LINKS BETWEEN LOCAL CONTROLLABILITY AND LOCAL CONTINUOUS STABILIZATION

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Abstract: We prove that a control system which satisfies well known sufficient conditions for small time local controllability — for example the Hermes Condition — can be dynamically locally asymptotically stabilized by means of a continuous time-varying feedback law. For special systems (including systems without drift) we get local stabilization in finite time by means of a continuous time-varying feedback law.

Keywords: Nonlinear control systems, local controllability, local stabilization, time-varying feedback law.

1. Introduction
For $f$ in $C^\infty(\mathbb{R}^n \times \mathbb{R}^m; \mathbb{R}^n)$ we consider the control system
\begin{equation}
\dot{x} = f(x, u)
\end{equation}
where $u$ in $\mathbb{R}^m$ is the control. H. Sussmann and V. Jurjdevic have proved in [SJ] that the set of reachable points from $x_0$ in small time and with small controls has $x_0$ in the closure of its interior if (and only if $f$ is analytic)
\begin{equation}
f(x_0, 0) = 0,
\end{equation}
\begin{equation}
\left\{ h(x_0); h \in \text{Lie} \left\{ \frac{\partial^{\alpha}}{\partial x^\alpha} (\cdot, 0); \alpha \in \mathbb{N}^m \right\} \right\} = \mathbb{R}^n
\end{equation}
where, for a family $\mathcal{F}$ of vector fields on $\mathbb{R}^n$, Lie $\mathcal{F}$ denotes the Lie algebra generated by the vector fields in $\mathcal{F}$. Condition (1.3) for all $x_0$ in $\mathbb{R}^n$ implies, for special $f$, the complete controllability of $\dot{x} = f(x, u)$. This is in particular the case if
\begin{equation}
f(x, u) = \sum_{i=1}^m u_i f_i(x).
\end{equation}

Let us recall that H. Sussmann has proved in [Su1] that, if $f$ is analytic, the complete controllability implies that $\dot{x} = f(x, u)$ can be steered to the origin by means of a piecewise analytic feedback law: $u = u(x)$. Let us also mention that in [DMK] and [Kaw2] it is proved that if $n = 2, m = 1$ and $f$ is affine then local controllability implies that $\dot{x} = f(x, u)$ is asymptotically stabilizable by means of a continuous feedback law. Unfortunately, as it has been shown by R. Brock-ett in [B], the complete controllability does not imply in more general situations - even if (1.4) holds - that $\dot{x} = f(x, u)$ can be asymptotically stabilized by means of a continuous feedback law. For example
\begin{equation}
\dot{x}_1 = u_1, \quad \dot{x}_2 = u_2, \quad \dot{x}_3 = x_1 u_2 - x_2 u_1
\end{equation}
is a control system which satisfies (1.3) for all $x_0$ in $\mathbb{R}^n$ and (1.4). Therefore it is completely controllable, but it is proved in [B] that it cannot be asymptotically stabilized by means of a continuous feedback law. In [Sa], C. Samson has proved that (1.5) can be globally asymptotically stabilized by means of a smooth time-varying feedback law $u = u(x, t)$. It turns out to be true in general under conditions (1.4) and (1.3) for all $x_0$ in $\mathbb{R}^n \setminus \{0\}$. More precisely it is proved in [C1] (see [F] and [Se] for special cases but with explicit feedback laws; see also [CPo] and [So3]).

Theorem 1.1. Assume (1.3) for all $x_0$ in $\mathbb{R}^n \setminus \{0\}$ and (1.4). Then, for any positive $T$, there exists $u$ in $C^\infty(\mathbb{R}^n \times \mathbb{R}; \mathbb{R}^m)$ such that
\begin{equation}
u(0, t) = 0 \quad \text{for all } t \in \mathbb{R},
\end{equation}
\begin{equation}u(x, t + T) = u(x, t) \quad \forall (x, t) \in \mathbb{R}^n \times \mathbb{R},
\end{equation}
\begin{equation}u(x, t + T) = u(x, t) \quad \forall (x, t) \in \mathbb{R}^n \times \mathbb{R}.
\end{equation}
Theorem 1.1 can be slightly generalized in the following

Proposition 1.2. Assume that (1.3) holds for all $x_0$ in $\mathbb{R}^n \setminus \{0\}$ and that there exists $\varphi$ in $C^\infty(\mathbb{R}^n; \mathbb{R}^m)$ such that
\begin{equation}\varphi(0) = 0
\end{equation}
\begin{equation}f(x, \varphi(u)) = -f(x, u) \quad \forall (x, u) \in \mathbb{R}^n \times \mathbb{R}^m.
\end{equation}
Then the conclusion of theorem 1.1 holds.

Examples 1.3. a) Choosing $\varphi(u) = -u$ we get Theorem 1.1. b) If $f(x, u) = u_1 g(x, u_2), u_1 \in \mathbb{R}, u_2 \in \mathbb{R}^{m-1}$ we can take $\varphi(u_1, u_2) = (-u_1, u_2)$.

We will sketch in Section 2 the modifications of the proof given in [C1] in order to get Proposition 1.2. Let us notice that (1.9) and (1.10) implies $f(x, 0) = 0$ and therefore do not allow a drift term. In presence of a drift term many studies have been carried out on sufficient conditions for the Small Time Local Controllability (STLC) — this means that the attainable set from $x_0$ at time $t > 0$ contains $x_0$ in its interior for all $t > 0$; see e.g. the recent nice survey [Kaw1] by M. Kawski.
on this question. An important tool to study STLC is the local approximation cones of the attainable set and the associated families of admissible control variations, see [Kaw1]. For stabilization it seems natural to modify the definition of $p$-th order tangent vector to the attainable set at zero in the following way (we assume $f(0,0)=0$)

**Definition 1.4.** For a positive integer $p$, let $D^p$ be the set of vectors $\xi \in \mathbb{R}^n$ such that there exists $u$ in $C^0([0,1]; L^1((0,1); \mathbb{R}^m))$ such that

$$|u(s)(t)| \leq s \quad \text{for all } (s,t) \in [0,1] \times [0,1],$$

and

$$\psi(u,s) = s^p \xi + o(s^p) \quad \text{as } s \to 0$$

where $\psi(u,s)$ denotes the value at time 0 of the solution of $\dot{x} = f(x,u(s)(t))$, $x(0) = 1$.

In order to state our main results on the stabilization of systems with a drift term let us introduce some other definitions

**Definition 1.5.** The system $\dot{x} = f(x,u)$ is locally asymptotically stabilizable by means of a $T$-periodic feedback law if there exists $u : \mathbb{R} 	imes \mathbb{R} \to \mathbb{R}^m$ such that

$$u \in C^\infty((\mathbb{R} \setminus \{0\}) \times \mathbb{R}; \mathbb{R}^m) \cap C^0(\mathbb{R}; \mathbb{R}^2),$$

$$u(0,t) = 0 \quad \text{for all } t \in \mathbb{R},$$

$$u(x,t+T) = u(x,t) \quad \forall x, t \in \mathbb{R} \times \mathbb{R}.$$  

$$3\delta > 0 \text{ such that for } |x_0| < \delta, t_0 \leq t \leq t_1 \text{ there exists one and only one solution on } [t_0,t_1],$$

and

$$\left\{ \begin{array}{l}
\dot{x} = f(x,u(x,t)) \quad x(t_0) = x_0 \\
0 \in \mathbb{R}^n \quad \text{is a locally asymptotically stable point of } \dot{x} = f(x,u(x,t))
\end{array} \right\}$$

If such a $u$ exists we will say that $\dot{x} = f(x,u)$ is $T$-LAS. If, moreover, for all small enough $x_0$, we have

$$\left\{ \begin{array}{l}
\dot{x} = f(x,u(x,t)) \quad \text{and } x(0) = x_0 \\
\implies x(T) = 0
\end{array} \right\}$$

we will say that $\dot{x} = f(x,u)$ is $T$-Locally Stabilizable ($T$-LS).

**Definition 1.6.** Let $k$ be an integer: $\dot{x} = f(x,u)$ is $k$-dynamically $T$-LAS (resp. $T$-LS) if the system $\dot{x} = f(x,u)$, $\dot{y} = v$ where the control is $(u,v) \in \mathbb{R}^m \times \mathbb{R}^k$, is $T$-LAS (resp. $T$-LS). Note that

$$T\text{-LS} \implies T\text{-LAS},$$

$$T\text{-LAS} \implies k\text{-dynamically } T\text{-LAS},$$

$$\left\{ \begin{array}{l}
k\text{-dynamically } T\text{-LAS and } k \leq k' \\
\implies k'\text{-dynamically } T\text{-LAS}
\end{array} \right\}.$$  

Until the end of this paper we will assume

$$f(0,0) = 0,$$

$$\left\{ h(0) : h \in \text{Lie \{0\}} \right\} = \mathbb{R}^n.$$  

Let us remark that (1.23) and (1.24) for the system $\dot{x} = f(x,u)$ are equivalent to (1.23) and (1.24) for the system $\dot{x} = f(x,y)$, $\dot{y} = u \in \mathbb{R}^m$. Let $D = \cup_{\delta > 1} D^\delta$ and let $\text{int} (D)$ its interior. We will prove in Section 3

**Theorem 1.7.** Assume

$$0 \in \text{int } D.$$  

Then, for all positive real number $T$, $\dot{x} = f(x,u)$ is $n$-dynamically $T$-LAS.

In Section 5 we will make some remarks concerning (1.25). In particular we will see that the Hermes Condition ([Su2; Section 7.3]) implies (1.25). Let us recall that this condition for $m = 1$ and $f(x,u) = f_0(x) + u f_1(x)$ (see Section 5 for the general case) means that any iterated Lie bracket of $f_0$ and $f_1$ with an even number of $f_1$ and an odd number of $f_0$ can be expressed at 0 as the sum of iterated Lie brackets containing fewer $f_1$. Let us notice that the interest of time-varying feedback law (resp. dynamical stabilization) for systems with drift has already been pointed out in [SS] [resp. [CP]].

It would be interesting to know if (1.25) implies that $\dot{x} = f(x,u)$ is $T$-LS: our next theorem is a partial result in this direction [see [C3] for other cases; see also [SCW] for different cases and $T$-LAS, instead of $T$-LS, but with explicit feedback laws].

**Theorem 1.8.** Let $u = (u_1, u_2) \in \mathbb{R}^{m-1} \times \mathbb{R}$ and $x = (x_1, x_2) \in \mathbb{R}^{m-1} \times \mathbb{R}$. Assume $n \geq 4$. (1.25).

$$f(x,u) = (f_1(x,u), u_2) \in \mathbb{R}^{m-1} \times \mathbb{R} \simeq \mathbb{R}^n,$$

$$f_1((0,x_2),(0,u_2)) = 0 \quad \forall (x_2,u_2) \in \mathbb{R} \times \mathbb{R}.$$  

Then $\dot{x} = f(x,u)$ is $T$-LS for all positive $T$.

For $n \geq 3$ our next proposition is a consequence of Theorem 1.8.

**Proposition 1.9.** Assume (1.25) holds. Then for all positive $T$, $\dot{x} = f(x,u)$ is $1$-dynamically $T$-LS.

**Proof.** For $n \geq 3$ apply Theorem 1.8 to the system $\dot{x} = f(x,u)$, $\dot{y} = v \in \mathbb{R}$. For $n \leq 2$ see Remark 4.1 b) or [C3].

Our next proposition is a corollary of Theorem 1.8 if $n \geq 4$ and is proved in [C3] if $n \leq 3$

**Proposition 1.10.** Assume $f(x,u) = \sum_{i=1}^m u_i f_i(x)$. Then $\dot{x} = f(x,u)$ is $T$-LS for all positive $T$.

**Proof.** Since $f(x,u) = \sum_{i=1}^m u_i f_i(x)$, $\dot{x} = f(x,u)$ satisfies the Hermes Condition and therefore (1.25). Moreover without loss of generality we may assume $f_m = e_m$ and $f_i, e_m = 0$ for all $i \in [1, m-1]$. The conclusion follows from Theorem 1.8 if $n \geq 4$; if $n \leq 3$ see Remark 4.2 b) or [C3].

Some of the proofs are only sketched. The details of these proofs are given in [C2] and [C3]. Section 3 deals with Theorem 1.7, Section 4 deals with Theorem 1.8. One of the tools we use is the study of the controllability of the linearized equations around trajectories. Roughly speaking we will see in Section 2 that, under some Lie algebra rank condition, the linearized

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equations around a generic family of trajectories near \( x_0 = 0 \) and with \( u \) "small" are locally controllable with impulsive controls. This is a generalization of a result contained in [C1].

Finally let us mention that many results on continuous feedback stabilization have been found recently. For a recent nice survey on this subject see [So2].

2. Study of the linearized equation

Throughout this paper "manifold" always means finite dimensional, Hausdorff second countable manifold of class \( C^\infty \). Unless otherwise specified the manifolds have no boundary. For two manifolds \( V \) and \( W \), and for \( p \) in \( \mathbb{N} \cup \{0\} \), \( C^p(V;W) \) is provided with the Whitney topology (see e.g. [GC:pp.42]). On \( C^\infty(V;W) \) we define a topology, called the \( C^\infty \)-topology, in the following way. For an integer \( k \), let \( J^k(V;W) \) be the set of \( k \)-jets of \( C^\infty \)-mappings from \( V \) into \( W \). Let \( (K_i; i \in \mathbb{N}) \) be a sequence of compact subsets of \( V \) such that \( K_i \subseteq K_{i+1} \) for all integer \( i \), \( \bigcup_{i \in \mathbb{N}} K_i = V \), and \( K_0 = \phi \). For a sequence \( k = (k_i; i \in \mathbb{N}) \) of integers and for a sequence \( U = (U_i; i \in \mathbb{N}) \) where \( U_i \) is an open subset of \( J^k_i(V;W) \) for all integer \( i \), let \( O(k,U) \) be the set of \( u \) in \( C^\infty(V;W) \) such that \( J^k_i(V \setminus K_i) \subseteq U_i \) for all integer \( i \). Our topology is the topology whose basis is the family of set \( O(k,U) \) where \( k \) and \( U \) are as above. This topology is independent of the choice of \( (K_i; i \in \mathbb{N}) \) and is finer than the Whitney \( C^\infty \)-topology if \( V \) is not compact. Note also that \( C^\infty(V;W) \) with our topology, as \( C^\infty(V;W) \) with the Whitney \( C^\infty \)-topology, is a Baire space (adapt the proof of [GG:Proposition II.3.13]). For a \( C^\infty \)-smooth fibration \( p : W \to V \), \( C^\infty(V) \) denotes the set of the \( C^\infty \)-smooth sections of this fibration. Let \( N \) and \( \Lambda \) be two manifolds and let \( U \) be an open set of \( \mathbb{R}^m \).

We denote by \( \pi : TN \to N \) the tangent bundle of \( N \) and by \( C^\infty(T \times U \times \Lambda) \) the \( C^\infty \)-smooth sections of the fibration \( \pi : TN \times U \times \Lambda \to N \times U \times \Lambda \). Let \( E \) be a vector subbundle of the tangent bundle of \( N \). Let \( g \) be in \( C^\infty(T \times U \times \Lambda) \). Throughout this section we assume that, for any \( (x_0,u_0,\lambda_0,\alpha) \) in \( N \times U \times \Lambda \times \mathbb{N}^m \)

\[
\frac{\partial g}{\partial u}(x_0,u_0,\lambda_0,\alpha) \in E(x_0). \tag{2.1}
\]

We will say that \( g \) satisfies hypothesis \( H(k) \) at \( (x_0,\lambda_0) \) if

\[
\text{Span} \left\{ \frac{\partial g}{\partial u}(x_0,0,\lambda_0,\alpha) \mid 1 \leq |\alpha| \leq k \right\} \cup \text{Br}_k \left( \frac{\partial g}{\partial u_0}(.,0,\lambda_0,\alpha) \mid 1 \leq |\alpha| \leq k \right)(x_0) = E(x_0) \tag{2.2}
\]

where \( \text{Br}_k F \) denotes the set of iterated Lie brackets of vectors in \( F \) of total length between \( k \) and \( k \) and where \( \text{Br}_k F(x_0) = \{ h(x) \mid h \in \text{Br}_k F \} \). Let, for \( \lambda_0 \) chosen in \( \Lambda \) and for \( u \) in \( C^\infty([0,T];U) \), \( : [0,T] \to N \) be a solution of

\[
\dot{\gamma} = g(\gamma,u(t),\lambda_0). \tag{2.3}
\]

The linearized control system around \( \gamma \) is:

\[
y(t) = A(t)y(t) + \sum_{i=1}^m v_i b_i(t) \tag{2.4}
\]

where \( v \in \mathbb{R}^m \) is the control and

\[
A(t) = \frac{\partial g}{\partial x} (\gamma(t),u(t),\lambda_0), b_i(t) = \frac{\partial g}{\partial u_i} (\gamma(t),u(t),\lambda_0). \tag{2.5}
\]

We will say that \( \gamma \) is \( (E,r) \)-controllable (and \( r \)-controllable if \( E = TM \)) at time \( t \) if

\[
\text{Span} \left\{ \left( \frac{d}{dt} - A(r) \right)^j b_i \right\}_{r=1} \mid 1 \leq i \leq m, 0 \leq j \leq r \right\} = E(\gamma(t)). \tag{2.6}
\]

For the reason of this definition, see e.g. [SM] or [Kai: p.614]. Finally we will say that \( \gamma \) is \( (E,N) \)-controllable on \( (0,T) \) (and \( N \)-controllable if \( E = TM \)) if, for any \( t \) in \( (0,T) \), there exists an integer \( r \) such that \( \gamma \) is \( (E,r) \)-controllable at time \( t \).

Let \( \theta \) in \( C^\infty(\Lambda;M) \) be such that the Cauchy problem

\[
\frac{\partial \hat{x}}{\partial t} = g(\hat{x},0,\lambda) \quad \text{and} \quad \hat{x}(0,\lambda) = \theta(\lambda) \tag{2.7}
\]

has a solution on \( [0,T] \times \Lambda \). Let \( d \) be a metric on \( N \times \Lambda \). Then we have

Theorem 2.1. For any neighborhood \( \Omega \) of 0 in \( C^\infty([0,T] \times \Lambda;\mathbb{R}^m) \) there exists \( \hat{u} \) in \( \Omega \cap C^\infty([0,T] \times \Lambda;\mathbb{R}^m) \) such that

(i) the solution of the Cauchy problem

\[
\frac{\partial \hat{x}}{\partial t} = g(\hat{x},\hat{u}(t,\lambda),\lambda) \quad \text{and} \quad \hat{x}(0,\lambda) = \theta(\lambda) \tag{2.8}
\]

is defined on \([0,T] \times \Lambda\);

(ii) for all \( (t,\lambda) \) in \((0,T) \times \Lambda \) and all integer \( k, \) if \( g \) satisfies \( t,\ldots,\lambda \) at \( (x,\lambda), \) for all \( (x,\lambda) \) in \( N \times \Lambda \) such that \( d((x,\lambda), (\hat{x}(t,\lambda),\lambda)) \leq 1 \) then

\[
\hat{x}(t,\lambda) \text{ is } (E,N) \text{ - controllable at time } t. \tag{2.9}
\]

Let us sketch the proof of Theorem 2.1 when \( \Lambda \) is reduced to a point \( \lambda_0 \) (the same proof works when \( \Lambda \) is compact) and \( \Omega \) is a neighborhood of 0 in \( C^p([0,T] \times \{\lambda_0\};\mathbb{R}^m) \) (for the general case see [C2; Corollary 1.8]). From now on we will omit \( \lambda_0 \). One first notices that Theorem 2.1 for \( \hat{x} = g(x,y), \hat{y} = v \) and \( \Omega \) a neighborhood of 0 in the \( C^p_{-1}([0,T];\mathbb{R}^m) \)-topology implies Theorem 2.1 for \( \hat{x} = g(x,u) \) and \( \Omega \) a neighborhood of 0 in the \( C^p([0,T];\mathbb{R}^m) \)-topology. Hence we may assume that \( g(x,u) = g_0(x) + \sum_{i=1}^m u_i g_i(x). \)

The idea is to choose \( \hat{u}(t) = b(\mu(t))b(\mu(t)) \) where \( b = (b_1,\ldots,b_m) \in C^\infty(\mathbb{R},\mathbb{R}^m) \) is \( T \)-periodic and \( \mu \) is a large integer. The real number \( 1/\mu \) plays the role of \( a \) in [C1; (4.1)]; it allows to "neglect" iterated Lie brackets of too large length if it is small. Proceeding in a similar way as in [C1] one can prove that, for generic \( b, \) if \( \mu \) is large enough \( \hat{u} \) is suitable; the only main modification is that we have, for \( t \) given and with the notations of [C1], to increase \( q \) in such a way that the condition
\[ \text{rank}(C_p(I); t: 1 \leq p \leq q, |I| \leq \ell) \]

\[ < q^*(\ell) = (m + 1)\frac{(m + 1)^r - 1}{m} \quad (2.11) \]

is now of codimension 2 — instead of codimension 1 in [C1] — in the space of jets \( \{b_j^p(t); j \leq q - 1, 1 \leq i \leq m\} \); note that now \( I \) may contain the index 0 and \( b_0 = 1 \).

**Remark 2.2.** a) One of the reasons for not having assumed \( E = T \) in order to allow time varying system — this is useful, for example, for step 1 in section 4: in order to study systems like \( \dot{x} = g(x, t, u, \lambda) \) one has just to apply theorem 2.1 to \( \dot{x} = g(x, s, u, \lambda) \), \( s = 1 \) which is a system on \( M \times R \); clearly for \( T'(s) \) equal to \( T'(s) \) in \( \mathcal{M} \times \{0\} \).

b) It follows from Theorem 2.1 (see [C2, Corollary 1.8]) that there exists an open neighborhood \( \Omega \) of 0 in \( C^\infty([0, T] \times \Lambda; \mathbb{R}^m) \) and \( \rho \) such that any \( u \in \Omega \) satisfies (2.8) and such that the set of \( u \) in \( \Omega \), satisfying (2.8) and (2) is generic in \( \Omega \) (for the \( C^\infty([0, T] \times \Lambda; \mathbb{R}^m) \)-topology).

c) Theorem 2.1 still holds even if (2.1) does not hold and if one replaces in (2.2) and (2.6) by \( \Lambda \) : see again [C2; Corollary 1.8].

d) Theorem 2.1 is related to a previous paper due to E.D. Sontag [So1]; the main novelty of our result is the \( (E, N) \)-controllability and the smoothness with respect to \( \lambda \).

e) After our paper has been completed E.D. Sontag has obtained in [So3] an interesting result related to Theorem 2.1; using his method one can also get Theorem 2.1 if \( N \) is an open subset of \( \mathbb{R}^n \), \( g \) is analytic with respect to \( x, \) and \( E = T N \). Note that using our method we can get [So3; Thm 2] with even controllability with impulsive controls, \( N \) any manifold and \( g \) only \( C^\infty \). (In this case one replaces the strong accessibility condition by \( E = T N \); see [C2] for more details.

As an application of Theorem 2.1 we explain how to modify the proof of Theorem 1.1 given in [C1] in order to get Proposition 1.2. We apply Theorem 2.1 with \( N = \Lambda = \mathbb{R}^n \times [0, \frac{T}{2}]; \mathbb{R}^m \) such that, on \( \{0\} \times \left[ \frac{T}{2}, \frac{T}{2} \right] \) and \( \{0\} \times \left[ \frac{T}{2}, \frac{T}{2} \right] \),

\[ \frac{\partial^\alpha}{\partial t^\alpha} \frac{\partial^\beta}{\partial x^\beta} u = 0 \quad \forall (x, \alpha) \in \mathbb{N}^n \times \mathbb{N} \quad (2.11) \]

and, for any \( x_0 \) in \( \mathbb{R}^n \{0\} \), the solution of \( \dot{x} = f(x, u(x, t)) \), \( x(0) = x_0 \) is \( \mathbb{N} \)-controllable on \( (0, T) \). Moreover we can impose that \( \bar{u} \) is small enough (for the \( C^\infty ([0, \frac{T}{2}]; \mathbb{R}^m) \)-topology) in such a way that, for all \( t \) in \( [0, \frac{T}{2}] \), \( x_0 \rightarrow x(x_0, t) \) is a diffeomorphism of \( \mathbb{R}^n \). We extend \( \bar{u} \) to \( \mathbb{R}^n \times \mathbb{R} \) by requiring (as in [C1]):

\[ \bar{u}(x, t-T) = \varphi(\bar{u}(x, t)) \quad \forall (x, t) \in \mathbb{R}^n \times \left[ \frac{T}{2}, T \right] \quad (2.12) \]

and

\[ \bar{u}(x, t+T) = \bar{u}(x, t) \quad \forall (x, t) \in \mathbb{R}^n \times \mathbb{R} \quad (2.13) \]

By (1.9), (2.11), (2.12) and (2.13) \( \bar{u} \in C^\infty(\mathbb{R}^n \times \mathbb{R}; \mathbb{R}^m) \). Moreover, by (1.10) and (1.12), if \( \dot{z} = f(z, \bar{u}(x_0, t)) \) and \( z(0) = x_0 \), then \( \bar{z}(T) = x_0 \). The remaining part of the proof of Proposition 1.2 is similar to the one given in [C1; Section 5].

### 3. Proof of Theorem 1.7

Let, for \( \epsilon > 0, B_\epsilon \) be the open ball of \( \mathbb{R}^n \) of radius \( \epsilon \) and let \( T \) be a positive real number. We assume (1.25). Our first step comes from [Kaw; Appendix] see also [H1].

**Lemma 3.1.** There exist \( \epsilon > 0 \) and \( u \) in \( C^\infty(B_\epsilon; L^1([0, T]; \mathbb{R}^m)) \) such that:

\[ \text{Sup}\{ |u(a)(t)| : t \in [0, T] \} \longrightarrow 0 \quad \text{as} \quad |a| \longrightarrow 0 \quad (3.1) \]

\[ x(T, a; u) = 0 \quad \text{for all} \quad a \in B_\epsilon \quad (3.2) \]

where \( x \) is defined by \( x(0, a; u) = a \) and

\[ \frac{\partial^\alpha}{\partial t^\alpha} x(t, a; u) = f(x(t, a; u), u(a)(t)) \quad (3.3) \]

**Proof:** In [Kaw, Appendix], M. Kawski has proved the existence of \( u : B_\epsilon \rightarrow L^1([0, T]; \mathbb{R}^m) \) satisfying (3.1) and (3.2). His \( u \) is not continuous — even if (1.25) holds — but a very slight modification of his proof gives a continuous \( u \). Let \( x_i, 1 \leq i \leq n \) be the usual basis of \( \mathbb{R}^n \) and let \( e_i = -e_i, a_i \) for \( i \) in \( [n + 1, 2n] \). By (1.25) and noting that \( D^p \subset D^{p+1} \) we may assume, after possibly a change of scale, that for some \( p \geq 1 \) \( e_i \in D^p \) for all \( i \) in \( [1, 2n] \). Let \( x_i = x_i(a, u) \). Hence, for a small enough,

\[ \mu(x_i) \leq \mu(a)/2 \quad (3.5) \]

where \( x_i = x_i(a, u) \). Hence, for a small enough,

\[ \mu(x_i) \leq \mu(a)/2 \quad (3.6) \]

We now define \( u(a) \) on \( \{u(a), \mu(a) + \mu(x_i)\} \) by

\[ u(a)(t) = x_i(t, a) \quad (3.7) \]

We have, if \( a \) is small enough, \( \mu(x_2) \leq \mu(x_1)/2 \) with \( x_2 = x_i(a) \). We keep going and define in this way \( u(a) \) on \( [0, \mu(a) + \sum_{i=1}^n \mu(x_i)] \). Note that, if \( a \) is small enough, \( \mu(a) + \sum_{i=1}^n \mu(x_i) \leq 2\mu(a) \leq T \). We extend \( u(a) \) on \( [0, T] \) by \( u(a)(t) = 0 \) if \( t \in [\mu(a) + \sum_{i=1}^n \mu(x_i), T] \); \( u(a) \) satisfies all the required properties.

For \( \epsilon \) in \( [0, \infty] \) let \( B_\epsilon = B_\epsilon(0) \) be \( \{x \in \mathbb{R}^n : 0 <\)
\( |z| < \varepsilon \) and let \( C^\infty_0(B'_t \times [0, T]; \mathbb{R}^n) \) be the set of functions \( u \) in \( C^\infty_0(B'_t \times [0, T]; \mathbb{R}^n) \cap C^\infty(B'_t \times [0, T]; \mathbb{R}^n) \) such that for all \( \alpha \in \mathbb{N}^{n+1} \)

\[
\partial^\alpha u = 0 \text{ on } B'_t \times \{0, T\},
\]

\[
u(0, t) = 0 \text{ for all } t \in [0, T].
\]

Our next statement is a corollary of Lemma 3.1 and Theorem 2.1. We do not need it for the proof of Theorem 1.7, but it will be useful in Section 4 (see also the end of this section).

**Corollary 3.2.** In Lemma 3.1 \( u \) can be chosen in \( C^\infty_0(B'_t \times [0, T]; \mathbb{R}^n) \).

**Proof:** By Theorem 2.1 there exist \( \delta > 0 \) and \( \tilde{u} \) in \( C^\infty_0(B'_t \times [0, T]; \mathbb{R}^n) \) such that the solutions of \( \dot{z} = f(\tilde{x}, \tilde{u}(a, t)) \), \( \tilde{z}(0, a; \gamma) = a \in B'_t \) are \( \mathbb{N} \)-controllable on \( (0, T/3) \). Let \( u \) be as in Lemma 3.1 but with \( T/3, 2T/3 \) instead of \( [0, T] \); we extend \( u \) on \( (T/3, T) \) by \( \gamma(a, t) = u(\tilde{z}(T/3, a; \tilde{u}), t) \) for \( t \in (T/3, 2T/3) \) and by \( 0 \) for \( t \in (2T/3, T) \). This makes sense if \( |u| < \eta \) with \( 0 < \eta \) small enough. Then \( u \) satisfies the conclusion of Lemma 3.1; this map has the regularity required by Corollary 3.2 but the \( \mathbb{N} \)-controllability allows to smooth it in a map in \( C^\infty_0(B'_t \times [0, T]; \mathbb{R}^n) \) satisfying (3.2) (proceed as in [C1]; see [C3] for more details).

Now the proof of Theorem 1.7 goes as follows. Let \( \tilde{u} \) be as in Lemma 3.1 but with \( T/3 \) instead of \( T \). Let \((u, y) : \mathbb{R}^2 \times \mathbb{R}^n \rightarrow \mathbb{R}^n \rightarrow \mathbb{R}^m \rightarrow \mathbb{R}^m \) be \( T \)-periodic in \( t \) and such that for \( t \in [0, T] \) and \(|x| + |y| \) small enough

\[
u(0, y, y, t) = \tilde{u}(y, (y - (T/3)) \text{ if } t \in (T/3, 2T/3) \]

\[
u(0, y, y, t) = 0 \text{ if } t \in (T/3, 2T/3) \]

\[
u(0, y, y, t) = -|y - \nu(x, y, t)|^{1/2} \text{ if } t \in [0, T/3) \cup (2T/3, T) \]


Step 1. Using Corollary 3.2 and Proposition 2.1 one can prove that there exist \( \varepsilon > 0 \) and \( u_1 \) in \( C^\infty_0(B'_t \times [0, T]; \mathbb{R}^m) \) such that

\[
\dot{x}(t, a; u_1) = 0 \text{ for all } a \in B'_t.
\]

the trajectories \( t \rightarrow x(t, a; u_1) \) are \( \mathbb{N} \)-controllable on \( (0, T) \) for all \( a \in B'_t \).

Let \( u_2 \) be the restriction of \( u \) to \( (0, \varepsilon_1 a) \times [0, T] \) and let \( C^\infty_0((0, \varepsilon_1 a) \times [0, T]; \mathbb{R}^m) \) be the set of maps \( u \) in \( C^\infty_0((0, \varepsilon_1 a) \times [0, T]; \mathbb{R}^m) \cap C^\infty((0, \varepsilon_1 a) \times [0, T]; \mathbb{R}^m) \) such that

\[
\partial^\alpha u = 0 \text{ on } (0, \varepsilon_1 a) \times \{0, T\} \text{ for all } \alpha \in \mathbb{N}^2 \text{.}
\]

Note that \( u_2 \in C^\infty_0((0, \varepsilon_1 a) \times [0, T]; \mathbb{R}^m) \). We now use \( n \geq 4 \); the next step is wrong for \( n \leq 3 \).

Step 2. Perturbing \( u_1 \) slightly and in a suitable way, if necessary, we obtain a new \( u \) satisfying again the properties of Step 1 such that the corresponding \( u_2 \), called \( u_3 \), is such that, for all \( t \in (0, T) \) the map

\[
(0, \varepsilon) \rightarrow \mathbb{R}^m, \quad a_n \rightarrow x(t, a_n; u_2) \text{ is an embedding).
\]

Note that by (4.2), for all \( a_n \in (0, \varepsilon) \), the trajectories \( t \rightarrow x(t, a_n; u_2) \) are \( \mathbb{N} \)-controllable on \( (0, T) \). The proof of this relies essentially on (4.2) and on the classical proof of Whitney's embedding theorem; see e.g. [GG:II.5]. One could alternatively slightly perturb only \( u_2 \) (instead of \( u_1 \)) and use ideas due to M. Gromov [G; (E) p. 1211] as well as [GG:II.5].

Step 3. Using the \( \mathbb{N} \)-controllability of the trajectories \( t \rightarrow x(t, a_n; u_3) \) on \( (0, T) \) for all \( a_n \) in \((0, \varepsilon) \) and the above embedding property one can prove that there exists an open neighborhood \( \mathcal{N}_1 \) of \((0, \varepsilon_1 a/2) \) and \( u_4 \) in \( C^\infty_0([0, T]; \mathbb{R}^m) \) such that, for all \( t \in [0, T] \), the map \( a \in \mathcal{N}_1 \rightarrow x(t, a; u_4) \) in \( \mathbb{R}^m \) is an embedding and \( x(T, a; u_4) = 0 \) for all \( a \in \mathcal{N}_1 \). From this we get that there exists a neighborhood \( \mathcal{N}_2(N, \mathcal{N}_1) \) of \((0, \varepsilon_1 a/4) \) and \( u_5 \) in \( C^\infty_0((0, T]; \mathbb{R}^m) \) such that (1.17) holds with \( u = u_5, 0 \leq t_0 \leq t_1 \leq T \) and that (1.19) holds with \( u = u_5 \) and \( x(0) \in \mathcal{N}_2 \cup \{0\} \).

Step 4. Finally we replace in the above steps \([0, T] \) by \([T/2, T] \) and define \( u \) on \( \mathbb{R}^2 \times (T/2, T) \) by \( u = u_5 \) on this set. On \( \mathbb{R}^2 \times (0, T/2) \) we choose \( u \) in \( C^\infty_0((0, T]; \mathbb{R}^m) \) such that (1.17) holds for \( 0 \leq t_0 \leq t_1 \leq T/2 \) and there exists \( \delta > 0 \) such that if \( |x(0)| < \delta_1 \) and \( \dot{z} = f(z, u(x, t)) \) then \( x(T/2) \in \mathcal{N}_2 \cup \{0\} \).

The existence of such \( u \) follows from (1.26) and (1.27). The map \( u \), extended by \( T \)-periodicity (in time) on all \( \mathbb{R}^2 \times \mathbb{R} \), satisfies (1.14) to (1.19).

**Remark 4.1.** a. We use \( n \geq 4 \) only at Step 2.

The existence of \( u_3 \) as in Step 2 can be proved (see [C3]) in a different manner if \( f_1(x, \langle -u_1, -u_2 \rangle) = -f_1(x, \langle u_1, u_2 \rangle) \) or \( f_1(x, \langle u_1, u_2 \rangle) = \tilde{f}_1(x, u_1) \): in these cases we do not need \( n \geq 4 \).

b. Assumptions (1.26) and (1.27) can be omitted, see [C3].
5. Links between the Sussmann condition and \(0 \in \text{Int } D\)

Let us assume, for the time being, that

\[ f(x, u) = f_0(x) + \sum_{i=1}^{m} u_i f_i(x) . \]

Let \( \text{Br}(f) \) be the set of iterated Lie brackets of \( \{f_0, f_1, \ldots, f_m\} \). For \( h \in \text{Br}(f) \) let \( \delta(h) \) the number of times that \( f_i \) appears in \( h \). Recall (see [Su2; Section 7]) that, for \( \theta \in [0, +\infty) \), \( \dot{z} = f_0(x) + \sum_{i=1}^{m} u_i f_i(x) \) satisfies the Sussmann condition \( S(\theta) \) if whenever \( h \in \text{Br}(f) \) with \( \delta_0(h) \) odd and \( \delta_i(h) \) even for all \( i \) in \( [1, m] \) then \( h(0) \) is the Span of the \( g(h) \) where the \( g \)'s are in \( \text{Br}(f) \) and satisfy

\[ \theta \delta_0(g) < \sum_{i=1}^{m} \delta_i(g) < \theta \delta_0(h) + \sum_{i=1}^{m} \delta_i(h) . \] (5.1)

with the convention that when \( \theta = +\infty \), (5.1) is replaced by \( \delta_0(g) < \delta_0(h) \). H. Sussmann has proved:

**Theorem 5.1.** [Su2; Thm. 7.3]. If, for some \( \theta \in [0, 1] \), \( \dot{z} = f_0(x) + \sum_{i=1}^{m} u_i f_i(x) \) satisfies \( S(\theta) \) then it is STL. Moreover it follows directly from the proof of [Su2; Thm. 7.3] that we have

**Proposition 5.2.** Under the hypothesis of Theorem 5.1, \( 0 \in \text{Int } (D) \).

Let us notice that one can check

**Proposition 5.3.** Let \( \theta \in [0, 1] \). Then \( \dot{z} = f_0(x) + \sum_{i=1}^{m} u_i f_i(x) \) satisfies \( S(\theta) \) if and only if \( \dot{z} = f_0(x) + \sum_{i=1}^{m} u_i f_i(x), \dot{y} = u \) satisfies \( S(\theta / (1 - \theta)) \).

This proposition allows us to extend \( S(\theta) \) to \( \dot{z} = f(x, u) \) in the following way

**Definition 5.4.** Let \( \theta \in [0, 1] \); we will say that \( \dot{z} = f(x, u) \) satisfies \( S(\theta) \) if \( \dot{z} = f(z, \dot{y}), \dot{y} = u \) satisfies \( S(\theta / (1 - \theta)) \).

What we have called in this paper the Hermes condition is \( S(0) \); the true Hermes condition is in fact more restrictive (see [H2] or [Su2; Section 7.3]). Moreover it follows from [Su2] that

**Proposition 5.5.** If, for some \( \theta \in [0, 1] \), \( \dot{z} = f(x, u) \) satisfies \( S(\theta) \) then it is STL.

**Proof:** Apply [Su2] to \( \dot{z} = f(z, \dot{y}), \dot{y} = u \) with the constraint \( f_0(u(s)) ds \leq 1 \) (instead of \(|u| \leq 1 \)).

**Remark 5.6.** Similar comments can be done for the sufficient condition for STL due to R. Bianchini and G. Stefani [BS]. In particular the hypothesis of [BS; Corollary p. 970] implies also \( 0 \in \text{Int } (D) \).

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