NUMERICAL METHODS FOR WEAKLY IONIZED GAS

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Abstract. This paper is devoted to the numerical approximation of the discontinuous solutions of the Euler equations for weakly ionized mixtures of reacting gases. The main difficulty stems from the non conservative formulation of these equations due to a widely used physical assumption. We show how to derive a well-posed conservative reformulation of the equations from the analysis of the associated full convective-diffusive system. We then propose an exact Roe-type linearization for the equivalent system of conservation laws on the basis of an original Lemma for averagings. Our results can be seen as an extension of the classical Roe average, for nonlinearities that cannot be recast under quadratic form.

1. Introduction

This work treats the numerical approximation of the solutions of the convective diffusive system governing ionized mixtures of reacting gases in thermal nonequilibrium. We write it in short form as

\[ \partial_t \mathbf{u} + A(\mathbf{u}) \partial_x \mathbf{u} - \partial_x (D(\mathbf{u}) \partial_x \mathbf{u}) = \omega(\mathbf{u}), \quad t > 0, x \in \mathbb{R}. \]  

(1)

Such plasmas are studied here in the context of large Mach number flows. This situation can be encountered in hypersonic flows around reentry bodies. At high speeds, the flow reaches extremely high temperatures near the

vehicle. These temperatures are high enough to induce vibrational excitation, dissociation of diatomic molecules and ionization. For a better description of the flow, it is necessary to introduce different temperatures, that are another way of expressing the value of energy. Whereas it is a good assumption to take translational and rotational temperatures of heavy species in equilibrium, vibrational temperatures of polyatomic molecules and translational temperatures of electron gas have to be characterized by their own temperatures.

The solutions we are interested in, are thus mainly driven by the underlying first order system. However and as we shall put forward below, the extracted first order system turns out to be ill-posed because of a simplifying assumption in the physical modelling. The present work is thus focused on the derivation of a well-posed first order system with special emphasis put on its numerical approximation. Plasmas have already been studied for their mathematical properties but in a distinct setting. We refer in particular to the work by Cordier et al. [6] devoted to large scale plasmas: e.g. with $D(u) = 0$. Here, we also address more general thermodynamic closure equations that yield, in particular, unexpected difficulties in the numerical analysis.

The main properties of the extracted first order system

\[ \partial_t u + A(u) \partial_x u = 0, \quad t > 0, x \in \mathbb{R}, \]  

(2)

are reported in this paper. This nonlinear system is seen to be hyperbolic so that its solutions are known to develop, generally speaking, discontinuities in a finite time. Since the flow Mach numbers to be dealt with are large, the discontinuous solutions of (2) are obviously of special interest to us. But when dealing with discontinuous solutions of (2), a major difficulty arises: there does not exist a flux function, say $f$, such that $A(u) = \nabla_u f(u)$. In other words, the hyperbolic system (2) is under non conservation form. It is known that the non conservative products involved in $A(u) \partial_x u$ have no classical sense at the location of a shock since they cannot be given a unique definition within the standard framework of distributions. For this reason, it must be recognized that additional information is required in order to specify the definition, e.g. the value of the non conservative product $A(u) \partial_x u$ at shocks. A closure equation for defining the shock solutions is therefore needed and of course, its mathematical definition must match the underlying physics. The key feature is that the definition of shock solutions heavily depends on the shape of the diffusive tensor $D(u)$ (see [9],[13]).

Once the discontinuous solutions of the non conservative hyperbolic system (2) are defined, the first order system is well-posed and its numerical approximation could be tackled. However, two difficulties arise in that way. First, a formula in closed form for shock solutions is in general not available.
Furthermore, even when explicitly available, we have illustrated [4] that the error in the discrete capture of shock solutions unacceptably grows with the strength of the shock. To overcome these two difficulties at the same time, we propose to study the existence of a conservative formulation for system (2) that is compatible with the diffusive tensor $D$. For this equivalent system of conservation laws, we then derive an exact Roe-type linearization [11]. A two-dimensional problem is displayed to numerically illustrate the method. Numerical results are compared to the RAM-C flights experiments.

2. Modeling and Equations

2.1. CONVECTIVE-DIFFUSIVE SYSTEM

This work treats mixtures of gases made of electrons and $n$ heavy species, $n_i$, $1 \leq n_i \leq n$ of them being ionized. The classical but central assumption we adopt is local charge neutrality. Its deep consequences, we put forward below, are in fact responsible for the ill-posedness of our problem. However, such an assumption turns out to be necessary as we shall briefly point out. To account for the smallness of the mass ratios $M_e/M_\alpha << 1$, $\alpha \in \{1,...,n\}$, the electron gas is endowed with a temperature $T_e$ distinct from the temperature $T$ of the heavy species mixture. Moreover, $n_v$, $1 \leq n_v \leq n$, molecular species have their own vibrational temperature $T_{v,\beta}$, $\beta \in \{1,...,n_v\}$. The second order diffusive convective system governing the mixture under consideration can be written as:

\begin{align}
\partial_t \rho Y_\alpha + \partial_x (\rho Y_\alpha v - \rho D \partial_x Y_\alpha) &= m_\alpha^+, \quad 1 \leq \alpha \leq n, \\
\partial_t \rho v + \partial_x (\rho v^2 + p + p_e) - \partial_x (\nu \partial_x v) &= 0, \\
\partial_t \rho E + \partial_x (\rho E + p + p_e) v - \partial_x (\nu v \partial_x v + \kappa \partial_x T) - \partial_x (\kappa_e \partial_x T_e + \sum_\beta \kappa_{v,\beta} \partial_x T_{v,\beta} + \sum_\alpha \rho D h_\alpha) &= 0 \\
\partial_t \rho Y_{\beta} c_{v,\beta} + \partial_x \rho Y_{\beta} c_{v,\beta} v - \partial_x (\kappa_{v,\beta} \partial_x T_{v,\beta}) &= S_{t-v;\beta} + S_{e-v;\beta} \\
&+ m_{\beta}^+ c_{v,\beta}, \\
1 \leq \beta \leq n_v \\
\partial_t \rho Y_e E_e + \partial_x (\rho Y_e E_e + p_e) v - N_{e,q_e} E v - \partial_x (\kappa_e \partial_x T_e) &= S_{t-e}.
\end{align}

Here, $Y_\alpha$, $\alpha \in \{1,...,n\}$, denotes the mass fraction of species $\alpha$. The conservation laws (4),(5) respectively govern the momentum and total energy $\rho E$ of the mixture of the heavy species plus the electron gas. Conservation laws (6) refer to the vibrational energies $\rho Y_{\beta} c_{v,\beta}$ of the $nv$ molecular species.
that are assumed to be in thermal nonequilibrium. These \((n + n_v + 2)\) conservation laws are supplemented by a balance equation (7): the expected conservation law for the electron gas energy \(\rho Y_e E_e\) must be, in fact, balanced by the work of the electric field \(\mathcal{E}\).

\(\nu, \kappa\) and \(D\) respectively stand for the viscosity, the thermal conductivity and the diffusion coefficient of the mixture of the heavy species. \(\kappa_{\nu, \beta}\) denotes the thermal conductivity of the \(\beta^{th}\) molecular species in thermal nonequilibrium. The modeling of transport phenomena for the electron gas only considers its thermal conductivity \(k_e\), its viscosity being negligible. \(S_{l-v;\beta}\) expresses the vibrational energy exchange of species \(\beta\) with the translational mode of heavy species, \(S_{e-v;\beta}\) the exchange with the translational mode of the electron gas. \(m_\alpha^+\) is the mass production per unit volume of species \(\alpha\). \(S_{l-e}\) accounts for the change in electron energy caused by elastic collisions.

The required closure equations are as follow. The electron gas pressure \(p_e\) obeys:

\[
p_e = (\gamma_e - 1)(\rho Y_e E_e - \frac{1}{2} Y_e \rho v^2), \quad \gamma_e \in [1, 3],
\]

while the pressure law \(p\) for the mixture of heavy species is defined by:

\[
p = (\gamma_{tr} - 1) \left( \rho E - \frac{1}{2} \rho v^2 \sum_{\beta} \rho Y_{\beta} c_{\nu;\beta} \right. \\
- \sum_{\alpha} \rho Y_{\alpha} (h_{\alpha}^0 + c_{\alpha}(T)) - \frac{p_e}{\gamma_e - 1}) \quad \gamma_{tr} \in [1, 3].
\]

Here, \(c_{\alpha}(T)\) refers to the energy of the internal modes of species \(\alpha\) at equilibrium with the temperature \(T\) and \(h_{\alpha}^0\) denotes its heat of formation. The temperatures \(T\) and \(T_e\) are defined using a perfect gas assumption.

The last closure equation is to be specified deals with the electric field \(\mathcal{E}\). The assumption of local charge neutrality yields \(\mathcal{E}\) from the electron gas momentum equation:

\[
N_{el} q_e \mathcal{E} = \partial_t p_e + \{ \partial_t Y_e \rho v + \partial_x (Y_e \rho v^2) \} - M_{e, elas}^+ \tag{10}
\]

with

\[
Y_e = M_e \sum_{\alpha=1}^{ni} z_{\alpha} \frac{Y_{\alpha}}{M_{\alpha}}.
\]

Here, \(z_{\alpha}\) refers to the electric charge of the \(\alpha^{th}\) ionized species. \(M_{e, elas}^+\) denotes the momentum source term for electrons due to elastic collisions. The second term in the right hand side of (10) is traditionally neglected, considering the smallness of \(Y_e\) (\(M_e/M_{\alpha} \ll 1\)). But in the present work, this term is kept and will turn out to play a crucial role in the analysis.
One may raise the natural question of giving up the assumption of local charge neutrality. Indeed, the expected closure equation for $E$ should be the Poisson law, written here in a dimensionless form:

$$
\varepsilon \partial_x E = \sum_{\alpha=1}^{n} \{ z_{\alpha} \frac{\rho Y_{\alpha}}{M_{\alpha}} - \frac{\rho Y}{M_c} \}.
$$  \hspace{1cm} (12)

In (12), $\varepsilon$ refers to a parameter proportional to the Debye length. This parameter actually yields a rough estimate of the spatial resolution that is required to approximate (12); namely $O(\varepsilon^{-1})$. However in our applications, this parameter turns out to be extremely small and therefore, the needed spatial resolution remains far beyond the capabilities of available supercomputers. To make the numerical approximation tractable, we are led to let $\varepsilon$ go to zero. Doing so, the Poisson law degenerates into the so-called local charge neutrality condition (see the definition of $Y_e$ (11) ) and the closure equation for $E$ is classically recovered from (10).

2.2. EXTRACTED FIRST ORDER SYSTEM

We are interested in the numerical approximation to the system (3)–(7). In that aim, it is well known that an essential step consists in the derivation of a suitable approximation to the underlying inviscid equations. Here, we shall only focus ourselves on that issue. The extracted first order system of equations to be solved reads

$$
\partial_t \rho Y_{\alpha} + \partial_x (\rho Y_{\alpha} v) = 0, \quad 1 \leq \alpha \leq n, \hspace{1cm} (13)
$$

$$
\partial_t \rho v + \partial_x (\rho v^2 + p + p_e) = 0, \hspace{1cm} (14)
$$

$$
\partial_t \rho E + \partial_x (\rho E + p + p_e) v = 0 \hspace{1cm} (15)
$$

$$
\partial_t \rho Y_{\beta} e_{\nu;\beta} + \partial_x \rho Y_{\beta} e_{\nu;\beta} v = 0, \quad 1 \leq \beta \leq nw, \hspace{1cm} (16)
$$

$$
\partial_t \rho Y_e E_{\alpha} + \partial_x (\rho Y_e E_c + p_e) v
$$

$$
- v \partial_x p_e + v Y_e \partial_x (p + p_e) = 0, \hspace{1cm} (17)
$$

since we easily get from (10)

$$
N_e q_e E = \partial_x p_e - Y_e \partial_x (p + p_e). \hspace{1cm} (18)
$$

The work of the electric field $N_e q_e E \times v$ in (17) is clearly a product involving first order and zero order terms and furthermore, it cannot be put under divergence form. The system is therefore under non conservation form and for the reasons put forward in the introduction, this system is ill-posed. The analysis we perform in what follows, adresses the issue of well-posedness. First, let us stress that the product $N_e q_e E \times v$ cannot be regarded, as it is
commonly handled (see [1] for instance), as a source term, e.g. a function depending solely on zero order terms. On the contrary, such a product is a constitutive part of the first order system. The assumption of local charge neutrality is responsible for the closure equation (18) and it is therefore responsible for the non conservation form of the first order system.

3. Analysis for well-posedness of the extracted first order system

3.1. BASIC PROPERTIES OF THE EXTRACTED FIRST ORDER SYSTEM

In this section, we state some of the main properties of the first order system (13)-(17). The results given below intend to provide a deeper insight into the system under consideration and in particular to shed light on its relationships with the standard $3 \times 3$ Euler equations. The phase space associated with (13)-(17) is the following subset of $\mathbb{R}^p$, $p = n + nv + 3$:

$$\Omega = \{ u \in \mathbb{R}^p; \rho > 0, \: v \in \mathbb{R}, \: p > 0, \: p_e > 0, \: 0 \leq Y_i \leq 1 \text{ and } Y_e + \Sigma_i Y_i = 1 \}. \quad (19)$$

We begin with the following result devoted to the smooth solutions of (13)-(17).

**Lemma 1.** Let $u : \mathbb{R} \times \mathbb{R}^+ \to \Omega$ be a $C^1$ solution of (13)-(17). Furthermore, $u$ satisfies the following balance equations in non conservation form:

$$\partial_t Y_\alpha + v \partial_x Y_\alpha = 0, \quad 1 \leq \alpha \leq n - 1, \quad (20)$$
$$\partial_t \rho + v \partial_x \rho + \rho \partial_x v = 0, \quad (21)$$
$$\partial_t v + v \partial_x v + \rho^{-1} \partial_x (p + p_e) = 0, \quad (22)$$
$$\partial_t p + \gamma p \partial_x v + v \partial_x p = 0, \quad (23)$$
$$\partial_t p_e + \gamma_e p_e \partial_x v + v \partial_x p_e = 0, \quad (24)$$
$$\partial_t Y_\beta v_{\nu, \beta} + v \partial_x Y_\beta v_{\nu, \beta} = 0, \quad 1 \leq \beta \leq n. \quad (25)$$

Dropping the electron gas pressure $p_e$ in (22), the three equations (21) to (23) are easily recognized to coincide with the ones governing the smooth solutions of the $3 \times 3$ Euler system. Here, the heavy species mixture and the electron gas equally contribute to the velocity balance equation: each with their own pressure. The pressure balance equations (23) and (22) read the same but the key point remains in their complete decoupling. This decoupling stems from the particular form of the closure equation (10),(18) we have considered : e.g. with no simplification. We stress that this decoupling is actually at the basis of all the desirable properties stated in what follows. More specifically, we have

**Proposition 1.**

i) The first order system (13)-(17) is hyperbolic in the phase space $\Omega$, $A(u) : \Omega \to \text{Mat}(\mathbb{R}^p)$ admits the following three real eigenvalues:

$$v - c, \: v, \: v + c, \quad c^2 = (\gamma p + \gamma_e p_e) / \rho. \quad (26)$$
where the eigenvalue $v$ has an order of multiplicity $(n + n_v + 1)$.

ii) Under standard thermodynamic assumptions, the fields associated with
the eigenvalues $v \pm c$ are genuinely nonlinear. They admit the following
common Riemann invariants $Y_\alpha$, $i \in \{1, \ldots, n\}$; $s$ (the specific mixture
entropy for heavy species); $p_e/p^*e$; $Y_{\beta e}v, \beta \in \{1, \ldots, n_v\}$. This list is
supplemented by an additional Riemann invariant whose differential form
reads, respectively for $v \pm c$:

$$
\rho c d_\xi v \mp d_\xi (p + p_c) = 0. \tag{27}
$$

iii) The fields associated with the eigenvalue $v$ are linearly degenerate. Their
Riemann invariants are given by $v$ and $(p + p_c)$.

iv) For contact discontinuities, the closure equation (12) for $N_{e0}eE$ never
involves a product of a Dirac mass against a Heaviside function and is thus
well-defined.

v) The non-conservative product $N_{e0}eE \times v$ has no classical sense for the
shock solutions of (13)-(17).

3.2. SHOCK SOLUTIONS

A well known consequence of hyperbolicity is that the solutions to (13)-(17)
may develop discontinuities in a finite time. According to v) above, the
shock solutions of (13)-(17) stay unknown within the standard framework
of distributions. The nonconservative products involved in $A(u)\partial_x u$ can
nevertheless receive a definition in the recent setting proposed by DalMaso-LeFloch-Murat [7]. Indeed, such products can be defined thanks to a given
fixed family of paths, denoted $\Phi$, that are subject to some consistency and
smoothness conditions (for the details, we refer to [7]). For a discontinuity
separating the two states $u_L$ and $u_R$, the non conservative product is
defined by:

$$
[A(u)\partial_x u]_\Phi = \int_0^1 A(\Phi(s; u_L, u_R))\partial_s \Phi(s; u_L, u_R)ds. \tag{28}
$$

Note that since $A$ is not a jacobian matrix, the definition (28) entirely
depends on the choice of $\Phi$. After LeFloch [9] and Sainsaulieu [13], the relevant choice of $\Phi$ comes from the study of the full second order convective
diffusive system (1). Indeed, the shock solutions of (13)-(17) can be defined
as the limit when diffusion is neglected in the travelling wave solutions of
(1). More precisely, we have

Definition 1.

A function $u : \mathbb{R} \times \mathbb{R}^+ \to \Omega$ is a travelling wave solution of (1) connecting
the states \( \mathbf{u}_L \) and \( \mathbf{u}_R \) with speed \( \sigma \), if \( \mathbf{u} \) is a bounded function in the form:

\[
\mathbf{u}(x, t) = \hat{\mathbf{u}}(x - \sigma t)
\]

with \( d_\xi \hat{\mathbf{u}} \in L^1(\mathbb{R}) \) such that:

\[
-\sigma d_\xi \hat{\mathbf{u}} + A(\hat{\mathbf{u}})d_\xi \hat{\mathbf{u}} - d_\xi (\mathcal{D}(\hat{\mathbf{u}})d_\xi \hat{\mathbf{u}}) = 0,
\]

\[
\lim_{\xi \to \infty} \hat{\mathbf{u}}(\xi) = \mathbf{u}_L, \quad \lim_{\xi \to -\infty} \hat{\mathbf{u}}(\xi) = \mathbf{u}_R.
\]

(29)

(30)

Note that (29) is nothing else but the system (1) written in the frame of the travelling wave \( \hat{\mathbf{u}} \). Next let be given a small parameter \( \epsilon > 0 \) and let us consider the function \( \hat{\mathbf{u}}(\frac{x - \sigma t}{\epsilon}) \). It can be easily seen that this function is also a travelling wave solution but of the convective-diffusive system (1) with vanishing viscosity:

\[
-\sigma d_\xi \hat{\mathbf{u}}_\epsilon + A(\hat{\mathbf{u}}_\epsilon)d_\xi \hat{\mathbf{u}}_\epsilon - \epsilon d_\xi (\mathcal{D}(\hat{\mathbf{u}}_\epsilon)d_\xi \hat{\mathbf{u}}_\epsilon) = 0,
\]

\[
\lim_{\xi \to \infty} \hat{\mathbf{u}}_\epsilon(\xi) = \mathbf{u}_L, \quad \lim_{\xi \to -\infty} \hat{\mathbf{u}}_\epsilon(\xi) = \mathbf{u}_R.
\]

(31)

(32)

Furthermore, since \( d_\xi \hat{\mathbf{u}}_\epsilon \in L^1(\mathbb{R}) \) with \( \|d_\xi \hat{\mathbf{u}}_\epsilon\|_{L^1} = \|d_\xi \hat{\mathbf{u}}\|_{L^1} \), we deduce that \( \{\hat{\mathbf{u}}_\epsilon\}_{\epsilon > 0} \) tends almost everywhere as \( \epsilon \) goes to zero to the step function \( \mathbf{u}_L + (\mathbf{u}_R - \mathbf{u}_L)\mathcal{H}(x - \sigma t) \) while for any given \( \epsilon > 0 \), the following jump-like conditions hold true:

\[
\sigma(\mathbf{u}_R - \mathbf{u}_L) = \int_R A(\hat{\mathbf{u}}(\xi))\hat{\mathbf{u}}'(\xi)d\xi = \int_R A(\hat{\mathbf{u}}(\xi))\hat{\mathbf{u}}'(\xi)d\xi.
\]

(33)

Turning back to a fixed frame and motivated by the above calculations, LeFloch [9] suggests the following (see also Sainsaulieu [13]):

**Definition 2.**

Let be given \( \hat{\mathbf{u}} : \mathbb{R} \to \Omega \) a travelling wave solution of (1) connecting \( \mathbf{u}_L \) to \( \mathbf{u}_R \) at speed \( \sigma \). Then the step function defined by

\[
\hat{\mathbf{u}}(x, t) = \begin{cases} 
\mathbf{u}_L, & \text{if } x < \sigma t, \\
\mathbf{u}_R, & \text{if } x > \sigma t,
\end{cases}
\]

is said to be a discontinuous solution of the first order extracted system (2) that is compatible with the diffusive tensor \( \mathcal{D} \).

It is essential to notice that by contrast with conservative hyperbolic systems, the travelling wave solutions and therefore the step function \( \hat{\mathbf{u}} \) depend on the shape of \( \mathcal{D} \). Indeed, two non proportional diffusive tensors generally yield distinct shock solutions for the underlying convective system.

The first order extracted system (13)-(17) can be closed according to the above framework and its numerical approximation can be tackled. However and as explained in the Introduction, two difficulties arise when dealing with the numerical approximation of the discontinuous solutions of (13)-(17). Namely, if its shock solutions can be theoretically defined, a formula
in closed form for the jump relations is in general not available while their knowledge is basically needed in the derivation of approximate Riemann solvers. Further more, even if available, these jump relations cannot be correctly restored at the discrete level at least for sufficiently strong shocks. In the companion paper [4], such a failure is rigorously exemplified and is also given numerical illustrations. The way we propose to overcome these two difficulties at the same time is provided by the following statement:

**Proposition 2** (Sainsaulieu-Raviart [11]).
Let be given a $C^1$ diffeomorphism $\Psi : \Omega \to \Omega$. Assume that for any given smooth solution $u : \mathbb{R} \times \mathbb{R}^1 \to \Omega$ of the non conservative convective diffusive system:

$$\partial_t u + A\partial_x u - \partial_x (D(u)\partial_x u) = 0,$$

(35)

the change of variables $v = \Psi(u)$ yields a solution of the following system of conservation laws:

$$\partial_t v + \partial_x F(v) - \partial_x (\mathcal{C}(v)\partial_x v) = 0.$$

(36)

Then, the shock solutions of the extracted first order system of (35), e.g. the ones compatible with $D$, do coincide with the shock solutions of the extracted conservative first order system of (36):

$$\partial_t u + \left[ A(u)\partial_x u \right]_{\Psi(u)} = 0 \quad \iff \quad \partial_t v + \partial_x F(v) = 0.$$

(37)

The benefit of this equivalence is twofold: first the shock wave solutions to (13)-(17) are now explicitly given by the Rankine-Hugoniot relations associated with

$$\partial_t v + \partial_x F(v) = 0,$$

$$\text{e.g.} \quad -\sigma(v(u_R) - v(u_L)) + \left( F(v(u_R)) - F(v(u_L)) \right) = 0,$$

(38)

and second by, upwind methods in conservation form can be applied to (38) in order to approximate the equivalent weak solutions of (13)-(17) that are compatible with $D$. Motivated by these strong benefits, we have laid in a related work [3] the foundations for a systematic characterization of all the $C^1$-diffeomorphisms $\Psi$ that bring (36) from (35). The starting point stems from the following observation: the equivalence stated in (37) requires in fact the fulfillment of two distinct conditions. Namely, not only the first order system must meet a conservation form (P1) but also that the second order operator must stay under conservation form (P2). The study of the admissible changes of variables $\Psi$ that satisfy both requirements (P1) and (P2) is detailed in [].
To summarize:

**Theorem 1.**

i) If the thermal conductivity $\kappa_e$ of the electron gas is neglected, then the discontinuous solutions of (13)-(17) that are compatible with (30) are equivalent weak solutions of the following hyperbolic systems in conservation form, where $g$ denotes a strictly increasing function:

\[
\begin{align*}
\partial_t \rho Y_\alpha + \partial_x \rho Y_\alpha v &= 0, \quad 1 \leq \alpha \leq n, \\
\partial_t (\rho v) + \partial_x (\rho v^2 + p + p_e) &= 0, \\
\partial_t (\rho E) + \partial_x (\rho E + p + p_e)v &= 0, \\
\partial_t s_e + \partial_x s_e u &= 0, \quad s_e = \rho g(p_e/Y_e \rho^\gamma_e), \\
\partial_t \rho Y_\beta v_{\beta} + \partial_x \rho Y_\beta v_{\beta} v &= 0, \quad 1 \leq \beta \leq nv.
\end{align*}
\]

ii) Provided that $g$ satisfies a set of relevant conditions [4], the associated non strictly hyperbolic system does admit entropy pairs that are strictly convex in the sense of Lax.

In what follows, we shall denote the pressure law for the electron gas:

\[
p_e = (\gamma_e - 1) Y_e \rho^\gamma_e h(s_e/\rho), \quad h = g^{-1}(> 0).
\]

4. Numerical experiments

Under the assumptions given in Theorem 1, a Roe-type linearization was built in an exact way, for the conservative formulation we propose [4]. In this section, this method is applied to the calculation of the flow around an experimental probe flying at approximately satellite speed. This probe, called the RAM-C II test, is a sphere-cone configuration with a 0.1524m nose radius, 9° cone half-angle and a total length of 1.295m. It was instrumented to measure electron number densities at four axial locations using microwave reflectometers and in the boundary layer using an electrostatic rake.

Calculations were performed to replicate the flight conditions at an altitude of 61 kilometers. The corresponding free-stream velocity was 7650 m/s, the temperature 254K, the pressure 19.7 Pa, the Mach number 23.9. The wall was assumed to be fully non-catalytic and its temperature was fixed at 1500K. The computations were performed on a mesh with 85 points along the sphere-cone and 50 points in the flowfield normal to the body. The second order of accuracy is achieved using the MUSCL approach. The CFL condition has been set at the constant value 10.0.

As shown in Figure 1, the ionized zone is restricted to the vicinity of the sphere. In Figure 2 are plots of mass fraction on the stagnation streamline.
As diatomic oxygen and nitrogen dissociate, there is a significant production of NO just behind the shock. But NO also rapidly dissociates near the wall. The computed results are now compared to the peak electron number density measured axially along the body (Figure 3). Although the numerical results slightly underestimate experimental values, agreement is rather good. In order to evaluate the validity of our approach in neglecting the thermal conductivity of electrons, this last one was compared to the translational and rotational conductivity of the heavy species. Figure 4 validates our approximation.

*Figure 1. NO$^+$ mass fraction contours*
Figure 2. Mass fractions along the stagnation streamline

Figure 3. Axial peak electron number density
Figure 4. Thermal conductivities along the stagnation streamline

References

5. Coquel F. and Marmignon C. work in preparation