Abstract. This paper is concerned with boundary dissipative conditions that guarantee the exponential stability of classical solutions of one-dimensional quasi-linear hyperbolic systems. We present a comprehensive review of the results that are available in the literature. The main result of the paper is then to supplement these previous results by showing how a new Lyapunov stability approach can be used for the analysis of boundary conditions that are known to be dissipative for the $C^1$-norm.

Key words. hyperbolic systems, Lyapunov stability

1. Introduction. We are concerned with one-dimensional quasi-linear strictly hyperbolic systems of the form

\begin{equation}
\frac{du}{dt} + A(u)u_x = 0, \quad x \in [0,1], \quad t \in [0,\infty),
\end{equation}

where $u : [0,\infty) \times [0,1] \to \mathbb{R}^n$ and $A : \mathbb{R}^n \to \mathcal{M}_{n,n}(\mathbb{R})$, $\mathcal{M}_{n,n}(\mathbb{R})$ denoting, as usual, the set of $n \times n$ real matrices.

Since the system is strictly hyperbolic, the matrix $A(0)$ has, by definition, $n$ real eigenvalues denoted $\Lambda_1, \ldots, \Lambda_n$ satisfying

\begin{equation}
\Lambda_i \neq \Lambda_j, \quad \forall i,j \in \{1, \ldots, n\} \text{ such that } i \neq j.
\end{equation}

We assume that none of these eigenvalues is zero. Then, possibly after a suitable linear change of variables, we may assume that there exists $m \in \{0,1,\ldots,n\}$ such that

$\Lambda_i > 0, \forall i \in \{1, \ldots, m\}$, \hspace{0.5cm} $\Lambda_i < 0, \forall i \in \{m+1, \ldots, n\}$,

and that $A(0)$ is a diagonal matrix:

$A(0) \triangleq \text{diag}\{\Lambda_1, \Lambda_2, \ldots, \Lambda_n\}$.

Our concern is to analyse the asymptotic convergence of the classical solutions of the system (1.1) under a boundary condition of the form

\begin{equation}
\begin{pmatrix}
    u_+(t,0) \\
    u_-(t,0)
\end{pmatrix} = G \begin{pmatrix}
    u_+(t,0) \\
    u_-(t,0)
\end{pmatrix}, \quad t \in [0,\infty),
\end{equation}

where the map $G : \mathbb{R}^n \to \mathbb{R}^n$ vanishes at $0$, while $u_+ \in \mathbb{R}^m$, $u_- \in \mathbb{R}^{n-m}$ are defined by requiring that $u \triangleq (u^T_+, u^T_-)^T$. The challenge is to give explicit conditions on the
map $G$ such that the boundary condition (1.3) is exponentially dissipative, i.e. implies that the equilibrium solution $u \equiv 0$ of system (1.1) with the boundary condition (1.3) is exponentially stable.

In order to state our results, we first introduce the functions $\rho_p : M_{n,n}(\mathbb{R}) \to \mathbb{R}$ defined by

$$
\rho_p(M) \triangleq \inf \left\{ \| \Delta M \Delta^{-1} \|_p, \; \Delta \in D_n^+ \right\}, \; 1 \leq p \leq \infty,
$$

where $D_n^+$ denotes the set of diagonal $n \times n$ real matrices with strictly positive diagonal entries and with

$$
\| \xi \|_p \triangleq \left[ \sum_{i=1}^{n} |\xi_i|^p \right]^{\frac{1}{p}}, \; \| \xi \|_\infty \triangleq \max \{ |\xi_i| : i \in \{1, \ldots, n\} \}, \; \forall \xi \triangleq (\xi_1, \ldots, \xi_n)^T \in \mathbb{R}^n,
$$

and

$$
\| M \|_p \triangleq \max_{\| \xi \|_p = 1} \| M \xi \|_p \forall M \in M_{n,n}(\mathbb{R}).
$$

In this paper, our main result (Section 3) is a new proof, using a Lyapunov function approach, that a sufficient condition for the exponential stability of the steady-state $u \equiv 0$ for the $C^1$-norm is to have a map $G$ such that:

$$
\rho_\infty(G'(0)) < 1,
$$

where $G'$ denotes the Jacobian matrix of the map $G$.

This new proof supplements the previous results which are already available in the literature and are reviewed in Section 2.

2. Literature review. The well-posedness of the Cauchy problem associated to nonlinear hyperbolic systems of the kind (1.1) and (1.3) has been studied in 1985 by Li and Yu in the framework of $C^1$-solutions. For such systems, the issue of finding sufficient dissipative boundary conditions has been addressed in the literature for more than thirty years. To our knowledge, first results were published by Slemrod in 1983 and by Greenberg and Li in 1984 for the special case of systems of size $n = 2$. A generalization to systems of size $n$ was then progressively elaborated by the Ta-Tsien Li school, in particular by Qin in 1985 and by Zhao in 1986. All these contributions deal with the particular case of ‘local’ boundary conditions having the specific form

$$
(2.1) \quad u_+(t, 0) = G_0(u_-(t, 0)), \quad u_-(t, 1) = G_1(u_+(t, 1)).
$$

With these boundary conditions, the analysis can be based on the method of characteristics which can exploit an explicit computation of the ‘reflection’ of the solutions at the boundaries along the characteristic curves. This has given rise to the sufficient condition

$$
(2.2) \quad \rho_\infty \left( \begin{pmatrix} 0 & G_0'(0) \\ G_1'(0) & 0 \end{pmatrix} \right) < 1,
$$

for the dissipativity of the boundary conditions (2.1) for the $C^1$-norm. This result is given for instance by Li [23, Theorem 1.3, page 173] in his seminal book of 1994 on the stability of the classical solutions of quasi-linear hyperbolic systems. Finally, by using an appropriate dummy doubling of the system size, de Halleux et al. [12, Theorem 4] have shown in 2003 how the general dissipative boundary condition $\rho_\infty(G'(0)) < 1$ can be established for systems with the general ‘non local’ boundary condition (1.3). This
dummy doubling has also been used by Li et al. [24] to prove the well-posedness of the Cauchy problem associated to (1.1) and (1.3) still in the framework of \( C^1 \)-solutions.

Another approach of the analysis of dissipative boundary conditions is based on the use of Lyapunov functions. The first attempts were in using entropies as Lyapunov functions, as done for instance by Coron et al. [8] in 1999 or by Leugering and Schmidt [22] in 2002. The drawback of this approach was however that the time derivatives of such entropy-based Lyapunov functions are necessarily only semi-definite negative. Hence one would like to conclude with the LaSalle invariant set principle. However, this principle requires the precompactness of the trajectories, a property which is difficult to get in the case of nonlinear partial differential equations.

In order to overcome this difficulty, Coron et al. [9] in 2007 have proposed, for systems of size \( n = 2 \), a strict Lyapunov function whose time derivative is strictly negative definite when

\[
\rho_\infty(G'(0)) < 1.
\]

The advantage is that the proof is less elaborated than the one using the method of characteristics because it uses more direct computations. Furthermore, another advantage of a Lyapunov analysis is to directly induce robustness properties with respect to small uncertainties and disturbances. Then, in the paper [7], Coron et al. have generalized in 2008 this Lyapunov approach to general nonlinear hyperbolic systems of the kind (1.1) and (1.3). In particular, they emphasize a new weaker dissipative boundary condition which is formulated as follows:

A sufficient condition for the exponential stability of the steady-state \( u \equiv 0 \) for the \( H^2 \)-norm is to have a map \( G \) such that:

\[
\rho_2(G'(0)) < 1.
\]

Moreover, it is also shown in the same paper that \( \rho_2(G'(0)) \leq \rho_\infty(G'(0)) \) \( \forall n \geq 1 \) and that this inequality can be strict if \( n \geq 2 \). However, it has been proved in [10] that \( \rho_2(G'(0)) < 1 \) is not a sufficient condition for the exponential stability of the steady-state \( u \equiv 0 \) for the \( C^1 \)-norm.

Various recent contributions and extensions of the previous results are also worth to be mentioned.

- Dos Santos and Maschke [37] in 2009 have given an Hamiltonian perspective to the stabilization of systems of two conservation laws.
- Using the Lyapunov approach, Castillo et al. [5] in 2013 give sufficient conditions for the existence of exponentially stable observers in the case where all eigenvalues \( \lambda_i \) are positive.
- Perrollaz and Rosier [29] have shown in 2013 that there exist boundary dissipative conditions for systems of size \( n = 2 \) that achieve finite-time stabilization.
- Using an approach via time delay equations, Coron and Nguyen [10] have shown in 2014 that \( \rho_p(G'(0)) < 1 \) implies the exponential stability in the Sobolev norm \( W^{2,p} \).

For so-called inhomogeneous quasi-linear hyperbolic systems (i.e. with additional zero-order terms):

\[
u_t + A(u)u_x + B(u) = 0,
\]
the analysis of dissipative boundary conditions is much more intricate and only very partial results are known.

- Using the method of characteristics, Prieur et al. [33] in 2008 have shown that the stability condition (2.2) holds for inhomogeneous systems when \( \|B(u)\| \) is small enough. Prieur [30] has shown in 2009 how this result can be extended to deal with differential or integral boundary errors.


- For inhomogeneous systems of size \( n = 2 \) with \( m = 1 \), Coron et al. [11] have shown in 2012 the existence of full-state feedback law which achieve exponential stability for the \( H^2 \)-norm. The proof uses a backstepping transformation (see [21]) to find new variables for which a strict Lyapunov function can be constructed.

**Linear hyperbolic systems.** A linear hyperbolic system,

\[
\begin{align*}
\frac{\partial u}{\partial t} + A\frac{\partial u}{\partial x} &= 0, & x &\in [0, 1], & t &\in [0, +\infty),
\end{align*}
\]

with linear boundary conditions

\[
\begin{bmatrix}
 u_+(t, 0) \\
 u_-(t, 1)
\end{bmatrix} = K \begin{bmatrix}
 u_+(t, 1) \\
 u_-(t, 0)
\end{bmatrix}, & t &\in [0, +\infty),
\]

is a special case of (1.1) and (1.3) where the matrix \( A \in \mathcal{D}_n(\mathbb{R}) \) with non-zero distinct diagonal entries, \( \mathcal{D}_n(\mathbb{R}) \) denoting the set of \( n \times n \) diagonal real matrices, and the matrix \( K \in \mathcal{M}_{n,n}(\mathbb{R}) \). As it can be expected, the analysis of dissipative boundary conditions is both simpler and more comprehensive for linear than for quasi-linear systems.

The linear hyperbolic system (2.5)-(2.6) is equivalent to a system of interconnected scalar time-delays \( \tau_i \triangleq \frac{1}{\Lambda_i} \). For such systems, it follows from a theorem by Silkowski [38] (quoted in [19, Chapter 9, Theorem 6.1], see also [28]), which relies on the *Kronecker density theorem* (e.g. [5]), that the boundary conditions (2.6) are, for any \( L^p \)-norm, robustly dissipative with respect to arbitrary small perturbations on the \( \Lambda_i \)'s if and only if

\[
\bar{\rho}(K) \triangleq \max\{\rho(\text{diag}(e^{i\theta_1}, \ldots, e^{i\theta_n})K; (\theta_1, \ldots, \theta_n)^T \in \mathbb{R}^n) < 1,
\]

where \( i \in \mathbb{C} \) is such that \( i^2 = -1 \) and \( \rho(M) \) is the spectral radius of \( M \in \mathcal{M}_{n,n}(\mathbb{C}) \). It is shown in [3, 7] that \( \bar{\rho}(K) = \rho_2(K) \) if there exists a permutation matrix \( P \) such that the matrix \( K = PKP^{-1} \) is a block diagonal matrix \( K = \text{diag}\{K_1, K_2, \ldots, K_p\} \) where each block \( K_i \) is a real \( n_i \times n_i \) matrix with \( n_i \in \{1, 2, 3, 4, 5\} \). Moreover, when \( n > 5 \), there always exist \( K \) such that \( \bar{\rho}(K) < \rho_2(K) \).

The exponential stability of non-uniform inhomogeneous linear hyperbolic systems has also received a lot of attention in the literature for a long time and can be
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traced back up to Rauch and Taylor [35, Section 4] in 1974 or Russell [36, Section 3] in 1978. These systems are written as

\[ u_t + A(x)u_x + B(x)u = 0, \]

where \( A(x) \in D_n(\mathbb{R}) \) and \( B(x) \in \mathcal{M}_{n,n}(\mathbb{R}), \forall x \in [0,1]. \) Using the Lyapunov approach (as for instance Xu and Sallet [43] in 2002 or Diagne et al. [13] in 2011 for the uniform case), it can be shown that the equilibrium \( u \equiv 0 \) is exponentially stable for the \( L_2 \)-norm if there exist a positive diagonal matrix \( P \) such that:

(a) \( P \begin{pmatrix} |A_+ (1)| & 0 \\ 0 & |A_- (0)| \end{pmatrix} - K^T P \begin{pmatrix} |A_+ (0)| & 0 \\ 0 & |A_- (1)| \end{pmatrix} K \) is positive definite;

(b) \( PB(x) + B^T(x)P - P \partial_x |A(x)| \) is semi positive definite for all \( x \in [0,1], \)

where, for every \( x \in [0,1], A_+ (x) \in D_m(\mathbb{R}) \) and \( A_- (x) \in D_{n-m}(\mathbb{R}) \) are defined by requiring

\( A(x) = \begin{pmatrix} A_+ (x) & 0 \\ 0 & A_- (x) \end{pmatrix}. \)

Obviously conditions (a) and (b) are rather conservative. Bastin and Coron [415] have given in 2011 a less restrictive sufficient condition for inhomogeneous linear systems of size \( n = 2 \) with \( m = 1 \) under the form of the existence of the solution of an associated ordinary differential equation.

Other recent significant contributions are worth to be mentioned.

- Tchouso et al. [41] have shown in 2009 how the Lyapunov approach can be extended to inhomogeneous linear hyperbolic systems of higher spatial dimension.
- Litrico and Fromion [26] in 2009 use a frequency domain method for the boundary control of homogeneous systems of size \( n = 2. \) They show that the transfer functions belong to the Callier-Desoer algebra, which opens the way to necessary and sufficient conditions for closed loop stability and the use of a Nyquist type test.
- Amin et al. [2] and Prieur et al. [31] have shown in 2012 how the Lyapunov approach can be extended to switched linear hyperbolic systems.
- Prieur and Mazenc [22] in 2012 have shown how time varying strict Lyapunov functions can be defined to get input-to-state stability for time varying inhomogeneous linear hyperbolic systems.
- Dick et al. [14] in 2012 address the feedback stabilization of quasi-linear hyperbolic systems with varying delays.
- The use of a backstepping method for boundary stabilization by full state feedback and observer design for non uniform inhomogeneous linear systems has also been extensively studied recently: Krstic et al. [20] in 2008 and Smyshlyaev et al. [40] in 2010 for unstable wave equations, Vazquez et al. [42] in 2011 for general systems of size \( n = 2 \) with \( m = 1, \) Di Meglio et al. [27] in 2012 for systems with \( m = n - 1 \) and a single controller at the right boundary, and Aamo [1] in 2013 for the rejection of differential boundary disturbances.
- Using a frequency domain approach, Bastin et al. [4] in 2014 give necessary and sufficient conditions for the stability of linear density-flow systems under proportional-integral control.
Our contribution in this paper fills a small gap in this panorama of the analysis of boundary dissipative conditions for one-dimensional linear and quasi-linear hyperbolic systems. We show how the Lyapunov approach can be used to establish the dissipativity of the boundary condition \( \rho_\infty(G'(0)) < 1 \).

3. Main result. We consider hyperbolic systems of the form

\[
(3.1) \quad u_t + A(u)u_x = 0, \quad x \in [0, 1], \quad t \in [0, +\infty),
\]

under a boundary condition of the form

\[
(3.2) \quad \begin{pmatrix} u_+(t, 0) \\ u_-(t, 1) \end{pmatrix} = G \begin{pmatrix} u_+(t, 1) \\ u_-(t, 0) \end{pmatrix}, \quad t \in [0, +\infty).
\]

For \( \sigma \), let \( B_\sigma \) be the open ball with radius \( \sigma \) in \( \mathbb{R}^n \) for the norm

\[
(3.3) \quad |u|_0 \triangleq \max\{|u_j|; j \in \{1, \ldots, n\}\}, \quad u = (u_1, \ldots, u_n)^T \in \mathbb{R}^n.
\]

We assume that, for some \( \sigma \in (0, +\infty) \), \( A : B_\sigma \to \mathcal{M}_{n,n}(\mathbb{R}) \) and \( G : B_\sigma \to \mathbb{R}^n \) are of class \( C^1 \). We define \( A_+(u) \in \mathcal{M}_{n,n}(\mathbb{R}) \), \( A_-(u) \in \mathcal{M}_{(n-m),n}(\mathbb{R}) \), \( G_+(u) \in \mathbb{R}^m \) and \( G_-(u) \in \mathbb{R}^{n-m} \) by requiring

\[
(3.4) \quad u(0, x) = u^0(x), \quad x \in (0, 1)
\]

which satisfies the compatibility conditions

\[
(3.5) \quad \begin{pmatrix} u_+^0(0) \\ u_-^0(1) \end{pmatrix} = G \begin{pmatrix} u_+^0(1) \\ u_-^0(0) \end{pmatrix},
\]

\[
(3.6) \quad A_+(u^0(0))u_+^0(0) = \left[ G'_{u_+} \begin{pmatrix} u_+^0(1) \\ u_-^0(0) \end{pmatrix} \right] A_+(u^0(1))u_+^0(1) \\
+ \left[ G'_{u_-} \begin{pmatrix} u_+^0(1) \\ u_-^0(0) \end{pmatrix} \right] A_-(u^0(0))u_+^0(0),
\]

\[
(3.7) \quad A_-(u^0(1))u_-^0(1) = \left[ G'_{u_+} \begin{pmatrix} u_+^0(1) \\ u_-^0(0) \end{pmatrix} \right] A_+(u^0(1))u_-^0(1) \\
+ \left[ G'_{u_-} \begin{pmatrix} u_+^0(1) \\ u_-^0(0) \end{pmatrix} \right] A_-(u^0(0))u_-^0(0).
\]

Let us now define some norms which will be useful to state and prove our results. For \( u \in C^0([0, 1]; \mathbb{R}^n) \) (resp. \( C^0([0, 1] \times [T_1, T_2]; \mathbb{R}^n) \), one defines

\[
(3.8) \quad |u|_0 \triangleq \max\{|u(x)|_0; x \in [0, 1]\},
\]

\[
(3.9) \quad (\text{resp. } |u|_0 \triangleq \max\{|u(t,x)|_0; t \in [T_1, T_2], x \in [0, 1]\}).
\]
For \( u \in C^1([0,1]; \mathbb{R}^n) \) (resp. in \( C^1([T_1, T_2] \times [0,1]; \mathbb{R}^n) \)), we define
\[
|u|_1 \triangleq |u|_0 + |u'|_0, \\
(3.10)
\]
(resp. \( |u|_1 \triangleq |u|_0 + |\partial_t u|_0 + |\partial_x u|_0 \)).
\[
(3.11)
\]

Let us first recall the following theorem, due to Li and Yu \[25\], on the well-posedness of the Cauchy problem \((3.1)–(3.2)–(3.4)\).

**Theorem 3.1.** Let \( T > 0 \). There exists \( C_1 > 0 \) and \( \varepsilon_1 > 0 \) such that, for every \( u^0 \in C^1([0,1]; \mathbb{R}^n) \) satisfying the compatibility conditions \((3.5)\) to \((3.7)\) and such that
\[
|u^0|_1 \leq \varepsilon_1, \\
(3.12)
\]
there exists one and only one solution on \([0,T] \times [0,1]\) to the Cauchy problem \((3.1)–(3.2)–(3.4)\). Moreover, this solution satisfies
\[
|u|_1 \leq C_1 |u^0|_1. \\
(3.13)
\]
(In fact, \[25\] deals with the case of local boundary conditions \((2.1)\); however the general case follows from this particular case by using the dummy doubling of the system size introduced in de Halleux et al.\[12\], already mentioned in Section 2: see \[24\].)

Our definition of exponential stability for the \( C^1 \)-norm is as follows.

**Definition 3.2.** The steady state \( x \in [0,1] \mapsto 0 \in \mathbb{R}^n \) of the system \((3.1)–(3.2)\) is exponentially stable for the \( C^1 \)-norm if there exist \( \varepsilon > 0, \nu > 0 \) and \( C > 0 \) such that, for every \( u^0 \) such that \( |u^0|_1 < \varepsilon \) and satisfying the compatibility conditions \((3.5)\) to \((3.7)\), the Cauchy problem \((3.1)–(3.2)–(3.4)\) has a unique \( C^1 \)-solution which satisfies
\[
|u(t,.)|_1 \leq Ce^{-\nu t} |u^0|_1, \quad \forall t \in [0, +\infty).
(3.14)
\]

The main goal of this article is to give a new proof of the following theorem due to Qin\[34\] and Zhao\[44\] (see also Li\[23\], Theorem 1.3, page 173] and de Halleux et al.\[12\], Theorem 4) .

**Theorem 3.3.** If
\[
\rho_{\infty}(G'(0)) < 1,
(3.14)
\]
the steady state \( x \in [0,1] \mapsto 0 \in \mathbb{R}^n \) of the system \((3.1)–(3.2)\) is exponentially stable for the \( C^1 \)-norm.

In the remaining part of this section we give our proof of Theorem 3.3. This proof is divided in two parts:
- In Section 3.1 we study the case where all the \( \Lambda_i \) are positive, i.e. the case where \( m = n \).
- In Section 3.2 we explain how to modify the Lyapunov function introduced in Section 3.1 in order to treat the general case.
3.1. Proof of Theorem [3.3] in the special case where \( m = n \). For the clarity of the demonstration, we shall first prove the theorem in the special case where \( m = n \), which means the matrix \( A(0) \) is the positive diagonal matrix \( \text{diag}\{\Lambda_1, \ldots, \Lambda_n\} \) with \( \Lambda_i > 0 \) \( \forall i = 1, \ldots, n \). In that case, the boundary condition (3.2) and the compatibility conditions (3.5) to (3.7) are simply rewritten

\[
(u(t, 0) = G(u(t, 1)), \quad u'(0) = G(u^o(1)), \quad A(u^o(0))u_x^o(0) = G'(u^o(1))A(u^o(1))u_x^o(1).
\]

As already mentioned above, our proof is based on a Lyapunov approach. In order to define an appropriate Lyapunov function for the analysis, we need the following technical classical lemma, for which our assumption (1.2) is crucial and which also holds if \( m \neq n \).

**Lemma 3.4.** Let \( D(u) \) be the diagonal matrix whose diagonal entries are the eigenvalues \( \lambda_i(u), i = 1, \ldots, n \), of the matrix \( A(u) \). There exist a positive real number \( \eta < \sigma \) and a map \( M : B_\eta \rightarrow M_{n,n}(\mathbb{R}) \) of class \( C^1 \) such that

\[
M(u)A(u) = D(u)M(u), \quad \forall u \in B_\eta, \quad M(0) = Id_n,
\]

where \( Id_n \) is the identity matrix of \( M_{n,n}(\mathbb{R}) \).

We use this slight abuse of (useful) notation (3.21) at other places in this article.

Throughout all this section, \( u : [0, T] \rightarrow \mathbb{R}^n \) denotes a \( C^1 \)-solution of (3.1) and (3.15). We define \( W_1 : [0, T] \rightarrow \mathbb{R} \) and \( W_2 : [0, T] \rightarrow \mathbb{R} \) by

\[
W_1(t) \triangleq W_1(u(t)), \quad W_2(t) \triangleq W_2(u(t)), \quad \forall t \in [0, T].
\]

In (3.22) and in the following, for \( t \in [0, T] \), \( u(t) : [0, 1] \rightarrow \mathbb{R}^n \) is defined by

\[
u(t)(x) \triangleq u(t, x), \quad \forall x \in [0, 1].
\]

Of course \( W_1(t) \) and \( W_2(t) \) depend also on \( u \). So we should in fact write \( W_1'u(t) \) and \( W_2'u(t) \). But, to simplify the notations, we omit this dependence on \( u \) in the notations \( W_1(t) \) and \( W_2(t) \).
The proof of the theorem will be based on two preliminary lemmas. These lemmas provide estimates on $dW_1/dt$ and $dW_2/dt$.

**Lemma 3.5.** If $\rho_{\infty}(G'(0)) < 1$, there exist $p_i > 0$ for $i = 1, \ldots, n$, positive real constants $\alpha$, $\beta_1$ and $\delta_1$ such that, for every $\mu \in (0, \delta_1)$, for every $p \in (1/\delta_1, +\infty)$, for every $C^1$-solution $u : [0, T] \times [0, 1] \to \mathbb{R}^n$ of (3.1) and (3.15) satisfying $u|_0 < \delta_1$, we have

$$
dW_1 \equiv \frac{dW_1}{dt}(t) \leq \left(-\mu \alpha + \beta_1 \|u_x(t)\|_0\right)W_1(t), \forall t \in [0, T].
$$

**Proof.** Let $u : [0, T] \times [0, 1] \to \mathbb{R}^n$ be a $C^1$-solution of (3.1) and (3.15). The time derivative of $W_1$ is:

$$
\frac{dW_1}{dt} = \frac{1}{2p} W_1^{1-2p} \int_0^1 \sum_{i=1}^n 2p \int_0^T \left(\sum_{j=1}^n m_{ij}(u)u_j\right)^{2p-1}

\left[\sum_{j=1}^n m_{ij}(u)\partial_t u_j + \sum_{j=1}^n (\partial_t m_{ij}(u))u_j\right] e^{-2p\mu x} dx.
$$

Using (3.1), the term between brackets can be written as

$$
\sum_{j=1}^n m_{ij}(u)\partial_t u_j = -\sum_{j=1}^n m_{ij}(u)\left(\sum_{k=1}^n a_{jk}(u)\partial_x u_k\right)

= -\sum_{k=1}^n \left(\sum_{j=1}^n m_{ij}(u)a_{jk}(u)\partial_x u_k\right),
$$

where $a_{jk}(u)$ is the $(j, k)$-th entry of the matrix $A(u)$. Now, from (3.17), we have

$$
\sum_{j=1}^n m_{ij}(u)a_{jk}(u)\partial_x u_k = \sum_{j=1}^n d_{ij}(u)m_{jk}(u)\partial_x u_k = \lambda_i(u)m_{ik}(u)\partial_x u_k,
$$

where $d_{ij}(u)$ is the $(i, j)$-th entry of the matrix $D(u)$. From (3.26) and (3.27), we have

$$
\sum_{j=1}^n m_{ij}(u)\partial_t u_j = -\lambda_i(u)\sum_{k=1}^n m_{ik}(u)\partial_x u_k = -\lambda_i(u)\sum_{j=1}^n m_{ij}(u)\partial_x u_j.
$$

By substituting this expression for the term between brackets in (3.25), we get

$$
\frac{dW_1}{dt} = \frac{1}{2p} W_1^{1-2p} \int_0^1 \sum_{i=1}^n 2p \int_0^T \left(\sum_{j=1}^n m_{ij}(u)u_j\right)^{2p-1}

\left(-\lambda_i(u)\sum_{j=1}^n m_{ij}(u)\partial_x u_j + \left(\sum_{j=1}^n (\partial_t m_{ij}(u))u_j\right)\right) e^{-2p\mu x} dx,
$$

where

$$\lambda_i(u) = \rho_{\infty}(u) \left(\int_0^t \sum_{j=1}^n d_{ij}(u)m_{jk}(u)\partial_x u_k\right), \forall i = 1, \ldots, n.$$
which leads to

\[
\frac{dW_t}{dt} = \frac{1}{2p} W_t^{1-2p} \left[ \int_0^1 - \sum_{i=1}^n p_i^p \lambda_i(u) \left( \sum_{j=1}^n m_{ij}(u)u_j \right) e^{-2p\mu x} dx \right. \\
+ \int_0^1 2p \sum_{i=1}^n p_i^p \left( \sum_{j=1}^n m_{ij}(u)u_j \right)^{2p-1} \left( \lambda_i(u) \sum_{j=1}^n (\partial_x m_{ij}(u))u_j \right) \\
\left. + \left( \sum_{j=1}^n (\partial_t m_{ij}(u))u_j \right) e^{-2p\mu x} dx \right].
\]

Using integrations by parts, we now get

\[
\frac{dW_t}{dt} = T_1 + T_2 + T_3,
\]

with

\[
T_1 = \frac{W_t^{1-2p}}{2p} \left[ - \sum_{i=1}^n p_i^p \lambda_i(u) \left( \sum_{j=1}^n m_{ij}(u)u_j e^{-\mu x} \right)^2 \right]_0^1,
\]

\[
T_2 = -\mu W_t^{1-2p} \int_0^1 \sum_{i=1}^n p_i^p \lambda_i(u) \left( \sum_{j=1}^n m_{ij}(u)u_j \right)^{2p} e^{-2p\mu x} dx,
\]

\[
T_3 = W_t^{1-2p} \int_0^1 \sum_{i=1}^n p_i^p \left( \sum_{j=1}^n m_{ij}(u)u_j \right)^{2p-1} \left( \lambda_i(u) \sum_{j=1}^n (\partial_x m_{ij}(u))u_j \right) \\
+ \left( \sum_{j=1}^n (\partial_t m_{ij}(u))u_j \right) + \frac{1}{2p} \left( \sum_{j=1}^n m_{ij}(u)u_j \right) \frac{\partial \lambda_i}{\partial u} (u) \partial_x u \right) e^{-2p\mu x} dx.
\]

**Analysis of the first term** $T_1$. From (3.31), we have

\[
T_1 = - \frac{W_t^{1-2p}}{2p} \left[ \sum_{i=1}^n p_i^p \lambda_i(u(t, 1)) \left( \sum_{j=1}^n m_{ij}(u(t, 1))u_j(t, 1) e^{-\mu} \right)^{2p} \\
- \sum_{i=1}^n p_i^p \lambda_i(u(t, 0)) \left( \sum_{j=1}^n m_{ij}(u(t, 0))u_j(t, 0) \right)^{2p} \right].
\]

Let $K \triangleq G'(0)$. By the definition of $\rho_\infty$ (see (1.4)) and (3.14), there exist $\Delta_i > 0$, $i \in \{1, \ldots, n\}$, such that

\[
\theta \triangleq \sum_{j=1}^n |K_{ij}| \frac{\Delta_i}{\Delta_j} < 1.
\]

The parameters $p_i$ are selected such that

\[
p_i^p \Lambda_i = \Delta_i^{2p}, \quad i = 1, \ldots, n.
\]
We define \( \xi_i : [0, T] \to \mathbb{R}, i = 1, \ldots, n, \) by
\[
(3.37) \quad \xi_i(t) \triangleq \Delta_i u_i(t, 1), \quad \forall t \in [0, T].
\]

From (3.34), (3.36) and (3.37), we have
\[
(3.38) \quad T_1 = -W_1^{1-2p} \left[ \sum_{i=1}^{n} \frac{\lambda_i(u(t, 1))}{\Lambda_i} \left( \sum_{j=1}^{n} m_{ij}(u(t, 1)) \frac{\Delta_j \xi_j(t)}{\Delta_j} e^{-\mu} \right)^{2p} - \sum_{i=1}^{n} \frac{\lambda_i(u(0))}{\Lambda_i} \left( \sum_{j=1}^{n} m_{ij}(u(0)) \Delta_j u_j(t, 0) \right)^{2p} \right].
\]

Let \( t \in [0, T]. \) Without loss of generality, we may assume that
\[
(3.39) \quad \xi_1^2(t) = \max\{\xi_i^2(t), i = 1, \ldots, n\}.
\]

Let us denote by \( \delta \) and \( C \) various positive constants which may vary from place to place but are independent of \( t \in [0, T] \), \( u \) and \( p \in \mathbb{N}_+. \) From (3.35), (3.39), we have, for \( |u(t, 1)|_0 \leq \delta, \)
\[
(3.40) \quad \sum_{i=1}^{n} \frac{\lambda_i(u(t, 1))}{\Lambda_i} \left( \sum_{j=1}^{n} m_{ij}(u(t, 1)) \frac{\Delta_j \xi_j(t)}{\Delta_j} e^{-\mu} \right)^{2p} \geq \lambda_1(u(t, 1)) \left( \sum_{j=1}^{n} m_{1j}(u(t, 1)) \frac{\Delta_j \xi_j(t)}{\Delta_j} e^{-\mu} \right)^{2p} \geq e^{-2p\mu} (1 - C|\xi_1(t)|) (|\xi_1(t)| - C|\xi_1(t)|^2)^{2p} = e^{-2p\mu} (1 - C|\xi_1(t)|)^{2p+1} (\xi_1(t))^{2p}.
\]

From (3.15), (3.35), (3.37) and (3.39), we have, for \( |u(t, 0)|_0 \leq \delta, \)
\[
(3.41) \quad \sum_{i=1}^{n} \frac{\lambda_i(u(0))}{\Lambda_i} \left( \sum_{j=1}^{n} m_{ij}(u(0)) \Delta_j u_j(t, 0) \right)^{2p} \leq (1 + C|\xi_1(t)|) \left( \sum_{i=1}^{n} \left( C|\xi_1(t)|^2 + \sum_{j=1}^{n} |K_{ij}| \frac{\Delta_j |\xi_j(t)|}{\Lambda_j} \right)^{2p} \right) \leq n (1 + C|\xi_1(t)|) (\theta|\xi_1(t)| + C|\xi_1(t)|^2)^{2p}.
\]

From (3.35), (3.38), (3.39) and (3.41), there exists \( \delta_{11} \in (0, 1), \) independent of \( u, \) such that, for every \( \mu \in (0, \delta_{11}), \) for every \( p \in (1/\delta_{11}, +\infty) \cap \mathbb{N}_+ \) and for every \( u, \) we have
\[
(3.42) \quad T_1(t) \leq 0 \text{ if } |u(t)|_0 < \delta_{11}.
\]

**Analysis of the second term** \( T_2. \) Let
\[
(3.43) \quad \alpha \triangleq \min(\Lambda_1, \ldots, \Lambda_n)/2.
\]
From (3.19) and (3.32) there is a $\delta_{12} \in (0, \eta]$ such that, for every $\mu \in (0, +\infty)$, for every $p \in \mathbb{N}_+$ and for every $u$,

$$
\beta \mu \in (3.18), \ (3.19) \text{ and } (3.45) \text{ one gets the existence of } \beta_1 > 0 \text{ such that, for every } \mu \in (0, +\infty), \text{ for every } p \in \mathbb{N}_+ \text{ and for every } u,$$

$$
T_3 \leq -\mu_1 W_1 \text{ if } |u_0| < \delta_{12}.
$$

**Analysis of the third term $T_3$.** Using (3.1) and (3.33), we have

$$
(\beta \mu \in (3.45) \quad T_3 = W_1^{1-2p} \int_0^1 \sum_{i=1}^n p_i \left( \sum_{j=1}^n m_{ij}(u) u_j \right)^{2p-1}
$$

$$
\left( \frac{1}{2p} \frac{\partial \lambda_i}{\partial u} u \right) \sum_{j=1}^n m_{ij}(u) u_j \right) + \sum_{j=1}^n \left( \frac{\partial m_{ij}}{\partial u} (A(u) + \lambda_i(u)) u_j \right) e^{-2\mu u x} dx.
$$

From (3.18), (3.19) and (3.45) one gets the existence of $\beta_1 > 0$ and $\delta_{12} > 0$ such that, for every $\mu \in (0, +\infty)$, for every $p \in \mathbb{N}_+$ and for every $u$,

$$
T_3 \leq \beta_1 |u_0| W_1 \text{ if } |u_0| < \delta_{12}.
$$

Let $\delta_1 \triangleq \min\{\delta_{11}, \delta_{12}, \delta_{13}\}$. From (3.30), (3.42), (3.44) and (3.46), we conclude that

$$
\frac{dW_1}{dt} \leq -\mu_1 W_1 + \beta_1 |u_0| W_1
$$

provided that $u$ is such that $|u_0| < \delta_1$, that $p \in (1/\delta_1, +\infty) \cap \mathbb{N}_+$ and that $\mu \in (0, \delta_1)$. This completes the proof of Lemma 3.5.

**Lemma 3.6.** Let $p_i$ ($i = 1, \ldots, n$) be given by (3.36). If $\rho_{\infty}(G'(0)) < 1$, there exist $\beta_2$ and $\delta_2$ such that, for every $\mu \in (0, \delta_2)$, for every $p \in (1/\delta_2, +\infty) \cap \mathbb{N}_+$ and for every $C^1$-solution $u : [0, T] \times [0, 1] \to \mathbb{R}^n$ of (3.1) and (3.15) such that $|u_0| < \delta_2$, we have, in the sense of distributions in $(0, T)$,

$$
\frac{dW_2}{dt} \leq \left( -\mu_1 + \beta_2 |u_0(t)| \right) W_2(t), \forall t \in [0, T],
$$

with $\alpha$ defined by (3.43).

**Proof.** We first deal with the case where $A$, $G$ and $u$ are of class $C^2$. By time differentiation of (3.1) and (3.15), we see that $u$ satisfy the following hyperbolic dynamics for $t \in [0, T]$ and $x \in [0, 1]$: $A'(u, u_t) \leq \frac{\partial a_{ij}(\nabla u) u_t}{\partial u} i \in \{1, \ldots, n\}, j \in \{1, \ldots, n\}$. Using (3.47)–(3.48), we see that the time derivative of $W_2$ is:

$$
\frac{dW_2}{dt} = \frac{1}{2p} W_2^{1-2p} \int_0^1 \sum_{i=1}^n \left( \sum_{j=1}^n m_{ij}(u) \partial_t u_j \right)^{2p-1}
$$

$$
\left( \sum_{j=1}^n m_{ij}(u) \partial_t u_j \right) + \sum_{j=1}^n \left( \partial_{t m_{ij}}(u) \partial_t u_j \right) e^{-2\mu_2 x} dx.
$$
Using integration by parts as in the proof of Lemma 3.5, we get

\[
\sum_{j=1}^{n} m_{ij}(u) \partial_t u_j = -\lambda_i(u) \sum_{j=1}^{n} m_{ij}(u) \partial_x (\partial_t u_j) + \sum_{j=1}^{n} m_{ij}(u) \left( \sum_{k=1}^{n} \tilde{a}_{jk}(u, u) \partial_t u_k \right),
\]

where \( \tilde{a}_{ij}(u, u) \) is the \((i, j)\)-th entry of the matrix \( \tilde{A}(u, u) \triangleq A'(u, u) A^{-1}(u) \). Then, by substituting this expression for the term between brackets in (3.49), we get

\[
\frac{dW_t}{dt} = \frac{1}{2p} W_t^{1-2p} \int_0^1 \sum_{i=1}^{n} \sum_{j=1}^{n} m_{ij}(u) \partial_x (\partial_t u_j) \left( -\lambda_i(u) \sum_{j=1}^{n} m_{ij}(u) \partial_x (\partial_t u_j) 
\right. \left. + \sum_{j=1}^{n} \left( \sum_{k=1}^{n} m_{ik}(u) \tilde{a}_{kj}(u, u) + \partial_t m_{ij}(u) \right) \partial_t u_j \right) e^{-2p\mu x} dx.
\]

Using integration by parts as in the proof of Lemma 3.5, we get

\[
\frac{dW_t}{dt} = U_1 + U_2 + U_3,
\]

with

\[
U_1 \triangleq \frac{W_t^{1-2p}}{2p} \left[ -\sum_{i=1}^{n} p_i \lambda_i(u) \left( \sum_{j=1}^{n} m_{ij}(u) (\partial_t u_j) e^{-\mu x} \right) e^{-2p\mu x} \right]_0^1,
\]

\[
U_2 \triangleq -\mu W_t^{1-2p} \int_0^1 \sum_{i=1}^{n} p_i \lambda_i(u) \left( \sum_{j=1}^{n} m_{ij}(u) (\partial_t u_j) \right) e^{-2p\mu x} dx,
\]

\[
U_3 \triangleq W_t^{1-2p} \int_0^1 \sum_{i=1}^{n} p_i \lambda_i(u) \left( \sum_{j=1}^{n} m_{ij}(u) (\partial_t u_j) \right) e^{-2p\mu x} dx.
\]

**Analysis of the first term \( U_1 \).** Using the boundary conditions (3.15) and (3.48), we have

\[
U_1 = -\frac{W_t^{1-2p}}{2p} \left[ \sum_{i=1}^{n} p_i \lambda_i(u(t, 1)) \left( \sum_{j=1}^{n} m_{ij}(u(t, 1)) (\partial_t u_j(t, 1)) \right) e^{-2p\mu x} 
\right. \left. - \sum_{i=1}^{n} p_i \lambda_i(G(u(t, 1))) \left( \sum_{j=1}^{n} m_{ij}(G(u(t, 1))) \frac{\partial G_j(u(t, 1))}{\partial u(t, 1)} \partial_t u(t, 1) \right) \right].
\]
Then, in a way similar to the analysis of $T_1$ in the proof of Lemma \[3.5\] we can show that, since $\rho_\infty(K) < 1$, there exists $\delta_1 \in (0,1)$, such that, for every $\mu \in (0, \delta_2)$, for every $p \in (1/\delta_2, +\infty) \cap \mathbb{N}_+$ and for every $u$, we have $U_1 \leq 0$ provided that $|u|_0 < \delta_2$.

**Analysis of the second term $U_2$.** Proceeding as in the proof of \[3.44\], we get the existence of $\delta_2 \in (0, \eta]$ such that, for every $\mu \in (0, +\infty)$, for every $p \in \mathbb{N}_+$ and for every $u$,

\[
U_2 \leq -\mu a W_2 \text{ if } |u|_0 < \delta_2,
\]

with $\alpha$ defined as in \[3.43\].

**Analysis of the third term $U_3$.** Proceeding as in the proof of \[3.46\], we get the existence of $\beta_2 > 0$ and $\delta_3 > 0$ such that, for every $\mu \in (0, +\infty)$, for every $p \in \mathbb{N}_+$ and for every $u$,

\[
U_3 \leq \beta_2 |u|_0 W_2 \text{ if } |u|_0 < \delta_3.
\]

From the analysis of $U_1$, $U_2$ and $U_3$, we conclude that, with $\delta \triangleq \min\{\delta_1, \delta_2, \delta_3\}$,

\[
\frac{dW_2}{dt} \leq -\mu a W_2 + \beta_2 |u|_0 W_2
\]

for all $u$ such that $|u|_0 < \delta$ provided that $\mu \in (0, \delta_2)$ and that $p \in (1/\delta_2, +\infty) \cap \mathbb{N}_+$.

The above estimates were obtained for $A$, $G$ and $u$ of class $C^2$. But their proofs show that they do not depend on the $C^2$-norm of $A$, $G$ and $u$. Hence, by density arguments, they remain valid with $A$, $G$ and $u$ only of class $C^1$. This completes the proof of Lemma \[3.6\] $\Box$

**Proof.** We now prove Theorem \[3.3\]. Let us choose $\mu \in \mathbb{R}$ such that

\[
0 < \mu < \min\{\delta_1, \delta_2\},
\]

where $\delta_1$ and $\delta_2$ are as in Lemma \[3.5\] and Lemma \[3.6\] respectively. Let us define two functionals $V_1 : C^1([0,1];\mathbb{R}^n) \to \mathbb{R}$ and $V_2 : C^1([0,1];\mathbb{R}^n) \to \mathbb{R}$ by

\[
V_1(u) \triangleq \left| \begin{array}{c}
\Delta_1^2 \sum_{j=1}^n m_{1j}(u) u_j e^{-\mu x}, \ldots, \Delta_n^2 \sum_{j=1}^n m_{nj}(u) u_j e^{-\mu x}
\end{array} \right|^T,
\]

\[
V_2(u) \triangleq \left| \begin{array}{c}
\Delta_1^2 \sum_{j=1}^n m_{1j}(u) (\partial_t u_j) e^{-\mu x}, \ldots, \Delta_n^2 \sum_{j=1}^n m_{nj}(u) (\partial_t u_j) e^{-\mu x}
\end{array} \right|^T,
\]

for every $u \in C^1([0,1];\mathbb{R}^n)$ with $|u|_0 < \varepsilon_0$, where $\varepsilon_0 > 0$ is chosen small enough. In \[3.53\], we again use \[3.21\] for every $u \in C^1([0,1];\mathbb{R}^n)$ with $|u|_0 < \varepsilon_0$. This is a valid definition if, again, $\varepsilon_0 > 0$ is chosen small enough. We consider the Lyapunov function candidate $V : C^1([0,1];\mathbb{R}^n) \to \mathbb{R}$ defined by

\[
V(u) \triangleq V_1(u) + V_2(u), \forall u \in C^1([0,1];\mathbb{R}^n)
\]

such that $|u|_0 < \varepsilon_0$.

Still if $\varepsilon_0 > 0$ is small enough, which will be always assumed, we can select $\gamma \in (1, +\infty)$ such that

\[
\frac{1}{\gamma} V(u) \leq |u|_1 \leq \gamma V(u) \quad \text{for every } u \in C^1([0,1];\mathbb{R}^n)
\]

such that $|u|_0 < \varepsilon_0$. 


Let us now choose $T > 0$ large enough so that

\begin{equation}
\gamma^2 e^{-\mu_0 T/2} \leq \frac{1}{2}
\end{equation}

with $\alpha$ defined by (3.43). Let $\varepsilon_2 \in (0, +\infty)$ be such that

\begin{equation}
\varepsilon_2 < \min \left\{ \frac{\delta_1}{C_1}, \frac{\delta_2}{C_1}, \frac{\varepsilon_0}{C_1}, \varepsilon_1 \right\},
\end{equation}

where, again, $\delta_1$ and $\delta_2$ are as in Lemma 3.5 and Lemma 3.6 respectively. Let $u^o$ be in $C^1([0,1]; \mathbb{R}^n)$, satisfy the compatibility conditions (3.16) and be such that

\begin{equation}
|u^o|_1 \leq \varepsilon_2.
\end{equation}

Let $u : [0,T] \times [0,1] \to \mathbb{R}^n$ be the solution of class $C^1$ of (3.1) and (3.15) satisfying the initial condition $u(0, \cdot) = u^o(\cdot)$. By Theorem 3.1 (3.57) and (3.58), $u$ is well defined and it satisfies

\begin{equation}
|u|_0 < \min\{\delta_1, \delta_2, \varepsilon_0\}.
\end{equation}

In order to emphasize the dependence of $W_1$ and $W_2$ on $p \in \mathbb{N}_+$, we now write them $W_{1,p}$ and $W_{2,p}$ respectively. For $t \in [0,T]$, let us define

\begin{align}
V(t) &= \mathbf{V}(u(t)), \quad V_1(t) = \mathbf{V}_1(u(t)), \quad V_2(t) = \mathbf{V}_2(u(t)), \\
W_{1,p}(t) &= \mathbf{W}_{1,p}(u(t)), \quad W_{2,p}(t) = \mathbf{W}_{2,p}(u(t)).
\end{align}

Let us point out that, for every continuous function $f = (f_1, \ldots, f_n)^T : [0,1] \to \mathbb{R}^n$, we have

\begin{align}
|f|_0 &= \lim_{p \to \infty} \left( \int_0^1 \sum_{i=1}^n |f_i(x)|^{2p} \, dx \right)^{1/(2p)}, \\
\left( \int_0^1 \sum_{i=1}^n |f_i(x)|^{2p} \, dx \right)^{1/(2p)} &\leq n^{1/(2p)} |f|_0, \forall p \in [1, +\infty).
\end{align}

By the definition of $W_1$ and $W_2$, (3.60), (3.61), (3.62) and (3.63), we have

\begin{align}
V_1(t) &= \lim_{p \to \infty} W_{1,p}(t) \quad \text{and} \quad V_2(t) = \lim_{p \to \infty} W_{2,p}(t), \forall t \in [0,T], \\
\exists M > 0 \text{ such that } W_{1,p}(t) + W_{2,p}(t) &\leq M|u|_1, \forall p \in [1, +\infty), \forall t \in [0,T].
\end{align}

In particular

\begin{align}
W_{1,p} &\rightharpoonup V_1 \text{ in the weak* topology } \sigma(L^\infty(0,T), L^1(0,T)) \text{ as } p \to +\infty, \\
W_{2,p} &\rightharpoonup V_2 \text{ in the weak* topology } \sigma(L^\infty(0,T), L^1(0,T)) \text{ as } p \to +\infty.
\end{align}

(3.62) 
Let us recall that, for a bounded sequence $(\varphi_p)_{p \in \mathbb{N}_+}$ of elements of $L^\infty(0,T)$ and $\varphi \in L^\infty(0,T)$ one says that $\varphi_p \rightharpoonup \varphi$ in the weak* topology $\sigma(L^\infty(0,T), L^1(0,T))$ if, for every $\psi \in L^1(0,T)$,

\begin{equation}
\int_0^T \varphi_p(t)\psi(t) \, dt \to \int_0^T \varphi(t)\psi(t) \, dt \text{ as } p \to +\infty.
\end{equation}
From Lemmas 3.5 and 3.6, (3.59), (3.66) and (3.67), we have, in the distribution sense in \((0, T)\),

\[
\frac{dV_1}{dt} \leq -\mu V_1 + \beta_1 |u_x|_0 V_1,
\]
\[
\frac{dV_2}{dt} \leq -\mu V_2 + \beta_2 |u_x|_0 V_2.
\]

Summing (3.68) and (3.69), we get, in the distribution sense in \((0, T)\),

\[
\frac{dV}{dt} \leq -\mu V + \beta |u|_1 V,
\]

with \(\beta \triangleq \max \{\beta_1, \beta_2\}\). Let us impose on \(\varepsilon_2\), besides (3.57), that

\[
\varepsilon_2 \leq \frac{\mu \alpha}{2\beta C_1}.
\]

From (3.13), (3.58) and (3.71), we get that

\[
\beta |u|_1 \leq \frac{\mu \alpha}{2}.
\]

From (3.70) and (3.72), we have, in the distribution sense in \((0, T)\),

\[
\frac{dV}{dt} \leq -\frac{\mu}{2} V,
\]

which implies that

\[
V(T) \leq e^{-\alpha \mu T/2} V(0).
\]

From (3.55) and (3.74), we obtain that

\[
|u(T)|_1 \leq \gamma e^{-\alpha \mu T/2} |u(0)|_1,
\]

which, together with (3.56), implies that

\[
|u(T)|_1 \leq \frac{1}{2} |u(0)|_1.
\]

This completes the proof of Theorem 3.3.

**Remark 3.7.** Instead of using time derivatives in the distribution sense as, for instance in (3.70), we could have directly derived inequality (3.74) on \(V(t)\) by establishing the same inequality for \(W(t) \triangleq W_1(t) + W_2(t)\) and let \(p \to +\infty\).

### 3.2. Proof of Theorem 3.3 in the general case where \(0 < m < n\).

In this section, we explain the modifications of the proof that must be used to deal with the case \(0 < m < n\) (of course, the case \(m = 0\) is equivalent to the case \(m = n\) by considering \(u(t, 1-x)\) instead of \(u(t, x)\).

The functionals \(W_1\) and \(W_2\) are now defined as follows:

\[
W_1(u) \triangleq \left( \int_0^1 \left[ \sum_{i=1}^m p_i \left( \sum_{j=1}^n m_{ij}(u) u_j \right)^{2p} e^{-2p\mu x} \right] dx \right)^{\frac{1}{2p}},
\]

where

\[
W_2(u) \triangleq \left( \int_0^1 \left[ \sum_{i=m+1}^n p_i \left( \sum_{j=1}^n m_{ij}(u) u_j \right)^{2p} e^{2p\mu x} \right] dx \right)^{\frac{1}{2p}}.
\]
Let $u : [0, T] \times [0, 1] \to \mathbb{R}^n$ be a solution of (3.1) - (3.2) of class $C^1$. With computations similar to those of Lemma 3.5, it is readily seen that the time derivative of $W_1(t) \triangleq W_1(u(t))$ is given by:

$$
\frac{dW_1}{dt} = T_1 + T_2 + T_3
$$

with

$$
T_1 \triangleq \frac{W_1^{1-2p}}{2p} \left[ - \sum_{i=1}^{m} p_i^p \lambda_i(u) \left( \sum_{j=1}^{n} m_{ij}(u) u_j e^{-\mu x} \right)^{2p} - \sum_{i=m+1}^{n} p_i^p \lambda_i(u) \left( \sum_{j=1}^{n} m_{ij}(u) u_j e^{\mu x} \right)^{2p} \right]_0^1,
$$

$$
T_2 \triangleq -\mu W_1^{1-2p} \int_0^1 \left[ \sum_{i=1}^{m} p_i^p \lambda_i(u) \left( \sum_{j=1}^{n} m_{ij}(u) u_j \right)^{2p} e^{-2\mu x} + \sum_{i=m+1}^{n} p_i^p |\lambda_i(u)| \left( \sum_{j=1}^{n} m_{ij}(u) u_j \right)^{2p} e^{2\mu x} \right] dx,
$$

$$
T_3 \triangleq W_1^{1-2p} \int_0^1 \left[ \sum_{i=1}^{m} p_i^p \left( \sum_{j=1}^{n} m_{ij}(u) u_j \right)^{2p-1} \lambda_i(u) \sum_{j=1}^{n} \left( \partial_x m_{ij}(u) \right) u_j + \frac{1}{2p} \left( \sum_{j=1}^{n} m_{ij}(u) u_j \right) \frac{\partial \lambda_i}{\partial u} (u) \partial_x u \right] e^{-2\mu x} dx
$$

$$
+ \sum_{i=m+1}^{n} p_i^p \left( \sum_{j=1}^{n} m_{ij}(u) u_j \right)^{2p-1} \lambda_i(u) \sum_{j=1}^{n} \left( \partial_x m_{ij}(u) \right) u_j + \frac{1}{2p} \left( \sum_{j=1}^{n} m_{ij}(u) u_j \right) \frac{\partial \lambda_i}{\partial u} (u) \partial_x u \right] e^{2\mu x} dx.
$$

The parameters $p_i > 0$ are now selected such that (to be compared with (3.30))

$$
(3.77) \quad p_i^p |\lambda_i| = \Delta_i^{2p}, \quad i = 1, \ldots, n.
$$

Concerning the $\xi_i$, $i = 1, \ldots, n$, they are now defined by (to be compared with (3.37))

$$
\xi_i(t) \triangleq \Delta_i u_i(t, 1) \quad \text{for} \quad i = 1, \ldots, m \quad \text{and} \quad \xi_i(t) \triangleq \Delta_i u_i(t, 0) \quad \text{for} \quad i = m + 1, \ldots, n.
$$
It is then a straightforward exercise to verify that Theorem 3.3 can be established for the case \(0 < m < n\) in a manner completely parallel to the one we have followed in the case \(m = n\).

**Remark 3.8.** Looking at our proof of Theorem 3.3 we see that, for every 

\[ (3.78) \quad \nu \in (0, -\min(|\Lambda_1|, \ldots, |\Lambda_n|) \ln (\rho (|G'(0)|))) , \]

there exist \(\varepsilon > 0\) and \(C > 0\) such that, for every \(u^o\) such that \(|u^o|_1 < \varepsilon\) and satisfying the compatibility conditions (3.5) to (3.7), the Cauchy problem (3.1)–(3.2)–(3.4) has a unique \(C^1\)-solution which satisfies

\[ |u(t, .)|_1 \leq Ce^{-\nu t}|u^o|_1, \quad \forall t \in [0, +\infty). \]

See in particular (3.40) and (3.41), replace the definition of \(\alpha\) given in (3.43) by the condition

\[ (3.79) \quad \alpha \in (0, \min(|\Lambda_1|, \ldots, |\Lambda_n|)) \]

and modify a little bit some arguments in order to deal with (3.79) instead of (3.43).

**4. Conclusion and final remark.** In this article, we have addressed the issue of exponential stability for the \(C^1\)-norm of quasi-linear hyperbolic systems of the form (3.1), (3.2). Our main result (Section 3) has been to give a new proof, using a strict Lyapunov function, that a sufficient condition for the exponential stability of the steady-state is to have a map \(G\) such that \(\rho_\infty (G'(0)) < 1\) for the boundary conditions (3.2).

The approach followed in the paper can be easily adapted to the exponential stability for the \(C^\ell\)-norm with \(\ell \in \mathbb{N}_+\) provided the compatibility conditions (3.5), (3.6), (3.7) are adapted accordingly (see [10] for details) and that \(A\) and \(G\) are of class \(C^\ell\). The analysis requires the following extensions of formulas (3.19) or (3.20) and (3.52) or (3.53):

\[ (4.1) \quad W_k (u) \triangleq \left( \int_0^1 \sum_{i=1}^n \rho_i^p \left( \sum_{j=1}^n m_{ij} (u) \partial_t^{k-1} u_j \right)^{2p} e^{-2p\mu x} dx \right)^{\frac{1}{2p}}, \]

\[ (4.2) \quad V_k (u) \triangleq \left| \Delta_t^2 \sum_{j=1}^n m_{1j} (u) (\partial_t^{k-1} u_j) e^{-\mu x}, \ldots, \Delta_t^2 \sum_{j=1}^n m_{nj} (u) (\partial_t^{k-1} u_j) e^{-\mu x} \right|^T \]

where \(\partial_t^{k-1} u\) is defined by differentiating formally (1.1) with respect to time. For example

\[ \partial_t^2 u \triangleq (A'(u) (A(u) \partial_x u)) \partial_t u + A(u) (A'(u) \partial_x u) \partial_x u + A(u) (A(\partial_x^2 u)), \forall u \in C^2([0, 1]; \mathbb{R}^n) \] with \(|u|_0\) small enough.

Then, using the Lyapunov function

\[ (4.3) \quad V(u) = \sum_{k=1}^{\ell+1} V_k (u), \]

the following theorem holds.
Theorem 4.1. Let assume that $A$ and $G$ are of class $C^\ell$. Then, if

\begin{equation}
\rho_\infty(G'(0)) < 1,
\end{equation}

the steady state $x \in [0, 1] \mapsto 0 \in \mathbb{R}^n$ of the system (3.1)–(3.2) is exponentially stable for the $C^\ell$-norm.

REFERENCES


