Electromagnetic waves in magnetized plasma
The dispersion relation

Bruno Després (LJLL-UPMC)

Vlasov-Maxwell

Vectors are in **bold**.

For each species $s = 1, \ldots,$, solve

$$\partial_t f_s + \mathbf{v} \cdot \nabla_x f_s + \frac{q_s}{m_s} (\mathbf{E} + \mathbf{v} \wedge \mathbf{B}) \cdot \nabla_v f_s = 0$$

for the kinetic unknown $f_s = f_s(t, x, v)$, and couple with the Maxwell's equations

$$\begin{cases} -\frac{1}{c^2}\partial_t \mathbf{E} + \nabla \wedge \mathbf{B} = \mu_0 \mathbf{J}, & \nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}, \\ \partial_t \mathbf{B} + \nabla \wedge \mathbf{E} = 0, & \nabla \cdot \mathbf{B} = 0, \\ \mathbf{J} = \sum_s q_s \int \mathbf{v} f_s dv, & \\ \rho = \sum_s q_s \int f_s dv. & \end{cases}$$

Expensive.

Euler-Maxwell

Simplify

For $s = 1, \dots,$ solve for $n_s = n_s(t,x)$ and $\mathbf{u}_e = \mathbf{u}_e(t,x)$

$$\begin{cases} \partial_t n_s + \nabla \cdot (n_s \mathbf{u}_s) = 0, \\ \partial_t (n_s \mathbf{u}_s) + \nabla \cdot (n_s \mathbf{u}_s \otimes \mathbf{u}_s + \frac{\nabla p_s}{m_s}) = q_s n_s \left(\mathbf{E} + \mathbf{u}_s \wedge \mathbf{B} \right), \end{cases}$$

coupled with the Maxwell's equations

$$\begin{cases} -\frac{1}{c^2}\partial_t \mathbf{E} + \nabla \wedge \mathbf{B} = \mu_0 \mathbf{J}, & \nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}, \\ \partial_t \mathbf{B} + \nabla \wedge \mathbf{E} = 0, & \nabla \cdot \mathbf{B} = 0, \\ \mathbf{J} = \sum_s q_s n_s \mathbf{u}_s, & \\ \rho = \sum_s q_s n_s. & \end{cases}$$

In the cold plasma approximation the pressure p_s is negligible

$$\frac{\nabla p_s}{m_s} \approx 0.$$

This is a closed system. Still expensive.

Simplify further

For each species $s = 1, \ldots,$, solve

$$m_s \left(\partial_t \mathbf{u}_s + \mathbf{u}_s \cdot \nabla \mathbf{u}_s \right) = q_s \left(\mathbf{E} + \mathbf{u}_s \wedge \mathbf{B} \right),$$

coupled with the Maxwell's equations

$$\begin{cases} -\frac{1}{c^2}\partial_t \mathbf{E} + \nabla \wedge \mathbf{B} = \mu_0 \mathbf{J}, \\ \partial_t \mathbf{B} + \nabla \wedge \mathbf{E} = 0, \qquad \nabla \cdot \mathbf{B} = 0, \\ \mathbf{J} = \sum_s q_s n_s \mathbf{u}_s. \end{cases}$$

Notice that n_s has to be provided to close the system.

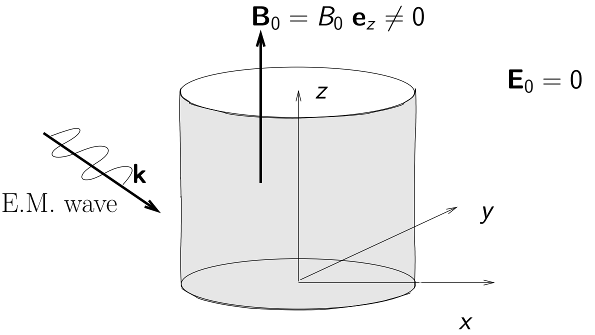
Models

Dispersion relation for one species
All coefficients constant

Dispersion relation for multispecies

Group velocity

Geometry and constant coefficients



$$\mathbf{B}_0 = B_0 \mathbf{e}_z \text{ and } \mathbf{k} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)\,.$$

Invariance by rotation around $\mathbf{e}_z$: $\varphi = 0$ without restriction.

$$\mathbf{k} = (\sin\theta, 0, \cos\theta)\,.$$

Linearization

One linearizes

$$\begin{cases} \mathbf{E} = 0 & +\mathbf{E}_1 + \dots, \\ \mathbf{B} = \mathbf{B}_0 & +\mathbf{B}_1 + \dots, \\ \mathbf{u}^s = 0 & +\mathbf{u}_1^s + \dots. \end{cases}$$

One obtains

$$m_s \partial_t \mathbf{u}_1^s = q_s \left(\mathbf{E}_1 + \mathbf{u}_1^s \wedge \mathbf{B}_0 \right),$$

coupled with

$$\begin{cases} -\frac{1}{c^2} \partial_t \mathbf{E}_1 + \nabla \wedge \mathbf{B}_1 = \mu_0 \mathbf{J}_1, \\ \partial_t \mathbf{B}_1 + \nabla \wedge \mathbf{E}_1 = 0, \\ \mathbf{J}_1 = \sum_s q_s n_s \mathbf{u}_1^s. \end{cases}$$

Here the densities n_s are given.
The system is closed.

$$\begin{cases} -\frac{1}{c^2}\partial_t \mathbf{E}_1 + \nabla \wedge \mathbf{B}_1 = \mu_0 q_e N_e(x) \mathbf{u}_1^e, \\ \partial_t \mathbf{B}_1 + \nabla \wedge \mathbf{E}_1 = 0, \\ m_s \partial_t \mathbf{u}_1^e = q_e \left(\mathbf{E}_1 + \mathbf{u}_1^e \wedge \mathbf{B}_0 \right). \end{cases}$$

Stix : *A general analysis of this model is able to provide a surprisingly comprehensive view of plasma waves.*

If you discuss this topic with a plasma physicist, just keep in mind that it is a model.
With this respect, the situation is very different from Maxwell's equation in the vaccum.

Dispersion relation

Consider simple solutions

$$\mathbf{E}_1 = \mathbf{E}(\mathbf{x}) e^{-\mathbf{i}\omega t} \text{ with } \mathbf{E}(\mathbf{x}) = \mathbf{e}\ e^{\mathbf{i}(\mathbf{k},\mathbf{x})}, \text{ and } \mathbf{e} \in \mathbb{C}^3, \quad \ldots$$

Propagative properties can be locally understood checking wether

$$\omega \in \mathbb{R} \text{ for } \mathbf{k} \in \mathbb{R}^3.$$

The phase velocity of a wave $e^{\mathbf{i}((\mathbf{k},\mathbf{x}-\omega t))}$ is

$$v_\varphi = \frac{\omega}{|\mathbf{k}|}.$$

Definition The accessible domain (in the phase space) is

$$\mathcal{A}(\omega) = \left\{ (N_e, |\mathbf{B}_0|), \exists\ \mathbf{k} \in \mathbf{R}^3 \text{ and a non trivial sol.} \right\}.$$

Derivation

Plugging the dependancy with respect to ω in the cold plasma model, one gets the linear system

$$\begin{cases} \frac{\mathbf{i}\omega}{c^2}\mathbf{E} + \nabla \wedge \mathbf{B} = \mu_0 q_e N_e \mathbf{u}_e, \\ -\mathbf{i}\omega \mathbf{B} + \nabla \wedge \mathbf{E} = 0, \\ -\mathbf{i}\omega m_e \mathbf{u}_e = q_e \left(\mathbf{E} + \mathbf{u}_e \wedge \mathbf{B}_0\right) \end{cases}$$

Elimination of the magnetic field yields ($\mu_0 \varepsilon_0 c^2 = 1$)

$$\nabla \wedge \nabla \wedge \mathbf{E} - \frac{\omega^2}{c^2}\mathbf{E} = \mathbf{i}\frac{\omega}{c^2 \varepsilon_0} q_e N_e \mathbf{u}_e.$$

It remains to compute the velocity $\mathbf{u}_e$ with respect to the electric field.

Models

Dispersion relation for one species
All coefficients constant

Dispersion relation for multispecies

Group velocity

Then $\mathbf{u}_e = -\frac{q_e}{\mathbf{i}\omega me}\mathbf{E}$. One gets

$$\nabla \wedge \nabla \wedge \mathbf{E} - \frac{\omega^2}{c^2}\mathbf{E} = -\frac{q_e^2 N_e}{c^2 \varepsilon_0 m_e}\mathbf{E}$$

which is conveniently rewritten as

$$\nabla \wedge \nabla \wedge \mathbf{E} - \frac{\omega^2}{c^2}\left(1 - \frac{\omega_p^2}{\omega^2}\right)\mathbf{E} = 0$$

where the plasma frequency is

$$\omega_p^2 = \frac{q_e^2 N_e}{c^2 \varepsilon_0 m_e}.$$

Spatial dependancy

Remind the Ansatz

$$\mathbf{E}(\mathbf{x}) = \mathbf{e}\ e^{\mathbf{i}(\mathbf{k},\mathbf{x})}, \quad \mathbf{B}(\mathbf{x}) = \mathbf{b}\ e^{\mathbf{i}(\mathbf{k},\mathbf{x})}, \qquad \mathbf{e}, \mathbf{b} \in \mathbb{C}^3,$$

where a priori $\mathbf{k} \in \mathbb{R}^3$.

A possibility is to write

$$\mathbf{k} = k\mathbf{n}$$

where $k > 0$ and

$$\mathbf{n} \in \mathbb{R}^3, \quad |\mathbf{n}| = 1.$$

is the direction in space of the wave.

With this Ansatz

$$\nabla \wedge \nabla \wedge \mathbf{E} - \frac{\omega^2}{c^2}\left(1 - \frac{\omega_p^2}{\omega^2}\right)\mathbf{E} = 0$$

becomes

$$-k^2 \mathbf{n} \wedge \mathbf{n} \wedge \mathbf{e} - \frac{\omega^2}{c^2}\left(1 - \frac{\omega_p^2}{\omega^2}\right)\mathbf{e} = 0$$

or also

$$-\mathbf{n} \wedge \mathbf{n} \wedge \mathbf{e} = \frac{\omega^2}{k^2 c^2}\left(1 - \frac{\omega_p^2}{\omega^2}\right)\mathbf{e}.$$

Interpretation

Notice that $-\mathbf{n} \wedge \mathbf{n} \wedge \mathbf{e} = M(\mathbf{n})\mathbf{e}$ where the matrix is symetric non negative

$$M(\mathbf{n}) = -\mathbf{n} \wedge \mathbf{n} \wedge = I - \mathbf{n} \otimes \mathbf{n} = M(\mathbf{n})^t \geq 0.$$

The whole problem reduces to

$$M(\mathbf{n})\mathbf{e} = \lambda \mathbf{e}, \quad \lambda = \frac{\omega^2}{k^2 c^2} \left(1 - \frac{\omega_p^2}{\omega^2} \right) = \frac{\omega^2 - \omega_p^2}{k^2 c^2}.$$

This is an classical eigenproblem for a <u>symetric real matrix</u> :
the eigenvector is $\mathbf{e}$;
the eigenvalue is λ.

Discussion

The characteristic polynomial is

$$\det(M(\mathbf{n}) - \lambda I) = (\lambda - 1)^2 \lambda = 0.$$

- The eigenspace $\mathbf{k}^\perp$ is associated to the same eigenvalue $\lambda = 1$ with multiplicity 2. One has the relation

$$k^2 c^2 = \omega^2 - \omega_p^2.$$

- The eigenvector $\mathbf{e} = \mathbf{k}$ is associated to $\lambda = 0$ with multiplicity 1. One gets

$$0 = \omega^2 - \omega_p^2.$$

Warnig : the classical discussion systematically disregard the zero eigenvalue.

Consequence for propagation

Corollary : Propagation ($\omega, n \in \mathbb{R}$) is possible iff

$$\omega^2 \geq \omega_p^2 = \frac{q_e^2 N_e}{c^2 \varepsilon_0 m_e}.$$

The phase velocity of the wave $e^{\mathbf{i}k\left((\mathbf{n},\mathbf{x})-\frac{\omega}{k}t\right)}$ is

$$v_\varphi = \frac{\omega}{k} = \underbrace{c\sqrt{1+\frac{\omega_p^2}{k^2c^2}}}_{v_\varphi(k)} = c\underbrace{\frac{1}{\sqrt{1-\frac{\omega_p^2}{\omega^2}}}}_{=v_\varphi(\omega)}.$$

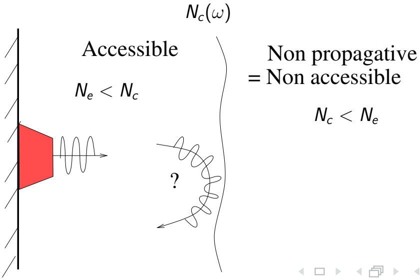
Accessibility in physical space

Consider a problem where $N_e = N_e(\mathbf{x})$.

Definition : The accessible domain in physical space is

$$\mathcal{B}_0(\omega) = \left\{ \mathbf{x} \in \mathbb{R}^d ; N_e(\mathbf{x}) \in \mathcal{A}_0(\omega) \right\} \subset \mathbb{R}^d$$

An antenna with given frequency ω is facing a fusion plasma with increasing electronic density $\frac{d}{d\omega} N_c > 0$:
"the accessible domain" = "the prop. region" is near the wall.



Principal case : $\mathbf{B}_0 \neq 0$

The linear system writes

$$-\mathbf{i}\omega m_e \begin{pmatrix} 1 & \mathbf{i}\frac{\omega_c}{\omega} & 0 \\ -\mathbf{i}\frac{\omega_c}{\omega} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix} = q_e \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$$

where the cycloton frequency is $\omega_c = \frac{|q_e|B_0}{m_e}$.

Notice $\begin{vmatrix} 1 & \mathbf{i}\frac{\omega_c}{\omega} \\ -\mathbf{i}\frac{\omega_c}{\omega} & 1 \end{vmatrix} = 1 - \frac{\omega_c^2}{\omega^2} = \frac{\omega^2 - \omega_c^2}{\omega^2}$ and

$$\begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix} = -\frac{q_e}{\mathbf{i}\omega} \begin{pmatrix} \frac{\omega^2}{\omega^2-\omega_c^2} & \mathbf{i}\frac{\omega\omega_c}{\omega^2-\omega_c^2} & 0 \\ -\mathbf{i}\frac{\omega\omega_c}{\omega^2-\omega_c^2} & \frac{\omega^2}{\omega^2-\omega_c^2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} \qquad (1)$$

The elimination of the velocity yields the **generalized eigenvalue problem**

$$M(\mathbf{n})\mathbf{e} = \frac{\omega^2}{k^2c^2}\underbrace{\left(I - \frac{\omega_p^2}{\omega^2}\begin{pmatrix} \frac{\omega^2}{\omega^2-\omega_c^2} & \mathbf{i}\frac{\omega\omega_c}{\omega^2-\omega_c^2} & 0 \\ -\mathbf{i}\frac{\omega\omega_c}{\omega^2-\omega_c^2} & \frac{\omega^2}{\omega^2-\omega_c^2} & 0 \\ 0 & 0 & 1 \end{pmatrix}\right)}_{=\mathcal{M}=\mathcal{M}^\star}\mathbf{e}.$$

- If $\mathbf{B}_0 = 0$, then $\omega_c = 0$ and $\mathcal{M} = I$.
- The parallel direction $\mathbf{e}_z$ is not modified by ω_c.
- Singular for $\omega = \omega_c$.

One writes usually

$$\mathcal{M} = \begin{pmatrix} S & -iD & 0 \\ iD & S & 0 \\ 0 & 0 & P \end{pmatrix}$$

with $S = \frac{1}{2}(R + L)$, $D = \frac{1}{2}(R - L)$ and

$$\begin{cases} R = 1 - \frac{\omega_p^2}{\omega^2}\frac{\omega}{\omega - \omega_c} \\ L = 1 - \frac{\omega_p^2}{\omega^2}\frac{\omega}{\omega + \omega_c} \\ P = 1 - \frac{\omega_p^2}{\omega^2}. \end{cases}$$

Discussion *à la* Stix

As before the eigenproblem that one consider writes

$$\det\left(M(\mathbf{n}) - \lambda \mathcal{M}\right) = 0, \qquad \lambda = \frac{\omega^2}{k^2 c^2}$$

Following Stix we study instead

$$\det\left(\mathcal{M} - \mu M(\mathbf{n})\right) = 0, \qquad \mu = \frac{1}{\lambda} = \frac{k^2 c^2}{\omega^2}.$$

The dispersion relation reduces to the second order polynomial equation

$$A\mu^2 - B\mu + C = 0$$

with

$$A = S\sin^2\theta + P\cos^2\theta, \quad B = RL\sin^2\theta + PS(1+\cos^2\theta)$$

and

$$C = \det(\mathcal{M}) = PRL.$$

Expansion of the dispersion relation

The dispersion relation $A\mu^2 - B\mu + C = 0$ can be rewritten as

$$\frac{c^4}{\omega^4}\left(S(\omega)k_1^2 + P(\omega)k_3^2\right)\left(k_1^2 + k_3^3\right)$$

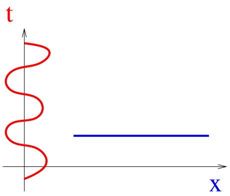
$$-\frac{c^2}{\omega^2}\left(R(\omega)L(\omega)k_1^2 + P(\omega)S(\omega)\left(k_1^2 + 2k_3^2\right)\right) + C(\omega) = 0$$

where $S = \frac{1}{2}(R + L)$, $C = PRL$ and

$$\begin{cases} R = 1 - \frac{\omega_p^2}{\omega^2}\frac{\omega}{\omega - \omega_c} \\ L = 1 - \frac{\omega_p^2}{\omega^2}\frac{\omega}{\omega + \omega_c} \\ P = R = 1 - \frac{\omega_p^2}{\omega^2}. \end{cases}$$

Notice that $B_0 = 0$ turns into $\omega_c = 0$ and $S(\omega) = P(\omega)$.

Cutoff



Definition : Cutoff$= \{k=0\}$.

That is $PRL = 0$.

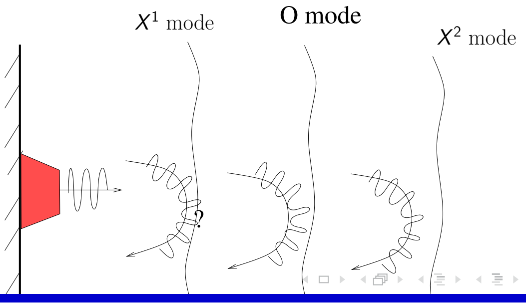
- $P=0$ is the previous cutoff : $\omega = \pm\omega_p$.
- $R=0$ yields
$$\omega^3 - \omega^2\omega_c - \omega\omega_p^2 = 0$$
that is $\omega = \frac{\omega_c \pm \sqrt{\omega_c^2+4\omega_p^2}}{2}$.
- $L=0$ yields
$$\omega^3 + \omega^2\omega_c - \omega\omega_p^2 = 0$$
that is $\omega = \frac{-\omega_c \pm \sqrt{\omega_c^2+4\omega_p^2}}{2}$.

Cutoff and accessibility ($B_0 \neq 0$)

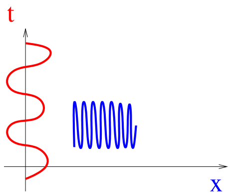
Assume ω and ω_c are given. Boundaries of the accesibility domain are given by

$$\omega_p^2 = \omega^2 + \eta\omega\omega_c, \qquad \eta = -1, 0, 1.$$

The antenna with given frequency ω still facing the fusion plasma with increasing electronic density : the prop. region is near the wall ($\frac{d}{d\omega}N_c > 0$). The cyclotron frequency is assumed constant.



Resonance : $\mathbf{B}_0 \neq 0$ necessary



Definition : Resonance$= \{k = \infty\}$.

Therefore $A = 0$ and $\tan^2\theta = -\frac{P}{S}$. Two major cases of interest follow.

- If $\theta = 0$, it yields $S = \infty$ which is true for $\omega^2 = \omega_c^2$. This is the **cyclotron resonnance**. Clearly the velocity (of electrons) goes to infinity, see (1).
- $\theta = \frac{\pi}{2}$, it yields

$$S = 0.$$

This is the more subtle **hybrid resonance**. We will see that the velocity (of electrons) also goes to infinity.

The resonant cone K_r

Definition : it is $K_r \subset \mathbb{R}^2$ the set of (k_1, k_3) such that

$$S(\omega)k_1^2 + P(\omega)k_3^2 = 0.$$

- If $\omega_c = 0$ or is small enough, $S(\omega)P(\omega) > 0$. So $K_r = (0, 0)$.
- If $|B_0|$ is large enough, so $S(\omega)P(\omega) < 0$. So K_r is the union of 2 straight lines : at infinity one gets a solution of the dispersion relation.
- The case $SP = 0$ is described before.

A short summary

- The dispersion relation of the cold plasma theory exhibits many features of waves in magnetic plasmas, with surprising accuracy (Stix).
  The algebra is tricky and needs time to be understood.
- Main features
  - Accessible domain
  - cut-off
  - resonance
  - resonant cone
  - dependance with respect to the principal plasma parameters which are ω_c and ω_p

Additivity principle

The equations are linear. Additivity principle yields

$$\mathcal{M} = \begin{pmatrix} S & -iD & 0 \\ iD & S & 0 \\ 0 & 0 & P \end{pmatrix}$$

with $S = \frac{1}{2}(R+L)$, $D = \frac{1}{2}(R-L)$ and

$$\begin{cases} R = 1 - \sum_s \frac{(\omega_p^s)^2}{\omega^2} \frac{\omega}{\omega - \varepsilon^s \omega_c^s} \\ L = 1 - \sum_s \frac{(\omega_p^s)^2}{\omega^2} \frac{\omega}{\omega + \varepsilon^s \omega_c^s} \\ P = R = 1 - \sum_s \frac{(\omega_p^s)^2}{\omega^2}, \end{cases}$$

where $\varepsilon^s = \pm 1$ depending on the charge.
Different combinations yields an enormous number of different modes.
This feature is the main justification of the method used.

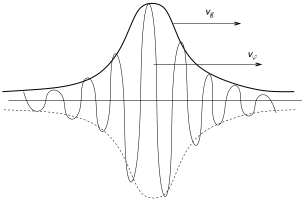
Definition

Phase velocity (Reminder) : By definition $v_\varphi = \frac{\omega}{k}$.

Group velocity : It is defined by

$$\mathbf{v}_g = \nabla_{\mathbf{k}} \omega \in \mathbb{R}^3.$$

It is the velocity of wave packets : a priori $v_g \neq |\mathbf{v}_\varphi|$.



If $\mathbf{B}_0 = 0$, it is easy to check that in 1D $\frac{d\omega}{dk} = \frac{c}{\sqrt{1+\frac{\omega_p^2}{c^2k^2}}}$.

Symetrization

Rewrite the eigenvalue problem without elimination

$$\begin{pmatrix} 0 & -\mathbf{i}c^2\mathbf{k}\wedge & \mu_0 c^2 q_e N_e I \\ \mathbf{i}\mathbf{k}\wedge & 0 & 0 \\ -\frac{q_e}{m_e} I & 0 & \omega_c \mathbf{e}_z\wedge \end{pmatrix} \begin{pmatrix} \mathbf{e} \\ \mathbf{b} \\ \mathbf{u}_e \end{pmatrix} = \mathbf{i}\omega \begin{pmatrix} \mathbf{e} \\ \mathbf{b} \\ \mathbf{u}_e \end{pmatrix}$$

Symetrize

$$\begin{pmatrix} 0 & c\mathbf{k}\wedge & -\mathbf{i}\omega_p I \\ -c\mathbf{k}\wedge & 0 & 0 \\ \mathbf{i}\omega_p I & 0 & -\mathbf{i}\omega_c \mathbf{e}_z\wedge \end{pmatrix} \begin{pmatrix} \mathbf{e} \\ c\mathbf{b} \\ \omega_p\mathbf{u}_e \end{pmatrix} = \omega \begin{pmatrix} \mathbf{e} \\ c\mathbf{b} \\ \omega_p\mathbf{u}_e \end{pmatrix}$$

That is $A(\mathbf{k})U = \omega U$ where $A(\mathbf{k}) = A(\mathbf{k})^\star$.

Consider $\mathbf{k} = \mathbf{k}_0 + \alpha\mathbf{d}$: $\widetilde{A}(\alpha) = A(\mathbf{k}_0 + \alpha\mathbf{d})$ is linear in α.
By theorem (Kato) : all eigenvalues $\omega_j(\alpha)$ are real analytic with respect to α. Orthonormal eigenvectors are analytic.

A simple consequence

$\forall \mathbf{d}$ such that $|\mathbf{d}| = 1$

$$\left|\omega_j'(0)\right| \leq c, \qquad \forall j.$$

Proof :

$$1) \qquad A(\alpha)u_j(\alpha) = \omega_j(\alpha)u_j(\alpha)$$

$$2) \qquad A'(\alpha)u_j(\alpha) + A(\alpha)u_j'(\alpha) = \omega_j'(\alpha)u_j(\alpha) + \omega_j(\alpha)u_j'(\alpha)$$

$$3) \qquad \left(A'(\alpha)u_j(\alpha), u_j(\alpha)\right) = \omega_j'(\alpha)\left(u_j(\alpha), u_j(\alpha)\right) = \omega_j'(\alpha)$$

Here $A'(\alpha)$ is easy to compute. And it is clear that $\|A'(0)\| \leq c$. Therefore

Prop. : The group velocity is bounded by c.

Sign of $\mathbf{k} \cdot \mathbf{v}_g$

Take $\mathbf{d} = \mathbf{k}$. So

$$A'(0) = A(\mathbf{k}) - \underbrace{\begin{pmatrix} 0 & 0 & -\mathbf{i}\omega_p I \\ 0 & 0 & 0 \\ \mathbf{i}\omega_p I & 0 & -\mathbf{i}\omega_c \mathbf{e}_z \wedge \end{pmatrix}}_{H} \begin{pmatrix} \mathbf{e} \\ c\mathbf{b} \\ \omega_p \mathbf{u}_e \end{pmatrix}$$

Therefore : $(A'(0)u_j(0), u_j(0)) = ((\omega \mathbf{I} - H)u_j(0), u_j(0))$.

Prop : Assume $\omega_p^2 < \omega^2 - \omega\omega_c$, then $\omega\mathbf{I} - H > 0$. Therefore the group velocity is in the direction of $\mathbf{k}$, that is

$$\omega_j'(0) = \mathbf{k} \cdot \nabla_k \omega_j > 0.$$

Some references

Models

Dispersion relation for one species
All coefficients constant

Dispersion relation for multispecies

Group velocity

- Stix : The theory of plasma waves McGraw-Hill, 1962.
- Stix : Waves in plasmas, AIP, 1992.
- Swanson : Plasma Waves, Academic Press, 1989.
- Miyamoto : Plasma physics and controled nuclear fusion, Springer, 2005.
- Brambilla : Kinetic Theory of Plasma Waves-Homogeneous Plasmas, Clarendon, 1998
- Freidberg : Plasma physics and fusion energy, Cambridge University Press, 2007.
- Kato : Perturbation theory of linear operators, Springer, 1980.