Recent Results In Positivity Preserving Polynomials

(motivated by Finite Difference Methods)

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For some PDEs, the solution u satisfies bounds like

$$\min_{x\in\mathbb{R}} u_0(x) \le u(t,x) \le \max_{x\in\mathbb{R}} u_0(x), \qquad t>0,\ x\in\mathbb{R}.$$

High-order polynomial approximation of functions with low regularity
$\Rightarrow$ Gibbs phenomenon, violent oscillations and violation of the bounds.

• This is the case for entropy weak solution of conservation laws

$$\begin{cases} \partial_t u + \partial_x f(u) = 0, & x\in\mathbb{R},\ t>0, \\ u(0,x) = u_0(x), & x\in\mathbb{R}. \end{cases}$$

• Seminal contributions by VanLeer, Harten, Roe, Sweby, Shu-Osher ENO 88', P. Lax (Gibbs phenomena, 2006) . . .

• Example of a modern contribution: Shu, Bound-preserving high order accurate schemes Notes of the Canadian Mathematical Society 2013, . . .

Change of the angle of attack

Address a priori **high order+control of oscillations** with polynomial tools, old and new.

- Study the set

$$P_n^+ := \{p_n \in P_n : \quad p_n(x) \geq 0, \quad \forall x \in [0,1]\} \subset P_n,$$

  Numerical approximation, discretization of advection equation (new limiting strategies).

- Extend to the set

$$U_n := \left\{p_n \in P_n^+ \mid 1 - p_n \in P_n^+\right\}.$$

  Applications to control of Gibbs phenomena, discretization of advection equation (new limiting strategies).

- 2D.

- Szego *Orthogonal polynomials, 38'*
- Milovanovic, Mitrinovic, Rassias, Topics in polynomials: extremal problems, inequalities, zeros, 1994.
- J. Bochnak, M. Coste and M.-F. Roy, Real algebraic geometry, Springer 1998.
- Lasserre, Moments, Positive Polynomials and Their Applications, Imperial college press, 2010

Lukacs Theorem.

Characterization/parametrization of $P_n^+ = \{p_n \in P_n,\ p_n(x)\ 0 \le x \le 1\}$

- First case: $\mathbf{n = 2p}$. Then $p_n \in P_n^+$ **if and only if** there exists $(a_p, b_p) \in P_p \times P_{p-1}$ such that

$$p_n(x) = a_p(x)^2 + x(1-x)\, b_{p-1}(x)^2.$$

- Second case: $\mathbf{n = 2p+1}$. Then $p_n \in P_n^+$ **if and only if** there exists $a_p, b_p \in P_p$ such that

$$p_n(x) = x\, a_p(x)^2 + (1-x)\, b_p(x)^2.$$

Remark: Non-uniqueness of the representation

$$1 = 1^2 + 0^2 x(1-x) = (1-2x)^2 + 2^2 x(1-x).$$

Proofs in the literature
- Szego *Orthogonal polynomials, 38'* (complex algebra),
- another proof in Achiezer *The classical moment problem 65'* (real algebra).

Related
- works by Peter Lax on degenerate matrices and **sum of squares**,
- 4-th order moments of non negative measures and applications, Francfort-Murat-Tartar, 1995.

A simple direct proof

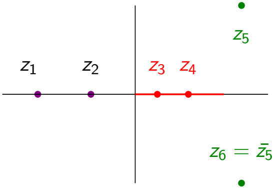
- $n = 2: \ p_2(x) = \Bigg(\underbrace{\sqrt{p_2(0)}(1-x) - \sqrt{p_2(1)}x}_{\text{oscillating polynomial}}\Bigg)^2 + x(1-x)\underbrace{b_0^2}_{\geq 0}.$
- $n \in 2\mathbb{N}$:

$$p_n(x) = \prod_{j=1}^{n/2} p_j(x), \ \text{with } p_j \in P_2^+$$

$$= \prod_{j=1}^{n/2} \left| a_{1,j}(x) + \mathbf{i}\sqrt{x(1-x)}b_{0,j} \right|^2$$

$$= \left| \prod_{j=1}^{n/2} \left(a_{1,j}(x) + \mathbf{i}\sqrt{x(1-x)}b_{0,j} \right) \right|^2$$

$$= \left| a_{n/2}(x) + \mathbf{i}\sqrt{x(1-x)}b_{n/2-1} \right|^2 = a_{n/2}(x)^2 + x(1-x)b_{n/2-1}^2$$



- $n \in 2\mathbb{N}+1: \ xp_n(x) = \hat{p}_{n+1}(x) = \hat{a}_{(n+1)/2}(x)^2 + x(1-x)\hat{b}_{(n+1)/2-1}^2 = (xa_{(n-1)/2}(x))^2 + x(1-x)b_{(n-1)/2}^2$. Simplify by x.

Positive interpolation

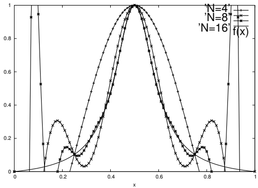
Introduction

P_n^+

U_n

2D

Polynomial interpolation. Let $f \in C_0^+[0,1]$. Take $n+1$ quadrature points and solve for $p_n \in P_n$: $p_n(x_i) = f(x_i), \quad 0 \le i \le n$.



Positive polynomial interpolation (Charles+Campos-Pinto+D., 2017): same problem with $p_n \in P_n^+$

$$p_n(x) = x a_p(x)^2 + (1-x) b_p(x)^2.$$

- We add **freedom to adjust the interpolation points**.
- Conveniently change $f(x_i)$ **by** $f(hx_i)$ where $0 < h \le 1$.
- In a FD/FV scheme, $h = r\Delta x$ is proportional to the mesh size.

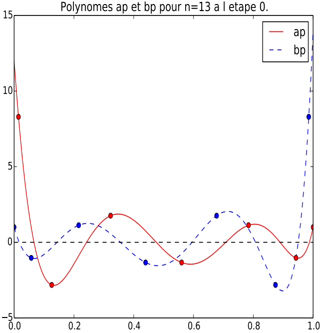
Calculation of a_p and b_p (case $n = 2p + 1$)

Finds points $0 = \beta_0 < \alpha_0 < \beta_1 < \cdots < \beta_p < \alpha_p = 1$ such that

$$p_n(x) = x a_p(x)^2 + (1 - x) b_p(x)^2 \in P_n^+$$

interpolates $f_h = f(h\,\cdot)$ at these points.

Assume polynomials $a_p, b_p \in \mathcal{P}_p$ satisfy $b_p(\alpha_i) = a_p(\beta_{i+1}) = 0$ for $0 \leq i \leq p - 1$.
Use oscillating polynomials with alternating signs $a_p(\alpha_i) = (-1)^{i+p}\sqrt{\frac{f(h\alpha_i)}{\alpha_i}}$ and $b_p(\beta_i) = (-1)^{i+p}\sqrt{\frac{f(h\beta_i)}{1-\beta_i}}$.



Fixed point problem

- Let $I_p = \{(x_1, \dots x_p) \subset (0,1)^p,\ 0 < x_1 < \cdots < x_p < 1\}$ and
$$(\alpha, \beta) = (\alpha_0, \dots, \alpha_{p-1};\ \beta_1, \dots, \beta_p) \in I_p^2.$$
- Let $a_{p,h}[\alpha]$ and $b_{p,h}[\beta]$ be oscillating polynomials

$$a_{p,h}[\alpha](x) = \sum_{0 \le i \le p} (-1)^{i+p} \sqrt{\frac{f(h\alpha_i)}{\alpha_i}} \prod_{0 \le j \ne i \le p} \frac{x - \alpha_j}{\alpha_i - \alpha_j} \quad (1)$$

and

$$b_{p,h}[\beta](x) = \sum_{0 \le i \le p} (-1)^{i+p} \sqrt{\frac{f(h\beta_i)}{1 - \beta_i}} \prod_{0 \le j \ne i \le p} \frac{x - \beta_j}{\beta_i - \beta_j}. \quad (2)$$

The equation to solve writes

$$\Theta_{p,h}(\alpha, \beta) = 0 \quad (3)$$

where the unknowns are the **interpolation points** (α, β). and $\Theta_{p,h} : I_p^2 \longrightarrow \mathbb{R}^{2p}$

$$\Theta_{p,h}(\alpha, \beta) = \left(b_{p,h}[\beta](\alpha_0), \dots, b_{p,h}[\beta](\alpha_{p-1}), a_{p,h}[\alpha](\beta_1), \dots, a_{p,h}[\alpha](\beta_p)\right). \quad (4)$$

Simplified Newton-Raphson algorithm

• Given a starting point $X^0 \in I_p^2$, compute

$$X^{m+1} = X^m - J_p(X^0)^{-1}\Theta_{p,h}(X^m) \quad (5)$$

where

$$J_p(X^0) = \nabla\Theta_{p,0}(X^0) = \begin{pmatrix} \nabla_\alpha b_{p,0}[\beta](\alpha) & \nabla_\beta b_{p,0}[\beta](\alpha) \\ \nabla_\alpha a_{p,0}[\alpha](\beta) & \nabla_\beta a_{p,0}[\alpha](\beta) \end{pmatrix}\Bigg|_{(\alpha,\beta)=X^0} \in \mathbb{R}^{2p\times 2p}.$$

It yields a sequence of **moving quadrature** points.

• Definition of a good starting point X^0.
Idea : a good starting point should be exact for small/vanishing h.

• For the limit case, $h = 0$, it amounts to seek two polynomials $a_p, b_p \in P_p$ such that

$$xa_p(x)^2 + (1-x)b_p(x)^2 = 1 \text{ for all } x \in [0,1]$$

which are actually the third and fourth kind Chebychef polynomials.

Convergence

Strong contraction properties (for small h).

Theorem. Let $f \in W_+^{1,\infty}[0,1]$. There exist $h_0 > 0$ such that for all $0 \leq h \leq h_0$ the following properties hold: the sequence $(X^m)_{m\geq 0}$ converges towards a fixed point of G_h in the ball $B(X^0, \varepsilon)$

$$\|X_h^\infty - X^m\| \leq C\left(\frac{h}{2h_0}\right)^{m+1} \qquad (6)$$

Theorem. Let $f \in W^{q,\infty}(0,1)$ for $1 \leq q \leq n+1$, $0 \leq h \leq h_0$ and $m \geq 0$. The reconstructed polynomial $p_n^m(x) := xa_p^m(x)^2 + (1-x)b_p^m(x)^2$ satisfies

$$\|p_n^m - f_h\| \leq Ch^{\min(q,2(m+1))}. \qquad (7)$$

Corollary. If $f \in P_n^+$, then $p_n^\infty = f$.

Introduction
P_n^+
U_n
2D

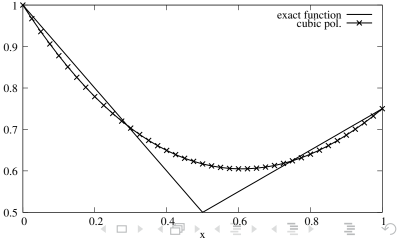
- For a given $h > 0$, one must guarantee that the approximated nodes stay away from each other. This is performed with an elementary separation/rescaling algorithm $S_{p,\varepsilon}$.
- For a given $h > 0$, beware that $f(0) \neq f^+ = \max_{0 \le x \le 1} f(x)$
- Using $\sqrt{\max(f(x),0)}$ instead of $\sqrt{f(x)}$, redefine $\Theta_{p,h}$.

- One obtains the algorithm

$$X^{m+1} = S_{p,\varepsilon}\left(X^m - \frac{1}{\sqrt{f^+}}\begin{pmatrix} D_\alpha & 0 \\ 0 & D_\beta \end{pmatrix}^{-1} \Theta_{p,h}(X^m)\right)$$

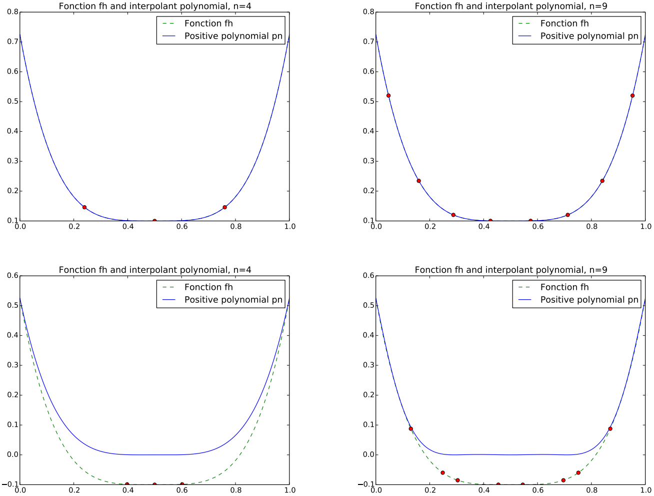
with X^0 given by the roots of the (modified) Tchebycheff polynomials.

m	α^m	β^m
0	0.250000,	0.750000
1	0.290569	0.750000
2	0.290569	0.747017
3	0.290678	0.747017
4	0.290678	0.747013
...	0.290678	0.747013



Constructive certificate of positivity

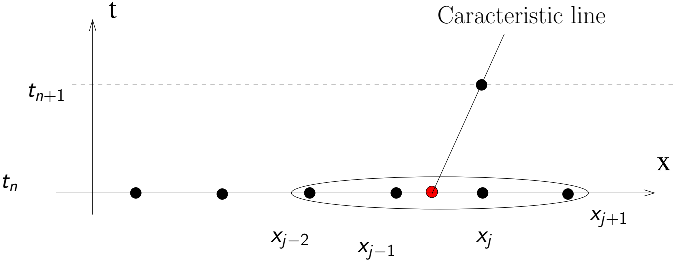
An algorithm which tells you wether a given polynomial is in P_n^+ or not: objective is $q_\lambda(x) = 10(x-1/2)^4 + \lambda$ with $\lambda = \pm 0.1$



New limiting strategies for SL schemes

$$\begin{cases} \partial_t u + a\partial_x u = 0, & x \in \mathbb{R}, \quad t > 0, \quad a = 1, \\ u(x,0) = u_0(x), & x \in \mathbb{R}, \end{cases}$$

Classical SL (Semi-Lagrangian) strategy:
a) standard Lagrange interpolant $p_n^{\text{lag.}}$ of u_r^{old} for $j - k_- \le r \le j + k_+$
b) update $u_j^{\text{new}} = p_n^{\text{lag.}}(j\Delta x - a\Delta t)$.



New algo.: use the certificate of positivity

a) standard Lagrange interpolant $p_n^{\text{lag.}}$ of u_r^{old} for $j - k_- \le r \le j + k_+$,
b) positive interpolation $p_n^{\text{pos.pol.}} \longleftarrow p_n^{\text{lag.}}$ with opt. num. of iterations,
c) update $u_j^{\text{new}} = p_n^{\text{pos.pol.}}(j\Delta x - a\Delta t)$.

$\Delta x = h$	$n = 1$	$n = 3$	$n = 5$	$n = 7$
20	0.195	0.0045767	0.00020966477	0.000016399842
40	0.109	0.0005707	0.00001083749	0.000000551818
80	0.058	0.0000725	0.00000039502	0.000000006955
160	0.029	0.0000091	0.00000001303.	0.000000000065
320	0.015	0.0000011	0.00000000041	$\varepsilon_{mac.}$
order	≈ 1	≈ 3	≈ 5	≈ 7

Above: Error for the non conservative semi-lagrangian interpolation.
Below: Error for conservative ENO-inspired method. Schemes are conservative.

$\Delta x = h$	$n = 1$	$n = 3$	$n = 5$	$n = 7$
20	0.03791	0.000800990	0.000022346530	0.000001284228
40	0.00964	0.000052541	0.000000460477	0.000000023856
80	0.00242	0.000003355	0.000000008021	0.000000000168
160	0.00060	0.000000211	0.000000000132	$\varepsilon_{mac.}$
320	0.00015	0.000000013	$\varepsilon_{mac.}$	$\varepsilon_{mac.}$
order	≈ 2	≈ 4	≈ 6	≈ 8

Algebra in U_n (here with $n = 2k$)

- $p \in P_n^+$: the Lukacs theorem yields the representation

$$p = a^2 + b^2 w \tag{8}$$

where a and b are polynomials with convenient degree and $w(x) = x(1-x)$ is the weight.

- $1 - p \in P_n^+$: so

$$1 - p = c^2 + d^2 w \tag{9}$$

where c and d are polynomials with convenient degree.

- That is $p \in U_n$ iff there exists 4 polynomials $(a, b, c, d,)$ such that

$$1 = a^2 + b^2 w + c^2 + d^2 w.$$

Set

$$\mathcal{U}_n = \{(a, b, c, d) \in P_n \times P_{n-1} \times P_n \times P_{n-1}$$

$$\text{such that } 1 = a^2 + b^2 w + c^2 + d^2 w\Big\}.$$

Beware U_n and $\mathcal{U}_n$ are fully different objects even though $U_{2n+1} \approx \mathcal{U}_n$.

A quaternion algebra

Consider the algebra

$$\begin{cases} A &= a\alpha &+wb\beta &+c\gamma &+wd\delta, \\ B &= a\beta &-b\alpha &+c\delta &-d\gamma, \\ C &= a\gamma &-wb\delta &-c\alpha &+wd\beta, \\ D &= a\delta &+b\gamma &-c\beta &-d\alpha. \end{cases}$$

One has the weighted 4-squares Euler identity

$$A^2 + B^2 w + C^2 + D^2 w = \left(a^2 + b^2 w + c^2 + d^2 w\right)\left(\alpha^2 + \beta^2 w + \gamma^2 + \delta^2 w\right)$$

Proof: take quaternions $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$, set

$$\begin{cases} e(x) &= a(x) &+\mathbf{i}\sqrt{w(x)}b(x) &+\mathbf{j}c(x) &+\mathbf{k}\sqrt{w(x)}d(x), \\ \epsilon(x) &= \alpha(x) &+\mathbf{i}\sqrt{w(x)}\beta(x) &+\mathbf{j}\gamma(x) &+\mathbf{k}\sqrt{w(x)}\delta(x), \\ E(x) &= A(x) &+\mathbf{i}\sqrt{w(x)}B(x) &+\mathbf{j}C(x) &+\mathbf{k}\sqrt{w(x)}D(x), \end{cases}$$

and check $E = \overline{e}\epsilon$. It yields $|E|^2 = |e|^2|\varepsilon|^2$.

- J. Itard, Les nombres premiers (on Euler 4-squares identity).
- D., Polynomials with bounds and numerical approximation, Numer. Algo., 2017.

Properties

• Assume $(a,b,c,d) \in \mathcal{U}_n$ and $(\alpha,\beta,\gamma,\delta) \in \mathcal{U}_m$. Then $(A,B,C,D) \in \mathcal{U}_{n+m}$.

• $\mathcal{U}_1$: all $(a,b,c,d) \in \mathcal{U}_1$ can represented with 3 "angles" $(\theta,\varphi,\mu) \in \mathbb{R}^3$

$$\begin{cases} a(x) &= \cos\theta x + \cos\varphi(1-x), \\ b &= R\cos\mu, \\ c(x) &= \sin\theta x + \sin\varphi(1-x), \\ d &= R\sin\mu. \end{cases} \qquad R = 2\sin\left(\frac{\theta-\varphi}{2}\right) \tag{10}$$

Theorem: Let $n \in 2\mathbb{N}$ being even. There exists a smooth function from $\mathbb{R}^{3n/2}$ onto U_n. The smooth function is made explicit by a constructive algorithm and is 2π-periodic with respect to all its arguments.

• For simplicity assume the result $E = \Pi_{j=1}^{n} e_j$ with $e_j \in \mathcal{U}_1 \iff \overline{e}^n E = \Pi_{j=1}^{n-1} e_j$. The proof boils down to find e^n such that $\overline{e}^n E \in \mathcal{U}_{n-1}$. A priori of course $\overline{e}^n E \in \mathcal{U}_{n+1}$.

• D.: Polynomials with bounds and numerical approximation, Numer. algorithms 2017.

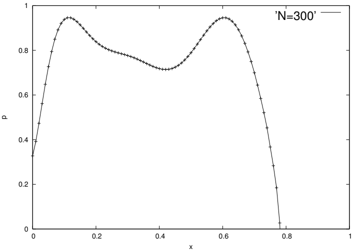
An unstable Matlab implementation

```
for k=1:n
theta=theta(k) ; phi=phi(k) ; mu=mu(k) ; R=2.*sin((theta-phi)/2.);
alpha=[cos(theta)-cos(phi) cos(phi)]; beta = [R*cos(mu)];
gamma=[sin(theta)-sin(phi) sin(phi)]; delta=[R*sin(mu)];

A=conv(a,alpha)+conv(conv(b,gamma),w) +conv(c,beta)+conv(conv(d,delta),w);
B=conv(a,beta)-conv(b,alpha)+conv(c,delta)-conv(d,gamma);
C=conv(a,gamma)-conv(conv(b,delta),w) -conv(c,alpha)+conv(conv(d,beta),w);
D=conv(a,delta)+conv(b,gamma)-conv(c,beta)-conv(d,alpha);

a=A; b=B; c=C; d=D;
end

p=conv(a,a)+conv(conv(b,b),w);
```



Hint: $\widehat{T}_n(x) := \frac{\cos(n \arccos(x))+1}{2} = 2^{n-1}x^n + \text{low order terms}$

A stable implementation at interpolation points

Set the diagonal matrix $D(x) := \text{diag}\,(1, w(x), 1, w(x)) \geq 0$ for $w(x) = x(1-x) \geq 0$.
Set $\alpha = x\cos\theta + (1-x)\cos\varphi$, $\beta = R\cos\mu$, $\gamma = x\sin\theta + (1-x)\sin\varphi$, $\delta = R\sin\mu$ with $R = 2\sin\left(\frac{\theta-\varphi}{2}\right)$.
Set the iteration matrix

$$M_k(x) := M(x;\theta,\varphi,\mu) := \begin{pmatrix} \alpha & w\beta & \gamma & w\delta \\ \beta & -\alpha & \delta & -\gamma \\ \gamma & -w\delta & -\alpha & w\beta \\ \delta & \gamma & -\beta & -\alpha \end{pmatrix}(x).$$

The loop at points is

$$\begin{pmatrix} a(x) \\ b(x) \\ c(x) \\ d(x) \end{pmatrix} = M_n(x)M_{n-1}(x)\dots M_2(x)M_1(x)X_0, \quad X_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

It is stable since $M_k(x)^t D(x) M_k(x) = D(x) \geq 0$ for $0 \leq x \leq 1$.

$\Longrightarrow$ Stability up to degree $n = 1000$ and more has been observed.

$U_n \Rightarrow$ control of Gibbs phenomenon

Problem: given quadrature points and numerical values which may be the output of an hyperbolic code for example, interpolate a polynomial.

Constraints: the polynomial must satisfied 2 bounds (0 and 1) and the function might be discontinuous.

- Some problems can be written like

$$\text{Find } p_n \in U_n \text{ such that } J(p_n) \leq J(q_n) \quad \forall q_n \in U_n$$

where J is some functional. It can be the L^2 norm between p_n and a given objective function f: in this case $J(q_n) = \int_0^1 |f(x) - q_n(x)|^2 dx$.
Below it is the L^1 norm

$$J(q_n) = \int_0^1 |f(x) - q_n(x)| dx$$

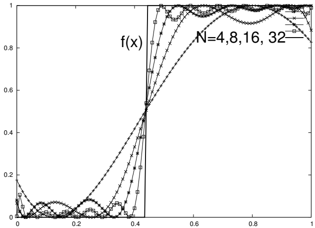
- Functional $J(q_n)$ is in practice discretized as

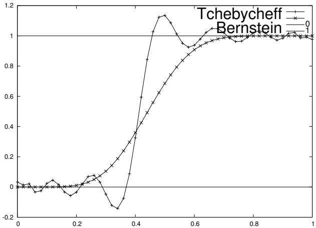
$$\mathcal{J}_h(\alpha) = \sum_i \omega_i |f(x_i) - q_n(x_i; \alpha))|, \quad \alpha \in \mathbb{R}^{3n}.$$

- Tests are performed within a Matlab test code with the `fminunc` function.

Step function

Minimization of L^1 norm to the objective function which is a step function $f_4(x) = 0$ for $x < 0.4$ and $z(x) = 1$ for $0.4 < x$.



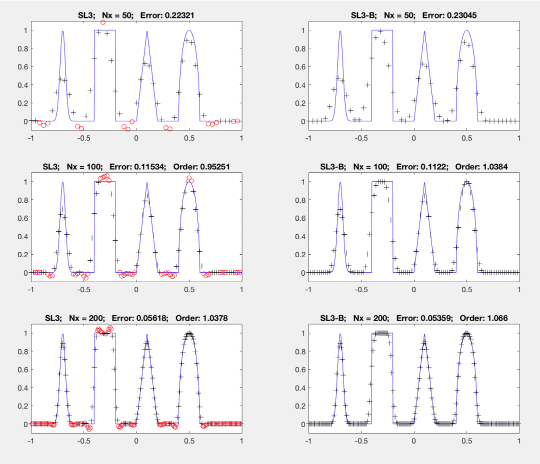


n	4	8	16	32
$m = n/2$	2	4	8	16
discrete L^1 error	0.041	0.021	0.0117	0.00518

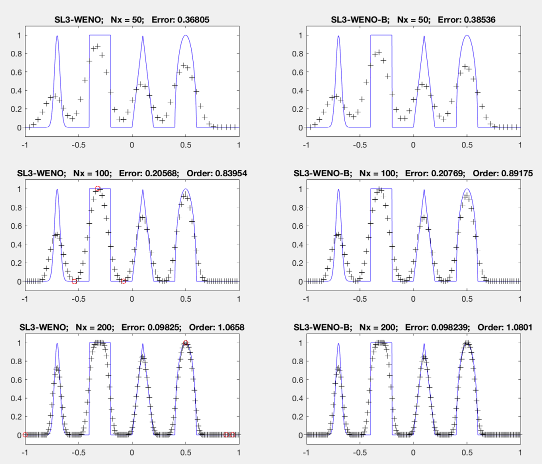
Numerically, the convergence is first order in L^1 (optimal for BV functions).

A new limiter strategy: SL3, SL3-B

First compute a and b with the algo. page 15.
Then "project" onto U_1.



Comparison with SL3-WENO, SL3-WENO-B



U_n: A bivariate version with 2 bounds on the square

$$\mathcal{U}_n^{2D} = \left\{ \text{tot. deg.} p_n \le n, \quad 0 \le p_n(x,y) \le 1 \ \ \forall (x,y) \in [0,1]^2 \right\}.$$

• Take the weights $w_1 = 1$, $w_2 = 1 - x^2$, $w_3 = 1 - y^2$, $w_4 = w_2 w_3$ and the representation

$$p(x,y) = a(x,y)^2 w_1(x) + c(x,y)^2 w_2(x) + e(x,y)^2 w_3(y) + g(x,y)^2 w_4(x)$$

$$1 - p(x,y) = b(x,y)^2 w_1(x) + d(x,y)^2 w_2(x) + f(x,y)^2 w_3(y) + h(x,y)^2 w_4(x).$$

Note the redundancy due to w_4.

• It yields a 8-squares formula

$$1 = \left(a(x,y)^2 + b(x,y)^2 \right) w_1(x) + \left(c(x,y)^2 + d(x,y)^2 \right) w_2(x)$$

$$+ \left(e(x,y)^2 + f(x,y)^2 \right) w_3(y) + \left(g(x,y)^2 + h(x,y)^2 \right) w_4(x).$$

Octonions

• The 1-2-4-8 Hurwiz theorem yields an obstruction to many squares identity at any order. In particular no 6-squares identity available: this is the reason of the redundancy with w_4.

• For $n = 8$, the solution is given in the form of the Degen identity

$$
\begin{aligned}
&(a^2+b^2+c^2+d^2+e^2+f^2+g^2+h^2)(m^2+n^2+o^2+p^2+q^2+r^2+s^2+t^2) \\
&= (am-bn-co-dp-eq-fr-gs-ht)^2 + (bm+an+do-cp+fq-er-hs+gt)^2 \\
&+(cm-dn+ao+bp+gq+hr-es-ft)^2 + (dm+cn-bo+ap+hq-gr+fs-et)^2 \\
&+(em-fn-go-hp+aq+br+cs+dt)^2 + (fm+en-ho+gp-bq+ar-ds+ct)^2 \\
&+(gm+hn+eo-fp-cq+dr+as-bt)^2 + (hm-gn+fo+ep-dq-cr+bs+at)^2.
\end{aligned}
$$

• It is convenient to rewrite it as an Euler identity with complex numbers. We define $(i^2=-1)$ $u = a+ib, \quad v = c+id, \quad w = e+if, \quad z = g+ih$ and $\alpha = m+in, \quad \beta = o+ip, \quad \gamma = q+ir, \quad \delta = s+it.$
The Degen identity rewrites as

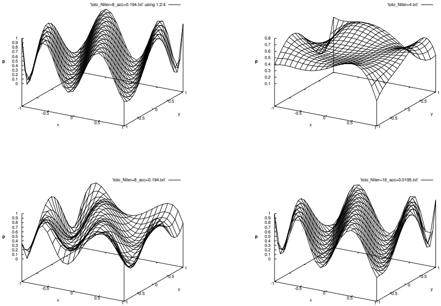
$$
|A|^2+|B|^2+|C|^2+|D|^2 = \left(|u|^2+|v|^2+|w|^2+|z|^2\right)\left(|\alpha|^2+|\beta|^2+|\gamma|^2+|\delta|^2\right)
$$

with

$$
\left\{
\begin{array}{l}
A = u\alpha - v^*\beta - w\gamma^* - z^*\delta, \\
B = v\alpha + u^*\beta + z\gamma^* - w^*\delta, \\
C = w\alpha - z\beta^* + u^*\gamma + v^*\delta, \\
D = z\alpha^* + w\beta - v\gamma + u\delta.
\end{array}
\right. \tag{11}
$$

2D: smooth objective function

Objective function $f_5(x,y) = \frac{T_8((2x+y)/3)+1}{2}$, 11×11 interpolation points.

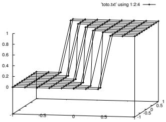


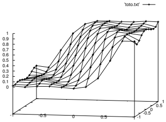
total degree	8	16	32
partial degree: x or y	4	8	16
iterations	93	291	340
L^2 error	0.34	0.10	0.027

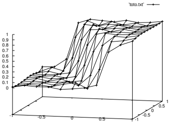
2D control Gibbs phenomenon

Minimization of the discrete L^2 distance to the objective function $f_6(x, y) = H(2x + y)$, $f_6 \in BV$.

The L^2 distance is better in terms of the smoothness and speed of convergence. The number of interpolation points is 11×11 and $n = 10$ then 20.







Conclusion

- Lukacs Theorem, that is SOS formulas like
  $p_n(x) = xa_p(x)^2 + (1-x)b_p(x)^2$,
  provides a rigorous basis for **positive interpolation with optimal polynomial degree and optimal error estimates**.
- Two bounds with quaternions algebra:
  quaternions algebra yield control of the Gibbs phenomenon
- **Application to the design of new families of high-order methods with a priori satisfaction of bounds.**
- Potential interest for SL discretization of transport equations and for DG is clear.
  Potentialities for the numerical discretization of non linear equations is still to demonstrate.

References

Classical literature on polynomial theory

- Szego *Orthogonal polynomials, 38'*
- Milovanovic, Mitrinovic, Rassias, Topics in polynomials: extremal problems, inequalities, zeros, World Scientific, 1994.
- J. Bochnak, M. Coste and M.-F. Roy, Real algebraic geometry, Springer 1998.
- Lasserre, Moments, Positive Polynomials and Their Applications, Imperial college press, 2010

- D. 2017, Polynomials with bounds and numerical approximation, Numer. Algo.
- Charles+Campos-Pinto+D., Algorithms for positive polynomial approximation, Hal preprint server 2017 (almost published).
- Herda+Charles+D.: in preparation.

- D.+Perthame: Uncertainty propagation; intrusive kinetic formulations of scalar conservation laws, 2016.
- D.+ Trelat: Two sided space-time L1 approximation and optimal control of polynomial systems, Siam J. Applied Math. and Optm., 2018.