On a Nonlinear Elliptic Equation Involving the Critical Sobolev Exponent: The Effect of the Topology of the Domain

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1. Introduction

Let $\Omega$ be a bounded regular and connected open set in $\mathbb{R}^N$ with $N \geq 3$. We are looking for a map $u$ from $\Omega$ into $\mathbb{R}$ such that

$$-\Delta u = u^{(N+2)/(N-2)} \quad \text{in} \quad \Omega,$$

$$u > 0 \quad \text{in} \quad \Omega,$$

$$u = 0 \quad \text{on} \quad \partial \Omega.$$

(1)

We shall denote by $H_d(\Omega; \mathbb{Z})$ the homology of dimension $d$ of $\Omega$ with $\mathbb{Z}$-coefficients.

Our main result is the following:

THEOREM 1. If there exists a positive integer $d$ such that $H_d(\Omega; \mathbb{Z}) \neq 0$, then (1) has a solution.

Note that if $N = 3$ and $\Omega$ is not contractible, then $H_1(\Omega; \mathbb{Z})$ or $H_1(\Omega; \mathbb{Z})$ is not trivial. Thus Theorem 1 implies

COROLLARY 2. If $N = 3$ and $\Omega$ is not contractible, then (1) has a solution.

Remarks 3.

a. Trudinger [24] has proved that any $H^1(\Omega)$-solution of (1) is in $L^\infty(\Omega)$ (and therefore in $C^\infty(\Omega)$).

b. Pokhozhaev [17] has proved that if $\Omega$ is star-shaped, then (1) has no solution.

c. Kazdan-Warner [9] have pointed out that if $\Omega$ is an annulus, then (1) has a solution.

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d. It has been proved in [6] that if $\Omega$ has a "small hole" (see [8] for the precise statement), then (1) has a solution.

e. Corollary 2 has been announced in [4] with a sketch of a proof.

We start the proof of Theorem 1 by recalling some well known facts.

2. Well Known Facts

2.1. The Palais-Smale condition. We first introduce some notations. Let, for $u$ in $H_0^2(\Omega)$, $\|u\| = (\int \nabla u^2)^{1/2}$, where the integration is on $\Omega$. Let

$$
\Sigma = \{ u \in H_0^2(\Omega) \mid \|u\| = 1 \},
$$

$$
\Sigma_+ = \{ u \in \Sigma \mid u \geq 0 \},
$$

$$
p = \frac{N + 2}{N - 2},
$$

$$
J(u) = \frac{1}{\|u\|^{p+1}} \quad \text{for} \quad u \in \Sigma.
$$

If $u$ is a critical point of $J$ in $\Sigma_+$, then $J(u)^{1/(p+1)}u$ is a solution of (1). $\Sigma_+$ is invariant by the flow associated to $-J'$. $J$ does not satisfy the Palais-Smale condition on $\Sigma_+$, but the sequences which violate the Palais-Smale condition are known. In order to describe them, let us introduce some notation. Let, for $a$ in $\mathbb{R}^N$ and $\lambda$ in $(0, \infty)$, $\delta(a, \lambda)$ be the function from $\mathbb{R}^N$ into $(0, \infty)$ defined by

$$
(\delta(a, \lambda))(x) = \frac{\lambda}{1 + \lambda|x - a|^2}^{(N - 2)/2},
$$

where $c_0$ is such that $\int_{\mathbb{R}^N} \nabla \delta(a, \lambda)^2 = 1$ ($c_0$ is independent of $a$ and $\lambda$).

For $\epsilon > 0$ and $n$ in $\mathbb{N}^*$ we denote by $V(\epsilon, n)$ the set of functions $u$ in $\Sigma$ such that: there exist

$$(a_1, a_2, \ldots, a_n) \in \Omega^n \quad \text{and} \quad (\lambda_1, \lambda_2, \ldots, \lambda_n) \in (0, \infty)^n$$

such that

$$
\|u - \frac{1}{n} \sum_{i=1}^n \delta(a_i, \lambda_i)\| < \epsilon,
$$

$$
\lambda_i d(a_i, \partial \Omega) > \epsilon^{-1} \quad \text{for all} \quad i,
$$

$$
\frac{\lambda_i}{\lambda_j} + \frac{\lambda_j}{\lambda_i} + \lambda_i \lambda_j |a_i - a_j|^2 > \epsilon^{-1} \quad \text{for all} \quad (i, j) \text{ with } i \neq j,
$$

where $P$ is the projection on $H_0^1(\Omega)$ (i.e., $P \varphi = \varphi - h$ with $\Delta h = 0$ in $\Omega$ and
\( h = \psi \) on \( \partial \Omega \) and \( d(a_\lambda, \partial \Omega) \) is the distance from \( a_\lambda \) to \( \partial \Omega \). Let

\[
S = \frac{\int \delta(a, \lambda)^{p+1}}{\int \delta(a, \lambda)}
\]

\( S \) does not depend on \( a \) and \( \lambda \). It is known, see [6], that

\[
\inf_{\hat{u} \in \hat{\mathcal{H}}} J(\hat{u}) = S
\]

and that this infimum is not achieved. Let \( b_\lambda = \delta^{p+1} \).  

We shall prove Theorem 1 by contradiction and, therefore, assume throughout the whole paper that (1) has no solution.

**Proposition 4.** Let \( u_k \) be a sequence in \( \mathcal{S}_\lambda \) such that \( J(u_k) \to 0 \) and \( J(u_k) \) is bounded; then there exists a positive integer \( n \) and a sequence \( (e_k) \) with \( e_k > 0 \) and \( \lim_{k \to \infty} e_k = 0 \) such that, for a subsequence of the \( e_k, u_k \in V(e_k, e_k) \).

Conversely, let \( n \) be a positive integer, let \( (e_k) \) be a sequence in \( (0, \infty) \) with \( \lim_{k \to \infty} e_k = 0 \) and let \( u_k \) be a sequence in \( \mathcal{S}_\lambda \) such that \( u_k \in V(n, e_k) \); then \( J(u_k) \to 0 \) and \( J(u_k) \to 0 \).

Sacks-Uhlenbeck [16] and Wente [27] pioneered this kind of conclusion; the paper [16] deals with harmonic maps, and [27] with H-systems. Improvements have been obtained by Meeks-Yau [13] and Siu-Yau [19] for harmonic maps and by Brezis and Coron [6] for H-systems. A similar description has been obtained, using Uhlenbeck's papers [25], [26], by Taubes [21], [22] for the Yang-Mills and the Yang-Mills-Higgs equations (see also Donaldson-Kron [10] and S Kidane [18]). Struwe [20] has obtained a result which is very close to Proposition 4. The conditions (4) and (5) appear for the first time in [6]. Lions' paper [11] is also related to Proposition 4.

In order to prove Proposition 4 one can introduce the functional

\[
E(u) = \frac{1}{2} \int \nabla u^2 + \frac{N - 2}{2N} \int (u)^{2N/(N - 2)}, \quad u \in H^1_0(\Omega).
\]

Note that if \( u \in \mathcal{S}_\lambda \), then \( J(u_k) \to 0 \) if and only if \( E(u_k)^{1/2} \to 0 \). To get Proposition 4 one can now follow [6] step by step using the functional \( E \). Note that the proof of [6] is inspired by the method of concentration compactness due to Lions [11].

For \( c \) in \( (0, \infty) \), let \( \mathcal{J}_c = \{ u \in \mathcal{S}_\lambda, J(u) \leq c \} \). It follows from Proposition 4 that if \( c_1, c_2 \) are two real numbers such that \( b_\lambda \leq c_1 \leq c_2 \leq b_\lambda \), for some integer \( n \), then \( J_{c_2} \) is a strong deformation retract of \( J_{c_1} \). In the following we set \( H_{c_2} = J_{c_2} \).

**Remarks.** a. Note that, if \( J(u_k) \to S \), then \( J(u_k) \to 0 \). Using this fact (or Lions [11] as in [8]) and Proposition 4 one can easily see that \( \Omega \) is homeomorphic
to a retract of $W_0$. This has been noticed and used in [8] (we also use it here—see (27)). It explains why the topology of $\Omega$ can play a role in the existence of a solution to (1). It has been conjectured in [8] that, if $\Omega$ is not contractible, then (1) has a solution. Corollary 2 solves this conjecture when $N = 3$ and Theorem 1 gives a partial answer when $N \geq 4$.

b. In [2], [3] the orbits in $\mathcal{F}(n, e)$ are studied and the "critical points at infinity", i.e., the orbits of $-J'$ which stay in $\mathcal{F}(n, e)$ are described. Their description involves the Green's function and its regular part, which indicates that the geometry of the domain should also be important for the existence of a solution to (1), and leads to the formula for the topology of $W'/W_{e-1}$ given in [4]. Even though we do not need this formula, it has helped us to find the topological argument described in Section 3, as one can see by looking at its proof sketched in [4]. In this sketch we use the formula for the topology of $W'/W_{e-1}$; it makes the topological argument more transparent. A similar method (i.e., to find the critical points at infinity and try to prove, in the absence of a solution, that there is a topological contradiction) has been used in [1].

c. A similar method to the one used in this present paper has very recently allowed to prove a result more general than the Yamabe conjecture on manifolds of dimension strictly less than six (see A. Bahri-H. Brezis [29]).

We continue Section 2 by presenting a classical deformation argument (see e.g. [14]).

2.2. A classical deformation argument. In this subsection $n$ is a positive integer which is fixed. Let $\Theta$ and $\dot{e}$ be two strictly positive real numbers; first $\Theta$ will be fixed large enough and then $\dot{e}$ small enough. Let $\mu$ be a function in $C^\infty(0, \infty); \mathbb{R}^+\}$ such that

$$\mu(0) = \dot{e},$$
$$-\frac{2}{r} \leq \mu' \leq 0,$$
$$\mu(r) = 0 \quad \text{for} \quad r \in [\Theta, +\infty].$$

Let now $F: \Sigma \to \mathbb{R}$ be defined by

$$F(u) = \begin{cases} J(u) - \mu(|J(u)|^2) & \text{for} \quad J(u) \leq \left( n + \frac{1}{2} \right)^{1/(n-1/2)}, \\ J(u) & \text{elsewhere}. \end{cases}$$

$F$ is $C^1$ (if $\Theta$ is small enough—use Proposition 4); let

$$K(u) = |J'(u)|^2.$$

An easy computation shows that there exists a constant $M$ such that

$$|K'(v) \cdot J'(v)| \leq M |J'(v)|^2$$

for all $v \in \Sigma$, with $|J'(v)| \leq b_{n+1}$. 


We now fix $\theta > 2M$. It follows from (6) and (7) that
\begin{equation}
F(v) \cdot J'(v) > 0 \quad \text{for all} \quad v \in \Sigma_+ \quad \text{with} \quad J(v) \leq b_{r+1}.
\end{equation}

Let
\[ F_e^* = \{ u \in \Sigma_+, |F(u) \leq e\}. \]

We assert the following (if $\varepsilon$ is small enough, see below):

**Proposition 6.** The pair $(F_e^*, W_{e-\varepsilon})$ is a strong deformation retract of the pair $(W_0, W_{e-\varepsilon})$.

Proof of Proposition 6: Let $f: [0, \infty) \times \Sigma \to \Sigma$ be the solution of
\begin{equation}
\frac{\partial}{\partial t}f(t, u) = -J'(f(t, u)),
\end{equation}

\[ f(0, u) = u. \]

(In [14], $f$ is defined by $\frac{\partial f}{\partial t} = -F'(f)$, $f(0, u) = u$. This modification has been suggested to us by Z. R. Jin)

Using Proposition 4 we have
\[ \{ t \geq 0 | F(f(t, u)) \leq b_e \} \neq \emptyset \quad \text{for all} \quad u \in W_e. \]

Let, for $u$ in $W_e$,
\[ T(u) = \min \{ t \geq 0 | F(f(t, u)) \leq b_e \}. \]

It follows from (8) and (9) that $T$ is continuous. Moreover, since $W_{e-1} \subset F_e^*$, we have
\begin{equation}
T(u) = 0 \quad \text{for all} \quad u \in W_{e-1}.
\end{equation}

We now define $\beta: [0, 1] \times W_e \to W_0$ by
\begin{align*}
\beta(t, u) =
\begin{cases}
\frac{t}{1-t} f(u) & \text{if } t \leq T(u) \quad \text{and} \quad t \neq 1, \\
T(u) & \text{if } T(u) = T(u) \quad \text{and} \quad t = 1, \\
f(T(u), u) & \text{if } T(u) = 1.
\end{cases}
\end{align*}

Then $\beta$ is continuous, $\beta(0, u) = u$ for any $u$ in $W_e$, $\beta(1, u) \in F_e^*$ for any $u$ in $W_e$, and finally $\beta(t, u) = u$ for any $u$ in $F_e^*$. This proves Proposition 6.

In Section 3 we conclude the proof of Theorem 2. In order not to interrupt the main thread of the topological argument we have placed many of the estimates needed in appendices.
3. The Topological Argument

First let us remark that, with the notations of Section 2 and \( n \) fixed, we have, using Proposition 4:

\[
\text{for all } \varepsilon > 0 \text{ there exists } \epsilon_1 > 0 \text{ such that } 0 < \epsilon < \epsilon_1 \Rightarrow F^b_n \setminus W_{n-1} \subseteq V(n, \epsilon).
\]

Hence (for \( \varepsilon \) small enough, \( \varepsilon \) being given), if we denote by \( i \) the inclusion map

\[
(F^b_n \cap V(n, \epsilon), W_{n-1} \cap V(n, \epsilon)) \to (F^b_n, W_{n-1}),
\]

then

\[
i_* \text{ is an isomorphism.}
\]

We are now going to give a parametrization of \( V(n, \epsilon) \). Let

\[
\psi: (0, \infty)^n \times \Omega^* \times (0, \infty)^n \to \Sigma,
\]

be defined by

\[
\psi(a, x, \lambda) = \left( \sum_{i=1}^{n} a_i \delta(x_i, \lambda_i) \right) \bigg/ \left( \sum_{i=1}^{n} a_i \delta(x_i, \lambda_i) \right),
\]

where \( a = (a_1, \cdots, a_n), x = (x_1, \cdots, x_n) \) and \( \lambda = (\lambda_1, \cdots, \lambda_n) \), and let \( B_{n} \) be the set of \((a, x, \lambda) \) in \((0, \infty)^n \times \Omega^* \times (0, \infty)^n\) such that

\[
\lambda_i d(x_i, \partial \Omega) > \varepsilon^{-1} \text{ for all } i, \quad \frac{\lambda_i + \lambda_j}{\lambda_i} + \lambda_i |x_i - x_j|^2 > \varepsilon^{-1} \text{ for all } i, j \text{ with } i \neq j, \quad \frac{1}{L^{2n}} < a_i < 2 \text{ for all } i.
\]

Let

\[
\varepsilon \sim \left( \frac{1}{L}, \frac{1}{L}, \cdots, \frac{1}{L} \right) \in (0, \infty)^n.
\]

Then

\[
V(n, \epsilon) = \{ u \in \Sigma \mid \text{there exist } (x, \lambda) \text{ with } (e, x, \lambda) \in B_{n} \text{ such that } ||u - \psi(e, x, \lambda)|| \leq \epsilon \}.
\]
We have the following proposition:

**Proposition 7.** For all \( n \), there is a \( c_n > 0 \) such that, for any \( u \) in \( V(n, c_n) \), the problem

\[
\min \| u - \psi(a, x, \lambda) \| \quad \text{for} \quad (a, x, \lambda) \in B_{c_n}
\]

has a unique solution (up to permutations).

The proof of Proposition 7 is given in Appendix A.

For a function \( u \) in \( V(n, c_n) \) let \((a, x, \lambda)\) be the unique solution (up to permutations) of the minimization problem in Proposition 7. Let \( X : V(n, c_n) \to \Omega^*/\nu \) be the map defined by \( X(u) = x \). Note that since \( \nu \) has uniqueness only up to permutations, \( X(u) \) is not in \( \Omega^* \) but in \( \Omega^*/\nu \) (an usual \( \nu \) denotes the group of permutations of \( \{1, \ldots, n\} \)).

Let \( K^* \) be a compact set in \( \Omega^* \), and let

\[
\Delta_{n-1} = \left\{ (t_1, \ldots, t_n) | t_i \in [0, 1] \text{ for all } i \text{ and } \sum_{i=1}^n t_i = 1 \right\},
\]

\[
B_\delta(K) = \left\{ \sum_i \delta_{x_i} | (x_1, \ldots, x_n) \in K^*, (t_1, \ldots, t_n) \in \Delta_{n-1} \right\},
\]

where \( \delta_{x_i} \) is the (true) Dirac mass at the point \( x_i \). We provide \( B_\delta(K) \) with the weak topology of measures, \( B_\delta(K) \), with its topology, can also be viewed as the quotient of \( K^* \times \Delta_{n-1} \), with its usual topology, by some equivalence relation that we shall denote \( \sim \). For example, when \( n = 2 \), \((x_1, x_2, t_1, t_2) \sim (x_1, x_2, t_2, t_1) \) and \((x_1, x_2, 0, 1) \sim (x_1, x_2, 1, 0) \).

Let

\[
R : H^2(\Omega) \setminus \{0\} \to \Sigma,
\]

\[
Ru = \frac{u}{\|u\|},
\]

and let \( g_* : K^* \times \Delta_{n-1} \to \Sigma \) be defined by

\[
g_*((x_1, \ldots, x_n), (a_1, \ldots, a_n)) = R \left( \sum_{i=1}^n a_i P(x_i, \lambda) \right),
\]

where \( \lambda \) is fixed in \((0, \infty)\) (\( \lambda \) will be taken large). Two elements of \( K^* \times \Delta_{n-1} \) which are equivalent for \( \sim \) have the same image under \( g_* \), hence \( g_* \) defines a map \( g_* : B_\delta(K) \to \Sigma \). It follows from Corollary B.3 in Appendix B that, if \( \lambda \) is large enough, \( g_* (B_\delta(K)) \subset W_* \). Moreover, Proposition B.1 tells us that the following is true.
Proposition 8. There exists a positive integer $n_0$ and $\lambda_2$ in $(0, \infty)$ such that, if $\lambda \geq \lambda_0$, $\mathcal{S}_\lambda(K(K)) \subset \mathcal{W}_\lambda - 1$. Throughout this section we shall denote by $H_{S_{\lambda+1}}(\Sigma)$ (respectively $H_{S_{\lambda+1}}(K)$) the homology (respectively the cohomology) with $Z_2$-coefficients. By convention, $\mathcal{B}_\lambda(K)$ will be the empty set (note that $\mathcal{W}_0$ is also the empty set) and we shall assume that $K$ is a regular manifold (possibly with boundary). Let

$$S_{\lambda+1} = \{ x \in K^* | \text{there exist } i \in [1, n], \ j \in [1, n] \text{ with } x_i \neq x_j \text{ and } i \neq j \},$$

and let $T_\lambda$ be an open neighborhood of $S_{\lambda}$ in $K^*$ which is invariant by $S_{\lambda}$ and such that (in order to construct such a $T_\lambda$, one can proceed as in Appendix C)

$$K^*_\lambda = K^* \setminus T_\lambda$$

(14)

is a manifold (with boundary).

(15) $S_{\lambda}$ is a strong $S_{\lambda}$-equivariant deformation retract of $T_\lambda$.

Statement (15) means that there exists a strong deformation retraction map of $T_\lambda$ to $S_{\lambda}$ which is $S_{\lambda}$-equivariant. Note that $S_{\lambda}$ acts on $K^* \times \Delta_{n-1}$, $T_\lambda \times \Delta_{n-1} \cup K^* \times \partial \Delta_{n-1}$ and $S_{\lambda} \times \Delta_{n-1} \cup K^* \times \partial \Delta_{n-1}$ by

$$\tau((x_1, \ldots, x_n), (s_1, \ldots, s_n)) = ((x_1, \ldots, x_n), (s_{\tau(1)}, \ldots, s_{\tau(n)}))$$

for $\tau \in S_{\lambda}$;

we shall denote by $K^* \times \Delta_{n-1}, \ K^* \times \partial \Delta_{n-1}, \ S_{\lambda} \times \Delta_{n-1}, \ S_{\lambda} \times \partial \Delta_{n-1}$ the quotient spaces. Note that for any $(x, s)$ in $K^* \times \Delta_{n-1}$ and any $\tau$ in $S_{\lambda}$ we have $(\tau(x), s) = \tau(x, s)$ and hence there exists a natural projection

$$h_\lambda: K^* \times \Delta_{n-1} \to B_\lambda(K);$$

$h_\lambda$ maps the pair $(K^* \times \Delta_{n-1}, S_{\lambda} \times \Delta_{n-1})$ into the pair $(B_\lambda(K), B_\lambda(K))$ and so defines a map

$$h_\lambda: H_{\lambda}(K^* \times \Delta_{n-1}, S_{\lambda} \times \Delta_{n-1}) \to \text{Ker}(k)$$

(16)

into $H_{\lambda}(B_\lambda(K), B_\lambda(K))$. Note that $h_\lambda$ is an isomorphism.

Indeed, $h_\lambda$ defines a homeomorphism between

$$K^* \times \Delta_{n-1} \setminus \{ S_{\lambda} \times \Delta_{n-1} \cup K^* \times \partial \Delta_{n-1} \} \to B_\lambda(K) \setminus B_\lambda(K)$$

and $B_\lambda(K) \setminus B_\lambda(K)$. 

and $S_\infty \Delta_{-1}$ is a strong deformation retract of one of its closed neighborhoods in $K^\infty \times \Delta_{-1}$.

The cap product

$$H^*(K^\infty \times \Delta_{-1}) \otimes H_\infty(K^\infty \times \Delta_{-1}, S_\infty \times \Delta_{-1}) \bigcup K^\infty \times \partial \Delta_{-1})$$

$$\to H_\infty(K^\infty \times \Delta_{-1}, S_\infty \times \Delta_{-1}) \bigcup K^\infty \times \partial \Delta_{-1})$$

provides $H_\infty(B_\varepsilon(K), B_{-\varepsilon}(K))$ with a structure of an $H^*(\Omega^n/\alpha)$-module via the isomorphism $b_\varepsilon$ and the homomorphism $a_\varepsilon^*$: $H^*(\Omega^n/\alpha) \to H^*(K^\infty \times \Delta_{-1})$ defined by the map

$$a_\varepsilon: K^\infty \times \Delta_{-1} \to \Omega^n/\alpha, a_\varepsilon(x, a) = x.$$ 

We shall denote this cap product by $\beta$. We shall denote by $\beta$ this cap product.

The map $\varepsilon_\infty$ defines a map $(B_\varepsilon(K), B_{-\varepsilon}(K)) \to (W_\varepsilon, W_{-\varepsilon})$ and hence a map

$$\beta_\infty: H_\infty(B_\varepsilon(K), B_{-\varepsilon}(K)) \to H_\infty(W_\varepsilon, W_{-\varepsilon}).$$

Our next proposition is

**Proposition 9.** The homology $H_\infty(W_\varepsilon, W_{-\varepsilon})$ has a natural structure of an $H^*(\Omega^n/\alpha)$-module and $\beta_\infty$ is $H^*(\Omega^n/\alpha)$-linear.

**Proof of Proposition 9:** The cap product

$$H^*(F_\varepsilon \cap V(n, e_0), W_{-\varepsilon} \cap V(n, e_0))$$

$$\to H^*(F_\varepsilon \cap V(n, e_0), W_{-\varepsilon} \cap V(n, e_0))$$

induces, by Proposition 6 and (12), the structure of an $H^*(\Omega^n/\alpha)$-module on $H_\infty(W_\varepsilon, W_{-\varepsilon})$. Moreover, using Proposition 7, we have defined a map $X: V(n, e_0) \to \Omega^n/\alpha$. Therefore, $H_\infty(W_\varepsilon, W_{-\varepsilon})$ is also, via the homomorphism

$$[X]_{V(n, e_0) \cap \Delta_{-1}}$$

an $H^*(\Omega^n/\alpha)$-module. We shall denote this cap product also by $\beta$. We are now going to prove that $\beta_\infty$ is $H^*(\Omega^n/\alpha)$-linear. Let, for $\eta$ in $(0, 1),

$$\Delta_{-1} = \{\eta(a_0 - \frac{1}{n}) + \frac{1}{n}, \eta(a_0 - \frac{1}{n}) + \in \Delta_{-1}\},$$

$$\eta \Delta_{-1} = \{\eta(a_0 - \frac{1}{n}) + \frac{1}{n}, \eta(a_0 - \frac{1}{n}) + \in \Delta_{-1}\},$$

$$\Delta_{-1} = \Delta_{-1} \setminus \eta \Delta_{-1}.$$
Let \( \tilde{g}_\lambda = g_\lambda \ast b_\nu \), \( d(T_\lambda) = \max_{\lambda, \mu} \min_{x, y} |x - y| \). It follows from the regularity of \( K \) that for any \( d > 0 \) there exists \( T_\lambda \) satisfying (14) and (15) and such that \( d(T_\lambda) < d \). Note that

\[
\lim_{\lambda \to \infty} \left( e \sum_{\lambda} \mu \left( \sum_{\lambda} \| f \|_{L^p}^p \right)^{1/p} \right)^{1/p} = S \left( \sum_{\lambda} \| f \|_{L^p}^p \right)^{1/p}\]

whenever \((a, x) \in \Delta_{\lambda, -1} \times K^t \) and satisfies \( x_i = x_j \) for all \( i \neq j \). Therefore, using Lemma B.7 and Lemma B.4, \( \mu \) can select \( \eta \) in \( (0, 1) \), \( d \) small enough, and then \( \lambda \) large enough in such a way that

\[
\tilde{g}_\lambda \left( K^t \times \Delta_{\lambda, -1} \right) \subset W_{\lambda, -1},
\]

\[
\tilde{g}_\lambda \left( K^t \times \Delta_{\lambda, -1} \right) \subset F_{\lambda, -1},
\]

\[
\tilde{g}_\lambda \left( K^t \times \Delta_{\lambda, -1} \right) \subset F_{\lambda, -1},
\]

where we have chosen \( T_\lambda \) satisfying (14)–(15) and such that \( d(T_\lambda) \leq d \). Hence the following diagram is commutative:

\[
\begin{array}{ccc}
K^t \times \Delta_{\lambda, -1}, & K^t \times \Delta_{\lambda, -1} & \rightarrow (P_{\lambda, -1}, W_{\lambda, -1}) \\
\downarrow i & & \downarrow j \\
K^t \times \Delta_{\lambda, -1} & \rightarrow (P_{\lambda, -1} \cap W_{\lambda, -1} \cap V(n, \epsilon)) \\
\end{array}
\]

where \( i, j \) and \( \iota \) are inclusion maps. Note that \( i_\lambda \) and \( i_\eta \) are isomorphisms (see (12) and use (34)). Moreover, if

\[
i_\eta: \left( K^t \times \Delta_{\lambda, -1}, \Delta_{\lambda, -1} \times \Delta_{\lambda, -1} \right) \rightarrow (K^t \times \Delta_{\lambda, -1} \Delta_{\lambda, -1} \times \Delta_{\lambda, -1})
\]

is the inclusion map, then $k$ follows from (15) that $k_*$ is an isomorphism; hence Proposition 9 follows from the commutativity of (17).

Since $H_k(\Omega) \neq 0$, it follows from Thom [23] that there exists a $d$-dimensional compact connected $C^\infty$-manifold without boundary, $\gamma$, and a continuous map $h: \gamma \to \Omega$ such that if we denote by $[\gamma]$ the class of orientation (modulo 2) of $\gamma$, then $h_*[\gamma] \neq 0$. Clearly, there exists a compact $C^\infty$-manifold with boundary $K$ such that $h(\gamma) \subset K \subset \Omega$. We define $B_K(V)$ in the same way as we have defined $B_\Omega(V)$. We define also

$$S_1^* = \{ x \in V^* \mid \text{there exist } i \in [1, n], j \in [1, n] \text{ such that } x_i = x_j \text{ and } i \neq j \}$$

$$k_*: V^* \times \Delta_{n-1} \to K^* \times \Delta_{n-1}, \quad k_*(x, a) = (h(x_1), \ldots, h(x_n), a),$$

$$g_*: B_\Omega(V) \to W_\Omega, \quad g_*\left( \sum a_i \delta_{a_i} \right) = \sum a_i \delta_{h(a_i)},$$

$$a_*^*: V^* \times \Delta_{n-1} \to \Omega^*/\sigma_n, \quad a_*^*(x, a) = (h(x_1), \ldots, h(x_n)),$$

and finally,

$$b_*: \left(V^* \times \Delta_{n-1}, S_1^* \times \Delta_{n-1} \cup V^* \times \partial \Delta_{n-1} \right) \to (B_\Omega(V), B_{\Omega,1}(V))$$

is the natural projection. As above (see (16))

$$b_*$ is an isomorphism.

The cap product

$$H^*(V^* \times \Delta_{n-1}) \otimes H_\Omega(V^* \times \Delta_{n-1}, S_1^* \times \Delta_{n-1} \cup V^* \times \partial \Delta_{n-1})$$

$$\to H_\Omega(V^* \times \Delta_{n-1}, S_1^* \times \Delta_{n-1} \cup K^* \times \partial \Delta_{n-1})$$

provides a structure of an $H^*(V^* \times \Delta_{n-1})$-module via the isomorphism $b_*^*$. We shall denote by $\ast$ this product. This product provides $H_\Omega(B_\Omega(V), B_{\Omega,1}(V))$ with a structure of an $H^*(\Omega^*/\sigma_n)$-module via the homomorphism $a_*(-)$; we shall denote by $\ast$ this new product. Note that $g_*$ maps the pair $(B_\Omega(V), B_{\Omega,1}(V))$ into the pair $(W_\Omega, W_{\Omega,1})$—we agree to assume that $B_\Omega(V) = \varnothing$. We have

$$g_*: H_\Omega(B_\Omega(V), B_{\Omega,1}(V)) \to H_\Omega(W_\Omega, W_{\Omega,1})$$

is $H_\Omega(\Omega^*/\sigma_n)$-linear.
Indeed we have the following commutative diagram:

\[
\begin{array}{ccc}
\Omega^*/\mathcal{A}_1 & \xrightarrow{\text{identity}} & \Omega^*/\mathcal{A}_1 \\
\kappa^* \downarrow & & \kappa^* \downarrow \\
(V^* \times \Delta_{n-1}, S^*_n \times \Delta_{n-1}, \bigcup_{\kappa_n} V^* \times \partial \Delta_{n-1}) & \xrightarrow{\mathcal{L}_n} & (K^* \times \Delta_{n-1}, S^*_n \times \Delta_{n-1}, \bigcup_{\kappa_n} K^* \times \partial \Delta_{n-1})
\end{array}
\]

\[
\begin{array}{ccc}
\mathcal{L}_n \downarrow & & \mathcal{L}_n \downarrow \\
(B_0(V), B_{n-1}(V)) & \xrightarrow{\mathcal{E}_n} & (W_0, W_{n-1}) \xrightarrow{\mathcal{E}_n} (B_0(K), B_{n-1}(K)).
\end{array}
\]

Hence (19) follows from Proposition 9. Let \( T'_n \) be an open neighborhood of \( S'_n \) in \( V^* \), \( \alpha_n \)-invariant and such that

\[
\nu_n = V^* \setminus T'_n \text{ is a manifold with boundary,}
\]

(20) \( S'_n \) is a strongly \( \alpha_n \)-equivariant deformation retract of \( T'_n \).

(See Appendix C for an example of \( T'_n \).)

Let

\[
\begin{align*}
\mathcal{L}_n^* & : \left( V^*_n \times \Delta_{n-1}, \partial \left( V^*_n \times \Delta_{n-1} \right) \right) \\
& \rightarrow \left( V^*_n \times \Delta_{n-1}, T^*_n \times \Delta_{n-1}, \bigcup_{\kappa_n} V^* \times \partial \Delta_{n-1} \right), \\
\mathcal{L}_n^* & : \left( V^*_n \times \Delta_{n-1}, S^*_n \times \Delta_{n-1}, \bigcup_{\kappa_n} V^* \times \partial \Delta_{n-1} \right) \\
& \rightarrow \left( V^*_n \times \Delta_{n-1}, T^*_n \times \Delta_{n-1}, \bigcup_{\kappa_n} V^* \times \partial \Delta_{n-1} \right)
\end{align*}
\]

be inclusion maps. It follows from (20) and (21) that \( \mathcal{L}_n^* \) and \( \mathcal{L}_n^* \) are isomorphisms. Let

\[
k_n^* : H_\alpha(B_0(V), B_{n-1}(V)) \rightarrow H_\alpha \left( V^*_n \times \Delta_{n-1}, \partial \left( V^*_n \times \Delta_{n-1} \right) \right)
\]

be defined by

\[
k_n^* = (\mathcal{L}_n^*)^{-1} \mathcal{L}_n^*(\mathcal{E}_n)^{-1};
\]

\( k_n \) is an isomorphism.
Note that $V^*_\Delta_{n-1}$ is a manifold with boundary; let $[V^*_\Delta_n \Delta_{n-1}]$ be the (modulo 2) orientation class of this manifold and let

$$
[V^*_\Delta_n \Delta_{n-1}] = \delta(V^*_\Delta_n \Delta_{n-1}) \in H_{d+1}(B_\lambda(V), B_{\lambda-1}(V)).
$$

We are going to prove, by induction on $n$, that

$$
s_{\Delta_n}([B_\lambda(V), B_{\lambda-1}(V)]) \neq 0 \quad \text{for all} \quad n \in \mathbb{N} \setminus \{0\}
$$

which is in contradiction with Proposition 8. Let $\omega \in H^d(\Sigma)$ be such that $(\omega, h_q([F])) = 1$ and let $\omega_p = h_q(\omega)$. We denote by $\sigma_\chi \times \sigma_{n-1}$ the subgroup of $\sigma_n$ which contains the permutations of $(1, \ldots, n)$ leaving 1 invariant. The transfer—we shall denote it by $tr$—defines (see e.g. Breconn [5]) a map from $H^d(\Sigma/\sigma_\chi \times \sigma_{n-1})$ into $H^d(\Sigma)$ and a map from $H^d(\Sigma/\sigma_\chi \times \sigma_{n-1})$ into $H^d(\Sigma/\sigma_\chi \times \sigma_{n-1})$. Let $\sigma_\chi \times \sigma_{n-1} \rightarrow \sigma_n$ be the projection on the first factor of $\Sigma$ and let $p: V^*_\Delta_n \Delta_{n-1} \rightarrow V$ be also the projection on the first factor of $V^*_\Delta_n \Delta_{n-1}$. Let us consider the following commutative diagram:

$$
\begin{array}{ccc}
H_\lambda(B_\lambda(V), B_{\lambda-1}(V)) & \xrightarrow{\delta} & H_\lambda(W_\lambda, W_{\lambda-1}) \\
\downarrow \quad s \quad \downarrow \quad s \\
H_\lambda(B_{\lambda-1}(V), B_{\lambda-1}(V)) & \xrightarrow{\delta_{\lambda-1}} & H_\lambda(W_{\lambda-1}, W_{\lambda-2})
\end{array}
$$

(24)

where the $\delta$ are the usual connecting homomorphisms. In Appendix C we prove that

$$
\delta((tr \, p^*\omega_p) \ast [B_\lambda(V), B_{\lambda-1}(V)]) = [B_{\lambda-1}(V), B_{\lambda-2}(V)].
$$

Using (19), (24), (25) and the functoriality of the transfer (see [5]) we have

$$
\delta((tr \, \pi^*\omega) \ast [B_\lambda(V), B_{\lambda-1}(V)]) = (\omega \ast [B_{\lambda-1}(V), B_{\lambda-2}(V)]).
$$

Let $e$ be the canonical generator of $H_\lambda(\Sigma)$ and using (19) once more we have

$$
s_{\Delta_n}([B_{\lambda}(V), B_{\lambda}(V)]) = \omega \ast s_{\Delta_n}([V]).
$$

and, therefore, since $s_{\Delta_n}([V]) \neq 0$ and $[V] = [B_\lambda(V), B_{\lambda}(V)]$,

$$
s_{\Delta_n}([B_\lambda(V), B_{\lambda}(V)]) \neq 0;
$$

(23) follows from (26) and (27) by induction on $n$. 

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COMMENTS 10. a. An important point in our proof is the "interaction" between the "particles" (i.e., the functions $P_0(a, \lambda)$). This interaction is computed in Appendix B (see in particular Proposition B.5) and it leads to Proposition 8. This interaction phenomenon has been used by Suu-Yan [9]. It has also been computed by Taubes [22] for the Yang-Mills-Higgs equations on $\mathbb{R}^3$, and allowed him to prove that for these equations the functional is a "good Morise function" (see [22] for the definition). This is also the case for our equation but only in the case $\Sigma \setminus J^+$ with $c$ large (this $c$ depends on $\Omega$; see [44]). In addition, in [21], Taubes computed the interaction between two particles for the Yang-Mills equations $S^4$, he has used it to prove that the analogue of $J^+ \cap \Sigma^+$ (which is not empty for these equations) is connected.

b. It follows from the universal-coefficients formula that $H_2(\Omega; \mathbb{Q}) \neq 0$ implies $H_2(\Omega; \mathbb{Z}) \neq 0$. When $d$ is odd and $H_2(\Omega; \mathbb{Q}) \neq 0$, one can prove the existence of a solution to (1) without using the transfer (see Appendix D).

c. One can find a different presentation of the topological argument in [3].

Appendix A

In this appendix we give a proof of Proposition 7. We recall that $B_\epsilon$ is the set of $(a', x, \lambda)$ in $\mathbb{R}^n \times \Omega^a \times (0, \infty)^s$ such that

(A.1) \[ \lambda_i, d(x_i, a) > \epsilon^{-1} \quad \text{for all } i, \]

(A.2) \[ \lambda_i / \lambda_j + \lambda_j / \lambda_i + \lambda_i / x_i - x_j^2 > \epsilon^{-1} \quad \text{for all } i \neq j, \]

(A.3) \[ \frac{1}{2s/n} - \alpha < 2 \quad \text{for all } i. \]

The symmetric group $S_n$ acts on $B_\epsilon$. We start with some lemmas.

LEMMMA A.1. Let $(a_k)$ be a sequence with $a_k \to 0$ and $\lim_{k \to +\infty} a_k = 0$ and let $(a', x', \lambda') \in B_\epsilon$. ($a^k, x^k, \lambda^k) \in B_\epsilon$ be such that

(A.4) \[ \lim_{k \to +\infty} \|\psi(a^k, x^k, \lambda') - \psi(a', x', \lambda')\| = 0. \]

Then (modulo permutations on $(a', x', \lambda')$),

(A.5) \[ \lim_{k \to +\infty} (\lambda_i \lambda'_i) = 1 \quad \text{for all } i \in [1, n], \]

(A.6) \[ \lim_{k \to +\infty} \lambda_i (x_i^k - x'_i)^2 = 0 \quad \text{for all } i \in [1, n], \]

(A.7) \[ \lim_{k \to +\infty} |a_i^k - a'_i| = 0 \quad \text{for all } i \in [1, n]. \]
Proof of Lemma A.1: Let $\bar{\delta}(a, \lambda) = P(\bar{\delta}_a, \lambda)$. Note that
\[
\lim_{\lambda \to \lambda^*} \|\bar{\delta}(a, \lambda)\| = 1
\]
and that
\[
\lim_{\lambda \to \lambda^*} \int \nabla \bar{\delta}(a, \lambda) \cdot \nabla \bar{\delta}(a', \lambda') = 0.
\]
It follows from (A.8) and (A.9) that there exists $c$ in $\mathbb{R}^+$ such that, for all $i \in \{1, n\}$ and all $k$, there exists $j$ such that
\[
\lambda^*_j \frac{a^*_j}{\lambda^*_k} + \lambda^*_j \frac{a^*_j}{\lambda^*_k} + \lambda^*_j \frac{a^*_j}{\lambda^*_k} \leq c,
\]
and, clearly, if $k$ is large enough, $i$ and $k$ being given, there exists one and only one $j$ which satisfies (A.10) (use the fact that $(a^*_1, a^*_2, \lambda^*_1)$ and $(\bar{a}^*_1, \bar{a}^*_2, \bar{\lambda}^*_1)$ are in $\mathcal{B}_k$). Without loss of generality we may assume that $j = i$. In the following we shall denote by $o(1)$ various sequences which tend to $0$ as $k$ goes to $\infty$ and we shall omit the index $k$. Using (A.4) and (A.9) we have for all $i \in \{1, n\}$,
\[
\int \nabla \bar{\delta}_i(x_i, \lambda_i) - \bar{\delta}_i(x_i, \lambda_i) = o(1).
\]
Hence using (A.8) we obtain (A.7) as well as: for all $i \in \{1, n\}$,
\[
\int \nabla \bar{\delta}_i(x_i, y_i) - \nabla \delta_i(x_i, y_i)/2 = o(1).
\]
Let
\[
\omega(x) = c_i \left( \frac{1}{1 + |x|^2} \right)^{\frac{n-2}{2}}
\]
then
\[
\int_{\mathbb{R}^n} |\nabla \omega - \nabla \delta \bar{\bar{\delta}}(x_i - x_i, \lambda_i)|^2 = o(1)
\]
\[
= \int_{\mathbb{R}^n} |\nabla \delta_i(x_i, y_i) - \nabla \delta_i(x_i, y_i)|^2 = o(1)
\]
and using (A.10) and (A.11) we deduce (A.6) and (A.7). Our next lemma is
LEMMA A.2. There exists $\epsilon_0 > 0$ such that, for any $u \in V(\sigma, \epsilon)$ with $\epsilon \leq \epsilon_0$,

$$\inf_{(a, x, \lambda) \in B_{\lambda_1}} \|u - \varphi(a, x, \lambda)\|$$

is achieved in $B_{\lambda_2}$ and is not achieved in $B_{\lambda_1} \setminus B_{\lambda_2}$.

Proof of Lemma A.2: We argue by contradiction and use Lemma A.1. Let us, for example, prove that the infimum cannot be achieved in $B_{\lambda_1} \setminus B_{\lambda_2}$. If $\epsilon_0$ is small enough. If this is not true, then there exist sequences $(\epsilon_k)$ with $\epsilon_k > 0$ and $\epsilon_k = o(1)$, $((x^k, \lambda_k))$ such that $(e, x^k, \lambda_k)$ is in $\mathcal{P}_\epsilon$ with $\epsilon = (1/\alpha_1, \ldots, 1/\alpha_n)$ in $[0, 1]^n$, and $(x^k, \lambda_k) \in B_{\lambda_2} \setminus B_{\lambda_1}$ such that

$$\|\varphi(a^k, x^k, \lambda_k) - \varphi(e, x^k, \lambda_k)\| = o(1).$$

Using Lemma A.2, we see that (modulo permutations)

(A.12) \[ x_i^k / \lambda_i^k = o(1) + 1 \text{ for all } i \in [1, n]. \]

(A.13) \[ |x_i^k - \lambda_i^k|/\lambda_i^k = o(1) \text{ for all } i \in [1, n]. \]

One easily checks that (A.12), (A.13), $(e, x^k, \lambda_k) \in B_{\lambda_1}$ and $(a^k, x^k, \lambda_k) \in B_{\lambda_2} \setminus B_{\lambda_1}$ are not compatible for $k$ large enough.

We are now going to prove Proposition 7. We argue by contradiction. If Proposition 7 is false, then, by Lemma B.3, there exist sequences $(\epsilon_k)$ with $\epsilon_k > 0$ and $\epsilon_k = o(1)$, $u^k \in V(\sigma, \epsilon_k, \lambda^k)$ and $(a^k, x^k, \lambda^k) \in B_{\lambda_{k+1}} \setminus B_{\lambda_k}$ for which the following properties hold:

first,

(A.14) \[ (a^k, x^k, \lambda^k) \neq (a^k, x^k, \lambda^k); \]

second, if $e^k = u^k - \varphi(a^k, x^k, \lambda^k)$, $\theta^k = u^k - \varphi(a^k, x^k, \lambda^k)$, then

(A.15) \[ 0 = \int \nabla u^k \cdot \nabla \lambda^k - \int \nabla \varphi(a^k, x^k, \lambda^k) \text{ for all } i \text{ and all } k, \]

(A.16) \[ 0 = \int \nabla \varphi(a^k, x^k, \lambda^k) \cdot \partial \lambda^k \text{ for all } i \text{ and all } k, \]

(A.17) \[ 0 = \int \nabla \varphi(a^k, x^k, \lambda^k) \cdot \partial \lambda^k \text{ for all } i \text{ and all } k, \]

(A.18) \[ 0 = \int \nabla \varphi(a^k, x^k, \lambda^k) \cdot \partial \lambda^k \text{ for all } i \text{ and all } k, \]

where

$$\delta^k = \delta(x^k, \lambda^k), \quad \delta^k = \delta(x^k, \lambda^k),$$
As before we shall omit the index $k$. Using Lemma A.1 we have (modulo permutations)

\[
\lambda_j / \tilde{\lambda}_j = 1 + o(1),
\]
\[
\lambda_j \tilde{\lambda}_j (x_j - \tilde{x}_j) = o(1),
\]
\[
|\eta_j - \tilde{\eta}_j| = o(1).
\]

From (A.15) and (A.17) we get

\[
\sum_j \left( \eta_j \nabla P \delta_j - \bar{\eta}_j \nabla P \tilde{\delta}_j \right) \nabla \delta_j - \int \nabla \delta_j (\nabla \delta_j - \nabla \tilde{\delta}_j).
\]

Let $a_j = \tilde{\lambda}_j (x_j - \tilde{x}_j)$, $\eta_j = \tilde{\lambda}_j / \lambda_j - 1$, $\mu_j = a_j - \bar{a}_j$. Note that $|a_j| = o(1)$, $|\eta_j| = o(1)$, $|\mu_j| = o(1)$. In the following, $c$ will denote various constants which do not depend on $k$. It is easy to see that

\[
|\delta_j (y) - \tilde{\delta}_j (y)| \leq c(|\eta_j| + |a_j|) \delta_j (y).
\]

and, since $-\Delta \delta_j \geq 0$, we have

\[
|\left( P \delta_j - P \tilde{\delta}_j \right) (y) | \leq c(|\eta_j| + |a_j|) \delta_j (y).
\]

Notice that

\[
\int \left( a_j \nabla ^2 \delta_j - \bar{a}_j \nabla P \tilde{\delta}_j \right) \nabla \delta_j = (a_j - \bar{a}_j) \int \nabla P \delta_j \nabla \delta_j
\]

\[
+ \bar{a}_j \delta_j \int \left( P \delta_j - P \tilde{\delta}_j \right).
\]

Relations (A.19), (A.21) and (A.22) yield (note that $\int |\nabla \delta_j |^2 = o(1)$)

\[
p_j + \bar{a}_j \delta_j \int \eta_j P \delta_j - \bar{\eta}_j \int \eta_j P \tilde{\delta}_j = o(1) \left( \sum_j (|\eta_j| + |a_j| + |\mu_j|) \right)
\]

\[
+ o(1) \int \left| \nabla \delta_j - \nabla \tilde{\delta}_j \right|^2^{1/2}.
\]

\[
\int \eta_j P \delta_j - \bar{\eta}_j \int \eta_j P \tilde{\delta}_j - \int \eta_j P (\delta_j - \tilde{\delta}_j).
\]
Using the maximum principle and (A.20), we see that

$$|\theta_r - \bar{\theta}_r| \leq c(|\eta_r| + |\eta|) \left( \frac{1}{\delta_r} \right)^{\frac{N-2}{2}}.$$  

Using (A.24), (A.25) and again (A.20), we obtain

$$\int \delta P(\theta_r - \bar{\theta}_r) = \eta + o(1)(|\eta_r| + |\eta|)$$

with

$$\eta = \int_{\mathbb{R}^n} \delta P(\theta_r - \bar{\theta}_r).$$

Moreover,

$$\eta = \int_{\mathbb{R}^n} \left( \frac{1}{1 + |\eta|^2} \right)^{\frac{N+2}{2}} \left( \frac{1}{1 + |\eta|^2} \right)^{\frac{N-2}{2}} \left( \frac{1 + \eta_r}{1 + |(1 + \eta_r)\eta + \eta|^2} \right)^{\frac{N-2}{2}} d\eta,$$

and

$$\left( \frac{1 + \eta_r}{1 + |(1 + \eta_r)\eta + \eta|^2} \right)^{\frac{N-2}{2}} = \left( \frac{1}{1 + |\eta|^2} \right)^{\frac{N-2}{2}} \left( 1 + \frac{N-2}{2} \eta_r \right) - \frac{|\eta|^2}{1 + |\eta|^2} - (N-2) \frac{\eta_r \eta}{1 + |\eta|^2} + O(|\eta|^2 = |\eta_r|^2).$$

where, as usual, $O(|\eta|^2 = |\eta_r|^2)$ denotes a sequence bounded by $c(|\eta_r|^2 + |\eta|^2)$

Hence,

$$\eta = \int_{\mathbb{R}^n} \left( \frac{1}{1 + |\eta|^2} \right)^{\frac{N-2}{2}} \left( 1 - \frac{2|\eta|^2}{1 + |\eta|^2} \right) d\eta + O(|\eta_r|^2 + |\eta|^2).$$

But

$$\int_0^\infty \frac{r^{N+1}}{(1 + r^2)^{N+1}} dr = \frac{1}{2N} \int_0^\infty \left( \frac{1}{1 + r^2} \right)^N r^N dr = \frac{1}{2} \int_0^\infty \frac{r^{N-1}}{(1 + r^2)^N} dr.$$
and, therefore,
\[(A.27)\]
\[\tau_i = O(|\alpha_i|^2 + |\eta_i|^2).\]

From (A.23), (A.26) and (A.27) we deduce
\[(A.28)\]
\[\mu_i = o(1) \left( \sum_{j} (|\eta_{ij}| + |\alpha_{ij}| + |\eta_{ji}|) \right) \text{ for all } i.

Using again (A.15) and (A.27) we have
\[
\begin{align*}
\sum_j f(a_j \nabla P \delta_j - \delta_j \nabla P \delta_j) \nabla \frac{\partial \delta}{\partial \lambda_i} &= (a_i - \delta_i) f \nabla P \delta_i \nabla \frac{\partial \delta_i}{\partial \lambda_i} \\
&\quad + \delta_i f \left( \nabla P \delta_i - \nabla P \delta_j \right) \nabla \frac{\partial \delta_j}{\partial \lambda_i},
\end{align*}
\]
and a similar computation as above leads to
\[(A.29)\]
\[0 = o(1) \left( \sum_j (|\eta_{ij}| + |\alpha_{ij}| + |\eta_{ji}|) \right) + \delta_i S f \left( P \delta_i - P \delta_j \right) \frac{\partial \delta_j}{\partial \lambda_i}.

Proceeding still as above one finds
\[(A.30)\]
\[f (P \delta_i - P \delta_j) \frac{\partial \delta_i}{\partial \lambda_i} = \tau'_i + o(1) \left( \frac{1}{\lambda_1} \right) (|\alpha_i| + |\eta_i|)
\]
with
\[
\tau'_i = \int_{\lambda} \frac{\partial \delta_j}{\partial \lambda_i} (\delta_i - \delta_j).
\]

We have
\[
\tau'_i = \frac{N + 2}{2\lambda_1} \int_{\Omega} \left( 1 - |\psi|^2 \right)^{\frac{N-2}{2}} \left( \frac{1}{1 + |\psi|^2} \right)^{\frac{(N-2)}{2}} \left( 1 + |\psi_{a_i,j}| \right)^{\frac{(N-2)}{2}} \text{d}x,
\]
and hence
\[(A.31)\]
\[\tau'_i = - \gamma_i \left( \frac{(N-2)(N+1)}{4\lambda_1} \right) \int_{\Omega} \left( 1 - |\psi|^2 \right)^{\frac{1}{2}} \left( \frac{1}{1 + |\psi|^2} \right)^{\frac{N-2}{2}} \text{d}x + \frac{1}{\lambda_1^2} O(|\alpha_i|^2 + |\eta_i|^2).
\]
It follows from (A.29), (A.30) and (A.31) that

$$\eta_i = o(1) \sum_j \left( |\eta_j| + |a_j| + |\mu_j| \right) \quad \text{for all } i.$$

Finally we use (A.16) and (A.18) and obtain

$$\sum_j \left( \sum_{\delta} \left( \sum_k \left( \delta \eta_k \right) \sum \delta x_k \right) \right) = \sum_j \left( \sum_{\delta} \left( \sum_k \left( \delta \eta_k \right) \sum \delta x_k \right) \right).$$

Computations similar to those above lead to

$$a_j = o(1) \left( \sum_j \left( |\eta_j| + |a_j| + |\mu_j| \right) \right).$$

From (A.28), (A.32) and (A.33) we deduce that, at least for $k$ large enough,

$$\eta_i = 0, \quad a_i = 0, \quad \mu_i = 0 \quad \text{for all } i \in \{1, n\},$$

in contradiction to (A.14).

**Appendix B**

In this section, $K$ is a $\delta$-closed compact set in $\Omega$; for $n = (a_1, \ldots, a_n)$ in $\Delta_{n-1}$, $z = (x_1, \ldots, x_n)$ in $K^*$, $\lambda$ in $(0, \infty)$ one defines

$$\bar{\psi}(a, x, \lambda) = J \left( \sum_{j=1}^n P_{j}(x_j, \lambda) \right).$$

Let $\psi(a, x, \lambda) = J(\psi(a, x, \lambda))$ for $(a, x, \lambda)$ in $\Delta_{n-1} \times K^* \times (0, \infty)$ and let $\psi(a, x, \lambda) = J(\psi(a, x, \lambda))$ for $(a, x, \lambda)$ in $\Delta_{n-1} \times K^* \times (0, \infty)$. In this appendix we are going to give some estimates on $\psi(a, x, \lambda)$ and $\phi(a, x, \lambda)$ In particular we shall prove

**Proposition B.1.** There exist a positive integer $n_0$ and a positive real number $\lambda_0$ such that

$$\lambda \geq \lambda_0 \Rightarrow \bar{\psi}(a, x, \lambda) \leq \delta |x|^{1/2} S \quad \text{for all } a \in \Delta_{n-1}, \quad \text{and all } x \in K^n.$$

For simplicity we write $\delta$, for $\delta(x, \lambda)$. We start with some lemmas.
Lemma B.2. \[ (B.2) \quad \psi(a, x, \lambda) \leq \frac{\int \left( \sum_{i=1}^{n} a_i \delta_i^{p-1} \right)^{p+1/2} \left( \sum_{i=1}^{n} \int a_i \delta_i^{p-1} \right)^{\lambda\gamma/(p+1)} \right) \] for all \( x \in \mathbb{R}^n \), all \( a \in \Delta_{n-1} \), and all \( \lambda \in (0, \infty)^n \), for all \( n \geq 1 \), where \[ a_i = \frac{\alpha_i \delta_i}{\sum_{j=1}^{n} a_j \delta_j}. \]

Proof of Lemma B.2: Let \[ u = \sum_{i=1}^{n} a_i \alpha_i \delta_i. \]

Notice that
\[ J(\partial u) = \frac{\int |\nabla u|^2}{\int u}^{p+1}. \]

For simplicity we shall write \( \Sigma_i \) instead of \( \Sigma_{n-1} \). We have
\[ \int |\nabla u|^2 = S \left( \int \left( \sum_{i=1}^{n} a_i \delta_i \right)^{p-1} \left( \sum_{i=1}^{n} \alpha_i \delta_i \right)^{\gamma/(p+1)} \right), \]

hence, by Hölder’s inequality,
\[ \int |\nabla u|^2 \leq S \left( \int \left( \sum_{i=1}^{n} a_i \delta_i \right)^{p-1} \left( \int \left( \sum_{i=1}^{n} \alpha_i \delta_i \right)^{\gamma/(p+1)} \right)^{p/(p+1)} \right)^{1/(p+1)}. \]

By the convexity of \( x \mapsto |x|^{p+1}/p \),
\[ \left( \sum_{i=1}^{n} a_i \delta_i \right)^{p+1} \leq \sum_{i=1}^{n} a_i \delta_i^{p+1}, \]
and, therefore,
\[
\left( \sum_{i} a_i h_i \right)^{p/(1+p)} \leq \left( \sum_{i} a_i h_i^{p/(1+p)} \right)^{1/(1+p)} \left( \sum_{i} a_i \right)^{p/(1+p)}
\]

Using now Hölder’s inequality,
\[
\left( \sum_{i} a_i h_i \right)^{p/(1+p)} \leq \left( \sum_{i} a_i \right)^{1/(1+p)}
\]
\[
\times \left( \int \left( \sum_{i} a_i h_i^{p/(1+p)} \right)^{(p/(1+p))} \right)^{1/(1+p)}
\]

(B.5)

By the convexity of \( x \rightarrow |x|^{p/(p-1)} \),
\[
\left( \sum_{i} a_i h_i^{p/(1+p)} \right)^{p/(p-1)} \leq \sum_{i} a_i h_i^{p-1},
\]

and, therefore, using (B.5) we have
\[
\left( \sum_{i} a_i h_i \right)^{p/(1+p)} \leq \left( \sum_{i} a_i \right)^{1/(1+p)} \left( \int \sum_{i} a_i h_i^{p-1} \right)^{1/(1+p)}
\]

(B.6)

Inequality (B.2) follows from (B.3), (B.4) and (B.6).

From Lemma B.2 we are now going to deduce

**Corollary B.3.** For all \( n > 0 \) and all \( \epsilon > 0 \), there exists \( \bar{\lambda} > 0 \) such that

\[
\lambda \in (\bar{\lambda}, \infty) \Rightarrow \phi(a, x, \lambda) \leq (n + \epsilon)^{p/(p-1)} S \quad \text{for all} \quad a \in \Delta_{n-1}
\]

and all \( x \in K^n \).

Proof of Corollary B.3: It follows from Lemma B.2 that, for \((a, x, \lambda) \in \Delta_{n-1} \times K^n \times (0, \infty)^n \),

\[
\phi(a, x, \lambda) \leq n^{p/(p-1)} S \left( \int \sum_{i} a_i \right)^{1/(1+p)}
\]

(B.7)
By the maximum principle we have

(B.8) \[ 0 \leq \delta_0 - P \delta_0 \leq \max \delta_i \leq \frac{c}{\lambda^{(n-2)/2}}. \]

where \( c \) is a constant (we recall that \( K \) is fixed). Corollary B.3 follows from (B.7) and (B.8).

We shall next prove

**Lemma B.4.** For any integer \( n \) in \([2, \infty)\), there exist a strictly positive real number \( \varepsilon \) and \( \lambda_2 \) in \((0, \infty)\) such that, for any \( x \in K^n \), for any \( \lambda \in [\lambda_2, \infty) \), and for any \( \varepsilon \) in \( \Delta_{\varepsilon, \lambda} \),

(B.9) \[ \text{there exists an } i \text{ with } a_i \leq \varepsilon \Rightarrow \varphi(a, x, \lambda) \leq n^{(n-1)/2}. \]

Proof of Lemma B.4: Let \( n \) be an integer in \([2, \infty)\). For \( x \in K^n \) and \( a \) in \( \Delta_{\varepsilon, \lambda} \) with \( a_i \neq 1 \), one defines \( \tilde{a} \) and \( \tilde{x} \) by

\[
\tilde{a} = \frac{1}{\sum_{i \geq 2} a_i} (a_2, \ldots, a_n) \in \Delta_{\varepsilon, \lambda},
\]

\[
\tilde{x} = (x_2, \ldots, x_n) \in K^{n-1}.
\]

Let \( \eta \) be in \((0, \infty)\); one easily sees that there exists \( \varepsilon \) in \((0, \infty)\) such that

for all \( a \in \Delta_{\varepsilon, \lambda} \), all \( x \in K^n \), and all \( \lambda \in [1, \infty) \),

(B.10) \[ a_i \leq \varepsilon \Rightarrow \varphi(a, x, \lambda) \leq \eta + \varphi(\tilde{a}, \tilde{x}, \tilde{\lambda}). \]

Lemma B.4 follows from Corollary B.3 and (B.10).

We shall give an expansion of \( \varphi(a, x, \lambda) \) when \( \min_j |x_j - x_j| \) is large. Let \( H(x, y) \) be the regular part of the Green's function, i.e.,

(B.11) \[ \Delta_x H(x, \cdot) = 0, \]

\[ H(x, y) = \frac{1}{|x - y|^{n-2}} \text{ if } y \not\in \partial \Omega, \]

and let \( G: \Omega \times \Omega \rightarrow \mathbb{R} \) be the Green's function:

\[ G(x, y) = \frac{1}{|x - y|^{n-2}} - H(x, y). \]
Let \( d = d(x) = \min_{r>0} |x_r - x| \) and \( \psi_n : \Delta_n \times K^\ast \times (0, \infty) \to \mathbb{R} \) be defined by

\[
\psi_n(x, \xi, \lambda) = g (\lambda \frac{r^{n+1}}{|a| r^{n+1}}) \left( 1 - \frac{c_1}{\lambda \frac{r^{n+1}}{|a| r^{n+1}}} \right) \sum_{r=1}^n \frac{H(x_r, \xi_r)}{|a| r^{n+1}} \left( 1 - \frac{2\lambda \frac{r^{n+1}}{|a| r^{n+1}}} 1 \right) + \sum_{(r,s) \neq (r, r)} \left( \frac{a_r a_s}{|a| r^{n+1}} - \frac{\alpha_r \alpha_s}{|a| r^{n+1}} \right) G(x_r, \xi_r) \right)
\]

with

\[
[a] = \left( \sum_{r=1}^n a_r^2 \right)^{1/2},
\]

\[
|a| = \left( \sum_{r=1}^n \frac{a_r^2}{r^{n+1}} \right)^{1/2},
\]

\[
c_1 = \frac{P + 1}{2} \int_0^{r^{n+1}} \frac{1}{|a| (1 + |y|^{2n+4/2})} dy,
\]

(see (2) for the definition of \( c_0 \)).

**Proposition B.5.** There exists a constant \( c(n) \) which depends only on \( n \) such that

\[
|\psi_n(x, \xi, \lambda) - \tilde{\psi}(x, \xi, \lambda)| \leq \frac{c(n)}{(\lambda d(x))^{n-1}}
\]

for any \( n \in \Delta_n \), any \( x \in K^\ast \) with \( d(x) > 0 \), and any \( \lambda \in (1, \infty) \).

Proof of Proposition B.5: Let \( (x, \xi, \lambda) \) be in \( \Delta_n \times K^\ast \times (1, \infty) \) and let \( \delta = P \delta_h, \lambda = \delta, \) and \( u = \sum_{r=1}^n \alpha_r \delta_r \). We start with the estimate of \( \int |\nabla u|^2 \).

We have

\[
\int \nabla \delta_h \nabla \delta_h = \int \nabla \delta_h \nabla \delta_h = S \int \delta_h \delta_h - h_j.
\]

\[
\int \delta_h \delta_h = \int_{K^\ast} \delta_h \delta_h - \int_{\partial K^\ast} \delta_h \delta_h = -1 - \int_{\partial K^\ast} \delta_h \delta_h.
\]

Let \( l = \text{dist}(K, \partial D) \) and let \( c \) be various constants which may depend on \( n \) but
only on \( n \) (we recall that \( K \) is fixed); \( O(\alpha) \) will denote functions such that \( |O(\alpha)| \leq c|\alpha| \). Note that, using Corollary B.3, we may assume that \( \lambda d(x) \geq 1 \).

We have
\[
\int_{\Omega \setminus \partial \Omega} \frac{8r^{-1}}{c} \leq c \int \left( \frac{\lambda}{1 + \lambda r^2} \right)^{n-1} r^{n-1} \, dr,
\]
and therefore
\[
\int_{\Omega \setminus \partial \Omega} 8r^{-1} \leq \frac{c}{\lambda^\nu}.
\]

On \( \partial \Omega \),
\[
h_\lambda(y) = e_{\Omega} \left( \frac{\lambda}{1 + \lambda |y - x|^2} \right)^{(N-1)/2},
\]
and hence
\[
h_\lambda(y) - \frac{e_{\Omega} \lambda^{(N-1)/2}}{|y - x|^{(N-2)/2}} \leq \frac{c}{\lambda^{N+2/2}}.
\]

Therefore, by the maximum principle,
\[
h_\lambda(y) - \frac{e_{\Omega}}{\lambda^{N+2/2}} H(y, x) \leq \frac{c}{\lambda^{N+2/2}} \text{ for all } y \in \Omega.
\]

Moreover,
\[
\int 8r \leq c \int_0^{2|y|} \left( \frac{\lambda}{1 + \lambda r^2} \right)^{N-1/2} r^{N-1} \, dr,
\]
and hence
\[
\int 8r \leq \frac{c}{\lambda^{N-2/2}},
\]
\[
\int 8r H(y, x) \, dy = \int_{B(x, \frac{1}{2})} 8r H(y, x) \, dy + \int_{\partial B(x, \frac{1}{2})} 8r H(y, x) \, dy,
\]
where \( B(x, \frac{1}{2}) = \{ y \in \mathbb{R}^N \mid |x - y| < \frac{1}{2} \} \) (and \( \xi = \delta(t/2) \)); hence, using (B.16),
\[
\int 8r H(y, x) \, dy = \int_{B(x, \frac{1}{2})} 8r H(y, x) \, dy + O \left( \frac{1}{\lambda^{N+2/2}} \right).
\]

Note that \( \Delta_y H(y, x) = 0 \); therefore, expanding \( H(y, x) \) near \( y = x \) and using
the symmetries of $\delta f$, we obtain
\[
\int_{\mathbb{R}^N} \delta_f H(y, x) \, dy = H(x, x) \int_{\mathbb{R}^N} \delta_f
\]
\[
+ O\left( \frac{\lambda^{N/2}}{1 + \lambda^2 x^2} \right)^{N+1/2} r^{N-1} dr.
\]
Consequently,
\[
\int_{\mathbb{R}^{N-1}/2} \delta_f H(y, x) \, dy \sim c_1 \frac{H(x, x)}{\lambda^{N-1/2}} + O\left( \frac{1}{\lambda^{N-1/2}} \right)
\]
with
\[
c_1 = c_1 \int_{\mathbb{R}^N} \left( \frac{1}{1 + |y|^2} \right)^{N+1/2} dy.
\]
Relations (B.15), (5.16) and (B.17) yield
\[
\int \delta_f h_i = c_1 \frac{H(x, x)}{\lambda^{N-1/2}} + O\left( \frac{1}{\lambda^{N-1/2}} \right),
\]
and finally, using (B.12), (B.13), (B.14) and (B.19), we have
\[
\int |\nabla \delta_i|^2 \sim \text{Scg}_i \frac{H(x, x)}{\lambda^{N-1/2}} + O\left( \frac{1}{\lambda^{N-1/2}} \right).
\]
Let now $i \neq j$; then
\[
\int \delta_f (\delta_i - h_i) = \int \delta_f h_i - \int \delta_f h_i = \int \delta_f h_i.
\]
Computations similar to those which lead to (B.19) give
\[
\int \delta_f h_i = c_2 \frac{H(x, x)}{\lambda^{N-1/2}} + O\left( \frac{1}{\lambda^{N-1/2}} \right).
\]
Moreover,
\[
\int \delta_f \delta_i \leq \int \delta_f \delta_i \left( \delta_f^{r+1} + \delta_f^{r+1} \right),
\]
and hence, by (B.14),
\[
\int \delta_f h_i \leq \frac{c}{\lambda^8}.
\]
Let $a_{ij} = x_i - x_j$ and $l = \int_{\Omega} \delta^l \, d\delta$. Then

$$f = c_l \int \left( \frac{1}{1 + |y|^2} \right)^{(N-2)/2} \left( \frac{1}{1 + |y - \lambda a_{ij}|^2} \right)^{(N-2)/2} \, d\phi.$$ 

We have also

$$1 + |y - \lambda a_{ij}|^2 = \left( 1 + \lambda^2 |a_{ij}|^2 \right) \left( 1 + \frac{|y|^2 - 2 \lambda y \cdot a_{ij}}{1 + \lambda^2 |a_{ij}|^2} \right);$$

hence

$$\left( 1 + |y - \lambda a_{ij}|^2 \right)^{-(N-2)/2} = \left( 1 + \lambda^2 |a_{ij}|^2 \right)^{-(N-2)/2} \left( 1 + \frac{(N-2)\lambda y \cdot a_{ij}}{1 + \lambda^2 |a_{ij}|^2} \right) + O\left( \frac{|y|^2}{1 + \lambda^2 |a_{ij}|^2} \right)$$

for $|y| \leq 2\lambda |a_{ij}|$.

Let

$$A(y) = \left( \frac{1}{1 + |y|^2} \right)^{(N-2)/2} \left( \frac{1}{1 + |y - \lambda a_{ij}|^2} \right)^{(N-2)/2}.$$ 

Then,

$$\int_{|y| < 2\lambda |a_{ij}|/4} A(y) \, d\phi = \frac{1}{(1 + \lambda^2 |a_{ij}|^2)^{N/2}} \times \left( \int_{|y| < 2\lambda |a_{ij}|/4} \left( \frac{1}{1 + |y|^2} \right)^{(N-2)/2} \, d\phi \right)$$

$$+ \frac{1}{(1 + \lambda^2 |a_{ij}|^2)} \left( \int_{|y| < 2\lambda |a_{ij}|/4} \frac{|y|^2}{(1 + |y|^2)^{(N-2)/2}} \, d\phi \right),$$

$$\int_{|y| < 2\lambda |a_{ij}|/4} \frac{|y|^2}{(1 + |y|^2)^{(N-2)/2}} \, d\phi = O(\log \lambda |a_{ij}|),$$

$$\int_{|y| < 2\lambda |a_{ij}|/4} \frac{\phi}{(1 + |y|^2)^{(N+2)/2}} = \frac{c_l}{c_0} \frac{1}{\lambda^2 |a_{ij}|^2} + O\left( \frac{1}{\lambda^2 |a_{ij}|^2} \right).$$
From (B.25), (B.26), (B.27) we obtain

(B.28) \[
\int_{||y|| \leq \lambda |a_j|^4/4} A(y) \, dy = \frac{c_5}{c_6} \frac{1}{\lambda^{N-1} |a_j|^{N-2}} + O\left(\frac{1}{\lambda^{N-1} |a_j|^{N-2}}\right).
\]

Let

\[ B_1 = \left\{ y \in \mathbb{R}^N \mid |y - \lambda a_j| \leq \frac{\lambda |a_j|}{4} \right\} \]

and

\[ B_2 = \left\{ y \in \mathbb{R}^N \mid |y| \leq \frac{\lambda |a_j|}{4} \right\}. \]

We have

\[
\int_{\mathbb{R}^N \setminus B_1 \cup B_2} A(y) \, dy \leq \frac{c}{\lambda^{N-2} |a_j|^{N-1}} \int_{|y - a_j|}^{\infty} \left(1 + r^2\right)^{N/2-1} \, dr.
\]

(B.29)

\[
\int_{\mathbb{R}^N \setminus B_1 \cup B_2} A(y) \, dy = O\left(\frac{1}{\lambda^{N/2} |a_j|^N}\right).
\]

\[
\int_{B_2} A(y) \, dy \leq \frac{c}{\lambda^{N-2} |a_j|^{N-1}} \int_0^{\frac{\lambda |a_j|}{4}} \left(1 + r^2\right)^{N/2-1} \, dr
\]

\[
- O\left(\frac{1}{\lambda^{N/2} |a_j|^N}\right).
\]

From (B.28), (B.29), (B.30) it follows that

(B.31)

\[
\int_{\mathbb{R}^N} A(y) \, dy = \frac{c_5}{c_6} \frac{1}{\lambda^{N-1} |a_j|^{N-2}} + O\left(\frac{1}{\lambda^{N-1} |a_j|^{N-2}}\right).
\]

Finally from (B.21), (B.22), (B.23) and (B.31) we obtain, with \( \zeta = d(x) \),

(B.32) \[
\int \nabla \bar{\zeta} \nabla \bar{\zeta} = \text{Sc}(d(x), x) \frac{1}{\lambda^{N/2}} + O\left(\frac{1}{\lambda^{N-1} \lambda^{N/2}}\right).
\]

Using now (B.32) and (B.20) we see that

\[
\int |\nabla u|^2 = |a|^2 - \frac{2}{p+1} \frac{c_1}{\lambda^{N-1}} \left(\sum_{i,j} a_i^2 H(x_i, x_j) - \sum_{i,j} a_i a_j G(x_i, x_j)\right)
\]

\[ + O\left(\frac{1}{\lambda^{N-1} \lambda^{N/2}}\right), \]
and therefore,
\[
\left( \int |\nabla u|^2 \right)^{(p+1)/2} = |a|^{p+1} \left \{ 1 - \frac{c_1}{|a|^{n-2}} \right \} \times \left[ \sum_{j \neq j_0} a_j^2 H(x_j, x_{j_0}) - \sum_{j \neq j_0} a_j a_{j_0} G(x_j, x_{j_0}) \right] + O\left( \frac{1}{|x \cdot a|^{n-2}} \right).
\]
(B.33)

We are now going to estimate \( J u \phi^{p+1} \). Let \( B_0 = \{ x \mid |x - y| < \min\{ \delta, l \} \} \).

Notice that
\[
\left( \int_{\Omega \cup \tilde{\Omega}} u \phi^{p+1} \right)^{\frac{1}{p+1}} \leq \left( \int_{x \cdot a \geq \lambda \min\{ \delta, l \}} \left( \frac{\lambda}{1 + \lambda r^2} \right)^N r^{N-1} \, dr \right)^{\frac{1}{N}} + O\left( \frac{1}{\lambda^N \delta^N} \right).
\]

Let \( d' = \min\{ \delta, l \} \). On \( B_0 \),
\[
u \phi^{p+1} - \phi^{p+1} \delta \phi^{-1} + (\phi + 1) \frac{\lambda}{1 + \lambda r^2} \left( \sum_{j \neq j_0} a_j \delta \frac{\bar{h}_j - a \bar{h}_i}{\delta r} \right)
\]
\[
+ O\left( \frac{\lambda^{N-2}}{(\lambda d')^{2N-2}} \delta \phi^{-1} \right).
\]

(B.35)

\[
\int_{R} \delta \phi^{-1} = \int_{0}^{\rho} \left( \frac{\lambda}{1 + \lambda r^2} \right)^{2N-1} r^{N-1} \, dr = \frac{1}{\lambda^{N-1}} \int_{0}^{\rho} \frac{1}{1 + r^2} \, dr,
\]
and then one easily sees that, for any \( N \geq 3 \),
\[
\frac{1}{(\lambda d')^{2N-2}} \int_{R} \left( \frac{\lambda}{1 + \lambda r^2} \right)^{2N-1} r^{N-1} \, dr = O\left( \frac{1}{\lambda d'} \right).
\]

Using (B.19), (B.22), (B.34), (B.35) and (B.36) we conclude that
\[
\int_{R} \nu \phi^{p+1} = \frac{\nu \phi^{p+1}}{2} - (\phi + 1) \frac{\lambda}{1 + \lambda r^2} \delta \phi^{-1} \frac{H(x_j, x_{j_0})}{\lambda^{N-1}}
\]
\[
+ \sum_{j \neq j_0} a_j a_{j_0} G(x_j, x_{j_0}) + O\left( \frac{1}{\lambda d'} \right).
\]

(B.37)
Finally, (B.37) and (B.34) imply that

$$\int u^{p+1} = \frac{\|u^{p+1}\|}{S} - \frac{2c_1}{\lambda^{k/2}} \left( \sum_{i,j} u^{p+1} H(x_i, x_j) - \sum_{i,j} n_0 G(x_i, x_j) \right)$$

(B.38)

$$+ O\left( \frac{1}{(\lambda d(x))^{k/2}} \right).$$

Proposition B.5 follows from (B.33) and (B.38).

Note that there exist $\epsilon' > 0$ and $\nu' > 0$ such that

$$H(y, y) \leq \epsilon' \quad \text{for all} \quad y \in K,$$

$$G(y_1, x_2) \geq \nu' \quad \text{for all} \quad (y_1, x_2) \in K^2.$$

Hence from Proposition B.5 one easily derives

**Corollary B.6.** There exist two positive real numbers $\epsilon$ and $\nu$ such that, for any positive integer $n$, there exists a constant $c(n)$ such that, for any $\lambda \in [1, \infty)$ and any $x$ in $K^\star$ with $d(x) \neq 0$,

$$\max_{a \in A_{\lambda, 1}} \psi(a, x, \lambda) \leq n^{\nu - 1/2} \left[ S + \frac{2}{\lambda^{\nu/2}} (\epsilon - n\nu) \right] + \frac{c(x)}{\lambda d(x)}^{1/\nu}.$$

Next we shall prove

**Lemma B.7.** For any integer $n \in [2, \infty)$ and any $\epsilon$ in $(0, \infty)$, there exist $d_\epsilon$ in $(0, \infty)$ and $\lambda_\epsilon$ in $[1, \infty)$ such that

$$\psi(x, x, \lambda) \leq n^{\nu - 1/2} S$$

(B.39)

for all $a \in A_{\lambda, 1} \cap [\epsilon, 1]^n$, all $\lambda \in [\lambda_\epsilon, +\infty]$, and all $x \in K^\star$ with $d(x) \leq d_\epsilon$.

Proof of Lemma B.7: Clearly, we may assume that

$$|x_1 - x_2| = d(x).$$
Note also that since
\[
\lim_{(y_1, y_2) \to (x_1, x_2)} G(y_1, y_2) = +\infty,
\]
Proposition B.5 implies that there exist \( d_1 > 0 \) and \( C_1 > 0 \) such that, for all \( x \in K^\circ \), all \( a \in \Delta_{\lambda_a} \), and all \( \lambda \in [1, \infty) \),
\[
(\text{B.40}) \quad \hat{\psi}(a, x, \lambda) \leq n^{1/(p-1/2)} \frac{\lambda(\lambda^0 - 1)}{\lambda^0 - \lambda} \quad \text{if} \quad d(x) \leq d_1 \quad \text{and} \quad \lambda|\lambda_1 - x_1| \geq C_1.
\]
Using (B.15) and (B.2) one sees that there exists \( C_2 > 0 \) such that
\[
\hat{\psi}(a, x, \lambda) \leq n^{1/(p-1/2)} \left( 1 + \frac{C_1}{\lambda(\lambda^0 - \lambda)} \right) \frac{\delta_{a^0}^{1/p+1} + n^{-1} \lambda^{1/(p-1/2)}}{\delta_{a^0}^{1/p+1} + n^{-1} \lambda^{1/(p-1/2)}} ;
\]
but there exists \( \tau \) in \((0, \infty)\) such that
\[
(\text{B.42}) \quad \frac{\delta_{a^0}^{1/p+1} + n^{-1} \lambda^{1/(p-1/2)}}{\delta_{a^0}^{1/p+1} + n^{-1} \lambda^{1/(p-1/2)}} \leq \frac{1}{\delta_a} \left( 1 - \tau \right) \quad \text{if} \quad \lambda|\lambda_1 - x_1| \leq C_2.
\]
(Notice that by translation and dilation we may assume that \( x_1 = 0 \) and \( \lambda = 1 \).) Lemma B.7 follows from (B.40), (B.41) and (B.42).

We can now prove Proposition B.1. We first use Corollary B.6 and choose \( \eta > 0 \) such that
\[
(\text{B.43}) \quad \eta = n \bar{\eta} \eta < 0.
\]
We use first Lemma B.4 and then Lemma B.7: there exist \( \varepsilon > 0 \), \( d_1 > 0 \) and \( \lambda_a > 0 \) such that
\[
(\text{B.44}) \quad \hat{\psi}(a, x, \lambda) \leq n^{1/(p-1/2)} \frac{\lambda(\lambda^0 - 1)}{\lambda^0 - \lambda} \quad \text{for all} \quad x \in K^\circ, \quad a \in \Delta_{\lambda_a} \setminus \{x_1 \} \quad \text{and} \quad \lambda \in [\lambda_a, +\infty),
\]
and
\[
(\text{B.45}) \quad d(x) \leq d_1 \Rightarrow \hat{\psi}(a, x, \lambda) \leq n^{1/(p-1/2)} \frac{\lambda(\lambda^0 - 1)}{\lambda^0 - \lambda} \quad \text{for all} \quad x \in K^\circ, \quad a \in \Delta_{\lambda_a} \cap \{x_1\} \quad \text{and} \quad \lambda \in [\lambda_a, +\infty).
\]
We use Corollary 6.6 once more as well as (B.45); then there exists \( \lambda_{5} \) such that
\[
\text{for all } x \in \mathcal{K}^{\ast}, \quad \alpha \in \Delta_{\alpha}, \text{ and all } \lambda \in [\lambda_{5}, +\infty),
\]
\[
d(x) \geq d_{J_{\alpha}} \psi \left( a, x, \lambda \right) \geq n\|x\|^{1/2}\lambda^{1/2}. \tag{B.46}
\]
Let now \( \lambda_{0} = \max(\lambda_{6}, \lambda_{5}) \), using (B.44), (B.45) and (B.46) we have
\[
\psi \left( a, x, \lambda \right) \leq \|x\|^{1/2}\lambda_{0}^{1/2}, \tag{B.47}
\]
hence Proposition 6.1 holds.

**Comments.**
1. The regular part \( H \) of the Green's function appears in the expression of \( \psi(a, x, \lambda) \); originally it came out of expansions along the gradient flow (see [2]–[3] for further precisions). The role of the regular part of the Green's function in connection with the critical Sobolev exponents has been pointed out for the first time by McLeod [12] for a Dirichlet problem and by Schoen [17] in the framework of the Yamabe conjecture. (The computations in [2]–[3] were made independently of [12] and [17].)
2. More generally, one finds the following expansion of \( \psi \):
\[
\psi \left( a, x, \lambda \right) \leq \|x\|^{1/2}\lambda \left[ 1 - \sum_{i} \frac{H(x, s_{i})}{\lambda_{i}^{1/2}} - \frac{a}{\|x\|^{1/2}} \right]
\]
\[
+ \sum_{(i, j) \in \mathcal{E}_{\alpha}} \frac{\lambda_{i}^{1/2} a_{i}}{\|x\|^{1/2}} - \frac{a_{i} s_{i}}{\|x\|^{1/2}}, \tag{6.14}
\]
\[
\leq c(\pi, K) \left( \sum_{i} \frac{1}{\lambda_{i}^{1/2}} + \sum_{(i, j) \in \mathcal{E}_{\alpha}} \frac{a_{i}^{\alpha_{i} - 1}}{\|x\|^{1/2}} \right) \quad \text{for } x \in \mathcal{K}^{\ast}, \text{ with }
\]
\[
e_{i} = \left( \frac{\lambda_{i}}{\lambda_{j}} \right)^{1/2} + \left( \frac{\lambda_{i} \lambda_{j}}{G'(x_{i}, x_{j})} \right)^{1/2} \quad \text{for all } i \neq j.
\]
Appendix C

This appendix is due to J. Lannes. We use here the notations of Section 3 and we prove

**Proposition C.1.**

\[
\partial \left( \left( \omega \star \omega \right) \star \left[ B_{\omega}(V), B_{\omega}(V) \right] \right) = \left[ B_{\omega}(V), B_{\omega}(V) \right].
\]

Proof of Proposition C.1: For simplicity we shall write $B_t$ instead of $B_t(V)$.

Let $\xi$ be a fixed point in $V$, and let $CB_{\omega, \xi}$ be the subset of $B_{\omega, \xi}$ defined by

\[
CB_{\omega, \xi} = \left\{ \sum_{i=1}^{n} \alpha_i \xi_i \in B_{\omega, \xi} \mid \text{there exist } i \in \{1, \ldots, n\} \text{ such that } x_i = \xi \cup B_{\omega, \xi} \right\}.
\]

$CB_{\omega, \xi}$ is contractible in itself and, therefore, $H_0(CB_{\omega, \xi}, B_{\omega, \xi}) = H_0(B_{\omega, \xi})$.

Let $\tau$ be the natural injection of $CB_{\omega, \xi}$ into $B_{\omega}$, $\tau$ maps the pair $(CB_{\omega, \xi}, B_{\omega, \xi})$ into the pair $(B_{\omega}, B_{\omega, \xi})$ and the following diagram is commutative ($\gamma$ and $\bar{\gamma}$ are the usual derivations)

\[
\begin{array}{c}
\xymatrix{ H_0(CB_{\omega, \xi}, B_{\omega, \xi}) \ar[r]^-{\gamma} \ar[d]_{\tau} & H_0(B_{\omega, \xi}) \ar[d]^{	ext{identity}} \\ H_0(B_{\omega}, B_{\omega, \xi}) & H_0(B_{\omega, \xi}) \ar[l]^-{\bar{\gamma}} }
\end{array}
\]

(C.2)

Let $p_i: V^a \times_{\alpha_i} \alpha_i \rightarrow V^a$ be the projection on the first factor, and let $\nu: V^a / \alpha_i \times_{\alpha_i} \alpha_i \rightarrow V^a$ be also the projection on the first factor. We choose an open neighborhood $U_{\xi}^a$ of $\xi^a$ in $V^a$, $\alpha_i$-invariant, satisfying (20)–(21), and such that

(C.3) \quad $\text{Ker}(d\nu)(x) + T_x U_{\xi}^a$ for all $x \in \partial U_{\xi}^a$ with $x_i = \xi$.

(C.4) \quad $\xi^a$ is a strong $\alpha_i$-equivariant deformation retract of $U_{\xi}^a$,

where, in (C.3), $(d\nu)(x)$ denotes the differential of $x$ at $x$ and $T_x U_{\xi}^a$ the tangent space of $\partial U_{\xi}^a$ at $x$ and where, in (C.4),

\[
\begin{align*}
S_{\xi}^a &= \{ x \in U_{\xi}^a \mid \text{there exists } i \in \{1, \ldots, n\} \text{ with } x_i = \xi \}, \\
T_{\xi}^a &= \{ x \in U_{\xi}^a \mid \text{there exists } i \in \{1, \ldots, n\} \text{ with } x_i = \xi \}.
\end{align*}
\]

We give at the end of this appendix an example of such a $\xi^a$. 


Note that it follows from (C.3) that $p^n_i(x)$ is a manifold (with boundary). In Section 3 we have defined an isomorphism $\delta$ between $H^k(B_0, B_{a-1})$ and $H^k(V^n_i \times \Delta_{a-1} \cup V^n_i \times \partial \Delta_{a-1})$ (see (22). In a similar way we are going to define an isomorphism $\delta^n_i$ between $H^k(CB_0, B_{a-1})$ and $H^k(p^n_i(x), \partial(p^n_i(x)))$.

Let

$$V^n_i = \{x \in V^n | \text{there exists } i \in [1, a] \text{ with } x_i = \xi\}$$

and let

$$\delta^n_i : \left( V^n_i \times \Delta_{a-1}, \gamma^n_i \times \Delta_{a-1} \cup V^n_i \times \partial \Delta_{a-1} \right) \rightarrow (CB_0, B_{a-1})$$

be the natural projection. As in Section 3 (see (16)), one easily proves that

(C.5) $\delta^n_i$ is an isomorphism.

Let now

$$\delta^n_i : \left( V^n_i \times \Delta_{a-1}, \gamma^n_i \times \Delta_{a-1} \cup V^n_i \times \partial \Delta_{a-1} \right) \rightarrow \left( V^n_i \times \Delta_{a-1}, \gamma^n_i \times \Delta_{a-1} \cup V^n_i \times \partial \Delta_{a-1} \right);$$

it follows from (C.4) that

(C.6) $\delta^n_i$ is an isomorphism.

Let

$$\iota^n_i : (p^n_i(x), \partial(p^n_i(x))) \rightarrow \left( V^n_i \times \Delta_{a-1}, V^n_i \times \partial \Delta_{a-1} \cup T^n_i \times \Delta_{a-1} \right)$$

be the restriction of the projection: $V^n_i \times \gamma^n_i \times \Delta_{a-1} \rightarrow V^n_i \times \gamma^n_i \times \Delta_{a-1}; \iota^n_i$ defines an homeomorphism between

$$p^n_i(x) \setminus (p^n_i(x))$$

and

$$V^n_i \times \Delta_{a-1} \setminus \left( V^n_i \times \partial \Delta_{a-1} \cup T^n_i \times \Delta_{a-1} \right).$$

Moreover, $\partial(p^n_i(x))$ is a strong deformation retract of one of its closed
neighborhoods in $p_0^*\{\xi\}$, and therefore,
\begin{equation}
\label{C.7}
\delta^*_s \text{ is an isomorphism.}
\end{equation}

We define
\begin{equation}
J_s = (f_s')^{-1}j_s(b_s')^{-1}.
\end{equation}

We next remark that the following diagram is commutative:
\begin{equation}
\label{C.8}
\begin{array}{c}
\hat{B}_s \times \hat{B}_s \xrightarrow{\varphi_s} (p_0^*\{\xi\}, \lambda(p_0^*\{\xi\})) \\
\downarrow \varphi \downarrow \quad \downarrow \rho \downarrow \quad \downarrow \nu \downarrow \quad \downarrow \phi \downarrow \\
V_s \times \Delta_{-1} \quad \vartheta (V_s 	imes \Delta_{-1}) \quad \vartheta (V_s 	imes \Delta_{-1}) \quad \vartheta (V_s 	imes \Delta_{-1}) \quad \vartheta (V_s 	imes \Delta_{-1})
\end{array}
\end{equation}

where $\hat{B}_s = V_s \times \Delta_{-1}$, $\varphi$ is the natural projection, and $\nu$ is the inclusion map.

We have
\begin{equation}
\gamma_s = \nu_s = k_s = k_{-1}.
\end{equation}

Indeed (C.9) is a consequence of the commutativity of the following diagrams:
\begin{equation}
\begin{array}{c}
(V_s \times \Delta_{-1}, \mathcal{S}_s \times \Delta_{-1} \cup V_s \times \Delta_{-1}) \xrightarrow{\rho_s} (B_s, B_{-1}) \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
(V_s \times \Delta_{-1}, \mathcal{S}_s \times \Delta_{-1} \cup V_s \times \Delta_{-1}) \xrightarrow{\rho_s} (B_s, B_{-1}) \\
\end{array}
\end{equation}

where the maps which are not labeled are inclusion maps.

Since $B_{-1}$ is contractible in $B_{-2}$, the map $\hat{x}_d(B_{-1}) \to H_d(B_{-1}, B_{-2})$ of the reduced homology sequence of $(B_{-1}, B_{-2})$ is one to one; moreover (see
(22)), $H_{t_0} = \pi_{t_0}^{-1}([B_{t_0}, B_{t_0}^2]) = Z_3$. Hence, 

(C.10) \[ \theta_t([t_0])^{-1}([p_{t_0}^{-1}([\xi]), r(p_{t_0}^{-1}([\xi]))]) = [B_{t_0}, B_{t_0}]. \]

where $[p_{t_0}^{-1}([\xi]), r(p_{t_0}^{-1}([\xi]))]$ is the orientation class (modulo $Z_3$) of the manifold with boundary $p_{t_0}^{-1}([\xi])$.

We denote by $\cap$ cap product. We are going to prove that

(C.11) \[ s_*([p_{t_0}^{-1}([\xi]), r(p_{t_0}^{-1}([\xi]))]) = (tr_0 p_{t_0}^{-1}([\xi]), B_{t_0}^*, \delta B_{t_0}^*), \]

where $tr_0$ is the transfer map $H^q(V^* \times_{r_1, x_{t_0}} 0, 0) \rightarrow H^q(V^* \times_{r_1, x_{t_0}} 0, 0)$. Note that (C.1) follows from (C.3), (C.9), (C.10), (C.11) and the functoriality of the transfer (see [5]). Since $q: V^* \times_{r_1, x_{t_0}} 0, 0 \rightarrow V^* \times_{r_1, x_{t_0}} 0, 0$ is a covering between two manifolds, $tr_0$ is Gysin's homomorphism; hence, for any $u$ in $H^q(V^* \times_{r_1, x_{t_0}} 0, 0)$, \[
(tr_0 u) \cap [B_{t_0}, \delta B_{t_0}] = q_*u \cap [V^* \times_{a_{t_0}, x_{t_0}} 0, \delta(V^* \times_{a_{t_0}, x_{t_0}} 0)].
\]

In particular, \[
(tr_0 p_{t_0} u) \cap [B_{t_0}, \delta B_{t_0}] = q_*u \cap [p_{t_0}^{-1}([\xi]), \delta(p_{t_0}^{-1}([\xi]))],
\]

but

\[
p_{t_0}(u) \cap [V^* \times_{a_{t_0}, x_{t_0}} 0, \delta(V^* \times_{a_{t_0}, x_{t_0}} 0)] = q_*([p_{t_0}^{-1}([\xi]), \delta(p_{t_0}^{-1}([\xi]))])
\]

and hence

\[
(tr_0 p_{t_0} u) \cap [B_{t_0}, \delta B_{t_0}] = q_*([p_{t_0}^{-1}([\xi]), \delta(p_{t_0}^{-1}([\xi]))]),
\]

which gives (C.11).

Finally, we give an example of an open neighborhood $T'_1$ of $S^1$ in $V^*$, $a_{t_0}$-invariant, satisfying (20), (21), (C.3) and (C.4). We provide $V$ with a $C^\infty$ Riemannian metric and denote by $d(x_1, x_2)$ the geodesic distance between two points $x_1$ and $x_2$ in $V$. Let $A: V^* \rightarrow R$ be a $C^\infty$ map such that

\[
A(x_1, x_2) = d^2(x_1, x_2) \quad \text{in a neighborhood of } S^1,
\]

\[
> 0 \quad \text{if } (x_1, x_2) \in V^* \setminus S^1
\]

Let $\epsilon$ be in $(0, \infty)$ and let

\[ T^\epsilon = \{ x \in V^* | d(x_1, x_2) < \epsilon \}. \]
$\mathcal{T}_\varepsilon$ is open, $\sigma_0$-invariant and contains $\mathcal{S}_\mathcal{E}$. Moreover, one easily verifies that if $\varepsilon$ is small enough, $\mathcal{T}_\varepsilon$ satisfies (20), (21), (C.3) and (C.4).

**Appendix D**

In this appendix we give a proof—which does not rely on the transfer—of existence of a solution to (3) when there exists some odd integer $d$ such that $H_d(\mathcal{G}; \mathcal{Q}) \neq 0$. We shall consider here only rational homology and cohomology; we shall write $H_d^*(\mathcal{G})$, $H^*(\mathcal{G})$ instead of $H_d(\mathcal{G}; \mathcal{Q})$, $H^*(\mathcal{G}; \mathcal{Q})$.

Let $\mathcal{K}$ be a compact in $\mathcal{G}$; in (13) we have defined a map $e_{x,\lambda} : K^* \times \Delta_{x,\lambda} \to \Sigma_+$ which depends on some parameter $\lambda$ in $(0, \infty)$. If $\lambda$ is large enough, $e_{x,\lambda}$ maps the pair $(K^* \times \Delta_{x,\lambda}; K^* \times \partial \Delta_{x,\lambda})$ into the pair $(W_+, W_-)$ and it is clear that

$$
\phi_{x,\lambda} : H_d(K^* \times \Delta_{x,\lambda}; K^* \times \partial \Delta_{x,\lambda}) \to H_d(W_+, W_-)
$$

is independent of the choice of $\lambda$ provided that $\lambda$ is large enough. On the other hand, the homology of $(K^* \times \Delta_{x,\lambda}; K^* \times \partial \Delta_{x,\lambda})$ is the direct limit of the homology of $(K^* \times \Delta_{x,\lambda}; K^* \times \partial \Delta_{x,\lambda})$, where the $K$ are compact sets in $\mathcal{G}$; hence one can define a natural map

$$
\phi_\ast : H_d(K^* \times \Delta_{x,\lambda}; K^* \times \partial \Delta_{x,\lambda}) \to H_d(W_+, W_-).
$$

Notice that

$$
H_d(K^* \times \Delta_{x,\lambda}; K^* \times \partial \Delta_{x,\lambda}) = H_d(K^*) \oplus H_d(\Delta_{x,\lambda}; \partial \Delta_{x,\lambda}).
$$

Let $e_{x,\lambda}$ be the canonical generator of $H_d(\Delta_{x,\lambda}; \partial \Delta_{x,\lambda})$. Let

$$
D : H_d(K^* \times \Delta_{x,\lambda}; K^* \times \partial \Delta_{x,\lambda}) \to H_d(K^* \times \Delta_{x,\lambda}; K^* \times \partial \Delta_{x,\lambda})
$$

be defined by

$$
D(f \times e_{x,\lambda}) = (-1)^{\deg f} \sum \left\{ (-1)^{i-1} \langle p_i f \rangle e_{x,\lambda} \right\} \times e_{x,\lambda},
$$

where $p : K^* \to K^* \times \Delta_{x,\lambda}$ is the projection

$$
p(x_1, \ldots, x_n) = (x_1, \ldots, x_n, x_{n+1}, \ldots, x_\xi)
$$

and where $\deg f$ is the degree of $f$.

Our first lemma is

**Lemma D.1.** The following diagram is commutative:

$$
\begin{array}{ccc}
H_d(K^* \times \Delta_{x,\lambda}; K^* \times \partial \Delta_{x,\lambda}) & \xrightarrow{\phi_{x,\lambda}} & H_d(W_+, W_-) \\
\downarrow{\sim} & & \downarrow{\phi} \\
H_d(K^* \times \Delta_{x,\lambda}; K^* \times \partial \Delta_{x,\lambda}) & \xrightarrow{\sim} & H_d(W_+, W_-)
\end{array}
$$
Proof of Lemma D.1: The lemma is a consequence of the commutativity of the diagram

\[
\begin{array}{c}
K^* \times \Delta_{n+1} \xrightarrow{id \times f} K^* \times \Delta_{n+1} \xrightarrow{\delta_{n+1}} W_{n+1} \\
\downarrow \rho \circ id \\
K^{n+1} \times \Delta_{n+1} \xrightarrow{\delta_{n+1}} W_{n+1}
\end{array}
\]

where \( f(t_1, \ldots, t_{n+1}) = (t_1, \ldots, t_{n-1}, 0, t_{n+1}, \ldots, t_{n+1}) \) and \( Id \) is the identity on \( \Delta_{n+1} \).

The cap product

\[
H^*(\Omega^* \times \Delta_{n+1}) \oplus H_*(\Omega^* \times \Delta_{n+1}, \Omega^* \times \partial \Delta_{n+1}) \\
\longrightarrow H_*(\Omega^* \times \Delta_{n+1}, \Omega^* \times \partial \Delta_{n+1})
\]

provides \( H_*(\Omega^* \times \Delta_{n+1}, \Omega^* \times \partial \Delta_{n+1}) \) with a structure of an \( H^*(\Omega^*) \)-module and hence a structure of an \( H^*(\Omega^*/\alpha_\ell) \)-module via the homomorphism \( \pi_* : H^*(\Omega^*/\alpha_\ell) \longrightarrow H^*(\Omega^*) \), where \( \pi \) is the projection \( \Omega^* \longrightarrow \Omega^*/\alpha_\ell \). We denote by \( \cdot \) this product.

We have seen in Proposition 9 that \( H_*(W_\ell, W_{n+1}) \) has also a structure of an \( H^*(\Omega^*/\alpha_\ell) \)-module. Our next lemma is

**Lemma D.2.** The map \( f_\ell \) is \( H^*(\Omega^*/\alpha_\ell) \)-linear.

Proof of Lemma D.2: We give a direct proof (one could also use Proposition 9). Let \( \xi \) be a compact in \( \Omega \) and let \( \bar{g}_\xi : K^* \times \Delta_{n+1} \longrightarrow \Sigma^* \), be defined by

\[
\bar{g}_\xi(x, a) = R \left( \sum_{i=1}^{n} u_i P \left( g(x, x_i) \right) \right).
\]

Note that

\[
(D.1) \quad \lim_{\lambda \to \infty} f(\bar{g}_\xi(x, a)) = S \left( \frac{\sum_{i=1}^{n} (x_i)^{n+1}}{\Sigma_{i=1}^{n+1}} \right) \text{ for all } (x, a) \in K^* \times \Delta_{n+1},
\]

and hence (if \( \lambda \) is large enough) \( \bar{g}_\xi \) maps the pair \((K^* \times \Delta_{n+1}, K^* \times \partial \Delta_{n+1})\) into the pair \((W_\ell, W_{n+1})\). We prove first that (if \( \lambda \) is large enough)

\[
(D.2) \quad \bar{g}_\xi = g_\xi.
\]
Let \( h_c : [0, 1] \times K^* \times \Delta_{n-1} \rightarrow \Sigma_c \) be defined by
\[
h_c \left( t, x, a \right) = R \left( \sum_{i=1}^d a_i \partial_i \delta(x, i \lambda + (1-t) \lambda) \right);
\]
h\(_c\) is continuous and we have
\[
(D.3) \quad h_c(0, x, a) = \tilde{g}_c(x, a) \quad \text{for all } (x, a) \in K^* \times \Delta_{n-1},
\]
\[
(D.4) \quad h_c(1, x, a) = g_c(x, a) \quad \text{for all } (x, a) \in K^* \times \Delta_{n-1}.
\]
Moreover, by Corollary B.3, if \( \lambda \) is large enough) for all \( t \in [0, 1]\),
\[
h_c(t; \cdot, \cdot) \text{ maps the pair } 
\]
\( (K^* \times \Delta_{n-1}, K^* \times \partial \Delta_{n-1}) \) into the pair \( (W_*, W_*^{+}) \).

The equality (D.2) follows from (D.3), (D.4) and (D.5).

We next remark (see in particular (D.1)) that there exist \( \eta_0 \) in \((0, \infty)\) and \( \lambda_2 \) in \((0, \infty)\) such that
\[
(D.6) \quad \lambda \geq \lambda_2 = \tilde{g}_c \left( K^* \times \Delta_{n-1, \eta_0} \right) \subset F_{\eta_0} \cap V(n, \eta_0),
\]
(\( \lambda = \lambda_{n-1, \eta} \) is defined in the proof of Proposition 9). It is also clear from (D.1) that (\( \eta_0 \) being now fixed), for \( \lambda \) large enough,
\[
(D.7) \quad \tilde{g}_c \left( K^* \times \left( \Delta_{n-1, \eta_0} \setminus \Delta_{n-1, \eta} \right) \right) \subset W_{*}^{-1}.
\]

Let \( b(x, a) = x \) for \((x, a) \in K^* \times \Delta_{n-1} \). Clearly, on \( K^* \times (\Delta_{n-1, \eta_0} \setminus \Delta_{n-1, \eta}) \),
\[
(D.8) \quad \nabla \times \sigma = q \circ b.
\]

It follows from (D.6), (D.7) and (D.8) that the diagram
\[
\begin{array}{ccc}
(K^* \times \Delta_{n-1, \eta_0}, K^* \times (\Delta_{n-1, \eta_0} \setminus \Delta_{n-1, \eta})) & \xrightarrow{\tilde{g}_c} & (F^*, W_{*}^{-1}) \\
\downarrow & & \downarrow \\
(K^* \times \Delta_{n-1, \eta_0}, K^* \times (\Delta_{n-1, \eta_0} \setminus \Delta_{n-1, \eta})) & \xrightarrow{b} & (F_{\eta_0} \cap V(n, \eta_0), W_{*}^{-1} \cap V(n, \eta_0)) \\
\downarrow & & \downarrow x \\
K^* & \xrightarrow{\pi} & \Omega/\Gamma_{n_0}
\end{array}
\]
is commutative. Lemma D.2 is a consequence of this commutativity and (D.2).
Let now \( z \in H_3(\Omega) \) and \( u \in H^4(\Omega) \) such that \( \langle u, z \rangle = 1 \). We are going to prove, by induction on \( n \), that if \( d \) is odd, then

\[
(I)_{n}(z^n \times e_{n-1}) \neq 0,
\]

where \( z^n = z \times z \times \cdots \times z \in H_3(\Omega^n) \), which is in contradiction with Proposition 8. First note that

\[
(I)_{1}(z^1 \times e_{0}) = 0.
\]

Indeed, let \( e \) be the canonical generator of \( H_3(\Omega) \). We know that \( I_2(z) \neq 0 \) and by Lemma D.2, \( I_3((z \times e_0)) = u \cdot I_3(z \times e_0) \) and \( u \cdot (z \times e_0) = u \); hence \((I)_{3}(z^3 \times e_{2}) = 0 \). Since we consider cohomology with rational coefficients, the map \( \pi^*: H^4(\Omega'/\alpha^*) \rightarrow H^4(\Omega^n) \) induces an isomorphism between \( H^4(\Omega'/\alpha^*_n) \) and the elements of \( H^4(\Omega^n) \) which are invariant under \( \alpha^*_n \) (see e.g. [5]). In particular, there exists a class, which we shall denote by \( \tilde{w} \), such that \( \pi^*(\tilde{w}) = w \) with

\[
w = (u \times I \times \cdots \times I) + (1 \times u \times \cdots \times I) + \cdots + (I \times \cdots \times I \times u)
\]

(\( w \in H^4(\Omega^n) \)),

where \( I \) denotes the unit element of \( H^4(\Omega) \).

We shall prove that

\[
(I)_{n}(z^n \times e_{n-1}) = (-1)^{n-1} \sum_{i=1}^{n} (-1)^{n-i} I_3(z^{n-i} \times e_{n-1})
\]

which, when \( d \) is odd, yields

\[
(I)_{d}(z^d \times e_{d-1}) = (-1)^{n} w_{n-1}(z^d \times e_{n-1});
\]

then \((I)_{9}(z^9 \times e_{8}) = 0 \) follows from \((I)_{10}(z^{10} \times e_{9}) = 0 \) when \( n \) is even.

In order to prove \((I)_{11}(z^{11} \times e_{10}) \), we remark that, in \( H_3(\Delta_{n-1}) \),

\[
w \cap z^n = \sum_{i=1}^{n} (-1)^{n-1} z^{n-i} \times e \times z^{n-i},
\]

and therefore, if we denote by \( I \) the unit element of \( H^3(\Delta_{n-1}) \), we have, in
\[ H_{\delta}(\Omega^{\ast} \times \Delta_{n-1}, \Omega^{\ast} \times \omega \Delta_{n-1}) \]

(D.13)

\[ (w \times 1) \cap z^{\ast} \times e_{\delta_{n-1}} = (1)^{1-1/d} \sum_{i=1}^{n-d} (1)^{1-1/d} z^{\ast} \times e \times z^{\ast} e_{\delta_{n-1}} \]

and (D.11) follows from (D.13), Lemma D.1 and Lemma D.2.

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Bibliography


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