2(Q)^N}\\ &+\|(\theta^*)'\Delta\psi_2\|_{L^2(Q)^N} + \|(\theta^*\psi_2)_t\|_{L^2(Q)^N} + \|((\theta^*)'\psi_2)_t\|_{L^2(Q)^N}\\ &+\|\theta^*\theta'\Delta\varphi\|_{L^2(Q)^N} + \|(\theta^*\theta'\varphi)_t\|_{L^2(Q)^N}).\end{aligned} \tag{3.48}$$

Let us remark that the previous power of T depends only on the data σ in (3.3), since σ determines the admissible values of l and k_3. In fact, from the fact that $2 < k_3 \le 4$, we find that $4/3 \le k_3' < 2$ and $0 < -1 + 2/k_3' \le 1/2$.

Then, we put together the inequalities (3.41) and (3.48) and we obtain :

$$\begin{aligned}&\iint_{\omega_2\times(0,T)} |\widehat{\theta}|^2|\varphi_t|^2\,dx\,dt\\ &\le C_{13}(1+\|\overline{y}\|_\infty^6)\|\overline{y}_t\|^2_{L^2(L^\sigma)}\;e^{C_{14}T\|\overline{y}\|_\infty^2}\Big(\lambda^{12}(1+T)(\|\theta g\|^2_{L^2(Q)^N}\\ &+\|\theta\varphi\|^2_{L^2(0,T;L^2(\omega_5)^N)} + \|\theta'\varphi\|^2_{L^2(0,T;L^2(\omega_5)^N)} + \|\theta\nabla\varphi\|^2_{L^2(0,T;L^2(\omega_5)^N)})\\ &+(1+T^{1/2})(\|\theta^*\theta''\varphi\|^2_{L^2(Q)^N} + \|(\theta^*)'\theta'\varphi\|^2_{L^2(Q)^N} + \|\theta^*\theta'\varphi_t\|^2_{L^2(Q)^N}\\ &+\|\theta^*\theta'\Delta\varphi\|_{L^2(Q)^N} + \|\theta^*\psi_{2,t}\|^2_{L^2(Q)^N} + \|(\theta^*)'\psi_{2,t}\|^2_{L^2(Q)^N}\\ &+\|(\theta^*)'\psi_2\|^2_{L^2(Q)^N} + \|(\theta^*)''\psi_2\|^2_{L^2(Q)^N} + \|\theta^*\Delta\psi_2\|^2_{L^2(Q)^N}\\ &+\|(\theta^*)'\Delta\psi_2\|^2_{L^2(Q)^N})\Big).\end{aligned} \tag{3.49}$$

STEP 7. *Last arrangements and conclusion.*

From the definition of $\psi_2 = -\psi_1 + \theta\varphi$ and (3.49), we get

$$\begin{aligned}
&\iint_{\omega_2\times(0,T)} |\widehat{\theta}|^2|\varphi_t|^2\,dx\,dt \\
&\leq C_{15}(1+\|\overline{y}\|_\infty^6)\|\overline{y}_t\|^2_{L^2(0,T;L^\sigma(\Omega)^N)}e^{C_{16}T\,\|\overline{y}\|^2_\infty} \\
&\Big(\lambda^{12}(1+T)(\|\theta g\|^2_{L^2(Q)^N}+\|\theta\varphi\|^2_{L^2(0,T;L^2(\omega_5)^N)} \\
&+\|\theta'\varphi\|^2_{L^2(0,T;L^2(\omega_5)^N)}+\|\theta\nabla\varphi\|^2_{L^2(0,T;L^2(\omega_5)^N)}) \\
&+(1+T^{1/2})(\|\theta^*\theta'\varphi\|^2_{L^2(Q)^N}+\|(\theta^*)'\theta'\varphi\|^2_{L^2(Q)^N}+\|(\theta^*)''\theta\varphi\|^2_{L^2(Q)^N} \\
&+\|\theta^*\theta''\varphi\|^2_{L^2(Q)^N}+\|\theta^*\theta\Delta\varphi\|^2_{L^2(Q)^N}+\|\theta^*\theta'\Delta\varphi\|_{L^2(Q)^N} \\
&+\|(\theta^*)'\theta\Delta\varphi\|^2_{L^2(Q)^N}+\|\theta^*\theta\varphi_t\|^2_{L^2(Q)^N}+\|\theta^*\theta'\varphi_t\|^2_{L^2(Q)^N} \\
&+\|(\theta^*)'\theta\varphi_t\|^2_{L^2(Q)^N}+\|(\theta^*)'\psi_1\|^2_{L^2(Q)^N}+\|(\theta^*)''\psi_1\|^2_{L^2(Q)^N} \\
&+\|\theta^*\Delta\psi_1\|^2_{L^2(Q)^N}+\|(\theta^*)'\Delta\psi_1\|^2_{L^2(Q)^N}+\|\theta^*\psi_{1,t}\|^2_{L^2(Q)^N} \\
&+\|(\theta^*)'\psi_{1,t}\|^2_{L^2(Q)^N}\Big)\,.
\end{aligned} \tag{3.50}$$

For all the terms concerning ψ_1, we can use estimate (3.33) since θ^*, $(\theta^*)'$ and $(\theta^*)''$ are bounded functions in $(0,T)$ for $s \geq s_6(T^4+T^8)$. Hence, we have

$$\begin{aligned}
&\iint_{\omega_2\times(0,T)} |\widehat{\theta}|^2|\varphi_t|^2\,dx\,dt \\
&\leq C_{17}(1+\|\overline{y}\|_\infty^6)\|\overline{y}_t\|^2_{L^2(0,T;L^\sigma(\Omega)^N)}e^{C_{18}T\|\overline{y}\|^2_\infty} \\
&\Big(\lambda^{12}(1+T)(\|\theta\, g\|^2_{L^2(Q)^N}+\|\theta\varphi\|^2_{L^2(0,T;L^2(\omega_5)^N)} \\
&+\|\theta'\varphi\|^2_{L^2(0,T;L^2(\omega_5)^N)}+\|\theta\nabla\varphi\|^2_{L^2(0,T;L^2(\omega_5)^N)}) \\
&+(1+T^{1/2})(\|\theta^*\theta'\varphi\|^2_{L^2(Q)^N}+\|(\theta^*)'\theta'\varphi\|^2_{L^2(Q)^N} \\
&+\|(\theta^*)''\theta\varphi\|^2_{L^2(Q)^N}+\|\theta^*\theta''\varphi\|^2_{L^2(Q)^N}+\|\theta^*\theta\Delta\varphi\|^2_{L^2(Q)^N} \\
&+\|\theta^*\theta'\Delta\varphi\|_{L^2(Q)^N}+\|(\theta^*)'\theta\Delta\varphi\|^2_{L^2(Q)^N}+\|\theta^*\theta\varphi_t\|^2_{L^2(Q)^N} \\
&+\|\theta^*\theta'\varphi_t\|^2_{L^2(Q)^N}+\|(\theta^*)'\theta\varphi_t\|^2_{L^2(Q)^N})\Big)\,.
\end{aligned} \tag{3.51}$$

Let us now estimate the global terms on φ, $\Delta\varphi$ and φ_t and check that they can be eliminated using the left hand side of (3.18). To this end, let us first write down some bounds for the weight functions :

$$|\theta^*\theta'|+|(\theta^*)'\theta| \leq C_{19}Ts^{-3/4}\lambda^{-4}e^{-s\alpha^*}\widehat{\xi}^{-1/2},$$

$$|(\theta^*)'\theta'|+|(\theta^*)''\theta|+|\theta^*\theta''| \leq C_{20}T^2s^{1/4}\lambda^{-4}e^{-s\alpha^*}(\xi^*)^{3/4}.$$

By virtue of these estimates, it is not difficult to see that, for all $0 < \beta \leq 1/2$, we have

$$T^{\beta}(|\theta^*\theta'| + |(\theta^*)'\theta|) \leq C_{21} s^{-1/2}\lambda^{-4} e^{-s\alpha^*}\widehat{\xi}^{-1/2},$$

and

$$T^{\beta}(|(\theta^*)'\theta'| + |(\theta^*)''\theta| + |\theta^*\theta''|) \leq C_{22} s^{3/2}\lambda^{-4} e^{-s\alpha^*}(\xi^*)^{3/2}$$

for $s \geq s_7(T^4 + T^8)$.

Combining this and (3.51), we get

$$\begin{aligned}
&\iint_{\omega_2\times(0,T)} |\widehat{\theta}|^2|\varphi_t|^2\,dx\,dt \\
&\quad \leq C_{23}(1 + \|\overline{y}\|_\infty^6)\|\overline{y}_t\|^2_{L^2(0,T;L^\sigma(\Omega)^N)} e^{C_{24}T\|\overline{y}\|^2_\infty}\Big(\lambda^{12}(1+T)(\|\theta g\|^2_{L^2(Q)^N} \\
&\quad +\|\theta\varphi\|^2_{L^2(0,T;L^2(\omega_5)^N)} + \|\theta'\varphi\|^2_{L^2(0,T;L^2(\omega_5)^N)} + \|\theta\nabla\varphi\|^2_{L^2(0,T;L^2(\omega_5)^N)}) \\
&\quad +s^3\lambda^{-8}\iint_Q e^{-2s\alpha^*}(\xi^*)^3|\varphi|^2\,dx\,dt \\
&\quad +s^{-1}\lambda^{-8}\iint_Q e^{-2s\alpha^*}\widehat{\xi}^{-1}(|\varphi_t|^2 + |\Delta\varphi|^2)\,dx\,dt\Big),
\end{aligned}$$

for $s \geq s_7(T^4 + T^8)$.

As $\alpha^*(t) = \max_{x\in\overline{\Omega}}\alpha(x,t)$, taking $\lambda \geq \lambda_6(1 + \|\overline{y}\|_\infty + \|\overline{y}_t\|^2_{L^2(0,T;L^\sigma(\Omega)^N)} + e^{\lambda_7 T\|\overline{y}\|^2_\infty})$, we obtain

$$\begin{aligned}
&C_4C_6\iint_{\omega_2\times(0,T)} |\widehat{\theta}|^2|\varphi_t|^2\,dx\,dt \\
&\quad \leq C_{25}\lambda^{20}(1+T)(\|\theta g\|^2_{L^2(Q)^N} + \|\theta\varphi\|^2_{L^2(0,T;L^2(\omega_5)^N)} + \|\theta\varphi\|^2_{L^2(0,T;L^2(\omega_5)^N)} \\
&\quad +\|\theta\nabla\varphi\|^2_{L^2(0,T;L^2(\omega_5)^N)}) + \frac{1}{2}I(s,\lambda;\varphi).
\end{aligned}$$

Let us finally combine this inequality, (3.18), (3.19) and (3.29). We find :

$$\begin{aligned}
I(s,\lambda;\varphi) \leq C_{26}\lambda^{20}(1+T)\Big(&\|\theta g\|^2_{L^2(Q)^N} + \|\theta\varphi\|^2_{L^2(0,T;L^2(\omega_5)^N)} \\
&+\|\theta\varphi\|^2_{L^2(0,T;L^2(\omega_5)^N)} + \|\theta\nabla\varphi\|^2_{L^2(0,T;L^2(\omega_5)^N)}\Big),
\end{aligned} \tag{3.52}$$

for $s \geq s_8(T^4 + T^8)$ and $\lambda \geq \lambda_8(1 + \|\overline{y}\|_\infty + \|\overline{y}_t\|^2_{L^2(0,T;L^\sigma(\Omega)^N)} + e^{\lambda_9 T\|\overline{y}\|^2_\infty})$.

It only remains to get rid of the local term on $\nabla\varphi$ in the right hand side of (3.52). Thus, let us introduce a cut-off function $\zeta \in C^2(\omega)$ such that

$$\zeta \equiv 1 \text{ in } \omega_5.$$

Then we have

$$\iint_{\omega_5\times(0,T)} |\theta|^2|\nabla\varphi|^2\,dx\,dt \le \iint_{\omega\times(0,T)} |\theta|^2\zeta|\nabla\varphi|^2\,dx\,dt$$
$$= \frac{1}{2}\iint_{\omega\times(0,T)} |\theta|^2\Delta\zeta|\varphi|^2\,dx\,dt - \iint_{\omega\times(0,T)} |\theta|^2\zeta\Delta\varphi\cdot\varphi\,dx\,dt.$$

Let us apply Young's inequality to the last integral. This gives

$$\iint_{\omega_5\times(0,T)} |\theta|^2|\nabla\varphi|^2\,dx\,dt \le C_{26}s\lambda^{20}(1+T)\iint_{\omega\times(0,T)} e^{2s\alpha^*}|\theta|^4\widehat{\xi}|\varphi|^2\,dx\,dt$$
$$+\frac{1}{2C_{25}}s^{-1}\lambda^{-20}(1+T)^{-1}\iint_{\omega\times(0,T)} e^{-2s\alpha^*}\widehat{\xi}^{-1}|\Delta\varphi|^2\,dx\,dt,$$

for a constant $C_{26}(\Omega,\omega)>0$ and, consequently, for any $\lambda \ge \lambda_9(1+\|\overline{y}\|_\infty + \|\overline{y}_t\|^2_{L^2(0,T;L^\sigma(\Omega)^N)} + e^{\lambda_{10}T\|\overline{y}\|^2_\infty})$ and $s\ge s_7(T^4+T^8)$, we obtain from (3.52)

$$I(s,\lambda;\varphi) \le C_{27}(1+T^2)\left(s^{15/2}\lambda^{20}\iint_Q e^{-4s\widehat{\alpha}+2s\alpha^*}\widehat{\xi}^{15/2}|g|^2\,dx\,dt\right.$$
$$\left.+s^{16}\lambda^{40}\iint_{\omega\times(0,T)} e^{-8s\widehat{\alpha}+6s\alpha^*}\widehat{\xi}^{16}|\varphi|^2\,dx\,dt\right),$$

which is exactly (3.6).

3 Null controllability of the linear system with a right hand side

In this section we will solve the null controllability problem for system (3.4) with a right hand side which decays exponentially as $t\to T^-$.

This result will be useful to deduce the local null controllability of (3.1) in the next section. Indeed, we would like to find $v\in L^2(\omega\times(0,T))^N$ such that the solution to

$$\begin{cases} Ly+\nabla p = f+v1_\omega & \text{in } Q,\\ \nabla\cdot y = 0 & \text{in } Q,\\ y=0 & \text{on } \Sigma,\\ y(0)=y^0 & \text{in } \Omega,\end{cases} \tag{3.53}$$

where

$$Ly = y_t - \Delta y + \nabla\cdot(y\otimes\overline{y}) + \nabla\cdot(\overline{y}\otimes y), \tag{3.54}$$

verifies

$$y(T)=0 \quad \text{in } \Omega. \tag{3.55}$$

Moreover, it will be convenient to prove the existence of a solution of the previous problem in an appropriate weighted space which depends on the spatial dimension. Before introducing this space, we will deduce a Carleman inequality with weight functions that do not vanish at $t=0$. More precisely, let us consider the function

$$\ell(t) = \begin{cases} \dfrac{T^2}{4} & \text{for } 0 \le t \le T/2, \\ t(T-t) & \text{for } T/2 \le t \le T \end{cases}$$

and the following associated weight functions :

$$\beta(x,t) = \frac{e^{5/4\lambda m\|\eta^0\|_\infty} - e^{\lambda(m\|\eta^0\|_\infty + \eta^0(x))}}{\ell(t)^4},$$
$$\widehat{\beta}(t) = \min_{x\in\overline{\Omega}} \beta(x,t), \quad \beta^*(t) = \max_{x\in\overline{\Omega}} \beta(x,t),$$
$$\gamma(x,t) = \frac{e^{\lambda(m\|\eta^0\|_\infty + \eta^0(x))}}{\ell(t)^4},$$
$$\widehat{\gamma}(t) = \max_{x\in\overline{\Omega}} \gamma(x,t), \quad \gamma^*(t) = \min_{x\in\overline{\Omega}} \gamma(x,t).$$

Lemma 4 *With the previous notation, for any $\overline{y}$ verifying* (3.3)*, there exists a positive constant C depending on T, s and λ, such that every solution to* (3.5) *verifies*

$$\begin{aligned} &\iint_Q e^{-2s\beta}\gamma^3|\varphi|^2\,dx\,dt + \iint_Q e^{-2s\beta}\gamma|\nabla\varphi|^2\,dx\,dt + \|\varphi(0)\|^2_{L^2(\Omega)^N} \\ &\qquad \le C\left(\iint_Q e^{-4s\widehat{\beta}+2s\beta^*}\widehat{\gamma}^{15/2}\,|g|^2\,dx\,dt \right. \\ &\qquad\quad \left. + \iint_{\omega\times(0,T)} e^{-8s\widehat{\beta}+6s\beta^*}\widehat{\gamma}^{16}|\varphi|^2\,dx\,dt\right), \end{aligned} \tag{3.56}$$

where s and λ are taken like in theorem 5.

Proof : We start with a simple a priori estimate for the Stokes system (3.5) :

$$\begin{aligned} &\|\varphi\|_{L^2(0,T/2;V)} + \|\varphi\|_{L^\infty(0,T/2;H)} \\ &\qquad \le C(\|g\|_{L^2(0,3T/4;L^2(\Omega)^N)} + \frac{1}{T}\|\varphi\|_{L^2(T/2,3T/4;L^2(\Omega)^N)}). \end{aligned} \tag{3.57}$$

To prove this, it suffices to introduce a function $\eta \in C^1([0,T])$ with

$$\eta = 1 \quad \text{in } [0,T/2], \quad \eta \equiv 0 \text{ in } [3T/4,T], \quad |\eta'| \le C/T$$

and use the classical energy estimates verified by $\eta\varphi$, that solves, together with $\eta\pi$, a Stokes problem. In particular, we have

$$\begin{aligned} &\|\varphi\|^2_{L^2(0,T/2;H^1(\Omega)^N)} + \|\varphi\|^2_{L^\infty(0,T/2;H)} \\ &\qquad \le C\left(\|g\|^2_{L^2(0,3T/4;L^2(\Omega)^N)} + \frac{1}{T^2}\|\varphi\|^2_{L^2(T/2,3T/4;L^2(\Omega)^N)}\right), \end{aligned} \tag{3.58}$$

which leads to (3.57). As a consequence, we can obtain a first estimate in $\Omega \times (0, T/2)$:

$$\begin{aligned}
&\int_0^{T/2}\!\!\int_\Omega e^{-2s\beta}\,\gamma^3\,|\varphi|^2\,dx\,dt + \int_0^{T/2}\!\!\int_\Omega e^{-2s\beta}\,\gamma\,|\nabla\varphi|^2\,dx\,dt \\
&\quad + \|\varphi(0)\|^2_{L^2(\Omega)^N} \le C(T,s,\lambda)\left(\int_0^{3T/4}\!\!\int_\Omega e^{-4s\widehat{\beta}+2s\beta^*}\,\widehat{\gamma}^{15/2}\,|g|^2\,dx\,dt\right. \\
&\quad \left. + \int_{T/2}^{3T/4}\!\!\int_\Omega e^{-2s\beta}\,\gamma^3\,|\varphi|^2\,dx\,dt\right).
\end{aligned} \tag{3.59}$$

On the other hand, since $\alpha = \beta$ in $\Omega \times (T/2, T)$, we have

$$\begin{aligned}
&\int_{T/2}^{T}\!\int_\Omega e^{-2s\beta}\,\gamma^3\,|\varphi|^2\,dx\,dt + \int_{T/2}^{T}\!\int_\Omega e^{-2s\beta}\,\gamma\,|\nabla\varphi|^2\,dx\,dt \\
&= \int_{T/2}^{T}\!\int_\Omega e^{-2s\alpha}\,\xi^3\,|\varphi|^2\,dx\,dt + \int_{T/2}^{T}\!\int_\Omega e^{-2s\alpha}\,\xi\,|\nabla\varphi|^2\,dx\,dt \\
&\le C I(s,\lambda;\varphi)
\end{aligned}$$

so, by virtue of the Carleman inequality (3.6), we have :

$$\begin{aligned}
&\int_{T/2}^{T}\!\int_\Omega e^{-2s\beta}\,\gamma^3\,|\varphi|^2\,dx\,dt + \int_{T/2}^{T}\!\int_\Omega e^{-2s\beta}\,\gamma\,|\nabla\varphi|^2\,dx\,dt \\
&\qquad \le C(T,s,\lambda)\left(\iint_Q e^{-4s\widehat{\alpha}+2s\alpha^*}\,\widehat{\xi}^{15/2}\,|g|^2\,dx\,dt\right. \\
&\qquad \left. + \iint_{\omega\times(0,T)} e^{-8s\widehat{\alpha}+6s\alpha^*}\,\widehat{\xi}^{16}\,|\varphi|^2\,dx\,dt\right).
\end{aligned}$$

Finally, from the definition of β, $\widehat{\beta}$, β^*, γ and $\widehat{\gamma}$, we get

$$\begin{aligned}
&\int_{T/2}^{T}\!\int_\Omega e^{-2s\beta}\,\gamma^3\,|\varphi|^2\,dx\,dt + \int_{T/2}^{T}\!\int_\Omega e^{-2s\beta}\,\gamma\,|\nabla\varphi|^2\,dx\,dt \\
&\qquad \le C(T,s,\lambda)\left(\iint_Q e^{-4s\widehat{\beta}+2s\beta^*}\,\widehat{\gamma}^{15/2}\,|g|^2\,dx\,dt\right. \\
&\qquad \left. + \iint_{\omega\times(0,T)} e^{-8s\widehat{\beta}+6s\beta^*}\,\widehat{\gamma}^{16}\,|\varphi|^2\,dx\,dt\right)
\end{aligned}$$

which, together with (3.59), provides the desired inequality (3.56).

Once we have got (3.56), we are ready to solve (3.53)–(3.55). In fact, we will prove two controllability results : first, we will obtain a null controllability result for (3.53) with no supplementary regularity for the control and the state (see proposition 8 below) ; then, we will prove (3.53)–(3.55) with a more regular velocity field y (see proposition 9).

Let us start with the first result.

Proposition 8 *Let $y^0 \in H$ and $e^{s\beta^*}(\gamma^*)^{-1/2} f \in L^2(0,T;H^{-1}(\Omega)^N)$. Then, we can find $v \in L^2(\omega \times (0,T))^N$ such that (3.53)–(3.55) is satisfied.*

Proof : The argument that leads from (3.56) to this result is classical, but it will be sketched here for the sake of completeness. We will first establish the approximate controllability of (3.53)–(3.55) using the method of [6].

Let us consider, for each $\psi^0 \in H$, the solution to (3.5) with zero right hand side, i. e. :

$$\begin{cases} -\psi_t - \Delta\psi - D\psi\,\overline{y} + \nabla q = 0 & \text{in } Q, \\ \nabla \cdot \psi = 0 & \text{in } Q, \\ \psi = 0 & \text{on } \Sigma, \\ \psi(T) = \psi^0 & \text{in } \Omega. \end{cases} \tag{3.60}$$

Let us then introduce, for each $\varepsilon > 0$, the following functional :

$$\begin{cases} J_\varepsilon(\psi^0) = \dfrac{1}{2} \displaystyle\iint_{\omega\times(0,T)} |\psi|^2 \, dx\, dt + \varepsilon \|\psi^0\|_H + \int_\Omega \psi(0)\cdot y^0 \, dx + \int_0^T \langle f, \psi\rangle \, dt \\ \forall \psi^0 \in H. \end{cases}$$

Here, $\langle \cdot\,,\cdot\rangle$ denotes the usual duality product between $H^{-1}(\Omega)^N$ and $H_0^1(\Omega)^N$.

From Carleman inequality (3.56), it is immediate that, for every $\varepsilon > 0$, J_ε is coercive and possesses a unique minimum $\psi_\varepsilon^0 \in H$. Let us denote by ψ_ε the corresponding solution to (3.60).

Then, we set $v_\varepsilon = \psi_\varepsilon 1_\omega$ and we denote by y_ε the associated solution to (3.53). From the fact that $J_\varepsilon(\psi_\varepsilon^0) \le 0$ and using (3.56), we find

$$\|\psi_\varepsilon 1_\omega\|_{L^2(Q)^N} \le C(\|y^0\|_H + \|e^{s\beta^*}(\gamma^*)^{-1/2} f\|_{L^2(0,T;H^{-1}(\Omega)^N)}) \tag{3.61}$$

and, in particular, v_ε is uniformly bounded in $L^2(\omega \times (0,T))^N$.

On the other hand, writing explicitly the necessary condition satisfied by J_ε at its minimum ψ_ε^0 and the duality between y_ε and ψ, we deduce

$$\|y_\varepsilon(T)\|_H \le \varepsilon. \tag{3.62}$$

Combining (3.61) and (3.62), we finally get the existence of a control v (the weak limit of a subsequence of v_ε in $L^2(\omega\times(0,T))^N$) such that the associated solution to (3.53) verifies (3.55).

We will now present a second null controllability result for (3.53) where we look for a more regular solution y. This will be crucial to deduce controllability properties for the nonlinear system (3.1) in the last section.

To this end, we proceed to the definition of the space where (3.53)–(3.55) is solved. Since this space depends on the dimension, we denote it by E_N. It is the following :

$$\begin{aligned} E_2 = \{ &(y,v) \in E_0 : \exists p \ \text{ such that} \\ &e^{s\beta^*}(\gamma^*)^{-1/2}(Ly + \nabla p - v1_\omega) \in L^2(0,T;H^{-1}(\Omega)^2)\} \end{aligned}$$

and

$$E_3 = \{(y,v) \in E_0 : e^{s\beta^*/2}(\gamma^*)^{-1/4} y \in L^4(0,T;L^{12}(\Omega)^3),$$
$$\exists p : e^{s\beta^*}(\gamma^*)^{-1/2}(Ly + \nabla p - v1_\omega) \in L^2(0,T;W^{-1,6}(\Omega)^3)\},$$

where

$$E_0 = \{(y,v) : e^{2s\widehat{\beta}-s\beta^*}\widehat{\gamma}^{-15/4} y,\ e^{4s\widehat{\beta}-3s\beta^*}\widehat{\gamma}^{-8} v1_\omega \in L^2(Q)^N,$$
$$e^{s\beta^*/2}(\gamma^*)^{-1/4} y \in L^2(0,T;V) \cap L^\infty(0,T;H)\}.$$

Of course, E_2 and E_3 are Banach spaces for the norms

$$\begin{aligned}\|(y,v)\|_{E_2} = \Big(&\|e^{2s\widehat{\beta}-s\beta^*}\widehat{\gamma}^{-15/4} y\|^2_{L^2(Q)^2} + \|e^{4s\widehat{\beta}-3s\beta^*}\widehat{\gamma}^{-8} v1_\omega\|^2_{L^2(Q)^2}\\ &+\|e^{s\beta^*/2}(\gamma^*)^{-1/4} y\|^2_{L^2(0,T;V)\cap L^\infty(0,T;H)}\\ &+\|e^{s\beta^*}(\gamma^*)^{-1/2}(Ly + \nabla p - v1_\omega)\|^2_{L^2(0,T;H^{-1}(\Omega)^2)}\Big)^{1/2}\end{aligned}$$

and

$$\begin{aligned}\|(y,v)\|_{E_3} = \Big(&\|e^{2s\widehat{\beta}-s\beta^*}\widehat{\gamma}^{-15/4} y\|^2_{L^2(Q)^3} + \|e^{4s\widehat{\beta}-3s\beta^*}\widehat{\gamma}^{-8} v1_\omega\|^2_{L^2(Q)^3}\\ &+\|e^{s\beta^*/2}(\gamma^*)^{-1/4} y\|^2_{L^2(0,T;V)\cap L^\infty(0,T;H)}\\ &+\|e^{s\beta^*/2}(\gamma^*)^{-1/4} y\|^2_{L^4(0,T;L^{12}(\Omega)^3)}\\ &+\|e^{s\beta^*}(\gamma^*)^{-1/2}(Ly + \nabla p - v1_\omega)\|^2_{L^2(0,T;W^{-1,6}(\Omega)^3)}\Big)^{1/2},\end{aligned}$$

respectively.

Remark 6 If $(y,v) \in E_N$, then $y(T) = 0$, so that y and v solve, together with some p, a null controllability problem for system (3.4) with an appropriate right hand side f.

Then, we have the following result :

Proposition 9 *Let us assume that $\overline{y}$ satisfies* (3.3) *and the following hypotheses on the initial condition and the right hand side hold :*
- *If $N = 2$: $y^0 \in H$, $e^{s\beta^*}(\gamma^*)^{-1/2} f \in L^2(0,T;H^{-1}(\Omega)^2)$.*
- *If $N = 3$: $y^0 \in H \cap L^4(\Omega)^3$, $e^{s\beta^*}(\gamma^*)^{-1/2} f \in L^2(0,T;W^{-1,6}(\Omega)^3)$.*

Then, there exists a control $v \in L^2(\omega \times (0,T))^N$ such that, if y is (together with some p) the associated solution to (3.53)*, one has $(y,v) \in E_N$. In particular,* (3.55) *holds.*

Proof : Let us first give an intuitive idea of the way we can find the couple (y,v).

Following the arguments in [11] (which are in fact adapted from a general method described in [9]), let us introduce the extremal problem

$$\begin{cases} \inf \dfrac{1}{2}\left(\displaystyle\iint_Q e^{4s\widehat{\beta}-2s\beta^*}\widehat{\gamma}^{-15/2}|y|^2 dxdt + \int_0^T\!\!\int_\omega e^{8s\widehat{\beta}-6s\beta^*}\widehat{\gamma}^{-16}|v|^2 dxdt\right) \\ \text{subject to } v\in L^2(Q)^N,\ \text{supp } v\subset \omega\times(0,T) \text{ and} \\ \qquad\begin{cases} Ly+\nabla p = f + v1_\omega & \text{in } Q, \\ \nabla\cdot y = 0 & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0)=y^0, \quad y(T)=0 & \text{in } \Omega. \end{cases} \end{cases} \tag{3.63}$$

Assume that (3.63) possesses a unique solution $(\widehat{y},\widehat{v})$. Then, in view of Lagrange's principle, there exist dual variables $\widehat{z}$ and $\widehat{q}$ such that

$$\begin{cases} \widehat{y} = e^{-4s\widehat{\beta}+2s\beta^*}\widehat{\gamma}^{15/2}(L^*\widehat{z}+\nabla\widehat{q}), \quad \nabla\cdot\widehat{z}=0 & \text{in } Q, \\ \widehat{v} = -e^{-8s\widehat{\beta}+6s\beta^*}\widehat{\gamma}^{-16}\,\widehat{z} & \text{in } \omega\times(0,T), \\ \widehat{z}=0 & \text{on } \Sigma, \end{cases} \tag{3.64}$$

where L^* is the adjoint operator of L, i. e.

$$L^*z = -z_t - \Delta z - Dz\,\overline{y}.$$

Let us now set

$$P_0 = \{(w,h)\in C^\infty(\overline{Q})^{N+1} : \nabla\cdot w = 0,\ w=0 \text{ on } \Sigma,\ \int_\omega h(x,t)\,dx = 0\}$$

and

$$a((\widehat{z},\widehat{q}),(w,h)) = \iint_Q e^{-4s\widehat{\beta}+2s\beta^*}\widehat{\gamma}^{15/2}\,(L^*\widehat{z}+\nabla\widehat{q})(L^*w+\nabla h)\,dx\,dt$$
$$+\iint_{\omega\times(0,T)} e^{-8s\widehat{\beta}+6s\beta^*}\widehat{\gamma}^{16}\,\widehat{z}\,w\,dx\,dt \qquad \forall (w,h)\in P_0.$$

Then, if the functions $\widehat{y}$ and $\widehat{v}$, given by (3.64), satisfy (3.53)–(3.55) (together with some $\widehat{p}$), we must have

$$a((\widehat{z},\widehat{q}),(w,h)) = \langle \ell,(w,h)\rangle \qquad \forall (w,h)\in P_0, \tag{3.65}$$

where we have used the notation

$$\langle \ell,(w,h)\rangle = \int_0^T \langle f(t),w(t)\rangle_{H^{-1},H_0^1}\,dt + \int_\Omega y^0\,w(0)\,dx. \tag{3.66}$$

The key idea in this proof is to demonstrate that there exists exactly one $(\widehat{z},\widehat{q})$ satisfying (3.65) in an appropriate class. We will then define $\widehat{y}$ and $\widehat{v}$ using (3.64) and we will check that $(\widehat{y},\widehat{v})$ fulfills the desired properties.

Thus, consider the linear space P_0 and the bilinear form $a(\cdot\,,\cdot)$ on P_0 :

$$\begin{aligned} a((z,q),(w,h)) = \iint_Q e^{-4s\widehat{\beta}+2s\beta^*}\widehat{\gamma}^{15/2}\,(L^*z+\nabla q)(L^*w+\nabla h)\,dx\,dt \\ + \iint_{\omega\times(0,T)} e^{-8s\widehat{\beta}+6s\beta^*}\widehat{\gamma}^{16}\,z\,w\,dx\,dt \quad \forall (z,q),(w,h)\in P_0. \end{aligned} \tag{3.67}$$

Observe that the Carleman inequality (3.56) holds for all $(w,h)\in P_0$, i. e.

$$\begin{aligned} \iint_Q e^{-2s\beta}\gamma^3|\varphi|^2\,dx\,dt + \iint_Q e^{-2s\beta}\gamma|\nabla\varphi|^2\,dx\,dt + \|\varphi(0)\|^2_{L^2(\Omega)^N} \\ \leq Ca((w,h),(w,h)) \qquad \forall (w,h)\in P_0. \end{aligned} \tag{3.68}$$

From the unique continuation property for the Stokes-like system

$$\begin{cases} L^*z+\nabla q = 0 & \text{in } Q,\\ \nabla\cdot z = 0 & \text{in } Q,\\ z=0 & \text{on } \Sigma, \end{cases}$$

we see that $a(\cdot\,,\cdot)$ is a scalar product in P_0.

Let us now consider the space P, given by the completion of P_0 for the norm associated to $a(\cdot\,,\cdot)$ (which we denote by $\|\cdot\|_P$). This is a Hilbert space and $a(\cdot\,,\cdot)$ is a continuous and coercive bilinear form on P.

Let us also introduce the linear form ℓ, given by (3.66) for all $(w,h)\in P$. After a simple computation, we see that

$$\begin{aligned} |\langle \ell,(w,h)\rangle| \leq \|e^{s\beta^*}(\gamma^*)^{-1/2}f\|_{L^2(0,T;H^{-1}(\Omega)^N)}\|e^{-s\beta^*}(\gamma^*)^{1/2}w\|_{L^2(0,T;H^1_0(\Omega)^N)} \\ +\|y^0\|_H\|w(0)\|_H \qquad \forall (w,h)\in P \end{aligned}$$

and, in particular, using (3.68) and the density of P_0 in P, we find :

$$\left\{ \begin{array}{l} |\langle \ell,(w,h)\rangle| \leq C(\|e^{s\beta^*}(\gamma^*)^{-1/2}f\|_{L^2(0,T;H^{-1}(\Omega)^N)} + \|y^0\|_H)\,\|(w,h)\|_P \\ \forall (w,h)\in P. \end{array} \right\}$$

In other words, ℓ is a bounded linear form on P. Consequently, in view of Lax-Milgram's lemma, there exists one and only one $(\widehat{z},\widehat{q})$ satisfying

$$\begin{cases} a((\widehat{z},\widehat{q}),(w,h)) = \langle \ell,(w,h)\rangle \\ \forall (w,h)\in P, \quad (\widehat{z},\widehat{q})\in P. \end{cases} \tag{3.69}$$

Let us set

$$\widehat{y} = e^{-4s\widehat{\beta}+2s\beta^*}\widehat{\gamma}^{15/2}(L^*\widehat{z}+\nabla\widehat{q}) \quad \text{and} \quad \widehat{v} = -e^{-8s\widehat{\beta}+6s\beta^*}\widehat{\gamma}^{16}\,\widehat{z}1_\omega. \tag{3.70}$$

and let us see that $(\widehat{y}, \widehat{v})$ verifies

$$\iint_Q e^{4s\widehat{\beta}-2s\beta^*}\widehat{\gamma}^{-15/2}|\widehat{y}|^2 dx\,dt + \int_0^T\!\!\int_\omega e^{8s\widehat{\beta}-6s\beta^*}\widehat{\gamma}^{-16}|\widehat{v}|^2 dx\,dt < +\infty$$

and is solution of Stokes system in (3.63) for some pressure $\widehat{p}$.

The first property is easy to check, since $(\widehat{z}, \widehat{q}) \in P$ and

$$\iint_Q e^{4s\widehat{\beta}-2s\beta^*}\widehat{\gamma}^{-15/2}|\widehat{y}|^2 dx\,dt + \int_0^T\!\!\int_\omega e^{8s\widehat{\beta}-6s\beta^*}\widehat{\gamma}^{-16}|\widehat{v}|^2 dx\,dt = a((\widehat{z}, \widehat{q}), (\widehat{z}, \widehat{q})).$$

Notice that, in particular, $\widehat{y} \in L^2(Q)^N$ and $\widehat{v} \in L^2(\omega \times (0,T)^N)$. Then, we introduce the (weak) solution $(\widetilde{y}, \widetilde{p})$ to the Stokes system

$$\begin{cases} L\widetilde{y} + \nabla\widetilde{p} = f + \widehat{v}1_\omega & \text{in } Q, \\ \nabla \cdot \widetilde{y} = 0 & \text{in } Q, \\ \widetilde{y} = 0 & \text{on } \Sigma, \\ \widetilde{y}(0) = y^0 & \text{in } \Omega. \end{cases} \tag{3.71}$$

Clearly, $\widetilde{y}$ is also the unique solution of (3.71) defined by transposition. Of course, this means that $\widetilde{y}$ is the unique function in $L^2(Q)^N$ satisfying

$$\begin{cases} \displaystyle\iint_Q \widetilde{y} \cdot b\,dx\,dt = \int_0^T \langle f(t), w(t)\rangle_{H^{-1},H_0^1}\,dt + \iint_Q \widehat{v}1_\omega \cdot w\,dx\,dt \\ \displaystyle\qquad + \int_\Omega y^0 \cdot w(0)\,dx \qquad \forall b \in L^2(Q)^N, \end{cases} \tag{3.72}$$

where w is, together with some h, the solution to

$$\begin{cases} L^*w + \nabla h = b & \text{in } Q, \\ \nabla \cdot w = 0 & \text{in } Q, \\ w = 0 & \text{on } \Sigma, \\ w(T) = 0 & \text{in } \Omega. \end{cases}$$

From (3.69) and (3.70), we see that $\widehat{y}$ also satisfies (3.72). Consequently, $\widehat{y} = \widetilde{y}$ and $\widehat{y}$ is, together with $\widehat{p} = \widetilde{p}$, the solution to the Stokes system in (3.63).

Finally, we must see that $(\widehat{y}, \widehat{v}) \in E_N$. We already know that

$$e^{2s\widehat{\beta}-s\beta^*}\widehat{\gamma}^{-15/4}\widehat{y},\ e^{4s\widehat{\beta}-3s\beta^*}\widehat{\gamma}^{-8}\widehat{v}1_\omega \in L^2(Q)^N,$$

$$e^{s\beta^*}(\gamma^*)^{-1/2}(L\widehat{y} + \nabla\widehat{p} - \widehat{v}1_\omega) \in L^2(0,T; H^{-1}(\Omega)^2) \quad \text{if } N = 2$$

and

$$e^{s\beta^*}(\gamma^*)^{-1/2}(L\widehat{y} + \nabla\widehat{p} - \widehat{v}1_\omega) \in L^2(0,T; W^{-1,6}(\Omega)^3) \quad \text{if } N = 3.$$

Thus, it remains to check only that

$$e^{s\beta^*/2}(\gamma^*)^{-1/4}\widehat{y} \in L^2(0,T;V) \cap L^\infty(0,T;H)$$

and $e^{s\beta^*/2}(\gamma^*)^{-1/4}\widehat{y} \in L^4(0,T;L^{12}(\Omega)^3)$ in dimension 3. To this end, let us introduce the functions $y^* = e^{s\beta^*/2}(\gamma^*)^{-1/4}\widehat{y}$, $p^* = e^{s\beta^*/2}(\gamma^*)^{-1/4}\widehat{p}$ and $f^* = e^{s\beta^*/2}(\gamma^*)^{-1/4}(f+\widehat{v}1_\omega)$. Then (y^*, p^*) satisfies

$$\begin{cases} Ly^* + \nabla p^* = f^* + (e^{s\beta^*/2}(\gamma^*)^{-1/4})_t\widehat{y} & \text{in } Q, \\ \nabla \cdot y^* = 0 & \text{in } Q, \\ y^* = 0 & \text{on } \Sigma, \\ y^*(0) = e^{s\beta^*(0)}\gamma^*(0)^{-1/4}y^0 & \text{in } \Omega. \end{cases} \tag{3.73}$$

Since $f^* \in L^2(0,T;H^{-1}(\Omega)^N)$, $(e^{s\beta^*/2}(\gamma^*)^{-1/4})_t\widehat{y} \in L^2(Q)^N$ and $y^0 \in H$, we have

$$y^* \in L^2(0,T;V) \cap L^\infty(0,T;H).$$

At this point, the proof of proposition 9 is finished in dimension 2.

Our last task will be to deduce that $y^* \in L^4(0,T;L^{12}(\Omega)^3)$ when $N = 3$. To this end, let us consider, for each $g \in L^{4/3}(0,T;L^{12/11}(\Omega)^3)$, the Stokes system

$$\begin{cases} L^*z + \nabla q = g & \text{in } Q, \\ \nabla \cdot z = 0 & \text{in } Q, \\ z = 0 & \text{on } \Sigma, \\ z(T) = 0 & \text{in } \Omega. \end{cases} \tag{3.74}$$

We have the following result :

Lemma 5 *Let $N = 3$ and $\overline{y} \in L^\infty(Q)^3$. Then, for each $g \in L^{4/3}(0,T;L^{12/11}(\Omega)^3)$, there exists a unique solution (z,q) to the Stokes system (3.74) satisfying*

$$z \in L^2(0,T;W_0^{1,6/5}(\Omega)^3) \cap C^0([0,T];L^{4/3}(\Omega)^3)$$

that depends continuously on g in these spaces.

Let us accept that lemma 5 holds. Then y^* must be identical to the solution by transposition of (3.73), namely, the unique function $y_* \in L^4(0,T;L^{12}(\Omega)^3)$ satisfying

$$\begin{cases} \displaystyle\iint_Q y_* \cdot g \, dx \, dt = \int_\Omega e^{s\beta^*(0)}\gamma^*(0)^{-1/4}y^0 \cdot z(0)\, dx + \langle F, z\rangle_{W^{-1,6},W_0^{1,6/5}} \\ \forall g \in L^{4/3}(0,T;L^{12/11}(\Omega)^3). \end{cases}$$

Here, F stands for the function

$$F = f^* + (e^{s\beta^*/2}(\gamma^*)^{-3/4})_t\widehat{y} - \nabla \cdot (y^* \otimes \overline{y}) - \nabla \cdot (\overline{y} \otimes y^*)$$

and (z, q) is the solution to (3.74) associated to g. Remark that, as we already had that $y^* \in L^2(0, T; L^6(\Omega)^3)$, all the terms of the previous definition make sense by virtue of lemma 5 and the assumption $y^0 \in L^4(\Omega)^3$.

Therefore, $y^* \in L^4(0, T; L^{12}(\Omega)^3)$. This ends the proof of proposition 9.

Let us finally give the proof of lemma 5, which is based on interpolation arguments.

Proof of lemma 5 : Let us first prove that $z \in L^2(0, T; W_0^{1,6/5}(\Omega)^3)$. Indeed, following [8], we deduce that

$$z \in L^{4/3}(0, T; W^{2,12/11}(\Omega)^3 \cap W_0^{1,12/7}(\Omega)^3), \; z_t \in L^{4/3}(0, T; L^{12/11}(\Omega)^3).$$

Just taking advantage of the fact that

$$z \in L^{4/3}(0, T; W^{2,12/11}(\Omega)^3) \cap L^{\infty}(0, T; L^{12/11}(\Omega)^3), \tag{3.75}$$

we can use interpolation arguments to deduce that

$$z \in L^2(0, T; W^{4/3,12/11}(\Omega)^3)$$

(see [18] for more details). In fact, from (3.75), we find that

$$z(t) \in (W^{2,12/11}(\Omega)^3, L^{12/11}(\Omega)^3)_{1/3,12/11} = W^{4/3,12/11}(\Omega)^3$$

and

$$\|z(t)\|_{W^{4/3,12/11}(\Omega)^3} \leq C \|z(t)\|^{2/3}_{W^{2,12/11}(\Omega)^3} \|z(t)\|^{1/3}_{L^{12/11}(\Omega)^3}$$

for almost all $t \in (0, T)$. The Sobolev embedding theorem tells that

$$W^{4/3,12/11}(\Omega)^3 \subset W^{1,36/29}(\Omega)^3 \subset W^{1,6/5}(\Omega)^3.$$

Consequently, we have

$$\|z(t)\|_{W^{1,6/5}(\Omega)^3} \leq \widetilde{C} \|z(t)\|^{2/3}_{W^{2,12/11}(\Omega)^3} \|z(t)\|^{1/3}_{L^{12/11}(\Omega)^3} \quad \text{a.e. } t \in (0, T). \tag{3.76}$$

Let us also remark that, since $z \in L^{4/3}(0, T; W_0^{1,12/7}(\Omega)^3)$,

$$z(t) \in W_0^{1,6/5}(\Omega)^3 \quad \text{a.e. } t \in (0, T). \tag{3.77}$$

Combining (3.76), (3.77) and (3.75), we conclude that

$$z \in L^2(0, T; W_0^{1,6/5}(\Omega)^3).$$

On the other hand, from interpolation arguments which can be found in [18] we also know that, if

$$z \in L^{4/3}(0, T; W^{2,12/11}(\Omega)^3) \subset L^{4/3}(0, T; L^4(\Omega)^3)$$

and

$$z_t \in L^{4/3}(0, T; L^{12/11}(\Omega)^3),$$

then

$$z \in C^0([0,T];(L^4(\Omega)^3, L^{12/11}(\Omega)^3)_{3/4,4/3}).$$

Moreover, the interpolation space

$$(L^4(\Omega)^3, L^{12/11}(\Omega)^3)_{3/4,4/3}$$

coincides with the Lorentz space $L^{4/3,4/3}(\Omega)^3 = L^{4/3}(\Omega)^3$. Consequently, we also have $z \in C^0([0,T];L^{4/3}(\Omega)^3)$.

4 Exact controllability to trajectories

In this section we give the proof of theorem 6 using similar arguments to those employed in [11]. We will see that the results obtained in the previous section allow us to locally invert a nonlinear equation. In fact, the regularity deduced for the solution to the linearized system (3.53) will be sufficient to apply a suitable inverse mapping theorem (see theorem 7 below).

Thus, let us set $y = \overline{y} + z$ and $p = \overline{p} + q$ and let us use these equalities in (3.1). Taking into account that $(\overline{y}, \overline{p})$ solves (3.2), we find :

$$\begin{cases} Lz + \nabla \cdot (z \otimes z) + \nabla q = v1_\omega & \text{in } Q, \\ \nabla \cdot z = 0 & \text{in } Q, \\ z = 0 & \text{on } \Sigma, \\ z(0) = y^0 - \overline{y}^0 & \text{in } \Omega \end{cases} \tag{3.78}$$

(recall that L is given by (3.54)).

This way, we have reduced our problem to a local null controllability result for the solution (z,q) to the *nonlinear* problem (3.78). We will use the following inverse mapping theorem (see [1]) :

Theorem 7 *Let E and G be two Banach spaces and let $\mathcal{A}3 : E \mapsto G$ satisfy $\mathcal{A} \in C^1(E;G)$. Assume that $e_0 \in E$, $\mathcal{A}(e_0) = h_0$ and $\mathcal{A}'(e_0) : E \mapsto G$ is surjective. Then, there exists $\delta > 0$ such that, for every $h \in G$ satisfying $\|h - h_0\|_G < \delta$, there exists a solution of the equation*

$$\mathcal{A}(e) = h, \quad e \in E.$$

In our setting, we use this theorem with the spaces $E = E_N$ and

$$G = \begin{cases} L^2(e^{s\beta^*}(\gamma^*)^{-1/2}; H^{-1}(\Omega)^2) \times H & \text{if } N = 2 \\ L^2(e^{s\beta^*}(\gamma^*)^{-1/2}; W^{-1,6}(\Omega)^3) \times (L^4(\Omega)^3 \cap H) & \text{if } N = 3 \end{cases}$$

and the operator

$$\mathcal{A}(z,v) = (Lz + \nabla \cdot (z \otimes z) + \nabla q - v1_\omega, z(0)) \quad \forall (z,v) \in E_N.$$

for $(z,v) \in E_N$.

In the following proposition, we check that the previous framework fits the regularity required to apply theorem 7.

Proposition 10 *Let us assume that* $\overline{y} \in L^\infty(Q)^N$*. Then,* $\mathcal{A} \in C^1(E;G)$*.*

Proof : We start by noticing that all the terms arising in the definition of $\mathcal{A}$ are linear (and consequently C^1), except for $\nabla \cdot (z \otimes z)$. However, the operator

$$((z_1, v_1), (z_2, v_2)) \mapsto \nabla \cdot (z_1 \otimes z_2) \tag{3.79}$$

is bilinear, so it suffices to prove its continuity from $E \times E$ into W, where

$$W = \begin{cases} L^2(e^{s\beta^*}(\gamma^*)^{-1/2}; H^{-1}(\Omega)^2) & \text{if } N = 2 \\ L^2(e^{s\beta^*}(\gamma^*)^{-1/2}; W^{-1,6}(\Omega)^3) & \text{if } N = 3 \end{cases}$$

In fact, for $N = 2$ we can use that $e^{s\beta^*/2}(\gamma^*)^{-1/4} z \in L^4(Q)^2$ for any $(z, v) \in E$ and we get

$$\begin{aligned} &\|\nabla \cdot (z_1 \otimes z_2)\|_{L^2(e^{s\beta^*}(\gamma^*)^{-1/2}(0,T);H^{-1}(\Omega)^2)} \\ &\leq C\|z_1 \otimes z_2\|_{L^2(e^{s\beta^*}(\gamma^*)^{-1/2}(0,T);L^2(\Omega)^2)} \\ &\leq C\|e^{s\beta^*/2}(\gamma^*)^{-1/4} z_1\|_{L^4(Q)^2} \|e^{s\beta^*/2}(\gamma^*)^{-1/4} z_2\|_{L^4(Q)^2}. \end{aligned}$$

On the other hand, for $N = 3$ we find that

$$\begin{aligned} &\|\nabla \cdot (z_1 \otimes z_2)\|_{L^2(e^{s\beta^*}(\gamma^*)^{-1/2}(0,T);W^{-1,6}(\Omega)^3)} \\ &\leq C\||z_1||z_2|\|_{L^2(e^{s\beta^*}(\gamma^*)^{-1/2}(0,T);L^6(\Omega)^3)} \\ &\leq C\|z_1\|_{L^4(e^{s\beta^*/2}(\gamma^*)^{-1/4}(0,T);L^{12}(\Omega)^3)} \|z_2\|_{L^4(e^{s\beta^*/2}(\gamma^*)^{-1/4}(0,T);L^{12}(\Omega)^3)}. \end{aligned}$$

Therefore, in both cases the continuity of (3.79) is established.

This proves proposition 10.

As a consequence of this result, we can apply theorem 7 for $e_0 = 0 \in \mathbf{R}^N$ and $h_0 = 0$. In fact, $\mathcal{A}'(0,0) : E \mapsto G$ is given by

$$\mathcal{A}'(0,0)(z, v) = (Lz + \nabla q - v 1_\omega, z(0)) \quad \forall (z, v) \in E$$

and is surjective in view of the null controllability result for the linearized system (3.53) given in proposition 9.

As a conclusion, an application of theorem 7 gives the existence of $\delta > 0$ such that, if $\|z(0)\|_{L^{2N-2}(\Omega)^N} \leq \delta$, then we find a control v such that the associated solution to (3.78) verifies $z(T) = 0$ in Ω.

This concludes the proof of theorem 6.

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Chapitre 4

Local exact controllability to the trajectories of the Navier-Stokes system with nonlinear Navier-slip boundary conditions

Local exact controllability to the trajectories of the Navier-Stokes system with nonlinear Navier-slip boundary conditions

S. Guerrero

Abstract

In this paper we deal with the local exact controllability of the Navier-Stokes system with nonlinear Navier-slip boundary conditions and distributed controls supported in small sets. In a first step, we prove a Carleman inequality for the linearized Navier-Stokes system, which leads to null controllability of this system at any time $T > 0$. Then, fixed point arguments lead to the deduction of a local result concerning the exact controllability to the trajectories of the Navier-Stokes system.

Résumé

Dans cet article on étudie le problème de la contrôlabilité exacte locale pour les équations de Navier-Stokes avec conditions frontières de glissement de Navier. Dans un premier temps, on montre une inegalité de Carleman pour le système de Navier-Stokes linearisé, qui nous amenera à la contrôllabilité nulle pour tout $T > 0$. Ensuite, un argument de point fix sera utilisé pour déduire un resultat locale concernant la contrôllabilité exacte aux trajectoires du système de Navier-Stokes.

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Introduction

Let $\Omega \subset \mathbf{R}^N$ ($N = 2$ or 3) be a bounded connected open set whose boundary $\partial\Omega$ is regular enough. Let $\omega \subset \Omega$ be a (small) nonempty open subset and let $T > 0$. We will use the notation $Q = \Omega \times (0, T)$ and $\Sigma = \partial\Omega \times (0, T)$ and we will denote by $n(x)$ the outward unit normal to Ω at the point $x \in \partial\Omega$.

On the other hand, we will denote by C a generic positive constant (usually depending on Ω and ω).

Let us consider the controlled Navier-Stokes system with nonlinear Navier slip boundary conditions. Given a nonlinear 'regular' function $f : \mathbf{R}^N \mapsto \mathbf{R}^N$ and an initial state y^0, we consider the following system :

$$\begin{cases} y_t - \nabla\cdot(Dy) + (y,\nabla)y + \nabla p = v1_\omega & \text{in } Q,\\ \nabla\cdot y = 0 & \text{in } Q,\\ y\cdot n = 0, \quad (\sigma(y,p)\cdot n)_{tg} + f(y)_{tg} = 0 & \text{on } \Sigma,\\ y(\cdot,0) = y^0(\cdot) & \text{in } \Omega, \end{cases} \tag{4.1}$$

where

$$\sigma(y,p) = -p\,Id + Dy$$

is the stress tensor and

$$(Dy)_{i,j} = \partial_j y^i + \partial_i y^j \quad (1\le i,j\le N)$$

is the deformation tensor. Here v denotes the control function which acts over system (4.1) trough ω during the time T. On the other hand, the subscript 'tg' stands for the tangential component of the corresponding vector field, i.e. :

$$w_{tg} = w - (w\cdot n)n.$$

The first of the boundary conditions is the *slip condition* and says that the particles in the fluid do not penetrate the boundary. On the other hand and in order to explain the second one, let us remark that $\sigma(y,p)\cdot n$ is the force exerted by the fluid over the solid wall. Thus, a condition like

$$(\sigma(y,p)\cdot n)_{tg} = -ky_{tg} \quad k \text{ constant}$$

mean that the tangential component of this force is proportional to the velocity field on the boundary. But k may not depend on $|y|$ linearly, so

$$(\sigma(y,p)\cdot n)_{tg} = -(f(y))_{tg}$$

stands for a more general condition. Let us also point out that this kind of conditions appear in turbulence. In such a context, we cannot expect to exactly find neither y nor p, not even approximatively. Then, one is led to 'average' the Navier-Stokes equations and solve them. When solid walls are concerned, experimental analysis show that these averaged variables behave like

$$y\cdot n = 0 \quad \text{and} \quad (\sigma(y,p)\cdot n)_{tg} = -f(y)_{tg}$$

near the boundary. This is known as 'wall law'. For further literature on this subject, see for instance [16, 4, 15].

In the context of controllability, this problem has not been studied up to now, so their properties are unknown.

A related linear control system which will be useful in this paper is

$$\begin{cases} w_t - \nabla\cdot(Dw) + (a(x,t)+b(x,t),\nabla)w + (w,\nabla)b(x,t) & \\ \qquad\qquad\qquad\qquad\qquad\qquad +\nabla q = v1_\omega & \text{in } Q,\\ \nabla\cdot w = 0 & \text{in } Q,\\ w\cdot n = 0, \quad (\sigma(w,q)\cdot n)_{tg} + (A(x,t)w)_{tg} = 0 & \text{on } \Sigma,\\ w(\cdot,0) = w^0(\cdot) & \text{in } \Omega, \end{cases} \tag{4.2}$$

with $A \in L^\infty(\mathbf{R}^N;\mathbf{R}^N)$ a $N \times N$ matrix function and a and b free-divergence vector field functions. Problems of this kind have already been studied in [5], where the author proved an approximate controllability result for system (4.2) in dimension 2 with $a \equiv b \equiv A \equiv 0$.

Let us define the concepts of controllability which will be concerned in this paper. For system (4.2), we will say that it is *null controllable* if for any (suitable) y^0 there exists a control v such that the associated solution to (4.2) verifies

$$w(\cdot,T) = 0 \ \text{ in } \Omega. \tag{4.3}$$

For system (4.1), we will say that it is *locally exactly controllable* to the trajectories if for a suitable trajectory $\overline{y}$ of system (4.1), there exists $\delta > 0$ such that

$$\|y^0 - \overline{y}(0)\|_E \le \delta \Rightarrow \exists v : y(\cdot,T) = \overline{y}(\cdot,T) \ \text{ in } \Omega,$$

for some Banach space E. In fact, $\overline{y}$ will satisfy

$$\begin{cases} \overline{y}_t - \Delta\overline{y} + (\overline{y},\nabla)\overline{y} + \nabla\overline{p} = 0 & \text{in } Q, \\ \nabla\cdot\overline{y} = 0 & \text{in } Q, \\ \overline{y}\cdot n = 0,\ (\sigma(\overline{y},\overline{p})\cdot n)_{tg} + (f(\overline{y}))_{tg} = 0 & \text{on } \Sigma. \end{cases} \tag{4.4}$$

The strategy of this paper will be, in a first step, to establish a null controllability result for (4.2) (see theorem 8), which can actually be seen as a linearization of system (4.1) around appropriate trajectories $\overline{y}$ of (4.1). Then, using this result and a fixed point argument, the local exact controllability to 'regular enough' trajectories of system (4.1) will be deduced (see theorem 9).

Let us introduce several spaces which are usual in the context of problems modelling incompressible fluids :

$$H = \left\{y \in L^2(\Omega)^N : \nabla\cdot y = 0 \text{ in } \Omega,\ y\cdot n = 0 \text{ on } \partial\Omega\right\} \tag{4.5}$$

and

$$W = \{y \in H^1(\Omega)^N : \nabla\cdot y = 0 \text{ in } \Omega,\ y\cdot n = 0 \text{ on } \partial\Omega\}. \tag{4.6}$$

In the sequel, some regularity assumptions will be imposed on the previous potentials and matrix functions. Let $0 < \ell < 1/2$ arbitrarily close to 1/2. For vector fields $d(x,t)$, we will assume certain regularity hypothesis :

$$d \in L^\infty(Q)^N, \quad d_t \in L^2(0,T;L^r(\Omega)^N) \quad \begin{pmatrix} r = 6 & \text{if } N = 3 \\ r = 4 & \text{if } N = 2 \end{pmatrix}, \tag{4.7}$$

while for a matrix function A the following will be imposed :

$$A \in L^\infty(\Sigma)^{N\times N}, \tag{4.8}$$

$$A \in H^{1-\ell}(0,T;W^{\nu_1,\nu_1+1}(\partial\Omega)^{N\times N}), \tag{4.9}$$

$$A \in H^{(3-\ell)/2}(0,T;H^{\nu_2}(\partial\Omega)^{N\times N}), \tag{4.10}$$

where $\nu_1 > 1$ (arbitrarily small) in dimension 3 and $\nu_1 = 1$ in dimension 2 and $\nu_2 = 1/2(3 - N) + (1-\ell)(N-2)$. In order to make this more comprehensible, observe that for instance a function $A \in C^{3/2}(\overline{\Sigma})^{N\times N}$ would fulfill the previous properties.

As announced, the first main result of this paper concerns the null controllability of system (4.2) and is presented in the following theorem.

Theorem 8 *Let $w^0 \in H$ and let us suppose that A verifies* (4.8)-(4.10) *and a, b are free-divergence vector fields verifying* (4.7)*.*

Then, there exist controls $v \in H^1(0,T;L^2(\omega)^N) \cap C^0([0,T];H^1(\omega)^N)$ such that the corresponding solution verifies (4.3)*.*

Furthermore, there exists a positive constant C depending on Ω, ω, T, $\|a\|_\infty$, $\|b\|_\infty$, $\|a_t\|_{L^2(L^r)}$, $\|b_t\|_{L^2(L^r)}$, $\|A\|_{H^{1-\ell}(W^{\nu_1,\nu_1+1})}$ and $\|A\|_{H^{(3-\ell)/2}(H^{\nu_2})}$, such that

$$\|v\|_{H^1(L^2)} + \|v\|_{L^\infty(H^1)} \le C\|w^0\|_H.$$

The proof of theorem 8 is based on the obtention of the so-called *observability inequality* for a backwards system associated to (4.2). In fact, we will consider the adjoint system

$$\begin{cases} -\varphi_t - \nabla\cdot(D\varphi) - (a(x,t),\nabla)\varphi - D\varphi\, b(x,t) + \nabla\pi = 0 & \text{in } Q, \\ \nabla\cdot\varphi = 0 & \text{in } Q, \\ \varphi\cdot n = 0, \quad (\sigma(\varphi,\pi)\cdot n)_{tg} + (A(x,t)^t\varphi)_{tg} = 0 & \text{on } \Sigma, \\ \varphi(\cdot,T) = \varphi^0(\cdot) & \text{in } \Omega. \end{cases} \tag{4.11}$$

The usual tools to obtain the observability for (4.11) are *global Carleman inequalities*. This was popularized by O. Yu. Imanuvilov and A. V. Fursikov in [9] and several advances have been made since then (see, for instance, [13, 12]). The proof of the corresponding Carleman inequality for this system will be divided in two steps :

• We first obtain a Carleman inequality for a heat system associated to (4.11). Precisely, we consider a function φ verifying

$$\begin{cases} -\varphi_t - \nabla\cdot(D\varphi) = G \in L^2(Q)^N, \quad \nabla\cdot\varphi = 0 & \text{in } Q, \\ \varphi\cdot n = 0, \quad (D\varphi\cdot n)_{tg} + (A(x,t)\varphi)_{tg} = 0 & \text{on } \Sigma. \end{cases}$$

Similar techniques to those developed in [9] are employed in order to get the desired estimate. More details will be given in subsection 3.1.

• Then, following the general ideas of [11, 7], a Carleman inequality for system (4.11) is established. Let us remark here the particular difficulty an estimate of this kind contains due to the coupling boundary conditions. The details are given in subsection 3.2.

This usually provides L^2 controls leading to the null controllability of the velocity vector field solution of (4.2). However, in order to perform a fixed point argument and extract some controllability properties for the nonlinear system (4.1), a more regular control is needed. The regularization process we use here was introduced in [2].

The second main result of this paper concerns the local exact controllability to the trajectories of (4.1). Several regularity hypotheses have to be assumed for the trajectories in order to be able to approach them :

$$\overline{y} \in L^\infty(Q)^N, \quad \overline{y} \in H^{1-\ell}(0,T;W^{\nu_1,\nu_1+1}(\partial\Omega)^N), \\ \overline{y} \in H^{(3-\ell)/2}(0,T;H^{\nu_2}(\partial\Omega)^N), \tag{4.12}$$

$$\overline{y}(\cdot,0) \in H^3(\Omega)^N \cap W, \\ (D\overline{y}(\cdot,0)\cdot n)_{tg} + (f(\overline{y}(\cdot,0)))_{tg} = 0 \text{ on } \partial\Omega. \tag{4.13}$$

Observe that, with a suitable initial condition, $\overline{y} \in C^{3/2}(\overline{\Sigma})^N$ would also suffice here to assure the the above properties. On the other hand, we will impose regularity to the nonlinearities appearing on the boundary condition :

$$f \in C^3(\mathbf{R}^N;\mathbf{R}^N). \tag{4.14}$$

Theorem 9 *Let f verify* (4.14), *and let $y^0 \in H^3(\Omega)^N \cap W$ satisfy the compatibility condition*

$$(Dy^0 \cdot n)_{tg} + (f(y^0))_{tg} = 0 \ \text{ on } \partial\Omega. \tag{4.15}$$

Then, the exact controllability to the trajectories of (4.1) *satisfying* (4.12)-(4.13) *holds, i.e., there exists $\delta > 0$ such that if $\|y^0 - \overline{y}(\cdot,0)\|_{H^3\cap W} \le \delta$, we can find controls v such that the corresponding solutions y to* (4.1) *satisfy*

$$y(\cdot,T) = \overline{y}(\cdot,T) \ \text{ in } \Omega.$$

Furthermore, these controls belong to

$$H^1(0,T;L^2(\omega)^N) \cap C([0,T];H^1(\omega)^N).$$

A fixed point technique is used to prove this result. This tool has successfully been used in this context several times ; see, for instance, [20, 6, 8]. We apply here *Kakutani's theorem.*

The main difficulty arising in this situation turns to be the restrictive spaces one is forced to deal with when proving compactness results for linear systems like (4.2). There, the regularity results which will be proved in section 2, are crucial.

In spite of this positive controllability result for system (4.1), this is still far from what would be desirable for systems of this kind. It would be interesting to know whether one has local exact controllability to all bounded trajectories. However, this seems to be a very complicated question.

The paper is organized as follows : in section 2 some previous and technical regularity results for systems of this kind are established. Section 3 will contain the proofs of the Carleman inequalities. Finally, the controllability results are proved in section 4 (theorems 1 and 2).

1 Previous results

In this section, we will prove several technical results which will be used later on. More precisely, we present two regularity results concerning the Stokes system with linear Navier-slip boundary conditions.

The first one concerns the existence of *strong* solutions, i.e. , solutions (u,θ) belonging to the space

$$(L^2(0,T;H^2(\Omega)^N\cap W)\cap H^1(0,T;H))\times L^2(0,T;H^1(\Omega)).$$

We give it in the following proposition :

Proposition 11 *Let A verify (4.8), $u^0\in H$, $f_1\in L^2(0,T;W')$, $f_2\in L^2(0,T;H^{-1/2}(\partial\Omega)^N)$ and let u be the weak solution of the system*

$$\begin{cases} u_t-\nabla\cdot(Du)+\nabla\theta=f_1 & in\ Q,\\ \nabla\cdot u=0 & in\ Q,\\ u\cdot n=0,\quad (\sigma(u,\theta))_{tg}+(A(x,t)u)_{tg}=f_2 & on\ \Sigma,\\ u(\cdot,0)=u^0(\cdot) & in\ \Omega,\end{cases}\tag{4.16}$$

namely, the function u satisfying

$$\begin{cases} \displaystyle\int_\Omega u_t(t)\cdot v\,dx+\frac12\int_\Omega Du(t):Dv\,dx+\int_{\partial\Omega}Au(t)\cdot v\,d\sigma\\ \displaystyle=\int_\Omega f_1(t)\cdot v\,dx+\int_{\partial\Omega}f_2(t)\cdot v\,d\sigma\qquad a.\ e.\ t\in(0,T)\quad\forall v\in W,\\ u(\cdot,0)=u^0(\cdot)\ \ in\ \Omega.\end{cases}$$

Then, if we also suppose that A verifies (4.9), $u^0\in W$ and

$$f_1\in L^2(Q)^N,\ f_2\in L^2(0,T;H^{1/2}(\partial\Omega)^N),\ f_2\in H^{(1-\ell)/2}(0,T;H^{\ell-1/2}(\partial\Omega)^N),$$

u is actually, together with a pressure θ, the strong solution of (4.16), i.e. ,

$$\begin{gathered} u\in L^2(0,T;H^2(\Omega)^N\cap W),\quad u_t\in L^2(0,T;H),\quad u\in L^\infty(0,T;W),\\ \theta\in L^2(0,T;H^1(\Omega)).\end{gathered}\tag{4.17}$$

Furthermore, there exists a positive constant C such that

$$\begin{aligned} &\|u_t\|^2_{L^2(Q)^N}+\|u\|^2_{L^2(H^2\cap W)}+\|u\|^2_{L^\infty(H^1)}+\|\theta\|^2_{L^2(H^1)}\\ &\quad\le Ce^{CT\|A\|^2_P}(1+\|A\|^4_P)(\|f_1\|^2_{L^2(Q)^N}+\|f_2\|^2_{L^2(H^{1/2})}\\ &\quad+\|f_2\|^2_{H^{(1-\ell)/2}(H^{\ell-1/2})}+\|u^0\|^2_{H^1(\Omega)^N}),\end{aligned}\tag{4.18}$$

where the space P is given by

$$P=H^{1-\ell}(0,T;W^{\nu_1,\nu_1+1}(\partial\Omega)^{N\times N}).\tag{4.19}$$

Proof : We remember here that the definitions of H and W were given in (4.5) and (4.6) at the beginning of the paper. A classical Galerkin's method can be employed in order to prove the existence and uniqueness and obtain estimates of u as weak solution of (4.16). In fact, multiplying the equation in (4.16) by $u(t)$ and integrating in Ω, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u(t)|^2\,dx + \frac{1}{2}\int_\Omega |Du(t)|^2\,dx + \int_{\partial\Omega}(Au)(t)\cdot u(t)\,d\sigma$$
$$= \langle f_1(t), u(t)\rangle_{W',W} + \langle f_2(t), u(t)\rangle_{\partial\Omega}$$

for a.e. $t \in (0,T)$. From the fact that $\|Du\|_{L^2(\Omega)^N}$ is a norm in W equivalent to that of $H^1(\Omega)^N$ (Korn's inequality ; see, for instance, [17]) and the trace inequality

$$\int_{\partial\Omega} |u(t)|^2\,d\sigma \le C\|u(t)\|_{L^2(\Omega)^N}\|u(t)\|_{H^1(\Omega)^N},$$

we find

$$\begin{aligned}&\|u\|^2_{L^\infty(L^2)} + \|u\|^2_{L^2(H^1)}\\ &\qquad \le Ce^{CT\|A\|^2_\infty}(\|f_1\|^2_{L^2(W')} + \|f_2\|^2_{L^2(H^{-1/2})} + \|u^0\|^2_{L^2(\Omega)^N}).\end{aligned} \tag{4.20}$$

Let us now multiply the equation in (4.16) by u_t and integrate in Ω. This yields

$$\begin{aligned}&\int_\Omega |u_t(t)|^2\,dx + \frac{1}{4}\frac{d}{dt}\int_\Omega |Du(t)|^2\,dx - \langle f_2(t), u_t(t)\rangle_{\partial\Omega}\\ &= -\int_{\partial\Omega}(Au(t))\cdot u_t(t)\,d\sigma + \int_\Omega f_1(t)\cdot u_t(t)\,dx \quad \text{a. e. } t\in(0,T).\end{aligned} \tag{4.21}$$

Using $f_1 \in L^2(Q)^N$, we have

$$\begin{aligned}&\frac{1}{2}\int_\Omega |u_t(t)|^2\,dx + \frac{1}{4}\frac{d}{dt}\int_\Omega |Du(t)|^2\,dx - \langle f_2(t), u_t(t)\rangle_{\partial\Omega}\\ &\le \frac{1}{2}\int_\Omega |f_1(t)|^2\,dx - \int_{\partial\Omega}(Au(t))\cdot u_t(t)\,d\sigma \quad \text{a. e. } t\in(0,T).\end{aligned} \tag{4.22}$$

Next, we see (4.16) like a stationary system, that is to say,

$$\begin{cases} -\Delta u(t) + \nabla\theta(t) = f_1(t) - u_t(t) & \text{in } \Omega,\\ \nabla\cdot u(t) = 0 & \text{in } \Omega,\\ u(t)\cdot n = 0, \quad (\sigma(u(t),\theta(t)))_{tg} + (A(x,t)u(t))_{tg} = f_2(t) & \text{on } \partial\Omega,\end{cases} \tag{4.23}$$

for almost every $t\in(0,T)$. The goal will be to prove that the weak solution (u,θ) of the system

$$\begin{cases} -\Delta u + \nabla\theta = g & \text{in } \Omega,\\ \nabla\cdot u = 0 & \text{in } \Omega,\\ u\cdot n = 0, \quad (\sigma(u,\theta))_{tg} = f_3 & \text{on } \partial\Omega,\end{cases} \tag{4.24}$$

actually belongs to

$$(H^2(\Omega)^N \cap W)\times H^1(\Omega)$$

with $g \in L^2(\Omega)^N$ and suitable f_3. The proof we develop here follows the ideas of [3].

Let us first remark that the weak solution of (4.24) verifies

$$\|u\|_{H^1(\Omega)^N} + \|\theta\|_{L^2(\Omega)} \leq C(\|g\|_{W'} + \|f_3\|_{H^{-1/2}(\partial\Omega)^N}), \tag{4.25}$$

for a positive constant C.

The interior regularity readily follows from the corresponding result with Dirichlet conditions and that can be founded in [19], for instance. Then, for every $\Omega' \subset\subset \Omega$, we have $u \in H^2(\Omega')^N$, $\theta \in H^1(\Omega')$ and

$$\|u\|_{H^2(\Omega')^N} + \|\theta\|_{H^1(\Omega')} \leq C\|g\|_{L^2(\Omega)^N}, \tag{4.26}$$

for some positive constant $C(\Omega', \Omega)$.

Let $x^0 \in \partial\Omega$ and let U_0 be a neighborhood of x^0. Then, we will prove that $u \in H^2(\Omega \cap \tilde{U})^N$ and $\theta \in H^1(\Omega \cap \tilde{U})$, for every $\tilde{U} \subset\subset U_0$. To this end, let ψ be a $W^{2,\infty}$ diffeomorphism which sends the set

$$C_0 = \{(\xi', \xi_N) \in \mathbf{R}^N : |\xi_i| < \alpha_0 \quad i = 1, ..., N-1,\ |\xi_N| < \beta_0\}$$

into and onto U_0 and which verifies $\psi(C_0^+) = \Omega \cap U_0$ and $\psi(\Delta_{\alpha_0}) = \partial\Omega \cap U_0$. Here, we have denoted $C_0^+ = C_0 \cap \mathbf{R}^N_+$ and $\Delta_{\alpha_0} = \partial\mathbf{R}^N_+ \cap C_0$. Let us now introduce a cut-off function $\zeta \in C^2(U_0)$ such that

$$\zeta \equiv 1 \ \text{ in } \tilde{U} \quad \text{and} \quad \operatorname{supp}\zeta \subset U_1 \subset\subset U_0,$$

where U_1 is a regular open set.

Then, let us set $z = \zeta u$, $h = \zeta\theta$. They verify :

$$\begin{cases} -\nabla\cdot(Dz) + \nabla h = g^* & \text{in } \Omega \cap U_0, \\ \nabla \cdot z = g_1^* & \text{in } \Omega \cap U_0, \\ z \cdot n = 0, \quad (\sigma(z,h)\cdot n)_{tg} = g_2^* & \text{on } \partial\Omega \cap U_0, \\ z = 0 & \text{on } \Omega \cap \partial U_0, \end{cases} \tag{4.27}$$

with

$$g^* = -\zeta g - 2\nabla\zeta\cdot\nabla u - \nabla\zeta\cdot\nabla^t u - \Delta\zeta u - \nabla\nabla\zeta\cdot u + \theta\nabla\zeta \in L^2(\Omega\cap U_0)^N,$$

$$g_1^* = \nabla\zeta\cdot u \in H^1(\Omega \cap U_0) \quad \text{and} \quad g_2^* = \zeta f_3 + \frac{\partial\zeta}{\partial n} u \in H^{1/2}(\partial\Omega \cap U_0)^N.$$

The weak solution of (4.27) is given by $(z, h) \in X \times L^2(\Omega \cap U_0)$ satisfying

$$\begin{cases} \displaystyle\int_{\Omega\cap U_0} Dz : \nabla v\, dx - \int_{\Omega\cap U_0} h\nabla\cdot v\, dx \\ \qquad\qquad = \displaystyle\int_{\Omega\cap U_0} g^*\cdot v\, dx + \int_{\partial\Omega\cap U_0} g_2^*\cdot v\, d\sigma \quad \forall v \in X, \\ \nabla\cdot z = g_1^* \ \text{ in } \Omega\cap U_0. \end{cases}$$

Here, we have denoted

$$X = \{v \in H^1(\Omega \cap U_0)^N : v = 0 \ \text{ on } \partial U_0 \cap \Omega, v\cdot n = 0 \ \text{ on } \partial\Omega \cap U_0\}.$$

Let us now perform the change of variables $x = \psi(\xi)$. If we define $\tilde{z} = z \circ \psi$ and $\tilde{h} = h \circ \psi$, they verify :

$$\left\{\begin{array}{l} \displaystyle\sum_{i,j} \int_{C_0^+} (\nabla \tilde{z}^i \cdot \partial_j \psi^{-1} + \nabla \tilde{z}^j \cdot \partial_i \psi^{-1})(\nabla \tilde{v}^i \cdot \partial_j \psi^{-1})(\xi)|J(\psi)|(\xi)\, d\xi \\ \displaystyle -\int_{C_0^+} \tilde{h}(\nabla \tilde{v} : \nabla^t \psi^{-1})(\xi)|J(\psi)|(\xi)\, d\xi = \int_{C_0^+} (\tilde{g}^* \cdot \tilde{v})(\xi)|J(\psi)|(\xi)\, d\xi \\ \displaystyle +\int_{\Delta_{\alpha_0}} (\tilde{g}_2^* \cdot \tilde{v})(\xi', 0)|J(\psi)|(\xi', 0)\, d\xi' \quad \forall \tilde{v} \in \tilde{X}, \\ \nabla \tilde{z} : \nabla^t \psi^{-1} = \tilde{g}_1^* \text{ in } C_0^+, \end{array}\right. \tag{4.28}$$

with

$$\tilde{X} = \{\tilde{v} \in H^1(C_0^+)^N : \tilde{v} = 0 \quad \text{on } \partial C_0^+ \cap \mathbf{R}_+^N,\ \tilde{v} \cdot n = 0 \text{ on } \Delta_{\alpha_0}\}$$

and where we have denoted

$$\tilde{g}^* = g^* \circ \psi, \quad \tilde{g}_1^* = g_1^* \circ \psi, \quad \tilde{g}_2^* = g_2^* \circ \psi$$

and $|J(\psi)|$ the determinant of the Jacobian of ψ.

Let us introduce $C_1 = \psi^{-1}(U_1)$ and $d = dist(\partial C_0^+, \partial C_1^+)$. Then, for every function $\tilde{v} \in H^1(C_1^+)^N$ verifying

$$\tilde{v} = 0 \text{ on } \partial C_1^+ \cap \mathbf{R}_+^N \quad \text{and} \quad \tilde{v} \cdot n \text{ on } \partial C_1^+ \cap \partial \mathbf{R}_+^N = \Delta_{\alpha_1}$$

(that is to say, $\tilde{v} \in \tilde{X}_1$), we have $\delta_k^{-m} \tilde{v} \in \tilde{X}$, for $k \in \{1, ..., N-1\}$ and $|m| \leq d/2$. We remind that, by definition,

$$\delta_k^m(x) = \frac{v(x + me_k) - v(x)}{m}.$$

This allows us to plug $\delta_k^{-m} \tilde{v}$ into expression (4.28) :

$$\left\{\begin{array}{l} \displaystyle\sum_{i,j} \int_{C_1^+} \delta_k^m [|J(\psi)|(\nabla \tilde{z}^i \cdot \partial_j \psi^{-1} + \nabla \tilde{z}^j \cdot \partial_i \psi^{-1})\partial_j \psi^{-1}] \cdot \nabla \tilde{v}^i\, d\xi \\ \displaystyle -\int_{C_1^+} \delta_k^m (|J(\psi)|\tilde{h} \nabla^t \psi^{-1}) : \nabla \tilde{v}\, d\xi \\ \displaystyle \quad = \int_{C_1^+} \delta_k^m (|J(\psi)|\tilde{g}^*) \cdot \tilde{v}\, d\xi + \int_{\Delta_{\alpha_1}} \delta_k^m (|J(\psi)|\tilde{g}_2^*) \cdot \tilde{v}\, d\xi' \quad \forall \tilde{v} \in \tilde{X}_1, \\ \nabla \tilde{z} : \nabla^t \psi^{-1} = \tilde{g}_1^* \text{ in } C_1^+. \end{array}\right. \tag{4.29}$$

Let us compute each one of the previous integrals, taking into account (4.25) and the formula :

$$\delta_k^m(v_1 v_2)(x) = v_1(x) \delta_k^m v_2(x) + \delta_k^m v_1(x) v_2(x + me_k).$$

First, we have

$$\sum_{i,j} \int_{C_1^+} \delta_k^m [|J(\psi)|(\nabla \tilde{z}^i \cdot \partial_j \psi^{-1} + \nabla \tilde{z}^j \cdot \partial_i \psi^{-1})\partial_j \psi^{-1}] \cdot \nabla \tilde{v}^i\, d\xi$$

$$= \sum_{i,j} \int_{C_1^+} |J(\psi)|(\nabla(\delta_k^m \tilde{z}^i) \cdot \partial_j \psi^{-1} + \nabla(\delta_k^m \tilde{z}^j) \cdot \partial_i \psi^{-1})(\nabla \tilde{v}^i \cdot \partial_j \psi^{-1})\, d\xi + I_1$$

with

$$|I_1| \leq C\|\psi\|_{W^{2,\infty}(C_0^+)^N}\|\tilde{z}\|_{H^1(C_1^+)^N}\|\tilde{v}\|_{H^1(C_1^+)^N}$$
$$\leq C(\|g\|_{W'} + \|f_3\|_{H^{-1/2}(\partial\Omega)^N})\|\tilde{v}\|_{H^1(C_1^+)^N}.$$

Additionally, we find

$$-\int_{C_1^+} \delta_k^m(|J(\psi)|\tilde{h}\nabla^t\psi^{-1}) : \nabla\tilde{v}\, d\xi = -\int_{C_1^+} |J(\psi)|\delta_k^m\tilde{h}\nabla^t\psi^{-1} : \nabla\tilde{v}\, d\xi + I_2,$$

with

$$|I_2| \leq C\|\tilde{h}\|_{L^2(C_1^+)}\|\tilde{v}\|_{H^1(C_1^+)^N} \leq C(\|g\|_{W'} + \|f_3\|_{H^{-1/2}(\partial\Omega)^N})\|\tilde{v}\|_{H^1(C_1^+)^N}.$$

Furthermore,

$$I_3 = \int_{C_1^+} \delta_k^m(|J(\psi)|\tilde{g}^*)\cdot\tilde{v}\, d\xi \leq C\|\tilde{g}^*\|_{L^2(C_1^+)^N}\|\tilde{v}\|_{\tilde{X}_1}$$
$$\leq C(\|g\|_{L^2(\Omega)^N} + \|f_3\|_{H^{-1/2}(\partial\Omega)^N})\|\tilde{v}\|_{H^1(C_1^+)^N},$$

where we have denoted

$$\|\tilde{v}\|_{\tilde{X}_1}^2 = \sum_{j=1}^{N-1}\int_{C_1^+} |\partial_j\tilde{v}|^2\, dx + \int_{C_1^+} |\partial_N\tilde{v} + \nabla\tilde{v}_N|^2\, dx.$$

Finally, we have

$$I_4 = \int_{\Delta_{\alpha_1}} \delta_k^m(|J(\psi)|\tilde{g}_2^*)\cdot\tilde{v}\, d\xi' = \int_{\Delta_{\alpha_1+d/2}} \delta_k^m(|J(\psi)|\tilde{g}_2^*)\cdot\tilde{v}\, d\xi'$$
$$\leq C\|\delta_h^m\tilde{g}_2^*\|_{H_{00}^{1/2}(\Delta_{\alpha_1+d/2})'}\|\tilde{v}\|_{H_{00}^{1/2}(\Delta_{\alpha_1+d/2})}$$
$$\leq C\|\tilde{g}_2^*\|_{H^{1/2}(\Delta_{\alpha_1})^N}\|\tilde{v}\|_{H^{1/2}(\Delta_{\alpha_1})^N}$$
$$\leq C(\|g\|_{L^2(\Omega)^N} + \|f_3\|_{H^{1/2}(\partial\Omega)^N})\|\tilde{v}\|_{H^1(C_1^+)^N}.$$

Here, we have employed the notation

$$H_{00}^{1/2}(\Delta) = \{v \in H^{1/2}(\Delta) : \rho_\Delta^{-1/2} v \in L^2(\Delta)\} \quad \text{with} \quad \rho_\Delta(x) = dist(x, \partial\Delta)$$

and we have used the fact that

$$\frac{\partial}{\partial x_i} \in \mathcal{L}(H^{1/2}, (H_{00}^{1/2})')$$

(see [14] for more details).

Consequently, we obtain an equivalent formulation of (4.29) :

$$\begin{cases} a_0(\delta_k^m\tilde{z}, \tilde{v}) + b_0(\tilde{v}, \delta_k^m\tilde{h}) = -(I_1 + I_2 + I_3 + I_4) \quad \forall\tilde{v} \in \tilde{X}_1, \\ b_0(\delta_k^m\tilde{z}, \tilde{f}_0) = -I_5 \quad \forall\tilde{f}_0 \in L^2(C_1^+), \end{cases} \tag{4.30}$$

with

$$a_0(v_1, v_2) = \sum_{i,j}\int_{C_1^+} |J(\psi)|(\nabla v_1^i\cdot\partial_j\psi^{-1} + \nabla v_1^j\cdot\partial_i\psi^{-1})(\partial_j\psi^{-1}\cdot\nabla v_2^i)\, d\xi,$$

$$b_0(v_2, v_3) = -\int_{C_1^+} |J(\psi)| v_3 (\nabla^t \psi^{-1} : \nabla v_2)$$

for v_1, $v_2 \in \tilde{X}_1$ and $v_3 \in L^2(C_1^+)$ and

$$I_5 = \int_{C_1^+} |J(\psi)| \tilde{f}_0 (\nabla^t \psi^{-1} : \nabla(\delta_k^m \tilde{z}))\, d\xi.$$

From the last condition in (4.29), we find

$$I_5 = \int_{C_1^+} |J(\psi)| \tilde{f}_0 \delta_k^m \tilde{g}_1^* \, d\xi - \int_{C_1^+} |J(\psi)| \tilde{f}_0 (\delta_k^m (\nabla \psi^{-1}) : \nabla \tilde{z}(\xi + m e_k))\, d\xi,$$

so

$$\begin{aligned} |I_5| &\leq C(\|\tilde{g}_1^*\|_{H^1(C_1^+)} + \|\tilde{z}\|_{H^1(C_1^+)^N}) \|\tilde{f}_0\|_{L^2(C_1^+)} \\ &\leq C(\|g\|_{W'} + \|f_3\|_{H^{-1/2}(\partial\Omega)^N}) \|\tilde{f}_0\|_{L^2(C_1^+)}. \end{aligned}$$

The mixed problem (4.29) will posses a unique solution $(\delta_k^m \tilde{z}, \delta_k^m \tilde{h}) \in \tilde{X}_1 \times L^2(C_1^+)$ if we prove that a_0 is continuous and coercive in $\tilde{X}_1$ and that b_0 is continuous and verifies the *inf-sup* condition in $\tilde{X}_1 \times L^2(C_1^+)$.

The continuity of a_0 and b_0 are trivial. Let us prove that a is coercive; for $\tilde{v}_1 \in \tilde{X}_1$, we have

$$\begin{aligned} a_0(\tilde{v}_1, \tilde{v}_1) &= \sum_{i,j} \int_{C_1^+} |J(\psi)| (\nabla \tilde{v}_1^i \cdot \partial_j \psi^{-1} + \nabla \tilde{v}_1^j \cdot \partial_i \psi^{-1})(\nabla \tilde{v}_1^i \cdot \partial_j \psi^{-1})\, d\xi \\ &= \frac{1}{2} \sum_{i,j} \int_{C_1^+} |J(\psi)| |\nabla \tilde{v}_1^i \cdot \partial_j \psi^{-1} + \nabla \tilde{v}_1^j \cdot \partial_i \psi^{-1}|^2 \, d\xi = \frac{1}{2} \int_{\Omega \cap U_1} |D v_1|^2 \, dx \\ &\geq C \|v_1\|^2_{H^1(\Omega \cap U_1)^N} \geq \tilde{C} \|\tilde{v}_1\|^2_{H^1(C_1^+)^N}. \end{aligned}$$

Finally, the *inf-sup* condition for b_0 tells that

$$\sup_{\tilde{v} \in \tilde{X}_1 \setminus \{0\}} \frac{b_0(\tilde{v}, \tilde{f}_0)}{\|\tilde{v}\|_{H^1(C_1^+)^N}} \geq C_2 \|\tilde{f}_0\|_{L^2(C_1^+)} \quad \forall \tilde{f}_0 \in L^2(C_1^+).$$

To prove this, we first observe that

$$b_0(\tilde{v}, \tilde{f}_0) = \int_{\Omega \cap U_1} f_0 \nabla \cdot v \, dx.$$

Now, for $f_0 \in L^2(\Omega \cap U_1) \setminus \{0\}$, we consider $v \in H_0^1(\Omega \cap U_1)^N$ such that $\nabla \cdot v = f_0$ and

$$\|v\|_{H^1(\Omega \cap U_1)^N} \leq C_3 \|f_0\|_{L^2(\Omega \cap U_1)}$$

(see, for instance, [10]). Consequently,

$$\begin{aligned} \sup_{\tilde{v} \in \tilde{X}_1 \setminus \{0\}} \frac{b_0(\tilde{v}, \tilde{f}_0)}{\|\tilde{v}\|_{H^1(C_1^+)^N}} &\geq C_4 \frac{\displaystyle\int_{\Omega \cap U_1} f_0 \nabla \cdot v \, dx}{\|v\|_{H^1(\Omega \cap U_1)^N}} = C_4 \frac{\|f_0\|^2_{L^2(\Omega \cap U_1)}}{\|v\|_{H^1(\Omega \cap U_1)^N}} \\ &\geq C_3^{-1} C_4 \|f_0\|_{L^2(\Omega \cap U_1)} \geq C_2 \|\tilde{f}_0\|_{L^2(C_1^+)}, \end{aligned}$$

as we wanted to see.

As a conclusion, there exists a unique solution $(\delta_k^m \tilde{z}, \delta_k^m \tilde{h}) \in \tilde{X}_1 \times L^2(C_1^+)$ of (4.30) and

$$\|\delta_k^m \tilde{z}\|_{\tilde{X}_1} + \|\delta_k^m \tilde{h}\|_{L^2(C_1^+)} \leq C(\|g\|_{L^2(\Omega)^N} + \|f_3\|_{H^{1/2}(\partial\Omega)^N})$$

for $k \in \{1, ..., N-1\}$. This tells that $(\partial_k \tilde{z}, \partial_k \tilde{h}) \in H^1(C_1^+)^N \times L^2(C_1^+)$, so that $(\partial_k z, \partial_k h) \in H^1(\Omega \cap \tilde{U})^N \times L^2(\Omega \cap \tilde{U})$. Finally, from the divergence free condition (in $\Omega \cap \tilde{U}$) and the differential equation verified by (z, h), we have $\partial_{NN} z \in L^2(\Omega \cap \tilde{U})^N$ which implies that $\partial_N h \in L^2(\Omega \cap \tilde{U})$.

Therefore, $(u, \theta) \in H^2(\Omega \cap \tilde{U})^N \times H^1(\Omega \cap \tilde{U})$ for every $\tilde{U} \subset\subset U$ and

$$\|u\|_{H^2(\Omega \cap \tilde{U})^N} + \|\theta\|_{H^1(\Omega \cap \tilde{U})} \leq C(\|g\|_{L^2(\Omega)^N} + \|f_3\|_{H^{1/2}(\partial\Omega)^N}).$$

Combining this estimate with the local one (4.26), we obtain the estimate for the solution of (4.23), say :

$$\begin{aligned} &\|u(t)\|_{H^2(\Omega)^N} + \|\theta(t)\|_{H^1(\Omega)} \leq (\|f_1(t)\|_{L^2(\Omega)^N} + \|u_t(t)\|_{L^2(\Omega)^N} \\ &+\|f_2(t)\|_{H^{1/2}(\partial\Omega)^N} + \|Au(t)\|_{H^{1/2}(\partial\Omega)^N}) \quad \text{a. e. } t \in (0, T). \end{aligned}$$

Let us now put this estimate together with (4.22). We obtain

$$\begin{aligned} &\|u_t(t)\|^2_{L^2(\Omega)^N} + \|u(t)\|^2_{H^2(\Omega)^N} + \frac{d}{dt}\int_\Omega |Du(t)|^2\,dx + \|\theta(t)\|^2_{H^1(\Omega)} \\ &\leq C\Big(\|f_1(t)\|^2_{L^2(\Omega)^N}\,dx + \|f_2(t)\|^2_{H^{1/2}(\partial\Omega)^N} + |\langle f_2(t), u_t(t)\rangle_{\partial\Omega}| \\ &+\|Au(t)\|^2_{H^{1/2}(\partial\Omega)^N} + \int_{\partial\Omega} |Au(t)||u_t(t)|\,d\sigma\Big), \end{aligned}$$

for almost every $t \in (0, T)$. A classical argument based on Gronwall's lemma leads to the absorption of the fourth term in the right hand side :

$$\begin{aligned} &\|u_t\|^2_{L^2(Q)^N} + \|u\|^2_{L^2(H^2)} + \|u\|^2_{L^\infty(H^1)} + \|\theta\|^2_{L^2(H^1)} \\ &\leq Ce^{CT\|A\|^2_{L^\infty(W^{1,\nu})}}\Big(\|f_1\|^2_{L^2(Q)^N}\,dx + \|f_2\|^2_{L^2(H^{1/2})} + \|u^0\|^2_{H^1(\Omega)^N} \\ &+ \int_0^T |\langle f_2(t), u_t(t)\rangle_{\partial\Omega}|\,dt + \iint_\Sigma |A||u||\partial_t u|\,d\sigma\,dt\Big), \end{aligned} \tag{4.31}$$

with $\nu > N-1$ arbitrarily small. Indeed, this readily follows from the fact

$$W^{1,\nu}(\partial\Omega) \cdot H^{1/2}(\partial\Omega) \subset H^{1/2}(\partial\Omega) \quad \text{continuously.}$$

In order to estimate the last term on (4.31), observe that for a vector valued function e^0 verifying $e^0_t \in L^2(Q)^N$ and $e^0 \in L^2(0, T; H^1(\Omega)^N)$, we have

$$e^0 \in H^{\ell'}(0, T; H^{1-\ell'}(\Omega)^N)$$

for every $0 < \ell' < 1$ and there exists a constant $C > 0$ such that

$$\|e^0\|_{H^{\ell'}(H^{1-\ell'})} \leq C\|e_t^0\|_{L^2(Q)^N}^{\ell'}\|e^0\|_{L^2(H^1)}^{1-\ell'}.$$

In particular, one can check that

$$\langle \partial_t(\gamma_0 e^0), e^1(\gamma_0 e^0)\rangle < +\infty$$

for $e^1 \in H^{1-\ell}(0,T; H^{\nu/2}(\partial\Omega)^N)$. Besides, we get

$$|\langle \partial_t(\gamma_0 e^0), e^1(\gamma_0 e^0)| \leq C\|e^1\|_{H^{1-\ell}(H^{\nu/2})}\|e_t^0\|_{L^2(Q)^N}\|e^0\|_{L^2(H^1)}.$$

This way, the last term in (4.31) can be estimated as follows :

$$\begin{aligned}
&\iint_\Sigma |A||u||\partial_t u|\, d\sigma\, dt \\
&\qquad \leq \varepsilon\|u_t\|_{L^2(Q)^N}^2 + C(1 + \|A\|_{H^{1-\ell}(H^{-3/2+N/2+2\ell'})}^2)\|u\|_{L^2(H^1)}^2,
\end{aligned}$$

for a small positive constant $\varepsilon(\Omega)$. Then,

$$\begin{aligned}
&\|u_t\|_{L^2(Q)^N}^2 + \|u\|_{L^2(H^2)}^2 + \|u\|_{L^\infty(H^1)}^2 + \|\theta\|_{L^2(H^1)}^2 \\
&\leq Ce^{CT\|A\|_{L^\infty(W^{1,\nu})}^2}\Big(\|f_1\|_{L^2(Q)^N}^2 + \|f_2\|_{L^2(H^{1/2})}^2 + \|u^0\|_{H^1(\Omega)^N}^2 \\
&+ \int_0^T |\langle f_2(t), u_t(t)\rangle_{\partial\Omega}|\, dt + (1 + \|A\|_{H^{1-\ell}(H^{\nu/2})}^2)\|u\|_{L^2(H^1)}^2\Big)
\end{aligned} \tag{4.32}$$

which, combined with (4.20), yields

$$\begin{aligned}
&\|u_t\|_{L^2(Q)^N}^2 + \|u\|_{L^2(H^2)}^2 + \|u\|_{L^\infty(H^1)}^2 + \|\theta\|_{L^2(H^1)}^2 \\
&\leq Ce^{CT\|A\|_{H^{1-\ell}(W^{\nu_1,\nu_1+1})}^2}(1 + \|A\|_{H^{1-\ell}(W^{1,\nu_1})}^2)\Big(\|f_1\|_{L^2(Q)^N}^2 \\
&+\|f_2\|_{L^2(H^{1/2})}^2 + \|u^0\|_{H^1(\Omega)^N}^2 + \int_0^T |\langle f_2(t), u_t(t)\rangle_{\partial\Omega}|\, dt\Big).
\end{aligned} \tag{4.33}$$

Here, we have used the fact (recall that ℓ is close to $1/2$ and the definition of ν_1)

$$H^{1-\ell}(0,T; W^{\nu_1,\nu_1+1}(\partial\Omega)) \subset L^\infty(0,T, W^{1,\nu} \cap H^{\nu/2}(\partial\Omega)) \quad \text{continuously.}$$

Let us now combine $u_t \in L^2(Q)^N$ and $u \in L^2(0,T; H^2(\Omega)^N)$ to obtain an estimate of $\langle f_2, u_t\rangle_{\partial\Omega}$ in appropriate spaces. More precisely, we have

$$u \in H^{(1+\ell)/2}(0,T; H^{1-\ell}(\Omega)^N) \tag{4.34}$$

and

$$\|u\|_{H^{(1+\ell)/2}(H^{1-\ell})} \leq C\|u_t\|_{L^2(Q)^N}^{(1+\ell)/2}\|u\|_{L^2(H^2)}^{(1-\ell)/2}. \tag{4.35}$$

We find :

$$\int_0^T |\langle f_2(t), u_t(t)\rangle_{\partial\Omega}|\, dt \leq C\|f_2\|_{H^{(1-\ell)/2}(H^{\ell-1/2})}\|u_t\|_{L^2(Q)^N}^{(1+\ell)/2}\|u\|_{L^2(H^2)}^{(1-\ell)/2}$$
$$\leq C(1+\|A\|^2_{H^{1-\ell}(W^{\nu_1,\nu_1+1})})\|f_2\|^2_{H^{(1-\ell)/2}(H^{\ell-1/2})}$$
$$+\varepsilon_1(1+\|A\|^2_{H^{1-\ell}(W^{\nu_1,\nu_1+1})})^{-1}\|u_t\|^{1+\ell}_{L^2(Q)^N}\|u\|^{1-\ell}_{L^2(H^2)}$$
$$\leq C(1+\|A\|^2_{H^{1-\ell}(W^{\nu_1,\nu_1+1})})\|f_2\|^2_{H^{(1-\ell)/2}(H^{\ell-1/2})}$$
$$+\varepsilon_2(1+\|A\|^2_{H^{1-\ell}(W^{\nu_1,\nu_1+1})})^{-1}(\|u_t\|^2_{L^2(Q)^N}+\|u\|^2_{L^2(H^2)}),$$

where ε_1 and ε_2 are small positive constants depending on Ω and ℓ. Plugging this into (4.33), we find

$$\|u_t\|^2_{L^2(Q)^N}+\|u\|^2_{L^2(H^2\cap W)}+\|u\|^2_{L^\infty(H^1)}+\|\theta\|^2_{L^2(H^1)}$$
$$\leq Ce^{CT\|A\|^2_{H^{1-\ell}(W^{\nu_1,\nu_1+1})}}(1+\|A\|^4_{H^{1-\ell}(W^{\nu_1,\nu_1+1})})(\|f_1\|^2_{L^2(Q)^N}$$
$$+\|f_2\|^2_{L^2(H^{1/2})}+\|f_2\|^2_{H^{(1-\ell)/2}(H^{\ell-1/2})}+\|u^0\|^2_{H^1(\Omega)^N}),$$

which is exactly (4.18).

This ends the proof of proposition 11.

Finally, we establish a further regularity result when the data is supposed to be more regular. This will be used when proving the local null controllability to the trajectories of system (4.1) in the last section. More precisely, it concerns a linear Stokes system similar to (4.16) but with null f_2 :

$$\begin{cases} u_t - \nabla\cdot(Du)+\nabla\theta = f_4 & \text{in } Q,\\ \nabla\cdot u = 0 & \text{in } Q,\\ u\cdot n = 0,\ (\sigma(u,\theta)\cdot n)_{tg}+(A(x,t)u)_{tg}=0 & \text{on } \Sigma,\\ u(\cdot,0)=u^0(\cdot) & \text{in } \Omega. \end{cases} \tag{4.36}$$

Proposition 12 *Let* $f_4 \in H^1(0,T;L^2(\Omega)^N)\cap C([0,T];H^1(\Omega)^N)$ *with* $f_4\cdot n = 0$, $u^0\in H^3(\Omega)^N\cap W$ *satisfying the compatibility condition*

$$(Du^0\cdot n)_{tg}+(A(x,0)u^0)_{tg}=0 \text{ on } \partial\Omega \tag{4.37}$$

and let A *satisfy* (4.8)-(4.10).

Then, the strong solution u *of* (4.36) *actually verifies* $u\in H^1(0,T;H^2(\Omega)^N\cap W)\cap H^2(0,T;H)$ *and*

$$\|u\|^2_{H^1(H^2\cap W)}+\|u\|^2_{H^2(L^2)}\leq C(\Omega,A)(\|f_4\|^2_{H^1(L^2)\cap L^\infty(H^1)}+\|u^0\|^2_{H^3\cap W}), \tag{4.38}$$

where C *is a positive constant.*

Proof : From proposition 11, we already know that u is a strong solution of (4.36) (i.e., it verifies (4.17)). In particular, $u\in H^1(0,T;H)$ and

$$\|u\|^2_{H^1(L^2)}\leq C(\Omega,\|A\|_P)(\|u^0\|^2_W+\|f_4\|^2_{L^2(Q)^N}).$$

Next, we are going to give sense to $\partial_t u$ as the strong solution of the following system :

$$\begin{cases} \tilde{u}_t - \nabla \cdot (D\tilde{u}) + \nabla\tilde{\theta} = \partial_t f_4 & \text{in } Q, \\ \nabla \cdot \tilde{u} = 0 & \text{in } Q, \\ \tilde{u} \cdot n = 0,\ (D\tilde{u} \cdot n)_{tg} + (A(x,t)\tilde{u})_{tg} = -(A(x,t))_t u & \text{on } \Sigma, \\ \tilde{u}(\cdot,0) = \partial_t u(\cdot,0) & \text{in } \Omega. \end{cases} \tag{4.39}$$

To this end, we must first check that

$$A_t\gamma_0 u \in L^2(0,T;H^{-1/2}(\partial\Omega)^N)$$

and

$$\partial_t u(\cdot,0) \in H. \tag{4.40}$$

From the fact that

$$u \in L^2(0,T;H^2(\Omega)^N) \cap H^1(0,T;L^2(\Omega)^N) \subset H^{1-\ell}(0,T;H^{2\ell-1/2}(\partial\Omega)^N)$$

continuously and (4.10), we have

$$\begin{aligned} A_t\gamma_0 u \in L^2(0,T;H^{1-2\ell}(\partial\Omega)^{N^2}) \cdot H^{1-\ell}(0,T;H^{2\ell-1/2}(\partial\Omega)^N) \\ \subset L^2(0,T;H^{-1/2}(\partial\Omega)^N) \ \text{ continuously,} \end{aligned}$$

so

$$\|A_t\gamma_0 u\|^2_{L^2(H^{-1/2})} \le C\|A\|^2_{H^1(H^{1-2\ell})}(\|u\|^2_{H^1(L^2)} + \|u\|^2_{L^2(H^2)}).$$

In order to prove (4.40), we use the differential equation in (4.36) and we find

$$\begin{aligned} &\int_\Omega |\partial_t u(0)|^2\,dx = \int_\Omega \partial_t u(0) \cdot (\nabla \cdot (Du^0) - \nabla\theta(0) + f_4(0))\,dx \\ &= \int_\Omega \partial_t u(0) \cdot (\nabla \cdot (Du^0) + f_4(0))\,dx \\ &\le \frac{1}{2}\int_\Omega |\partial_t u(0)|^2\,dx + C(\|u^0\|^2_{H^2\cap W} + \|f_4\|^2_{L^\infty(L^2)}). \end{aligned}$$

Consequently, we have that (4.39) has a unique weak solution $\tilde{u} \in L^2(0,T;W) \cap L^\infty(0,T;H)$, which must coincide with $\partial_t u$. Therefore, $\partial_t u \in L^2(0,T;W) \cap L^\infty(0,T;H)$ and

$$\|\partial_t u\|^2_{L^2(H^1)\cap L^\infty(L^2)} \le C(\Omega,A)(\|f_4\|^2_{H^1(L^2)} + \|u^0\|^2_{H^2\cap W}).$$

Finally and by virtue of proposition 11, we must check that

$$A_t\gamma_0 u \in L^2(0,T;H^{1/2}(\partial\Omega)^N) \cap H^{(1-\ell)/2}(0,T;H^{-1/2+\ell}(\partial\Omega)^N)$$

and

$$\partial_t u(\cdot,0) \in W. \tag{4.41}$$

The first fact follows from (4.10) and

$$u \in H^1(0,T;H^1(\Omega)^N) \cap L^2(0,T;H^2(\Omega)) \subset H^{1-\ell}(0,T;H^{\ell+1/2}(\partial\Omega)^N)$$

continuously. Indeed, it is not difficult to see that

$$\begin{aligned} A_t\gamma_0 u &\in L^2(0,T;H^{\nu_2}(\partial\Omega)^{N^2}) \cdot H^{1-\ell}(0,T;H^{\ell+1/2}(\partial\Omega)^N) \\ &\subset L^2(0,T;H^{1/2}(\partial\Omega)^N) \ \text{ continuously} \end{aligned}$$

and

$$\begin{aligned} A_t\gamma^0 u &\in H^{(1-\ell)/2}(0,T;H^{(3-N)(\ell-1/2)}(\partial\Omega)^{N^2})H^{1-\ell}(0,T;H^{\ell+1/2}(\partial\Omega)^N) \\ &\subset H^{(1-\ell)/2}(0,T;H^{-1/2+\ell}(\partial\Omega)^N) \ \text{ continuously.} \end{aligned} \tag{4.42}$$

Besides,

$$\begin{aligned} &\|A_t\gamma_0 u\|^2_{L^2(H^{1/2})} + \|A_t\gamma_0 u\|^2_{H^{(1-\ell)/2}(H^{-1/2+\ell})} \\ &\qquad \le C\|A\|^2_{H^{(3-\ell)/2}(H^{\nu_2})}(\|u\|^2_{H^1(H^1)} + \|u\|^2_{L^2(H^2)}), \end{aligned}$$

for a positive constant $C = C(\Omega)$.

Let us now prove (4.41). For this, we first realize that $\theta(0) \in H^2(\Omega)^N$ from the elliptic system

$$\begin{cases} \Delta\theta(0) = \nabla \cdot f_4(0) & \text{in } \Omega, \\ \dfrac{\partial}{\partial n}\theta(0) = \Delta u^0 \cdot n & \text{on } \partial\Omega, \end{cases}$$

which satisfies the compatibility condition

$$\int_{\partial\Omega} \Delta u^0 \cdot n \, d\sigma = \int_\Omega \nabla \cdot f_4(0)\, dx = 0.$$

Hence, $\theta(0) \in H^2(\Omega)^N$ and

$$\|\theta(0)\|^2_{H^2(\Omega)} \le C(\|f_4\|^2_{L^\infty(H^1)} + \|u^0\|^2_{H^3\cap W}).$$

Again from the differential equation satisfied by u, we find

$$\partial_i\partial_t u(0) = \partial_i\Delta u^0 - \partial_i\nabla\theta(0) + \partial_i(f_4(0)) \in L^2(\Omega)^N,$$

so that

$$\|\partial_i\partial_t u(0)\|^2_{L^2(\Omega)^N} \le C(\|f_4\|^2_{L^\infty(H^1)} + \|u^0\|^2_{H^3\cap W})$$

for every $i \in \{1, ..., N\}$. Consequently, $\partial_t u(0) \in H^1(\Omega)^N$ and

$$\|\partial_t u(0)\|^2_{H^1} \le C(\|f_4\|^2_{L^\infty(H^1)} + \|f_4\|^2_{H^1(L^2)} + \|u^0\|^2_{H^3\cap W}).$$

As a conclusion, $\tilde{u} \in L^2(0,T;H^2(\Omega)^N \cap W) \cap H^1(0,T;H)$, so it has to be the case of $\partial_t u$ as well since u^0 satisfies (4.37). Furthermore,

$$\|\partial_t u\|^2_{L^2(H^2\cap W)\cap H^1(L^2)} \le C(\Omega, A)(\|f_4\|^2_{L^\infty(H^1)} + \|f_4\|^2_{H^1(L^2)} + \|u^0\|^2_{H^3\cap W}).$$

From this, (4.38) is readily deduced.

Remark 7 *One could keep the explicit dependence of $C(\Omega, A)$ in (4.38) with respect to the norms of A but this will not be needed, so me omit it for the sake of simplicity.*

2 Carleman inequality for the adjoint system

In this section, we will prove a Carleman inequality for system (4.11). In order to do this, some weight functions are needed :

$$
\begin{aligned}
&\alpha(x,t) = \frac{e^{2\lambda\|\eta^0\|_\infty} - e^{\lambda\eta^0(x)}}{t^4(T-t)^4}, \quad \xi(x,t) = \frac{e^{\lambda\eta^0(x)}}{t^4(T-t)^4},\\
&\tilde{\alpha}(x,t) = \frac{e^{2\lambda\|\eta^0\|_\infty} - e^{-\lambda\eta^0(x)}}{t^4(T-t)^4}, \quad \tilde{\xi}(x,t) = \frac{e^{-\lambda\eta^0(x)}}{t^4(T-t)^4},\\
&\alpha^*(t) = \max_{x\in\overline{\Omega}} \alpha(x,t) = \alpha_{|\partial\Omega}(x,t), \quad \widehat{\alpha}(t) = \min_{x\in\overline{\Omega}} \alpha(x,t).
\end{aligned}
\tag{4.43}
$$

Here, $\eta^0 \in C^2(\overline{\Omega})$ verifies

$$
\eta^0 > 0 \text{ in } \Omega, \quad \eta^0 \equiv 0 \text{ on } \partial\Omega, \quad |\nabla\eta^0| > 0 \text{ in } \overline{\Omega\setminus\omega'} \tag{4.44}
$$

with $\omega' \subset\subset \omega$ an open set. Let us remark that functions of this kind were first introduced in [9] in order to obtain Carleman inequalities for the heat system. The existence of such a function is also proved in that reference.

2.1 Carleman inequality for the heat system

In this paragraph, we will deduce a Carleman inequality for a (vector valued) function φ verifying

$$
\left\{
\begin{array}{ll}
-\varphi_t - \nabla\cdot(D\varphi) = G \in L^2(Q)^N, \quad \nabla\cdot\varphi = 0 & \text{in } Q,\\
\varphi\cdot n = 0, \quad (D\varphi\cdot n)_{tg} + (A(x,t)\varphi)_{tg} = 0 & \text{on } \Sigma,
\end{array}
\right.
\tag{4.45}
$$

It is the following :

Proposition 13 *Let A verify* (4.8)-(4.9)*. Then, there exist three positive constants C, $\overline{s}$ and $\overline{\lambda}$ only depending on Ω and ω, such that*

$$
I(s,\lambda;\varphi) \le C\left(s^3\lambda^4\int_0^T\int_{\omega_0} e^{-2s\alpha}\xi^3|\varphi|^2\,dx\,dt + \lambda\iint_Q e^{-2s\alpha}|G|^2\,dx\,dt\right) \tag{4.46}
$$

for any $\lambda \ge \overline{\lambda}e^{\overline{\lambda}T\|A\|_P^2}(1+\|A\|_P^5)$, any $s \ge e^{4\lambda\|\eta^0\|_\infty}\overline{s}(T^6+T^8)$ and any φ verifying (4.45) *(recall that P was defined in* (4.19)*). Here, we have denoted*

$$
\begin{aligned}
I(s,\lambda;\varphi) = s^3\lambda^4 \iint_Q e^{-2s\alpha}\,\xi^3\,|\varphi|^2\,dx\,dt + s\lambda^2\iint_Q e^{-2s\alpha}\,\xi\,|\nabla\varphi|^2\,dx\,dt\\
+s^{-1}\iint_Q e^{-2s\alpha}\,\xi^{-1}\,(|\varphi_t|^2 + |\Delta\varphi|^2)\,dx\,dt
\end{aligned}
\tag{4.47}
$$

for each s, $\lambda > 0$ and ω_0 is an open set verifying

$$
\omega' \subset\subset \omega_0 \subset\subset \omega. \tag{4.48}
$$

Proof : In this proof, we follow the ideas developed in [9]. More precisely, we will set

$$\psi = e^{-s\alpha}\varphi, \quad \tilde{\psi} = e^{-s\tilde{\alpha}}\varphi$$

and we will make several computations to deduce the desired inequality. To this end, let us split the proof in three steps : in the first step, we will obtain a Carleman inequality for ψ, in the second one a similar inequality will be deduced for $\tilde{\psi}$, while some simplifications and the conclusion will be given in the last step.

– STEP 1 : A Carleman inequality for ψ.

From the equation verified by φ we find, after some computations :

$$M_1\psi + M_2\psi = G_{s,\lambda},$$

with

$$M_1\psi = -\nabla\cdot(\nabla\psi + \nabla^t\psi) - s^2\lambda^2|\nabla\eta^0|^2\xi^2\psi - s\lambda\xi\nabla^t\psi\cdot\nabla\eta^0 - s\alpha_t\psi,$$

$$M_2\psi = -\psi_t + 2s\lambda\xi\nabla\psi\cdot\nabla\eta^0 + 2s\lambda\xi\nabla^t\psi\cdot\nabla\eta^0 + 2s\lambda^2|\nabla\eta^0|^2\xi\psi$$

and

$$G_{s,\lambda} = e^{-s\alpha}G - s\lambda\Delta\eta^0\xi\psi - s\lambda\xi\nabla\nabla\eta^0\cdot\psi + s\lambda^2|\nabla\eta^0|^2\xi\psi - s\lambda^2\nabla\eta^0\xi\nabla\eta^0\cdot\psi.$$

This gives

$$\|M_1\psi\|^2_{L^2(Q)^N} + \|M_2\psi\|^2_{L^2(Q)^N} + 2(M_1\psi, M_2\psi)_{L^2(Q)^N} = \|G_{s,\lambda}\|^2_{L^2(Q)^N}. \tag{4.49}$$

Let us first develop the double product term. Then, we will conveniently get profit of the positiveness of $\|M_1\psi\|^2_{L^2(Q)^N}$ and $\|M_2\psi\|^2_{L^2(Q)^N}$.

First, we have

$$((M_1\psi)_1, (M_2\psi)_1)_{L^2(Q)^N} = \iint_Q (\nabla\cdot(\nabla\psi + \nabla^t\psi))\cdot\psi_t\,d\sigma\,dt$$

$$= \iint_\Sigma ((\nabla\psi + \nabla^t\psi)\cdot n)_{tg}\cdot\psi_t\,d\sigma\,dt - \iint_Q (\nabla\psi + \nabla^t\psi) : \nabla\psi_t\,dx\,dt = A_1 + A_2,$$

where

$$A_2 = -\frac{1}{2}\iint_Q \frac{d}{dt}(|\nabla\psi|^2 + \nabla\psi : \nabla^t\psi)\,dx\,dt = 0.$$

Here, we have used the fact that $\psi\cdot n = 0$ and the exponential decay of ψ (and its derivatives) close to $t = 0$ and $t = T$.

Additionally,

$$((M_1\psi)_1,(M_2\psi)_2)_{L^2(Q)^N} = -2s\lambda \iint_Q \xi(\nabla\cdot(\nabla\psi+\nabla^t\psi))\cdot(\nabla\psi\cdot\nabla\eta^0)\,dx\,dt$$

$$= -2s\lambda \iint_\Sigma \xi\frac{\partial\eta^0}{\partial n}((\nabla\psi+\nabla^t\psi)\cdot n)\cdot(\nabla\psi\cdot n)\,d\sigma\,dt$$

$$+2s\lambda \iint_Q \xi(\nabla\nabla\eta^0\cdot\nabla\psi):(\nabla\psi+\nabla^t\psi)\,dx\,dt$$

$$+2s\lambda^2 \iint_Q \xi((\nabla\psi+\nabla^t\psi)\cdot\nabla\eta^0)\cdot(\nabla\psi\cdot\nabla\eta^0)\,dx\,dt$$

$$+2s\lambda \iint_Q \xi(\nabla\psi+\nabla^t\psi):(\nabla\nabla\psi\cdot\nabla\eta^0)\,dx\,dt = B+\mathcal{C}+D+E,$$

where the properties of η^0 and, more precisely,

$$\nabla\eta^0 = \frac{\partial\eta^0}{\partial n}n$$

have been employed (see (4.44)). Let us now compute the terms D and E :

$$D = 2s\lambda^2 \iint_Q \xi|\nabla\psi\cdot\nabla\eta^0|^2\,dx\,dt$$
$$+2s\lambda^2 \iint_Q \xi(\nabla^t\psi\cdot\nabla\eta^0)\cdot(\nabla\psi\cdot\nabla\eta^0)\,dx\,dt = D_1+D_2.$$

Two integration by parts in x give the following for D_2 :

$$D_2 = -2s\lambda^2 \iint_Q (\nabla^t(\nabla\eta^0\nabla\eta^0\xi)\cdot\nabla\psi)\cdot\psi\,dx\,dt$$
$$-2s^2\lambda^3 \iint_Q \xi\nabla\eta^0\cdot\nabla(\xi\nabla\eta^0\cdot\psi)\nabla\eta^0\cdot\psi\,dx\,dt$$
$$= -2s\lambda^2 \iint_Q (\nabla^t(\nabla\eta^0\nabla\eta^0\xi)\cdot\nabla\psi)\cdot\psi\,dx\,dt$$
$$+s^2\lambda^3 \iint_Q \Delta\eta^0|\xi\nabla\eta^0\cdot\psi|^2\,dx\,dt = D_{21}+D_{22}.$$

For E, we have :

$$E = s\lambda \iint_Q \xi\nabla\eta^0\cdot\nabla|\nabla\psi|^2\,dx\,dt$$
$$+2s\lambda \iint_Q \xi\nabla^t\psi:(\nabla\nabla\psi\cdot\nabla\eta^0)\,dx\,dt = E_1+E_2.$$

Integrations by parts yield :

$$E_1 = s\lambda \iint_\Sigma \xi \frac{\partial \eta^0}{\partial n} |\nabla \psi|^2 \, d\sigma \, dt - s\lambda \iint_Q \Delta\eta^0 \xi |\nabla \psi|^2 \, dx \, dt$$

$$-s\lambda^2 \iint_Q |\nabla \eta^0|^2 \xi |\nabla \psi|^2 \, dx \, dt = E_{11} + E_{12} + E_{13}$$

and

$$E_2 = 2s\lambda \iint_\Sigma \xi \frac{\partial \eta^0}{\partial n} \nabla\psi : \nabla^t \psi \, d\sigma \, dt - 2s\lambda \iint_Q \Delta\eta^0 \xi \nabla\psi : \nabla^t\psi \, dx \, dt$$

$$-2s\lambda^2 \iint_Q |\nabla\eta^0|^2 \xi \nabla\psi : \nabla^t\psi \, dx \, dt - 2s\lambda \iint_Q \xi \nabla\eta^0 \cdot (\nabla\nabla^t\psi : \nabla\psi) \, dx \, dt$$

$$= E_{21} + E_{22} + E_{23} + E_{24}.$$

Let us deal again with the last term :

$$E_{24} = -2s\lambda \iint_\Sigma \xi \frac{\partial \eta^0}{\partial n} (\nabla\psi \cdot n) \cdot (\nabla^t\psi \cdot n) \, d\sigma \, dt$$

$$+2s\lambda \iint_Q \xi (\nabla\nabla\eta^0 \cdot \nabla^t\psi) \cdot \nabla\psi \, dx \, dt + 2s\lambda^2 \iint_Q \xi (\nabla^t\psi \cdot \nabla\eta^0) \cdot (\nabla\psi \cdot \nabla\eta^0) \, dx \, dt$$

$$+2s^2\lambda^2 \iint_Q \xi^2 (\nabla\nabla\eta^0 \cdot \psi) \cdot (\nabla\psi \cdot \nabla\eta^0) \, dx \, dt$$

$$+s^2\lambda^3 \iint_Q \xi^2 \nabla\eta^0 \cdot (\nabla(\psi\psi) \cdot \nabla\eta^0 \cdot \nabla\eta^0) \, dx \, dt$$

$$+2s^2\lambda^2 \iint_Q \xi^2 (\nabla\psi \cdot \nabla\eta^0) \cdot (\nabla^t\psi \cdot \nabla\eta^0) \, dx \, dt$$

$$= E_{241} + E_{242} + D_2 + E_{243} + E_{244} + E_{245}.$$

Here, we have used the fact that $\nabla \cdot \psi = s\lambda\xi\nabla\eta^0 \cdot \psi$. Finally, since $\eta^0 \in C^2(\overline{\Omega})$, we obtain

$$E_{244} \geq -C(1+\lambda)s^2\lambda^3 \iint_Q \xi^2 |\psi|^2 \, dx \, dt$$

and

$$E_{245} = -2s^2\lambda^2 \iint_Q \xi^2 (\nabla(\nabla\eta^0\nabla\eta^0) \cdot \nabla^t\psi) \cdot \psi \, dx \, dt - 2E_{244}$$

$$-s^3\lambda^3 \iint_Q \xi\nabla\eta^0 \cdot \nabla|\xi\psi \cdot \nabla\eta^0|^2 \, dx \, dt$$

$$= -2s^2\lambda^2 \iint_Q \xi^2 (\nabla(\nabla\eta^0\nabla\eta^0) \cdot \nabla^t\psi) \cdot \psi \, dx \, dt - 2E_{244}$$

$$+s^3\lambda^3 \iint_Q \Delta\eta^0 \xi^3 |\psi \cdot \nabla\eta^0|^2 \, dx \, dt + s^3\lambda^4 \iint_Q |\nabla\eta^0|^2 \xi^3 |\psi \cdot \nabla\eta^0|^2 \, dx \, dt$$

$$= E_{2451} - 2E_{244} + E_{2452} + E_{2453}.$$

We also have

$$((M_1\psi)_1,(M_2\psi)_3)_{L^2(Q)^N} = -2s\lambda\iint_Q \xi\nabla\cdot(\nabla\psi+\nabla^t\psi)\cdot(\nabla^t\psi\cdot\nabla\eta^0)\,dx\,dt$$

$$= -2s\lambda\iint_\Sigma \xi\frac{\partial\eta^0}{\partial n}((\nabla\psi+\nabla^t\psi)\cdot n)\cdot(\nabla^t\psi\cdot n)\,d\sigma\,dt$$

$$+2s\lambda\iint_Q \xi(\nabla\psi+\nabla^t\psi):(\nabla\nabla\eta^0\cdot\nabla^t\psi)\,dx\,dt$$

$$+2s\lambda^2\iint_Q \xi(\nabla\psi\cdot\nabla\eta^0)\cdot(\nabla^t\psi\cdot\nabla\eta^0)\,dx\,dt + 2s\lambda^2\iint_Q \xi|\nabla^t\psi\cdot\nabla\eta^0|^2\,dx\,dt$$

$$+4s\lambda\iint_Q \xi(\nabla\psi:(\nabla\eta^0\cdot\nabla^t\nabla\psi))\,dx\,dt$$

$$= F_1+F_2+G_1+G_2+H.$$

Using $\psi\cdot n=0$ and the expression of $\nabla\cdot\psi$, we obtain the following for G_1 :

$$G_1 = -2s\lambda^2\iint_Q \xi(\nabla\nabla\eta^0:\nabla\psi)(\psi\cdot\nabla\eta^0)\,dx\,dt$$

$$-2s\lambda^3\iint_Q \xi((\nabla\psi\cdot\nabla\eta^0)\cdot\nabla\eta^0)(\psi\cdot\nabla\eta^0) - s^2\lambda^3\iint_Q \nabla\eta^0\cdot\nabla|\xi\psi\cdot\nabla\eta^0|^2\,dx\,dt$$

$$= G_{11}+G_{12}+D_{22}.$$

On the other hand,

$$H = 4s\lambda\iint_\Sigma \xi\frac{\partial\eta^0}{\partial n}|\nabla^t\psi\cdot n|^2\,d\sigma\,dt - 4s\lambda\iint_Q \xi(\nabla\nabla\eta^0\cdot\nabla^t\psi)\cdot\nabla^t\psi\,dx\,dt$$

$$-4s\lambda^2\iint_Q \xi|\nabla^t\psi\cdot\nabla\eta^0|^2\,dx\,dt - 4s^2\lambda^2\iint_Q \xi\nabla(\xi\psi\cdot\nabla\eta^0)\cdot(\nabla^t\psi\cdot\nabla\eta^0)\,dx\,dt$$

$$= H_1+H_2+H_3+H_4.$$

Let us then rewrite the fourth term like this :

$$H_4 = -4s^2\lambda^2\iint_Q \xi^2(\nabla\nabla\eta^0\cdot\psi)\cdot(\nabla^t\psi\cdot\nabla\eta^0)\,dx\,dt$$

$$-4s^2\lambda^3\iint_Q \xi^2(\psi\cdot\nabla\eta^0)\nabla\eta^0\cdot(\nabla^t\psi\cdot\nabla\eta^0)\,dx\,dt - 4s^2\lambda^2\iint_Q \xi^2|\nabla^t\psi\cdot\nabla\eta^0|^2\,dx\,dt$$

$$= H_{41} - 2E_{244} + H_{42}.$$

To finish with the double products of the first term in $M_1\psi$, let us compute the following :

$$((M_1\psi)_1,(M_2\psi)_4)_{L^2(Q)^N} = -2s\lambda^2\iint_Q |\nabla\eta^0|^2\xi(\nabla\cdot(\nabla\psi+\nabla^t\psi))\cdot\psi\,dx\,dt$$

$$= -2s\lambda^2\iint_\Sigma \xi\left|\frac{\partial\eta^0}{\partial n}\right|^2((\nabla\psi+\nabla^t\psi)\cdot n)_{tg}\cdot\psi\,d\sigma\,dt$$

$$+4s\lambda^2\iint_Q \xi(\nabla\nabla\eta^0\nabla\eta^0)\cdot((\nabla\psi+\nabla^t\psi)\cdot\psi)\,dx\,dt$$

$$+2s\lambda^3\iint_Q |\nabla\eta^0|^2\xi(\nabla\eta^0\cdot(\nabla\psi+\nabla^t\psi))\cdot\psi\,dx\,dt$$

$$+2s\lambda^2\iint_Q |\nabla\eta^0|^2\xi|\nabla\psi|^2\,dx\,dt + 2s\lambda^2\iint_Q |\nabla\eta^0|^2\xi\nabla\psi:\nabla^t\psi\,dx\,dt$$

$$= I_1+I_2+I_3+I_4+I_5.$$

Consequently, we obtain

$$\begin{aligned}
&((M_1\psi)_1, M_2\psi)_{L^2(Q)^N} = \\
&((M_1\psi)_1,(M_2\psi)_1+(M_2\psi)_2+(M_2\psi)_3+(M_2\psi)_4)_{L^2(Q)^N} \\
&= A_1+B+\mathcal{C}+D_1+2D_{21}+3D_{22}+E_{11}+E_{12}+E_{13}+E21+E_{22} \\
&+E_{23}+E_{241}+E_{242}+E_{243}-3E_{244}+E_{2452}+E_{2453}+F_1+F_2+G_{11} \\
&+G_{12}+H_1+H_2+H_3+H_{41}+H_{42}+I_1+I_2+I_3+I_4+I_5.
\end{aligned} \tag{4.50}$$

Watching carefully the expressions of these integrals, we observe that

$$\mathcal{C},\ E_{12},\ E_{22},\ E_{242},\ F_2,\ H_2 \le Cs\lambda\iint_Q \xi|\nabla\psi|^2\,dx\,dt,$$

$$D_{21},\ F_2,\ G_{11},\ I_2 \le C\left(s\lambda\iint_Q \xi|\nabla\psi|^2\,dx\,dt + s\lambda^3\iint_Q \xi|\psi|^2\,dx\,dt\right),$$

$$D_{22} \le Cs^2\lambda^3\iint_Q \xi^2|\psi|^2\,dx\,dt,$$

$$E_{13}+I_4 = s\lambda^2\iint_Q |\nabla\eta^0|^2\xi|\nabla\psi|^2\,dx\,dt,$$

$$E_{243},\ H_{41} \le C\left(s\lambda\iint_Q \xi|\nabla\psi|^2\,dx\,dt + s^3\lambda^3\iint_Q \xi^3|\psi|^2\,dx\,dt\right),$$

$$E_{2452} \le Cs^3\lambda^3\iint_Q \xi^3|\psi|^2\,dx\,dt,$$

$$G_{12},\ I_3 \le C\left(\lambda^2\iint_Q \xi|\nabla\psi|^2\,dx\,dt + s^2\lambda^4\iint_Q \xi|\psi|^2\,dx\,dt\right),$$

$$G_2+H_3 = -2s\lambda^2\iint_Q \xi|\nabla^t\psi\cdot\nabla\eta^0|^2\,dx\,dt$$

and $D_1 \geq 0$, $E_{23} + I_4 = 0$. All this, together with the estimate of E_{244}, provides from (4.50) :

$$
\begin{aligned}
((M_1\psi)_1, M_2\psi)_{L^2(Q)^N} \geq & \iint_\Sigma ((\nabla\psi + \nabla^t\psi)\cdot n)_{tg} \cdot \psi_t \, d\sigma\, dt \\
& -2s\lambda \iint_\Sigma \xi \frac{\partial \eta^0}{\partial n}((\nabla\psi + \nabla^t\psi)\cdot n)\cdot(\nabla\psi\cdot n)\, d\sigma\, dt \\
& +s\lambda \iint_\Sigma \xi \frac{\partial \eta^0}{\partial n}|\nabla\psi|^2\, d\sigma\, dt + 2s\lambda \iint_\Sigma \xi \frac{\partial \eta^0}{\partial n} \nabla\psi : \nabla^t\psi\, d\sigma\, dt \\
& -2s\lambda \iint_\Sigma \xi \frac{\partial \eta^0}{\partial n}(\nabla\psi\cdot n)\cdot(\nabla^t\psi\cdot n)\, d\sigma\, dt \\
& -2s\lambda \iint_\Sigma \xi \frac{\partial \eta^0}{\partial n}((\nabla\psi + \nabla^t\psi)\cdot n)\cdot(\nabla^t\psi\cdot n)\, d\sigma\, dt \\
& +4s\lambda \iint_\Sigma \xi \frac{\partial \eta^0}{\partial n}|\nabla^t\psi\cdot n|^2\, d\sigma\, dt \\
& -2s\lambda^2 \iint_\Sigma \xi \left|\frac{\partial \eta^0}{\partial n}\right|^2 ((\nabla\psi + \nabla^t\psi)\cdot n)_{tg}\cdot\psi\, d\sigma\, dt \\
& +s\lambda^2 \iint_Q |\nabla\eta^0|^2\xi|\nabla\psi|^2\, dx\, dt + s^3\lambda^4 \iint_Q |\nabla\eta^0|^2\xi^3|\psi\cdot\nabla\eta^0|^2\, dx\, dt \\
& -2s\lambda^2 \iint_Q \xi|\nabla^t\psi\cdot\nabla\eta^0|^2\, dx\, dt - 4s^2\lambda^2 \iint_Q \xi^2|\nabla^t\psi\cdot\nabla\eta^0|^2\, dx\, dt \\
& -C\left(s\lambda \iint_Q \xi|\nabla\psi|^2\, dx\, dt + s\lambda^3(s+s^2+\lambda)\iint_Q \xi^3|\psi|^2\, dx\, dt\right).
\end{aligned}
\tag{4.51}
$$

On the other hand, we have

$$
\begin{aligned}
((M_1\psi)_2, (M_2\psi)_1)_{L^2(Q)^N} &= \frac{1}{2}s^2\lambda^2 \iint_Q |\nabla\eta^0|^2\xi^2\frac{\partial}{\partial t}|\psi|^2\, dx\, dt \\
&= -s^2\lambda^2 \iint_Q |\nabla\eta^0|^2\xi\xi_t|\psi|^2\, dx\, dt = J.
\end{aligned}
$$

Additionally, we find that

$$
\begin{aligned}
&((M_1\psi)_2, (M_2\psi)_2)_{L^2(Q)^N} = -s^2\lambda^3 \iint_Q |\nabla\eta^0|^2\xi^3\nabla\eta^0\cdot\nabla|\psi|^2\, dx\, dt \\
&= -s^3\lambda^3 \iint_\Sigma \xi^3\left(\frac{\partial\eta^0}{\partial n}\right)^3 |\psi|^2\, d\sigma\, dt + s^3\lambda^3 \iint_Q \xi^3\nabla\cdot(|\nabla\eta^0|^2\nabla\eta^0)|\psi|^2\, dx\, dt \\
&+3s^3\lambda^4 \iint_Q |\nabla\eta^0|^4\xi^3|\psi|^2\, dx\, dt = K_1 + K_2 + K_3.
\end{aligned}
$$

Let us know skip the product

$$((M_1\psi)_2, (M_2\psi)_3)_{L^2(Q)^N}.$$

In fact, a computation of this term will not be crucial for the sequel, since it will be compensated with another one. More details will be given below.

The last product for $(M_1\psi)_2$ gives :

$$((M_1\psi)_2,(M_2\psi)_4)_{L^2(Q)^N} = -2s^3\lambda^4 \iint_Q |\nabla\eta^0|^4\xi^3|\psi|^2\,dx\,dt = L.$$

Consequently, we deduce that

$$\begin{aligned}&((M_1\psi)_2, M_2\psi)_{L^2(Q)^N}\\ &\qquad = ((M_1\psi)_2,(M_2\psi)_1+(M_2\psi)_2+(M_2\psi)_3+(M_2\psi)_4)_{L^2(Q)}\\ &\qquad = J+K_1+K_2+K_3+L+((M_1\psi)_2,(M_2\psi)_3)_{L^2(Q)^N}.\end{aligned}$$

Taking into account that $\eta^0>0$ in Ω, we find that

$$\xi_t \le CT\xi^{5/4}$$

and so

$$\begin{aligned}((M_1\psi)_2, M_2\psi)_{L^2(Q)^N} &\ge -s^3\lambda^3\iint_\Sigma \xi^3\left(\frac{\partial\eta^0}{\partial n}\right)^3|\psi|^2\,d\sigma\,dt\\ &+s^3\lambda^4\iint_Q|\nabla\eta^0|^4\xi^3|\psi|^2\,dx\,dt+((M_1\psi)_2,(M_2\psi)_3)_{L^2(Q)^N}\\ &-C\left(s^2\lambda^2T\iint_Q\xi^{9/4}|\psi|^2\,dx\,dt+s^3\lambda^3\iint_Q\xi^3|\psi|^2\,dx\,dt\right).\end{aligned} \tag{4.52}$$

Let us now consider the scalar products of the third term in $M_1\psi$. Firstly, we have :

$$\begin{aligned}&((M_1\psi)_3,(M_2\psi)_1)_{L^2(Q)^N} = s\,\lambda\iint_Q \xi(\nabla^t\psi\cdot\nabla\eta^0)\cdot\psi_t\,dx\,dt\\ &= -s\lambda\iint_Q\xi(\nabla\nabla\eta^0\cdot\psi)\cdot\psi_t\,dx\,dt - s\lambda^2\iint_Q\xi(\psi\cdot\nabla\eta^0)(\psi_t\cdot\nabla\eta^0)\,dx\,dt\\ &-s^2\lambda^2\iint_Q\xi(\psi\cdot\nabla\eta^0)\nabla\eta^0\cdot(\xi\psi)_t\,dx\,dt = M_1+M_2+M_3.\end{aligned}$$

Here, we must remark that, due to the exponential decay of ψ at $t=0$ and $t=T$, $M_3=0$, while

$$M_1 = \frac{1}{2}s\lambda\iint_Q \xi_t(\nabla\nabla\eta^0\cdot\psi)\cdot\psi\,dx\,dt$$

and

$$M_2 = \frac{1}{2}s\lambda^2\iint_Q\xi_t|\psi\cdot\nabla\eta^0|^2\,dx\,dt.$$

Furthermore,

$$((M_1\psi)_3,(M_2\psi)_2)_{L^2(Q)^N} = -2s^2\lambda^2 \iint_Q \xi^2(\nabla^t\psi\cdot\nabla\eta^0)\cdot(\nabla\psi\cdot\nabla\eta^0)\,dx\,dt$$

$$= 2s^2\lambda^2 \iint_Q \xi^2(\nabla^t\psi\cdot\nabla(\nabla\eta^0\nabla\eta^0))\cdot\psi\,dx\,dt$$

$$+4s^2\lambda^3 \iint_Q \xi^2(\psi\cdot\nabla\eta^0)(\nabla\psi\cdot\nabla\eta^0)\cdot\nabla\eta^0\,dx\,dt$$

$$+2s^3\lambda^3 \iint_Q \xi^2(\psi\cdot\nabla\eta^0)\nabla\eta^0\cdot\nabla(\xi\psi\cdot\nabla\eta^0)\,dx\,dt = N_1 + 2E_{244} + N_2.$$

Let us compute the last integral :

$$N_2 = s^3\lambda^3 \iint_Q \xi\nabla\eta^0\cdot\nabla|\xi\psi\cdot\nabla\eta^0|^2\,dx\,dt$$

$$= -s^3\lambda^3 \iint_Q \xi\Delta\eta^0|\xi\psi\cdot\nabla\eta^0|^2\,dx\,dt - s^3\lambda^4 \iint_Q |\nabla\eta^0|^2\xi^3|\psi\cdot\nabla\eta^0|^2\,dx\,dt$$

$$= N_{21} + N_{22}.$$

Furthermore, we have

$$((M_1\psi)_3,(M_2\psi)_3)_{L^2(Q)^N} = -2s^2\lambda^2 \iint_Q |\nabla\eta^0|^2\xi^2|\nabla^t\psi\cdot\nabla\eta^0|^2\,dx\,dt = O.$$

The last product for $(M_1\psi)_3$ gives

$$((M_1\psi)_3,(M_2\psi)_4)_{L^2(Q)^N} = -2s^2\lambda^3 \iint_Q |\nabla\eta^0|^2\xi^2(\nabla^t\psi\cdot\nabla\eta^0)\cdot\psi\,dx\,dt$$

$$= 2s^2\lambda^3 \iint_Q (\nabla(|\nabla\eta^0|^2\xi^2\nabla\eta^0)\cdot\psi)\cdot\psi\,dx\,dt$$

$$+2s^3\lambda^4 \iint_Q |\nabla\eta^0|^2\xi^3|\psi\cdot\nabla\eta^0|^2\,dx\,dt = P_1 + P_2.$$

We will use here, as we did before, the following estimate for P_1 :

$$P_1 \le Cs^2\lambda^3(1+\lambda) \iint_Q \xi^2|\psi|^2\,dx\,dt.$$

Putting all the products of $(M_1\psi)_3$ together, we find

$$((M_1\psi)_3, M_2\psi)_{L^2(Q)^N}$$

$$= ((M_1\psi)_3,(M_2\psi)_1 + (M_2\psi)_2 + (M_2\psi)_3 + (M_2\psi)_4)_{L^2(Q)^N}$$

$$= M_1 + M_2 + N_1 + 2E_{244} + N_{21} + N_{22} + O + P_1 + P_2.$$

Now, using the properties of η^0 and Holder's inequality, we have

$$\begin{aligned}((M_1\psi)_3, M_2\psi)_{L^2(Q)^N} \geq\ & s^3\lambda^4 \iint_Q |\nabla\eta^0|^2\xi^3|\psi\cdot\nabla\eta^0|^2\,dx\,dt \\ & -2s^2\lambda^2 \iint_Q |\nabla\eta^0|^2\xi^2|\nabla^t\psi\cdot\nabla\eta^0|^2\,dx\,dt \\ & -C\left(s\lambda T(1+\lambda)\iint_Q \xi^{5/4}|\psi|^2\,dx\,dt\right. \\ & +s\lambda\iint_Q \xi|\nabla\psi|^2\,dx\,dt + s^3\lambda^3\iint_Q \xi^3|\psi|^2\,dx\,dt \\ & \left.+s^2\lambda^3(1+\lambda)\iint_Q \xi^2|\psi|^2\,dx\,dt\right).\end{aligned} \tag{4.53}$$

Let us finally perform the computations for the fourth and last term in the expression of $M_1\psi$.

First, we have :

$$\begin{aligned}((M_1\psi)_4, (M_2\psi)_1)_{L^2(Q)^N} &= \frac{1}{2}s\iint_Q \alpha_t\frac{\partial}{\partial t}|\psi|^2\,dx\,dt \\ &= -\frac{1}{2}s\iint_Q \alpha_{tt}|\psi|^2\,dx\,dt = Q.\end{aligned}$$

Then,

$$\begin{aligned}((M_1\psi)_4, (M_2\psi)_2)_{L^2(Q)^N} &= -s^2\lambda\iint_Q \xi\alpha_t\nabla\eta^0\cdot|\psi|^2\,dx\,dt \\ = -s^2\lambda\iint_\Sigma \frac{\partial\eta^0}{\partial n}\xi\alpha_t|\psi|^2\,d\sigma\,dt &+ s^2\lambda\iint_Q \nabla\cdot(\xi\alpha_t\nabla\eta^0)|\psi|^2\,dx\,dt = R_1+R_2.\end{aligned}$$

Let us again skip the term

$$((M_1\psi)_4, (M_2\psi)_3)_{L^2(Q)^N}.$$

Finally, we have

$$((M_1\psi)_4, (M_2\psi)_4)_{L^2(Q)^N} = -2s^2\lambda^2\iint_Q |\nabla\eta^0|^2\xi\alpha_t|\psi|^2\,dx\,dt = S.$$

All the computations made for $(M_1\psi)_4$ yields

$$((M_1\psi)_4, M_2\psi)_{L^2(Q)^N} = Q + R_1 + R_2 + S + ((M_1\psi)_4, (M_2\psi)_3)_{L^2(Q)^N}.$$

Let us then deal with the expression of α_t :

$$\alpha_t = -4(T-2t)(e^{2\lambda\|\eta^0\|_\infty} - e^{\lambda\eta^0})(t(T-t))^{-5}.$$

From the properties of η^0, we deduce

$$|\alpha_t| \leq CTe^{2\lambda\|\eta^0\|_\infty}\xi^{5/4}. \tag{4.54}$$

On the other hand, it is not difficult to check these two other estimates :

$$|\nabla\alpha_t| \leq C\lambda T\xi^{5/4},$$

$$|\alpha_{tt}| \leq Ce^{2\lambda\|\eta^0\|_\infty}(T^2\xi^{3/2} + \xi^{5/4}).$$

Consequently, we find the following for $(M_1\psi)_4$:

$$\begin{aligned}((M_1\psi)_4, M_2\psi)_{L^2(Q)^N} \geq {} & -s^2\lambda\iint_\Sigma \frac{\partial\eta^0}{\partial n}\xi\alpha_t|\psi|^2\,d\sigma\,dt \\ & +((M_1\psi)_4,(M_2\psi)_3)_{L^2(Q)^N} \\ & -Ce^{2\lambda\|\eta^0\|_\infty}\left(s\iint_Q (T^2\xi^{3/2}+\xi^{5/4})|\psi|^2\,dx\,dt \right. \\ & \left. +s^2\lambda(1+\lambda)T\iint_Q \xi^{9/4}|\psi|^2\,dx\,dt\right)\end{aligned} \tag{4.55}$$

As a conclusion, taking into account (4.51), (4.52), (4.53), (4.55) and

$$\xi^{-1} \leq CT^8,$$

we obtain

$$
\begin{aligned}
2(M_1\psi, M_2\psi)_{L^2(Q)^N} \geq\ & 2\iint_\Sigma ((\nabla\psi + \nabla^t\psi)\cdot n)_{tg}\cdot \psi_t \, d\sigma \, dt \\
& -4s\lambda \iint_\Sigma \xi \frac{\partial\eta^0}{\partial n}((\nabla\psi + \nabla^t\psi)\cdot n)\cdot(\nabla\psi\cdot n)\, d\sigma\, dt \\
& +2s\lambda \iint_\Sigma \xi \frac{\partial\eta^0}{\partial n}|\nabla\psi|^2\, d\sigma\, dt + 4s\lambda \iint_\Sigma \xi\frac{\partial\eta^0}{\partial n}\nabla\psi : \nabla^t\psi\, d\sigma\, dt \\
& -4s\lambda \iint_\Sigma \xi \frac{\partial\eta^0}{\partial n}(\nabla\psi\cdot n)\cdot(\nabla^t\psi\cdot n)\, d\sigma\, dt \\
& -4s\lambda \iint_\Sigma \xi \frac{\partial\eta^0}{\partial n}((\nabla\psi + \nabla^t\psi)\cdot n)\cdot(\nabla^t\psi\cdot n)\, d\sigma\, dt \\
& -4s\lambda^2 \iint_\Sigma \xi \left|\frac{\partial\eta^0}{\partial n}\right|^2 ((\nabla\psi + \nabla^t\psi)\cdot n)_{tg}\cdot\psi\, d\sigma\, dt \\
& -2s^3\lambda^3 \iint_\Sigma \xi^3 \left(\frac{\partial\eta^0}{\partial n}\right)^3 |\psi|^2\, dx\, dt - 2s^2\lambda \iint_\Sigma \frac{\partial\eta^0}{\partial n}\xi\alpha_t|\psi|^2\, dx\, dt \\
& +8s\lambda \iint_\Sigma \xi\frac{\partial\eta^0}{\partial n}|\nabla^t\psi\cdot n|^2\, d\sigma\, dt + 4s^3\lambda^4 \iint_Q |\nabla\eta^0|^2\xi^3|\psi\cdot\nabla\eta^0|^2\, dx\, dt \\
& +2s\lambda^2 \iint_Q |\nabla\eta^0|^2\xi|\nabla\psi|^2\, dx\, dt + 2s^3\lambda^4 \iint_Q |\nabla\eta^0|^4\xi^3|\psi|^2\, dx\, dt \\
& -4s\lambda^2 \iint_Q \xi|\nabla^t\psi\nabla\eta^0|^2\, dx\, dt - 12s^2\lambda^2 \iint_Q \xi^2|\nabla^t\psi\nabla\eta^0|^2\, dx\, dt \\
& +2((M_1\psi)_2, (M_2\psi)_3)_{L^2(Q)^N} + 2((M_1\psi)_4, (M_2\psi)_3)_{L^2(Q)^N} \\
& -C\Bigg(s\lambda \iint_Q \xi|\nabla\psi|^2\, dx\, dt + s\lambda^3(s+s^2+\lambda+s\lambda)\iint_Q \xi^3|\psi|^2\, dx\, dt \\
& +sT^7e^{2\lambda\|\eta^0\|_\infty}(T^7+\lambda T^8+\lambda^2T^8+s\lambda+s\lambda^2)\iint_Q \xi^3|\psi|^2\, dx\, dt\Bigg).
\end{aligned}
\tag{4.56}
$$

From this inequality, we see that we have two leading positive terms in the right hand side, namely :

$$
2s^3\lambda^4 \iint_Q |\nabla\eta^0|^4\xi^3|\psi|^2\, dx\, dt \quad \text{and} \quad 2s\lambda^2 \iint_Q |\nabla\eta^0|^2\xi|\nabla\psi|^2\, dx\, dt.
$$

By virtue of the properties satisfied by η^0, we find

$$
\begin{aligned}
s^3\lambda^4 \iint_Q |\nabla\eta^0|^4\xi^3|\psi|^2\, dx\, dt \geq Cs^3\lambda^4 \Bigg(& \iint_Q \xi^3|\psi|^2\, dx\, dt \\
& - \iint_{\omega'\times(0,T)} \xi^3|\psi|^2\, dx\, dt\Bigg)
\end{aligned}
$$

and

$$s\lambda^2 \iint_Q |\nabla \eta^0|^2 \xi |\nabla \psi|^2 \, dx\, dt \geq C s\lambda^2 \left(\iint_Q \xi |\nabla \psi|^2 \, dx\, dt - \iint_{\omega' \times (0,T)} \xi |\nabla \psi|^2 \, dx\, dt \right).$$

In both cases, the global terms will stay in the left hand side, while the local ones will pass to the right.

Let us now make a proper choice of s and λ, so that the last integrals appearing in (4.56) are absorbed by the two global terms (with $s\lambda^2$ and $s^3\lambda^4$) we have kept in the left hand side. More precisely, let us take $\lambda \geq C(\Omega, \omega')$ and $s \geq C e^{2\lambda \|\eta^0\|_\infty}(T^7 + T^8)$. This way, we have

$$C \left(s\lambda \iint_Q \xi |\nabla \psi|^2 \, dx\, dt + s\lambda^3 (s + s^2 + \lambda + s\lambda) \iint_Q \xi^3 |\psi|^2 \, dx\, dt + sT^7 e^{(3\lambda/4)\|\eta^0\|_\infty} (T^7 + \lambda T^8 + \lambda^2 T^8 + s\lambda + s\lambda^2) \iint_Q \xi^3 |\psi|^2 \, dx\, dt \right) \leq \delta \left(s^3\lambda^4 \iint_Q \xi^3 |\psi|^2 \, dx\, dt + s\lambda^2 \iint_Q \xi |\nabla \psi|^2 \, dx\, dt \right)$$

for a constant $\delta(\Omega, \omega) > 0$ small enough.

Consequently, we obtain from (4.56)

$$
\begin{aligned}
2(M_1\psi, M_2\psi)_{L^2(Q)^N} \geq{} & 2\iint_\Sigma ((\nabla\psi + \nabla^t\psi)\cdot n)_{tg}\cdot \psi_t \, d\sigma\, dt \\
& -4s\lambda \iint_\Sigma \xi \frac{\partial \eta^0}{\partial n}((\nabla\psi + \nabla^t\psi)\cdot n)\cdot(\nabla\psi\cdot n)\, d\sigma\, dt \\
& +2s\lambda \iint_\Sigma \xi \frac{\partial \eta^0}{\partial n}|\nabla\psi|^2 \, d\sigma\, dt + 4s\lambda \iint_\Sigma \xi \frac{\partial \eta^0}{\partial n}\nabla\psi : \nabla^t\psi\, d\sigma\, dt \\
& -4s\lambda \iint_\Sigma \xi \frac{\partial \eta^0}{\partial n}(\nabla\psi\cdot n)\cdot(\nabla^t\psi\cdot n)\, d\sigma\, dt \\
& -4s\lambda \iint_\Sigma \xi \frac{\partial \eta^0}{\partial n}((\nabla\psi + \nabla^t\psi)\cdot n)\cdot(\nabla^t\psi\cdot n)\, d\sigma\, dt \\
& -4s\lambda^2 \iint_\Sigma \xi \left|\frac{\partial \eta^0}{\partial n}\right|^2 ((\nabla\psi + \nabla^t\psi)\cdot n)_{tg}\cdot\psi\, d\sigma\, dt \\
& -2s^3\lambda^3 \iint_\Sigma \xi^3 \left(\frac{\partial \eta^0}{\partial n}\right)^3 |\psi|^2\, dx\, dt - 2s^2\lambda \iint_\Sigma \frac{\partial \eta^0}{\partial n}\xi\alpha_t|\psi|^2\, dx\, dt \\
& +8s\lambda \iint_\Sigma \xi \frac{\partial \eta^0}{\partial n}|\nabla^t\psi\cdot n|^2\, d\sigma\, dt + 4s^3\lambda^4 \iint_Q |\nabla\eta^0|^2\xi^3|\psi\cdot\nabla\eta^0|^2\, dx\, dt \\
& +s\lambda^2 \iint_Q \xi|\nabla\psi|^2\, dx\, dt + s^3\lambda^4 \iint_Q \xi^3|\psi|^2\, dx\, dt \\
& -4s\lambda^2 \iint_Q \xi|\nabla^t\psi\nabla\eta^0|^2\, dx\, dt - 12s^2\lambda^2 \iint_Q \xi^2|\nabla^t\psi\nabla\eta^0|^2\, dx\, dt \\
& +2((M_1\psi)_2, (M_2\psi)_3)_{L^2(Q)^N} + 2((M_1\psi)_4, (M_2\psi)_3)_{L^2(Q)^N},
\end{aligned}
\tag{4.57}
$$

for any $\lambda \geq C(\Omega,\omega)$ and any $s \geq C(\Omega,\omega)e^{2\lambda\|\eta^0\|_\infty}(T^7+T^8)$.

On the other hand, we also have several negative terms (in Q) with high powers of the parameters s and λ. Those will be eliminated with the use of the positive terms appearing in the left hand side of (4.49), i.e.,

$$
\|M_1\psi\|^2_{L^2(Q)^N} \quad \text{and} \quad \|M_2\psi\|^2_{L^2(Q)^N}.
$$

Let us also observe that

$$
\|G_{s,\lambda}\|^2_{L^2(Q)^N} \leq C\left(\iint_Q e^{-2s\alpha}|G|^2\, dx\, dt + s^2\lambda^4 \iint_Q \xi^2|\psi|^2\, dx\, dt\right),
$$

for $\lambda \geq C(\Omega,\omega)$.

Therefore, from (4.49) and (4.57), we readily get :

$$\begin{aligned}
&\|M_1\psi\|^2_{L^2(Q)^N} + \|M_2\psi\|^2_{L^2(Q)^N} + s^3\lambda^4 \iint_Q \xi^3|\psi|^2\,dx\,dt \\
&+s\lambda^2 \iint_Q \xi|\nabla\psi|^2\,dx\,dt + 4s^3\lambda^4 \iint_Q |\nabla\eta^0|^2\xi^3|\psi\cdot\nabla\eta^0|^2\,dx\,dt \\
&-4s\lambda^2 \iint_Q \xi|\nabla^t\psi\nabla\eta^0|^2\,dx\,dt - 12s^2\lambda^2 \iint_Q \xi^2|\nabla^t\psi\nabla\eta^0|^2\,dx\,dt \\
&+2((M_1\psi)_2,(M_2\psi)_3)_{L^2(Q)^N} + 2((M_1\psi)_4,(M_2\psi)_3)_{L^2(Q)^N} \\
&+2\iint_\Sigma ((\nabla\psi+\nabla^t\psi)\cdot n)_{tg}\cdot\psi_t\,d\sigma\,dt \\
&-4s\lambda \iint_\Sigma \xi\frac{\partial\eta^0}{\partial n}((\nabla\psi+\nabla^t\psi)\cdot n)\cdot(\nabla\psi\cdot n)\,d\sigma\,dt \\
&+2s\lambda \iint_\Sigma \xi\frac{\partial\eta^0}{\partial n}|\nabla\psi|^2\,d\sigma\,dt + 4s\lambda\iint_\Sigma \xi\frac{\partial\eta^0}{\partial n}\nabla\psi:\nabla^t\psi\,d\sigma\,dt \\
&-4s\lambda \iint_\Sigma \xi\frac{\partial\eta^0}{\partial n}(\nabla\psi\cdot n)\cdot(\nabla^t\psi\cdot n)\,d\sigma\,dt \\
&-4s\lambda \iint_\Sigma \xi\frac{\partial\eta^0}{\partial n}((\nabla\psi+\nabla^t\psi)\cdot n)\cdot(\nabla^t\psi\cdot n)\,d\sigma\,dt \\
&-4s\lambda^2 \iint_\Sigma \xi\left|\frac{\partial\eta^0}{\partial n}\right|^2((\nabla\psi+\nabla^t\psi)\cdot n)_{tg}\cdot\psi\,d\sigma\,dt \\
&-2s^3\lambda^3 \iint_\Sigma \xi^3\left(\frac{\partial\eta^0}{\partial n}\right)^3|\psi|^2\,dx\,dt - 2s^2\lambda\iint_\Sigma \frac{\partial\eta^0}{\partial n}\xi\alpha_t|\psi|^2\,dx\,dt \\
&+8s\lambda \iint_\Sigma \xi\frac{\partial\eta^0}{\partial n}|\nabla^t\psi\cdot n|^2\,d\sigma\,dt \\
&\qquad\le C\left(\iint_Q e^{-2s\alpha}|G|^2\,dx\,dt + s^3\lambda^4\iint_{\omega'\times(0,T)}\xi^3|\psi|^2\,dx\,dt\right. \\
&\qquad\left.+s\lambda^2\iint_{\omega'\times(0,T)}\xi|\nabla\psi|^2\,dx\,dt\right),
\end{aligned} \tag{4.58}$$

for any $\lambda \ge C(\Omega,\omega)$ and any $s \ge C(\Omega,\omega)e^{2\lambda\|\eta^0\|_\infty}(T^7+T^8)$.

At this point, let us develop the term $M_1\psi$ in the following way :

$$M_1\psi = \overline{M}_1\psi - 2s\lambda\xi\nabla^t\psi\cdot\nabla\eta^0,$$

with

$$\overline{M}_1\psi = -\Delta\psi - s^2\lambda^2|\nabla\eta^0|^2\xi^2\psi - s\alpha_t\psi - s\lambda\xi(\nabla\nabla\eta^0\cdot\psi) - s\lambda^2\nabla\eta^0\xi(\psi\cdot\nabla\eta^0),$$

that is to say,

$$\overline{M}_1\psi = -\Delta\psi + (M_1\psi)_2 + (M_1\psi)_4 - s\lambda\xi(\nabla\nabla\eta^0\cdot\psi) - s\lambda^2\nabla\eta^0\xi(\psi\cdot\nabla\eta^0).$$

Hence,

$$\|M_1\psi\|^2_{L^2(Q)^N} = \|\overline{M}_1\psi\|^2_{L^2(Q)^N} + 4s^2\lambda^2 \iint_Q \xi^2 |\nabla^t\psi\cdot\nabla\eta^0|^2\,dx\,dt$$
$$+2(\overline{M}_1\psi, -2s\lambda\xi(\nabla^t\psi\cdot\nabla\eta^0))_{L^2(Q)^N}$$

Let us deal with the double product term. To this end, we first observe that it actually coincides with

$$2(\overline{M}_1\psi, -(M_2\psi)_3)_{L^2(Q)^N} = -2((M_1\psi)_2,(M_2\psi)_3) - 2((M_1\psi)_4,(M_2\psi)_3)_{L^2(Q)^N}$$
$$+4s\lambda \iint_Q \xi(\nabla^t\psi\cdot\nabla\eta^0)\cdot\Delta\psi\,dx\,dt$$
$$+4s^2\lambda^2 \iint_Q \xi^2(\nabla\nabla\eta^0\cdot\psi)\cdot(\nabla^t\psi\cdot\nabla\eta^0)\,dx\,dt$$
$$+4s^2\lambda^3 \iint_Q \xi^2(\psi\cdot\nabla\eta^0)\nabla\eta^0\cdot(\nabla^t\psi\cdot\nabla\eta^0)\,dx\,dt$$
$$= -2((M_1\psi)_2,(M_2\psi)_3) - 2((M_1\psi)_4,(M_2\psi)_3)_{L^2(Q)^N} + \mathcal{T} - H_{41} + 2E_{244}.$$

The next step will be to estimate the third term in the previous expression :

$$\mathcal{T} = -4s\lambda \iint_Q \xi(\nabla\nabla\eta^0\cdot\psi)\cdot\Delta\psi\,dx\,dt - 4s\lambda^2 \iint_Q \xi(\psi\cdot\nabla\eta^0)(\Delta\psi\cdot\nabla\eta^0)\,dx\,dt$$
$$-4s^2\lambda^2 \iint_Q \xi(\psi\cdot\nabla\eta^0)\Delta(\xi\psi\cdot\nabla\eta^0)\,dx\,dt = \mathcal{T}_1+\mathcal{T}_2+\mathcal{T}_3.$$

For $\mathcal{T}_1$, we have

$$\mathcal{T}_1 \le Cs\lambda \iint_Q \xi|\psi||\Delta\psi|\,dx\,dt,$$

while we must still arrange $\mathcal{T}_2$ and $\mathcal{T}_3$, making some integration by parts. Indeed, we have

$$\mathcal{T}_2 = 4s\lambda^2 \iint_Q \xi(\nabla(\nabla\eta^0\nabla\eta^0)\cdot\nabla\psi)\cdot\psi\,dx\,dt$$
$$+4s\lambda^3 \iint_Q \xi(\psi\cdot\nabla\eta^0)(\nabla\psi\cdot\nabla\eta^0)\cdot\nabla\eta^0\,dx\,dt$$
$$+4s\lambda^2 \iint_Q \xi|\nabla^t\psi\cdot\nabla\eta^0|^2\,dx\,dt = \mathcal{T}_{21}+\mathcal{T}_{22}+\mathcal{T}_{23}.$$

$\mathcal{T}_{23}$ will compensate the sixth term in the left hand side of (4.58) and $\mathcal{T}_{21}$, $\mathcal{T}_{22}$ verify

$$\mathcal{T}_{21}+\mathcal{T}_{22} \le C\left(s^2\lambda^4 \iint_Q \xi^2|\psi|^2\,dx\,dt + (1+\lambda^2)\iint_Q \xi|\nabla\psi|^2\,dx\,dt\right)$$

so they will be eliminated by taking $\lambda \geq 1$ and $s \geq CT^8$. Besides,

$$\begin{aligned}\mathcal{T}_3 = 4s^2\lambda^2 \iint_Q \xi(\nabla\nabla\eta^0 \cdot \psi) \cdot \nabla(\xi\psi \cdot \nabla\eta^0)\, dx\, dt\\ +2s^2\lambda^3 \iint_Q \nabla\eta^0 \cdot \nabla|\xi\psi \cdot \nabla\eta^0|^2\, dx\, dt\\ +4s^2\lambda^2 \iint_Q \xi(\nabla^t\psi \cdot \nabla\eta^0) \cdot \nabla(\xi\psi \cdot \nabla\eta^0)\, dx\, dt = \mathcal{T}_{31} + \mathcal{T}_{32} + \mathcal{T}_{33}.\end{aligned}$$

For $\mathcal{T}_{31}$ and $\mathcal{T}_{32}$ (after integration by parts), we have

$$\mathcal{T}_{31} + \mathcal{T}_{32} \leq C\left(s^2\lambda^2(1+s+\lambda) \iint_Q \xi^2|\psi|^2\, dx\, dt + s\lambda \iint_Q \xi|\nabla\psi|^2\, dx\, dt\right),$$

which are 'good' terms, provided we make a good choice of s and λ. For $\mathcal{T}_{33}$, we find

$$\begin{aligned}\mathcal{T}_{33} = 4s^2\lambda^2 \iint_Q \xi^2(\nabla^t\psi \cdot \nabla\eta^0) \cdot (\nabla\nabla\eta^0 \cdot \psi)\, dx\, dt\\ +4s^2\lambda^3 \iint_Q \xi^2(\nabla^t\psi \cdot \nabla\eta^0) \cdot \nabla\eta^0(\psi \cdot \nabla\eta^0)\, dx\, dt\\ +4s^2\lambda^2 \iint_Q \xi^2|\nabla^t\psi \cdot \nabla\eta^0|^2\, dx\, dt = -H_{41} + 2E_{244} + \mathcal{T}_{331}.\end{aligned}$$

Consequently,

$$\mathcal{T}_{33} \geq -C\left(s^2\lambda^3(1+\lambda) \iint_Q \xi^3|\psi|^2\, dx\, dt + s\lambda \iint_Q \xi|\nabla\psi|^2\, dx\, dt\right) + \mathcal{T}_{331}.$$

Combining all this and (4.58), we deduce the following inequality :

$$
\begin{aligned}
&\|\overline{M}_1\psi\|^2_{L^2(Q)^N} + \|M_2\psi\|^2_{L^2(Q)^N} + s^3\lambda^4 \iint_Q \xi^3|\psi|^2\,dx\,dt \\
&+s\lambda^2 \iint_Q \xi|\nabla\psi|^2\,dx\,dt + 4s^3\lambda^4 \iint_Q |\nabla\eta^0|^2\xi^3|\psi\cdot\nabla\eta^0|^2\,dx\,dt \\
&-4s^2\lambda^2 \iint_Q \xi^2|\nabla^t\psi\nabla\eta^0|^2\,dx\,dt + 2\iint_\Sigma ((\nabla\psi+\nabla^t\psi)\cdot n)_{tg}\cdot\psi_t\,d\sigma\,dt \\
&-4s\lambda \iint_\Sigma \xi\frac{\partial\eta^0}{\partial n}((\nabla\psi+\nabla^t\psi)\cdot n)\cdot(\nabla\psi\cdot n)\,d\sigma\,dt \\
&+2s\lambda \iint_\Sigma \xi\frac{\partial\eta^0}{\partial n}|\nabla\psi|^2\,d\sigma\,dt + 4s\lambda\iint_\Sigma \xi\frac{\partial\eta^0}{\partial n}\nabla\psi:\nabla^t\psi\,d\sigma\,dt \\
&-4s\lambda \iint_\Sigma \xi\frac{\partial\eta^0}{\partial n}(\nabla\psi\cdot n)\cdot(\nabla^t\psi\cdot n)\,d\sigma\,dt \\
&-4s\lambda \iint_\Sigma \xi\frac{\partial\eta^0}{\partial n}((\nabla\psi+\nabla^t\psi)\cdot n)\cdot(\nabla^t\psi\cdot n)\,d\sigma\,dt \\
&-4s\lambda^2 \iint_\Sigma \xi\left|\frac{\partial\eta^0}{\partial n}\right|^2((\nabla\psi+\nabla^t\psi)\cdot n)_{tg}\cdot\psi\,d\sigma\,dt \\
&-2s^3\lambda^3 \iint_\Sigma \xi^3\left(\frac{\partial\eta^0}{\partial n}\right)^3|\psi|^2\,dx\,dt - 2s^2\lambda\iint_\Sigma \frac{\partial\eta^0}{\partial n}\xi\alpha_t|\psi|^2\,dx\,dt \\
&+8s\lambda \iint_\Sigma \xi\frac{\partial\eta^0}{\partial n}|\nabla^t\psi\cdot n|^2\,d\sigma\,dt \\
&\qquad \le C\left(\iint_Q e^{-2s\alpha}|G|^2\,dx\,dt + s\lambda\iint_Q \xi|\psi||\Delta\psi|\,dx\,dt\right. \\
&\qquad \left. + s^3\lambda^4\iint_{\omega'\times(0,T)} \xi^3|\psi|^2\,dx\,dt + s\lambda^2\iint_{\omega'\times(0,T)}\xi|\nabla\psi|^2\,dx\,dt\right),
\end{aligned}
\tag{4.59}
$$

for any $\lambda \ge C(\Omega,\omega)$ and any $s \ge C(\Omega,\omega)e^{2\lambda\|\eta^0\|_\infty}(T^7+T^8)$.

Let us finally expand the term $M_2\psi$ as follows :

$$
M_2\psi = \overline{M}_2\psi + 2s\lambda\xi\nabla^t\psi\cdot\nabla\eta^0,
$$

with

$$
\overline{M}_2\psi = (M_2\psi)_1 + (M_2\psi)_2 + (M_2\psi)_4.
$$

This way, $\|M_2\psi\|^2_{L^2(Q)^N}$ can be written as follows :

$$\begin{aligned}\|M_2\psi\|^2_{L^2(Q)^N} = 4s^2\lambda^2 \iint_Q \xi^2 |\nabla^t\psi\cdot\nabla\eta^0|^2\,dx\,dt + \|\overline{M}_2\psi\|^2_{L^2(Q)^N}\\ -4\Big[((M_1\psi)_3,(M_2\psi)_1)_{L^2(Q)^N} + ((M_1\psi)_3,(M_2\psi)_2)_{L^2(Q)^N}\\ +((M_1\psi)_3,(M_2\psi)_4)_{L^2(Q)^N}\Big].\end{aligned}$$

Coming back to the computations we have already made for these products, we have :

$$\begin{aligned}-4\Big[((M_1\psi)_3,(M_2\psi)_1+(M_2\psi)_2+(M_2\psi)_4))_{L^2(Q)^N}\Big]\\ = -4(M_1+M_2+N_1+2E_{244}+N_{21}+N_{22}+P_1+P_2).\end{aligned}$$

Making now the same computations we did when we deduce (4.53), we find

$$\begin{aligned}\|M_2\psi\|^2_{L^2(Q)^N} \geq 4s^2\lambda^2 \iint_Q \xi^2|\nabla^t\psi\cdot\nabla\eta^0|^2\,dx\,dt + \|\overline{M}_2\psi\|^2_{L^2(Q)^N}\\ -4s^3\lambda^4\iint_Q |\nabla\eta^0|^2\xi^3|\psi\cdot\nabla\eta^0|^2\,dx\,dt\\ -C\Big(s\lambda T(1+\lambda)\iint_Q \xi^{5/4}|\psi|^2\,dx\,dt + s\lambda\iint_Q \xi|\nabla\psi|^2\,dx\,dt\\ +s^3\lambda^3\iint_Q \xi^3|\psi|^2\,dx\,dt + s^2\lambda^3(1+\lambda)\iint_Q \xi^2|\psi|^2\,dx\,dt\Big).\end{aligned}$$

This, together with (4.59), provides the desired Carleman estimate for ψ we wanted in the step 1, say :

$$
\begin{aligned}
&\|\overline{M}_1\psi\|^2_{L^2(Q)^N} + \|\overline{M}_2\psi\|^2_{L^2(Q)^N} + s^3\lambda^4 \iint_Q \xi^3|\psi|^2\,dx\,dt \\
&+s\lambda^2 \iint_Q \xi|\nabla\psi|^2\,dx\,dt + 2\iint_\Sigma ((\nabla\psi+\nabla^t\psi)\cdot n)_{tg}\cdot\psi_t\,d\sigma\,dt \\
&-4s\lambda \iint_\Sigma \xi\frac{\partial\eta^0}{\partial n}((\nabla\psi+\nabla^t\psi)\cdot n)\cdot(\nabla\psi\cdot n)\,d\sigma\,dt \\
&+2s\lambda \iint_\Sigma \xi\frac{\partial\eta^0}{\partial n}|\nabla\psi|^2\,d\sigma\,dt + 4s\lambda\iint_\Sigma \xi\frac{\partial\eta^0}{\partial n}\nabla\psi:\nabla^t\psi\,d\sigma\,dt \\
&-4s\lambda \iint_\Sigma \xi\frac{\partial\eta^0}{\partial n}(\nabla\psi\cdot n)\cdot(\nabla^t\psi\cdot n)\,d\sigma\,dt \\
&-4s\lambda \iint_\Sigma \xi\frac{\partial\eta^0}{\partial n}((\nabla\psi+\nabla^t\psi)\cdot n)\cdot(\nabla^t\psi\cdot n)\,d\sigma\,dt \\
&-4s\lambda^2 \iint_\Sigma \xi\left|\frac{\partial\eta^0}{\partial n}\right|^2((\nabla\psi+\nabla^t\psi)\cdot n)_{tg}\cdot\psi\,d\sigma\,dt \\
&-2s^3\lambda^3 \iint_\Sigma \xi^3\left(\frac{\partial\eta^0}{\partial n}\right)^3|\psi|^2\,dx\,dt - 2s^2\lambda\iint_\Sigma \frac{\partial\eta^0}{\partial n}\xi\alpha_t|\psi|^2\,dx\,dt \\
&+8s\lambda \iint_\Sigma \xi\frac{\partial\eta^0}{\partial n}|\nabla^t\psi\cdot n|^2\,d\sigma\,dt \\
&\qquad\le C\left(\iint_Q e^{-2s\alpha}|G|^2\,dx\,dt + s\lambda\iint_Q \xi|\psi||\Delta\psi|\,dx\,dt\right. \\
&\qquad\left.+s^3\lambda^4\iint_{\omega'\times(0,T)}\xi^3|\psi|^2\,dx\,dt + s\lambda^2\iint_{\omega'\times(0,T)}\xi|\nabla\psi|^2\,dx\,dt\right),
\end{aligned}
\tag{4.60}
$$

for any $\lambda \geq C(\Omega,\omega)$ and any $s \geq C(\Omega,\omega)e^{2\lambda\|\eta^0\|_\infty}(T^7+T^8)$.

– STEP 2 : A Carleman inequality for $\tilde{\psi}$:

The strategy we follow here will be analogous to that employed in the first step. Moreover, all the integration by parts will provide the same terms we obtained above up to the sign. Consequently, we will pass over the details and we will focus on the explicit expressions of the resulting integrals.

More precisely, the following equality holds :

$$M_3\tilde{\psi} + M_4\tilde{\psi} = \tilde{G}_{s,\lambda},$$

with

$$
\begin{aligned}
M_3\tilde{\psi} &= -\nabla\cdot(\nabla\tilde{\psi}+\nabla^t\tilde{\psi}) - s^2\lambda^2|\nabla\eta^0|^2\tilde{\xi}^2\tilde{\psi} + s\lambda\tilde{\xi}\nabla^t\tilde{\psi}\cdot\nabla\eta^0 - s\tilde{\alpha}_t\tilde{\psi}, \\
M_4\tilde{\psi} &= -\tilde{\psi}_t - 2s\lambda\tilde{\xi}\nabla\tilde{\psi}\cdot\nabla\eta^0 - 2s\lambda\tilde{\xi}\nabla^t\tilde{\psi}\cdot\nabla\eta^0 + 2s\lambda^2|\nabla\eta^0|^2\tilde{\xi}\tilde{\psi}
\end{aligned}
$$

and

$$\tilde{G}_{s,\lambda} = e^{-s\tilde{\alpha}}G + s\lambda\Delta\eta^0\tilde{\xi}\tilde{\psi} + s\lambda\tilde{\xi}\nabla\nabla\eta^0\cdot\tilde{\psi} + s\lambda^2|\nabla\eta^0|^2\tilde{\xi}\tilde{\psi} - s\lambda^2\nabla\eta^0\tilde{\xi}\nabla\eta^0\cdot\tilde{\psi}.$$

Similarly to (4.49), we find that

$$\|M_3\tilde{\psi}\|^2_{L^2(Q)^N} + \|M_4\tilde{\psi}\|^2_{L^2(Q)^N} + 2(M_3\tilde{\psi}, M_4\tilde{\psi})_{L^2(Q)^N} = \|\tilde{G}_{s,\lambda}\|^2_{L^2(Q)^N}. \tag{4.61}$$

Before proceeding with the computations of the double product term, let us point out several properties which are different in this case :

$$\nabla\tilde{\xi} = -\lambda\nabla\eta^0\tilde{\xi}, \quad \nabla\cdot\tilde{\psi} = -s\lambda\tilde{\xi}(\tilde{\psi}\cdot\nabla\eta^0).$$

First, we get

$$((M_3\tilde{\psi})_1, (M_4\tilde{\psi})_1)_{L^2(Q)^N} = \tilde{A}_1,$$

with

$$\tilde{A}_1 = \iint_\Sigma((\nabla\tilde{\psi} + \nabla^t\tilde{\psi})\cdot n)_{tg}\cdot\tilde{\psi}_t\,d\sigma\,dt.$$

Then,

$$\begin{aligned}((M_3\tilde{\psi})_1, (M_4\tilde{\psi})_2)_{L^2(Q)^N} = {}& \tilde{B} + \tilde{\mathcal{C}} + \tilde{D}_1 + \tilde{D}_2 + \tilde{D}_{21} + \tilde{D}_{22} + \tilde{E}_{11} + \tilde{E}_{12}\\ &+\tilde{E}_{13} + \tilde{E}_{21} + \tilde{E}_{22} + \tilde{E}_{23} + \tilde{E}_{241} + \tilde{E}_{242} + \tilde{E}_{243}\\ &-\tilde{E}_{244} + \tilde{E}_{2451} + \tilde{E}_{2452} + \tilde{E}_{2453},\end{aligned}$$

with

$$\tilde{B} = 2s\lambda\iint_\Sigma\tilde{\xi}\frac{\partial\eta^0}{\partial n}((\nabla\tilde{\psi} + \nabla^t\tilde{\psi})\cdot n)\cdot(\nabla\tilde{\psi}\cdot n)\,d\sigma\,dt,$$

$$\tilde{\mathcal{C}} = -2s\lambda\iint_Q\tilde{\xi}(\nabla\nabla\eta^0\cdot\nabla\tilde{\psi}) : (\nabla\tilde{\psi} + \nabla^t\tilde{\psi})\,dx\,dt,$$

$$\tilde{D}_1 = 2s\lambda^2\iint_Q\tilde{\xi}|\nabla\tilde{\psi}\cdot\nabla\eta^0|^2\,dx\,dt,$$

$$\tilde{D}_2 = 2s\lambda^2\iint_Q\tilde{\xi}(\nabla^t\tilde{\psi}\cdot\nabla\eta^0)\cdot(\nabla\tilde{\psi}\cdot\nabla\eta^0)\,dx\,dt,$$

$$\tilde{D}_{21} = -2s\lambda^2\iint_Q(\nabla^t(\nabla\eta^0\nabla\eta^0\tilde{\xi})\cdot\nabla\tilde{\psi})\cdot\tilde{\psi}\,dx\,dt,$$

$$\tilde{D}_{22} = -s^2\lambda^3\iint_Q\Delta\eta^0|\tilde{\xi}\nabla\eta^0\cdot\tilde{\psi}|^2\,dx\,dt,$$

$$\tilde{E}_{11} = s\lambda\iint_\Sigma\tilde{\xi}\frac{\partial\eta^0}{\partial n}|\nabla\tilde{\psi}|^2\,d\sigma\,dt, \quad \tilde{E}_{12} = s\lambda\iint_Q\Delta\eta^0\tilde{\xi}|\nabla\tilde{\psi}|^2\,dx\,dt,$$

$$\tilde{E}_{13} = -s\lambda^2\iint_Q|\nabla\eta^0|^2\tilde{\xi}|\nabla\tilde{\psi}|^2\,dx\,dt,$$

$$\tilde{E}_{21} = -2s\lambda \iint_\Sigma \tilde{\xi} \frac{\partial \eta^0}{\partial n} \nabla \tilde{\psi} : \nabla^t \tilde{\psi} \, d\sigma \, dt,$$

$$\tilde{E}_{22} = 2s\lambda \iint_Q \Delta \eta^0 \tilde{\xi} \nabla \tilde{\psi} : \nabla^t \tilde{\psi} \, dx \, dt,$$

$$\tilde{E}_{23} = -2s\lambda^2 \iint_Q |\nabla \eta^0|^2 \tilde{\xi} \nabla \tilde{\psi} : \nabla^t \tilde{\psi} \, dx \, dt,$$

$$\tilde{E}_{241} = 2s\lambda \iint_\Sigma \tilde{\xi} \frac{\partial \eta^0}{\partial n} (\nabla \tilde{\psi} \cdot n) \cdot (\nabla^t \tilde{\psi} \cdot n) \, d\sigma \, dt,$$

$$\tilde{E}_{242} = -2s\lambda \iint_Q \tilde{\xi} (\nabla \nabla \eta^0 \cdot \nabla^t \tilde{\psi}) \cdot \nabla \tilde{\psi} \, dx \, dt,$$

$$\tilde{E}_{243} = 2s^2\lambda^2 \iint_Q \tilde{\xi}^2 (\nabla \nabla \eta^0 \cdot \tilde{\psi}) \cdot (\nabla \tilde{\psi} \cdot \nabla \eta^0) \, dx \, dt,$$

$$\tilde{E}_{244} = -s^2\lambda^3 \iint_Q \tilde{\xi}^2 \nabla \eta^0 \cdot (\nabla (\tilde{\psi} \tilde{\psi}) \cdot \nabla \eta^0 \cdot \nabla \eta^0) \, dx \, dt,$$

$$\tilde{E}_{2451} = 2s^2\lambda^2 \iint_Q \tilde{\xi}^2 (\nabla (\nabla \eta^0 \nabla \eta^0) \cdot \nabla^t \tilde{\psi}) \cdot \tilde{\psi} \, dx \, dt,$$

$$\tilde{E}_{2452} = -s^3\lambda^3 \iint_Q \Delta \eta^0 \tilde{\xi}^3 |\tilde{\psi} \cdot \nabla \eta^0|^2 \, dx \, dt$$

and

$$\tilde{E}_{2453} = s^3\lambda^4 \iint_Q |\nabla \eta^0|^2 \tilde{\xi}^3 |\tilde{\psi} \cdot \nabla \eta^0|^2 \, dx \, dt.$$

We also have

$$((M_3\tilde{\psi})_1, (M_4\tilde{\psi})_3)_{L^2(Q)^N} = \tilde{D}_{22} - 2\tilde{E}_{244} + \tilde{F}_1 + \tilde{F}_2 + \tilde{G}_{11} + \tilde{G}_{12} + \tilde{G}_2 + \tilde{H}_1 + \tilde{H}_2 + \tilde{H}_3 + \tilde{H}_{41} + \tilde{H}_{42},$$

where

$$\tilde{F}_1 = -2s\lambda \iint_\Sigma \tilde{\xi} \frac{\partial \eta^0}{\partial n} ((\nabla \tilde{\psi} + \nabla^t \tilde{\psi}) \cdot n) \cdot (\nabla^t \tilde{\psi} \cdot n) \, d\sigma \, dt,$$

$$\tilde{F}_2 = -2s\lambda \iint_Q \tilde{\xi} (\nabla \tilde{\psi} + \nabla^t \tilde{\psi}) : (\nabla \nabla \eta^0 \cdot \nabla^t \tilde{\psi}) \, dx \, dt,$$

$$\tilde{G}_{11} = -2s\lambda^2 \iint_Q \tilde{\xi} (\nabla \nabla \eta^0 : \nabla \tilde{\psi})(\tilde{\psi} \cdot \nabla \eta^0) \, dx \, dt,$$

$$\tilde{G}_{12} = 2s\lambda^3 \iint_Q \tilde{\xi} ((\nabla \tilde{\psi} \cdot \nabla \eta^0) \cdot \nabla \eta^0)(\tilde{\psi} \cdot \nabla \eta^0) \, dx \, dt,$$

$$\tilde{G}_2 = 2s\lambda^2 \iint_Q \tilde{\xi} |\nabla^t \tilde{\psi} \cdot \nabla \eta^0|^2 \, dx \, dt,$$

$$\tilde{H}_1 = -4s\lambda \iint_\Sigma \tilde{\xi} \frac{\partial \eta^0}{\partial n} |\nabla^t \tilde{\psi} \cdot n|^2 \, d\sigma \, dt,$$

$$\tilde{H}_2 = 4s\lambda \iint_Q \tilde{\xi} (\nabla\nabla\eta^0 \cdot \nabla^t \tilde{\psi}) \cdot \nabla^t \tilde{\psi} \, dx \, dt,$$

$$\tilde{H}_3 = -4s\lambda^2 \iint_Q \tilde{\xi} |\nabla^t \tilde{\psi} \cdot \nabla\eta^0|^2 \, dx \, dt,$$

$$\tilde{H}_{41} = -4s^2\lambda^2 \iint_Q \tilde{\xi}^2 (\nabla\nabla\eta^0 \cdot \tilde{\psi}) \cdot (\nabla^t \tilde{\psi} \cdot \nabla\eta^0) \, dx \, dt$$

and

$$\tilde{H}_{42} = -4s^2\lambda^2 \iint_Q \tilde{\xi}^2 |\nabla^t \tilde{\psi} \cdot \nabla\eta^0|^2 \, dx \, dt.$$

The last double product of the first term of $M_3\tilde{\psi}$ is the following one :

$$((M_3\tilde{\psi})_1, (M_4\tilde{\psi})_4)_{L^2(Q)^N} = \tilde{I}_1 + \tilde{I}_2 + \tilde{I}_3 + \tilde{I}_4 + \tilde{I}_5,$$

with

$$\tilde{I}_1 = -2s\lambda^2 \iint_\Sigma \tilde{\xi} \left| \frac{\partial \eta^0}{\partial n} \right|^2 ((\nabla\tilde{\psi} + \nabla^t \tilde{\psi}) \cdot n)_{tg} \cdot \tilde{\psi} \, d\sigma \, dt,$$

$$\tilde{I}_2 = 4s\lambda^2 \iint_Q \tilde{\xi} (\nabla\nabla\eta^0 \nabla\eta^0) \cdot ((\nabla\tilde{\psi} + \nabla^t \tilde{\psi}) \cdot \tilde{\psi}) \, dx \, dt,$$

$$\tilde{I}_3 = -2s\lambda^3 \iint_Q |\nabla\eta^0|^2 \tilde{\xi} (\nabla\eta^0 \cdot (\nabla\tilde{\psi} + \nabla^t \tilde{\psi})) \cdot \tilde{\psi} \, dx \, dt,$$

$$\tilde{I}_4 = 2s\lambda^2 \iint_Q |\nabla\eta^0|^2 \tilde{\xi} |\nabla\tilde{\psi}|^2 \, dx \, dt$$

and

$$\tilde{I}_5 = 2s\lambda^2 \iint_Q |\nabla\eta^0|^2 \tilde{\xi} \nabla\tilde{\psi} : \nabla^t \tilde{\psi} \, dx \, dt.$$

Equivalently to (4.51), we find the next inequality :

$$
\begin{aligned}
((M_3\tilde{\psi})_1, M_4\tilde{\psi})_{L^2(Q)^N} \geq & \iint_\Sigma ((\nabla\tilde{\psi} + \nabla^t\tilde{\psi})\cdot n)_{tg}\cdot \tilde{\psi}_t \, d\sigma\, dt \\
& +2s\lambda \iint_\Sigma \tilde{\xi}\frac{\partial \eta^0}{\partial n}((\nabla\tilde{\psi} + \nabla^t\tilde{\psi})\cdot n)\cdot(\nabla\tilde{\psi}\cdot n)\, d\sigma\, dt \\
& -s\lambda \iint_\Sigma \tilde{\xi}\frac{\partial \eta^0}{\partial n}|\nabla\tilde{\psi}|^2\, d\sigma\, dt - 2s\lambda \iint_\Sigma \tilde{\xi}\frac{\partial \eta^0}{\partial n}\nabla\tilde{\psi} : \nabla^t\tilde{\psi}\, d\sigma\, dt \\
& +2s\lambda \iint_\Sigma \tilde{\xi}\frac{\partial \eta^0}{\partial n}(\nabla\tilde{\psi}\cdot n)\cdot(\nabla^t\tilde{\psi}\cdot n)\, d\sigma\, dt \\
& +2s\lambda \iint_\Sigma \tilde{\xi}\frac{\partial \eta^0}{\partial n}((\nabla\tilde{\psi} + \nabla^t\tilde{\psi})\cdot n)\cdot(\nabla^t\tilde{\psi}\cdot n)\, d\sigma\, dt \\
& -4s\lambda \iint_\Sigma \tilde{\xi}\frac{\partial \eta^0}{\partial n}|\nabla^t\tilde{\psi}\cdot n|^2\, d\sigma\, dt \\
& -2s\lambda^2 \iint_\Sigma \tilde{\xi}\left|\frac{\partial \eta^0}{\partial n}\right|^2((\nabla\tilde{\psi} + \nabla^t\tilde{\psi})\cdot n)_{tg}\cdot\tilde{\psi}\, d\sigma\, dt \\
& +s\lambda^2 \iint_Q |\nabla\eta^0|^2\tilde{\xi}|\nabla\tilde{\psi}|^2\, dx\, dt + s^3\lambda^4 \iint_Q |\nabla\eta^0|^2\tilde{\xi}^3|\tilde{\psi}\cdot\nabla\eta^0|^2\, dx\, dt \\
& -2s\lambda^2 \iint_Q \tilde{\xi}|\nabla^t\tilde{\psi}\cdot\nabla\eta^0|^2\, dx\, dt - 4s^2\lambda^2 \iint_Q \tilde{\xi}^2|\nabla^t\tilde{\psi}\cdot\nabla\eta^0|^2\, dx\, dt \\
& -C\left(s\lambda \iint_Q \tilde{\xi}|\nabla\tilde{\psi}|^2\, dx\, dt + s\lambda^3(s+s^2+\lambda)\iint_Q \tilde{\xi}^3|\tilde{\psi}|^2\, dx\, dt\right).
\end{aligned}
\tag{4.62}
$$

On the other hand, we have

$$((M_3\tilde{\psi})_2, (M_4\tilde{\psi})_1)_{L^2(Q)^N} = \tilde{J} = -s^2\lambda^2 \iint_Q |\nabla\eta^0|^2\tilde{\xi}\tilde{\xi}_t|\tilde{\psi}|^2\, dx\, dt.$$

Additionally, we obtain

$$((M_3\tilde{\psi})_2, (M_4\tilde{\psi})_2)_{L^2(Q)^N} = \tilde{K}_1 + \tilde{K}_2 + \tilde{K}_3,$$

where

$$\tilde{K}_1 = s^3\lambda^3 \iint_\Sigma \tilde{\xi}^3\left(\frac{\partial \eta^0}{\partial n}\right)^3|\tilde{\psi}|^2\, d\sigma\, dt,$$

$$\tilde{K}_2 = -s^3\lambda^3 \iint_Q \tilde{\xi}^3\nabla\cdot(|\nabla\eta^0|^2\nabla\eta^0)|\tilde{\psi}|^2\, dx\, dt$$

and

$$\tilde{K}_3 = 3s^3\lambda^4 \iint_Q |\nabla\eta^0|^4\tilde{\xi}^3|\tilde{\psi}|^2\, dx\, dt.$$

It will not be necessary to perform the product

$$((M_3\tilde{\psi})_2, (M_4\tilde{\psi})_3)_{L^2(Q)^N}$$

neither here.

The last product for $(M_3\tilde{\psi})_2$ yields :

$$((M_3\tilde{\psi})_2, (M_4\tilde{\psi})_4)_{L^2(Q)^N} = \tilde{L} = -2s^3\lambda^4 \iint_Q |\nabla\eta^0|^4\tilde{\xi}^3|\tilde{\psi}|^2\,dx\,dt.$$

Here, we must notice that

$$\tilde{\xi}_t \leq CTe^{(\lambda/4)\|\eta^0\|_\infty}\tilde{\xi}^{5/4}.$$

Consequently, we deduce that

$$\begin{aligned}
&((M_3\tilde{\psi})_2, M_4\tilde{\psi})_{L^2(Q)^N} \geq s^3\lambda^3 \iint_\Sigma \tilde{\xi}^3 \left(\frac{\partial\eta^0}{\partial n}\right)^3 |\tilde{\psi}|^2\,d\sigma\,dt \\
&+s^3\lambda^4 \iint_Q |\nabla\eta^0|^4\tilde{\xi}^3|\tilde{\psi}|^2\,dx\,dt + ((M_3,\tilde{\psi})_2, (M_4\tilde{\psi})_3)_{L^2(Q)^N} \\
&-C\left(s^2\lambda^2 Te^{(\lambda/4)\|\eta^0\|_\infty} \iint_Q \tilde{\xi}^{9/4}|\tilde{\psi}|^2\,dx\,dt + s^3\lambda^3 \iint_Q \tilde{\xi}^3|\tilde{\psi}|^2\,dx\,dt\right).
\end{aligned} \tag{4.63}$$

Let us now compute the scalar products of the third term of $M_3\tilde{\psi}$. Firstly, we have :

$$((M_3\tilde{\psi})_3, (M_4\tilde{\psi})_1)_{L^2(Q)^N} = \tilde{M}_1 + \tilde{M}_2,$$

with

$$\tilde{M}_1 = -\frac{1}{2}s\lambda \iint_Q \tilde{\xi}_t(\nabla\nabla\eta^0\cdot\tilde{\psi})\cdot\tilde{\psi}\,dx\,dt, \quad \tilde{M}_2 = \frac{1}{2}s\lambda^2 \iint_Q \tilde{\xi}_t|\tilde{\psi}\cdot\nabla\eta^0|^2\,dx\,dt.$$

Additionally,

$$((M_3\tilde{\psi})_3, (M_4\tilde{\psi})_2)_{L^2(Q)^N} = 2\tilde{E}_{244} + \tilde{N}_1 + \tilde{N}_{21} + \tilde{N}_{22},$$

where

$$\tilde{N}_1 = 2s^2\lambda^2 \iint_Q \tilde{\xi}^2(\nabla^t\tilde{\psi}\cdot\nabla(\nabla\eta^0\nabla\eta^0))\cdot\tilde{\psi}\,dx\,dt,$$

$$\tilde{N}_{21} = s^3\lambda^3 \iint_Q \tilde{\xi}\Delta\eta^0|\tilde{\xi}\tilde{\psi}\cdot\nabla\eta^0|^2\,dx\,dt$$

and

$$\tilde{N}_{22} = -s^3\lambda^4 \iint_Q |\nabla\eta^0|^2\tilde{\xi}^3|\tilde{\psi}\cdot\nabla\eta^0|^2\,dx\,dt.$$

Furthermore, we have

$$((M_3\tilde{\psi})_3, (M_4\tilde{\psi})_3)_{L^2(Q)^N} = \tilde{O} = -2s^2\lambda^2 \iint_Q |\nabla\eta^0|^2\tilde{\xi}^2|\nabla^t\tilde{\psi}\cdot\nabla\eta^0|^2\,dx\,dt.$$

The last product for $(M_3\tilde{\psi})_3$ gives :

$$((M_3\tilde{\psi})_3, (M_4\tilde{\psi})_4)_{L^2(Q)^N} = \tilde{P}_1 + \tilde{P}_2,$$

where

$$\tilde{P}_1 = -2s^2\lambda^3 \iint_Q (\nabla(|\nabla\eta^0|^2\tilde{\xi}^2\nabla\eta^0)\cdot\tilde{\psi})\cdot\tilde{\psi}\,dx\,dt$$

and

$$\tilde{P}_2 = 2s^3\lambda^4 \iint_Q |\nabla\eta^0|^2\tilde{\xi}^3|\tilde{\psi}\cdot\nabla\eta^0|^2\,dx\,dt.$$

Combining all the products of $(M_3\tilde{\psi})_3$, we find

$$\begin{aligned}((M_3\tilde{\psi})_3, M_4\tilde{\psi})_{L^2(Q)^N} \geq{}& s^3\lambda^4 \iint_Q |\nabla\eta^0|^2\tilde{\xi}^3|\tilde{\psi}\cdot\nabla\eta^0|^2\,dx\,dt \\ &-2s^2\lambda^2 \iint_Q |\nabla\eta^0|^2\tilde{\xi}^2|\nabla^t\tilde{\psi}\cdot\nabla\eta^0|^2\,dx\,dt \\ &-C\Bigg(s\lambda T(1+\lambda)e^{(\lambda/4)\|\eta^0\|_\infty}\iint_Q \tilde{\xi}^{5/4}|\tilde{\psi}|^2\,dx\,dt \\ &+s\lambda\iint_Q \tilde{\xi}|\nabla\tilde{\psi}|^2\,dx\,dt + s^3\lambda^3\iint_Q \tilde{\xi}^3|\tilde{\psi}|^2\,dx\,dt \\ &+s^2\lambda^3(1+\lambda)\iint_Q \tilde{\xi}^2|\tilde{\psi}|^2\,dx\,dt\Bigg).\end{aligned} \tag{4.64}$$

Let us finally consider the computations for $M_3\tilde{\psi}$.
First, we have :

$$((M_3\tilde{\psi})_4, (M_4\tilde{\psi})_1)_{L^2(Q)^N} = \tilde{Q} = -\frac{1}{2}s\iint_Q \tilde{\alpha}_{tt}|\tilde{\psi}|^2\,dx\,dt.$$

Then,

$$((M_3\tilde{\psi})_4, (M_4\tilde{\psi})_2)_{L^2(Q)^N} = \tilde{R}_1 + \tilde{R}_2,$$

with

$$\tilde{R}_1 = s^2\lambda\iint_\Sigma \frac{\partial\eta^0}{\partial n}\tilde{\xi}\tilde{\alpha}_t|\tilde{\psi}|^2\,d\sigma\,dt, \quad \tilde{R}_2 = -s^2\lambda\iint_Q \nabla\cdot(\tilde{\xi}\tilde{\alpha}_t\nabla\eta^0)|\tilde{\psi}|^2\,dx\,dt.$$

Let us again skip the term

$$((M_3\tilde{\psi})_4, (M_4\tilde{\psi})_3)_{L^2(Q)^N}.$$

Finally, we have

$$((M_3\tilde{\psi})_4, (M_4\tilde{\psi})_4)_{L^2(Q)^N} = \tilde{S} = -2s^2\lambda^2\iint_Q |\nabla\eta^0|^2\tilde{\xi}\tilde{\alpha}_t|\tilde{\psi}|^2\,dx\,dt.$$

This time, the expression of $\tilde{\alpha}_t$ is :

$$\tilde{\alpha}_t = -4(T-2t)(e^{2\lambda\|\eta^0\|_\infty} - e^{-\lambda\eta^0})(t(T-t))^{-5},$$

so

$$|\tilde{\alpha}_t| \leq CTe^{(13\lambda/4)\|\eta^0\|_\infty}\tilde{\xi}^{5/4}. \tag{4.65}$$

Furthermore, we have :

$$|\nabla(\tilde{\alpha}_t)| \leq C\lambda Te^{(\lambda/4)\|\eta^0\|_\infty}\tilde{\xi}^{5/4},$$

$$|\tilde{\alpha}_{tt}| \leq Ce^{(13\lambda/4)\|\eta^0\|_\infty}(T^2e^{(\lambda/4)\|\eta^0\|_\infty}\tilde{\xi}^{3/2} + \tilde{\xi}^{5/4}).$$

Consequently, we find the following for $(M_3\tilde{\psi})_4$:

$$\begin{aligned}((M_3\tilde{\psi})_4, M_4\tilde{\psi})_{L^2(Q)^N} \geq\ & s^2\lambda \iint_\Sigma \frac{\partial\eta^0}{\partial n}\tilde{\xi}\tilde{\alpha}_t|\tilde{\psi}|^2\,d\sigma\,dt \\ &+((M_3\tilde{\psi})_4, (M_4\tilde{\psi})_3)_{L^2(Q)^N} \\ &-Ce^{(13\lambda/4)\|\eta^0\|_\infty}\left(s\iint_Q (T^2e^{(\lambda/4)\|\eta^0\|_\infty}\tilde{\xi}^{3/2} + \tilde{\xi}^{5/4})|\tilde{\psi}|^2\,dx\,dt \right.\\ &\left.+s^2\lambda(1+\lambda)T\iint_Q \tilde{\xi}^{9/4}|\tilde{\psi}|^2\,dx\,dt\right)\end{aligned} \tag{4.66}$$

Therefore, taking into account (4.62), (4.63), (4.64), (4.66) and

$$\tilde{\xi}^{-1} \leq CT^8e^{\lambda\|\eta^0\|_\infty},$$

we obtain

$$
\begin{aligned}
2(M_3\tilde{\psi}, M_4\tilde{\psi})_{L^2(Q)^N} \geq\ & 2\iint_\Sigma ((\nabla\tilde{\psi} + \nabla^t\tilde{\psi})\cdot n)_{tg}\cdot \tilde{\psi}_t \, d\sigma\, dt \\
&+4s\lambda \iint_\Sigma \tilde{\xi}\frac{\partial \eta^0}{\partial n}((\nabla\tilde{\psi} + \nabla^t\tilde{\psi})\cdot n)\cdot(\nabla\tilde{\psi}\cdot n)\, d\sigma\, dt \\
&-2s\lambda \iint_\Sigma \tilde{\xi}\frac{\partial \eta^0}{\partial n}|\nabla\tilde{\psi}|^2\, d\sigma\, dt - 4s\lambda\iint_\Sigma \tilde{\xi}\frac{\partial \eta^0}{\partial n}\nabla\tilde{\psi} : \nabla^t\tilde{\psi}\, d\sigma\, dt \\
&+4s\lambda \iint_\Sigma \tilde{\xi}\frac{\partial \eta^0}{\partial n}(\nabla\tilde{\psi}\cdot n)\cdot(\nabla^t\tilde{\psi}\cdot n)\, d\sigma\, dt \\
&+4s\lambda \iint_\Sigma \tilde{\xi}\frac{\partial \eta^0}{\partial n}((\nabla\tilde{\psi} + \nabla^t\tilde{\psi})\cdot n)\cdot(\nabla^t\tilde{\psi}\cdot n)\, d\sigma\, dt \\
&-4s\lambda^2 \iint_\Sigma \tilde{\xi}\left|\frac{\partial \eta^0}{\partial n}\right|^2((\nabla\tilde{\psi} + \nabla^t\tilde{\psi})\cdot n)_{tg}\cdot\tilde{\psi}\, d\sigma\, dt \\
&+2s^3\lambda^3 \iint_\Sigma \tilde{\xi}^3\left(\frac{\partial \eta^0}{\partial n}\right)^3|\tilde{\psi}|^2\, dx\, dt + 2s^2\lambda\iint_\Sigma \frac{\partial \eta^0}{\partial n}\tilde{\xi}\tilde{\alpha}_t|\tilde{\psi}|^2\, dx\, dt \\
&-8s\lambda \iint_\Sigma \tilde{\xi}\frac{\partial \eta^0}{\partial n}|\nabla^t\tilde{\psi}\cdot n|^2\, d\sigma\, dt + 4s^3\lambda^4\iint_Q |\nabla\eta^0|^2\tilde{\xi}^3|\tilde{\psi}\cdot\nabla\eta^0|^2\, dx\, dt \\
&+2s\lambda^2 \iint_Q |\nabla\eta^0|^2\tilde{\xi}|\nabla\tilde{\psi}|^2\, dx\, dt + 2s^3\lambda^4\iint_Q |\nabla\eta^0|^4\tilde{\xi}^3|\tilde{\psi}|^2\, dx\, dt \\
&-4s\lambda^2 \iint_Q \tilde{\xi}|\nabla^t\tilde{\psi}\nabla\eta^0|^2\, dx\, dt - 12s^2\lambda^2\iint_Q \tilde{\xi}^2|\nabla^t\tilde{\psi}\nabla\eta^0|^2\, dx\, dt \\
&+2((M_3\tilde{\psi})_2, (M_4\tilde{\psi})_3)_{L^2(Q)^N} + 2((M_3\tilde{\psi})_4, (M_4\tilde{\psi})_3)_{L^2(Q)^N} \\
&-C\Bigg(s\lambda\iint_Q \tilde{\xi}|\nabla\tilde{\psi}|^2\, dx\, dt + s\lambda^3(s + s^2 + \lambda + s\lambda)\iint_Q \tilde{\xi}^3|\tilde{\psi}|^2\, dx\, dt \\
&+s\lambda T^7 e^{\lambda\|\eta^0\|_\infty}(s\lambda + (1+\lambda)T^8 e^{\lambda\|\eta^0\|_\infty})\iint_Q \tilde{\xi}^3|\tilde{\psi}|^2\, dx\, dt \\
&+sT^7 e^{4\lambda\|\eta^0\|_\infty}(T^7 e^{\lambda\|\eta^0\|_\infty} + s\lambda + s\lambda^2)\iint_Q \tilde{\xi}^3|\tilde{\psi}|^2\, dx\, dt\Bigg).
\end{aligned}
\tag{4.67}
$$

Working as in the case of ψ, we find

$$
\begin{aligned}
s^3\lambda^4 \iint_Q |\nabla\eta^0|^4\tilde{\xi}^3|\tilde{\psi}|^2\, dx\, dt \geq Cs^3\lambda^4 \Bigg(&\iint_Q \tilde{\xi}^3|\tilde{\psi}|^2\, dx\, dt \\
&- \iint_{\omega'\times(0,T)} \tilde{\xi}^3|\tilde{\psi}|^2\, dx\, dt\Bigg)
\end{aligned}
$$

and

$$s\lambda^2 \iint_Q |\nabla \eta^0|^2 \tilde{\xi} |\nabla \tilde{\psi}|^2 \, dx \, dt \geq C s\lambda^2 \left(\iint_Q \tilde{\xi} |\nabla \tilde{\psi}|^2 \, dx \, dt \right.$$
$$\left. - \iint_{\omega' \times (0,T)} \tilde{\xi} |\nabla \tilde{\psi}|^2 \, dx \, dt \right).$$

Making now the choice

$$\lambda \geq C(\Omega, \omega'), \quad s \geq C e^{4\lambda \|\eta^0\|_\infty} (T^7 + T^8),$$

we have

$$\begin{aligned}
&C \left(s\lambda \iint_Q \xi |\nabla \psi|^2 \, dx \, dt + s\lambda^3 (s + s^2 + \lambda + s\lambda) \iint_Q \xi^3 |\psi|^2 \, dx \, dt \right. \\
&+ s\lambda T^7 e^{\lambda \|\eta^0\|_\infty} (s\lambda + (1+\lambda) T^8 e^{\lambda \|\eta^0\|_\infty}) \iint_Q \tilde{\xi}^3 |\tilde{\psi}|^2 \, dx \, dt \\
&\left. + s T^7 e^{4\lambda \|\eta^0\|_\infty} (T^7 e^{\lambda \|\eta^0\|_\infty} + s\lambda + s\lambda^2) \iint_Q \tilde{\xi}^3 |\tilde{\psi}|^2 \, dx \, dt \right) \\
&\qquad \leq \tilde{\delta} \left(s^3 \lambda^4 \iint_Q \tilde{\xi}^3 |\tilde{\psi}|^2 \, dx \, dt + s\lambda^2 \iint_Q \tilde{\xi} |\nabla \tilde{\psi}|^2 \, dx \, dt \right)
\end{aligned}$$

for a constant $\tilde{\delta}(\Omega, \omega) > 0$ small enough.

Consequently, we get the following from (4.67)

$$\begin{aligned}
2(M_3\tilde{\psi}, M_4\tilde{\psi})_{L^2(Q)^N} \geq {} & 2\iint_\Sigma ((\nabla\tilde{\psi} + \nabla^t\tilde{\psi})\cdot n)_{tg}\cdot \tilde{\psi}_t \, d\sigma\, dt \\
& +4s\lambda \iint_\Sigma \tilde{\xi}\frac{\partial\eta^0}{\partial n}((\nabla\tilde{\psi} + \nabla^t\tilde{\psi})\cdot n)\cdot(\nabla\tilde{\psi}\cdot n)\, d\sigma\, dt \\
& -2s\lambda \iint_\Sigma \tilde{\xi}\frac{\partial\eta^0}{\partial n}|\nabla\tilde{\psi}|^2\, d\sigma\, dt - 4s\lambda\iint_\Sigma \tilde{\xi}\frac{\partial\eta^0}{\partial n}\nabla\tilde{\psi} : \nabla^t\tilde{\psi}\, d\sigma\, dt \\
& +4s\lambda \iint_\Sigma \tilde{\xi}\frac{\partial\eta^0}{\partial n}(\nabla\psi\cdot n)\cdot(\nabla^t\tilde{\psi}\cdot n)\, d\sigma\, dt \\
& +4s\lambda \iint_\Sigma \tilde{\xi}\frac{\partial\eta^0}{\partial n}((\nabla\tilde{\psi} + \nabla^t\tilde{\psi})\cdot n)\cdot(\nabla^t\tilde{\psi}\cdot n)\, d\sigma\, dt \\
& -4s\lambda^2 \iint_\Sigma \tilde{\xi}\left|\frac{\partial\eta^0}{\partial n}\right|^2((\nabla\tilde{\psi} + \nabla^t\tilde{\psi})\cdot n)_{tg}\cdot\tilde{\psi}\, d\sigma\, dt \\
& +2s^3\lambda^3 \iint_\Sigma \tilde{\xi}^3\left(\frac{\partial\eta^0}{\partial n}\right)^3|\tilde{\psi}|^2\, dx\, dt + 2s^2\lambda\iint_\Sigma \frac{\partial\eta^0}{\partial n}\tilde{\xi}\tilde{\alpha}_t|\tilde{\psi}|^2\, dx\, dt \\
& -8s\lambda \iint_\Sigma \tilde{\xi}\frac{\partial\eta^0}{\partial n}|\nabla^t\tilde{\psi}\cdot n|^2\, d\sigma\, dt + 4s^3\lambda^4\iint_Q |\nabla\eta^0|^2\tilde{\xi}^3|\tilde{\psi}\cdot\nabla\eta^0|^2\, dx\, dt \\
& +s\lambda^2\iint_Q \tilde{\xi}|\nabla\tilde{\psi}|^2\, dx\, dt + s^3\lambda^4\iint_Q\tilde{\xi}^3|\tilde{\psi}|^2\, dx\, dt \\
& -4s\lambda^2\iint_Q \tilde{\xi}|\nabla^t\tilde{\psi}\nabla\eta^0|^2\, dx\, dt - 12s^2\lambda^2\iint_Q\tilde{\xi}^2|\nabla^t\tilde{\psi}\nabla\eta^0|^2\, dx\, dt \\
& +2((M_3\tilde{\psi})_2, (M_4\tilde{\psi})_3)_{L^2(Q)^N} + 2((M_3\tilde{\psi})_4, (M_4\tilde{\psi})_3)_{L^2(Q)^N},
\end{aligned} \tag{4.68}$$

for any $\lambda \geq C(\Omega,\omega)$ and any $s \geq C(\Omega,\omega)e^{4\lambda\|\eta^0\|_\infty}(T^7 + T^8)$.

Let us also remark that

$$\|\tilde{G}_{s,\lambda}\|^2_{L^2(Q)^N} \leq C\left(\iint_Q e^{-2s\tilde{\alpha}}|G|^2\, dx\, dt + s^2\lambda^4\iint_Q \tilde{\xi}^2|\tilde{\psi}|^2\, dx\, dt\right),$$

for $\lambda \geq C(\Omega,\omega)$.

As a conclusion, we obtain the following inequality up to now :

$$\begin{aligned}
&\|M_3\tilde{\psi}\|^2_{L^2(Q)^N} + \|M_4\tilde{\psi}\|^2_{L^2(Q)^N} + s^3\lambda^4 \iint_Q \tilde{\xi}^3 |\tilde{\psi}|^2\,dx\,dt \\
&+s\lambda^2 \iint_Q \tilde{\xi}|\nabla\tilde{\psi}|^2\,dx\,dt + 4s^3\lambda^4 \iint_Q |\nabla\eta^0|^2\tilde{\xi}^3|\tilde{\psi}\cdot\nabla\eta^0|^2\,dx\,dt \\
&-4s\lambda^2 \iint_Q \tilde{\xi}|\nabla^t\tilde{\psi}\nabla\eta^0|^2\,dx\,dt - 12s^2\lambda^2 \iint_Q \tilde{\xi}^2|\nabla^t\tilde{\psi}\nabla\eta^0|^2\,dx\,dt \\
&+2((M_3\tilde{\psi})_2,(M_4\tilde{\psi})_3)_{L^2(Q)^N} + 2((M_3\tilde{\psi})_4,(M_4\tilde{\psi})_3)_{L^2(Q)^N} \\
&+2\iint_\Sigma ((\nabla\tilde{\psi}+\nabla^t\tilde{\psi})\cdot n)_{tg}\cdot\tilde{\psi}_t\,d\sigma\,dt \\
&+4s\lambda \iint_\Sigma \tilde{\xi}\frac{\partial\eta^0}{\partial n}((\nabla\tilde{\psi}+\nabla^t\tilde{\psi})\cdot n)\cdot(\nabla\tilde{\psi}\cdot n)\,d\sigma\,dt \\
&-2s\lambda \iint_\Sigma \tilde{\xi}\frac{\partial\eta^0}{\partial n}|\nabla\tilde{\psi}|^2\,d\sigma\,dt - 4s\lambda \iint_\Sigma \tilde{\xi}\frac{\partial\eta^0}{\partial n}\nabla\tilde{\psi}:\nabla^t\tilde{\psi}\,d\sigma\,dt \\
&+4s\lambda \iint_\Sigma \tilde{\xi}\frac{\partial\eta^0}{\partial n}(\nabla\tilde{\psi}\cdot n)\cdot(\nabla^t\tilde{\psi}\cdot n)\,d\sigma\,dt \qquad (4.69)\\
&+4s\lambda \iint_\Sigma \tilde{\xi}\frac{\partial\eta^0}{\partial n}((\nabla\tilde{\psi}+\nabla^t\tilde{\psi})\cdot n)\cdot(\nabla^t\tilde{\psi}\cdot n)\,d\sigma\,dt \\
&-4s\lambda^2 \iint_\Sigma \tilde{\xi}\left|\frac{\partial\eta^0}{\partial n}\right|^2((\nabla\tilde{\psi}+\nabla^t\tilde{\psi})\cdot n)_{tg}\cdot\tilde{\psi}\,d\sigma\,dt \\
&+2s^3\lambda^3 \iint_\Sigma \tilde{\xi}^3\left(\frac{\partial\eta^0}{\partial n}\right)^3|\tilde{\psi}|^2\,dx\,dt + 2s^2\lambda \iint_\Sigma \frac{\partial\eta^0}{\partial n}\tilde{\xi}\tilde{\alpha}_t|\tilde{\psi}|^2\,dx\,dt \\
&-8s\lambda \iint_\Sigma \tilde{\xi}\frac{\partial\eta^0}{\partial n}|\nabla^t\tilde{\psi}\cdot n|^2\,d\sigma\,dt \\
&\qquad\qquad \le C\left(\iint_Q e^{-2s\tilde{\alpha}}|G|^2\,dx\,dt + s^3\lambda^4\iint_{\omega'\times(0,T)}\tilde{\xi}^3|\tilde{\psi}|^2\,dx\,dt\right. \\
&\qquad\qquad \left. +s\lambda^2 \iint_{\omega'\times(0,T)}\tilde{\xi}|\nabla\tilde{\psi}|^2\,dx\,dt\right),
\end{aligned}$$

for any $\lambda \ge C(\Omega,\omega)$ and any $s \ge C(\Omega,\omega)e^{4\lambda\|\eta^0\|_\infty}(T^7+T^8)$.

Similar computations to those in step 1 must be made now for $M_3\tilde{\psi}$ and $M_4\tilde{\psi}$. As above, we just write all the integrals coming out from these calculations.

For $M_3\tilde{\psi}$, we have

$$M_3\tilde{\psi} = \overline{M}_3\tilde{\psi} + 2s\lambda\tilde{\xi}\nabla^t\tilde{\psi}\cdot\nabla\eta^0,$$

with

$$\overline{M}_3\tilde{\psi} = -\Delta\tilde{\psi} - s^2\lambda^2|\nabla\eta^0|^2\tilde{\xi}^2\tilde{\psi} - s\tilde{\alpha}_t\tilde{\psi} + s\lambda\tilde{\xi}(\nabla\nabla\eta^0\cdot\tilde{\psi}) - s\lambda^2\nabla\eta^0\tilde{\xi}(\tilde{\psi}\cdot\nabla\eta^0).$$

Developing again the L^2 norm of $M_3\tilde{\psi}$, we focus on the double product term :

$$2(\overline{M}_3\tilde{\psi}, s\lambda\tilde{\xi}\nabla^t\tilde{\psi}\cdot\nabla\eta^0)_{L^2(Q)^N} = -2(\overline{M}_3\tilde{\psi}, (M_4\tilde{\psi})_3)_{L^2(Q)^N} =$$
$$= -2((M_3\tilde{\psi})_2, (M_4\tilde{\psi})_3) - 2((M_3\tilde{\psi})_4, (M_4\tilde{\psi})_3)_{L^2(Q)^N} + 4\tilde{E}_{244} - 2\tilde{H}_{41}$$
$$+\tilde{\mathcal{T}}_1 + \tilde{\mathcal{T}}_{21} + \tilde{\mathcal{T}}_{22} + 2\tilde{\mathcal{T}}_{23} + \tilde{\mathcal{T}}_{31} + \tilde{\mathcal{T}}_{32} + \tilde{\mathcal{T}}_{331},$$

where

$$\tilde{\mathcal{T}}_1 = 4s\lambda\iint_Q \tilde{\xi}(\nabla\nabla\eta^0\cdot\tilde{\psi})\cdot\Delta\tilde{\psi}\,dx\,dt,$$

$$\tilde{\mathcal{T}}_{21} = 4s\lambda^2\iint_Q \tilde{\xi}(\nabla(\nabla\eta^0\nabla\eta^0)\cdot\nabla\tilde{\psi})\cdot\tilde{\psi}\,dx\,dt,$$

$$\tilde{\mathcal{T}}_{22} = -4s\lambda^3\iint_Q \tilde{\xi}(\tilde{\psi}\cdot\nabla\eta^0)(\nabla\tilde{\psi}\cdot\nabla\eta^0)\cdot\nabla\eta^0\,dx\,dt,$$

$$\tilde{\mathcal{T}}_{23} = +4s\lambda^2\iint_Q \tilde{\xi}|\nabla^t\tilde{\psi}\cdot\nabla\eta^0|^2\,dx\,dt$$

$$\tilde{\mathcal{T}}_{31} = 4s^2\lambda^2\iint_Q \tilde{\xi}(\nabla\nabla\eta^0\cdot\tilde{\psi})\cdot\nabla(\tilde{\xi}\tilde{\psi}\cdot\nabla\eta^0)\,dx\,dt,$$

$$\tilde{\mathcal{T}}_{32} = -2s^2\lambda^3\iint_Q \nabla\eta^0\cdot\nabla|\tilde{\xi}\tilde{\psi}\cdot\nabla\eta^0|^2\,dx\,dt$$

and

$$\tilde{\mathcal{T}}_{331} = 4s^2\lambda^2\iint_Q \tilde{\xi}^2|\nabla^t\tilde{\psi}\cdot\nabla\eta^0|^2\,dx\,dt.$$

Thus, from (4.69) we deduce the following inequality :

$$
\begin{aligned}
&\|\overline{M}_3\tilde{\psi}\|^2_{L^2(Q)^N} + \|M_4\tilde{\psi}\|^2_{L^2(Q)^N} + s^3\lambda^4 \iint_Q \tilde{\xi}^3|\tilde{\psi}|^2\,dx\,dt \\
&+s\lambda^2 \iint_Q \tilde{\xi}|\nabla\tilde{\psi}|^2\,dx\,dt + 4s^3\lambda^4 \iint_Q |\nabla\eta^0|^2\tilde{\xi}^3|\tilde{\psi}\cdot\nabla\eta^0|^2\,dx\,dt \\
&-4s^2\lambda^2 \iint_Q \tilde{\xi}^2|\nabla^t\tilde{\psi}\nabla\eta^0|^2\,dx\,dt + 2\iint_\Sigma ((\nabla\tilde{\psi}+\nabla^t\tilde{\psi})\cdot n)_{tg}\cdot\tilde{\psi}_t\,d\sigma\,dt \\
&+4s\lambda \iint_\Sigma \tilde{\xi}\frac{\partial\eta^0}{\partial n}((\nabla\tilde{\psi}+\nabla^t\tilde{\psi})\cdot n)\cdot(\nabla\tilde{\psi}\cdot n)\,d\sigma\,dt \\
&-2s\lambda \iint_\Sigma \tilde{\xi}\frac{\partial\eta^0}{\partial n}|\nabla\tilde{\psi}|^2\,d\sigma\,dt - 4s\lambda\iint_\Sigma \tilde{\xi}\frac{\partial\eta^0}{\partial n}\nabla\tilde{\psi}:\nabla^t\tilde{\psi}\,d\sigma\,dt \\
&+4s\lambda \iint_\Sigma \tilde{\xi}\frac{\partial\eta^0}{\partial n}(\nabla\tilde{\psi}\cdot n)\cdot(\nabla^t\tilde{\psi}\cdot n)\,d\sigma\,dt \\
&+4s\lambda \iint_\Sigma \tilde{\xi}\frac{\partial\eta^0}{\partial n}((\nabla\tilde{\psi}+\nabla^t\tilde{\psi})\cdot n)\cdot(\nabla^t\tilde{\psi}\cdot n)\,d\sigma\,dt \\
&-4s\lambda^2 \iint_\Sigma \tilde{\xi}\left|\frac{\partial\eta^0}{\partial n}\right|^2((\nabla\tilde{\psi}+\nabla^t\tilde{\psi})\cdot n)_{tg}\cdot\tilde{\psi}\,d\sigma\,dt \\
&+2s^3\lambda^3 \iint_\Sigma \tilde{\xi}^3\left(\frac{\partial\eta^0}{\partial n}\right)^3|\tilde{\psi}|^2\,dx\,dt + 2s^2\lambda\iint_\Sigma \frac{\partial\eta^0}{\partial n}\tilde{\xi}\tilde{\alpha}_t|\tilde{\psi}|^2\,dx\,dt \\
&-8s\lambda \iint_\Sigma \tilde{\xi}\frac{\partial\eta^0}{\partial n}|\nabla^t\tilde{\psi}\cdot n|^2\,d\sigma\,dt \\
&\qquad\le C\left(\iint_Q e^{-2s\tilde{\alpha}}|G|^2\,dx\,dt + s\lambda\iint_Q \tilde{\xi}|\tilde{\psi}||\Delta\tilde{\psi}|\,dx\,dt\right. \\
&\qquad\left.+s^3\lambda^4\iint_{\omega'\times(0,T)}\tilde{\xi}^3|\tilde{\psi}|^2\,dx\,dt + s\lambda^2\iint_{\omega'\times(0,T)}\tilde{\xi}|\nabla\tilde{\psi}|^2\,dx\,dt\right),
\end{aligned}
\tag{4.70}
$$

for any $\lambda \geq C(\Omega,\omega)$ and any $s \geq C(\Omega,\omega)e^{4\lambda\|\eta^0\|_\infty}(T^7+T^8)$.

On the other hand, for $M_4\tilde{\psi}$, we have :

$$M_4\tilde{\psi} = \overline{M}_4\tilde{\psi} - 2s\lambda\tilde{\xi}\nabla^t\tilde{\psi}\cdot\nabla\eta^0,$$

with

$$\overline{M}_4\tilde{\psi} = (M_4\tilde{\psi})_1 + (M_4\tilde{\psi})_2 + (M_4\tilde{\psi})_4.$$

Employing now similar estimates as in the first case, we find

$$\begin{aligned}\|M_4\tilde\psi\|^2_{L^2(Q)^N} \geq{}& 4s^2\lambda^2 \iint_Q \tilde\xi^2|\nabla^t\tilde\psi\cdot\nabla\eta^0|^2\,dx\,dt + \|\overline{M}_4\tilde\psi\|^2_{L^2(Q)^N}\\ &-4s^3\lambda^4\iint_Q|\nabla\eta^0|^2\tilde\xi^3|\tilde\psi\cdot\nabla\eta^0|^2\,dx\,dt\\ &-C\left(s\lambda T(1+\lambda)e^{(\lambda/4)\|\eta^0\|_\infty}\iint_Q\tilde\xi^{5/4}|\tilde\psi|^2\,dx\,dt + s\lambda\iint_Q\tilde\xi|\nabla\tilde\psi|^2\,dx\,dt\right.\\ &\left.+s^3\lambda^3\iint_Q\tilde\xi^3|\tilde\psi|^2\,dx\,dt + s^2\lambda^3(1+\lambda)\iint_Q\tilde\xi^2|\tilde\psi|^2\,dx\,dt\right).\end{aligned}$$

This, together with (4.70), provides the estimate we searched in this second step :

$$\begin{aligned}&\|\overline{M}_3\tilde\psi\|^2_{L^2(Q)^N} + \|\overline{M}_4\tilde\psi\|^2_{L^2(Q)^N} + s^3\lambda^4\iint_Q\tilde\xi^3|\tilde\psi|^2\,dx\,dt\\ &+s\lambda^2\iint_Q\tilde\xi|\nabla\tilde\psi|^2\,dx\,dt + 2\iint_\Sigma((\nabla\tilde\psi+\nabla^t\tilde\psi)\cdot n)_{tg}\cdot\psi_t\,d\sigma\,dt\\ &+4s\lambda\iint_\Sigma\tilde\xi\frac{\partial\eta^0}{\partial n}((\nabla\tilde\psi+\nabla^t\tilde\psi)\cdot n)\cdot(\nabla\tilde\psi\cdot n)\,d\sigma\,dt\\ &-2s\lambda\iint_\Sigma\tilde\xi\frac{\partial\eta^0}{\partial n}|\nabla\tilde\psi|^2\,d\sigma\,dt - 4s\lambda\iint_\Sigma\tilde\xi\frac{\partial\eta^0}{\partial n}\nabla\tilde\psi:\nabla^t\tilde\psi\,d\sigma\,dt\\ &+4s\lambda\iint_\Sigma\tilde\xi\frac{\partial\eta^0}{\partial n}(\nabla\tilde\psi\cdot n)\cdot(\nabla^t\tilde\psi\cdot n)\,d\sigma\,dt\\ &+4s\lambda\iint_\Sigma\tilde\xi\frac{\partial\eta^0}{\partial n}((\nabla\tilde\psi+\nabla^t\tilde\psi)\cdot n)\cdot(\nabla^t\tilde\psi\cdot n)\,d\sigma\,dt\\ &-4s\lambda^2\iint_\Sigma\tilde\xi\left|\frac{\partial\eta^0}{\partial n}\right|^2((\nabla\tilde\psi+\nabla^t\tilde\psi)\cdot n)_{tg}\cdot\tilde\psi\,d\sigma\,dt\\ &+2s^3\lambda^3\iint_\Sigma\tilde\xi^3\left(\frac{\partial\eta^0}{\partial n}\right)^3|\tilde\psi|^2\,dx\,dt + 2s^2\lambda\iint_\Sigma\frac{\partial\eta^0}{\partial n}\tilde\xi\tilde\alpha_t|\tilde\psi|^2\,dx\,dt\\ &-8s\lambda\iint_\Sigma\tilde\xi\frac{\partial\eta^0}{\partial n}|\nabla^t\tilde\psi\cdot n|^2\,d\sigma\,dt\\ &\quad\leq C\left(\iint_Q e^{-2s\tilde\alpha}|G|^2\,dx\,dt + s\lambda\iint_Q\tilde\xi|\tilde\psi||\Delta\tilde\psi|\,dx\,dt\right.\\ &\quad\left.+s^3\lambda^4\iint_{\omega'\times(0,T)}\tilde\xi^3|\tilde\psi|^2\,dx\,dt + s\lambda^2\iint_{\omega'\times(0,T)}\tilde\xi|\nabla\tilde\psi|^2\,dx\,dt\right),\end{aligned}\tag{4.71}$$

for any $\lambda \geq C(\Omega,\omega)$ and any $s \geq C(\Omega,\omega)e^{4\lambda\|\eta^0\|_\infty}(T^7+T^8)$.

– STEP 3 : Last arrangements and conclusion.

In this paragraph we will combine the inequalities obtained in steps 1 and 2 ((4.60) and (4.71)). Firstly, we deal with the integrals on the boundary. Then, we will eliminate the local term of $\nabla\psi$ (and $\nabla\tilde{\psi}$). Finally, we will turn back to the original variable φ and we will deduce the inequality (4.46).

Let us thus study the terms on Σ. To this end, we set some relations which will be useful in the sequel, namely :

$$\left.\begin{array}{l} \nabla\psi^i = e^{-s\alpha}(\nabla\varphi^i + s\lambda\nabla\eta^0\xi\varphi^i) \\ \nabla\tilde{\psi}^i = e^{-s\alpha}(\nabla\varphi^i - s\lambda\nabla\eta^0\xi\varphi^i) \end{array}\right\} \text{ on } \Sigma, \text{ for } i = 1, ..., N.$$

These come directly from the fact that

$$\xi = \tilde{\xi}, \quad \alpha = \tilde{\alpha} \quad \text{and} \quad \psi = \tilde{\psi} \quad \text{on } \Sigma.$$

Consequently, since we have $\varphi \cdot n = 0$ on Σ, we obtain

$$\left\{\begin{array}{ll} \nabla\psi \cdot n = e^{-s\alpha}(\nabla\varphi \cdot n + s\lambda\dfrac{\partial\eta^0}{\partial n}\xi\varphi), & \nabla^t\psi \cdot n = e^{-s\alpha}(\nabla^t\varphi \cdot n) \\ \nabla\tilde{\psi} \cdot n = e^{-s\alpha}(\nabla\varphi \cdot n - s\lambda\dfrac{\partial\eta^0}{\partial n}\xi\varphi), & \nabla^t\tilde{\psi} \cdot n = e^{-s\alpha}(\nabla^t\varphi \cdot n). \end{array}\right.$$

Let us start computing $2A_1 + 2\tilde{A}_1$:

$$\begin{aligned} 2A_1 + 2\tilde{A}_1 = {} & 2\iint_\Sigma ((\nabla\psi + \nabla^t\psi) \cdot n)_{tg} \cdot \psi_t \, d\sigma \, dt \\ & +2\iint_\Sigma ((\nabla\tilde{\psi} + \nabla^t\tilde{\psi}) \cdot n)_{tg} \cdot \tilde{\psi}_t \, d\sigma \, dt \\ & = 2\iint_\Sigma e^{-s\alpha}(\nabla\varphi \cdot n + \nabla^t\varphi \cdot n + s\lambda\frac{\partial\eta^0}{\partial n}\xi\varphi)_{tg} \cdot (e^{-s\alpha}\varphi)_t \, dx \, dt \\ & +2\iint_\Sigma e^{-s\alpha}(\nabla\varphi \cdot n + \nabla^t\varphi \cdot n - s\lambda\frac{\partial\eta^0}{\partial n}\xi\varphi)_{tg} \cdot (e^{-s\alpha}\varphi)_t \, dx \, dt \\ & = 2\iint_\Sigma e^{-s\alpha}(A\varphi) \cdot (e^{-s\alpha}\varphi)_t \, dx \, dt \\ & \leq C\|A\|_{H^{(1-\ell)/2}(H^{-3/2+N/2+2\ell})}\|e^{-s\alpha^*}\varphi\|^2_{H^{(1+\ell)/2}(H^{1/2-\ell})}. \end{aligned}$$

To prove the last estimate, it suffices to realize that the product of two functions, one of them $(e^{-s\alpha^*}\varphi)$ belonging to the space $H^{(1+\ell)/2}(0,T;H^{1/2-\ell}(\partial\Omega)^N)$ and the other one (A) belonging to $H^{(1-\ell)/2}(0,T;H^{-3/2+N/2+2\ell}(\partial\Omega)^{N^2})$, is actually in

$$H^{(1-\ell)/2}(0,T;H^{-1/2+\ell}(\partial\Omega)).$$

Here, we have used the hypothesis (4.9) (observe that (4.9) implies that the previous norm of A is finite) and the fact that $\alpha_{|\Sigma}(x,t) = \alpha^*(t)$ (see (4.43) above)

In order to estimate the last expression, let us introduce the function $\varphi^* = e^{-s\alpha^*}\varphi$. It verifies :

$$\begin{cases} -\varphi^*_t - \nabla\cdot(D\varphi^*) = G^* & \text{in } Q,\\ \varphi^*\cdot n = 0, \quad (D\varphi^*\cdot n)_{tg} + (A(x,t)\varphi^*)_{tg} = 0 & \text{on } \Sigma,\\ \varphi^*(\cdot,T) = 0 & \text{in } \Omega,\end{cases}$$

with

$$G^* = e^{-s\alpha^*}G - (e^{-s\alpha^*})_t\varphi.$$

For this system, one can perform a similar proof to that of proposition 11 (using Lax-Milgram's lemma instead of a mixed problem) and obtain a the corresponding estimate equivalent to (4.18). Besides, the interpolation inequality (4.35) tells us that the term

$$\|\varphi^*\|^2_{H^{(1+\ell)/2}(H^{1/2-\ell})}$$

can be estimated in terms of its norm as a strong solution (i.e. , $\varphi^*_t \in L^2$ and $\varphi^* \in L^2(H^2)$). More precisely, we have

$$\|\varphi^*\|^2_{H^{(1+\ell)/2}(H^{1/2-\ell})} \le C\|\varphi^*_t\|^{1+\ell}_{L^2(Q)^N}\|\varphi^*\|^{1-\ell}_{L^2(H^2)} \le C(\|\varphi^*_t\|^2_{L^2(Q)^N} + \|\varphi^*\|^2_{L^2(H^2)}).$$

Let us now use estimate (4.18). This yields

$$\begin{aligned}\|e^{-s\alpha^*}\varphi\|^2_{H^{(1+\ell)/2}(H^{1/2-\ell})} &\le Ce^{CT\|A\|^2_P}(1+\|A\|^4_P)\times\\ &\times(\|e^{-s\alpha^*}G\|^2_{L^2(Q)^N} + \|(e^{-s\alpha^*})_t\varphi\|^2_{L^2(Q)^N}).\end{aligned} \tag{4.72}$$

Similarly to (4.54), we find

$$|\alpha^*_t| \le CTe^{2\lambda\|\eta^0\|_\infty}(\xi^*)^{5/4}.$$

Consequently, we obtain from (4.72)

$$\begin{aligned}&2A_1 + 2\tilde{A}_1 \le e^{CT\|A\|^2_P}(1+\|A\|^5_P)\times\\ &\times\left(\iint_Q e^{-2s\alpha}|G|^2\,dx\,dt + s^2e^{4\lambda\|\eta^0\|_\infty}T^2\iint_Q e^{-2s\alpha}\xi^{5/2}|\varphi|^2\,dx\,dt\right).\end{aligned}$$

Choosing $\lambda \ge Ce^{CT\|A\|^2_P}(1+\|A\|^5_P)$ and $s \ge Ce^{4\lambda\|\eta^0\|}(T^6+T^8)$, we will be able to absorb the last term, while the first one is bounded by

$$\lambda\iint_Q e^{-2s\alpha}|G|^2\,dx\,dt.$$

Additionally,

$$\begin{aligned}2B + 2F_1 + 2\tilde{B} + 2\tilde{F}_1 &= -4s\lambda\iint_\Sigma \xi\frac{\partial\eta^0}{\partial n}|\nabla\psi\cdot n + \nabla^t\psi\cdot n|^2\,d\sigma\,dt\\ &+4s\lambda\iint_\Sigma \xi\frac{\partial\eta^0}{\partial n}|\nabla\tilde{\psi}\cdot n + \nabla\tilde{\psi}\cdot n|^2\,d\sigma\,dt\\ &= -16s^2\lambda^2\iint_\Sigma e^{-2s\alpha}\left|\frac{\partial\eta^0}{\partial n}\right|^2\xi^2\left(\nabla\varphi\cdot n + \nabla^t\varphi\cdot n\right)_{tg}\cdot\varphi\,d\sigma\,dt.\end{aligned}$$

Moreover,

$$2E_{11}+2\tilde{E}_{11}=2s\lambda\iint_Q \xi\frac{\partial\eta^0}{\partial n}|\nabla\psi|^2\,d\sigma\,dt-2s\lambda\iint_Q \xi\frac{\partial\eta^0}{\partial n}|\nabla\tilde{\psi}|^2\,dx\,dt$$

$$=8s^2\lambda^2\iint_\Sigma e^{-2s\alpha}\left|\frac{\partial\eta^0}{\partial n}\right|^2\xi^2(\nabla\varphi\cdot n)_{tg}\cdot\varphi\,d\sigma\,dt.$$

Besides,

$$2E_{21}+2\tilde{E}_{21}=4s\lambda\iint_\Sigma \xi\frac{\partial\eta^0}{\partial n}\nabla\psi:\nabla^t\psi\,d\sigma\,dt-4s\lambda\iint_\Sigma \xi\frac{\partial\eta^0}{\partial n}\nabla\tilde{\psi}:\nabla^t\tilde{\psi}\,d\sigma\,dt$$

$$=16s^2\lambda^2\iint_\Sigma e^{-2s\alpha}\xi^2\left|\frac{\partial\eta^0}{\partial n}\right|^2(\nabla^t\varphi\cdot n)_{tg}\cdot\varphi\,d\sigma\,dt.$$

Then,

$$2E_{241}+2\tilde{E}_{241}=-4s\lambda\iint_\Sigma \xi\frac{\partial\eta^0}{\partial n}(\nabla\psi\cdot n)\cdot(\nabla^t\psi\cdot n)\,d\sigma\,dt$$

$$+4s\lambda\iint_\Sigma \xi\frac{\partial\eta^0}{\partial n}(\nabla\tilde{\psi}\cdot n)\cdot(\nabla^t\tilde{\psi}\cdot n)\,d\sigma\,dt$$

$$=-8s^2\lambda^2\iint_\Sigma e^{-2s\alpha}\xi^2\left|\frac{\partial\eta^0}{\partial n}\right|^2(\nabla^t\varphi\cdot n)_{tg}\cdot\varphi\,d\sigma\,dt.$$

For I_1 and $\tilde{I}_1$, we have

$$2I_1+2\tilde{I}_1=-4s\lambda^2\iint_\Sigma \xi\left|\frac{\partial\eta^0}{\partial n}\right|^2((\nabla\psi+\nabla^t\psi)\cdot n)_{tg}\cdot\psi\,d\sigma\,dt$$

$$-4s\lambda^2\iint_\Sigma \xi\left|\frac{\partial\eta^0}{\partial n}\right|^2((\nabla\tilde{\psi}+\nabla^t\tilde{\psi})\cdot n)_{tg}\cdot\tilde{\psi}\,d\sigma\,dt$$

$$=8s\lambda^2\iint_\Sigma e^{-2s\alpha}\xi\left|\frac{\partial\eta^0}{\partial n}\right|^2(A\varphi)_{tg}\cdot\varphi\,d\sigma\,dt.$$

Finally, we observe that

$$H_1=-\tilde{H}_1,\quad K_1=-\tilde{K}_1\quad\text{and}\quad R_1=-\tilde{R}_1.$$

As a conclusion, an addition of the boundary terms in (4.60) and (4.71) yields an expression that can be bounded as follows :

$$C\left(s^2\lambda^2\|A\|_\infty^2\iint_\Sigma e^{-2s\alpha}\xi^2|\varphi|^2\,d\sigma\,dt+\lambda\iint_Q e^{-2s\alpha}|G|^2\,dx\,dt\right)$$

$$+\varepsilon s^3\lambda^4\iint_Q e^{-2s\alpha}\xi^3|\varphi|^2\,dx\,dt,$$

for a constant $\varepsilon=\varepsilon(\Omega,\omega)>0$ small enough and where we have taken $\lambda\geq Ce^{CT\|A\|_P^2}(1+\|A\|_P^5)$ and $s\geq Ce^{4\lambda\|\eta^0\|}(T^6+T^8)$.

A simple computation proves that the first term can also be absorbed. In fact,

$$-s^2\lambda^2\iint_Q e^{-2s\alpha}\xi^2\nabla\eta^0\cdot\nabla|\varphi|^2\,dx\,dt = s^2\lambda^2\iint_\Sigma e^{-2s\alpha}\xi^2\left|\frac{\partial\eta^0}{\partial n}\right||\varphi|^2\,d\sigma\,dt$$
$$+s^2\lambda^2\iint_Q \nabla(e^{-2s\alpha}\xi^2)\cdot\nabla\eta^0|\varphi|^2\,dx\,dt,$$

so

$$s^2\lambda^2\iint_\Sigma e^{-2s\alpha}\xi^2|\varphi|^2\,d\sigma\,dt$$
$$\le C\left(s^3\lambda^3\int_Q e^{-2s\alpha}\xi^3|\varphi|^2\,dx\,dt + s\lambda\iint_Q e^{-2s\alpha}\xi|\nabla\varphi|^2\,dx\,dt\right),$$

for $s \ge CT^8$ and $\lambda \ge C$.

Consequently, if we take $s \ge CT^8$ and $\lambda \ge C(1+\|A\|_\infty)$, we find the following from (4.60) and (4.71) :

$$\begin{aligned}
&\|\overline{M}_1\psi\|^2_{L^2(Q)^N} + \|\overline{M}_2\psi\|^2_{L^2(Q)^N} + \|\overline{M}_3\tilde\psi\|^2_{L^2(Q)^N} + \|\overline{M}_4\tilde\psi\|^2_{L^2(Q)^N}\\
&+s^3\lambda^4\iint_Q(\xi^3|\psi|^2+\tilde\xi^3|\tilde\psi|^2)\,dx\,dt + s\lambda^2\iint_Q(\xi|\nabla\psi|^2+\tilde\xi|\nabla\tilde\psi|^2)\,dx\,dt\\
&+s^2\lambda^2\iint_\Sigma e^{-2s\alpha}\xi^2|\varphi|^2\,d\sigma\,dt \le C\left(\lambda\iint_Q(e^{-2s\alpha}+e^{-2s\tilde\alpha})|G|^2\,dx\,dt\right.\\
&\qquad +s\lambda\iint_Q(\xi|\psi||\Delta\psi|+\tilde\xi|\tilde\psi||\Delta\tilde\psi|)\,dx\,dt\\
&\qquad +s\lambda^2\iint_{\omega'\times(0,T)}(\xi|\nabla\psi|^2+\tilde\xi|\nabla\tilde\psi|^2)\,dx\,dt\\
&\qquad \left.+s^3\lambda^4\iint_{\omega'\times(0,T)}(\xi^3|\psi|^2+\tilde\xi^3|\tilde\psi|^2)\,dx\,dt\right),
\end{aligned}\tag{4.73}$$

for any $\lambda \ge Ce^{CT\|A\|_P^2}(1+\|A\|_\infty+\|A\|_P^5)$ and any $s \ge Ce^{4\lambda\|\eta^0\|_\infty}(T^6+T^8)$. Recall that $P\subset L^\infty(\Omega)$ continuously, so it suffices to take $\lambda \ge Ce^{CT\|A\|_P^2}(1+\|A\|_P^5)$.

The next step will be to eliminate the local terms of $\nabla\psi$ and $\nabla\tilde\psi$ as well as the terms of $|\psi||\Delta\psi|$ and $|\tilde\psi||\Delta\tilde\psi|$. To this end, we are going to add integrals of $\Delta\psi$ and $\Delta\tilde\psi$ in the left hand side of (4.73). This will be done taking advantage of the presence of $\overline{M}_1\psi$ and $\overline{M}_3\tilde\psi$. Indeed, from

$$\Delta\psi = -\overline{M}_1\psi - s^2\lambda^2|\nabla\eta^0|^2\xi^2\psi - s\alpha_t\psi - s\lambda\xi(\nabla\nabla\eta^0\cdot\psi) - s\lambda^2\nabla\eta^0\xi(\psi\nabla\eta^0)$$

and

$$\Delta\tilde\psi = -\overline{M}_3\tilde\psi - s^2\lambda^2|\nabla\eta^0|^2\tilde\xi^2\tilde\psi - s\tilde\alpha_t\tilde\psi + s\lambda\tilde\xi(\nabla\nabla\eta^0\cdot\tilde\psi) - s\lambda^2\nabla\eta^0\tilde\xi(\tilde\psi\nabla\eta^0),$$

we find

$$
\begin{aligned}
s^{-1} & \iint_Q (\xi^{-1}|\Delta\psi|^2 + \tilde{\xi}^{-1}|\Delta\tilde{\psi}|^2)\,dx\,dt \\
& \leq C \Bigg(s^{-1} \iint_Q (\xi^{-1}|\overline{M}_1\psi|^2 + \tilde{\xi}^{-1}|\overline{M}_3\tilde{\psi}|^2)\,dx\,dt \\
& + s^3\lambda^4 \iint_Q (\xi^3|\psi|^2 + \tilde{\xi}^3|\tilde{\psi}|^2)\,dx\,dt \\
& + sT^2 e^{(13/2)\lambda\|\eta^0\|_\infty} \iint_Q (\xi^{3/2}|\psi|^2 + \tilde{\xi}^{3/2}|\tilde{\psi}|^2)\,dx\,dt \\
& + s\lambda^2 \iint_Q (\xi|\psi|^2 + \tilde{\xi}|\tilde{\psi}|^2)\,dx\,dt + s\lambda^4 \iint_Q (\xi|\psi|^2 + \tilde{\xi}|\tilde{\psi}|^2)\,dx\,dt \Bigg).
\end{aligned}
$$

Here, we have employed the bounds of α_t and $\tilde{\alpha}_t$ given in (4.54) and (4.65), respectively. As a consequence, taking again $\lambda \geq C$ and $s \geq Ce^{4\lambda\|\eta^0\|_\infty}(T^7 + T^8)$, we deduce from (4.73) that

$$
\begin{aligned}
& \|\overline{M}_2\psi\|^2_{L^2(Q)^N} + \|\overline{M}_4\tilde{\psi}\|^2_{L^2(Q)^N} + s^3\lambda^4 \iint_Q (\xi^3|\psi|^2 + \tilde{\xi}^3|\tilde{\psi}|^2)\,dx\,dt \\
& + s\lambda^2 \iint_Q (\xi|\nabla\psi|^2 + \tilde{\xi}|\nabla\tilde{\psi}|^2)\,dx\,dt \\
& + s^{-1} \iint_Q (\xi^{-1}|\Delta\psi|^2 + \tilde{\xi}^{-1}|\Delta\tilde{\psi}|^2)\,dx\,dt + s^2\lambda^2 \iint_\Sigma e^{-2s\alpha}\xi^2|\varphi|^2\,d\sigma\,dt \\
& \qquad \leq C \Bigg(\lambda \iint_Q (e^{-2s\alpha} + e^{-2s\tilde{\alpha}})|G|^2\,dx\,dt \\
& \qquad + s\lambda \iint_Q (\xi|\psi||\Delta\psi| + \tilde{\xi}|\tilde{\psi}||\Delta\tilde{\psi}|)\,dx\,dt \\
& \qquad + s\lambda^2 \iint_{\omega'\times(0,T)} (\xi|\nabla\psi|^2 + \tilde{\xi}|\nabla\tilde{\psi}|^2)\,dx\,dt \\
& \qquad + s^3\lambda^4 \iint_{\omega'\times(0,T)} (\xi^3|\psi|^2 + \tilde{\xi}^3|\tilde{\psi}|^2)\,dx\,dt \Bigg).
\end{aligned}
\tag{4.74}
$$

Similar computations lead to an estimate of the term

$$
s^{-1} \iint_Q (\xi^{-1}|\psi_t|^2 + \tilde{\xi}^{-1}|\tilde{\psi}_t|^2)\,dx\,dt
$$

in terms of the left hand side of (4.74), so it can be added there as well.

On the other hand, we have

$$
\begin{aligned}
s\lambda \iint_Q (\xi|\psi||\Delta\psi| + \tilde{\xi}|\tilde{\psi}||\Delta\tilde{\psi}|)\,dx\,dt
&\le C\left(s^3\lambda^3 \iint_Q (\xi^3|\psi|^2 + \tilde{\xi}^3|\tilde{\psi}|^2)\,dx\,dt\right.\\
&\left. + s^{-1}\lambda^{-1}\iint_Q (\xi^{-1}|\Delta\psi|^2 + \tilde{\xi}^{-1}|\Delta\tilde{\psi}|^2)\,dx\,dt\right).
\end{aligned}
$$

This tells that the second term in the right hand side of (4.74) can be absorbed provided that we take $\lambda \ge C$. Therefore, we obtain

$$
\begin{aligned}
&s^3\lambda^4 \iint_Q (\xi^3|\psi|^2 + \tilde{\xi}^3|\tilde{\psi}|^2)\,dx\,dt + s\lambda^2 \iint_Q (\xi|\nabla\psi|^2 + \tilde{\xi}|\nabla\tilde{\psi}|^2)\,dx\,dt\\
&+ s^{-1}\iint_Q (\xi^{-1}(|\psi_t|^2 + |\Delta\psi|^2) + \tilde{\xi}^{-1}(|\tilde{\psi}_t|^2 + |\Delta\tilde{\psi}|^2))\,dx\,dt\\
&+ s^2\lambda^2 \iint_\Sigma e^{-2s\alpha}\xi^2|\varphi|^2\,d\sigma\,dt \le C\left(\lambda \iint_Q (e^{-2s\alpha} + e^{-2s\tilde{\alpha}})|G|^2\,dx\,dt\right.\\
&\qquad + s\lambda^2 \iint_{\omega'\times(0,T)} (\xi|\nabla\psi|^2 + \tilde{\xi}|\nabla\tilde{\psi}|^2)\,dx\,dt\\
&\qquad \left. + s^3\lambda^4 \iint_{\omega'\times(0,T)} (\xi^3|\psi|^2 + \tilde{\xi}^3|\tilde{\psi}|^2)\,dx\,dt\right)
\end{aligned}
\tag{4.75}
$$

for any $\lambda \ge Ce^{CT\|A\|_P^2}(1 + \|A\|_P^5)$ and any $s \ge Ce^{4\lambda\|\eta^0\|_\infty}(T^6 + T^8)$.

Let us now operate with the local term in $\nabla\psi$ (analogous computations can be performed for $\nabla\tilde{\psi}$). For this, we introduce a function $\rho \in C^2(\Omega)$ such that

$$
\rho \equiv 1 \ \text{ in } \omega', \quad \operatorname{supp}\rho \subset \omega_0.
$$

(remind that ω_0 was introduced in (4.48)). Then,

$$\begin{aligned}
& s\lambda^2 \iint_{\omega'\times(0,T)} \xi|\nabla\psi|^2\,dx\,dt \le \iint_{\omega_0\times(0,T)} \rho\xi|\nabla\psi|^2\,dx\,dt \\
& = -s\lambda^2 \iint_{\omega_0\times(0,T)} \xi(\nabla\rho\cdot\nabla\psi)\psi\,dx\,dt \\
& -s\lambda^3 \iint_{\omega_0\times(0,T)} \rho\xi(\nabla\eta^0\cdot\nabla\psi)\psi\,dx\,dt \\
& -s\lambda^2 \iint_{\omega_0\times(0,T)} \rho\xi\Delta\psi\psi\,dx\,dt \le C\left(\lambda^2 \iint_{\omega_0\times(0,T)} |\nabla\psi|^2\,dx\,dt\right. \\
& \left. s^2\lambda^4 \iint_{\omega_0\times(0,T)} \xi^2|\psi|^2\,dx\,dt + s^3\lambda^4 \iint_{\omega_0\times(0,T)} \xi^3|\psi|^2\,dx\,dt\right) \\
& +\varepsilon s^{-1} \iint_{\omega_0\times(0,T)} \xi^{-1}|\Delta\psi|^2\,dx\,dt,
\end{aligned} \tag{4.76}$$

for a positive constant $\varepsilon = \varepsilon(\Omega,\omega)$ small enough.

Furthermore, from the definition of the weight functions (see (4.43) above), we find

$$e^{-2s\tilde\alpha} \le e^{-2s\alpha}, \quad \tilde\xi \le \xi, \quad |\tilde\psi| \le |\psi| \text{ in } Q.$$

Combining all this with (4.75), we get

$$\begin{aligned}
& s^3\lambda^4 \iint_Q \xi^3|\psi|^2\,dx\,dt + s\lambda^2 \iint_Q \xi|\nabla\psi|^2\,dx\,dt \\
& +s^{-1} \iint_Q \xi^{-1}(|\psi_t|^2 + |\Delta\psi|^2)\,dx\,dt + s^2\lambda^2 \iint_\Sigma e^{-2s\alpha}\xi^2|\varphi|^2\,d\sigma\,dt \\
& \le C\left(\lambda \iint_Q e^{-2s\alpha}|G|^2\,dx\,dt + s^3\lambda^4 \iint_{\omega_0\times(0,T)} \xi^3|\psi|^2\,dx\,dt\right),
\end{aligned} \tag{4.77}$$

for any $\lambda \ge Ce^{CT\|A\|_P^2}(1+\|A\|_P^5)$ and any $s \ge Ce^{4\lambda\|\eta^0\|_\infty}(T^6+T^8)$.

Finally, let us go back to the original variable φ. For the moment, we have

$$\begin{aligned}
& s^3\lambda^4 \iint_Q e^{-2s\alpha}\xi^3|\varphi|^2\,dx\,dt + s\lambda^2 \iint_Q \xi|\nabla\psi|^2\,dx\,dt \\
& +s^{-1} \iint_Q \xi^{-1}(|\psi_t|^2 + |\Delta\psi|^2)\,dx\,dt + s^2\lambda^2 \iint_\Sigma e^{-2s\alpha}\xi^2|\varphi|^2\,d\sigma\,dt \\
& \le C\left(\lambda \iint_Q e^{-2s\alpha}|G|^2\,dx\,dt + s^3\lambda^4 \iint_{\omega_0\times(0,T)} e^{-2s\alpha}\xi^3|\varphi|^2\,dx\,dt\right),
\end{aligned} \tag{4.78}$$

for $\lambda \ge Ce^{CT\|A\|_P^2}(1+\|A\|_P^5)$ and $s \ge Ce^{4\lambda\|\eta^0\|_\infty}(T^6+T^8)$. Next, from the expressions

$$\nabla\varphi = e^{s\alpha}(-s\lambda\nabla\eta^0\xi\psi + \nabla\psi),$$

$$\Delta\varphi = e^{s\alpha}(s^2\lambda^2|\nabla\eta^0|^2\xi^2\psi - 2s\lambda\xi(\nabla\eta^0\nabla\psi) + \Delta\psi - s\lambda\Delta\eta^0\xi\psi - s\lambda^2|\nabla\eta^0|^2\xi\psi),$$

and

$$\varphi_t = e^{s\alpha}(s\alpha_t\psi + \psi_t),$$

we find the following estimates :

$$s\lambda^2 \iint_Q e^{-2s\alpha}\xi|\nabla\varphi|^2\,dx\,dt \\ \leq C\left(s^3\lambda^4\iint_Q \xi^3|\psi|^2\,dx\,dt + s\lambda^2\iint_Q \xi|\nabla\psi|^2\,dx\,dt\right),$$

$$s^{-1}\iint_Q e^{-2s\alpha}\xi^{-1}|\Delta\varphi|^2\,dx\,dt \\ \leq C\left(s^3\lambda^4\iint_Q \xi^3|\psi|^2\,dx\,dt + s\lambda^2\iint_Q \xi|\nabla\psi|^2\,dx\,dt\right. \\ \left. + s^{-1}\iint_Q \xi^{-1}|\Delta\psi|^2\,dx\,dt + +s\lambda^2\iint_Q \xi|\psi|^2\,dx\,dt + s\lambda^4\iint_Q \xi|\psi|^2\,dx\,dt\right)$$

and

$$s^{-1}\iint_Q e^{-2s\alpha}\xi^{-1}|\varphi_t|^2\,dx\,dt \\ \leq C\left(sT^2e^{4\lambda\|\eta^0\|_\infty}\iint_Q \xi^{3/2}|\psi|^2\,dx\,dt + s^{-1}\iint_Q \xi^{-1}|\psi_t|^2\,dx\,dt\right).$$

Now, it is not difficult to see that the bounds of these three terms can be estimated by the left hand side of (4.77) if we take $\lambda \geq C$ and $s \geq Ce^{2\lambda\|\eta^0\|_\infty}(T^7 + T^8)$. As a conclusion, these considerations and the inequality (4.78) provide the desired estimate (4.46), namely :

$$s^3\lambda^4\iint_Q e^{-2s\alpha}\xi^3|\varphi|^2\,dx\,dt + s\lambda^2\iint_Q \xi|\nabla\psi|^2\,dx\,dt \\ + s^{-1}\iint_Q \xi^{-1}(|\psi_t|^2 + |\Delta\psi|^2)\,dx\,dt + s^2\lambda^2\iint_\Sigma e^{-2s\alpha}\xi^2|\varphi|^2\,d\sigma\,dt \\ \leq C\left(\lambda\iint_Q e^{-2s\alpha}|G|^2\,dx\,dt + s^3\lambda^4\iint_{\omega_0\times(0,T)} e^{-2s\alpha}\xi^3|\varphi|^2\,dx\,dt\right),$$

for $\lambda \geq Ce^{CT\|A\|_P^2}(1 + \|A\|_P^5)$ and $s \geq C(\Omega,\omega)e^{4\lambda\|\eta^0\|_\infty}(T^6 + T^8)$.

Remark 8 *The computations made in the last step of the proof of proposition 13 prove that, under the 'natural' assumption on A to be an uniformly elliptic matrix function, a positive boundary integral comes out when combining all the boundary terms appearing in* (4.60) *and* (4.71).

2.2 Carleman inequality for the Stokes system

In this subsection and as a consequence of inequality (4.46), we will deduce a Carleman kind estimate for the system (4.11). This will serve us to prove the null controllability of system (4.2) in the next section.

Proposition 14 *Let us suppose that a and b satisfy hypothesis* (4.7) *and A satisfy* (4.8)-(4.10). *Then, there exist three positive constants $\tilde{\lambda}$, $\tilde{s}$ and $\tilde{C}$ depending on Ω and ω such that,*

$$I(s,\lambda;\varphi) \le C(1+T)s^{15/2}\lambda^8 \iint_{\omega\times(0,T)} e^{-4s\widehat{\alpha}+2s\alpha^*}\widehat{\xi}^{15/2}|\varphi|^2\,dx\,dt \tag{4.79}$$

for any $\lambda \ge \tilde{\lambda}e^{\tilde{\lambda}T(\|a\|_\infty^2+\|b\|_\infty^2+\|A\|_P^2)}(1+\|a\|_\infty^{10/3}+\|b\|_\infty^{10/3}+\|a_t\|_{L^2(L^r)}^{10/3}+\|b_t\|_{L^2(L^r)}^{10/3}+\|A\|_P^5+\|A\|_{H^{(3-\ell)/2}(H^{\nu_2})}^{10/3})$, any $s \ge \tilde{s}e^{8\lambda\|\eta^0\|_\infty}(T^4+T^8)$ and any $\varphi^0 \in H$, where φ is the corresponding solution to (4.11). *Here, we remind that the definition of $I(s,\lambda;\varphi)$ was given in* (4.47).

Proof : Let us remark that this proof follows the same ideas of the Carleman inequalities for Stokes systems with Dirichlet conditions, which have been proved in [11, 7].

We start applying inequality (4.46) with a right hand side

$$G = -\nabla\pi + (a,\nabla)\varphi + D\varphi\, b.$$

This yields

$$I(s,\lambda;\varphi) \le \left(s^3\lambda^4 \iint_{\omega_0\times(0,T)} e^{-2s\alpha}\xi^3|\varphi|^2\,dx\,dt + \lambda\iint_Q e^{-2s\alpha}|\nabla\pi|^2\,dx\,dt \right. \\ \left. +\lambda(\|a\|_\infty^2+\|b\|_\infty^2)\iint_Q e^{-2s\alpha}|\nabla\varphi|^2\,dx\,dt\right),$$

for $\lambda \ge Ce^{CT\|A\|_P^2}(1+\|A\|_P^5)$ and $s \ge \overline{s}e^{4\lambda\|\eta^0\|_\infty}(T^6+T^8)$.

Now, we take $\lambda \ge C(\|a\|_\infty^2+\|b\|_\infty^2)$, so that the last term can be eliminated using the term in $s\lambda^2$ of $I(s,\lambda;\varphi)$. Consequently, we have

$$I(s,\lambda;\varphi) \le C\left(s^3\lambda^4 \iint_{\omega_0\times(0,T)} e^{-2s\alpha}\xi^3|\varphi|^2\,dx\,dt \right. \\ \left. +\lambda\iint_Q e^{-2s\alpha}|\nabla\pi|^2\,dx\,dt\right), \tag{4.80}$$

for any $\lambda \ge Ce^{CT\|A\|_P^2}(1+\|a\|_\infty^2+\|b\|_\infty^2+\|A\|_P^5)$ and any $s \ge Ce^{4\lambda\|\eta^0\|_\infty}(T^6+T^8)$.

The next step will be to localize the pressure term. This will be made by means of an elliptic Carleman inequality which has been proved in [12]. Indeed, let us take the divergence operator in the equation verified by φ. Then,

$$\Delta\pi(t) = \nabla\cdot((a,\nabla)\varphi + D\varphi\, b)(t) \ \text{ in } \Omega \ \text{ a.e. } t\in(0,T). \tag{4.81}$$

We see the expression in the right hand side as a H^{-1} term and we apply that result. Thus, there exist two constants $\tilde{\tau}$ and $\tilde{\lambda}$ greater than 1, such that

$$\begin{aligned}&\int_\Omega e^{2\tau\eta}|\nabla\pi(t)|^2\,dx+\tau^2\lambda^2\int_\Omega e^{2\tau\eta}\eta^2|\pi(t)|^2\,dx\\&\leq C\left(\tau(\|a\|_\infty^2+\|b\|_\infty^2)\int_\Omega e^{2\tau\eta}\eta|\nabla\varphi(t)|^2\,dx+\tau^{1/2}e^{2\tau}\|\pi(t)\|^2_{H^{1/2}(\partial\Omega)}\right.\\&\left.+\tau^2\lambda^2\int_{\omega'}e^{2\tau\eta}\eta^2|\pi(t)|^2\,dx+\int_{\omega'}e^{2\tau\eta}|\nabla\pi(t)|^2\,dx\right),\end{aligned}$$

for $\tau\geq\tilde{\tau}$ and $\lambda\geq\tilde{\lambda}$. Here, the function η is given by

$$\eta(x)=e^{\lambda\eta^0(x)}\quad\text{in }\Omega$$

for each $\lambda>0$. We remind that the function η^0 was introduced in (4.44).

Let us now set

$$\tau=\frac{s}{t^4(T-t)^4}$$

and multiply the previous inequality by

$$\exp\left\{-2s\frac{e^{2\lambda\|\eta^0\|_\infty}}{t^4(T-t)^4}\right\}.$$

Then, an integration in time gives

$$\begin{aligned}&\iint_Q e^{-2s\alpha}|\nabla\pi|^2\,dx\,dt+s^2\lambda^2\iint_Q e^{-2s\alpha}\xi^2|\pi|^2\,dx\,dt\\&\leq C\left(s(\|a\|_\infty^2+\|b\|_\infty^2)\iint_Q e^{-2s\alpha}\xi|\nabla\varphi|^2\,dx\,dt\right.\\&+s^{1/2}\int_0^T e^{-2s\alpha^*}(\xi^*)^{1/2}\|\pi(t)\|^2_{H^{1/2}(\partial\Omega)}\\&+s^2\lambda^2\iint_{\omega'\times(0,T)}e^{-2s\alpha}\xi^2|\pi|^2\,dx\,dt\\&\left.+\iint_{\omega'\times(0,T)}e^{-2s\alpha}|\nabla\pi|^2\,dx\,dt\right),\end{aligned}\tag{4.82}$$

for $\lambda\geq C$ and $s\geq CT^8$. We remind that the definitions of α^* and ξ^* were given in (4.43). Observe that taking $\lambda\geq C(\|a\|_\infty^2+\|b\|_\infty^2)$ and $s\geq CT^8$, the first term in the right will be absorbed when combined with (4.80). On the other hand, let us introduce the following functions, in order to estimate the term of the trace :

$$\widetilde{\varphi}=s^{1/4}e^{-s\alpha^*}(\xi^*)^{1/4}\varphi,\quad\widetilde{\pi}=s^{1/4}e^{-s\alpha^*}(\xi^*)^{1/4}\pi.$$

They verify

$$\begin{cases} -\widetilde{\varphi}_t - \Delta\widetilde{\varphi} - (a(x,t),\nabla)\widetilde{\varphi} - D\widetilde{\varphi}\, b(x,t) + \nabla\widetilde{\pi} \\ \qquad\qquad\qquad\qquad = -(s^{1/4}e^{-s\alpha^*}(\xi^*)^{1/4})_t(t)\varphi & \text{in } Q, \\ \nabla\cdot\widetilde{\varphi} = 0 & \text{in } Q, \\ \widetilde{\varphi}\cdot n = 0, \quad (\sigma(\widetilde{\varphi},\widetilde{\pi}))_{tg} + (A(x,t)\widetilde{\varphi})_{tg} = 0 & \text{on } \Sigma, \\ \widetilde{\varphi}(\cdot,T) = 0 & \text{in } \Omega. \end{cases} \tag{4.83}$$

Let us first watch $\widetilde{\varphi}$ as the weak solution to (4.83). In particular, it satisfies

$$\|\widetilde{\varphi}\|^2_{L^2(H^1)} \le e^{CT(\|a\|^2_\infty+\|b\|^2_\infty+\|A\|^2_\infty)}\|s^{1/4}(e^{-s\alpha^*}(\xi^*)^{1/4})_t\varphi\|^2_{L^2(Q)^N}.$$

Applying now proposition 11 and taking into account that $P\subset L^\infty(\Omega)$ continuously, we find

$$\begin{aligned}&\|\widetilde{\pi}\|^2_{L^2(H^1)} \le Ce^{CT(\|a\|^2_\infty+\|b\|^2_\infty+\|A\|^2_P)}(1+\|A\|^4_P)\times\\ &\times(1+\|a\|^2_\infty+\|b\|^2_\infty)s^{5/2}T^2e^{4\lambda\|\eta^0\|_\infty}\iint_Q e^{-2s\alpha^*}(\xi^*)^3|\varphi|^2\,dx\,dt.\end{aligned} \tag{4.84}$$

Plugging this into (4.82) and taking $\lambda \ge Ce^{CT(\|a\|^2_\infty+\|b\|^2_\infty+\|A\|^2_P)}(1+\|a\|^2_\infty+\|b\|^2_\infty+\|A\|^5_P)$ and $s\ge Ce^{8\lambda\|\eta^0\|_\infty}T^4$, we get

$$\begin{aligned}&\iint_Q e^{-2s\alpha}|\nabla\pi|^2\,dx\,dt + s^2\lambda^2\iint_Q e^{-2s\alpha}\xi^2|\pi|^2\,dx\,dt\\ &\le C\left(s(\|a\|^2_\infty+\|b\|^2_\infty)\iint_Q e^{-2s\alpha}\xi|\nabla\varphi|^2\,dx\,dt\right.\\ &\left.+s^2\lambda^2\iint_{\omega'\times(0,T)} e^{-2s\alpha}\xi^2|\pi|^2\,dx\,dt + \iint_{\omega'\times(0,T)} e^{-2s\alpha}|\nabla\pi|^2\,dx\,dt\right)\\ &+\varepsilon s^3\lambda^4\iint_Q e^{-2s\alpha}\xi^3|\varphi|^2\,dx\,dt,\end{aligned}$$

for a small positive constant $\varepsilon(\Omega,\omega)$.

Combining this with (4.80), we have

$$\begin{aligned}I(s,\lambda;\varphi) \le C&\left(s^3\lambda^4\iint_{\omega_0\times(0,T)} e^{-2s\alpha}\xi^3|\varphi|^2\,dx\,dt\right.\\ &\left.+s^2\lambda^3\iint_{\omega'\times(0,T)} e^{-2s\alpha}\xi^2|\pi|^2\,dx\,dt + \lambda\iint_{\omega'\times(0,T)} e^{-2s\alpha}|\nabla\pi|^2\,dx\,dt\right)\end{aligned} \tag{4.85}$$

for $\lambda\ge Ce^{CT(\|a\|^2_\infty+\|b\|^2_\infty+\|A\|^2_P)}(1+\|a\|^2_\infty+\|b\|^2_\infty+\|A\|^5_P)$ and $s\ge Ce^{8\lambda\|\eta^0\|_\infty}(T^4+T^8)$.

The last task will be to bound the local terms involving the pressure. To do this, we will follow the same ideas developed in [7], so we may sometimes give few details along the proof. Indeed, let us take the pressure π to satisfy

$$\int_{\omega'} \pi(t)\,dx = 0 \ \text{ a.e. } t \in (0,T).$$

Then, by virtue of Poincare-Wirtinger's inequality and the definition of $\widehat{\alpha}$ and $\widehat{\xi}$ (see (4.43)), we find

$$\begin{aligned} s^2\lambda^3 \iint_{\omega'\times(0,T)} e^{-2s\alpha}\xi^2|\pi|^2\,dx\,dt &\le s^2\lambda^3 \iint_{\omega'\times(0,T)} e^{-2s\widehat{\alpha}}\widehat{\xi}^2|\pi|^2\,dx\,dt \\ &\le Cs^2\lambda^3 \iint_{\omega'\times(0,T)} e^{-2s\widehat{\alpha}}\widehat{\xi}^2|\nabla\pi|^2\,dx\,dt, \end{aligned}$$

for some positive constant $C(\Omega,\omega)$.

Moreover, from the differential equation in (4.11), we have

$$\nabla\pi = \varphi_t + \Delta\varphi + (a(x,t),\nabla)\varphi + D\varphi\, b(x,t) \ \text{ in } Q,$$

which connected with (4.85), gives

$$\begin{aligned} I(s,\lambda;\varphi) \le C \Bigg(& s^3\lambda^4 \iint_{\omega_0\times(0,T)} e^{-2s\alpha}\xi^3|\varphi|^2\,dx\,dt \\ & + s^2\lambda^3 \iint_{\omega'\times(0,T)} e^{-2s\widehat{\alpha}}\widehat{\xi}^2|\varphi_t|^2\,dx\,dt \\ & + s^2\lambda^3 \iint_{\omega'\times(0,T)} e^{-2s\widehat{\alpha}}\widehat{\xi}^2|\Delta\varphi|^2\,dx\,dt \\ & + s^2\lambda^3(\|a\|_\infty^2 + \|b\|_\infty^2) \iint_{\omega'\times(0,T)} e^{-2s\widehat{\alpha}}\widehat{\xi}^2|\nabla\varphi|^2\,dx\,dt \Bigg), \end{aligned} \tag{4.86}$$

for $\lambda \ge Ce^{CT(\|a\|_\infty^2+\|b\|_\infty^2+\|A\|_P^2)}(1+\|a\|_\infty^2+\|b\|_\infty^2+\|A\|_P^5)$ and $s \ge Ce^{8\lambda\|\eta^0\|_\infty}(T^4+T^8)$.

Let us denote

$$\widehat{\theta}(t) = s\lambda^{3/2}e^{-\widehat{\alpha}(t)}\widehat{\xi}(t)$$

At this point, we use a local estimate of $\Delta\varphi$ proved in [7] (see step 4 in the proof of theorem 1), namely :

$$\begin{aligned} \iint_{\omega'\times(0,T)} |\widehat{\theta}|^2|\Delta\varphi|^2\,dx\,dt \le C(1+T) \Bigg(& \iint_{\omega_0\times(0,T)} |\widehat{\theta}'|^2|\varphi|^2\,dx\,dt \\ & + \iint_{\omega_0\times(0,T)} |\widehat{\theta}|^2(|(a,\nabla)\varphi|^2 + |D\varphi\, b|^2 + |\varphi|^2)\,dx\,dt \Bigg). \end{aligned}$$

Let us combine this inequality with (4.86). We obtain

$$
\begin{aligned}
I(s,\lambda;\varphi) \le C(1+T)\Bigg(&s^4\lambda^4T^2e^{4\lambda\|\eta^0\|_\infty}\iint_{\omega'\times(0,T)} e^{-2s\widehat{\alpha}}\widehat{\xi}^{9/2}|\varphi|^2\,dx\,dt\\
&+s^2\lambda^3\iint_{\omega'\times(0,T)} e^{-2s\widehat{\alpha}}\widehat{\xi}^2|\varphi_t|^2\,dx\,dt\\
&+s^2\lambda^3(\|a\|_\infty^2+\|b\|_\infty^2)\iint_{\omega_1\times(0,T)} e^{-2s\widehat{\alpha}}\widehat{\xi}^2|\nabla\varphi|^2\,dx\,dt\Bigg),
\end{aligned}
\tag{4.87}
$$

for $\lambda \ge Ce^{CT(\|a\|_\infty^2+\|b\|_\infty^2+\|A\|_P^2)}(1+\|a\|_\infty^2+\|b\|_\infty^2+\|A\|_P^5)$ and $s \ge Ce^{8\lambda\|\eta^0\|_\infty}(T^4+T^8)$.

It only rests to estimate φ_t. For this, we integrate by parts with respect to t :

$$
\begin{aligned}
s^2\lambda^3&\iint_{\omega'\times(0,T)} e^{-2s\widehat{\alpha}}\widehat{\xi}^2|\varphi_t|^2\,dx\,dt\\
&=\frac{1}{2}s^2\lambda^3\iint_{\omega'\times(0,T)}(e^{-2s\widehat{\alpha}}\widehat{\xi}^2)_{tt}|\varphi|^2\,dx\,dt\\
&-s^2\lambda^3\iint_{\omega'\times(0,T)} e^{-2s\widehat{\alpha}}\widehat{\xi}^2\varphi\cdot\varphi_{tt}\,dx\,dt.
\end{aligned}
\tag{4.88}
$$

It is not difficult to see that $(e^{-2s\widehat{\alpha}}\widehat{\xi}^2)_{tt}$ is a function bounded by

$$
Cs^2T^2e^{-2s\widehat{\alpha}}\widehat{\xi}^{9/2}.
$$

On the other hand, let us apply Hölder's inequality in the last term :

$$
\begin{aligned}
s^2\lambda^3&\iint_{\omega'\times(0,T)} e^{-2s\widehat{\alpha}}\widehat{\xi}^2\varphi\cdot\varphi_{tt}\,dx\,dt\\
&\le\Bigg(s^{15/2}\lambda^8\iint_{\omega'\times(0,T)} e^{-4s\widehat{\alpha}+2s\alpha^*}\widehat{\xi}^{-15/2}|\varphi|^2\,dx\,dt\\
&+\iint_{\omega'\times(0,T)}|\theta^*|^2|\varphi_{tt}|^2\,dx\,dt\Bigg),
\end{aligned}
\tag{4.89}
$$

where we have denoted

$$
\theta^* = s^{-7/4}\lambda^{-1}e^{-s\alpha^*}\widehat{\xi}^{-7/4}.
$$

Let us now introduce the functions $u=\theta^*\varphi_t$ and $h=\theta^*\pi_t$. They verify

$$
\begin{cases}
-u_t-\Delta u-(a(x,t),\nabla)u-Dub(x,t)+\nabla h = G & \text{in } Q,\\
\nabla\cdot u = 0 & \text{in } Q,\\
u\cdot n = 0,\ (\sigma(u,h)\cdot n)_{tg}+(A(x,t)u)_{tg} = -\theta^*A_t(x,t)\varphi & \text{on } \Sigma,\\
u(\cdot,T)=0 & \text{in } \Omega,
\end{cases}
\tag{4.90}
$$

with

$$G = -\theta_t^* \varphi_t + \theta^*(a_t, \nabla)\varphi + \theta^* D\varphi\, b_t.$$

The idea we develop now was already presented in proposition 11, but in a different way. On the other hand, we will keep an explicit dependence on the coefficients a, b and A.

We first define u as the weak solution of (4.90). In view of (4.20), it suffices to have $G \in L^2(0,T;H^{-1}(\Omega)^N)$ and $\theta^* A_t \varphi \in L^2(0,T;H^{-1/2}(\partial\Omega)^N)$. We already have that φ is a strong solution, i.e.,

$$\varphi \in L^2(0,T;H^2(\Omega)^N) \cap H^1(0,T;L^2(\Omega)^N).$$

Then, the hypothesis (4.7) on a and b and (4.10) (on A) readily imply that $G \in L^2(0,T;H^{-1}(\Omega)^N)$ and that $\theta^* A_t \varphi \in L^2(0,T;H^{-1/2}(\partial\Omega)^N)$. Moreover,

$$\begin{aligned} &\|u\|^2_{L^2(H^1)} \\ &\qquad \le Ce^{CT(\|a\|^2_\infty + \|b\|^2_\infty + \|A\|^2_\infty)}(\|G\|^2_{L^2(H^{-1})} + \|\theta^* A_t\varphi\|^2_{L^2(H^{-1/2})}). \end{aligned} \tag{4.91}$$

Next, we see u as the strong solution of (4.90). By virtue of proposition 11, we must verify that $G \in L^2(Q)^N$ and $\theta^* A_t \varphi \in H^{(1-\ell)/2}(0,T;H^{\ell-1/2}(\partial\Omega)^N) \cap L^2(0,T;H^{1/2}(\partial\Omega)^N)$.

Let us first show that $G \in L^2(Q)^N$. The only terms to study are $\theta^*(a_t,\nabla)\varphi$ and $\theta^* D\varphi\, b_t$. Since $u \in L^2(0,T;H^1(\Omega)^N)$, we have $\theta^*\nabla\varphi \in H^1(0,T;L^2(\Omega)^{N\times N})$. This, together with $\theta^*\nabla\varphi \in L^2(0,T;H^1(\Omega)^{N\times N})$, gives (see [18] for more details)

$$\theta^*\nabla\varphi \in L^\infty(0,T;H^{1/2}(\Omega)^{N\times N}).$$

Then, (4.7) readily yields $\theta^*(a_t,\nabla)\varphi \in L^2(Q)^N$, $\theta^* D\varphi\, b_t \in L^2(Q)^N$ and

$$\begin{aligned} &\|\theta^*(a_t,\nabla)\varphi\|^2_{L^2(Q)^N} + \|\theta^* D\varphi\, b_t\|^2_{L^2(Q)^N} \\ &\le C(\|a_t\|^2_{L^2(L^r)} + \|b_t\|_{L^2(L^r)})(\|\theta^*\varphi\|^2_{L^2(H^2)} + \|\theta_t^*\varphi\|^2_{L^2(W)} + \|u\|^2_{L^2(W)}). \end{aligned}$$

Moreover, from (4.91) we have $\theta^*\varphi \in H^1(0,T;H^{1/2}(\partial\Omega)^N)$ which combined with (4.10) gives

$$\theta^* A_t\varphi \in H^{(1-\ell)/2}(0,T;H^{\nu_2}(\partial\Omega)^N) \subset H^{(1-\ell)/2}(0,T;H^{\ell-1/2}(\partial\Omega)^N)$$

(ν_2 was defined in the introduction) and

$$\|\theta^* A_t\varphi\|^2_{H^{(1-\ell)/2}(H^{\ell-1/2})} \le C\|A\|^2_{H^{(3-\ell)/2}(H^{\nu_2})}(\|\theta_t^*\varphi\|^2_{L^2(H^1)} + \|u\|^2_{L^2(H^1)}).$$

Additionally, from $\theta^*\varphi \in L^2(0,T;H^2(\Omega)^N) \cap H^1(0,T;H^1(\Omega)^N)$ we find that $\theta^*\varphi$ belongs to $H^{1/4}(0,T;H^{5/4}(\partial\Omega)^N)$. Thus, using $A_t \in H^{(1-\ell)/2}(0,T;H^{1/2}(\partial\Omega)^{N\times N})$ (deduced from (4.10)), we have $\theta^* A_t\varphi \in L^2(0,T;H^{1/2}(\partial\Omega)^N)$ and

$$\begin{aligned} &\|\theta^* A_t\varphi\|^2_{L^2(H^{1/2})} \\ &\qquad \le C\|A\|^2_{H^{(3-\ell)/2}(H^{1/2})}(\|\theta^*\varphi\|^2_{L^2(H^2)} + \|\theta_t^*\varphi\|^2_{L^2(H^1)} + \|u\|^2_{L^2(H^1)}). \end{aligned}$$

As a conclusion, we deduce that (in particular)

$$u \in H^1(0,T;L^2(\Omega)^N)$$

and

$$\|u\|^2_{H^1(L^2)} \le Ce^{CT\|A\|^2_P}(1+\|A\|^4_P)(1+\|A\|^2_{H^{(3-\ell)/2}(H^{\nu_2})})\times$$
$$\times(\|G\|^2_{L^2(Q)^N}+\|(a,\nabla)u\|^2_{L^2(Q)^N}+\|Dub\|^2_{L^2(Q)^N}+\|\theta^*A_t\varphi\|^2_{H^{(1-\ell)/2}(H^{\ell-1/2})}$$
$$+\|\theta^*A_t\varphi\|^2_{L^2(H^{1/2})}) \le Ce^{CT\|A\|^2_P}(1+\|A\|^4_P)(1+\|A\|^2_{H^{(3-\ell)/2}(H^{\nu_2})})\times$$
$$\Big[(1+\|a_t\|^2_{L^2(L^r)}+\|b_t\|^2_{L^2(L^r)})(\|\theta^*_t\varphi_t\|^2_{L^2(Q)^N}+\|\theta^*\varphi\|^2_{L^2(H^2)}+\|\theta^*_t\varphi\|^2_{L^2(H^1)}$$
$$+\|u\|^2_{L^2(H^1)})+(\|a\|^2_\infty+\|b\|^2_\infty)\|u\|^2_{L^2(H^1)}+\|A\|^2_{H^{(3-\ell)/2}(H^{\nu_2})}(\|\theta^*_t\varphi\|^2_{L^2(H^1)}$$
$$+\|u\|^2_{L^2(H^1)}+\|\theta^*\varphi\|^2_{L^2(H^2)})\Big].$$

Combining this with (4.91), we obtain

$$\|\theta^*\varphi_{tt}\|^2_{L^2(Q)^N} \le Ce^{CT(\|a\|^2_\infty+\|b\|^2_\infty+\|A\|^2_P)}(1+\|A\|^4_P)\times$$
$$\times(1+\|A\|^2_{H^{(3-\ell)/2}(H^{\nu_2})})(1+\|a\|^2_\infty+\|b\|^2_\infty+\|a_t\|^2_{L^2(L^r)}+\|b_t\|^2_{L^2(L^r)}$$
$$+\|A\|^2_{H^{(3-\ell)/2}(H^{\nu_2})})(\|\theta^*_t\varphi_t\|^2_{L^2(Q)^N}+\|\theta^*\varphi\|^2_{L^2(H^2)}+\|\theta^*_t\varphi\|^2_{L^2(H^1)}).$$

Taking now $\lambda \ge Ce^{CT(\|a\|^2_\infty+\|b\|^2_\infty+\|A\|^2_\infty)}(1+\|a\|^{10/3}_\infty+\|b\|^{10/3}_{L^\infty(Q)^N}+\|a_t\|^{10/3}_{L^2(L^r)}+\|b_t\|^{10/3}_{L^2(L^r)}+\|A\|^5_P+\|A\|^{10/3}_{H^{(3-\ell)/2}(H^{\nu_2})})$, we have

$$\begin{aligned}
&\|\theta^*\varphi_{tt}\|^2_{L^2(Q)^N} \le C\lambda^2(\|\theta^*_t\varphi_t\|^2_{L^2(Q)^N}+\|\theta^*\varphi\|^2_{L^2(H^2)}+\|\theta^*_t\varphi\|^2_{L^2(W)})\\
&\le C\lambda^2\left(\lambda^{-2}T^2s^{-3/2}e^{4\lambda\|\eta^0\|_\infty}\iint_Q e^{-2s\alpha^*}\widehat{\xi}^{-1}|\varphi_t|^2\,dx\,dt+\right.\\
&\left.+\lambda^{-2}T^2s^{-3/2}e^{4\lambda\|\eta^0\|_\infty}\iint_Q e^{-2s\alpha^*}\widehat{\xi}^{-1}|\nabla\varphi|^2\,dx\,dt+\|\theta^*\varphi\|^2_{L^2(H^2)}\right).
\end{aligned} \tag{4.92}$$

In order to estimate the last term, let us set $(\widehat{\varphi},\widehat{\pi})=\theta^*(\varphi,\pi)$. They fulfill

$$\begin{cases}
-\widehat{\varphi}_t-\Delta\widehat{\varphi}-(a(x,t),\nabla)\widehat{\varphi}-D\widehat{\varphi}\,b(x,t)+\nabla\widehat{\pi}=-\theta^*_t\varphi & \text{in } Q,\\
\nabla\cdot\widehat{\varphi}=0 & \text{in } Q,\\
\widehat{\varphi}\cdot n=0,\ (\sigma(\widehat{\varphi},\widehat{\pi})\cdot n)_{tg}+(A(x,t)\widehat{\varphi})_{tg}=0 & \text{on } \Sigma,\\
\widehat{\varphi}(\cdot,T)=0 & \text{in } \Omega.
\end{cases}$$

Similarly as we did in (4.84), we find :

$$\|\widehat{\varphi}\|^2_{H^1(L^2)\cap L^2(H^2)} \le Ce^{CT(\|a\|^2_\infty+\|b\|^2_\infty+\|A\|^2_P)}(1+\|A\|^4_P)\times$$
$$\times(1+\|a\|^2_\infty+\|b\|^2_\infty)\|\theta^*_t\varphi\|^2_{L^2(Q)^N}$$
$$\le Ce^{CT(\|a\|^2_\infty+\|b\|^2_\infty+\|A\|^2_P)}(1+\|A\|^4_P)(1+\|a\|^2_\infty+\|b\|^2_\infty)\times$$
$$\times s^{-3/2}\lambda^{-2}T^2e^{4\lambda\|\eta^0\|_\infty}\iint_Q e^{-2s\alpha^*}(\widehat{\xi})^{-1}|\varphi|^2\,dx\,dt.$$

Taking $\lambda \geq Ce^{CT(\|a\|_\infty^2+\|b\|_\infty^2+\|A\|_P^2)}(1+\|a\|_\infty^{10/3}+\|b\|_\infty^{10/3}+\|A\|_P^5)$ and $s \geq Ce^{8\lambda\|\eta^0\|_\infty}(T^4+T^8)$, we obtain from (4.92)

$$\|\theta^*\varphi_{tt}\|^2_{L^2(Q)^N} \leq \varepsilon s^{-1} \iint_Q e^{-2s\alpha^*}\widehat{\xi}^{-1}(|\varphi_t|^2+|\nabla\varphi|^2+|\varphi|^2)\,dx\,dt$$

for a small positive constant $\varepsilon(\Omega,\omega)$.

Once $\theta^*\varphi_{tt}$ has been bounded, we come back to expressions (4.88) and (4.89) and we deduce

$$\begin{aligned} s^2\lambda^3 &\iint_{\omega'\times(0,T)} e^{-2s\widehat{\alpha}}\widehat{\xi}^2|\varphi_t|^2\,dx\,dt \\ &\leq C\left(s^{15/2}\lambda^8 \iint_{\omega'\times(0,T)} e^{-4s\widehat{\alpha}+2s\alpha^*}\widehat{\xi}^{15/2}|\varphi|^2\,dx\,dt \right.\\ &\left. +\varepsilon s^{-1}\iint_Q e^{-2s\alpha^*}\widehat{\xi}^{-1}(|\varphi_t|^2+|\nabla\varphi|^2+|\varphi|^2)\,dx\,dt\right), \end{aligned}$$

for

$\lambda \geq Ce^{CT(\|a\|_\infty^2+\|b\|_\infty^2+\|A\|_P^2)}(1+\|a\|_\infty^{10/3}+\|b\|_\infty^{10/3}+\|a_t\|_{L^2(L^r)}^{10/3}+\|b_t\|_{L^2(L^r)}^{10/3}+\|A\|_P^5+\|A\|_{H^{(3-\ell)/2}(H^{\nu_2})}^{10/3})$

and $s \geq Ce^{8\lambda\|\eta^0\|_\infty}(T^4+T^8)$.

Therefore, by virtue of (4.87), we find (see (4.47) for the expression of $I(s,\lambda;\varphi)$)

$$\begin{aligned} I(s,\lambda;\varphi) \leq C(1+T)&\left(s^{15/2}\lambda^8 \iint_{\omega'\times(0,T)} e^{-4s\widehat{\alpha}+2s\alpha^*}\widehat{\xi}^{15/2}|\varphi|^2\,dx\,dt\right.\\ &\left.+s^2\lambda^3(\|a\|_\infty^2+\|b\|_\infty^2)\iint_{\omega_0\times(0,T)} e^{-2s\widehat{\alpha}}\widehat{\xi}^2|\nabla\varphi|^2\,dx\,dt\right)+\varepsilon I(s,\lambda;\varphi), \end{aligned} \tag{4.93}$$

for

$\lambda \geq Ce^{CT(\|a\|_\infty^2+\|b\|_\infty^2+\|A\|_P^2)}(1+\|a\|_\infty^{10/3}+\|b\|_\infty^{10/3}+\|a_t\|_{L^2(L^r)}^{10/3}+\|b_t\|_{L^2(L^r)}^{10/3}+\|A\|_P^5+\|A\|_{H^{(3-\ell)/2}(H^{\nu_2})}^{10/3})$

and $s \geq Ce^{8\lambda\|\eta^0\|_\infty}(T^4+T^8)$.

Finally, the last term can be eliminated, first using a cut-off function $\rho \in C^2(\overline{\omega})$ with

$$\rho \equiv 1 \text{ in } \omega_0, \text{ supp } \rho \subset\subset \omega$$

and then the term

$$s^{-1}\iint_Q e^{-2s\alpha}\xi^{-1}|\Delta\varphi|^2\,dx\,dt,$$

appearing in the expression of $I(s,\lambda;\varphi)$ (as we did in (4.76)). Indeed, we have

$$\begin{aligned} s^2\lambda^3(\|a\|_\infty^2+\|b\|_\infty^2)\iint_{\omega_0\times(0,T)} e^{-2s\widehat{\alpha}}\widehat{\xi}^2|\nabla\varphi|^2\,dx\,dt \leq \varepsilon s^{-1}\iint_Q e^{-2s\alpha^*}\widehat{\xi}^{-1}|\Delta\varphi|^2\,dx\,dt \\ +Cs^5\lambda^6(\|a\|_\infty^4+\|b\|_\infty^4)\iint_{\omega\times(0,T)} e^{-4s\widehat{\alpha}+2s\alpha^*}\widehat{\xi}^5|\varphi|^2\,dx\,dt, \end{aligned}$$

for a small positive constant $\varepsilon(\Omega,\omega)$. From this and (4.93), it easily follows inequality (4.79).

This ends the proof of proposition 14.

3 Local null controllability of the Navier-Stokes system

In this section, theorem 8 and theorem 9 will be proved. In a first paragraph, we will provide the proof of the null controllability result for system (4.2), while the local exact controllability to the trajectories of (4.1) will be given in the last paragraph.

3.1 Proof of theorem 8

Proof : Let us remark that the proof of this result follows the ideas of [2]. Let us first consider, for each $\varepsilon > 0$, the extremal problem

$$\inf_{(w,v)\, solve\, (4.2)} \frac{1}{2} \iint_{\omega\times(0,T)} \rho(t)|v|^2\, dx\, dt + \frac{1}{2\varepsilon}\|w(T)\|^2_{L^2(\Omega)^N}. \tag{4.94}$$

Here, we have denoted

$$\rho(t) = e^{4s\widehat{\alpha}-2s\alpha^*}\widehat{\xi}^{-15/2},$$

where s and λ are chosen like in proposition 14 and the definition of $\widehat{\alpha}$, α^* and $\widehat{\xi}$ was given in (4.43). Furthermore, w is the solution of (4.2) associated to v.

Let us suppose that there exists a (unique) solution of (4.94) $(w_\varepsilon, v_\varepsilon)$ with v_ε belonging to $L^2(\rho(t)(0,T); L^2(\omega)^N)$. The necessary condition of minimum yields

$$\iint_{\omega\times(0,T)} \rho(t) v_\varepsilon \cdot v\, dx\, dt + \frac{1}{\varepsilon}\int_\Omega w_\varepsilon(T)\cdot u(T)\, dx = 0 \quad \forall v \in L^2(\omega\times(0,T))^N, \tag{4.95}$$

where u is, together with certain pressure θ, solution of

$$\begin{cases} u_t - \nabla\cdot(Du) + (a(x,t),\nabla)u + (b(x,t),\nabla)u & \\ \qquad\qquad +(u,\nabla)b(x,t) + \nabla\theta = v1_\omega & \text{in } Q,\\ \nabla\cdot u = 0 & \text{in } Q,\\ u\cdot n = 0,\ (\sigma(u,\theta)\cdot n)_{tg} + (A(x,t)u)_{tg} = 0 & \text{on } \Sigma,\\ u(\cdot,0) = 0 & \text{in } \Omega. \end{cases} \tag{4.96}$$

Let us now introduce $(\varphi_\varepsilon, \pi_\varepsilon)$ solution of the following homogeneous adjoint system :

$$\begin{cases} -\varphi_{\varepsilon,t} - \nabla\cdot(D\varphi_\varepsilon) - (a(x,t),\nabla)\varphi_\varepsilon - D\varphi_\varepsilon b(x,t) + \nabla\pi_\varepsilon = 0 & \text{in } Q,\\ \nabla\cdot\varphi_\varepsilon = 0 & \text{in } Q,\\ \varphi_\varepsilon\cdot n = 0,\ (\sigma(\varphi_\varepsilon,\pi_\varepsilon)\cdot n)_{tg} + (A^t(x,t)\varphi_\varepsilon)_{tg} = 0 & \text{on } \Sigma,\\ \varphi_\varepsilon(\cdot,T) = -\dfrac{1}{\varepsilon} w_\varepsilon(\cdot,T) & \text{in } \Omega. \end{cases}$$

Then, the duality properties between φ_ε and w provides

$$-\frac{1}{\varepsilon}\int_\Omega w_\varepsilon(T)\cdot u(T)\, dx = \iint_{\omega\times(0,T)} v\cdot\varphi_\varepsilon\, dx\, dt,$$

which combined with (4.95), yields

$$\iint_{\omega\times(0,T)} v\cdot\varphi_\varepsilon\,dx\,dt = \iint_{\omega\times(0,T)} v\cdot(\rho(t)v_\varepsilon)\,dx\,dt \quad \forall v\in L^2(\omega\times(0,T))^N.$$

Consequently, we can identify v_ε :

$$v_\varepsilon = \rho(t)^{-1}\varphi_\varepsilon 1_\omega.$$

From the systems fulfilled by w_ε and φ_ε, we find

$$-\frac{1}{\varepsilon}\|w_\varepsilon(T)\|^2_{L^2(\Omega)^N} = \iint_{\omega\times(0,T)} \rho(t)^{-1}|\varphi_\varepsilon|^2\,dx\,dt + \int_\Omega \varphi_\varepsilon(0)\cdot w^0\,dx.$$

The Carleman inequality (4.79) used for φ_ε tells us that

$$\|\varphi_\varepsilon(0)\|^2_{L^2(\Omega)^N} \le C(\Omega,\omega,T,a,b,A)\iint_{\omega\times(0,T)} \rho(t)^{-1}|\varphi_\varepsilon|^2\,dx\,dt,$$

so

$$\frac{1}{\varepsilon}\|w_\varepsilon(T)\|^2_{L^2(\Omega)^N} + \|\rho(t)^{1/2}v_\varepsilon\|^2_{L^2(\omega\times(0,T))^N} \le C\|w^0\|^2_H \quad \forall\varepsilon>0. \tag{4.97}$$

Consequently, we deduce the existence of a control $v\in L^2(\rho(t)(0,T);L^2(\omega)^N)$ (whose corresponding solution we denote by w) such that (4.3) holds and

$$\|v\|_{L^2(\rho(t)(0,T);L^2(\omega)^N)} \le C(\Omega,\omega,T,a,b,A)\|w^0\|_H.$$

Let us finally bound the $H^1(L^2)$ and the $L^\infty(H^1)$ norms of this control. For this, let us introduce the functions $(\psi_\varepsilon,q_\varepsilon)=\rho(t)^{-1}(\varphi_\varepsilon,\pi_\varepsilon)$. They verify :

$$\begin{cases} -\psi_{\varepsilon,t}-\nabla\cdot(D\psi_\varepsilon)-(a,\nabla)\psi_\varepsilon-D\psi_\varepsilon b+\nabla q_\varepsilon = -(\rho(t)^{-1})_t\varphi_\varepsilon & \text{in } Q,\\ \nabla\cdot\psi_\varepsilon=0 & \text{in } Q,\\ \psi_\varepsilon\cdot n=0,\ (\sigma(\psi_\varepsilon,q_\varepsilon)\cdot n)_{tg}+(A^t(x,t)\psi_\varepsilon)_{tg}=0 & \text{on } \Sigma,\\ \psi_\varepsilon(\cdot,T)=0 & \text{in } \Omega. \end{cases}$$

Applying here the a priori estimate obtained in proposition 11, we have

$$\|\psi_{\varepsilon,t}\|_{L^2(Q)^N}+\|\psi_\varepsilon\|_{L^\infty(H^1)} \le C(\Omega,\omega,a,b,A)\|(\rho(t)^{-1})_t\varphi_\varepsilon\|_{L^2(Q)^N},$$

which combined with (4.79), implies

$$\|v_{\varepsilon,t}\|_{L^2(\omega\times(0,T))^N}+\|v_\varepsilon\|_{L^\infty(H^1)} \le C(\Omega,\omega,T,a,b,A)\|\rho(t)^{1/2}v_\varepsilon\|_{L^2(\omega\times(0,T))^N}.$$

From (4.97), we conclude that $v\in H^1(0,T;L^2(\omega)^N)$ and

$$\|v\|_{H^1(L^2)}+\|v\|_{L^\infty(H^1)} \le C(\Omega,\omega,T,a,b,A)\|y^0\|_H.$$

Remark 9 *The control previously found can be constructed in such a way that it acts over the system (4.2) in the form ζv instead of $v1_\omega$, where ζ is a cut-off function with support in ω. The proof of this would be exactly the same of proposition 8 except for the corresponding change in system (4.96).*

3.2 Proof of theorem 9

Proof : Let us subtract system (4.4) fulfilled by $\overline{y}$ from (4.1). Denoting $w = y - \overline{y}$, we have :

$$\begin{cases} w_t - \Delta w + (w,\nabla)w + (w,\nabla)\overline{y} + (\overline{y},\nabla)w + \nabla q = v1_\omega & \text{in } Q, \\ \nabla\cdot w = 0 & \text{in } Q, \\ w\cdot n = 0,\ (\sigma(w,q)\cdot n)_{tg} + (F(\overline{y};w)w)_{tg} = 0, & \text{on } \Sigma, \\ w(0) = y^0 - \overline{y}^0 = w^0 & \text{in } \Omega, \end{cases} \tag{4.98}$$

where

$$F(\overline{y};w) = \int_0^1 \nabla f(\overline{y} + lw)\,dl \in \mathbf{R}^N \times \mathbf{R}^N.$$

With this notation, our goal is to find a control v such that w solution to (4.98) satisfies (4.3). This would end the proof of theorem 9.

Let us introduce the Banach space

$$Z = \{z \in H^{(3-\ell)/2}(0,T;H^{\nu_2+1/2}(\Omega)^N \cap W) \cap H^{1-\ell}(0,T;W^{\nu_1+1/2,\nu_1+1}(\Omega)^N \cap W)\}$$

(ℓ, ν_1 and ν_2 were defined in the introduction, at the beginning of the paper) and the closed linear manifold

$$Z_0 = \{z \in Z : z(\cdot,0) = w^0 \text{ in } \Omega\}.$$

Then, for each $z \in Z_0$ and by virtue of theorem 8 and remark 9, there exists a control $v_z \in H^1(0,T;L^2(\omega)^N) \cap C^0([0,T];H^1(\omega)^N)$ such that the solution w_z of

$$\begin{cases} w_t - \nabla\cdot(Dw) + (z,\nabla)w + (w,\nabla)\overline{y} + (\overline{y},\nabla)w + \nabla p = \zeta v_z & \text{in } Q, \\ \nabla\cdot w = 0 & \text{in } Q, \\ w\cdot n = 0,\ (Dw\cdot n)_{tg} + (F(\overline{y};z)w)_{tg} = 0 & \text{on } \Sigma, \\ w(\cdot,0) = w^0(\cdot) & \text{in } \Omega \end{cases} \tag{4.99}$$

verifies (4.3). Observe that, since $F \in C^2(\mathbf{R}^N;\mathbf{R}^{N\times N})$ (see (4.14)) and $\overline{y}$ verifies (4.12), $F(\overline{y},z) \in Z$ for every $z \in Z_0$. Moreover, v_z can be built such that

$$\|v_z\|_{H^1(L^2)} + \|v_z\|_{L^\infty(L^2)} \le C(\Omega,\omega,T,\|z\|_Z,\|F(\overline{y};z)\|_Z)\|w^0\|_H. \tag{4.100}$$

Next, since the terms

$$(z,\nabla)w,\ (w,\nabla)\overline{y},\ (\overline{y},\nabla)w,\ \zeta v_z$$

belong to

$$L^\infty(0,T;H^1(\Omega)^N) \cap H^1(0,T;L^2(\Omega)^N),$$

they have null normal traces and $w^0 = y^0 - \overline{y}^0$ satisfies the compatibility condition

$$(Dw^0\cdot n)_{tg} + (F(\overline{y};z)(x,0)w^0)_{tg} = 0$$

(from (4.13) and (4.15)), we can apply proposition 12 and we obtain

$$w_z \in H^2(0,T;H) \cap H^1(0,T;H^2(\Omega)^N \cap W) \tag{4.101}$$

for each $z \in Z_0$.

Let us introduce the space

$$\tilde{Z} = H^1(0,T;H^2(\Omega) \cap W) \cap H^2(0,T;H).$$

Observe that $\tilde{Z}$ is compactly embedded into Z. Indeed, this can be deduced from $H^2(\Omega) \subset W^{\nu_1+1/2,\nu_1+1}(\Omega)$ compactly (recall that $\nu_1 > 1$ arbitrarily small) and

$$\tilde{Z} \subset H^{(3-(\ell-\varepsilon))/2}(0,T;H^{\nu_2+1/2+\varepsilon}(\Omega)^N) \cap H^1(0,T;H^2(\Omega)^N) \quad \text{continuously}$$

for positive ε arbitrarily small (in fact, we can even take $\varepsilon < 1/2$). This can be established by interpolation spaces arguments.

Let $\Lambda(z)$ be the set constituted by the controls $v_z \in H^1(0,T;L^2(\omega)^N) \cap C^0([0,T];H^1(\omega)^N)$ driving the solution w_z of system (4.99) to zero at time T and such that (4.100) holds. On the other hand, let us introduce their associated states

$$A(z) = \{w_z \text{ solution of (4.99)} : v_z \in \Lambda(z)\}.$$

Observe that $A(z) \subset \tilde{Z} \overset{c}{\subset} Z$. Furthermore, for every $z \in Z$,

$$\|w_z\|_Z \le \tilde{C}(\Omega,\omega,T,\|z\|_Z,\|F(\overline{y};z)\|_Z)\|w^0\|_{H^3\cap W}, \tag{4.102}$$

for certain positive constant $\tilde{C}$.

Our goal is to prove that the set-valued mapping

$$z \longmapsto A(z)$$

has a fixed point with the additional hypothesis $\|w^0\|_{H^3\cap W} \le \delta(\Omega,\omega,T,\ell,\nu)$. This would finish the proof of theorem 9. To this end, we will apply Kakutani's theorem (see [1]) : let

$$A : Z_0 \longmapsto Z_0$$

be a set-valued mapping such that

- $A(z)$ is a nonempty closed convex set of Z_0, for every $z \in Z_0$.
- There exists a convex compact set $K \subset Z_0$ such that $A(K) \subset K$.
- A is upper-hemicontinuous in Z_0, i.e., $\forall \lambda \in Z_0'$, the mapping

$$z \longmapsto \sup_{w \in A(z)} \langle \lambda, w \rangle_{Z_0' Z_0}$$

is upper semicontinuous.

Then, there exists $z \in K$ such that $z \in A(z)$.

The first item is readily satisfied. In order to prove the second one, let $M > 0$ be given and let us denote

$$C(M) = \sup_{\|z\|_Z \leq M} \tilde{C}(\Omega, \omega, T, \ell, \nu, \|z\|_Z, \|F(\overline{y}; z)\|_Z),$$

where $\tilde{C}$ was introduced in (4.102). Then, with a choice like $\delta = M/\tilde{C}(M)$, we have that A sends the closed convex set

$$\tilde{K} = \{z \in Z_0 : \|z\|_Z \leq M\}$$

in a compact set $K \subset \tilde{K}$. This comes from the fact that $\tilde{Z}$ is compactly embedded into Z.

Let us finally prove the upper-hemicontinuity of A. Let

$$z_k \to z \text{ in } Z.$$

From the compactness of $A(z_k)$, we have that

$$\sup_{w \in A(z_k)} \langle \lambda, w \rangle_{Z'Z} = \langle \lambda, w_k \rangle_{Z'Z}$$

for certain $w_k \in A(z_k)$. We choose $\{z_{k'}\} \subset \{z_k\}$ such that

$$\limsup_{k \to \infty} \sup_{w \in A(z_k)} \langle \lambda, w \rangle_{Z'Z} = \lim_{k' \to \infty} \langle \lambda, w_{k'} \rangle_{Z'Z}$$

and denote by $v_{k'}$ the controls belonging to $\Lambda(z_{k'})$ associated to $w_{k'}$, so that they fulfill the system

$$\begin{cases} \partial_t w_{k'} - \nabla \cdot (Dw_{k'}) + (z_{k'}, \nabla) w_{k'} + (z_{k'}, \nabla)\overline{y} + (\overline{y}, \nabla) z_{k'} + \nabla p_{k'} = v_{k'} \zeta & \text{in } Q, \\ \nabla \cdot w_{k'} = 0 & \text{in } Q, \\ w_{k'} \cdot n = 0, \ (Dw_{k'} \cdot n)_{tg} + (F(\overline{y}; z_{k'}) w_{k'})_{tg} = 0 & \text{on } \Sigma, \\ w_{k'}(\cdot, 0) = w^0(\cdot) & \text{in } \Omega. \end{cases}$$

Then, using $F(z_{k'}) \to F(z)$ in Z, estimates (4.102) and (4.100) and $\tilde{Z} \overset{c}{\subset} Z$, we find (at least for a subsequence)

$$w_{k'} \to w^* \text{ in } Z$$

and

$$v_{k'} \to v^* \text{ weakly in } H^1(0, T; L^2(\omega)^N) \cap L^\infty(0, T; L^2(\omega)^N).$$

It is not difficult then to deduce that $v^* \in \Lambda(z)$ and $w^* \in A(z)$. Hence,

$$\lim_{k' \to \infty} \sup_{w \in A(z_{k'})} \langle \lambda, w \rangle_{Z'Z} = \langle \lambda, w^* \rangle_{Z'Z} \leq \sup_{w \in A(z)} \langle \lambda, w \rangle_{Z'Z},$$

as we wanted to prove.

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