A quadratic Lyapunov function for hyperbolic density–velocity systems with nonuniform steady states

Georges Bastin a,*, Jean-Michel Coron b

a Department of Mathematical Engineering, ICTEAM, Université catholique de Louvain, 4, Avenue G. Lemaître, 1348 Louvain-La-Neuve, Belgium
b Laboratoire Jacques-Louis Lions, Université Pierre et Marie Curie, 4, Place Jussieu, 75252 Paris Cedex, France

ARTICLE INFO

Article history:
Received 5 October 2016
Received in revised form 14 January 2017
Accepted 31 March 2017
Available online 6 May 2017

Keywords:
Stability
Lyapunov
Hyperbolic
Euler equation
Partial differential equation

ABSTRACT

A new explicit Lyapunov function allows to study the exponential stability for a class of physical 2 by 2 hyperbolic systems with nonuniform steady states. In fluid dynamics, this class of systems involves isentropic Euler equations and Saint-Venant equations. The proposed quadratic Lyapunov function allows to analyze the local exponential stability of the system equilibria for suitable dissipative Dirichlet boundary conditions without additional conditions on the system parameters.

© 2017 Elsevier B.V. All rights reserved.

1. Introduction

Our purpose in this paper is to address the $H^2$-stability of the equilibria for a class of one-dimensional 2 × 2 hyperbolic systems. In fluid dynamics, this class of systems involves, for instance, isentropic Euler equations and Saint-Venant equations. These are systems of two balance laws with the density and the velocity of the considered fluid as state variables, and with source terms representing friction effects.

In [1–3] we have introduced a generic $H^2$-Lyapunov function, expressed in Riemann coordinates, to analyze the $H^2$-stability in the frictionless case when the system takes the form of conservation laws and has, therefore, a steady-state which is spatially uniform.

The problem which is addressed in this paper is more difficult because we consider the case where the friction is not negligible and the system is represented by balance laws with nonuniform steady-states. For that case, general implicit stability conditions are given in [4, Chapter 6] by using a weighted form of the $H^2$-Lyapunov function with a weighting matrix $Q(x)$ which is not a-priori given and has to be found for each particular application.

The issue addressed in this paper was previously considered in [5] (see also [6–8]) for the special case of isothermal Euler equations. In the paper [5], for a system with a non-uniform steady-state, the authors use a matrix $Q(x)$ which is related to the one proposed in [2] for conservation laws with uniform steady-state. This choice allows to obtain stability for suitable boundary conditions albeit with limitations depending on the values of some system parameters.

In this paper, we shall show how a more physical choice of the weighting matrix $Q(x)$ allows to remove the previous limitations. With this choice, we are able to prove the $H^2$ local exponential stability for suitable boundary conditions without any additional conditions on the system parameters.

2. Hyperbolic density–velocity systems

We are concerned with hyperbolic systems of two partial differential equations (PDEs) of the form

\[ H_t + (HV)_x = 0, \]

\[ V_t + \frac{c^2}{H} H_x + VV_x + f(H, V) = 0, \]

where $t \in [0, +\infty)$, $x \in [0, L]$, $H : [0, +\infty) \times [0, L] \to (0, +\infty)$, $V : [0, +\infty) \times [0, L] \to (0, +\infty)$, $c \in C^2([0, +\infty); [0, +\infty))$, $f \in C^1([0, +\infty)^2; \mathbb{R})$.

In many applications, the model (1) represents the one-dimensional propagation of some physical quantity, with density $H$, propagation velocity $V$, flow density $HV$ and friction $f(H, V)$. The first equation is a mass conservation law and the second equation is a momentum balance.

A very typical example is given by the isentropic Euler equations for the gas motion in pipelines where $H$ is the gas density. In the...
particular case of a polytropic gas, the gas pressure is the following function of the gas density:
\[ p(H) = \kappa H^\gamma \]  
with \( \kappa > 0 \) and \( \gamma > 1 \) real constants. In that case the function \( c(H) \) is the sound velocity defined as:
\[ c(H) = \sqrt{p(H)} = \sqrt{\kappa H^{\gamma-1}}. \]  
Another typical example is given by the Saint-Venant equations for the water motion in open channels where \( H \) is the water level and the function \( c(H) = \sqrt{gh} \), with \( g > 0 \) the gravity acceleration constant.

In Eq. (1b), the coefficient \( c^2(H)/H \) is the second order derivative of the internal energy \( U(H) \) such that:
\[ U''(H) = \frac{c^2(H)}{H} \]  
with \( U \in C^4((0, +\infty); \mathbb{R}) \). Multiplying the first equation (1a) by \( U'(H) \) and the second equation (1b) by \( HV \), then summing up the two resulting equations, we obtain the following energy balance law which is satisfied along the smooth solutions of the system (1):
\[ E(H, V)_t + F(H, V)_x + (HV)\dot{f}(H, V) = 0, \]  
with
\[ E(H, V) = \frac{1}{2} HV^2 + U(H), \quad F(H, V) = HV \left( U'(H) + \frac{1}{2} V^2 \right). \]  
The function \( E(H, V) \) is the energy density; the first term \( HV^2/2 \) is the kinetic energy density and the second term \( U(H) \) is the potential or internal energy density. Obviously, the total energy of the system (1) at the time \( t \) is given by the integral of \( E \) over the interval \([0, L]\):
\[ \phi(t) = \int_0^L E(H(t, x), V(t, x)) \, dx. \]  
In this paper, our purpose is to show how the energy \( \phi \) can be used to define a Lyapunov function for the exponential stability analysis of the steady-states of the system (1) when both the boundary conditions and the source term \( f(H, V) \) are dissipative.

A steady state (or equilibrium) of the system (1) is a time-invariant solution \((H^*, V^*) : [0, L] \rightarrow (0, +\infty)^2\). It satisfies the system of ordinary differential equations
\[ V^*_x = \frac{V^*(H^*, V^*)}{c^2(H^*)} - V^2*, \quad H^*_x = -\frac{H^*f(H^*, V^*)}{c^2(H^*)} - V^2*. \]  
In this paper, we consider the standard physical case where
1. the steady-state density \( H^*(x) > 0 \) and velocity \( V^*(x) > 0 \) are strictly positive for all \( x \in [0, L] \);
2. the steady state flow is subcritical, i.e. the denominators in (8) are strictly positive:
\[ c^2(H^*) - V^2 > 0 \quad \forall x \in [0, L]. \]  
In order to linearize the model, we define the deviations of the states \( H(t, x) \) and \( V(t, x) \) with respect to the steady states \( H^*(x) \) and \( V^*(x) \):
\[ h(t, x) \triangleq H(t, x) - H^*(x), \quad v(t, x) \triangleq V(t, x) - V^*(x). \]  
The linearization of the system (1) about the steady state is then
\[ \begin{pmatrix} h_t \\ v_t \end{pmatrix} + \begin{pmatrix} V^* \\ c^2(H^*) \\ H^* \end{pmatrix} \begin{pmatrix} h_x \\ v_x \end{pmatrix} + \begin{pmatrix} V^*_x \\ c^2(H^*) \end{pmatrix} \begin{pmatrix} h_x \\ v_x + f_x^* \end{pmatrix} (\begin{pmatrix} h_v \\ v \end{pmatrix} = 0. \]  
with the notations
\[ f^*_h = \frac{\partial f}{\partial H}(H^*, V^*), \quad f^*_v = \frac{\partial f}{\partial V}(H^*, V^*). \]

3. A physical quadratic Lyapunov function for the linearized system
In this section, we are concerned with the exponential stability of the \( L^2 \)-solutions of the linearized system (11) under local linear boundary conditions of the form
\[ v(t, 0) = -b_0 h(t, 0), \quad v(t, L) = b_1 h(t, L), \]  
with constants \( b_0 \in \mathbb{R}, b_1 \in \mathbb{R} \), and under an initial condition
\[ h(0, x) = h^0(x), \quad v(0, x) = v^0(x), \]  
such that
\[ (h^0, v^0) \in L^2((0, L); \mathbb{R}^2) \]  
The Cauchy problem (11), (13), (14) is well-posed (see [4, Appendix A]).

Definition 1. The system (11), (13) is exponentially stable (for the \( L^2 \)-norm) if there exist \( \nu > 0 \) and \( C > 0 \) such that, for every initial condition \((h^0, v^0) \in L^2((0, L); \mathbb{R}^2)\), the solution to the Cauchy problem (11), (13), (14) satisfies
\[ \| (h(t, \cdot), v(t, \cdot)) \|_{L^2} \leq Ce^{-\nu t} \| (h^0, v^0) \|_{L^2}. \]  
For this analysis, as a candidate Lyapunov function, we introduce the following quadratic functional:
\[ V = \int_0^L (\frac{c^2(H^*)}{H^*} h^2 + H^* v^2) \, dx \]  
\[ = \int_0^L (h^T v) \left( \frac{c^2(H^*)}{H^*} 0 \right) (h_v \, dx. \]
With the notations
\[ Y = (h_v), \quad A(x) = \left( \begin{array}{c} V^* \\ c^2(H^*) \\ H^* \end{array} \right), \quad B(x) = \left( \begin{array}{c} V^*_x \\ c^2(H^*) \end{array} \right), \]  
we compute the time derivative of \( V \) along the \( C^1 \)-solutions of the system (11), (13):
\[ \frac{dV}{dt} = \int_0^L (Y^T D(x) Y + Y^T D(x) Y) \, dx \]  
\[ = -\int_0^L (Y^T D(x) A(x) Y + B(x) Y) + (Y^T A^T(x) + Y^T B(x)) D(x) Y) \, dx. \]  
We observe that the matrix \( M(x) = D(x)A(x) \) is symmetric:
\[ M(x) = D(x)A(x) = A^T(x)D(x) = \left( \begin{array}{cc} \frac{c^2(H^*)}{H^*} V^* & c^2(H^*) \\ H^* & H^* V^* \end{array} \right). \]
Now, using (20), (21) and integration by parts, we have
\[
\frac{dV}{dt} = -\int_0^l (Y^TM(x)Y)_x dx - \int_0^l [Y^T(-M'(x) + B'(x)D(x) + D(x)B(x))]Y dx
\]
\[
= -[Y^TM(x)Y]_0^l - \int_0^l (Y^TN(x)Y) dx,
\]
with, using (8),
\[
N(x) = -M(x) + B'(x)D(x) + D(x)B(x)
\]
\[
= \left[2c(H^*)c(H^{*1})V^* - H^*_0V^*_0 \right] / 2H^*(V^*_0 + f^*_V).
\]
Hence, from Eq. (22), we see that \(dV/dt\) is a negative definite function along the solutions of the system (11), (13) if the two following conditions are satisfied:

(a) the boundary term is non negative (i.e. \(Y^TM(x)Y\) \(\geq 0\)) for every \(Y(t, 0) \in \mathbb{R}^2\) and every \(Y(t, L) \in \mathbb{R}^2\) such that \(v(t, 0) = -b_0 h(t, 0)\) and \(v(t, L) = b_1 h(t, L)\); and

(b) the matrix \(N(x)\) is positive definite for all \(x \in [0, L]\).

Since, for all \(x \in [0, L]\), the matrix \(N(x)\) is positive definite, there exists a positive real number \(V\) such that
\[
Y^TN(x)Y \geq 2\nu Y^TD^2(x)Y, \quad \forall x \in [0, L], \quad \forall Y \in \mathbb{R}^2.
\]
Then, it follows directly from the definition of \(V\) and from (22) that
\[
\frac{dV}{dt} \leq -2\nu V
\]
along the \(C^1\)-solutions of the system. However, since the \(C^1\)-solutions are dense in the set of \(L^2\)-solutions, inequality (26) is also satisfied in the sense of distributions for \(L^2\)-solutions (see [4, Section 2.1.3] for details). Consequently, \(V\) is an exponentially decaying Lyapunov function for the \(L^2\)-norm and there is a positive constant \(C\) such that the solutions satisfy inequality (16) of Definition 1.

In the next proposition, we now give conditions on the coefficients \(b_0\) and \(b_1\) such that the boundary condition (a) is fulfilled. For simplicity, we introduce the following notations:
\[
h_0(t) = h(t, 0), \quad H^*_0 = H^*(0), \quad V^*_0 = V^*(0).
\]
\[
h_1(t) = h(t, L), \quad H^*_1 = H^*(L), \quad V^*_1 = V^*(L).
\]
**Proposition.** Condition (a) is verified if and only if \(b_0\) and \(b_1\) satisfy the following inequalities:
\[
b_0 \in [b_0, b_0^*] \quad \text{with} \quad b_0^* = \frac{c^2(H^*_0)}{H^*_0} \left( \frac{1}{V^*_0} + \sqrt{\frac{1}{V^*_0^2} - \frac{1}{c^2(H^*_0)}} \right),
\]
\[
b_1 \in \mathbb{R} \setminus (b_1^*, b_1^*) \quad \text{with} \quad b_1^* = \frac{c^2(H^*_1)}{H^*_1} \left( \frac{1}{V^*_1} + \sqrt{\frac{1}{V^*_1^2} - \frac{1}{c^2(H^*_1)}} \right).
\]
**Proof.** We have
\[
[Y^TN(x)Y]_0^l = [Y^T(t, L)M(t,L)Y(t, L)] - [Y^T(t, 0)M(0)Y(t, 0)].
\]
Under the boundary condition (13), we have
\[
[Y^T(t, 0)M(0)Y(t, 0)] = (H^*_0V^*_0b^*_0 - 2c^2(H^*_0)b_0 + \frac{c^2(H^*_0)}{H^*_0}V^*_0)h^*_0(t).
\]
\(\Phi_0\) is a degree-2 polynomial in \(b_0\) with two positive real roots \(b^*_0, b^*_0\). Condition (29) holds for values of \(b_0\) located in the closed interval \([b^*_0, b^*_0]\) such that \(\Phi_0\) is nonpositive and, consequently, that \(Y^T(t, 0)M(0)Y(t, 0) \leq 0\) for all \(t\).

On the other hand, under the boundary condition (13), we have
\[
[Y^T(t, L)M(t,L)Y(t, L)] = (H^*_1V^*_1b^*_1 + 2c^2(H^*_1)b_1 + \frac{c^2(H^*_1)}{H^*_1}V^*_1)h^*_1(t).
\]
\(\Phi_1\) is a degree-2 polynomial in \(b_1\) with two negative real roots \(b^*_1, b^*_1\). Condition (30) holds for values of \(b_1\) located outside the open interval \((b^*_1, b^*_1)\) such that \(\Phi_1\) is nonnegative and, consequently, that \(Y^T(t, L)M(t,L)Y(t, L) \geq 0\) for all \(t\). This completes the proof of the proposition. □

In the next two subsections we compute the matrix \(N(x)\) and we show that condition (b) is satisfied for the examples of isentropic Euler equations and Saint-Venant equations.

**Isentropic Euler equations**

The isentropic Euler equations are typically used to describe the gas propagation in ducts, with a pressure \(p(H) = \kappa H^\gamma\) (where \(\gamma > 1\) is the adiabatic index) and a friction proportional to the square of the velocity:
\[
f(H, V) = kV^2, \quad k > 0.
\]
In this case, for all \(x \in [0, L]\), we have:
\[
c^2(H^*) = k\gamma H^{\gamma - 1} > 0,
\]
\[
2c(H^*)c(H^{*1}) = k\gamma (\gamma - 1)H^{\gamma - 2} > 0,
\]
\[
f(H^*, V^*) > 0, \quad f^*_{V} = 0, \quad f^*_{V^2} = 2kV^* > 0.
\]
Under the subcritical flow condition (9), it follows directly that \(V^*_* > 0\) and \(N(x)\) is positive definite for all \(x \in [0, L]\).

**Saint-Venant equations**

For a horizontal channel with a rectangular cross section and a unit width, the water propagation may be described by Saint-Venant equations that are a special case of the model (1) with
\[
c^2(H) = gH \quad \text{and} \quad f(H, V) = kV^2 / H, \quad k > 0.
\]
In this case, for all \(x \in [0, L]\), we have
\[
2c(H^*)c(H^{*1}) = g, \quad f^*_{V} = -kV^2 / H^2, \quad f^*_{V^2} = 2kV^* / H^2.
\]
Then, using (8), the matrix \(N(x)\) is
\[
N(x) = \left( \begin{array}{c} \frac{gkV^3}{H^3} - \frac{kV^2}{H^2} \\ \frac{2kV^3}{H^3} - 4kV^* \end{array} \right).
\]
Here the subcritical flow condition (9) implies that \(H^* > V^* > 0\). It follows that
\[
\det[N(x)] = \left( \frac{kV^2}{H^2} \right)^2 \left( \frac{2gH^*V^2}{(gH^* - V^2)^2} + \frac{4gH^*}{(gH^* - V^2)} - 1 \right) > 0,
\]
and therefore that the matrix \(N(x)\) is positive definite for all \(x \in [0, L]\).
Remark 1 (Comment on the Choice of the Lyapunov Function). Let us expand the energy function (7) in a neighborhood of the steady-state \( H^*, V^* \) for the Saint-Venant equations. We obtain:

\[
\delta = \frac{1}{2} \int_0^L \left( (H^* + h)(V^* + v)^2 + g(H^* + h)^2 \right) dx
\]

\[
= \delta_1 + \delta_2 + \delta_3,
\]

with

\[
\delta_1 = \frac{1}{2} \int_0^L \left( (V^* + v)(V^* + v)^2 + gV^*(V^* + v)^2 \right) dx,
\]

\[
\delta_2 = \frac{1}{2} \int_0^L (gh^2 + 2V^*hv + H^*v^2) dx,
\]

\[
\delta_3 = \frac{1}{2} \int_0^L (h^2) dx.
\]

We could believe that the quadratic term \( \delta_2 \) should be a natural candidate for a Lyapunov function for this system. Let us therefore compute the time derivative of \( \delta_2 \) along the solutions of (11):

\[
\frac{d\delta_2}{dt} = \int_0^L \left[ (gh + V^*v) \frac{dh}{dt} + (V^*h + H^*v) \frac{dh}{dt} + (h^2 + 2V^*hv + H^*v^2) \right] dx
\]

\[
= \int_0^L \left[ (gh + V^*v) \frac{dh}{dt} + (V^*h + H^*v) \frac{dh}{dt} + (h^2 + 2V^*hv + H^*v^2) \right] dx
\]

\[
= \int_0^L \left[ (gh + V^*v) \frac{dh}{dt} + (V^*h + H^*v) \frac{dh}{dt} + (h^2 + 2V^*hv + H^*v^2) \right] dx.
\]

Integrating by parts and using the equilibrium conditions (8) we get:

\[
\frac{d\delta_2}{dt} = B + \int_0^L \left[ \frac{kV^2}{H^2} - \frac{h}{H^*} - \frac{V^*}{V^*} \right] dx
\]

with

\[
B = \int_0^L \left[ gV^2 h^2 + (V^*h + H^*v)h + 2V^*hv + H^*v^2 \right] dx.
\]

In this case, we observe that the integral term can be positive (just take \( h > 0 \) and \( v = 0 \)) and consequently that \( \delta_2 \) may not be a Lyapunov function for our system. Here the interesting point is that, by ignoring the cross-term \( 2V^*hv \) in \( \delta_2 \), we get a valid candidate for the Lyapunov function as we have shown above.

Remark 2 (The Special Case of Isothermal Euler Equations). The special case of an isothermal gas flow is sometimes considered for modeling pipeline dynamics and control (e.g. [5–8]). In that case the Euler equations are said to be isothermal, the pressure is proportional to the density \( p(H) = \kappa H \) and the sound velocity is constant \( c = \sqrt{\kappa} \). It then follows that \( c^* = 0 \) and therefore that the matrix \( N(x) \) is not positive definite. In that case, the Lyapunov function introduced in this paper is not applicable and the use of Lyapunov functions expressed in Riemann coordinates with exponential weights as in [5–8] is not redundant.

4. Exponential stability of the steady-state of the nonlinear system

In this section, we shall now show that conditions (a) and (b) are also sufficient to guarantee the exponential stability in \( H^2 \) of the steady state \( H^*(x), V^*(x) \) of the nonlinear system (1). For this purpose it is first useful to transform the linearized system (11) into Riemann coordinates. Under the subcritical condition (9), the matrix \( \Lambda(x) \) has two real distinct eigenvalues:

\[
\lambda^+(x) = V^* + c(H^*) > 0, \quad \lambda^-(x) = V^* - c(H^*) < 0.
\]

Therefore, \( \Lambda(x) \) is diagonalizable with the invertible matrix \( S(x) \) defined as

\[
S(x) = \frac{1}{2} \begin{pmatrix} H^*(x) & -H^*(x) \\ c(H^*) & c(H^*) \end{pmatrix}
\]

such that

\[
\Lambda(x) = S^{-1}(x) \Lambda(x) S(x) \quad \text{with} \quad \Lambda(x) = \begin{pmatrix} \lambda^+(x) & 0 \\ 0 & \lambda^-(x) \end{pmatrix}.
\]

Then the vector of Riemann coordinates

\[
R = \begin{pmatrix} R^+ \\ R^- \end{pmatrix}
\]

is defined as

\[
R(t, x) = S^{-1}(x) Y(t, x)
\]

so that

\[
R^+ = v + \frac{c(H^*)}{H^*}, \quad R^- = v - \frac{c(H^*)}{H^*}.
\]

In these coordinates, the linearized system (11) is written

\[
\dot{R}_L + \Lambda(x) R + C(x) R = 0
\]

with

\[
C(x) = S^{-1}(x) \left[ A(x) S(x) + B(x) S(x) \right] = \begin{pmatrix} \gamma_1(x) & \delta_1(x) \\ \gamma_2(x) & \delta_2(x) \end{pmatrix}.
\]

Moreover, the boundary conditions (13) are written as follows in the Riemann coordinates (54):

\[
\begin{pmatrix} R^+(t, 0) \\ R^-(t, L) \end{pmatrix} = \begin{pmatrix} k_0 b_H \delta_H^* + c(H^*_0) \\ k_1 b_H \delta_H^* \end{pmatrix}, \quad \begin{pmatrix} b_H \delta_H^* \end{pmatrix} \begin{pmatrix} \delta_H^* \end{pmatrix} = c(H^*_0) - c(H^*_1) - c(H^*_2).
\]

The linear system (55), (57) is the linearization, around a steady-state \( H^*(x), V^*(x) \) of our initial nonlinear system (1)

\[
H_t + (HV)_x = 0,
\]

\[
V_t + \frac{c^2(H)}{H} H_x + V_x + f(H, V) = 0,
\]

with nonlinear boundary conditions

\[
V(t, 0) = \mathcal{H}_0(H(t, 0)), \quad V(t, L) = \mathcal{H}_1(H(t, L))
\]

s.t.

\[
b_0 = -\mathcal{H}_0^*(H(0)), \quad b_1 = \mathcal{H}_1^*(H(L)).
\]

Sufficient conditions for the exponential stability of the steady-state of the nonlinear system (59), (60) are then given in the following Theorem (\( \mathcal{D}_n \) denotes the set of \( n \times n \) real diagonal matrices with strictly positive diagonal elements).

Theorem 1. The steady state \( H^*(x), V^*(x) \) of the system (59), (60) is exponentially stable for the \( H^2 \)-norm if
We omit the argument is positive definite for all \( x \in [0, L] \),

(ii) the following inequalities are satisfied:

\[
\begin{align*}
 k_0^2 & \leq \frac{\lambda^-(0)}{\lambda^+(0)} \\
 k_1^2 & \leq \frac{\lambda^-(L)}{\lambda^+(L)}.
\end{align*}
\]

Proof. This theorem is a special case of Theorems 6.6 and 6.10 in [4].

Now, in the next two lemmas, we shall show that conditions (i) and (ii) are fulfilled with the matrix

\[
Q(x) = S^T(x)D(x)S(x) = \frac{1}{2} \begin{pmatrix} H^+(x) & 0 \\ 0 & H^-(x) \end{pmatrix}
\]

if the matrix \( N(x) \) is positive definite and if the linearized boundary conditions satisfy inequalities (60) (62) and (29), (30).

Lemma 1. If the matrix \( N(x) \) is positive definite for all \( x \in [0, L] \), then the matrix

\[-(Q(x)A(x)) + Q(x)(A(x) + C^T(x))Q(x)\]

is positive definite for all \( x \in [0, L] \).

Proof. We omit the argument \( x \) for notational simplicity. Using the definition (62) of \( Q \), the definition (56) of \( C \), the equality (51) \( SA = AS \), the equality (21) \( M = DA = A^TD \) and the definition (23) of \( N \), we have:

\[
- (Q(x)A(x) + Q(x)(A(x) + C^T(x))Q(x) = -(S^TDS)A(x) + S^T(DAS + BS) + (S^TBS + S^TAS + BS) \\
+ (S^TBS + S^TAS + BS)DS \\
= -(S^TDS)A(x) + S^T(DAS + BS) + (S^TBS + S^TAS + BS)DS \\
= -(S^TDS)A(x) + S^T(DAS + BS) + (S^TBS + S^TAS + BS)DS \\
= -(S^TDS)A(x) + S^T(DAS + BS) + (S^TBS + S^TAS + BS)DS \\
= S^T(M(x) + DB + B^TD)S \\
= S^TNS.
\]

Hence, since \( S \) is invertible, the positive definiteness of \( N \) implies the positive definiteness of \(- (Q(x)A(x) + Q(x)(A(x) + C^T(x))Q(x). \)

Moreover Lemma 2 states that conditions (29), (30) on \( b_0 \) and \( b_1 \) are equivalent to conditions (61) on \( k_0 \) and \( k_1 \).

Lemma 2. The real numbers \( b_0 \) and \( b_1 \) satisfy inequalities (29), (30) if and only if the real numbers \( k_0 \) and \( k_1 \) satisfy inequalities (61).

Proof. Let us denote \( c_0 = c(H^+_0) \). Then we have

\[
\begin{align*}
 k_0^2 & \leq \frac{\lambda^-(0)}{\lambda^+(0)} \\
 k_0 & \leq \frac{\lambda^-(0)}{\lambda^+(0)}
\end{align*}
\]

by (49) and (58)

\[
\begin{align*}
 & \leq (b_0H^+_0 - c_0)^2(c_0 + V^+_0) \leq (b_0H^+_0 + c_0)^2(c_0 - V^+_0) \\
 & \leq (b_0H^+_0)^2 + 2b_0c_0H^+_0 + c_0^2(c_0 + V^+_0) \\
 & \leq (b_0H^+_0)^2 + 2b_0c_0H^+_0 + c_0^2(c_0 - V^-_0) \\
 & \leq b_0^2H^+_0V^+_0 + 2b_0c_0H^+_0 + c_0^2V^-_0 \leq 0
\end{align*}
\]

The proof is similar for \( b_1 \).

