Multi-Trace Boundary Integral Formulation for Acoustic Scattering by Composite Structures

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Abstract

We study the scattering of an acoustic wave by an object composed of several adjacent parts with different material properties. For this problem we derive a new integral equation formulation of the first kind. This formulation involves two Dirichlet data and two Neumann data at each point of each material interface of the diffracting object. It is immune to spurious resonances, and it enjoys a stability property that ensures quasi-optimal convergence of conforming Galerkin boundary element discretization. In addition, the operator of this formulation satisfies a relation similar to the standard Calderón identity. © 2012 Wiley Periodicals, Inc.

Introduction

The simulation of wave propagation in a medium with piecewise constant wave number is of practical interest in many applications related to acoustics and electromagnetics. To tackle this type of problem, one possible approach consists of formulating the problem as a boundary integral equation. As regards integral formulations though, most of the literature deals with geometries where at most two different participating media are adjacent to each other. However, in practice, there are many relevant geometrical configurations where three or more different media are adjacent to each other; what we call multiple subdomain scattering is the study of wave propagation in arrangements with this type of geometry. The present article studies typical scalar wave propagation problems of the form

$$\begin{aligned}&\Delta u + \kappa_j^2 u = 0 \quad \text{in } \Omega_j,\ \forall j = 0,\dots,n,\\ &+ \text{transmission conditions at interfaces}\\ &+ \text{radiation condition at infinity}\end{aligned}$$

where $\kappa_j \in \mathbb{R}$ refers to the wavenumber in the subdomain Ω_j. In such a multi-subdomain setting, one difficulty for the solution of this problem is related to the presence of triple points, i.e., points where more than two subdomains are adjacent. Because of such geometrical features, the union of interfaces is not orientable. This

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precludes the use of powerful Calderón preconditioning techniques very popular in the standard case of interfaces separating only two media.

Concerning multi-subdomain scattering, a variational direct boundary integral formulation of the first kind derived from a representation formula has long been known; see [6, 14, 20]. It was first analyzed by von Petersdorff in [27] for scalar problems, and this analysis was extended to Maxwell's equations by Buffa [4]. In this approach the transmission conditions are taken into account directly via the choice of well-chosen variational spaces. Such a formulation turns out to be the generalization for multiple subdomain configurations of the classical direct first-kind formulation well-known for transmission problems where interfaces separate at most two different media. One interesting feature of this formulation is that, at each point of each interface, it involves only one Dirichlet datum and one Neumann datum. As a consequence, we call it single-trace formulation of the first kind. Note that no efficient preconditioner has been proposed so far for this formulation in the case of multiple subdomain scattering.

In [16], Steinbach and coworkers developed another formulation of the first kind involving only one Dirichlet datum and two Neumann data at each interface. Several variants of this formulation were proposed later; see [18, 26] and references therein. It employs a domain decomposition approach where part of the transmission conditions are imposed by means of Lagrange multipliers. Such a method can be readily preconditioned. However, it requires the inversion of a Steklov-Poincaré operator in each subdomain.

More recently, in [15], Jerez-Hanckes and one of the authors developed yet another integral formulation of the first kind for multi-subdomain scattering that also has good properties in terms of preconditioning possibilities. This approach is different since transmission conditions are not imposed through Lagrange multipliers. They named this formulation "multi-trace formulation," as all unknowns of the problem are doubled on each interface. This approach does not involve any Steklov-Poincaré operator, and it is amenable to Calderón preconditioning.

In the present article, starting from the classical formulation obtained by a representation formula, we derive a new formulation that involves two Dirichlet data and two Neumann data at each point of each interface. Hence it also qualifies as multi-trace. We prove existence and uniqueness of solutions for any choice of wavenumbers κ_j. We also show that it satisfies some coercivity property modulo a compact perturbation so that quasi-optimal convergence of conforming Galerkin methods can be established. Finally, we show that the square of the operator associated to this formulation is of the form "identity + compact"; it seems that no counterpart of such a result has ever been established for other known integral formulations for multi-subdomain scattering. An analogous multi-trace formulation with similar properties is even available for electromagnetic scattering [11].

The present paper is structured as follows. First we describe the problem under consideration. It addresses the scattering of a scalar wave in a medium with piecewise constant wave number. In Section 2 we introduce a functional setting

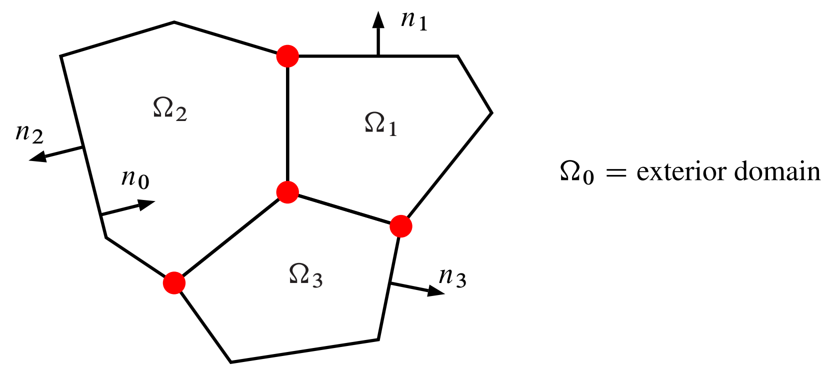


FIGURE 1.1. Example of a multi-subdomain geometry.

that is well adapted to our problem. In Section 3 we recall some classical result about integral formulations. In Section 4 we provide a brief review of the classical single-trace formulation of the first kind. In Section 5 we temporarily focus on the particular case of two separated scatterers in order to motivate and introduce the new formulation. Starting with its rigorous theoretical analysis, in Section 6 we establish some remarkable properties satisfied by the space of Cauchy data for the Helmholtz equation. In Sections 7 and 8, we rewrite the new boundary integral equations in a manner that takes advantage of the properties of the functional setting that we introduced before. In Section 9 we show that the classical single-trace formulation of the first kind is equivalent to a new formulation whose principal feature is that no constraint is imposed anymore on the trial space and on the test space. In Section 10 we prove that the bilinear form associated to this new formulation is coercive. In Section 11 we establish a property close to a Calderón identity for the operator associated with our new formulation. Finally, in Section 12 we present a few numerical results for a boundary element discretization of the new formulation.

1 Setting of the Problem

We consider a partition $\mathbb{R}^d = \bigcup_{i=0}^{n} \overline{\Omega}_i$ where $\bigcup_{i=1}^{n} \overline{\Omega}_i$ is bounded and each Ω_i is a connected Lipschitz domain; i.e., $\partial\Omega_i$ is locally the graph of a Lipschitz function; see [19, def. 3.28]. We also set $\Gamma = \bigcup_{i=1}^{n} \partial\Omega_i$. A typical example of such a geometrical configuration is represented in Figure 1.1 above. Note that there may exist points where three or more subdomains would be adjacent, which is precisely the situation that we wish to tackle. For each i the vector n_i refers to the normal vector on $\partial\Omega_i$ directed toward the *exterior* of Ω_i. The normal vector field n_i is defined almost everywhere on $\partial\Omega_i$ according to Rademacher's theorem; see [28, 11A, p. 272].

1.1 Elementary Functional Spaces

Let us recall the definition of some elementary functional spaces. We set

$$H^1(\omega) := \Big\{ v \in \mathrm{L}^2(\omega) \;\Big|\; \|v\|^2_{H^1(\omega)} := \int_\omega |v|^2 + |\nabla v|^2 \, d\mathbf{x} < +\infty \Big\},$$

$$H(\mathrm{div},\omega) := \Big\{ \mathbf{q} \in \mathrm{L}^2(\omega)^3 \;\Big|\; \|\mathbf{q}\|^2_{H(\mathrm{div},\omega)} := \int_\omega |\mathbf{q}|^2 + |\mathrm{div}(\mathbf{q})|^2 \, d\mathbf{x} < +\infty \Big\}.$$

We also define the space $H^1(\Delta,\omega) := \{v \in H^1(\omega) \mid \nabla v \in H(\mathrm{div},\omega)\}$ equipped with the norm $\|u\|^2_{\Delta,\omega} := \|u\|^2_{H^1(\omega)} + \|\Delta u\|^2_{\mathrm{L}^2(\omega)}$. If $H(\omega)$ is any one of the spaces $H^1(\omega)$, $H(\mathrm{div},\omega)$, or $H^1(\Delta,\omega)$, then we set $H_{\mathrm{loc}}(\overline{\omega}) := \{v \text{ such that } \varphi v \in H(\omega)\ \forall \varphi \in \mathscr{D}(\mathbb{R}^d)\}$ where $\mathscr{D}(\mathbb{R}^d)$ refers to the set of compactly supported C^∞ functions.

1.2 Trace Spaces

For an open set $\omega \subset \mathbb{R}^d$ with Lipschitz boundary, the trace $v \mapsto v|_{\partial\omega}$ defines a continuous map from $H^1(\omega)$ into $\mathrm{L}^2(\partial\omega)$ (see [24, theorem 2.6.8] and [19, theorem 3.37]), and the space $H^{1/2}(\partial\omega) := \{v|_{\partial\omega} \mid v \in H^1(\omega)\}$ equipped with the norm

$$\|v\|_{H^{1/2}(\partial\omega)} := \inf\{\|u\|_{H^1(\omega)} \text{ such that } u \in H^1(\omega),\ u|_{\partial\omega} = v\}$$

is a Banach space (this is a direct application of [22, theorem 1.41]). Let us denote $H^{-1/2}(\partial\omega)$ its topological dual, which we equip with the corresponding dual norm

$$\|q\|_{H^{-1/2}(\partial\omega)} := \sup_{v \in H^{1/2}(\partial\omega)} \frac{1}{\|v\|_{H^{1/2}(\partial\omega)}} \Big| \int_{\partial\omega} vq \, d\sigma \Big|.$$

If n denotes the normal vector field over $\partial\omega$, the trace $\mathbf{q} \mapsto n \cdot \mathbf{q}|_{\partial\omega}$ gives rise to a continuous map from $H(\mathrm{div},\omega)$ onto $H^{-1/2}(\partial\omega)$, and the trace $n \cdot \nabla v|_{\partial\omega}$ is well defined whenever $v \in H^1_{\mathrm{loc}}(\Delta,\overline{\omega})$; see [24, theorem 2.7.7] and [19, lemma 4.3].

1.3 Trace Operators

Recall that n_j refers to the normal vector on $\partial\Omega_j$ directed toward the exterior of Ω_j. For $j = 0,\dots,n$ and $v \in H^1_{\mathrm{loc}}(\Delta,\overline{\Omega}_j)$, define $\gamma^j_{\mathrm{D}}(v) = v|_{\partial\Omega_j}$ and $\gamma^j_{\mathrm{N}}(v) = n_j \cdot \nabla v|_{\partial\Omega_j}$ where these Dirichlet and Neumann traces are taken from the *interior* of Ω_j. The exterior Dirichlet and Neumann traces on $\partial\Omega_j$ will be denoted $\gamma^j_{\mathrm{D},c}$ and $\gamma^j_{\mathrm{N},c}$ (with normal vector still directed toward the exterior of Ω_j). We also consider mean and jump combinations of these operators

$$\begin{gathered} \gamma^j(u) = \big(\gamma^j_{\mathrm{D}}(u), \gamma^j_{\mathrm{N}}(u)\big)^\top, \quad \gamma^j_c(u) = \big(\gamma^j_{\mathrm{D},c}(u), \gamma^j_{\mathrm{N},c}(u)\big)^\top \quad \forall u \in H^1_{\mathrm{loc}}(\Delta,\overline{\Omega}_i), \\ [\gamma^j] := \gamma^j - \gamma^j_c, \quad \{\gamma^j\} := \frac{1}{2}\big(\gamma^j + \gamma^j_c\big). \end{gathered} \tag{1.1}$$

1.4 Transmission Conditions

In the present article, we shall study a problem of wave propagation in a medium containing subdomains Ω_j, each of which is characterized by material coefficients. One of these coefficients, $\mu_j \in (0, +\infty)$, comes into play in the transmission conditions that we have to impose across the interfaces $\partial\Omega_j \cap \partial\Omega_k$ for $j, k = 0, \dots, n$. These transmission conditions read

$$\forall j, k = 0, \dots, n \begin{cases} \gamma_{\mathrm{D}}^j(u) - \gamma_{\mathrm{D}}^k(u) = 0 & \text{on } \partial\Omega_j \cap \partial\Omega_k, \\ \mu_j^{-1}\gamma_{\mathrm{N}}^j(u) + \mu_k^{-1}\gamma_{\mathrm{N}}^k(u) = 0 & \text{on } \partial\Omega_j \cap \partial\Omega_k, \end{cases} \tag{1.2}$$

where $u \in H^1_{\mathrm{loc}}(\Delta, \overline{\Omega}_j)$, $j = 0, \dots, n$. The above equations shall be understood in the sense of distributions over $\partial\Omega_j \cap \partial\Omega_k$. To be more precise, set $\Gamma_{j,k} = \partial\Omega_j \cap \partial\Omega_k$. The first transmission condition, for example, precisely means

$$\int_{\Gamma_{j,k}} \big(\gamma_{\mathrm{D}}^j(u) - \gamma_{\mathrm{D}}^k(u)\big)\varphi \, d\sigma = 0 \quad \forall \varphi \in \mathscr{D}(\Gamma_{j,k})$$

where $\mathscr{D}(\Gamma_{j,k})$ stands for the set of traces on $\Gamma_{j,k}$ of C^∞-functions whose support is included in the interior of $\Gamma_{j,k}$ (so that these functions vanish in the neighborhood of $\partial\Gamma_{j,k}$). Although (1.2) is quite a standard way of writing transmission conditions (see [15, 16, 26, 27]), we choose another formulation of these conditions that better fits the functional framework that we shall introduce later. Set

$$\mu(\mathbf{x}) = \mu_j \quad \text{in } \Omega_j \ \text{ with } \mu_j \in (0, +\infty) \ \ \forall j = 0, \dots, n.$$

It is an elementary exercise on Sobolev spaces to show that (1.2) is strictly equivalent to the conditions $u \in H^1_{\mathrm{loc}}(\mathbb{R}^d)$ and $\mu^{-1}\nabla u \in H_{\mathrm{loc}}(\mathrm{div}, \mathbb{R}^d)$, which are the transmission conditions that we shall consider from now on.

1.5 The Scattering Problem

Let $u_{\mathrm{inc}} \in H^1_{\mathrm{loc}}(\mathbb{R}^d)$ satisfy $\Delta u_{\mathrm{inc}} + \kappa_0^2 u_{\mathrm{inc}} = 0$ in $\mathbb{R}^d$ for some $\kappa_0 \in \mathbb{R}$. This function will play the role of incident field. In the present article we study the following problem:

$$\begin{cases} \text{Find } u \in H^1_{\mathrm{loc}}(\mathbb{R}^d) \text{ such that} \\ \mu^{-1}\nabla u \in H_{\mathrm{loc}}(\mathrm{div}, \mathbb{R}^d) \\ \Delta u + \kappa_j^2 u = 0 \text{ in } \Omega_j, \ j = 0, \dots, n, \\ \mathrm{CI}_{\kappa_0}(u - u_{\mathrm{inc}}) = 0. \end{cases} \tag{1.3}$$

In the problem above, the constant $\kappa_j \in \mathbb{C}$ refers to the wavenumber inside the subdomain Ω_j. For the rest of this article, we will assume that it satisfies the conditions considered in assumption 2.1 in [27], namely,

$$\kappa_j \neq 0, \quad \Im\mathrm{m}\{\kappa_j\} \geq 0, \quad \Re\mathrm{e}\{\kappa_j\} \geq 0, \quad \forall j = 0, \dots, n. \tag{1.4}$$

In problem (1.3), the condition "$\mathrm{CI}_{\kappa_0}(u - u_{\mathrm{inc}}) = 0$" refers to a condition at infinity. It depends on κ_0 in the following manner:

- If $\Im\mathrm{m}\{\kappa\} > 0$, we set that $\mathrm{CI}_\kappa(v) = 0$ if and only if $v \in H^1(\mathbb{R}^d \setminus \overline{\mathrm{B}}_r)$ for any $r > 0$ such that $(\mathbb{R}^d \setminus \Omega_0) \subset \overline{\mathrm{B}}_r$ where $\mathrm{B}_r = \{\mathbf{x} \in \mathbb{R}^d \mid |\mathbf{x}| < r\}$.
- If $\kappa \in (0, +\infty)$, we set that $\mathrm{CI}_\kappa(v) = 0$ if and only if

$$\lim_{r\to\infty} \int_{\partial \mathrm{B}_r} |\partial_r v - i\kappa v|^2 \, d\sigma_r = 0 \quad \text{with } r = |\mathbf{x}|.$$

This is the Sommerfeld radiation condition; see [12, 19].

Under condition (1.4), with the definition of CI_κ that we consider, problem (1.3) is well posed. This result is already well-known for the case of a single scatterer (see, for example, [12, 17, 19]), and it was established in [27] for transmission problems with multi-subdomain geometry. As was pointed out in [17], problem (1.3) may be ill posed (in the sense that the corresponding homogeneous problem admits nontrivial solutions) if no such condition as (1.4) was imposed.

2 Adapted Functional Spaces

In this paragraph we describe trace spaces for later use. This setting was already considered in [10]. The spaces that we introduce are well adapted to problems of multi-subdomain scattering.

2.1 Multitrace Space

In order to formulate an integral equation of problem (1.3) posed over Γ, a natural functional setting consists in taking the Cartesian product of trace spaces, namely,

$$\mathbb{H}(\Gamma) := \prod_{j=0}^{n} \mathbb{H}(\partial\Omega_j) \quad \text{where } \mathbb{H}(\partial\Omega_j) := H^{1/2}(\partial\Omega_j) \times H^{-1/2}(\partial\Omega_j)$$

with the norm

$$\|U\|_{\mathbb{H}} := \Big(\sum_{j=0}^{n} \|u_j\|^2_{H^{1/2}(\partial\Omega_j)} + \|p_j\|^2_{H^{-1/2}(\partial\Omega_j)} \Big)^{1/2}$$

$$\text{when } U = \begin{pmatrix} u_j \\ p_j \end{pmatrix}_{j=0,\dots,n}$$

The space $\mathbb{H}(\Gamma)$ equipped with the norm $\|\cdot\|_{\mathbb{H}}$ is a Banach space. Observe that this space can be identified to its own dual by means of the following duality pairing:

$$\text{(2.1)} \qquad \mathrm{B}(U, V) := \sum_{i=0}^{n} \mathrm{B}_i(U_i, V_i) \quad \text{with } \mathrm{B}_i(U_i, V_i) := \int_{\partial\Omega_i} u_i q_i - p_i v_i \, d\sigma$$

where

$$U = (U_j) = \begin{pmatrix} u_j \\ p_j \end{pmatrix}_{j=0,\dots,n} \in \mathbb{H}(\Gamma), \quad V = (V_j) = \begin{pmatrix} v_j \\ q_j \end{pmatrix}_{j=0,\dots,n} \in \mathbb{H}(\Gamma).$$

Although there exist many possible choices for the definition of the duality pairing between $\mathbb{H}(\Gamma)$ and its dual, the pairing that we introduce in (2.1) has properties that will be crucial for the analysis that we present in what follows. As can be straightforwardly checked, the bilinear form $\mathrm{B}(\cdot,\cdot)$ is nondegenerate: if $\mathrm{B}(U,V) = 0$ $\forall V \in \mathbb{H}(\Gamma)$ then $U = 0$.

2.2 Single-Trace Space

Now we introduce spaces that seem more adapted to the treatment of transmission conditions. This setting is inspired by the work of Bendali and his coworkers on a boundary integral formulation derived from representation formulas in the context of scattering by composite objects; see [2, 3]. We set

$$\begin{aligned}
\mathbb{X}^{+1/2}(\Gamma) &:= \Big\{ (v_i) \in \prod_{i=0}^{n} H^{1/2}(\partial\Omega_i) \;\Big|\; \\
&\qquad \exists v \in H^1_{\mathrm{loc}}(\mathbb{R}^d) \text{ with } v\big|_{\partial\Omega_j} = v_j,\ \forall j = 0,\dots,n \Big\},\\
\mathbb{X}^{-1/2}(\Gamma) &:= \Big\{ (q_i) \in \prod_{i=0}^{n} H^{-1/2}(\partial\Omega_i) \;\Big|\; \\
&\qquad \exists \mathbf{q} \in H_{\mathrm{loc}}(\mathrm{div},\mathbb{R}^d) \text{ with } n_j\cdot\mathbf{q}\big|_{\partial\Omega_j} = q_j,\ \forall j = 0,\dots,n \Big\},\\
\mathbb{X}(\Gamma) &:= \left\{ \begin{pmatrix} v_j \\ q_j \end{pmatrix}_{j=0,\dots,n} \in \mathbb{H}(\Gamma) \;\middle|\; (v_j) \in \mathbb{X}^{+1/2}(\Gamma) \text{ and } (q_j) \in \mathbb{X}^{-1/2}(\Gamma) \right\}.
\end{aligned}$$

To get an intuition of these spaces, observe that in the case where $\mathbb{R}^d = \overline{\Omega}_0 \cup \overline{\Omega}_1$ so that $\Gamma = \partial\Omega_0 = \partial\Omega_1$, there holds $\mathbb{X}(\Gamma) = \{(v,q,v,-q) \mid v \in H^{1/2}(\Gamma),\ q \in H^{-1/2}(\Gamma)\}$. Here is another instructive remark. Take $j = 0,\dots,n$ and consider $v \in H^1_{\mathrm{loc}}(\Delta, \mathbb{R}^d \setminus \Omega_j)$, as well as $V = (V_q)_{0\le q\le n}$ such that $V_j = \gamma_c^j(v)$ and $V_q = \gamma^q(v)$ for $q \neq j$. Then $V \in \mathbb{X}(\Gamma)$. Recall also the following result, which was established in [10]:

PROPOSITION 2.1. *Let*

$$(u_j) \in \prod_{j=0}^{n} H^{+1/2}(\partial\Omega_j) \quad \textit{and} \quad (p_j) \in \prod_{j=0}^{n} H^{-1/2}(\partial\Omega_j).$$

We have the following:

(i) $(u_j) \in \mathbb{X}^{+1/2}(\Gamma) \iff \sum_{j=0}^{n} \int_{\partial\Omega_j} u_j q_j \, d\sigma = 0 \quad \forall (q_j) \in \mathbb{X}^{-1/2}(\Gamma)$,

(ii) $(p_j) \in \mathbb{X}^{-\frac{1}{2}}(\Gamma) \iff \sum_{j=0}^{n} \int_{\partial\Omega_j} p_j v_j \, d\sigma = 0 \quad \forall (v_j) \in \mathbb{X}^{+1/2}(\Gamma)$.

Clearly $\mathbb{X}(\Gamma)$ is closed in $\mathbb{H}(\Gamma)$ for $\|\cdot\|_{\mathbb{H}}$ since, according to the previous proposition, the constraints characterizing $\mathbb{X}^{1/2}(\Gamma)$ (respectively, $\mathbb{X}^{-1/2}(\Gamma)$) involve continuous functionals over $H^{1/2}(\partial\Omega_j)$ (respectively, $H^{-1/2}(\partial\Omega_j)$), $j = 0,\dots,n$. Moreover, one obvious consequence of the preceding proposition is that $\mathbb{X}(\Gamma)$ can be identified with its own polar set under the duality pairing $\mathrm{B}(\cdot,\cdot)$. More precisely, for any $U \in \mathbb{H}(\Gamma)$ we have

$$U \in \mathbb{X}(\Gamma) \iff \mathrm{B}(U,V) = 0 \quad \forall V \in \mathbb{X}(\Gamma). \tag{2.2}$$

Our motivation for introducing the space $\mathbb{X}(\Gamma)$ is that the transmission conditions contained in equation (1.3) can thus be recast with compact notation. Indeed, $(\gamma_{\mathrm{D}}^{j}(u))_{0\le j\le n} \in \mathbb{X}^{1/2}(\Gamma)$ implies that u admits no jump across any interface $\partial\Omega_j \cap \partial\Omega_k$, and $(\mu_j^{-1}\gamma_{\mathrm{N}}^{j}(u))_{0\le j\le n} \in \mathbb{X}^{-1/2}(\Gamma)$ implies that $\mu^{-1}\nabla u$ admits no normal jump across such interfaces. To sum up

$$u \text{ satisfies (1.2)} \iff \begin{cases} u|_{\Omega_j} \in H^1_{\mathrm{loc}}(\Delta,\overline{\Omega}_j), & j = 0,\dots,n,\\ \mathsf{T}_\mu(U) \in \mathbb{X}(\Gamma) & \text{with } U = \big(\gamma^j(u)\big)_{0\le j\le n}, \end{cases} \tag{2.3}$$

where we used the following notation:

$$\mathrm{Q}_j := \begin{bmatrix} 1 & 0 \\ 0 & 1/\mu_j \end{bmatrix},$$
$$\mathsf{T}_\mu(U) := (\mathrm{Q}_j(U_j))_{0\le j\le n} \quad \text{for } U = (U_j)_{0\le j\le n} \in \mathbb{H}(\Gamma).$$

In what follows, we shall also consider the operator T_0 constructed in the same manner as T_μ except that $\mu_j = \mu_0$ for $j = 1,\dots,n$. In other words we set $\mathsf{T}_0(U) = (\mathrm{Q}_0(U_j))_{j=0,\dots,n}$ for $U = (U_0,\dots,U_n) \in \mathbb{H}(\Gamma)$. Note the following property satisfied by T_0 (but not by T_μ for arbitrary $\mu_0,\dots\mu_n$):

$$\mathsf{T}_0(\mathbb{X}(\Gamma)) = \mathbb{X}(\Gamma). \tag{2.4}$$

3 Classical Results on Potential Operators

In this section we recall classical results related to integral formulations for the Helmholtz equation. Since what follows is already well-known, we do not provide any proof for these results and refer the reader to the textbooks [19, chaps. 6–7], [21, chaps. 3–4], and [24, chap. 3].

In what follows, $\mathscr{G}_\kappa(\mathbf{x})$ will denote the Green kernel of the operator $-\Delta - \kappa^2$ that satisfies the condition at infinity $\mathrm{CI}_\kappa(\mathscr{G}_\kappa(\cdot)) = 0$. For any subdomain Ω_j,

$$\forall V = \begin{pmatrix} v \\ q \end{pmatrix} \in \mathbb{H}(\partial\Omega_j) \quad \forall \mathbf{x} \in \mathbb{R}^d \setminus \partial\Omega_j \quad \text{we set}$$

$$\begin{aligned} \mathsf{SL}_\kappa^j(q)(\mathbf{x}) &:= \int_{\partial\Omega_j} \mathscr{G}_\kappa(\mathbf{x}-\mathbf{y})q(\mathbf{y})d\sigma(\mathbf{y}),\\ \mathsf{DL}_\kappa^j(v)(\mathbf{x}) &:= -\int_{\partial\Omega_j} n_j(\mathbf{y})\cdot(\nabla\mathscr{G}_\kappa)(\mathbf{x}-\mathbf{y})v(\mathbf{y})d\sigma(\mathbf{y}),\\ \mathsf{G}_\kappa^j(V)(\mathbf{x}) &:= -\mathsf{DL}_\kappa^j(v)(\mathbf{x}) + \mathsf{SL}_\kappa^j(q)(\mathbf{x}). \end{aligned} \tag{3.1}$$

The potential $\mathsf{G}_{\kappa}^{j}(V)(\mathbf{x})$ is a well-defined function over $\mathbb{R}^d \setminus \Gamma$. It induces continuous maps $\mathsf{G}_{\kappa}^{j} : \mathbb{H}(\partial\Omega_j) \to \sum_{q=0}^{n} H^1_{\mathrm{loc}}(\Delta, \overline{\Omega}_q)$. Let us recall a crucial result about these potential operators; see [19, theorem 7.7, 7.15], [21, theorem 3.1.1], and [24, sec. 3.1.1].

PROPOSITION 3.1. *Let $u \in H^1_{\mathrm{loc}}(\overline{\Omega}_j)$ such that $\Delta u + \kappa_j^2 u = 0$ in Ω_j and $\mathrm{CI}_{\kappa_0}(u) = 0$ if $j = 0$. We have the representation formula*

$$\mathsf{G}_{\kappa_j}^{j}(\gamma^j(u))(\mathbf{x}) = \begin{cases} u(\mathbf{x}) & \text{if } \mathbf{x} \in \Omega_j, \\ 0 & \text{if } \mathbf{x} \in \mathbb{R}^d \setminus \overline{\Omega}_j. \end{cases}$$

Similarly, let $u \in H^1_{\mathrm{loc}}(\mathbb{R}^d \setminus \Omega_j)$ such that $\Delta u + \kappa_j^2 u = 0$ in $\mathbb{R}^d \setminus \overline{\Omega}_j$ and $\mathrm{CI}_{\kappa_j}(u) = 0$ except for $j = 0$. The following formula holds:

$$\mathsf{G}_{\kappa_j}^{j}(\gamma_c^j(u))(\mathbf{x}) = \begin{cases} 0 & \text{if } \mathbf{x} \in \Omega_j, \\ -u(\mathbf{x}) & \text{if } \mathbf{x} \in \mathbb{R}^d \setminus \overline{\Omega}_j. \end{cases}$$

In what follows, we will also need the jump relations that describe the behavior of potentials across $\partial\Omega_j$ (recall that the jump operator $[\gamma^j]$ was defined in (1.1)). We have (cf. [19, theorem 6.11], [21, theorem 3.1.2], and [24, theorem 3.3.1])

$$[\gamma^j] \cdot \mathsf{G}_{\kappa_j}^{j}(V) = V \quad \forall V \in \mathbb{H}(\partial\Omega_j). \tag{3.2}$$

We shall say that $V \in H^{1/2}(\partial\Omega_j) \times H^{-1/2}(\partial\Omega_j)$ is a Cauchy datum of Ω_j whenever there exists $u \in H^1_{\mathrm{loc}}(\Delta, \overline{\Omega}_j)$ such that $\Delta u + \kappa_j^2 u = 0$ in Ω_j, with $\mathrm{CI}_{\kappa_j}(u) = 0$ if $j = 0$, and such that $\gamma^j(u) = V$. We set

$$\begin{aligned} \mathcal{C}_{\kappa_j}(\partial\Omega_j) &:= \{\text{Cauchy data of } \Omega_j \text{ for the wavenumber } \kappa_j\}, \\ \mathcal{C}_{\kappa}(\Gamma) &:= \prod_{j=0}^{n} \mathcal{C}_{\kappa_j}(\partial\Omega_j). \end{aligned}$$

The operator $\mathsf{G}_{\kappa_j}^{j}$ provides a convenient characterization of Cauchy data of Ω_j. The following result is once again classical; see [21, theorem 3.1.3] and [24, sec. 3.6].

PROPOSITION 3.2. *For any $j = 0, \dots, n$, the operator $\gamma^j \cdot \mathsf{G}_{\kappa_j}^{j} : \mathbb{H}(\partial\Omega_j) \to \mathbb{H}(\partial\Omega_j)$ is a projector, called the Calderón projector interior to Ω_j, and for any $V \in \mathbb{H}(\partial\Omega_j)$, we have*

$$V \in \mathcal{C}_{\kappa_j}(\partial\Omega_j) \iff \gamma^j \cdot \mathsf{G}_{\kappa_j}^{j}(V) = V.$$

Thus the set of Cauchy data can be characterized by means of Calderón projectors. Let us consider the continuous operator $\mathsf{C}_{\kappa_j}^{j} : \mathbb{H}(\partial\Omega_j) \to \mathbb{H}(\partial\Omega_j)$ defined by

$$\frac{\mathsf{Id}}{2} + \mathsf{C}_{\kappa_j}^{j} \; = \; \gamma^j \cdot \mathsf{G}_{\kappa_j}^{j}. \tag{3.3}$$

A simple consequence of (3.2) is that $\mathsf{C}^j_{\kappa_j} = \{\gamma^j\}\cdot\mathsf{G}^j_{\kappa_j}$. The operator $\mathsf{Id}/2+\mathsf{C}^j_{\kappa_j}$ is a Calderón projector: it satisfies $(\mathsf{Id}/2+\mathsf{C}^j_{\kappa_j})^2 = \mathsf{Id}/2+\mathsf{C}^j_{\kappa_j}$. By proposition 3.2 we see $\mathcal{R}(\mathsf{Id}/2+\mathsf{C}^j_{\kappa_j}) = \mathcal{C}_{\kappa_j}(\partial\Omega_j)$. As a consequence, for any $U \in \mathbb{H}(\Gamma)$, we have

$$U \in \mathcal{C}_\kappa(\Gamma) \iff (\mathsf{Id}/2+\mathsf{C}_\kappa)U = U$$
$$\text{where}\quad \mathsf{C}_\kappa = \begin{bmatrix} \mathsf{C}^0_{\kappa_0} & 0 & \cdots & 0 \\ 0 & \mathsf{C}^1_{\kappa_1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \mathsf{C}^n_{\kappa_n} \end{bmatrix}. \tag{3.4}$$

This shows that since $\mathcal{C}_\kappa(\Gamma) = \mathrm{Ker}(\mathsf{Id}/2-\mathsf{C}_\kappa)$, $\mathcal{C}_\kappa(\Gamma)$ is a closed subspace of $\mathbb{H}(\Gamma)$. In addition, we also have $V \in \mathcal{R}(\mathsf{Id}/2-\mathsf{C}^j_{\kappa_j})$ if and only if there exists $u \in H^1_{\mathrm{loc}}(\mathbb{R}^d\setminus\Omega_j)$ such that $\Delta u + \kappa_j^2 u = 0$ in $\mathbb{R}^d\setminus\overline{\Omega}_j$, $\mathrm{CI}_{\kappa_j}(u) = 0$ except if $j = 0$, and $\gamma^j_c(u) = V$.

4 Classical Single-Trace Formulation of the First Kind

In this section we give a brief review of the formulation that was analyzed by von Petersdorff in [27]. Note, however, that we state this formulation relying on a functional setting introduced by Bendali and his coworkers; see [2, 3]. First of all, set $U^{\mathrm{inc}} = (\mathrm{Q}_0\gamma^0(u_{\mathrm{inc}}), 0, \dots, 0)^\top$ and observe that, according to (2.3), problem (1.3) can be reformulated as

$$\begin{aligned}&\text{Find } U \in \mathbb{X}(\Gamma) \text{ such that}\\ &(\tfrac{\mathsf{Id}}{2}-\mathsf{C}_\kappa)\cdot\mathsf{T}_\mu^{-1}\cdot(U-U^{\mathrm{inc}}) = 0.\end{aligned} \tag{4.1}$$

In this formulation the unknown $U = (U_0,\dots,U_n)$ is related to the total field u by the relation $\mathrm{Q}_j\gamma^j(u) = U_j$ for $j = 0,\dots,n$. Equation (4.1) is well posed, as problem (1.3) is. Let us multiply equation (4.1) on the left by T_μ and set

$$\mathsf{A}_{\kappa,\mu} = \mathsf{T}_\mu\cdot\mathsf{C}_\kappa\cdot\mathsf{T}_\mu^{-1} \quad\text{and}\quad F^{\mathrm{inc}} = \mathsf{T}_\mu(\mathsf{Id}/2-\mathsf{C}_\kappa)\mathsf{T}_\mu^{-1}(U^{\mathrm{inc}}).$$

We encourage our reader to check that, during the analysis of the next sections, the only feature of F^{inc} that we use is that $F^{\mathrm{inc}} \in \mathcal{R}[\mathsf{T}_\mu(\mathsf{Id}/2-\mathsf{C}_\kappa)]$. In other words, our analysis is not restricted to the case where U^{inc} takes the particular form $(\mathrm{Q}_0(\gamma^0(u_{\mathrm{inc}})), 0, \dots, 0)^\top$. Recalling that $\mathrm{B}(U,V) = 0$ whenever both U and V belong to $\mathbb{X}(\Gamma)$, the solution U to problem (4.1) is also a solution to

$$\begin{aligned}&U \in \mathbb{X}(\Gamma) \text{ and}\\ &\mathrm{B}(\mathsf{A}_{\kappa,\mu}(U), V) = -\mathrm{B}(F^{\mathrm{inc}}, V) \quad \forall V \in \mathbb{X}(\Gamma).\end{aligned} \tag{4.2}$$

At first sight though, it is not clear whether a solution to (4.2) is necessarily a solution to (4.1). Actually both equations are equivalent, since formulation (4.2) admits a unique solution as well. For the sake of completeness the proof of the following proposition is given in the appendix.

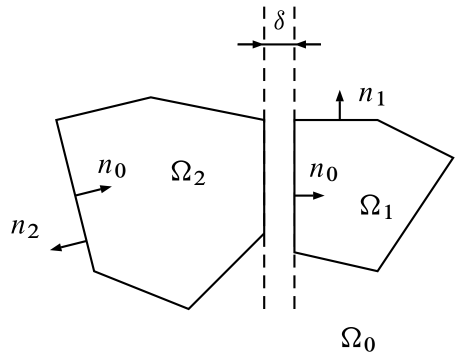


FIGURE 5.1. Simpler geometry of two disjoint scatterers.

PROPOSITION 4.1. *Assume that* $\mu_j > 0$, $j = 0, \dots, n$, *and that* $\kappa_0, \dots, \kappa_n$ *satisfy assumption* (1.4). *For any* $F \in \mathbb{H}(\Gamma)$, *there exists a unique* $U \in \mathbb{X}(\Gamma)$ *such that*

$$\mathrm{B}(\mathsf{A}_{\kappa,\mu}(U), V) = \mathrm{B}(F, V) \quad \forall V \in \mathbb{X}(\Gamma).$$

The operator $\mathsf{A}_{\kappa,\mu}$ admits the same structure as C_κ: it is block diagonal. In addition, $\mathsf{Id}/2 \pm \mathsf{A}_{\kappa,\mu}$ are projectors of the space $\mathbb{H}(\Gamma)$. The elements on the diagonal of $\mathsf{A}_{\kappa,\mu}$ will be denoted $\mathsf{A}^j_{\kappa_j,\mu_j}$, $j = 0, \dots, n$, and are defined by

$$\mathsf{A}^j_{\kappa_j,\mu_j} = \mathrm{Q}_j \cdot \mathsf{C}^j_{\kappa_j} \cdot \mathrm{Q}_j^{-1}.$$

Observe that $\mathsf{Id}/2 \pm \mathsf{A}^j_{\kappa_j,\mu_j} : \mathbb{H}(\partial\Omega_j) \to \mathbb{H}(\partial\Omega_j)$ are projectors since $\mathsf{Id}/2 \pm \mathsf{C}^j_{\kappa_j}$ are projectors as well. In what follows we shall also refer to spaces that are "generalized versions" of the spaces of Cauchy data

$$\begin{aligned} \mathcal{C}_{\kappa_j,\mu_j}(\partial\Omega_j) &= \mathrm{Q}_j(\mathcal{C}_{\kappa_j}(\partial\Omega_j)) = \mathcal{R}(\mathrm{Id}/2 + \mathsf{A}^j_{\kappa_j,\mu_j}), \\ \mathcal{C}_{\kappa,\mu}(\Gamma) &= \prod_{j=0}^{n} \mathcal{C}_{\kappa_j,\mu_j}(\partial\Omega_j) = \mathcal{R}(\mathrm{Id}/2 + \mathsf{A}_{\kappa,\mu}). \end{aligned} \tag{4.3}$$

5 The Gap Idea

In this section we retrace the discovery of our new formulation through a *heuristic* and geometric limit argument. The rigorous mathematical analysis is postponed to later sections. The considerations target a simple situation and can serve as a guide to the general results reported later.

To begin with, we consider two subdomains (that is, $n = 2$) separated by a small "gap" of width $1 \gg \delta > 0$ as illustrated in Figure 5.1. This permits us to orient the interfaces consistently with their induced orientation as boundaries of domains. For the sake of simplicity, in this section we assume that $\mu_0 = \mu_1 = \mu_2 = 1$.

5.1 The Case of Separated Scatterers

Let us examine the single-trace equations from (4.1) and (4.2) for the particular situation of Figure 5.1. Since $\Delta u_{\mathrm{inc}} + \kappa_0^2 u_{\mathrm{inc}} = 0$ in $\Omega_1 \cup \Omega_2$, we have $(\mathsf{Id}/2 + \mathsf{C}^0_{\kappa_0})\gamma^0(u_{\mathrm{inc}}) = 0$, so that $(\mathsf{Id}/2 - \mathsf{C}^0_{\kappa_0})\gamma^0(u_{\mathrm{inc}}) = \gamma^0(u_{\mathrm{inc}})$. As a consequence, equations (4.1) become

$$
\begin{gathered}
(\mathsf{Id}/2 - \mathsf{C}^0_{\kappa_0})\gamma^0(u) = \gamma^0(u_{\mathrm{inc}}), \quad (\mathsf{Id}/2 - \mathsf{C}^1_{\kappa_1})\gamma^1(u) = 0, \\
(\mathsf{Id}/2 - \mathsf{C}^2_{\kappa_2})\gamma^2(u) = 0.
\end{gathered}
\tag{5.1}
$$

In the current situation there are no triple points and all interfaces are complete boundaries of some subdomain. As we have already mentioned above, this present special arrangement of subdomains allows us to consistently orient the interfaces, which implies $n_j = -n_0$ on $\partial\Omega_j$, $j = 1, 2$. Thus, by the definition of the traces, we can recast the transmission conditions as

$$
\gamma^0_{\mathrm{D}}(u)\big|_{\partial\Omega_j} = \gamma^j_{\mathrm{D},c}(u), \quad \gamma^0_{\mathrm{N}}(u)\big|_{\partial\Omega_j} = -\gamma^j_{\mathrm{N},c}(u), \quad j = 1, 2. \tag{5.2}
$$

In addition, potentials can readily be split into contributions from $\partial\Omega_1$ and $\partial\Omega_2$, which yields

$$
-\mathsf{G}^0_{\kappa_0}\gamma^0(u) = \mathsf{G}^1_{\kappa_0}\gamma^1_c(u) + \mathsf{G}^2_{\kappa_0}\gamma^2_c(u)
$$

for the potentials defined in (3.1). By applying the decomposition into interface contributions, the first identity in (5.1) reads

$$
\begin{bmatrix} \gamma^1_c(u) \\ \gamma^2_c(u) \end{bmatrix} + \begin{bmatrix} \gamma^1_c \cdot \mathsf{G}^1_{\kappa_0} & \gamma^1_c \cdot \mathsf{G}^2_{\kappa_0} \\ \gamma^2_c \cdot \mathsf{G}^1_{\kappa_0} & \gamma^2_c \cdot \mathsf{G}^2_{\kappa_0} \end{bmatrix} \begin{bmatrix} \gamma^1_c(u) \\ \gamma^2_c(u) \end{bmatrix} = \begin{bmatrix} \gamma^1(u_{\mathrm{inc}}) \\ \gamma^2(u_{\mathrm{inc}}) \end{bmatrix}. \tag{5.3}
$$

Using Definition (3.3) and the jump relation (3.2), we find $\gamma^j_c \cdot \mathsf{G}^j_{\kappa_0} = -\mathsf{Id}/2 + \mathsf{C}^j_{\kappa_0}$. In addition, let us define the coupling integral operators

$$
\mathsf{R}^{k,l}_{\kappa_0} := \gamma^k \cdot \mathsf{G}^l_{\kappa_0} \quad \text{for } k \neq l. \tag{5.4}
$$

Observe as well that $\mathsf{R}^{k,l}_{\kappa_0} = \gamma^k_c \cdot \mathsf{G}^l_{\kappa_0}$ since $\partial\Omega_k \cap \partial\Omega_l = \varnothing$. With this new notation, we have

$$
\begin{bmatrix}
\mathsf{Id}/2 - \mathsf{C}^1_{\kappa_1} & 0 & 0 & 0 \\
0 & \mathsf{Id}/2 + \mathsf{C}^1_{\kappa_0} & \mathsf{R}^{1,2}_{\kappa_0} & 0 \\
0 & \mathsf{R}^{2,1}_{\kappa_0} & \mathsf{Id}/2 + \mathsf{C}^2_{\kappa_0} & 0 \\
0 & 0 & 0 & \mathsf{Id}/2 - \mathsf{C}^2_{\kappa_2}
\end{bmatrix} \cdot \begin{bmatrix} \gamma^1(u) \\ \gamma^1_c(u) \\ \gamma^2_c(u) \\ \gamma^2(u) \end{bmatrix}
= \begin{bmatrix} 0 \\ \gamma^1(u_{\mathrm{inc}}) \\ \gamma^2(u_{\mathrm{inc}}) \\ 0 \end{bmatrix}. \tag{5.5}
$$

In order to obtain the final form of the first-kind integral equations, we appeal to the transmission conditions $\gamma^1(u) = \gamma_c^1(u)$ and $\gamma^2(u) = \gamma_c^2(u)$. Hence, (5.5) can be condensed into

$$(5.6)\quad \widehat{\mathsf{A}}_\kappa \cdot \begin{bmatrix} \gamma^1(u) \\ \gamma^2(u) \end{bmatrix} := \begin{bmatrix} \mathsf{C}^1_{\kappa_0} + \mathsf{C}^1_{\kappa_1} & \mathsf{R}^{1,2}_{\kappa_0} \\ \mathsf{R}^{2,1}_{\kappa_0} & \mathsf{C}^2_{\kappa_0} + \mathsf{C}^2_{\kappa_2} \end{bmatrix} \cdot \begin{bmatrix} \gamma^1(u) \\ \gamma^2(u) \end{bmatrix} = \begin{bmatrix} \gamma^1(u_{\mathrm{inc}}) \\ \gamma^2(u_{\mathrm{inc}}) \end{bmatrix}.$$

This is merely a standard single-trace formulation written in an unusual way. As such it enjoys coercivity in $\mathbb{X}(\Gamma)$ up to a compact perturbation. However, (5.6) warrants attention, because *if the interfaces can be oriented consistently, then the single-trace operator* $\widehat{\mathsf{A}}_\kappa$ *is its own inverse up to compact perturbation*. To reveal this, we temporarily assume *equal wavenumbers* $\kappa_0 = \kappa_1 = \kappa_2 = \kappa$. Clearly, by Calderón identities we have $(\mathsf{C}^j_\kappa)^2 = \mathsf{Id}/4$, $j = 1, 2$. More interestingly, note that

$$\mathsf{R}^{1,2}_\kappa \mathsf{R}^{2,1}_\kappa = \big(\gamma^1 \mathsf{G}^2_\kappa\big)\big(\gamma^2 \mathsf{G}^1_\kappa\big) = \gamma^1\big(\mathsf{G}^2_\kappa(\gamma^2 \mathsf{G}^1)\big) = \gamma^1(0) = 0,$$

because by Proposition 3.1 G^2_κ vanishes outside Ω_2 when applied to Cauchy data on $\partial\Omega_2$. The same argument together with Proposition 3.2 confirms

$$\mathsf{R}^{2,1}_\kappa \mathsf{C}^1_\kappa = \big(\gamma^2 \mathsf{G}^1_\kappa\big)\big(\gamma^1 \mathsf{G}^1_\kappa - \mathsf{Id}/2\big) = 0 - \gamma^2 \mathsf{G}^1_\kappa/2 = -\mathsf{R}^{2,1}_\kappa/2,$$
$$\mathsf{C}^2_\kappa \mathsf{R}^{2,1}_\kappa = \big(\gamma^2 \mathsf{G}^2_\kappa - \mathsf{Id}/2\big)\big(\gamma^2 \mathsf{G}^1_\kappa\big) = \mathsf{Id}\big(\gamma^2 \mathsf{G}^1_\kappa\big) - \gamma^2 \mathsf{G}^1_\kappa/2 = \mathsf{R}^{2,1}_\kappa/2.$$

An immediate consequence is $(\widehat{\mathsf{A}}_\kappa)^2 = \mathsf{Id}$ for equal wavenumbers! Sloppily speaking, $\widehat{\mathsf{A}}_\kappa$ furnishes a preconditioner for itself, which is a very attractive property for a first-kind boundary integral equation.

5.2 The Case of Adjacent Scatterers

So far in this section, we have assumed that the scatterers are separated. Now consider the same geometry as in Figure 5.1 but with vanishing gap, i.e., $\delta = 0$. The key observation is that even for $\delta = 0$ all the operators in (5.6) still make sense.

This is clear for $\mathsf{C}^1_{\kappa_j}$ and $\mathsf{C}^2_{\kappa_j}$ with $j = 0, 1, 2$, because they comprise only standard boundary integral operators on the respective subdomain boundary. Yet a closer scrutiny reveals that the coupling operators $\mathsf{R}^{1,2}_{\kappa_0}$ and $\mathsf{R}^{2,1}_{\kappa_0}$ also remain meaningful for $\delta = 0$. To see this, note that the restriction of $\mathsf{G}^2_{\kappa_0}(V)$, $V \in \mathbb{H}(\partial\Omega_2)$, to Ω_1 belongs to $H^1_{\mathrm{loc}}(\Delta, \overline{\Omega}_1)$ even if $\partial\Omega_1 \cap \partial\Omega_2 \neq \varnothing$. Conversely, $\mathsf{G}^1_{\kappa_0}(\mathbb{H}(\partial\Omega_1))|_{\Omega_2} \subset H^1_{\mathrm{loc}}(\Delta, \overline{\Omega}_2)$; no problem is encountered when taking traces as required by definition (5.4). As a consequence, the operators $\mathsf{R}^{1,2}_{\kappa_0}$ and $\mathsf{R}^{2,1}_{\kappa_0}$ remain well defined.

Thus, (5.6) is a valid boundary integral formulation for $\delta = 0$, and it represents our *new multi-trace formulation* for the composite scattering problem in the case $n = 2$. Its unknowns will continue to be the formally independent traces $\gamma^1(u)$ and $\gamma^2(u)$. In the case $\delta = 0$, this means that we search a single pair of Dirichlet and Neumann data on the interface $\partial\Omega_0$ outside the "collapsed gap." Where there appears an interface between Ω_1 and Ω_2, we retain two pairs of Dirichlet and Neumann data, a total of four functions, as unknowns.

We expect this new formulation to inherit the intriguing properties of the classical first-kind single-trace formulation addressed above for $\delta > 0$. However, we could not convert the intuitive limit $\delta \to 0$ into a rigorous mathematical argument. Thus in the following sections we embark on a completely different approach to the mathematical analysis of (5.6) and its generalizations. The main idea will be to eliminate all the contributions involving U_0 by means of (2.2) from (4.2). As in (5.6), in our final formulation, the unknowns will be independent traces associated with the different bounded subdomains. Using some remarkable properties of $\mathcal{C}_{\kappa,\mu}(\Gamma)$, we will also prove a multi-subdomain counterpart of the coercivity property and the Calderón identity satisfied by the operator in (5.6).

6 Properties of the Space of Cauchy Data

We return to a general situation with n subdomains that may be adjacent to each other. In this section we would like to point out properties of the space $\mathcal{C}_{\kappa,\mu}(\Gamma)$ that will be very important for the forthcoming analysis. A first important property of the space of Cauchy data is that it yields a complement to $\mathbb{X}(\Gamma)$ in $\mathbb{H}(\Gamma)$.

PROPOSITION 6.1. *Assume that $\mu_j > 0$, $j = 0, \dots, n$, and that $\kappa_0, \dots, \kappa_n$ satisfy assumption* (1.4)*. Then we have the decomposition*

$$\mathbb{H}(\Gamma) = \mathbb{X}(\Gamma) \oplus \mathcal{C}_{\kappa,\mu}(\Gamma). \tag{6.1}$$

PROOF. First of all we have $\mathbb{X}(\Gamma) \cap \mathcal{C}_{\kappa,\mu}(\Gamma) = \{0\}$. Indeed, consider any $U \in \mathbb{X}(\Gamma) \cap \mathcal{C}_{\kappa,\mu}(\Gamma)$. Take $V_j \in \mathcal{C}_{\kappa_j}(\partial\Omega_j)$ such that $\mathsf{Q}_j(V_j) = U_j$ for $j = 0, \dots, n$. Define $u \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^d)$ by $u(\mathbf{x}) = \mathsf{G}^j_{\kappa_j}(V_j)(\mathbf{x})$ for $\mathbf{x} \in \Omega_j$. Then for each j we have $\gamma^j(u) = V_j$ so that u satisfies $\Delta u + \kappa_j^2 u = 0$ in Ω_j, $\mathrm{CI}_{\kappa_0}(u) = 0$, and $(\mathsf{Q}_j \gamma^j(u)) \in \mathbb{X}(\Gamma)$. To sum up u would satisfy problem (1.3) with no incident field. Since problem (1.3) is well posed, $u = 0$ and so $V_j = \gamma^j(u) = 0$, which finally implies $U = 0$.

Now let us prove that $\mathbb{X}(\Gamma) + \mathcal{C}_{\kappa,\mu}(\Gamma) = \mathbb{H}(\Gamma)$. Take any $U \in \mathbb{H}(\Gamma)$. According to Proposition 4.1 there exists a unique $W \in \mathbb{X}(\Gamma)$ such that

$$\mathrm{B}((\mathsf{Id}/2 + \mathsf{A}_{\kappa,\mu})W, V) = \mathrm{B}(U, V) \quad \forall V \in \mathbb{X}(\Gamma).$$

We have $(\mathsf{Id}/2 + \mathsf{A}_{\kappa,\mu})W \in \mathcal{C}_{\kappa,\mu}(\Gamma)$ according to Proposition 3.2 and (4.3). Set $W' = U - (\mathsf{Id}/2 + \mathsf{A}_{\kappa,\mu})W$. By construction we have $\mathrm{B}(W', V) = 0 \; \forall V \in \mathbb{X}(\Gamma)$. Hence $W' \in \mathbb{X}(\Gamma)$ according to (2.2). Since $U = W' + (\mathsf{Id}/2 + \mathsf{A}_{\kappa,\mu})W$, this ends the proof. □

Decomposition (6.1) implies that the space $\mathcal{C}_{\kappa,\mu}(\Gamma)$ is a candidate for representing the elements of the dual to $\mathbb{X}(\Gamma)$ by means of the duality pairing B. We also state another property that is an easy consequence of theorem 3.2 in [27]. However, for the sake of completeness we provide a detailed proof.

LEMMA 6.2. *For any $j = 0, \dots, n$, assume that $\mu_j > 0$ and that $\kappa_j \in \mathbb{C} \setminus \{0\}$ with $\Re\mathrm{e}\{\kappa_j\} \geq 0$, $\Im\mathrm{m}\{\kappa_j\} \geq 0$. For any $U \in \mathbb{H}(\partial\Omega_j)$ we have*

$$U \in \mathcal{C}_{\kappa_j,\mu_j}(\partial\Omega_j) \iff \mathrm{B}_j(U, V) = 0 \quad \forall V \in \mathcal{C}_{\kappa_j,\mu_j}(\partial\Omega_j).$$

PROOF. First of all, since $\mu_j \mathrm{B}_j(\mathrm{Q}_j(U), \mathrm{Q}_j(V)) = \mathrm{B}_j(U, V)$, it suffices to prove the lemma for the case where $\mu_j = 1$. For any $j = 0, \dots, n$, take arbitrary Cauchy data $U, V \in \mathcal{C}_{\kappa_j}(\partial\Omega_j)$. Consider $u, v \in H^1(\Omega_j)$ such that $\Delta u + \kappa_j^2 u = 0$ and $\Delta v + \kappa_j^2 v = 0$ in Ω_j and such that $\gamma^j(u) = U$ and $\gamma^j(v) = V$. In the case where $j \neq 0$, i.e., the case where Ω_j is bounded, we have

$$\begin{aligned} 0 &= \int_{\Omega_j} u\big(\Delta v + \kappa_j^2 v\big) - v\big(\Delta u + \kappa_j^2 u\big) d\mathbf{x} \\ &= \int_{\partial\Omega_j} \gamma_{\mathrm{D}}^j(u)\gamma_{\mathrm{N}}^j(v) - \gamma_{\mathrm{N}}^j(u)\gamma_{\mathrm{D}}^j(v) d\sigma = \mathrm{B}_j(U, V). \end{aligned}$$

The proof is the same for $j = 0$ in the case where $\Im\mathrm{m}\{\kappa_0\} > 0$. As a consequence, let us assume that $j = 0$ and that $\kappa_0 \in (0, +\infty)$. Let B_r refer to the open ball of center 0 and of radius r. Take r large enough to guarantee that $(\mathbb{R}^d \setminus \Omega_0) \subset \mathrm{B}_r$. We have

$$\begin{aligned} 0 &= \int_{\Omega_0 \cap \mathrm{B}_r} u\Delta v - v\Delta u \, d\mathbf{x} = \mathrm{B}_0(U, V) + \int_{\partial \mathrm{B}_r} u \,\partial_r v - v \,\partial_r u \, d\sigma_r, \\ \mathrm{B}_0(U, V) &= \int_{\partial \mathrm{B}_r} v\,(\partial_r u - i\kappa_0 u) - u\,(\partial_r v - i\kappa_0 v) d\sigma_r. \end{aligned}$$

It is well known (see formula (3.8) in chapter 3 of [13], for example) that there exists a constant $C > 0$ independent r such that $\int_{\partial \mathrm{B}_r} |u|^2 \, d\sigma_r \leq C$ for all $r > 0$ whenever u satisfies the Sommerfeld radiation condition for the Helmholtz equation. Since both u and v are assumed to satisfy Sommerfeld's radiation condition, we obtain the existence of $C > 0$ independent of r such that

$$|\mathrm{B}_0(U, V)|^2 \leq C\bigg[\int_{\partial \mathrm{B}_r} |\partial_r u - i\kappa_0 u|^2 \, d\sigma_r + \int_{\partial \mathrm{B}_r} |\partial_r v - i\kappa_0 v|^2 \, d\sigma_r\bigg] \underset{r\to\infty}{\longrightarrow} 0.$$

Now take arbitrarily any $j = 0, \dots, n$ and any $U = (u, p) \in H^{1/2}(\partial\Omega_j) \times H^{-1/2}(\partial\Omega_j)$ satisfying $\mathrm{B}_j(U, V) = 0 \; \forall V \in \mathcal{C}_{\kappa_j}(\partial\Omega_j)$. For any $\mathbf{x} \in \mathbb{R}^d \setminus \Omega_j$, the function $\mathbf{y} \mapsto \mathscr{G}_{\kappa_j}(\mathbf{x} - \mathbf{y})$ is a solution to the Helmholtz in Ω_j with wavenumber

κ_j, and $\mathrm{CI}_{\kappa_0}(\mathscr{G}_{\kappa_0}(\mathbf{x}-\cdot)) = 0$ in the case $j = 0$. As a consequence,

$$0 = -\int_{\partial\Omega_j} n_j(\mathbf{y}) \cdot (\nabla\mathscr{G}_{\kappa_j})(\mathbf{x}-\mathbf{y})\, u(\mathbf{y}) d\sigma - \int_{\partial\Omega_j} \mathscr{G}_{\kappa_j}(\mathbf{x}-\mathbf{y})\, p(\mathbf{y}) d\sigma \quad \forall \mathbf{x} \in \mathbb{R}^d \setminus \overline{\Omega}_j.$$

Taking the exterior Dirichlet and Neumann trace of the above expression on $\partial\Omega_j$, we deduce that $(\mathsf{Id}/2 - \mathsf{C}_{\kappa_j})U = 0$. According to (3.4), this means $U \in \mathcal{C}_{\kappa_j}(\partial\Omega_j)$. □

An obvious consequence of the preceding lemma is that, in the case where $\mu_0, \dots, \mu_n \in (0, +\infty)$ and $\kappa_0, \dots, \kappa_n$ satisfy (1.4), the space of Cauchy data over Γ satisfies a property very similar to (2.2), namely

$$U \in \mathcal{C}_{\kappa,\mu}(\Gamma) \iff \mathrm{B}(U, V) = 0 \quad \forall V \in \mathcal{C}_{\kappa,\mu}(\Gamma). \tag{6.2}$$

Note that Lemma 6.2 and (6.2) would not hold if we had chosen a different duality pairing $\mathrm{B}(\cdot,\cdot)$. Another interesting remark is that a counterpart of Lemma 6.2 also holds in the exterior of each Ω_j. We do not give the proof of the following lemma since it is nearly identical to the proof of Lemma 6.2.

LEMMA 6.3. *For any $j = 0, \dots, n$, assume that $\mu_j > 0$ and that $\kappa_j \in \mathbb{C} \setminus \{0\}$ with $\Re\mathrm{e}\{\kappa_j\} \geq 0$, $\Im\mathrm{m}\{\kappa_j\} \geq 0$. Consider any $j = 0, \dots, n$. For any $U \in \mathbb{H}(\partial\Omega_j)$ we have*

$$U \in \mathcal{R}(\mathsf{Id}/2 - \mathsf{A}^j_{\kappa_j,\mu_j}) \iff \mathrm{B}_j(U, V) = 0 \quad \forall V \in \mathcal{R}(\mathsf{Id}/2 - \mathsf{A}^j_{\kappa_j,\mu_j}).$$

7 New Functional Setting

To proceed further, we need to introduce new multi- and single-trace spaces that could be considered as the restriction to $\mathbb{R}^d \setminus \overline{\Omega}_0$ of the spaces defined in Section 2. Set

$$\begin{aligned} &\widehat{\mathbb{H}}(\Gamma) = \prod_{j=1}^{n} \mathbb{H}(\partial\Omega_j), \\ &\widehat{\mathcal{C}}_0(\Gamma) = \prod_{j=1}^{n} \mathcal{C}_{\kappa_0,\mu_0}(\partial\Omega_j), \quad \text{and} \\ &\widehat{\mathbb{X}}(\Gamma) = \{\widehat{U} \in \widehat{\mathbb{H}}(\Gamma) \mid \exists U_0 \in \mathbb{H}(\partial\Omega_0) \text{ such that } (U_0, \widehat{U}) \in \mathbb{X}(\Gamma)\}. \end{aligned} \tag{7.1}$$

Note that the space $\widehat{\mathbb{H}}(\Gamma)$ differs from $\mathbb{H}(\Gamma)$ as the index j in its definition ranges from 1 to n (not from 0 to n). Moreover, notice that in the definition of

$\widehat{\mathcal{C}}_0(\Gamma)$, all wavenumbers are equal to κ_0, and that only μ_0 is involved (and not μ_j for $j \neq 0$). It is clear from (7.1) and (6.1) considered in the case $\kappa_j = \kappa_0\ \forall j$ that

$$\widehat{\mathbb{X}}(\Gamma) + \widehat{\mathcal{C}}_0(\Gamma) = \widehat{\mathbb{H}}(\Gamma).$$

The sum above is not a direct sum as $\widehat{\mathbb{X}}(\Gamma) \cap \widehat{\mathcal{C}}_0(\Gamma) \neq \{0\}$. We equip the space $\widehat{\mathbb{H}}(\Gamma)$ with a norm denoted $\|\cdot\|$, and a duality pairing analogous to the one considered for $\mathbb{H}(\Gamma)$, setting

$$\|\widehat{U}\| = \Big(\sum_{j=1}^{n} \|u_j\|^2_{H^{1/2}(\partial\Omega_j)} + \|p_j\|^2_{H^{-1/2}(\partial\Omega_j)}\Big)^{1/2}$$

$$\text{for } \widehat{U} = \begin{pmatrix} u_j \\ p_j \end{pmatrix}_{j=1,\dots,n} \in \widehat{\mathbb{H}}(\Gamma).$$

$$\widehat{B}(U, V) = \sum_{j=1}^{n} \mathrm{B}_j(U_j, V_j) \quad \text{for } U, V \in \widehat{\mathbb{H}}(\Gamma).$$

Although $\widehat{\mathbb{X}}(\Gamma)$ may seem "smaller" than $\mathbb{X}(\Gamma)$ at first glance, both spaces are actually isomorphic, as pointed out by the following lemma:

LEMMA 7.1. *For $\widehat{U} \in \widehat{\mathbb{X}}(\Gamma)$, there is a unique $U_0 \in \mathbb{H}(\partial\Omega_0)$ such that $(U_0, \widehat{U}) \in \mathbb{X}(\Gamma)$.*

PROOF. For $\widehat{U} \in \widehat{\mathbb{X}}(\Gamma)$, consider $U_0, V_0 \in \mathbb{H}(\partial\Omega_0)$ such that $(U_0, \widehat{U}) \in \mathbb{X}(\Gamma)$ and $(V_0, \widehat{U}) \in \mathbb{X}(\Gamma)$. As a consequence, $(U_0 - V_0, 0, \dots, 0) \in \mathbb{X}(\Gamma)$. By definition of $\mathbb{X}(\Gamma)$, there exists $u \in H^1_{\mathrm{loc}}(\mathbb{R}^d)$ and $\mathbf{p} \in H_{\mathrm{loc}}(\mathrm{div}, \mathbb{R}^d)$ such that

$$(u|_{\partial\Omega_0}, n_0 \cdot \mathbf{p}|_{\partial\Omega_0})^\top = U_0 - V_0$$

and $u|_{\partial\Omega_j} = 0$, $n_j \cdot \mathbf{p}|_{\partial\Omega_j} = 0$ for $j = 1, \dots, n$.

Let us show that $u|_{\partial\Omega_0} = 0$. Take any $q_0 \in H^{-1/2}(\partial\Omega_0)$. There exists a compactly supported $\mathbf{q} \in H(\mathrm{div}, \mathbb{R}^d)$ such that $n_0 \cdot \mathbf{q} = q_0$. Since $u|_{\partial\Omega_j} = 0$ for $j = 1, \dots, n$, we have

$$\int_{\partial\Omega_0} u q_0 \, d\sigma = \sum_{j=0}^{n} \int_{\partial\Omega_j} u\mathbf{q} \cdot n_j \, d\sigma = \sum_{j=0}^{n} \int_{\Omega_j} \mathbf{q} \cdot \nabla u + u \,\mathrm{div}(\mathbf{q}) d\mathbf{x}$$
$$= \int_{\mathbb{R}^d} \mathbf{q} \cdot \nabla u + u \,\mathrm{div}(\mathbf{q}) d\mathbf{x} = 0.$$

Since q_0 is arbitrary, this implies that $u\big|_{\partial\Omega_0} = 0$. We prove in the same manner that $n_0 \cdot \mathbf{p}\big|_{\partial\Omega_0} = 0$. As a consequence, we finally have $U_0 - V_0 = 0$. □

We will also need a weak characterization of the space $\widehat{\mathbb{X}}(\Gamma)$. Although $\mathbb{X}(\Gamma)$ is its own polar set according to (2.2), such is not the case for $\widehat{\mathbb{X}}(\Gamma)$.

PROPOSITION 7.2. *Let* $\widehat{\mathbb{X}}_0(\Gamma) = \{\widehat{V} \in \widehat{\mathbb{X}}(\Gamma) \mid (0,\widehat{V}) \in \mathbb{X}(\Gamma)\}$. *For any* $\widehat{U} \in \widehat{\mathbb{H}}(\Gamma)$ *we have*

$$\widehat{U} \in \widehat{\mathbb{X}}(\Gamma) \iff \widehat{\mathrm{B}}(\widehat{U},\widehat{V}) = 0 \quad \forall \widehat{V} \in \widehat{\mathbb{X}}_0(\Gamma).$$

PROOF. Assume first that $\widehat{U} \in \widehat{\mathbb{X}}(\Gamma)$, and consider any $\widehat{V} \in \widehat{\mathbb{X}}_0(\Gamma)$. Let $U_0 \in \mathbb{H}(\partial\Omega_0)$ such that $U = (U_0,\widehat{U}) \in \mathbb{X}(\Gamma)$, and set $V = (0,\widehat{V})$ so that $V \in \mathbb{X}(\Gamma)$. Since $U, V \in \mathbb{X}(\Gamma)$ we have $0 = \mathrm{B}(U,V) = \widehat{\mathrm{B}}(\widehat{U},\widehat{V})$.

Now assume that $\widehat{U} \in \widehat{\mathbb{H}}(\Gamma)$ satisfies $\widehat{\mathrm{B}}(\widehat{U},\widehat{V}) = 0$ $\forall \widehat{V} \in \widehat{\mathbb{X}}_0(\Gamma)$. Let $(u_j) \in \sum_{j=1}^n H^{1/2}(\partial\Omega_j)$ and $(p_j) \in \sum_{j=1}^n H^{-1/2}(\partial\Omega_j)$ be such that $\widehat{U} = (u_j,p_j)^\top_{1\le j\le n}$. Let us show that there exists $u_0 \in H^{1/2}(\partial\Omega_0)$ such that $(u_0,u_1,\dots,u_n) \in \mathbb{X}^{1/2}(\Gamma)$. For any $j = 1,\dots,n$, there exists $v_j \in H^1(\Omega_j)$ such that $\gamma_{\mathrm{D}}^j(v_j) = u_j$. Define $v \in \mathrm{L}^2(\mathbb{R}^d \setminus \overline{\Omega}_0)$ by $v|_{\Omega_j} = v_j$ and $\mathbf{q} \in \mathrm{L}^2(\mathbb{R}^d \setminus \overline{\Omega}_0)$ such that $\mathbf{q}|_{\Omega_j} = \nabla v_j$, $j = 1,\dots,n$. Take any $\mathbf{s} \in H(\mathrm{div},\mathbb{R}^d \setminus \overline{\Omega}_0)$ such that $n_0\cdot\mathbf{s}|_{\partial\Omega_0} = 0$, and observe that for such a vector field we have $\widehat{V} = (0,s_j)^\top_{1\le j\le n} \in \widehat{\mathbb{X}}_0(\Gamma)$ where $s_j = n_j\cdot\mathbf{s}|_{\partial\Omega_j}$. As a consequence,

$$\begin{aligned}
\int_{\mathbb{R}^d\setminus\overline{\Omega}_0} v\,\mathrm{div}(\mathbf{s})d\mathbf{x} &= \sum_{j=1}^n \int_{\Omega_j} v_j\,\mathrm{div}(\mathbf{s})d\mathbf{x}\\
&= \sum_{j=1}^n \int_{\partial\Omega_j} u_j\,s_j\,d\sigma - \sum_{j=1}^n \int_{\Omega_j} \mathbf{s}\cdot\nabla v_j\,d\mathbf{x} \qquad (7.2)\\
&= -\int_{\mathbb{R}^d\setminus\overline{\Omega}_0} \mathbf{q}\cdot\mathbf{s}\,d\mathbf{x} + \widehat{\mathrm{B}}(\widehat{U},\widehat{V}) = -\int_{\mathbb{R}^d\setminus\overline{\Omega}_0} \mathbf{q}\cdot\mathbf{s}\,d\mathbf{x}.
\end{aligned}$$

We used the fact $\widehat{\mathrm{B}}(\widehat{U},\widehat{V}) = 0$ since $\widehat{V} \in \widehat{\mathbb{X}}_0(\Gamma)$. Since (7.2) holds for any $\mathbf{s} \in H(\mathrm{div},\mathbb{R}^d \setminus \overline{\Omega}_0)$ such that $n_0\cdot\mathbf{s}|_{\partial\Omega_0} = 0$, this shows that $v \in H^1(\mathbb{R}^d \setminus \overline{\Omega}_0)$. Extending v to Ω_0 properly, we may consider that $v \in H^1(\mathbb{R}^d)$. Set $u_0 = \gamma_{\mathrm{D}}^0(v)$. With such a choice, we have $(u_0,\dots,u_n) \in \mathbb{X}^{1/2}(\Gamma)$, which is what we wanted to show. We prove in the same manner, mutatis mutandis, that there exists $p_0 \in H^{-1/2}(\partial\Omega_0)$ such that $(p_0,\dots,p_n) \in \mathbb{X}^{-1/2}(\Gamma)$. To conclude, set $U_0 = (u_0,p_0)^\top$. The preceding construction shows that $(U_0,\widehat{U}) \in \mathbb{X}(\Gamma)$. □

8 Transformation of the Classical Single-Trace Formulation

The formulation that we are going to derive will be obtained by reshaping (4.2). The main idea consists in trying to eliminate all the integrals on $\partial\Omega_0$ in formulation (4.2) by taking explicitly into account that both U and V belong to $\mathbb{X}(\Gamma)$ and by using the following simple lemma:

LEMMA 8.1. $\sum_{j=0}^n \mathsf{G}_{\kappa_0}^j(U_j)(\mathbf{x}) = 0$ $\forall \mathbf{x} \in \Omega_0$, $\forall U = (U_j)_{0\le j\le n} \in \mathbb{X}(\Gamma)$.

PROOF. Pick an arbitrary $\mathbf{x} \in \mathbb{R}^d \setminus \Gamma$, and consider a C^∞ cutoff function $\chi : \mathbb{R}^d \to \mathbb{R}_+$ such that $\chi = 0$ over a neighborhood of $\mathbf{x}$, and $\chi = 1$ over a neighborhood of Γ. Let $v_{\mathbf{x}} \in C^\infty(\mathbb{R}^d)$ be defined by $v_{\mathbf{x}}(\mathbf{y}) = \chi(\mathbf{y})\mathscr{G}_{\kappa_0}(\mathbf{x}-\mathbf{y})$. Let $V = (\gamma^0(v_{\mathbf{x}}), \dots, \gamma^n(v_{\mathbf{x}})) \in \mathbb{X}(\Gamma)$. Since $v_{\mathbf{x}}(\mathbf{y}) = \mathscr{G}_{\kappa_0}(\mathbf{x}-\mathbf{y})$ for any $\mathbf{y}$ chosen sufficiently close to Γ, we have $\sum_{j=0}^n \mathsf{G}_{\kappa_0}^j(U_j)(\mathbf{x}) = \mathrm{B}(U, V) = 0\ \forall U \in \mathbb{X}(\Gamma)$. □

Now we apply Lemma 8.1 considering $\mathsf{T}_0^{-1}(U)$ with $U \in \mathbb{X}(\Gamma)$ (instead of just considering U). This is justified since $\mathsf{T}_0^{-1}(U) \in \mathbb{X}(\Gamma)$ whenever $U \in \mathbb{X}(\Gamma)$ according to (2.4). Hence

$$\sum_{j=0}^n \mathsf{G}_{\kappa_0}^j(\mathrm{Q}_0^{-1}(U_j))(\mathbf{x}) = 0 \quad \forall \mathbf{x} \in \Omega_0,\ \forall U = (U_j)_{0\le j\le n} \in \mathbb{X}(\Gamma).$$

Take the interior traces on $\partial\Omega_0$ of the preceding expression, multiply it on the left by Q_0, and test against any trace function belonging to $\mathbb{H}(\partial\Omega_0)$. This yields

$$\begin{aligned} &\mathrm{B}_0\big(\mathrm{Q}_0 \cdot \gamma^0 \cdot \mathsf{G}_{\kappa_0}^0\big(\mathrm{Q}_0^{-1}(U_0)\big), V_0\big) \\ &\quad = \mathrm{B}_0\big((\mathsf{Id}/2 + \mathsf{A}_{\kappa_0,\mu_0}^0)U_0, V_0\big) \\ &\quad = -\sum_{j=1}^n \mathrm{B}_0\big(\mathrm{Q}_0 \cdot \gamma^0 \cdot \mathsf{G}_{\kappa_0}^j\big(\mathrm{Q}_0^{-1}(U_j)\big), V_0\big) \\ &\qquad \forall U = (U_j) \in \mathbb{X}(\Gamma),\ \forall V_0 \in \mathbb{H}(\partial\Omega_0). \end{aligned} \tag{8.1}$$

Let us examine each term of the sum in the right-hand side above. Take an arbitrary $j = 1, \dots, n$ and define $W = (W_k) \in \mathbb{H}(\Gamma)$ by $W_k = \mathrm{Q}_0 \cdot \gamma^k \cdot \mathsf{G}_{\kappa_0}^j(\mathrm{Q}_0^{-1}(U_j))$ if $k \neq j$ and $W_j = \mathrm{Q}_0 \cdot \gamma_c^j \cdot \mathsf{G}_{\kappa_0}^j(\mathrm{Q}_0^{-1}(U_j))$. Since $\mathsf{G}_{\kappa_0}^j(\mathrm{Q}_0^{-1}(U_j)) \in H^1_{\mathrm{loc}}(\Delta, \mathbb{R}^d \setminus \Omega_j)$, we conclude that $\mathsf{T}_0^{-1}(W) \in \mathbb{X}(\Gamma)$ and therefore $W \in \mathbb{X}(\Gamma)$ according to (2.4). As a consequence, we have

$$\begin{aligned} &\mathrm{B}_0\big(\mathrm{Q}_0 \cdot \gamma^0 \cdot \mathsf{G}_{\kappa_0}^j\big(\mathrm{Q}_0^{-1}(U_j)\big), V_0\big) \\ &\quad = -\sum_{q\neq 0,j} \mathrm{B}_q\big(\mathrm{Q}_0 \cdot \gamma^q \cdot \mathsf{G}_{\kappa_0}^j\big(\mathrm{Q}_0^{-1}(U_j)\big), V_q\big) \\ &\qquad - \mathrm{B}_j\big(\mathrm{Q}_0 \cdot \gamma_c^j \cdot \mathsf{G}_{\kappa_0}^j\big(\mathrm{Q}_0^{-1}(U_j)\big), V_j\big) \quad \forall V = (V_j) \in \mathbb{X}(\Gamma). \end{aligned} \tag{8.2}$$

In the expression above, $\sum_{q\neq 0,j}$ means that we sum over $q = 1, \dots, n$ and $q \neq j$. Now we plug (8.2) into (8.1) and use it to rewrite formulation (4.2). Whenever both

U and V belong to $\mathbb{X}(\Gamma)$, since $\mathrm{B}(U,V)=0$, we have

$$\begin{aligned}
&\mathrm{B}(\mathsf{A}_{\kappa,\mu}U,V)\\
&\quad=\sum_{j=0}^{n}\mathrm{B}_j\big((\mathsf{Id}/2+\mathsf{A}^j_{\kappa_j,\mu_j})U_j,V_j\big)\\
&\quad=\sum_{j=1}^{n}\mathrm{B}_j\big((+\mathsf{Id}/2+\mathsf{A}^j_{\kappa_j,\mu_j})U_j,V_j\big)\\
&\qquad+\sum_{j=1}^{n}\mathrm{B}_j\big((-\mathsf{Id}/2+\mathsf{A}^j_{\kappa_0,\mu_0})U_j,V_j\big)\\
&\qquad+\sum_{j=1}^{n}\sum_{q\neq 0,j}\mathrm{B}_q\big(\mathsf{Q}_0\cdot\gamma^q\cdot\mathsf{G}^j_{\kappa_0}\big(\mathsf{Q}_0^{-1}(U_j)\big),V_q\big)\quad\forall U,V\in\mathbb{X}(\Gamma).
\end{aligned}\tag{8.3}$$

In the calculus above, we used that $\mathsf{Q}_0\cdot\gamma_c^j\cdot\mathsf{G}^j_{\kappa_0}(\mathsf{Q}_0^{-1}(U_j))=(-\mathsf{Id}/2+\mathsf{A}^j_{\kappa_0,\mu_0})U_j$. In the right-hand side of the last identity above, there is no integral over Ω_0 coming into play any longer.

8.1 Transformation of the Right-Hand Side

Let us handle the right-hand side of (4.2) in the same manner as in (8.3). Recall that $F^{\mathrm{inc}}=(F_0^{\mathrm{inc}},F_1^{\mathrm{inc}},\dots,F_n^{\mathrm{inc}})\in\mathcal{R}(\mathsf{Id}/2-\mathsf{A}_{\kappa,\mu})$. In particular, $F_0^{\mathrm{inc}}\in\mathcal{R}(\mathsf{Id}/2-\mathsf{A}^0_{\kappa_0,\mu_0})$ so that $F_0^{\mathrm{inc}}=-\mathsf{Q}_0\cdot\gamma_c^0\cdot\mathsf{G}^0_{\kappa_0}(\mathsf{Q}_0^{-1}(F_0^{\mathrm{inc}}))$. Set

$$\widetilde{F}_j^{\mathrm{inc}}=-\mathsf{Q}_0\cdot\gamma^j\cdot\mathsf{G}^0_{\kappa_0}\big(\mathsf{Q}_0^{-1}(F_0^{\mathrm{inc}})\big)\quad \textit{and}\quad \widetilde{F}^{\mathrm{inc}}=(\widetilde{F}_1^{\mathrm{inc}},\dots,\widetilde{F}_n^{\mathrm{inc}});$$

then $(F_0^{\mathrm{inc}},\widetilde{F}^{\mathrm{inc}})\in\mathbb{X}(\Gamma)$. As a consequence of (2.2), we can write

$$\begin{aligned}
&-\mathrm{B}(F^{\mathrm{inc}},V)\\
&\quad=-\mathrm{B}_0(F_0^{\mathrm{inc}},V_0)-\sum_{j=1}^{n}\mathrm{B}_j(F_j^{\mathrm{inc}},V_j)\\
&\quad=\sum_{j=1}^{n}\mathrm{B}_j(\widetilde{F}_j^{\mathrm{inc}}-F_j^{\mathrm{inc}},V_j)=\widehat{B}(\widehat{F}^{\mathrm{inc}},\widehat{V})\quad\forall V=(V_0,\widehat{V})\in\mathbb{X}(\Gamma).
\end{aligned}\tag{8.4}$$

$$\text{with } \widehat{F}^{\mathrm{inc}}\stackrel{\mathrm{def}}{=}(\widetilde{F}_1^{\mathrm{inc}}-F_1^{\mathrm{inc}},\dots,\widetilde{F}_n^{\mathrm{inc}}-F_n^{\mathrm{inc}}).$$

Before going further in the analysis we would like to point out the relation between F^{inc} and $\widehat{F}^{\mathrm{inc}}$: we have

$$\begin{aligned}
&\widehat{F}^{\mathrm{inc}}=-\big(F_j^{\mathrm{inc}}+\mathsf{Q}_0\cdot\gamma^j\cdot\mathsf{G}^0_{\kappa_0}\big(\mathsf{Q}_0^{-1}(F_0^{\mathrm{inc}})\big)\big)_{1\le j\le n}\\
&\text{where } F^{\mathrm{inc}}=(F_j^{\mathrm{inc}})_{0\le j\le n}\in\mathcal{R}(\mathsf{T}_\mu(\mathsf{Id}/2-\mathsf{C}_\kappa)).
\end{aligned}\tag{8.5}$$

8.2 Final Form of the Equation

Using the notation introduced in (8.4) as well as identity (8.3), we can rewrite the formulation presented in Section 4 in a new manner. Indeed, we have just proved that formulation (4.2) is equivalent to

$$\begin{aligned}&\text{Find } \widehat{U} \in \widehat{\mathbb{X}}(\Gamma) \text{ such that}\\ &\widehat{\mathsf{B}}(\widehat{\mathsf{A}}_{\kappa,\mu}(\widehat{U}), \widehat{V}) = \widehat{\mathsf{B}}(\widehat{F}^{\mathrm{inc}}, \widehat{V})\ \forall \widehat{V} \in \widehat{\mathbb{X}}(\Gamma),\end{aligned} \tag{8.6}$$

where the operator $\widehat{\mathsf{A}}_{\kappa,\mu}$ is a continuous operator mapping $\widehat{\mathbb{H}}(\Gamma)$ into $\widehat{\mathbb{H}}(\Gamma)$ and defined in accordance with (8.3) by

$$\widehat{\mathsf{A}}_{\kappa,\mu}(U) = \begin{bmatrix} \mathsf{A}^1_{\kappa_1,\mu_1} + \mathsf{A}^1_{\kappa_0,\mu_0} & \mathsf{R}^{1,2} & \cdots & \cdots & \mathsf{R}^{1,n} \\ \mathsf{R}^{2,1} & \mathsf{A}^2_{\kappa_2,\mu_2} + \mathsf{A}^2_{\kappa_0,\mu_0} & \cdots & \cdots & \mathsf{R}^{2,n} \\ \mathsf{R}^{3,1} & \mathsf{R}^{3,2} & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \mathsf{R}^{n-1,n} \\ \mathsf{R}^{n,1} & \mathsf{R}^{n,2} & \cdots & \cdots & \mathsf{A}^n_{\kappa_n,\mu_n} + \mathsf{A}^n_{\kappa_0,\mu_0} \end{bmatrix} \times \begin{bmatrix} U_1 \\ U_2 \\ \vdots \\ U_{n-1} \\ U_n \end{bmatrix}, \tag{8.7}$$

where we have set

$$\mathsf{R}^{q,j} := \mathrm{Q}_0 \cdot \gamma^q \cdot \mathsf{G}^j_{\kappa_0} \cdot \mathrm{Q}_0^{-1}. \tag{8.8}$$

Clearly, the operators $\mathsf{R}^{q,j}$ are continuous maps from $\mathbb{H}(\partial\Omega_j)$ into $\mathbb{H}(\partial\Omega_q)$. Note that, as a consequence of Proposition 3.1, we have $\mathsf{R}^{q,j} \cdot (\mathsf{Id}/2 + \mathsf{A}^j_{\kappa_0,\mu_0}) = 0$ and $(\mathsf{Id}/2 - \mathsf{A}^q_{\kappa_0,\mu_0}) \cdot \mathsf{R}^{q,j} = 0$ as well as $\mathsf{R}^{q,j} \cdot \mathsf{R}^{j,p} = 0$ whenever $q \neq j$. The previous derivation, as well as Lemma 7.1, leads to the following conclusion.

PROPOSITION 8.2. *Assume that F^{inc} and $\widehat{F}^{\mathrm{inc}}$ satisfy equation* (8.5). *Let $U = (U_0, \widehat{U}) \in \mathbb{X}(\Gamma)$ with $\widehat{U} \in \widehat{\mathbb{X}}(\Gamma)$. Then $\widehat{U}$ is a solution to* (8.6) *if U is a solution to* (4.2). *Reciprocally, if $\widehat{U}$ is a solution to* (8.6)*, then there exists $U_0 \in \mathbb{H}(\partial\Omega_0)$ such that $U = (U_0, \widehat{U}) \in \mathbb{X}(\Gamma)$ is a solution to* (4.2).

9 Decoupling of Traces

Formulations (4.2) or (8.6) are interesting because they are set on a single-trace space: somehow they involve as few unknowns as possible. Calderón preconditioning for such formulations would be highly desirable. Unfortunately. it is not clear how to apply this technique in a functional setting such as $\mathbb{X}(\Gamma)$ or $\widehat{\mathbb{X}}(\Gamma)$. This is our motivation for considering a formulation similar to (8.6) but with $\widehat{\mathbb{X}}(\Gamma)$ replaced by $\widehat{\mathbb{H}}(\Gamma)$: this will offer more flexibility, and the functional setting will be more prone to preconditioning.

THEOREM 9.1. *Assume that F^{inc} and $\widehat{F}^{\mathrm{inc}}$ satisfy equation* (8.5). *If $\widehat{U} \in \widehat{\mathbb{X}}(\Gamma)$ is a solution to* (8.6)*, then it satisfies the following multi-trace formulation of the first kind*:

$$\text{(9.1)}\qquad \begin{aligned} &\text{Find } \widehat{U} \in \widehat{\mathbb{H}}(\Gamma) \text{ such that} \\ &\widehat{\mathsf{B}}(\widehat{\mathsf{A}}_{\kappa,\mu}(\widehat{U}), \widehat{V}) = \widehat{\mathsf{B}}(\widehat{F}^{\mathrm{inc}}, \widehat{V}) \quad \forall \widehat{V} \in \widehat{\mathbb{H}}(\Gamma). \end{aligned}$$

PROOF. Assume that $\widehat{U} \in \widehat{\mathbb{X}}(\Gamma)$ is a solution to (8.6). Since $\widehat{\mathbb{H}}(\Gamma) = \widehat{\mathbb{X}}(\Gamma) + \widehat{\mathcal{C}}_0(\Gamma)$, in order to prove that U is a solution to (9.1), it is sufficient to prove that $\widehat{\mathsf{B}}(\widehat{\mathsf{A}}_{\kappa,\mu}(\widehat{U}), V) = \widehat{\mathsf{B}}(\widehat{F}_{\mathrm{inc}}, V)$ for any $V \in \widehat{\mathcal{C}}_0(\Gamma)$. Pick an arbitrary $\widehat{V} = (V_1, \ldots, V_n)$ in $\widehat{\mathcal{C}}_0(\Gamma)$, and observe first that $\mathsf{B}_j(\mathsf{R}^{j,q}(U_q), V_j) = 0$ for $j \neq q$ according to Lemma 6.2. As a consequence,

$$\text{(9.2)}\qquad \begin{aligned} &\widehat{\mathsf{B}}\big(\widehat{\mathsf{A}}_{\kappa,\mu}(\widehat{U}), \widehat{V}\big) \\ &\quad = \sum_{j=1}^{n} \mathsf{B}_j\big((\mathsf{A}^j_{\kappa_j,\mu_j} + \mathsf{A}^j_{\kappa_0,\mu_0})U_j, V_j\big) \\ &\quad = \sum_{j=1}^{n} \mathsf{B}_j\bigg(\bigg(\frac{\mathsf{Id}}{2} + \mathsf{A}^j_{\kappa_0,\mu_0}\bigg)U_j, V_j\bigg) - \sum_{j=1}^{n} \mathsf{B}_j\bigg(\bigg(\frac{\mathsf{Id}}{2} - \mathsf{A}^j_{\kappa_j,\mu_j}\bigg)U_j, V_j\bigg) \\ &\quad = -\sum_{j=1}^{n} \mathsf{B}_j\bigg(\bigg(\frac{\mathsf{Id}}{2} - \mathsf{A}^j_{\kappa_j,\mu_j}\bigg)U_j, V_j\bigg). \end{aligned}$$

For the calculation above, we used Lemma 6.2 and the fact that

$$V_j \in \mathcal{C}_{\kappa_0,\mu_0}(\partial\Omega_j) = \mathcal{R}(\mathsf{Id}/2 + \mathsf{A}^j_{\kappa_0,\mu_0})$$

to deduce that $\mathsf{B}_j((\mathsf{Id}/2 + \mathsf{A}^j_{\kappa_0,\mu_0})U_j, V_j) = 0$ for all $j = 1, \ldots, n$.

Consider $U_0 \in \mathbb{H}(\partial\Omega_0)$ such that $U = (U_0, \widehat{U}) \in \mathbb{X}(\Gamma)$. According to Proposition 8.2, U is a solution to (4.2), which implies, according to equation (4.1) and Proposition 4.1, that $(\mathsf{Id}/2 - \mathsf{A}^j_{\kappa_j,\mu_j})U_j = F^{\mathrm{inc}}$, $j = 1, \ldots, n$; hence

$$\widehat{\mathsf{B}}\big(\widehat{\mathsf{A}}_{\kappa,\mu}\widehat{U}, \widehat{V}\big) = -\sum_{j=1}^{n} \mathsf{B}_j\big(F_j^{\mathrm{inc}}, V_j\big) = \sum_{j=1}^{n} \mathsf{B}_j\big(\widetilde{F}_j^{\mathrm{inc}} - F_j^{\mathrm{inc}}, V_j\big) = \widehat{\mathsf{B}}(\widehat{F}^{\mathrm{inc}}, \widehat{V}).$$

For the second equality above, we used Lemma 6.2 and the fact that $\widetilde{F}_j^{\mathrm{inc}} \in \mathcal{C}_{\kappa_0,\mu_0}(\partial\Omega_j)$. Since the calculation above holds for any $\widehat{V} \in \widehat{\mathcal{C}}_0(\Gamma)$, this proves that $\widehat{U}$ is a solution to (9.1). □

THEOREM 9.2. *Assume that F^{inc} and $\widehat{F}^{\mathrm{inc}}$ satisfy equation* (8.5). *If $\widehat{U} \in \widehat{\mathbb{H}}(\Gamma)$ is a solution to* (9.1)*, then $\widehat{U}$ belongs to $\widehat{\mathbb{X}}(\Gamma)$ and it is a solution to* (8.6).

PROOF. Assume that $\widehat{U} = (U_1, \ldots, U_n) \in \widehat{\mathbb{H}}(\Gamma)$ is a solution to (9.1). We have to show that $\widehat{U} \in \widehat{\mathbb{X}}(\Gamma)$ necessarily. As a preliminary step, we prove that

$(\mathsf{Id}/2 - \mathsf{A}^j_{\kappa_j,\mu_j})U_j = F^{\mathrm{inc}}_j \ \forall j = 1,\dots,n$. Choose any $\widehat{V} \in \widehat{\mathcal{C}}_0(\Gamma)$. Setting $\widetilde{F}^{\mathrm{inc}}_j = -\mathsf{Q}_0 \cdot \gamma^j \cdot \mathsf{G}^0_{\kappa_0}(\mathsf{Q}_0^{-1}(F^{\mathrm{inc}}_0))$ so that $\widetilde{F}^{\mathrm{inc}}_j \in \mathcal{C}_{\kappa_0,\mu_0}(\partial\Omega_j)$ for $j = 1,\dots,n$, and since $\mathcal{R}(\mathsf{R}^{j,q}) \subset \mathcal{C}_{\kappa_0,\mu_0}(\partial\Omega_j)$, we have $\mathrm{B}_j(\mathsf{R}^{j,q}(U_q), V_j) = 0$ for $q \neq j$ and $\mathrm{B}_j(\widetilde{F}^{\mathrm{inc}}_j, V_j) = 0$ according to Lemma 6.2. As a consequence,

$$\widehat{\mathrm{B}}\big(\widehat{\mathsf{A}}_{\kappa,\mu}\widehat{U}, \widehat{V}\big) = -\sum_{j=1}^{n} \mathrm{B}_j\big((\mathsf{Id}/2 - \mathsf{A}^j_{\kappa_j,\mu_j})U_j, V_j\big),$$
$$\widehat{\mathrm{B}}\big(\widehat{F}^{\mathrm{inc}}\widehat{U}, \widehat{V}\big) = -\sum_{j=1}^{n} \mathrm{B}_j\big(F^{\mathrm{inc}}_j, V_j\big).$$

For the first equality above, we applied the same calculation as in (9.2). Since $\widehat{V}$ is arbitrarily chosen in $\widehat{\mathcal{C}}_0(\Gamma)$, applying (9.1) yields

$$\mathrm{B}_j\big((\mathsf{Id}/2 - \mathsf{A}^j_{\kappa_j,\mu_j})U_j, V_j\big) = \mathrm{B}_j(F^{\mathrm{inc}}_j, V_j) \quad \forall V_j \in \mathcal{C}_{\kappa_0,\mu_0}(\partial\Omega_j),\ \forall j = 1,\dots,n.$$

According to Lemma 6.3 and Lemma A.2 in the Appendix, since $F^{\mathrm{inc}}_j \in \mathcal{R}(\mathsf{Id}/2 - \mathsf{A}^j_{\kappa_j,\mu_j})$, this implies $(\mathsf{Id}/2 - \mathsf{A}^j_{\kappa_j,\mu_j})U_j = F^{\mathrm{inc}}_j$ for all $j = 1,\dots,n$.

Now let us take $\widehat{V} = (V_1,\dots,V_n) \in \widehat{\mathbb{X}}_0(\Gamma)$ arbitrarily, where $\widehat{\mathbb{X}}_0(\Gamma)$ was defined in Proposition 7.2. Since $V = (0, V_1,\dots,V_n) \in \mathbb{X}(\Gamma)$, equation (8.2) can be applied, which yields

$$\mathrm{B}_j\big(\mathsf{Q}_0 \cdot \gamma^j_c \cdot \mathsf{G}^j_{\kappa_0}\big(\mathsf{Q}_0^{-1}(U_j)\big), V_j\big) + \sum_{q\neq 0,j} \mathrm{B}_q(\mathsf{Q}_0 \cdot \gamma^q \cdot \mathsf{G}^j_{\kappa_0}\big(\mathsf{Q}_0^{-1}(U_j)\big), V_q) = 0 \quad \forall j = 1,\dots,n.$$

This can be rewritten as

$$\mathrm{B}_j\big((-\mathsf{Id}/2 + \mathsf{A}^j_{\kappa_0,\mu_0})U_j, V_j\big) + \sum_{q\neq 0,j} \mathrm{B}_q(\mathsf{R}^{q,j}U_j, V_q) = 0 \quad \forall j = 1,\dots,n.$$

Taking into account this identity, and the definitions of $\widehat{\mathsf{A}}_{\kappa,\mu}$ and of the Calderón projectors, for any $\widehat{V} \in \widehat{\mathbb{X}}_0(\Gamma)$ we have

$$\begin{aligned}\widehat{\mathrm{B}}(\widehat{\mathsf{A}}_{\kappa,\mu}\widehat{U}, \widehat{V}) &= \sum_{j=1}^{n} \mathrm{B}_j\big((\mathsf{Id}/2 + \mathsf{A}^j_{\kappa_j,\mu_j})U_j, V_j\big) \\ &= \sum_{j=1}^{n} \mathrm{B}_j(U_j, V_j) - \sum_{j=1}^{n} \mathrm{B}_j\big(F^{\mathrm{inc}}_j, V_j\big).\end{aligned} \tag{9.3}$$

In addition, $(F^{\mathrm{inc}}_0, \widetilde{F}^{\mathrm{inc}}_1, \dots \widetilde{F}^{\mathrm{inc}}_n) \in \mathbb{X}(\Gamma)$, so

$$\sum_{j=1}^{n} \mathrm{B}_j(\widetilde{F}^{\mathrm{inc}}_j, V_j) = 0 \quad \forall (V_1,\dots,V_n) \in \widehat{\mathbb{X}}_0(\Gamma).$$

As a consequence,

$$(9.4)\qquad \widehat{B}(\widehat{F}^{\mathrm{inc}},\widehat{V}) = -\sum_{j=1}^{n} \mathrm{B}_j(F_j^{\mathrm{inc}},V_j) \quad \forall \widehat{V}=(V_1,\dots,V_n)\in\widehat{\mathbb{X}}_0(\Gamma).$$

Gathering (9.3) and (9.4), and taking into account that $\widehat{U}$ is solution (9.1), we obtain that $\widehat{B}(\widehat{U},\widehat{V}) = 0\ \forall \widehat{V}\in\widehat{\mathbb{X}}_0(\Gamma)$. Thus it is a consequence of Proposition 7.2 that $\widehat{U}\in\widehat{\mathbb{X}}(\Gamma)$. This concludes the proof. □

COROLLARY 9.3. *Assume that* $\widehat{F}^{\mathrm{inc}}$ *and* F^{inc} *satisfy equation* (8.5). *Then formulation* (9.1) *admits a unique solution that coincides with the unique solution to* (4.1).

Note that this result of well-posedness relies on a strong assumption regarding the right-hand side $\widehat{F}^{\mathrm{inc}}$. In the next section, we show how to discard this restriction.

10 Coercivity

In this section we prove that the operator of formulation (9.1) satisfies a coercivity property similar to (3.22) in [5]. Before stating such a result, we define the operators $\Theta_j : \mathbb{H}(\partial\Omega_j)\to\mathbb{H}(\partial\Omega_j)$ and $\Theta : \widehat{\mathbb{H}}(\Gamma)\to\widehat{\mathbb{H}}(\Gamma)$ by

$$(10.1)\qquad \Theta_j\left(\begin{bmatrix} u_j \\ p_j \end{bmatrix}\right) = \begin{bmatrix} -\overline{u}_j \\ +\overline{p}_j \end{bmatrix} \quad\text{and}\quad \Theta(U_1,\dots,U_n) = (\Theta_1(U_1),\dots,\Theta_n(U_n)).$$

In order to prove a coercivity result for the operator $\widehat{\mathsf{A}}_\kappa$, we need two technical results. Let us first recall formula (17) of [27]. We state this result in a form adapted to our analysis.

PROPOSITION 10.1. *Assume that* $\kappa_* = \imath$, *and consider any* $\mu_* > 0$. *For any* $j = 1,\dots,n$, *take arbitrarily* $U_j\in\mathbb{H}(\partial\Omega_j)$ *and set*

$$\psi_j(\mathbf{x}) = \mathsf{G}^j_{\kappa_*}(\mathrm{Q}_*^{-1}(U_j))(\mathbf{x}) \quad \textit{with} \quad \mathrm{Q}_* = \begin{bmatrix} 1 & 0 \\ 0 & 1/\mu_* \end{bmatrix}.$$

Then we have

$$\mathrm{B}_j\big(\mathsf{A}^j_{\kappa_*,\mu_*}(U_j),\Theta_j(U_j)\big) = \frac{1}{\mu_*}\sum_{q=0}^{n}\int_{\Omega_q} |\nabla\psi_j|^2 + |\psi_j|^2\, d\mathbf{x}.$$

Here and in the following, "$\imath$" will refer to $\exp(i\pi/2)$ (we use this notation to avoid confusion with any index). Now we establish an extension of the preceding result that will allow us to deal with extradiagonal terms in the expression of $\widehat{\mathsf{A}}_\kappa$.

PROPOSITION 10.2. *Assume that $\kappa_* = \imath$ and consider any $\mu_* > 0$. For any $j,k \in \{1,\dots,n\}$, take arbitrarily $U_k \in \mathbb{H}(\partial\Omega_k)$ and $U_j \in \mathbb{H}(\partial\Omega_j)$. Let $\psi_k(\mathbf{x}) = \mathsf{G}^k_{\kappa_*}\big(\mathsf{Q}_*^{-1}(U_k)\big)(\mathbf{x})$ and $\psi_j(\mathbf{x}) = \mathsf{G}^j_{\kappa_*}\big(\mathsf{Q}_*^{-1}(U_j)\big)(\mathbf{x})$. Then we have*

$$\begin{aligned} &\Re\mathrm{e}\{\mathrm{B}_k(\mathsf{R}_*^{k,j}(U_j),\Theta_k(U_k)) + \mathrm{B}_j(\mathsf{R}_*^{j,k}(U_k),\Theta_j(U_j))\} \\ &\quad= \frac{2}{\mu_*}\sum_{q=0}^{n}\Re\mathrm{e}\Big\{\int_{\Omega_q}\nabla\psi_k\cdot\nabla\overline{\psi}_j + \psi_k\,\overline{\psi}_j\,d\mathbf{x}\Big\} \\ &\qquad \text{where } \mathsf{R}_*^{q,j} := \mathsf{Q}_*\cdot\gamma^q\cdot\mathsf{G}^j_\kappa\cdot\mathsf{Q}_*^{-1}. \end{aligned} \tag{10.2}$$

PROOF. First of all, observe that

$$\mathrm{B}_k\big(\mathsf{R}_*^{k,j}(U_j),\Theta_k(U_k)\big) = \frac{1}{\mu_*}\mathrm{B}_k\big(\gamma^k\cdot\mathsf{G}^j_{\kappa_*}(\mathsf{Q}_*^{-1}(U_j)),\Theta_k(\mathsf{Q}_*^{-1}(U_k))\big).$$

As a consequence, by considering $\mathsf{Q}_*^{-1}(U_j)$ instead of U_j and $\mathsf{Q}_*^{-1}(U_k)$ instead of U_k, it is sufficient to prove the proposition in the case where $\mu_* = 1$, which we will assume for the rest of the proof. Let us write explicitly each term in the left-hand side of (10.2). Recall that according to the jump relation (3.2), we have $U_k = \gamma^k(\psi_k) - \gamma^k_c(\psi_k)$. According to Definition (8.8) we have

$$\begin{aligned} &\mathrm{B}_k\big(\mathsf{R}_*^{k,j}(U_j),\Theta_k(U_k)\big) + \mathrm{B}_j\big(\mathsf{R}_*^{j,k}(U_k),\Theta_j(U_j)\big) \\ &\quad= \int_{\partial\Omega_k}\gamma^k_{\mathrm{D}}(\psi_j)\big(\gamma^k_{\mathrm{N}}(\overline{\psi}_k) - \gamma^k_{\mathrm{N},c}(\overline{\psi}_k)\big)d\sigma \\ &\qquad+ \int_{\partial\Omega_j}\gamma^j_{\mathrm{N}}(\psi_k)\big(\gamma^j_{\mathrm{D}}(\overline{\psi}_j) - \gamma^j_{\mathrm{D},c}(\overline{\psi}_j)\big)d\sigma \\ &\qquad+ \int_{\partial\Omega_k}\gamma^k_{\mathrm{N}}(\psi_j)\big(\gamma^k_{\mathrm{D}}(\overline{\psi}_k) - \gamma^k_{\mathrm{D},c}(\overline{\psi}_k)\big)d\sigma \\ &\qquad+ \int_{\partial\Omega_j}\gamma^j_{\mathrm{D}}(\psi_k)\big(\gamma^j_{\mathrm{N}}(\overline{\psi}_j) - \gamma^j_{\mathrm{N},c}(\overline{\psi}_j)\big)d\sigma. \end{aligned} \tag{10.3}$$

Let us deal with the terms containing $\gamma^k_{\mathrm{D}}(\overline{\psi}_j)$, $\gamma^j_{\mathrm{D}}(\psi_j)$, or $\gamma^j_{\mathrm{D},c}(\psi_j)$, i.e., the first line in the right-hand side of (10.3). Apply Green's formula in Ω_j and Ω_k and take into account that $\Delta\psi_k = \psi_k$ both in Ω_j and Ω_k. This yields

$$\begin{aligned}
&\Re\mathrm{e}\Big\{ \int_{\partial\Omega_k} \gamma_{\mathrm{D}}^k(\psi_j)\big(\gamma_{\mathrm{N}}^k(\overline{\psi}_k) - \gamma_{\mathrm{N},c}^k(\overline{\psi}_k)\big)d\sigma \\
&\quad + \int_{\partial\Omega_j} \gamma_{\mathrm{N}}^j(\psi_k)\big(\gamma_{\mathrm{D}}^j(\overline{\psi}_j) - \gamma_{\mathrm{D},c}^j(\overline{\psi}_j)\big)d\sigma \Big\} \\
&= \Re\mathrm{e}\Big\{ \int_{\Omega_k} \nabla\psi_k \cdot \nabla\overline{\psi}_j + \psi_k\,\overline{\psi}_j\,d\mathbf{x} + \int_{\Omega_j} \nabla\psi_k \cdot \nabla\overline{\psi}_j + \psi_k\overline{\psi}_j\,d\mathbf{x} \Big\} \\
&\quad - \Re\mathrm{e}\Big\{ \int_{\partial\Omega_k} \gamma_{\mathrm{D}}^k(\overline{\psi}_j)\gamma_{\mathrm{N},c}^k(\psi_k)d\sigma + \int_{\partial\Omega_j} \gamma_{\mathrm{D},c}^j(\overline{\psi}_j)\gamma_{\mathrm{N}}^j(\psi_k)d\sigma \Big\}.
\end{aligned} \tag{10.4}$$

Let us deal only with the last term in the right-hand side above. Consider $v \in H^1(\mathbb{R}^d)$ and $\mathbf{q} \in H(\mathrm{div}, \mathbb{R}^d)$ such that v coincides with ψ_j in $\mathbb{R}^d \setminus \overline{\Omega}_j$ and $\mathbf{q}$ coincides with $\nabla\psi_k$ in $\mathbb{R}^d \setminus \overline{\Omega}_k$. Given that $\int_{\mathbb{R}^d} \mathbf{q}\cdot\nabla v + v\,\mathrm{div}(\mathbf{q})d\mathbf{x} = 0$ ($|\psi_j(\mathbf{x})|$, $j \geq 1$, decays exponentially for $|\mathbf{x}| \to \infty$), by applying a Green formula inside both Ω_k and Ω_j, we obtain

$$\begin{aligned}
&- \int_{\partial\Omega_k} \gamma_{\mathrm{D}}^k(\overline{\psi}_j)\gamma_{\mathrm{N},c}^k(\psi_k)d\sigma - \int_{\partial\Omega_j} \gamma_{\mathrm{D},c}^j(\overline{\psi}_j)\gamma_{\mathrm{N}}^j(\psi_k)d\sigma \\
&\quad = - \int_{\partial\Omega_k} \overline{v}\,\mathbf{q}\cdot n_k\,d\sigma - \int_{\partial\Omega_j} \overline{v}\,\mathbf{q}\cdot n_j\,d\sigma \\
&\quad = - \int_{\Omega_k\cup\Omega_j} \mathbf{q}\cdot\nabla\overline{v} + \overline{v}\,\mathrm{div}(\mathbf{q})d\mathbf{x} \\
&\quad = \int_{\mathbb{R}^d\setminus(\overline{\Omega}_k\cup\overline{\Omega}_j)} \mathbf{q}\cdot\nabla\overline{v} + \overline{v}\,\mathrm{div}(\mathbf{q})d\mathbf{x} = \int_{\mathbb{R}^d\setminus(\overline{\Omega}_k\cup\overline{\Omega}_j)} \nabla\psi_k\cdot\nabla\overline{\psi}_j + \psi_k\,\overline{\psi}_j\,d\mathbf{x}.
\end{aligned} \tag{10.5}$$

Plugging (10.5) into (10.4), we obtain

$$\begin{aligned}
&\Re\mathrm{e}\Big\{ \int_{\partial\Omega_k} \gamma_{\mathrm{D}}^k(\psi_j)\big(\gamma_{\mathrm{N}}^k(\overline{\psi}_k) - \gamma_{\mathrm{N},c}^k(\overline{\psi}_k)\big)d\sigma + \int_{\partial\Omega_j} \gamma_{\mathrm{N}}^j(\psi_k)\big(\gamma_{\mathrm{D}}^j(\overline{\psi}_j) - \gamma_{\mathrm{D},c}^j(\overline{\psi}_j)\big)d\sigma \Big\} \\
&\quad = \Re\mathrm{e}\Big\{ \int_{\Omega_k} \nabla\psi_k \cdot \nabla\overline{\psi}_j + \psi_k\overline{\psi}_j\,d\mathbf{x} + \int_{\Omega_j} \nabla\psi_k \cdot \nabla\overline{\psi}_j + \psi_k\overline{\psi}_j\,d\mathbf{x} \Big\} \\
&\quad\quad + \Re\mathrm{e}\Big\{ \int_{\mathbb{R}^d\setminus(\overline{\Omega}_k\cup\overline{\Omega}_j)} \nabla\psi_k \cdot \nabla\overline{\psi}_j + \psi_k\overline{\psi}_j\,d\mathbf{x} \Big\}.
\end{aligned} \tag{10.6}$$

We can apply the same treatment to the second line in the right-hand side of (10.3), which yields the following identity:

$$
\begin{aligned}
&\Re\mathrm{e}\Big\{ \int_{\partial\Omega_k} \gamma_{\mathrm{N}}^{k}(\psi_j)\big(\gamma_{\mathrm{D}}^{k}(\overline{\psi}_k) - \gamma_{\mathrm{D},c}^{k}(\overline{\psi}_k)\big)d\sigma \\
&\quad + \int_{\partial\Omega_j} \gamma_{\mathrm{D}}^{j}(\psi_k)\big(\gamma_{\mathrm{N}}^{j}(\overline{\psi}_j) - \gamma_{\mathrm{N},c}^{j}(\overline{\psi}_j)\big)d\sigma\Big\} \\
&\quad = \Re\mathrm{e}\Big\{ \int_{\Omega_k} \nabla\psi_k\cdot\nabla\overline{\psi}_j + \psi_k\,\overline{\psi}_j\,d\mathbf{x} + \int_{\Omega_j} \nabla\psi_k\cdot\nabla\overline{\psi}_j + \psi_k\,\overline{\psi}_j\,d\mathbf{x}\Big\} \\
&\qquad + \Re\mathrm{e}\Big\{ \int_{\mathbb{R}^d\setminus(\overline{\Omega}_k\cup\overline{\Omega}_j)} \nabla\psi_k\cdot\nabla\overline{\psi}_j + \psi_k\,\overline{\psi}_j\,d\mathbf{x}\Big\}.
\end{aligned}
\tag{10.7}
$$

Gathering (10.6) and (10.7) and plugging them into (10.3) leads to the desired result. □

Now we establish a central result whose corollary will be the coercivity of $\widehat{\mathsf{A}}_{\kappa,\mu}$ modulo a compact perturbation.

PROPOSITION 10.3. *Assume that $\kappa_j = \imath$ and $\mu_j \in (0,+\infty)$ for all $j = 0, 1, \dots n$. Then there exists a constant $C > 0$ such that*

$$\Re\mathrm{e}\big\{\widehat{\mathsf{B}}\big(\widehat{\mathsf{A}}_{\kappa,\mu}U, \Theta(U)\big)\big\} \geq C\,\|U\|^2 \quad \forall U \in \widehat{\mathbb{H}}(\Gamma).$$

PROOF. We expand the expression of $\widehat{\mathsf{A}}_{\kappa,\mu}$ according to its definition (8.7) and use the preceding technical results. Consider any $U \in \widehat{\mathbb{H}}(\Gamma)$. Since $\kappa_j = \kappa_0$, we have

$$
\begin{aligned}
\Re\mathrm{e}\{\widehat{\mathsf{B}}(\widehat{\mathsf{A}}_{\kappa,\mu}U, \Theta(U))\} = {}& \sum_{j=1}^{n} \Re\mathrm{e}\big\{\mathrm{B}_j\big((\mathsf{A}^{j}_{\kappa_0,\mu_j} + \mathsf{A}^{j}_{\kappa_0,\mu_0})U_j, \Theta_j(U_j)\big)\big\} \\
& + \sum_{j=1}^{n}\sum_{k\neq 0,j} \Re\mathrm{e}\{\mathrm{B}_k(\mathsf{R}^{k,j}(U_j), \Theta_k(U_k))\}.
\end{aligned}
$$

In the expression above, $k \neq 0, j$ means that k ranges from 1 to n with $k \neq j$. For any $j = 1, \dots, n$, define

$$\psi_j(\mathbf{x}) = \mathsf{G}^{j}_{\kappa_0}(\mathrm{Q}_0^{-1}(U_j))(\mathbf{x}) \quad \text{and} \quad \xi_j(\mathbf{x}) = \mathsf{G}^{j}_{\kappa_0}(\mathrm{Q}_j^{-1}(U_j))(\mathbf{x}).$$

Apply Propositions 10.1 and 10.2. This yields

$$\begin{aligned}\Re\mathrm{e}\big\{\widehat{\mathsf{B}}\big(\widehat{\mathsf{A}}_{\kappa,\mu}\,U,\Theta(U)\big)\big\} &= \sum_{q=0}^{n}\sum_{j=1}^{n}\frac{1}{\mu_j}\int_{\Omega_q}|\nabla\xi_j|^2+|\xi_j|^2\,d\mathbf{x}\\ &\quad+\sum_{q=0}^{n}\Re\mathrm{e}\Big\{\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{1}{\mu_0}\int_{\Omega_q}\nabla\psi_j\cdot\nabla\overline{\psi}_k+\psi_j\overline{\psi}_k\,d\mathbf{x}\Big\}\\ &= \sum_{q=0}^{n}\sum_{j=1}^{n}\frac{1}{\mu_j}\int_{\Omega_q}|\nabla\xi_j|^2+|\xi_j|^2\,d\mathbf{x}\\ &\quad+\sum_{q=0}^{n}\frac{1}{\mu_0}\int_{\Omega_q}\Big|\nabla\Big(\sum_{j=1}^{n}\psi_j\Big)\Big|^2+\Big|\sum_{j=1}^{n}\psi_j\Big|^2\,d\mathbf{x},\end{aligned}$$

$$\begin{aligned}\Re\mathrm{e}\{\widehat{\mathsf{B}}(\widehat{\mathsf{A}}_{\kappa,\mu}\,U,\Theta(U))\} &\geq \sum_{q=0}^{n}\sum_{j=1}^{n}\frac{1}{\mu_j}\int_{\Omega_q}|\nabla\xi_j|^2+|\xi_j|^2\,d\mathbf{x}\\ &= \sum_{q=0}^{n}\sum_{j=1}^{n}\frac{1}{\mu_j}\|\xi_j\|^2_{H^1(\Omega_q)}.\end{aligned}$$

By continuity of the trace operators, there exists a constant $C > 0$ such that for any $j = 1,\dots,n$ we have

$$\|\gamma_{\mathrm{D}}^{q}(v)\|^2_{H^{1/2}(\partial\Omega_q)}+\mu_q^2\|\gamma_{\mathrm{N}}^{q}(v)\|^2_{H^{-1/2}(\partial\Omega_q)}\leq C\|v\|^2_{\Delta,\Omega_q}\quad\forall v\in H^1(\Delta,\Omega_q),$$

$$\|\gamma_{\mathrm{D},c}^{q}(v)\|^2_{H^{1/2}(\partial\Omega_q)}+\mu_q^2\|\gamma_{\mathrm{N},c}^{q}(v)\|^2_{H^{-1/2}(\partial\Omega_q)}\leq C\|v\|^2_{\Delta,\mathbb{R}^d\setminus\overline{\Omega}_q}\quad\forall v\in H^1(\Delta,\mathbb{R}^d\setminus\overline{\Omega}_q),$$

where $\|\cdot\|_{\Delta,\Omega_q}$ was defined at the beginning of Section 1. As a consequence, since $\Delta\xi_j = \xi_j$ in Ω_q for all $j,q = 1,\dots,n$, we have $\|\xi_j\|^2_{\Delta,\Omega_q}\leq 2\|\xi_j\|^2_{H^1(\Omega_q)}$. Hence, if $U_j = (u_j,p_j)^\top$ with $u_j\in H^{1/2}(\partial\Omega_j)$ and $p_j\in H^{-1/2}(\partial\Omega_j)$, then

$$\begin{aligned}&\Re\mathrm{e}\{\widehat{\mathsf{B}}(\widehat{\mathsf{A}}_{\kappa,\mu}U,\Theta(U))\}\\ &\quad\geq\frac{1}{2}\min_{j=1,\dots,n}(1/\mu_j)\sum_{q=0}^{n}\sum_{j=1}^{n}\|\xi_j\|^2_{\Delta,\Omega_q}\\ &\quad\geq\frac{1}{2C}\min_{j=1,\dots,n}(1/\mu_j)\sum_{j=1}^{n}\|u_j\|^2_{H^{1/2}(\partial\Omega_j)}+\|p_j\|^2_{H^{-1/2}(\partial\Omega_j)}\\ &\quad\geq\frac{1}{2C}\min_{j=1,\dots,n}(1/\mu_j)\|U\|^2.\end{aligned}$$

□

THEOREM 10.4. *For any choice of wavenumbers $\kappa_0, \dots, \kappa_n \in \mathbb{C}$ satisfying* (1.4) *and any $\mu_0, \dots, \mu_n \in (0, +\infty)$, there exists a compact operator $\mathsf{K} : \widehat{\mathbb{H}}(\Gamma) \to \widehat{\mathbb{H}}(\Gamma)$ and a constant $C > 0$ such that*

$$\Re\mathrm{e}\{\widehat{\mathsf{B}}((\widehat{\mathsf{A}}_{\kappa,\mu} + \mathsf{K})U, \Theta(U))\} \geq C \|U\|^2 \quad \forall U \in \widehat{\mathbb{H}}(\Gamma).$$

PROOF. Consider the operator $\widehat{\mathsf{A}}_{*,\mu}$ that is defined exactly in the same manner as $\widehat{\mathsf{A}}_{\kappa,\mu}$ except that in the expression of $\widehat{\mathsf{A}}_{*,\mu}$ all the wavenumbers are taken equal to $\kappa_* = \imath$ and set $\mathsf{K} = \widehat{\mathsf{A}}_{*,\mu} - \widehat{\mathsf{A}}_{\kappa,\mu}$. The operator K is compact: according to the arguments given in remark 3.1.3 of [24], this is the consequence of the fact that K is constructed with operators of the form $\gamma^j \cdot (\mathsf{G}_{\kappa_j} - \mathsf{G}_{\kappa_*})$. We conclude by applying Proposition 10.3 with $\widehat{\mathsf{A}}_{*,\mu}$ instead of $\widehat{\mathsf{A}}_{\kappa,\mu}$. □

A first important consequence of Theorem 10.4 is that the operator $\widehat{\mathsf{A}}_\kappa$ is of Fredholm type. A corollary of this result is that problems of the same form as (9.1) are systematically well posed, no matter what the right-hand side.

THEOREM 10.5. *For any $F \in \widehat{\mathbb{H}}(\Gamma)$ and any choice of wavenumbers $\kappa_j \in \mathbb{C}$ satisfying* (1.4) *and any $\mu_j \in (0, +\infty)$, $j = 0, \dots, n$, there exists a unique solution to the following problem*:

$$\text{(10.8)} \qquad \begin{aligned} &\text{Find } \widehat{U} \in \widehat{\mathbb{H}}(\Gamma) \text{ such that} \\ &\widehat{\mathsf{B}}(\widehat{\mathsf{A}}_{\kappa,\mu}(\widehat{U}), \widehat{V}) = \widehat{\mathsf{B}}(F, \widehat{V}) \ \forall \widehat{V} \in \widehat{\mathbb{H}}(\Gamma). \end{aligned}$$

PROOF. Since $\widehat{\mathsf{A}}_{\kappa,\mu}$ is of Fredholm type, it suffices to show that the only solution to (10.8) is $\widehat{U} = 0$ whenever $F = 0$. In addition, $F = 0$ fits the type of right-hand side that was considered in formulation (9.1). As a consequence, we can apply the results of Sections 8 and 9: by application of Proposition 8.2, if $\widehat{U} \in \widehat{\mathbb{H}}(\Gamma)$ satisfies $\widehat{\mathsf{B}}(\widehat{\mathsf{A}}_{\kappa,\mu}(\widehat{U}), \widehat{V}) = 0 \ \forall \widehat{V} \in \widehat{\mathbb{H}}(\Gamma)$, then there exists $U_0 \in \mathbb{H}(\partial\Omega_0)$ such that $U = (U_0, \widehat{U})$ belongs to $\mathbb{X}(\Gamma)$ and U is a solution to $\mathsf{B}(\mathsf{A}_{\kappa,\mu}(U), V) = 0$ for all $V \in \mathbb{X}(\Gamma)$. According to Proposition 4.1, this implies that $U = 0$ and hence $\widehat{U} = 0$. □

Another important consequence of Theorem 10.4 concerns the solvability of formulation (10.8) by means of a Galerkin approach: it guaranties a quasi-optimal convergence of the numerical solution toward the exact solution. Hence the following proposition is a direct application to formulation (10.8) of theorem 4.2.9 in [24].

PROPOSITION 10.6. *Let $(\widehat{\mathbb{H}}_h)_{0<h<1}$ be any dense sequence of finite-dimensional subspaces in $\widehat{\mathbb{H}}(\Gamma)$ satisfying $\Theta(\widehat{\mathbb{H}}_h) = \widehat{\mathbb{H}}_h$. For any $F \in \widehat{\mathbb{H}}(\Gamma)$, any choice of wavenumbers $\kappa_j \in \mathbb{C}$ satisfying* (1.4)*, and any $\mu_j \in (0, +\infty)$, $j = 0, \dots, n$, there exists $h_0 > 0$ such that the following problem admits a unique solution for*

$h \in (0, h_0)$:

$$\text{(10.9)} \qquad \begin{aligned} &\text{Find } U_h \in \widehat{\mathbb{H}}_h \text{ such that} \\ &\widehat{\mathsf{B}}(\widehat{\mathsf{A}}_{\kappa,\mu}(U_h), V_h) = \widehat{\mathsf{B}}(F, V_h) \quad \forall V_h \in \widehat{\mathbb{H}}_h. \end{aligned}$$

In addition, there exists a constant $C > 0$ *independent of h such that, if U is the unique solution to the continuous problem* (10.8)*, we have*

$$\|U - U_h\| \le C \inf_{V_h \in \widehat{\mathbb{H}}_h} \|U - V_h\| \quad \forall h \in (0, h_0).$$

11 Calderón Identity

In this section we present another property of the operator $\widehat{\mathsf{A}}_{\kappa,\mu}$ that was already suggested by the gap idea: whenever $\kappa_0 = \cdots = \kappa_n$ and $\mu_0 = \cdots = \mu_n$, it satisfies the Calderón identity $(\widehat{\mathsf{A}}_{\kappa,\mu})^2 = \mathsf{Id}$. As was established through the pioneering work of Steinbach and Wendland [25] and Christiansen and Nédélec [7, 8, 9], this identity is particularly interesting for deriving an efficient preconditioner for the effective numerical solution to (10.9).

THEOREM 11.1. *If* $\kappa_0 = \cdots = \kappa_n$ *with* $\kappa_0 \neq 0$ *and* $\Im\mathrm{m}\{\kappa_0^2\} \ge 0$*, and if* $\mu_0 = \cdots = \mu_n$ *with* $\mu_0 \in (0, +\infty)$*, then* $(\widehat{\mathsf{A}}_{\kappa,\mu})^2 = \mathsf{Id}$.

PROOF. Denote $\mathsf{J} = (\mathsf{J}_{j,p})_{1 \le j,p \le n} = (\widehat{\mathsf{A}}_{\kappa,\mu})^2$. Let us compute each term $\mathsf{J}_{j,p}$. We have to distinguish diagonal and extradiagonal terms. Let us first examine the case $j = p$. We have

$$\mathsf{J}_{p,p} = \big(2\mathsf{A}^p_{\kappa_0,\mu_0}\big)^2 + \sum_{q \neq 0,p} \mathsf{R}^{p,q} \cdot \mathsf{R}^{q,p}.$$

Recall that $(2\,\mathsf{A}^p_{\kappa_0,\mu_0})^2 = \mathsf{Id}$; this is the classical Calderón identity for the domain Ω_p; see, for example, formula (3.1.41) in [21]. In addition, we have $\mathsf{R}^{p,q} \cdot \mathsf{R}^{q,p} = 0$ if $p \neq q$. Indeed, $\mathcal{R}(\mathsf{R}^{q,p}) \subset \mathcal{R}[\mathsf{Q}_0(\mathsf{Id}/2 + \mathsf{C}^q_{\kappa_0})]$ and $\mathsf{R}^{p,q} V = 0 \ \forall V \in \mathcal{R}[\mathsf{Q}_0(\mathsf{Id}/2 + \mathsf{C}^q_{\kappa_0})]$ according to Proposition 3.1. To sum up, we have $\mathsf{J}_{p,p} = \mathsf{Id}$ for any $p = 1, \dots, n$.

Now take arbitrary $j, p \in \{1, \dots, n\}$ such that $j \neq p$. We have

$$\mathsf{J}_{j,p} = 2\big(\mathsf{A}^j_{\kappa_0,\mu_0} \cdot \mathsf{R}^{j,p} + \mathsf{R}^{j,p} \cdot \mathsf{A}^p_{\kappa_0,\mu_0}\big) + \sum_{q \neq 0,j,p} \mathsf{R}^{j,q} \cdot \mathsf{R}^{q,p}.$$

Using the same remark as in the first part of the proof, we see that $\mathsf{R}^{j,q} \cdot \mathsf{R}^{q,p} = 0$ for $q \neq 0, j, p$. It only remains to examine the term $\mathsf{A}^j_{\kappa_0,\mu_0} \cdot \mathsf{R}^{j,p} + \mathsf{R}^{j,p} \cdot \mathsf{A}^p_{\kappa_0,\mu_0}$. According to (3.3) and (8.8), we have

$$\begin{aligned} \mathsf{Q}_0^{-1} \cdot \mathsf{A}^j_{\kappa_0,\mu_0} \cdot \mathsf{R}^{j,p} \cdot \mathsf{Q}_0 &= \{\gamma^j\} \cdot \mathsf{G}^j_{\kappa_0} \cdot \gamma^j \cdot \mathsf{G}^p_{\kappa_0} = \frac{1}{2}\,\gamma^j \cdot \mathsf{G}^j_{\kappa_0} \cdot \gamma^j \cdot \mathsf{G}^p_{\kappa_0} \\ &= \frac{1}{2}[\gamma^j] \cdot \mathsf{G}^j_{\kappa_0} \cdot \gamma^j \cdot \mathsf{G}^p_{\kappa_0} = \frac{1}{2}\,\gamma^j \cdot \mathsf{G}^p_{\kappa_0} = \end{aligned}$$

$$= \frac{1}{2}\gamma^{j} \cdot \mathsf{G}^{p}_{\kappa_0} \cdot [\gamma^{p}] \cdot \mathsf{G}^{p}_{\kappa_0} = -\frac{1}{2}\gamma^{j} \cdot \mathsf{G}^{p}_{\kappa_0} \cdot \gamma^{p}_{c} \cdot \mathsf{G}^{p}_{\kappa_0}$$
$$= -\gamma^{j} \cdot \mathsf{G}^{p}_{\kappa_0} \cdot \{\gamma^{p}\} \cdot \mathsf{G}^{p}_{\kappa_0} = -\mathsf{Q}_0^{-1} \cdot \mathsf{R}^{j,p} \cdot \mathsf{A}^{p}_{\kappa_0,\mu_0} \cdot \mathsf{Q}_0.$$

In the calculation above, we used the fact that, according to Proposition 3.1, we have $\gamma_c^j \cdot \mathsf{G}^j_{\kappa_0} \cdot \gamma^j \cdot \mathsf{G}^p_{\kappa_0} = 0$ and $\gamma^j \cdot \mathsf{G}^p_{\kappa_0} \cdot \gamma^p \cdot \mathsf{G}^p_{\kappa_0} = 0$ when $j \neq p$. In conclusion, we have $\mathsf{A}^j_{\kappa_0,\mu_0} \cdot \mathsf{R}^{j,p} + \mathsf{R}^{j,p} \cdot \mathsf{A}^p_{\kappa_0,\mu_0} = 0$ when $j \neq p$. As a consequence, we have $\mathsf{J}_{j,p} = 0$ for $j \neq p$. This is sufficient to conclude that $\mathsf{J} = \mathsf{Id}$. □

COROLLARY 11.2. *Assume that* $\mu_0 = \cdots = \mu_n$ *with* $\mu_0 \in (0, +\infty)$, *and choose any wavenumbers* $\kappa_0, \ldots, \kappa_n \in \mathbb{C}$ *satisfying* (1.4); *there exists a compact operator* $\mathsf{K} : \widehat{\mathbb{H}}(\Gamma) \to \widehat{\mathbb{H}}(\Gamma)$ *such that*

$$(\widehat{\mathsf{A}}_{\kappa,\mu})^2 = \mathsf{Id} + \mathsf{K}.$$

PROOF. The proof relies on the same arguments as for Theorem 10.4. Consider the operator $\widehat{\mathsf{A}}_{*,\mu}$ that is defined exactly in the same manner as $\widehat{\mathsf{A}}_{\kappa,\mu}$ except that in the expression of $\widehat{\mathsf{A}}_{*,\mu}$ all wavenumbers are taken equal to $\kappa_* = \imath$, and set $\mathsf{K} = \widehat{\mathsf{A}}_{\kappa,\mu} - \widehat{\mathsf{A}}_{*,\mu}$. We have $(\widehat{\mathsf{A}}_{*,\mu})^2 = \mathsf{Id}$; hence

$$(\widehat{\mathsf{A}}_{\kappa,\mu})^2 = (\widehat{\mathsf{A}}_{*,\mu} + \mathsf{K})^2 = \mathsf{Id} + \widehat{\mathsf{A}}_{*,\mu} \cdot \mathsf{K} + \mathsf{K} \cdot \widehat{\mathsf{A}}_{*,\mu} + \mathsf{K}^2.$$

The operator K is compact; see remark 3.1.3 of [24]. Thus the operator $\widehat{\mathsf{A}}_{*,\mu} \cdot \mathsf{K} + \mathsf{K} \cdot \widehat{\mathsf{A}}_{*,\mu} + \mathsf{K}^2$ is compact as well. □

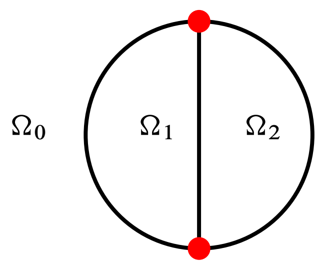
It is not clear to us whether it may be possible to prove a result similar to Corollary 11.2 for the general case of arbitrary $\mu_j \in (0, +\infty)$.

12 Numerical Results

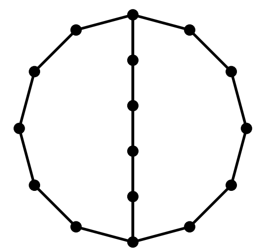
In this section we will present the results of a numerical experiment where we wish to compare the performances of both formulation (9.1) and the classical single-trace formulation, i.e., formulation (4.2), for a two-dimensional model problem. The geometry that we consider contains three parts:

$$\mathbb{R}^2 = \bigcup_{j=0}^{2} \overline{\Omega}_j \quad \text{with} \quad \overline{\Omega}_1 \cup \overline{\Omega}_2 = \overline{\mathrm{D}}(0,1).$$

We consider the scattering of an incident plane wave $u_{\text{inc}}(x, y) = \exp(-i\kappa_0 x)$ at a disk divided into two parts, as represented in the figure below:



Geometry for the numerical experiment



The mesh used

For discretization, we consider a uniform paneling $\Gamma \simeq \Gamma^h = \bigcup_{j=1}^{\mathrm{J}} \Gamma_j^h$ where $\mathrm{J} \in \mathbb{N}$ depends on the step of the mesh h. This induces a paneling of the boundary of each subdomain: for each $k = 0, 1, 2$ there exists $\mathcal{J}_k \subset \{1, \dots, \mathrm{J}\}$ such that $\partial\Omega_k \simeq \partial\Omega_k^h = \bigcup_{j \in \mathcal{J}_k} \Gamma_j^h$. Admittedly, Γ^h and $\partial\Omega_k^h$ are only approximations of Γ and $\partial\Omega_k$ and this induces an error, which we shall comment on later. We only considered meshes that admit the triple points of the geometry as nodes. We used uniform meshes, so that the total number of nodes of the mesh is proportional to $1/h$ where h is the characteristic length of the panels of the mesh.

12.1 Convergence Results

We use piecewise linear functions for approaching both Dirichlet and Neumann traces. As an approximation of the space $\widehat{\mathbb{H}}(\Gamma)$, we consider the discrete space $\widehat{\mathbb{H}}_h$ defined by

$$\widehat{\mathbb{H}}_h = \prod_{j=1}^{2} \mathbb{V}_h(\partial\Omega_j) \times \mathbb{V}_h(\partial\Omega_j),$$
$$\mathbb{V}_h(\partial\Omega_k) = \left\{ v \in C^0\big(\partial\Omega_k^h\big) \;\middle|\; v|_{\Gamma_j^h} \in \mathbb{P}_1 \text{ for } \Gamma_j^h \subset \partial\Omega_k^h \right\},$$

where, as usual, $\mathbb{P}_k$ refers to the set of polynomials of order k. Let $U_h^{(1)}$ refer to the numerical approximation that we compute for the solution to formulation (10.9). Thus $U_h^{(1)}$ is defined as the unique solution to

$$\begin{aligned} &U_h^{(1)} \in \widehat{\mathbb{H}}_h \text{ such that} \\ &\widehat{\mathsf{B}}\big(\widehat{\mathsf{A}}_{\kappa,\mu}\big(U_h^{(1)}\big), V_h\big) = \widehat{\mathsf{B}}(\widehat{F}^{\text{inc}}, V_h) \quad \forall V_h \in \widehat{\mathbb{H}}_h. \end{aligned} \tag{12.1}$$

For the discretization of the classical single-trace formulation, we consider a discrete counterpart of the single-trace space introduced in Section 2. Let us set

$$\mathbb{X}_h^{1/2} = \mathbb{X}^{1/2}(\Gamma) \cap \sum_{j=0}^{2} \mathbb{V}_h(\partial\Omega_j),$$

$$\mathbb{X}_h^{-1/2} = \left\{ (q_j) \in \sum_{j=0}^{2} \mathbb{V}_h(\partial\Omega_j) \;\middle|\; \sum_{j=0}^{2} \int_{\partial\Omega_j^h} v_j q_j \, d\sigma = 0 \quad \forall (v_j) \in \mathbb{X}_h^{1/2} \right\},$$

$$\mathbb{X}_h = \left\{ (v_j, q_j) \in \sum_{j=0}^{2} \mathbb{V}_h(\partial\Omega_j)^2 \;\middle|\; (v_j) \in \mathbb{X}_h^{1/2},\ (q_j) \in \mathbb{X}_h^{-1/2} \right\}.$$

Let $U_h^{(2)}$ refer to the numerical approximation that we compute for the solution to the classical single-trace formulation. The function $U_h^{(2)}$ is thus defined as the unique solution to the discrete problem

$$\begin{aligned} & U_h^{(2)} \in \mathbb{X}_h \text{ such that} \\ & \mathrm{B}\big(\mathsf{A}_{\kappa,\mu}\big(U_h^{(2)}\big), V_h\big) = -\mathrm{B}(F^{\mathrm{inc}}, V_h) \quad \forall V_h \in \mathbb{X}_h. \end{aligned} \tag{12.2}$$

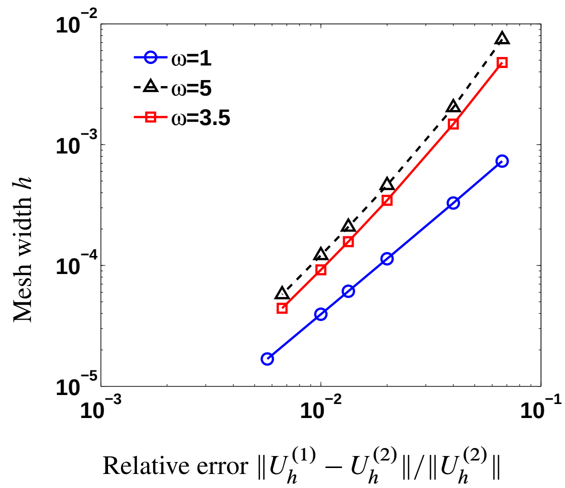
For the assembly of the matrix associated to formulation (12.1) or (12.2), we used the MATLAB® toolbox ie2m developed by A. Bendali. Note that, from the practical point of view of implementation, one advantage of formulation (12.1) is that the treatment of triple points does not require any special arrangement. On the other hand, formulation (12.1) requires more degrees of freedom, and hence more storage.

We solved problem (1.3) using both formulation (12.1) and (12.2) for different values of κ_j and μ_j, $j = 0, 1, 2$. For the results of the figures below, we chose the following values,

$$\kappa_0 = \omega,\ \mu_0 = 1, \qquad \kappa_1 = 2\,\omega,\ \mu_1 = \tfrac{1}{2}, \qquad \kappa_2 = 3\,\omega,\ \mu_2 = 2.$$

where ω, the pulsation of the wave, is the same in all media. In Figure 12.1 below, we represented the relative error $\|U_h^{(1)} - U_h^{(2)}\| / \|U_h^{(2)}\|$ for $h \to 0$. Since we approximate the boundaries $\partial\Omega_j$ by polygonal lines, the relative error between the exact solution of the problem and $U_h^{(2)}$ cannot be smaller than $O(h)$. In addition, Figure 12.1 shows that $\|U_h^{(1)} - U_h^{(2)}\| / \|U_h^{(2)}\| = O(h)$. This indicates that the rate of convergence of the error between the exact solution and $U_h^{(1)}$ is $O(h)$, which is optimal in the present case.

Moreover, we observe that the error $\|U_h^{(1)} - U_h^{(2)}\| / \|U_h^{(2)}\|$ deteriorates as ω grows. This suggests that our formulation is less accurate for higher frequencies, which is a standard feature as well.



For all numerical results we used the coefficients:

$\kappa_0 = \omega, \quad \mu_0 = 1$

$\kappa_1 = 2\,\omega, \quad \mu_1 = 1/2$

$\kappa_2 = 3\,\omega, \quad \mu_2 = 2$

FIGURE 12.1. Consistency results.

12.2 Calderón Preconditioning

In this paragraph we propose a preconditioner for the matrix associated to (12.1). Let $\mathbf{x}_{k,j}$, $j = 1, \dots, \mathfrak{J}_k$, refer to the nodes of $\partial\Omega_k^h$. Let $\varphi_{k,j}$ refer to the piecewise linear continuous function defined on $\partial\Omega_k^h$ such that $\varphi_{k,j}(\mathbf{x}_{k,l}) = 0$ if $l \neq j$ and $\varphi_{k,j}(\mathbf{x}_{k,j}) = 1$. Now we construct a basis (ψ_j) of $\widehat{\mathbb{H}}_h$ in the following manner: We set

$$\begin{aligned} \psi_j &= (\varphi_{1,j}, 0, \dots, 0) && \text{for } j = 1, \dots, \mathfrak{J}_1, \\ \psi_{\mathfrak{J}_1 + j} &= (0, \varphi_{1,j}, 0, \dots, 0) && \text{for } j = 1, \dots, \mathfrak{J}_1, \\ \psi_{2\mathfrak{J}_1 + j} &= (0, 0, \varphi_{2,j}, \dots, 0) && \text{for } j = 1, \dots, \mathfrak{J}_2, \\ &\vdots \end{aligned}$$

and so on, so that the ψ_j's are numbered for j ranging from 1 to $2\,\mathfrak{J}_{\text{tot}}$ where $\mathfrak{J}_{\text{tot}} = \mathfrak{J}_1 + \dots + \mathfrak{J}_n$. Let us consider the matrices

$$\begin{aligned} \widehat{\mathsf{A}}_h &= (\widehat{\mathsf{A}}_{i,j}) \quad \text{where} \quad \widehat{\mathsf{A}}_{i,j} = \mathrm{B}(\widehat{\mathsf{A}}_{\kappa,\mu}\psi_i, \psi_j), \\ \mathsf{M}_h &= (\mathsf{M}_{i,j}) \quad \text{where} \quad \mathsf{M}_{i,j} = \mathrm{B}(\psi_i, \psi_j). \end{aligned}$$

Following the Calderón preconditioning strategy introduced in [7, 8, 9, 25], we use the matrix $\mathsf{R}_h = \mathsf{M}_h^{-1} \cdot \widehat{\mathsf{A}}_h \cdot \mathsf{M}_h^{-1}$ as a preconditioner for $\widehat{\mathsf{A}}_h$. Note that if we had used a piecewise constant function for approximating Neumann traces, we would have had to use dual meshes for the multiplication by M_h^{-1}.

Corollary 11.2 does not apply if $\mu_0 = 1$, $\mu_1 = \frac{1}{2}$, and $\mu_2 = 2$, so that it may not seem so clear that $\mathsf{M}_h^{-1} \cdot \widehat{\mathsf{A}}_h \cdot \mathsf{M}_h^{-1}$ is a relevant preconditioner for such a case. In accordance with the conclusions of [1] though, numerical experiments show that

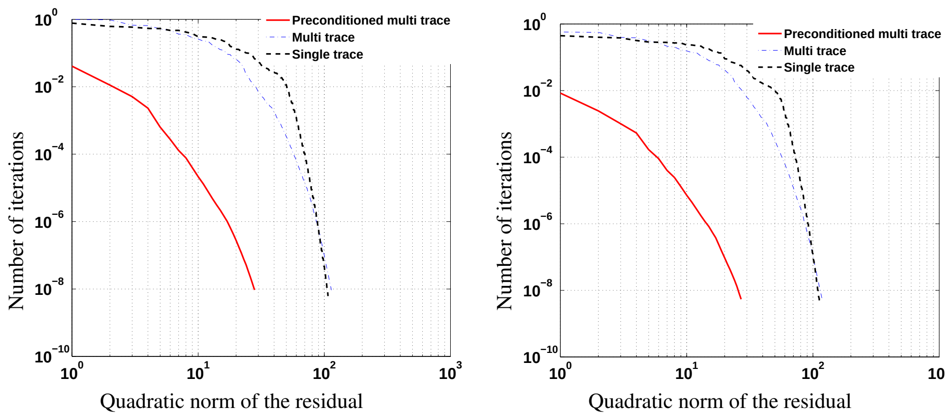


FIGURE 12.2. Convergence history of GMRES with no restart. We take the following values: $\omega = 2$, $\kappa_0 = \omega, \mu_0 = 1$, $\kappa_1 = 2\omega$, $\mu_1 = \frac{1}{2}$, $\kappa_2 = 3\omega$, $\mu_2 = 2$. We achieved the same reduction in the number of iterations with $h = 0.02$ (left) and for $h = 0.0066$ (right).

it is a relevant preconditioner even for the case where the μ_j's are not necessarily equal.

In Figure 12.2 we examine the convergence history of GMRES applied to formulation (12.2) without any preconditioning (single-trace formulation), to (12.1) without preconditioning (multi-trace formulation), and to (12.1) with Calderón preconditioning (preconditioned multi-trace formulation). For more details about the GMRES algorithm, we refer the reader to [23]. In Figure 12.2, we observe that the use of Calderón preconditioning significantly improves the convergence of GMRES.

Appendix

PROPOSITION A.1. *Assume that $\mu_j > 0$, $j = 0, \ldots, n$, and that $\kappa_0, \ldots, \kappa_n$ satisfy assumption* (1.4). *For any $F \in \mathbb{H}(\Gamma)$, there exists a unique $U \in \mathbb{X}(\Gamma)$ such that*

$$\mathsf{B}(\mathsf{A}_{\kappa,\mu}(U), V) = \mathsf{B}(F, V) \quad \forall V \in \mathbb{X}(\Gamma).$$

PROOF. According to (ii) §2.1 in [27], there exists a compact operator $\mathsf{K} : \mathbb{H}(\Gamma) \to \mathbb{H}(\Gamma)$ and a constant $\alpha > 0$ such that $\Re e\{\mathsf{B}((\mathsf{A}_{\kappa,\mu} + \mathsf{K})V, \Theta(V))\} \geq \alpha\|V\|\ \forall V \in \mathbb{H}(\Gamma)$ where Θ has been defined in (10.1). As a consequence, according to the Fredholm alternative, in order to prove the result, we only need to show that the only $U \in \mathbb{X}(\Gamma)$ satisfying $\mathsf{B}(\mathsf{A}_{\kappa,\mu}(U), V) = 0\ \forall V \in \mathbb{X}(\Gamma)$ is $U = 0$. In the remainder of this proof, we will assume that $\kappa_j \in \mathbb{R}$ for all $j = 0, \ldots, n$. The case where $\Im m\{\kappa_j\} > 0$ for some j can be treated in a similar way.

Take any $U = (U_0, \ldots, U_n)^\top \in \mathbb{X}(\Gamma)$ satisfying $\mathsf{B}(\mathsf{A}_{\kappa,\mu}(U), V) = 0\ \forall V \in \mathbb{X}(\Gamma)$. Define $\psi_j(\mathbf{x}) = \mathsf{G}^j_{\kappa_j}(\mathsf{Q}_j^{-1}(U_j))(\mathbf{x})$. First, let us prove that $\psi_j = 0$ in Ω_j

for all $j = 0, \dots, n$. Define $\varphi \in L^2_{\text{loc}}(\mathbb{R}^d)$ such that $\varphi|_{\Omega_j} = \psi_j$, and set $W_{\text{int}} = (\mathsf{Id}/2 + \mathsf{A}_{\kappa,\mu})U$. We have $W_{\text{int}} = (Q_0 \cdot \gamma^0(\varphi), \dots, Q_n \cdot \gamma^n(\varphi))$, and since $B(W_{\text{int}}, V) = B((\mathsf{Id}/2 + \mathsf{A}_{\kappa,\mu})U, V) = B(\mathsf{A}_{\kappa,\mu}U, V) = 0\ \forall V \in \mathbb{X}(\Gamma)$, we deduce that $W_{\text{int}} \in \mathbb{X}(\Gamma)$. Therefore we have

$$\begin{cases} \varphi \in H^1_{\text{loc}}(\mathbb{R}^d) \text{ such that} \\ \mu^{-1}\nabla\varphi \in H_{\text{loc}}(\mathbb{R}^d) \\ \Delta\varphi + \kappa_j^2\varphi = 0 \text{ in } \Omega_j,\ j = 0, \dots, n, \\ \mathrm{CI}_{\kappa_0}(\varphi) = 0 \text{ in } \Omega_0. \end{cases}$$

As a consequence φ is a solution to an homogeneous transmission problem that is well posed. Hence $\varphi = 0$, i.e., $\psi_j = 0$ in Ω_j for all $j = 0, \dots, n$.

Now let us show that $\psi_j = 0$ in $\mathbb{R}^d \setminus \overline{\Omega}_j$ for all $j = 0, \dots, n$. Set $W_{\text{ext}} = -(\mathsf{Id}/2 - \mathsf{A}_{\kappa,\mu})U$. We have

$$B(W_{\text{ext}}, V) = -B((\mathsf{Id}/2 - \mathsf{A}_{\kappa,\mu})U, V) = B(\mathsf{A}_{\kappa,\mu}U, V) = 0 \quad \forall V \in \mathbb{X}(\Gamma)$$

so that $W_{\text{ext}} \in \mathbb{X}(\Gamma)$ according to (2.2). Clearly

$$\Delta\psi_j + \kappa_j^2\psi_j = 0 \ \text{ in } \mathbb{R}^d \setminus \overline{\Omega}_j \quad \text{and} \quad \mathrm{CI}_{\kappa_j}(\psi_j) = 0 \ \text{ for } j \neq 0.$$

Since $W_{\text{ext}} \in \mathcal{R}(\mathsf{Id}/2 - \mathsf{A}_{\kappa,\mu})$, we have $W_{\text{ext}} = (Q_0 \cdot \gamma_c^0(\psi_0), \dots, Q_n \cdot \gamma_c^n(\psi_n)) \in \mathbb{X}(\Gamma)$. Take $r > 0$ large enough to ensure that

$$(\mathbb{R}^d \setminus \Omega_0) \subset B_r = \{\mathbf{x} \in \mathbb{R}^d \mid |\mathbf{x}| < r\}.$$

Applying Green formulas in each $B_r \setminus \overline{\Omega}_j$ we obtain

$$\frac{1}{\mu_j}\int_{\partial B_r} \psi_j\, \partial_r\overline{\psi}_j\, d\sigma = \frac{1}{\mu_j}\int_{B_r\setminus\overline{\Omega}_j} |\nabla\psi_j|^2 - \kappa_j^2\,|\psi_j|^2\, d\mathbf{x} + \frac{1}{\mu_j}\int_{\partial\Omega_j} \gamma^j_{\mathrm{D},c}(\psi_j)\gamma^j_{\mathrm{N},c}(\overline{\psi}_j)d\sigma \quad \forall j \neq 0,$$

$$0 = \frac{1}{\mu_0}\int_{B_r\setminus\overline{\Omega}_0} |\nabla\psi_0|^2 - \kappa_0^2\,|\psi_0|^2\, d\mathbf{x} + \frac{1}{\mu_0}\int_{\partial\Omega_0} \gamma^0_{\mathrm{D},c}(\psi_0)\gamma^0_{\mathrm{N},c}(\overline{\psi}_0)d\sigma.$$

In the equations above ∂_r refers to the radial derivative. Take the imaginary part of the identity above and sum over $j = 0, \dots, n$, taking into account that

$$\left(\gamma^j_{\mathrm{D},c}(\psi_j)\right)_{0\le j\le n} \in \mathbb{X}^{1/2}(\Gamma)$$

and

$$\left(\mu_j^{-1}\gamma^j_{\mathrm{N},c}(\psi_j)\right)_{0\le j\le n} \in \mathbb{X}^{-1/2}(\Gamma)$$

(since $W_{\text{ext}} \in \mathbb{X}(\Gamma)$). This yields

$$\sum_{j=1}^{n} \Im\text{m}\Big\{\frac{1}{\mu_j}\int_{\partial B_r} \psi_j\, \partial_r \overline{\psi}_j\, d\sigma\Big\} = \Im\text{m}\Big\{\sum_{j=0}^{n}\frac{1}{\mu_j}\int_{\partial\Omega_j} \gamma_{\mathrm{D},c}(\psi_j)\gamma_{\mathrm{N},c}(\overline{\psi}_j) d\sigma\Big\} = 0.$$

In the last equality above we used Proposition 2.1. Note that, by construction, $\mathrm{CI}_{\kappa_j}(\psi_j) = 0$. Combining this condition at infinity with the identity above for $j = 1,\dots,n$ yields

$$\sum_{j=1}^{n}\frac{1}{\mu_j}\int_{\partial B_r} |\partial_r\psi_j|^2 + \kappa_j^2|\psi_j|^2\, d\sigma =$$
$$\sum_{j=1}^{n}\frac{1}{\mu_j}\int_{\partial B_r} |\partial_r\psi_j - i\kappa_j\psi_j|^2\, d\sigma - \sum_{j=1}^{n}\Im\text{m}\Big\{\frac{1}{\mu_j}\int_{\partial B_r}\psi_j\,\partial_r\overline{\psi}_j d\sigma\Big\} \underset{r\to\infty}{\longrightarrow} 0.$$

This shows in particular that $\lim_{r\to\infty}\int_{\partial B_r}|\psi_j|^2 d\sigma = 0$ for all $j = 1,\dots,n$. As a consequence, we can apply the Rellich lemma (see lemma 2.11 in [12]), which implies that $\psi_j = 0$ in $\mathbb{R}^d \setminus \overline{\Omega}_j$, $j = 1,\dots,n$.

There only remains to deal with ψ_0. According to the transmission conditions satisfied by ψ_0 we have $\gamma_{\mathrm{D},c}(\psi_0) = 0$ and $\gamma_{\mathrm{N},c}(\psi_0) = 0$. Hence $-\psi_0 = \mathsf{DL}^0_{\kappa_0}\big(\gamma_{\mathrm{D},c}(\psi_0)\big) + \mathsf{SL}^0_{\kappa_0}\big(\gamma_{\mathrm{N},c}(\psi_0)\big) = 0$.

To conclude the proof, note that if $U = (U_0,\dots,U_n)$, we have $\mathrm{Q}_j^{-1}(U_j) = \gamma^j(\psi_j) - \gamma_c^j(\psi_j)$ for all $j = 0,\dots,n$. As a consequence, $U = 0$. □

LEMMA A.2. *For any $j = 0,\dots,n$, any $\mu_0,\mu_j > 0$, and any $\kappa_0,\kappa_j \in \mathbb{C}\setminus\{0\}$ such that $\Re\text{e}\{\kappa_0\} \geq 0$, $\Re\text{e}\{\kappa_j\}\geq 0$ and $\Im\text{m}\{\kappa_0\}\geq 0$, $\Im\text{m}\{\kappa_j\}\geq 0$, we have*

$$\mathcal{R}\big(\mathsf{Id}/2 - \mathsf{A}^j_{\kappa_j,\mu_j}\big) \oplus \mathcal{R}\big(\mathsf{Id}/2 + \mathsf{A}^j_{\kappa_0,\mu_0}\big) = \mathbb{H}(\partial\Omega_j).$$

PROOF. For $j = 0$, this result is the consequence of the fact that $\mathsf{Id}/2 + \mathsf{A}^0_{\kappa_0,\mu_0}$ is a projector (which is easy to check). Take any $j = 1,\dots,n$ and any $V_j = (v_j, q_j) \in \mathbb{H}(\partial\Omega_j)$. Let u be the unique function satisfying the following equations:

$$\begin{aligned} &u \in H^1(\Omega_j), \quad \Delta u + \kappa_0^2 u = 0 \quad \text{in } \Omega_j,\\ &u \in H^1_{\text{loc}}(\mathbb{R}^d\setminus\overline{\Omega}_j), \quad \Delta u + \kappa_j^2 u = 0 \quad \text{in } \mathbb{R}^d\setminus\overline{\Omega}_j \quad u \text{ outgoing radiating},\\ &\gamma^j_{\mathrm{D}}(u) - \gamma^j_{\mathrm{D},c}(u) = v_j \quad \text{and} \quad \mu_0^{-1}\gamma^j_{\mathrm{N}}(u) - \mu_j^{-1}\gamma^j_{\mathrm{N},c}(u) = q_j. \end{aligned} \tag{A.1}$$

This is a standard transmission problem that is classically well posed; see chapter 3 in [13]. In addition, we have $\gamma^j(u) \in \mathcal{R}(\mathsf{Id}/2 + \mathsf{C}^j_{\kappa_0})$ and $\gamma^j_c(u) \in \mathcal{R}(\mathsf{Id}/2 - \mathsf{C}^j_{\kappa_j})$. As a consequence

$$V_j = \mathrm{Q}_0(\gamma^j(u)) - \mathrm{Q}_j\big(\gamma^j_c(u)\big) \in \mathcal{R}\big(\mathsf{Id}/2 - \mathsf{A}^j_{\kappa_j,\mu_j}\big) + \mathcal{R}\big(\mathsf{Id}/2 + \mathsf{A}^j_{\kappa_0,\mu_0}\big).$$

To conclude the proof, it only remains to prove that $\mathcal{R}(\mathsf{Id}/2+\mathsf{A}^j_{\kappa_0,\mu_0})\cap\mathcal{R}(\mathsf{Id}/2-\mathsf{A}^j_{\kappa_j,\mu_j}) = \{0\}$. Take any $V_j \in \mathcal{R}(\mathsf{Id}/2+\mathsf{A}^j_{\kappa_0,\mu_0})\cap\mathcal{R}(\mathsf{Id}/2-\mathsf{A}^j_{\kappa_j,\mu_j})$. Since $\mathcal{R}(\mathsf{Id}/2+\mathsf{A}^j_{\kappa_0,\mu_0})$ is the image under Q_0 of the space of interior Cauchy data in Ω_j for the wave number κ_0, there exists $v_{\mathrm{int}} \in H^1(\Omega_j)$ such that $\Delta v_{\mathrm{int}} + \kappa_0^2 v_{\mathrm{int}} = 0$ in Ω_j, and such that $\mathrm{Q}_0(\gamma^j(v_{\mathrm{int}})) = V_j$. Similarly, $\mathcal{R}(\mathsf{Id}/2-\mathsf{A}^j_{\kappa_j,\mu_j})$ is the image under Q_j of the space of Cauchy data for the exterior of Ω_j associated with the wavenumber κ_j. Hence there exists $v_{\mathrm{ext}} \in H^1(\mathbb{R}^d\setminus\overline{\Omega}_j)$ such that $\Delta v_{\mathrm{ext}}+\kappa_j^2 v_{\mathrm{ext}} = 0$ in $\mathbb{R}^d\setminus\overline{\Omega}_j$ and v_{ext} outgoing radiating, and such that $\mathrm{Q}_j(\gamma_c^j(v_{\mathrm{ext}})) = V_j$. Finally, define $u \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^d)$ by $u|_{\Omega_j} = v_{\mathrm{int}}$ and $u|_{\mathbb{R}^d\setminus\overline{\Omega}_j} = v_{\mathrm{ext}}$. This function u satisfies equations (A.1) with v_j and q_j replaced by 0. Since problem (A.1) is classically well posed, this means that $u = 0$, and hence $V_j = \mathrm{Q}_0(\gamma^j(u)) = 0$. □

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Bibliography

[1] Antoine, X.; Boubendir, Y. An integral preconditioner for solving the two-dimensional scattering transmission problem using integral equations. *Int. J. Comput. Math.* **85** (2008), no. 10, 1473–1490.

[2] Bendali, A.; Fares, M'B.; Gay, J. A boundary-element solution of the Leontovitch problem. *IEEE Trans. Antennas and Propagation* **47** (1999), no. 10, 1597–1605.

[3] Boubendir, Y.; Bendali, A.; Fares, M. B. Coupling of a non-overlapping domain decomposition method for a nodal finite element method with a boundary element method. *Internat. J. Numer. Methods Engrg.* **73** (2008), no. 11, 1624–1650.

[4] Buffa, A. Remarks on the discretization of some noncoercive operator with applications to heterogeneous Maxwell equations. *SIAM J. Numer. Anal.* **43** (2005), no. 1, 1–18 (electronic).

[5] Buffa, A.; Hiptmair, R.; von Petersdorff, T.; Schwab, C. Boundary element methods for Maxwell transmission problems in Lipschitz domains. *Numer. Math.* **95** (2003), no. 3, 459–485.

[6] Chang, Y.; Harrington, R. A surface formulation for characteristic modes of material bodies. *IEEE Trans. Antennas and Propagation* **25** (1977), no. 6, 789–795.

[7] Christiansen, S. H.; Nédélec, J.-C. Des préconditionneurs pour la résolution numérique des équations intégrales de frontière de l'acoustique. *C. R. Acad. Sci. Paris Sér. I Math.* **330** (2000), no. 7, 617–622.

[8] Christiansen, S. H.; Nédélec, J.-C. Des préconditionneurs pour la résolution numérique des équations intégrales de frontière de l'électromagnétisme. *C. R. Acad. Sci. Paris Sér. I Math.* **331** (2000), no. 9, 733–738.

[9] Christiansen, S. H.; Nédélec, J.-C. A preconditioner for the electric field integral equation based on Calderon formulas. *SIAM J. Numer. Anal.* **40** (2002), no. 3, 1100–1135.

[10] Claeys, X. A single trace integral formulation of the second kind for acoustic scattering. Technical Report 2011-14, Seminar of Applied Mathematics, ETH Zürich, 2011.

[11] Claeys, X.; Hiptmair, R. Electromagnetic scattering at composite objects: a novel multi-trace boundary integral formulation. *ESAIM Math. Model. Numer. Anal.* **46** (2012), no. 6, 1421–1445.

[12] Colton, D.; Kress, R. *Inverse acoustic and electromagnetic scattering theory*. Second edition. Applied Mathematical Sciences, 93. Springer, Berlin, 1998.

[13] Colton, D. L.; Kress, R. *Integral equation methods in scattering theory*. Pure and Applied Mathematics (New York). Wiley, New York, 1983.
[14] Harrington, R. Boundary integral formulations for homogeneous material bodies. *J. Electromagnetic Waves and Applications* **3** (1989), no. 1, 1–15. doi:10.1163/156939389X00016
[15] Hiptmair, R.; Jerez-Hanckes, C. Multiple traces boundary integral formulation for Helmholtz transmission problems. *Adv. Comput. Math.* **37** (2012), no. 1, 39–91. doi:10.1007/s10444-011-9194-3
[16] Hsiao, G. C.; Steinbach, O.; Wendland, W. L. Domain decomposition methods via boundary integral equations. *J. Comput. Appl. Math.* **125** (2000), no. 1-2, 521–537.
[17] Kreß, R.; Roach, G. F. Transmission problems for the Helmholtz equation. *J. Mathematical Phys.* **19** (1978), no. 6, 1433–1437.
[18] Langer, U.; Steinbach, O. Boundary element tearing and interconnecting methods. *Computing* **71** (2003), no. 3, 205–228.
[19] McLean, W. *Strongly elliptic systems and boundary integral equations*. Cambridge University Press, Cambridge, 2000.
[20] Miller, E.; Poggio, A. *Integral equation solutions of three-dimensional scattering problems*. Computer Techniques for Electromagnetics, 7. Pergamon Press, Oxford–New York–Toronto, 1973.
[21] Nédélec, J.-C. *Acoustic and electromagnetic equations*. Applied Mathematical Sciences, 144. Springer, New York, 2001.
[22] Rudin, W. *Functional analysis*. Second edition. International Series in Pure and Applied Mathematics, McGraw-Hill, New York, 1991.
[23] Saad, Y. *Iterative methods for sparse linear systems*. Second edition. Society for Industrial and Applied Mathematics, Philadelphia, 2003.
[24] Sauter, S. A.; Schwab, C. *Boundary element methods*. Springer Series in Computational Mathematics, 39. Springer, Berlin, 2011.
[25] Steinbach, O.; Wendland, W. L. The construction of some efficient preconditioners in the boundary element method. *Adv. Comput. Math.* **9** (1998), no. 1-2, 191–216.
[26] Steinbach, O.; Windisch, M. Modified combined field integral equations for electromagnetic scattering. *SIAM J. Numer. Anal.* **47** (2009), no. 2, 1149–1167.
[27] von Petersdorff, T. Boundary integral equations for mixed Dirichlet, Neumann and transmission problems. *Math. Methods Appl. Sci.* **11** (1989), no. 2, 185–213.
[28] Whitney, H. *Geometric integration theory*. Princeton University Press, Princeton, N.J., 1957.

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