Branched transport limit of the Ginzburg-Landau functional

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Joint work with S. Conti, F. Otto and S. Serfaty
Superconductivity was first observed by Onnes in 1911 and has nowadays many applications.
In 1933, Meissner understood that superconductivity was related to the expulsion of the magnetic field outside the material sample.
Ginzburg Landau functional

In the 50’s Ginzburg and Landau proposed a phenomenological model (later derived from the BCS theory):

$$E(u, A) = \int_{\Omega} |\nabla_A u|^2 + \frac{\kappa^2}{2} (1 - \rho^2)^2 dx + \int_{\mathbb{R}^3} |\nabla \times A - B_{\text{ex}}|^2 dx$$

where $u = \rho e^{i\theta}$ is the order parameter, $B = \nabla \times A$ is the magnetic field, $B_{\text{ex}}$ is the external magnetic field, $\kappa$ is the Ginzburg-Landau constant and

$$\nabla_A u = \nabla u - iAu$$

is the covariant derivative.

$\rho \sim 0$ represents the normal phase and $\rho \sim 1$ the superconducting one.
The various terms in the energy

For $u = \rho e^{i\theta}$, $|\nabla_A u|^2 = |\nabla \rho|^2 + \rho^2 |\nabla \theta - A|^2$.

In $\rho > 0$ first term wants $A = \nabla \theta \implies \nabla \times A = 0$

That is

$$\rho^2 B \approx 0 \quad \text{(Meissner effect)}$$

and penalizes fast oscillations of $\rho$

Second term forces $\rho \approx 1$ (superconducting phase favored)

Last term wants $B \approx B_{\text{ex}}$. In particular, this should hold outside the sample.
Coherence and penetration length

Already two typical lengths, coherence length $\xi$ and penetration length $\lambda$.

In our units, $\lambda = 1$, $\kappa = \frac{1}{\xi}$
Our setting

We consider $\Omega = Q_L, T = [-L, L]^2 \times [-T, T]$ with periodic lateral boundary conditions and take $B_{ex} = b_{ex} e_3$.

We want to understand extensive behavior $L \gg 1$. 
First rescaling

We let
\[ \kappa T = \sqrt{2\alpha} \quad b_{ex} = \frac{\beta \kappa}{\sqrt{2}} \]

and then
\[
\hat{x} = T^{-1} x \\
\hat{u}(\hat{x}) = u(x) \\
\hat{A}(\hat{x}) = A(x) \\
\hat{B}(\hat{x}) = \nabla \times \hat{A}(\hat{x}) = TB(x)
\]

In these units,

- coherence length \( \simeq \alpha^{-1} \)
- penetration length \( \simeq T^{-1} \)

We are interested in the regime \( T \gg 1, \alpha \gg 1, \beta \ll 1 \).
The energy

The energy can be written as

\[
E_T(u, A) = \frac{1}{L^2} \int_{Q_{L,1}} |\nabla_{TA} u|^2 + (B_3 - \alpha(1 - \rho^2))^2 + |B'|^2 \\
+ \|B_3 - \alpha/\beta\|_{H^{-1/2}(x_3=\pm 1)}^2
\]

- **First term**: penalizes oscillations + \(\rho^2 B \simeq 0\) (Meissner effect)
The energy

The energy can be written as

$$E_T(u, A) = \frac{1}{L^2} \int_{Q_{L,1}} |\nabla T_A u|^2 + (B_3 - \alpha(1 - \rho^2))^2 + |B'|^2$$

$$+ \|B_3 - \alpha/\beta\|_{H^{-1/2}(x_3=\pm1)}^2$$

- First term: penalizes oscillations + $\rho^2 B \simeq 0$ (Meissner effect)
- Second term: degenerate double well potential.

If Meissner then:

$$(B_3 - \alpha(1 - \rho^2))^2 \simeq \alpha^2 \chi_{\{\rho > 0\}}(1 - \rho^2)^2$$

Rk: wants $B_3 = \alpha$ in $\{\rho = 0\}$

Similar features in mixtures of BEC (cf G. Merlet ’15)
Crash course on optimal transportation

For $\rho_0$, $\rho_1$ probability measures

$$W_2^2(\rho_0, \rho_1) = \inf \left\{ \int_{Q_L \times Q_L} |x - y|^2 d\Pi(x, y) : \Pi_1 = \rho_0, \Pi_2 = \rho_1 \right\}$$

\textbf{Theorem}

\begin{itemize}
  \item \textbf{(Benamou-Brenier)}
  \[
  W_2^2(\rho_0, \rho_1) = \inf_{\mu, B'} \left\{ \int_0^1 \int_{Q_L} |B'|^2 d\mu : \partial_3 \mu + \text{div}' B' \mu = 0, \mu(0, \cdot) = \rho_0, \mu(1, \cdot) = \rho_1 \right\}
  \]
  \item \textbf{(Brenier)} If $\rho_0 \ll dx$,
  \[
  W_2^2(\rho_0, \rho_1) = \min \left\{ \int_{Q_L} |x - T(x)|^2 d\rho_0 : T \# \rho_0 = \rho_1 \right\}
  \]
\end{itemize}
The energy continued

\[ E_T(u, A) = \frac{1}{L^2} \int_{Q_{L,1}} |\nabla_{TA} u|^2 + (B_3 - \alpha(1 - \rho^2))^2 + |\mathcal{B}'|^2 + \|B_3 - \alpha\beta\|_{H^{-1/2}(\pm 1)}^2 \]

- **Third term**: with Meissner and \( B_3 \simeq \alpha(1 - \rho^2) = \chi \),
  \( \text{div } B = 0 \) can be rewritten as

\[ \partial_3 \chi + \text{div}' \chi B' = 0 \]

Benamou-Brenier \( \implies \) Wasserstein energy of \( x_3 \rightarrow \chi(\cdot, x_3) \)
The energy continued

\[ E_T(u, A) = \frac{1}{L^2} \int_{Q_{L,1}} |\nabla_T A u|^2 + (B_3 - \alpha(1 - \rho^2))^2 + |B'|^2 \]

\[ + \|B_3 - \alpha\beta\|_{H^{-1/2}(x_3=\pm1)}^2 \]

- Third term: with Meissner and \( B_3 \approx \alpha(1 - \rho^2) = \chi \),
  \( \text{div } B = 0 \) can be rewritten as

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Benamou-Brenier \( \implies \) Wasserstein energy of \( x_3 \to \chi(\cdot, x_3) \)

- Last term: penalizes non uniform distribution on the boundary but negative norm \( \implies \) allows for oscillations
A non-convex energy regularized by a gradient term

If we forget the kinetic part of the energy, can make $B' = 0$ and

$$E_T(u, A) = \frac{1}{L^2} \int_{Q_{L,1}} (B_3 - \alpha(1 - \rho^2))^2 + \|B_3 - \alpha\beta\|_{H^{-1/2}(x_3=\pm1)}^2$$

\[
\begin{array}{c}
\rho=0 \\
\rho=1
\end{array}
\]

\[\implies\ \text{infinitely small oscillations of phases}\]
\[
\{\rho = 0, B_3 = \alpha\} \text{ and } \{\rho = 1, B_3 = 0\}
\]
with average volume fraction $\beta$.

the kinetic term $|\nabla_A u|^2$ fixes the lengthscale.
Branching is energetically favored

\[ \rho \approx 1 \]

\[ \| B_3 - \alpha \beta \|_{H^{-1/2}(x_3=\pm 1)}^2 \downarrow 0 \]

but interfacial energy \( \uparrow \infty \)

\[ x_3 = -1 \]

\[ x_3 = 1 \]

interfacial energy \( \downarrow \)

but \( \int_{Q_{L,1}} |B'|^2 \uparrow \).

Landau '43
Experimental results

Complex patterns at the boundary

Experimental pictures from Prozorov and al.

Limitations:
- Difficult to see the pattern inside the sample
- Hysteresis
Branching patterns in other related models

- Shape memory alloys (Kohn-Müller model) (Left, picture from Chu and James)
- Uniaxial ferromagnets (Right, picture from Hubert and Schäffer)

Schematic difference: in our problem $W_2^2$ replaces $H^{-1}$ norm
See works of Kohn, Müller, Conti, Otto, Choksi ...
Related functional: Ohta-Kawasaki
Theorem (Conti, Otto, Serfaty ’15, See also Choksi, Conti, Kohn, Otto ’08)

In the regime $T \gg 1$, $\alpha \gg 1$, $\beta \ll 1$,

$$\min E_T \simeq \min(\alpha^{4/3} \beta^{2/3}, \alpha^{10/7} \beta)$$

First regime: $E_T \sim \alpha^{4/3} \beta^{2/3}$
Uniform branching,
$$\|B_3 - \alpha \beta\|_{H^{-1/2}(x_3=\pm1)} = 0$$

Second regime: $E_T \sim \alpha^{10/7} \beta$
Non-Uniform branching,
$$\|B_3 - \alpha \beta\|_{H^{-1/2}(x_3=\pm1)} > 0$$
fractal behavior
Scaling law

Theorem (Conti, Otto, Serfaty ’15, See also Choksi, Conti, Kohn, Otto ’08)

In the regime $T \gg 1$, $\alpha \gg 1$, $\beta \ll 1$,

$$\min E_T \simeq \min(\alpha^{4/3} \beta^{2/3}, \alpha^{10/7} \beta)$$

We concentrate on the first regime (uniform branching)

$$\rho \simeq 1$$

$$\implies \alpha^{-2/7} \ll \beta.$$
From the upper bound construction, we expect
penetration length \ll \text{coherence length} \ll \text{domain size} \ll \text{sample size}
which amounts in our parameters to
\[ T^{-1} \ll \alpha^{-1} \ll \alpha^{-1/3} \beta^{1/3} \ll L. \]
Crash course in Γ-convergence

$F_n$ sequence of functionals on a metric space $(X, d)$. We say that $F_n$ Γ–converges to $F$ if

1. $\forall x_n \in X, F_n(x_n) \leq C \implies$ Compactness +

$$\lim_{n \to +\infty} F_n(x_n) \geq F(x)$$

2. $\forall x \in X, \exists x_n \to x$ with

$$\lim_{n \to +\infty} F_n(x_n) \leq F(x)$$

It implies

1. $\inf F_n \to \inf F$

2. if $x_n$ are minimizers of $F_n \implies x$ is a minimizer of $F$. 

Compactness and Lower bounds
First limit, $T \to +\infty$

Recall:

$$E_T(u, A) = \frac{1}{L^2} \int_{Q_{L,1}} |\nabla_T A u|^2 + (B_3 - \alpha(1 - \rho^2))^2 + |B'|^2$$

$$+ \|B_3 - \alpha\beta\|_{H^{-1/2}(x_3=\pm1)}^2$$

**Proposition**

If $E_T(u_T, A_T) \leq C$ then $\rho_T = |u_T| \to \rho$, $B_T = \nabla \times A_T \to B$ and

- $\rho^2 B = 0$, div $B = 0$ (Meissner effect)
- $\lim_{T} E_T(u_T, A_T) \geq F_{\alpha,\beta}(\rho, B)$ where

$$F_{\alpha,\beta}(\rho, B) = \frac{1}{L^2} \int_{Q_{L,1}} |\nabla \rho|^2 + (B_3 - \alpha(1 - \rho^2))^2 + |B'|^2$$

$$+ \|B_3 - \alpha\beta\|_{H^{-1/2}(x_3=\pm1)}^2$$
Second rescaling

In this limit, penetration length = 0, coherence length $\simeq \alpha^{-1}$, domain size $\alpha^{-1/3} \beta^{1/3}$.

In order to get sharp interface limit with finite domain size, we make the anisotropic rescaling

$$\begin{pmatrix} \hat{x}' \\ \hat{x}_3 \end{pmatrix} = \begin{pmatrix} 0 \\ \alpha^{-1/3} \end{pmatrix} x',
\begin{pmatrix} \hat{B}' \\ \hat{B}_3 \end{pmatrix} (\hat{x}) = \begin{pmatrix} \alpha^{-2/3} B' \\ \alpha^{-1} \end{pmatrix} (x),
\hat{F}_{\alpha,\beta} = \alpha^{-4/3} F_{\alpha,\beta},
\hat{\rho}(\hat{x}) = \rho(x),$$

In these variable: coherence length $\simeq \alpha^{-2/3} \ll 1$ and normal domain size $\simeq \beta^{1/3}$
Second limit, $\alpha \to +\infty$

Dropping the hats

$$L^2 F_{\alpha,\beta}(\rho, B) = \int_{Q_{L,1}} \alpha^{-2/3} \left| \left( \frac{\nabla' \rho}{\alpha^{-1/3} \partial_3 \rho} \right) \right|^2 + \alpha^{2/3} |B_3 - (1 - \rho^2)|^2 + |B'|^2$$

$$+ \alpha^{1/3} \|B_3 - \beta\|_{H^{-1/2}(x_3=\pm 1)}^2$$

and the Meissner condition

$$\text{div } B = 0 \quad \text{and} \quad \rho^2 B = 0$$

still holds

**Proposition**

If $F_{\alpha,\beta}(\rho_\alpha, B_\alpha) \leq C$, then $1 - \rho_\alpha^2 \to \chi \in \{0, 1\}$, $B'_\alpha \to B'$ and

- $\chi(\cdot, \pm 1) = \beta$, $\chi B' = B'$, $\partial_3 \chi + \text{div}' \chi B' = 0$
- $\lim_\alpha F_{\alpha,\beta}(\rho_\alpha, B_\alpha) \geq G_\beta(\chi, B')$ where

$$G_\beta(\chi, B') = \frac{1}{L^2} \int_{Q_{L,1}} \frac{4}{3} |\nabla' \chi| + |B'|^2$$
Comments on the proof

- Anisotropic rescaling \(\implies\) control only on the horizontal derivative.

- Thanks to Meissner, double well potential

\[
\alpha^{-2/3} \left| \left( \alpha^{-1/3} \partial_3 \rho \right) \right|^2 + \alpha^{2/3} |B_3 - (1 - \rho^2)|^2 \geq \\
\alpha^{-2/3} |\nabla' \rho|^2 + \alpha^{2/3} \chi_{\{\rho > 0\}} |1 - \rho^2|^2
\]

Recall Modica-Mortola

\[
\int \varepsilon |\nabla' \rho_\varepsilon|^2 + \varepsilon^{-1} \rho_\varepsilon^2 (1 - \rho_\varepsilon^2) \rightarrow C \int |\nabla' \chi|
\]
Last rescaling

We want to send $\beta \to 0$ and get 1 dimensional trees. We make another anisotropic rescaling:

$$\begin{pmatrix} \hat{x}' \\ \hat{x}_3 \end{pmatrix} = \begin{pmatrix} \beta^{1/6} x' \\ x_3 \end{pmatrix}, \quad \hat{\chi}(\hat{x}) = \beta^{-1} \chi(x) \in \{0, \beta^{-1}\}$$

$$\hat{B}'(\hat{x}) = \beta^{1/6} B'(x)$$

$$\hat{G}_\beta = \beta^{-2/3} G_\beta$$

Now, domain width $\simeq \beta^{1/2} \ll 1$, $L = 1$ and (dropping hats)

$$G_\beta(\chi, B') = \int_{Q_{1,1}} \frac{4}{3} \beta^{1/2} |\nabla' \chi| + \chi|B'|^2$$

with $\partial_3 \chi + \text{div}'(\chi B') = 0$, $\chi B' = \beta^{-1} B'$ and $\chi(\cdot, x_3) \to dx'$ when $x_3 \to \pm 1$. 
Limiting functional

Because of isoperimetric effects, on every slice

$$\chi \simeq \beta^{-1} \sum_i \chi_B(x_i, \beta^{1/2} r_i)$$

If $$\phi_i = \pi r_i^2$$,

$$\int_{[-1,1]^2} \chi \simeq \sum_i \phi_i$$

and

$$\int_{[-1,1]^2} \beta^{1/2} |\nabla\chi| \simeq 2\pi^{1/2} \sum_i \sqrt{\phi_i}$$

Hence $$\chi \rightharpoonup \sum_i \phi_i \delta_{x_i}$$
The limiting functional II

For $\mu$ a probability measure with $\mu_{x_3} = \sum_i \phi_i \delta_{x_i(x_3)}$ for a.e. $x_3$ and $\mu_{x_3} \rightharpoonup dx'$ when $x_3 \to \pm 1$, and $m \ll \mu$, with $\partial_3 \mu + \text{div}'m = 0$,

$$I(\mu, m) = \int_{-1}^{1} \frac{8\pi^{1/2}}{3} \sum_{x' \in Q_1} (\mu_{x_3}(x'))^{1/2} dx_3 + \int_{Q_1, 1} \left( \frac{dm}{d\mu} \right)^2 d\mu$$

Formally,

$$I(\mu) = \inf_m I(\mu, m) = \int_{-1}^{1} \sum_i \frac{8\pi^{1/2}}{3} \phi_i^{1/2} + \phi_i \dot{x}_i^2 dx_3$$
**Proposition**

If \( G_\beta(\chi_\beta, B'_\beta) \leq C \chi_\beta \rightarrow \mu, \chi_\beta B'_\beta \rightarrow m \) and

\[
\lim_{\beta} G_\beta(\chi_\beta, B'_\beta) \geq I(\mu, m)
\]
The limiting functional

- $I(\mu)$ reminiscent of branched transport models (see Bernot-Caselles-Morel). Our result, similar in spirit to Oudet-Santambrogio ’11.
- Minimizers of $I$, contain no loop, finite number of branching points away from boundary (with quantitative estimate), branches are linear between two branching points
- Every measure can be irrigated
Definition of regular measures

For $\varepsilon > 0$, we denote by $\mathcal{M}_R^\varepsilon(Q_{1,1})$ the set of regular measures, i.e., of measures such that:

(i) $\mu$ is finite polygonal.

(ii) All branching points are triple points. This means that any $x \in Q_{1,1}$ belongs to no more than three segments.

(iii) there is $\varepsilon^{1/2} \gg \lambda_\varepsilon \gg \varepsilon$ with $1/\lambda_\varepsilon \in \mathbb{N}$, such that for all $x_3 \in [1 - \varepsilon, 1]$ one has $\mu_{x_3} = \sum_{x' \in \lambda_\varepsilon \mathbb{Z}^2 \cap Q_1} \varphi_{x'} \delta_{x'}$, with $\varphi_{x'} = \lambda_\varepsilon^2$, and the same in $[-1, -1 + \varepsilon]$. 

A crucial density result

**Proposition**

For every $\mu$ with $I(\mu) < \infty$, $\exists \mu_\varepsilon \in M_\varepsilon^\varepsilon(Q_1,1)$ with $\mu_\varepsilon \rightharpoonup \mu$ and $\lim_\varepsilon I(\mu_\varepsilon) \leq I(\mu)$.

$\implies \simeq$ enough to make the construction for finite polygonal measures.
Idea of the proof

- Rescale $\mu$ to $[-1 + 2\varepsilon, 1 - 2\varepsilon]$
Idea of the proof

- Rescale $\mu$ to $[-1 + 2\varepsilon, 1 - 2\varepsilon]$
- in the boundary layer plug in a uniform branching construction
Idea of the proof

- Rescale $\mu$ to $[-1 + 2\varepsilon, 1 - 2\varepsilon]$
- in the boundary layer plug in a uniform branching construction
- remove small branches. Tool: notion of subsystem cf. Ambrosio-Gigli-Savare
Idea of the proof

- Rescale $\mu$ to $[-1 + 2\epsilon, 1 - 2\epsilon]$.
- In the boundary layer plug in a uniform branching construction.
- Discretize and minimize.
Recovery sequences
Recovery sequence, from trees to sharp interface

Want to enlarge the 1D trees. Far from branching points, easy (take a tube). At a branching point:

Need to transform a rectangle into disk with controlled energy
Recovery sequence, from sharp to diffuse interface

Need to reintroduce a smooth transition + vertical derivative.

To get smooth transition: use optimal profile (keeping Meissner) + careful estimate of the error terms
Can define \( A \) with \( \nabla \times A = B \). Need to define \( \theta \). Would be easy if quantization of flux \( \phi_i \in 2\pi\mathbb{Z} \).

\[
\Gamma_0 \leftrightarrow x, \quad \theta(x) = \int_{\Gamma} A \cdot \tau
\]

Quantization + Stokes \( \implies \) well defined + \( \nabla \theta = A \)

\( \implies \) Need to modify the fluxes to get quantization
Ongoing work and perspective

▶ Understand the cross-over regime $\alpha^{-2/7} \sim \beta$
▶ Go from GL to sharp interface when coherence $\sim$ penetration (more complex/non-local optimal profile problem)
▶ Understand minimizers of the limiting functional (self-similarity à la Conti)
▶ Investigate the non-uniform branching $\sim$ fractal behavior
“What in the name of Sir Isaac H. Newton happened here?”

Dr. Emmett 'Doc' Brown

Thank you! Any Question?