Integr. Equ. Oper. Theory
DOI 10.1007/s00020-013-2085-x

Integral Equations and Operator Theory

Integral Equations on Multi-Screens

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Abstract. In the present article, we develop a new functional framework for the study of scalar wave scattering by objects, called multi-screens, that are arbitrary arrangements of thin panels of impenetrable materials. From a geometric point of view, multi-screens are a priori non-orientable non-Lipschitz surfaces. We use our new framework to study boundary integral formulations of the scattering by such objects.

Mathematics Subject Classification (2010). Primary 45A05; Secondary 65R20.

Keywords. Screen, integral equation, scattering, wave propagation, Helmholtz, junction points.

1. Introduction

Numerical computation of acoustic wave scattering by complex arrangements of panels made of some sound-soft material is of great interest in applications. It often occurs that some of the pieces composing such an arrangement have a thickness much smaller than the wavelength, whereas they are large in the other directions; such pieces of material may then be considered infinitely thin, and we call them "screens".

In this article we aim to study integral equation formulations for strongly elliptic boundary value problems with particular focus on the scalar Helmholtz equation when a boundary condition is perscribed on a screen-like object that may consist of several panels. We call such an object a "multi-screen" and a typical representative is shown in Fig. 2.

Integral equations for acoustic scattering by screens have already been considered in numerous works, such as [1–3, 10–15, 22, 23]. These references provide full description of integral formulation for wave scattering by screens for the case where they can be described, from a geometrical point of view, as smooth manifolds with (smooth) boundary. In [4], the authors extended these results to the case where only Lipschitz regularity was assumed for

This work received financial support from Fondation ISAE, and from the French Ministry of Defense via DGA-MRIS.

Birkhäuser

Published online: 20 August 2013

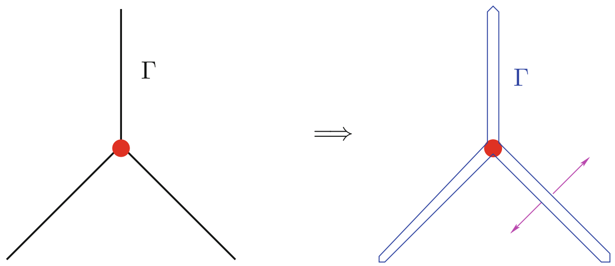

Figure 1. A two-dimensional screen structure (*black*) can be inflated to domain, whose boundary (*blue*) corresponds to the original screen. Obviously, each point on the screen is associated with two points on the new surface (color figure online)

the surface describing the screen. So far though, to our knowledge, it has always been assumed that the screens where represented by surfaces that are everywhere locally orientable i.e. the surface posseses two sides in the neighbourhood of any of its point. Unfortunately this assumption excludes a number of cases, certainly relevant for applications, where the surface would have three or more branches joining along a curve on the surface, see, for example, Fig. 2.

In the geometrical configurations represented in this figure, important theoretical difficulties arise from the presence of junction points located at the border of three or more panels. Such a geometrical feature has already been studied in the litterature on boundary integral equations for transmission problems with penetrable scatterers, see [7, 18] and references therein, although this is a different context compared to scattering by screens.

Because of junction points, the surfaces represented in Fig. 2 do not belong to the class of Lipschitz manifolds. As a consequence the result presented in [4, 8] are not directly reusable here. Adapting the results of [4, 8] to this type of geometry is the main purpose of the present document. Here we focus on Helmholtz equation. We will adress the case of Maxwell's equations in a forthcoming work.

For multi-screen we are going to recover results very similar to what is already known in simpler situations: Green's formula, representation theorem, etc.... In many respects the conventional theory can be adapted by treating the screens as objects of finite thickness, with one exception however: the jump formulas do not hold in the same form as in Lemma 4.1 of [8].

In order to establish these results, we need to construct a new functional framework, that allows to talk about traces on the surface of multi-screens. Buffa and Christiansen [4], also introduced a new functional framework adapted to the study of scattering by standard Lipschitz screens. The present approach is much different, though. The intuition behind it is to treat multi-screens as if they had an (infinitesimal) thickness so that, crudely speaking, they can be viewed as orientable Lipschitz manifolds without boundary, see Fig. 1.

The outline of this article is as follows. In the next section, we provide a precise definition of a multi-screen. In Sect. 3, we recall well known results concerning Sobolev spaces and trace spaces. In Sect. 4, we introduce Sobolev spaces of functions adapted to multi-screens. These functions may admit jump across the screens. In Sect. 5, we define trace spaces on multi-screens. These new trace spaces, called multi-trace spaces, generalize standard traces, and their definition guarantees that Green's formula holds. In Sect. 6, we introduce remarkable subspaces of the multi-trace spaces. We also exhibit close relationship between these remarkable subspaces, and standard trace spaces. In Sect. 7, we study boundary value problems set around a multi-screen, with boundary values prescribed at the multi-screen, and we also provide two useful density results. In Sect. 8, we introduce and study layer potentials adapted to multi-screens. We prove an analogue of the representation theorem, jump formulas, and show that the Dirichlet trace of the single layer potential, and the Neumann trace of the double layer potentials are isomorphisms.

Remark. Throughout this article, we systematically restrict the analysis to $\mathbb{R}^d$ with $d = 2$ or 3 only.

2. Geometry

Before providing a detailed definition for the geometries that we wish to consider, let us first recall the definition of a Lipschitz screen in $\mathbb{R}^3$ as proposed by Buffa and Christiansen [4].

Definition 2.1 (*Lipschitz screen*). A *Lipschitz screen* (in the sense of Buffa–Christiansen) is a subset $\Gamma \subset \mathbb{R}^3$ that satisfies the following properties:

- the set $\overline{\Gamma}$ is a compact Lipschitz two-dimensional sub-manifold with boundary,
- denoting $\partial\Gamma$ the boundary of $\overline{\Gamma}$, we have $\Gamma = \overline{\Gamma}\backslash\partial\Gamma$,
- there exists a finite covering of $\overline{\Gamma}$ with cubes such that, for each such cube C, denoting by a the length of its sides, we have
  - * if C contains a point of $\partial\Gamma$, there exists an orthonormal basis of $\mathbb{R}^3$ in which C can be identified with $(0, a)^3$ and there are Lipschitz continuous functions $\psi : \mathbb{R} \to \mathbb{R}$ and $\phi : \mathbb{R}^2 \to \mathbb{R}$ with values in $(0, a)$ such that

$$\begin{aligned} \Gamma \cap C &= \{(x, y, z) \in C \mid y < \psi(x),\ z = \phi(x, y)\}, \\ \partial\Gamma \cap C &= \{(x, y, z) \in C \mid y = \psi(x),\ z = \phi(x, y)\}, \end{aligned} \tag{2.1}$$

  - * if C contains no boundary point, there exists a Lipschitz open set $\Omega \subset \mathbb{R}^3$ such that we have $\Gamma \cap C = \partial\Omega \cap C$.

The definition of a Lipschitz screen in $\mathbb{R}^2$ is very similar, but simpler. The only difference compared to Definition 2.1 is that Condition (2.1) should be replaced by: *there is a Lipschitz continuous function* $\phi : \mathbb{R} \to \mathbb{R}$ *with values in* $(0, a)$ *and a constant* $a_0 \in (0, a)$ *such that*

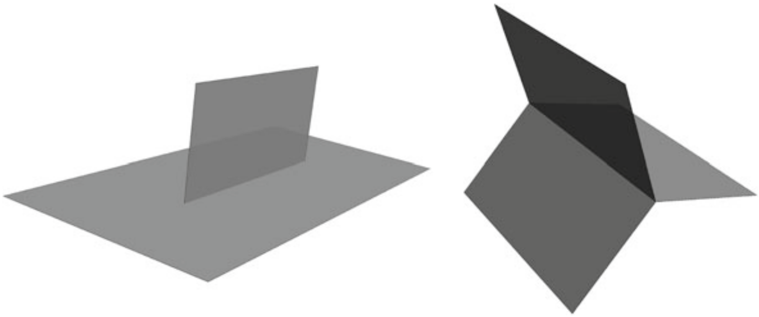

FIGURE 2. Two examples of multi-screen geometries

$$\Gamma \cap C = \{(x,y) \in C \mid x < a_0, y = \phi(x)\}$$
$$\text{and} \quad \partial\Gamma \cap C = \{(x,y) \in C \mid x = a_0,\ y = \phi(x)\}.$$

Now let us focus on potentially more complicated surfaces. In order to propose a convenient definition for surfaces shaped like screen with several branches, we first introduce an intermediary definition.

Definition 2.2 (*Lipschitz partition*). A *Lipschitz partition* of $\mathbb{R}^d$ is a finite collection of Lipschitz open sets $(\Omega_j)_{j=0\dots n}$ such that $\mathbb{R}^d = \cup_{j=0}^n \overline{\Omega}_j$ and $\Omega_j \cap \Omega_k = \emptyset$, if $j \neq k$.

Definition 2.3 (*Multi-screen*). A *multi-screen* is a subset $\Gamma \subset \mathbb{R}^d$ such that there exists a Lipschitz partition of $\mathbb{R}^d$ denoted $(\Omega_j)_{j=0\dots n}$ satisfying $\Gamma \subset \cup_{j=0}^n \partial\Omega_j$ and such that, for each $j = 0\dots n$, we have $\overline{\Gamma} \cap \partial\Omega_j = \overline{\Gamma}_j$ where $\Gamma_j \subset \partial\Omega_j$ is some Lipschitz screen (in the sense of Buffa–Christiansen).

Note that a Lipschitz screen, in the sense of Definition 2.1, is a multi-screen. The surfaces represented in Fig. 2 represent multi-screens that are not Lipschitz screens. Besides, the skeleton $\cup_{j=0\dots n}\partial\Omega_j$ of a Lipschitz partition $(\Omega_j)_{j=0\dots n}$ of $\mathbb{R}^d$ is a multi-screen.

A multi-screen is not a priori orientable which makes it more delicate to analyze compared to a more standard surface such as the boundary of a $C^\infty-$domain. For example, a Möbius strip is a Lipschitz screen in the sense of Buffa and Christiansen, as was pointed out in [4], although it is not globally orientable (Fig. 3).

Of course, the Möbius strip fits the definition of a multi-screen: as is shown in Fig. 4, one can find a Lipshitz partition that contains the Möbius strip in its skeleton.

Remark 2.4. A multi-screen according to Definition 2.3 may contain points where three or more "branches" meet so that, at these points, the multi-screen is not two-sided. This situation compounds difficulties and forces us to adopt an abstract point of view for concepts such as trace operators and trace spaces that are more straightforward in other contexts. Hence part of the present paper will focus on properly defining objects and results that are already very well known in other classical situations.

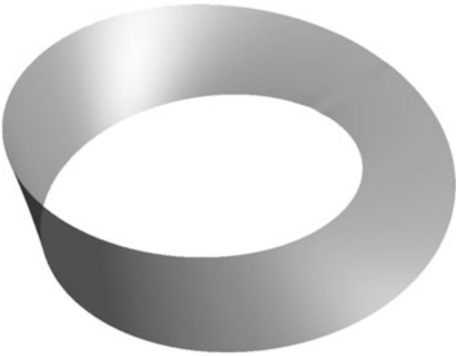

FIGURE 3. Möbius strip

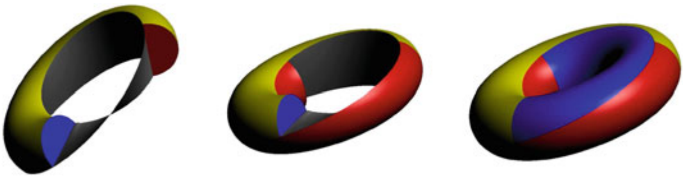

FIGURE 4. Lipschitz partition with Möbius strip in its skeleton

We end this section by stating precisely what we mean by "the boundary of a multi-screen". If Γ is multi-screen, define $\mathrm{int}(\Gamma)$ as the set of points $\mathbf{x} \in \Gamma$ such that there exists a ball $\mathrm{B}_{\mathbf{x}}$ centered at $\mathbf{x}$ and a Lipschitz partition $\mathbb{R}^d = \cup_{j=0}^{n} \overline{\Omega}_j$ satisfying $\mathrm{B} \cap \Gamma = \mathrm{B} \cap_{j=0}^{n} \partial\Omega_j$. We set

$$\partial\Gamma = \overline{\Gamma} \setminus \mathrm{int}(\Gamma).$$

This definition matches the classical definition of $\partial\Gamma$ in the case where Γ is a Lipschitz screen (in the sense of Buffa–Christiansen).

3. Standard Functional Framework

A significant part of the present article is devoted to extending already well established results related to Sobolev spaces and their traces to the case where the domain of definition of the functions under consideration excludes objects whose geometry may be as complex as in the previous section. Before deriving this extended functional setting though, let us recall precisely what we regard as "standard functional framework", at least in the context of integral equations for strongly elliptic operators. For further details about the content of this section, we refer the reader to [16, Chapter 3] or [20, Chapter 2].

In this section, we consider an arbitrary open bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$, and consider any Lipschitz screen $\Gamma \subset \partial\Omega$, see Definition 2.1.

3.1. Standard Dirichlet Traces

With the conventional notation $\mathrm{H}^1(\Omega) = \{v \in \mathrm{L}^2(\Omega) \mid \|v\|^2_{\mathrm{H}^1(\Omega)} := \int_\Omega |v|^2 + |\nabla v|^2\, d\mathbf{x} < \infty\ \}$, as usual we define $\mathrm{H}^1_{0,\Gamma}(\Omega)$ as the closure of $\mathrm{C}^\infty_{0,\Gamma}(\overline{\Omega}) := \{\varphi \in \mathrm{C}^\infty(\overline{\Omega}) \mid \varphi = 0$ in a neighbourhood of $\Gamma\ \}$ with respect to the norm $\|\ \ \|_{\mathrm{H}^1(\Omega)}$.

The point trace operator $\tau_{\mathrm{D},\Gamma} : v \mapsto v|_\Gamma$ induces a continuous map from $\mathrm{H}^1(\Omega)$ into $\mathrm{L}^2(\Gamma)$, see [20, Thm. 2.6.8] and [16, Thm. 3.37]. The following definitions of Hilbert spaces are standard:

$$\begin{aligned} \mathrm{H}^{\frac{1}{2}}(\Gamma) &:= \big\{u|_\Gamma \mid u \in \mathrm{H}^1(\Omega)\big\} = \mathrm{Range}(\tau_{\mathrm{D},\Gamma}), \\ \widetilde{\mathrm{H}}^{\frac{1}{2}}(\Gamma) &:= \Big\{u|_\Gamma \mid u \in \mathrm{H}^1_{0,\partial\Omega\setminus\overline{\Gamma}}(\Omega)\Big\}, \end{aligned} \tag{3.1}$$

where $\mathrm{H}^1_{0,\partial\Omega\setminus\overline{\Gamma}}(\Omega)$ is defined in the same manner as $\mathrm{H}^1_{0,\Gamma}(\Omega)$ (as $\partial\Omega \setminus \overline{\Gamma}$ is a Lipschitz screen as well). Customarily, definitions of these spaces are given based on local charts mapping functions from their respective parameter domains [20, Def. 2.4.1]. This yields "proper function spaces".

There is an alternative angle from which to view $\mathrm{H}^{\frac{1}{2}}(\Gamma)$. It relies on the (a priori non-trivial) result that $\mathrm{H}^1_{0,\Gamma}(\Omega) = \mathrm{Ker}(\tau_{\mathrm{D},\Gamma})$, see [16, Chap. 3], so that the trace operator $\tau_{\mathrm{D},\Gamma}$ induces an isomorphism from $\mathrm{H}^1(\Omega)/\mathrm{H}^1_{0,\Gamma}(\Omega)$ onto $\mathrm{H}^{1/2}(\Gamma)$. Through this isomorphism we can identify both spaces in the sequel and write

$$\mathrm{H}^{\frac{1}{2}}(\Gamma) = \mathrm{H}^1(\Omega)/\mathrm{H}^1_{0,\Gamma}(\Omega). \tag{3.2}$$

In fact thanks to the Lipschitz property of Γ and Sobolev extension theorems this definition is intrinsic in the sense that $\mathrm{H}^1(\mathbb{R}\setminus\overline{\Omega})/\mathrm{H}^1_{0,\Gamma}(\mathbb{R}\setminus\overline{\Omega})$ yields a Hilbert space with equivalent norm. Summing, up it is possible to introduce trace spaces as *quotient spaces* and this is the approach we are going to pursue in the sequel, because it can cope with multi-screens, which pose a challenge to chart based techniques.

3.2. Standard Neumann Traces

Similar results and definitions hold for Neumann traces. We recall $\mathrm{H}(\mathrm{div},\Omega) = \{\mathbf{q} \in \mathrm{L}^2(\Omega)^d \mid \ \|\mathbf{q}\|^2_{\mathrm{H}(\mathrm{div},\Omega)} := \int_\Omega |\mathbf{q}|^2 + |\mathrm{div}(\mathbf{q})|^2\, d\mathbf{x} < +\infty\}$, and define $\mathrm{H}_{0,\Gamma}(\mathrm{div},\Omega)$ as the closure of $\mathrm{C}^\infty_{0,\Gamma}(\overline{\Omega})^d$ with respect to $\|\ \ \|_{\mathrm{H}(\mathrm{div},\Omega)}$.

Denoting by $\mathbf{n}$ the normal vector to $\partial\Omega$ pointing toward the exterior of Ω, the normal component trace operator $\tau_{\mathrm{N},\partial\Omega} : \mathbf{q} \mapsto \mathbf{n}\cdot\mathbf{q}|_{\partial\Omega}$ induces a continuous and surjective mapping from $\mathrm{H}(\mathrm{div},\Omega)$ onto $\mathrm{H}^{-1/2}(\partial\Omega) := \mathrm{H}^{1/2}(\partial\Omega)'$ (the dual space to $\mathrm{H}^{1/2}(\partial\Omega)$), see [20, Thm. 2.7.7] and [16, Thm. 4.3]. In the usual way we introduce

$$\begin{aligned} \mathrm{H}^{-\frac{1}{2}}(\Gamma) &:= \{q|_\Gamma \mid q \in \mathrm{H}^{-\frac{1}{2}}(\partial\Omega)\}, \\ \widetilde{\mathrm{H}}^{-\frac{1}{2}}(\Gamma) &:= \{\mathbf{n}\cdot\mathbf{p}|_{\partial\Omega} \mid \mathbf{p} \in \mathrm{H}_{0,\partial\Omega\setminus\overline{\Gamma}}(\mathrm{div},\Omega)\}. \end{aligned} \tag{3.3}$$

where the symbol "$|_\Gamma$" in the definition of $\mathrm{H}^{-1/2}(\Gamma)$ should be understood as the restriction operator in the sense of distributions on $\partial\Omega$. Again, these are "proper function spaces". As above, quotient spaces offer an alternative, as we have $\mathrm{H}_{0,\Gamma}(\mathrm{div},\Omega) = \mathrm{Ker}(\tau_{\mathrm{N},\Gamma})$, so that the normal trace allows the following identification

$$\begin{aligned}\mathrm{H}^{-1/2}(\Gamma) &= \mathrm{H}(\mathrm{div},\Omega)/\mathrm{H}_{0,\Gamma}(\mathrm{div},\Omega)\\ &= \mathrm{H}(\mathrm{div},\mathbb{R}^d\setminus\overline{\Omega})/\mathrm{H}_{0,\Gamma}(\mathrm{div},\mathbb{R}^d\setminus\overline{\Omega}).\end{aligned} \tag{3.4}$$

For the remainder of this article we use the quotient space norms induced by (3.2) and (3.4) as norms on $\mathrm{H}^{\frac{1}{2}}(\Gamma)$ and $\mathrm{H}^{-1/2}(\Gamma)$, respectively.

4. Domain Based Function Spaces

We now consider the situation where the domain of definition of functions contains a multi-screen $\Gamma\subset\mathbb{R}^d$, see Definition 2.3. We aim at adapting the result of the previous section to domains of the form $\mathbb{R}^d\setminus\overline{\Gamma}$. As the geometry is non-standard, we elaborate many details in order to avoid any ambiguity. First, we focus on domain based functions.

The space $\mathrm{H}^1(\mathbb{R}^d\setminus\overline{\Gamma})$ will stand for the set of functions $u\in\mathrm{L}^2(\mathbb{R}^d)$ such that there exists $\mathbf{p}\in\mathrm{L}^2(\mathbb{R}^d)^d$ satisfying

$$\int\limits_{\mathbb{R}^d\setminus\overline{\Gamma}} u\,\mathrm{div}(\mathbf{q})d\mathbf{x} = -\int\limits_{\mathbb{R}^d\setminus\overline{\Gamma}} \mathbf{p}\cdot\mathbf{q}\,d\mathbf{x} \qquad \forall\mathbf{q}\in\mathscr{D}(\mathbb{R}^d\setminus\overline{\Gamma})^d,$$

where, for any open set $\omega\subset\mathbb{R}^d$, the space $\mathscr{D}(\omega)$ comprises functions $\varphi\in\mathrm{C}^\infty(\omega)$ such that $\mathrm{supp}(\varphi)\subset\omega$. By definition, we may write $\mathbf{p}=\nabla u|_{\mathbb{R}^d\setminus\overline{\Gamma}}$ (in the sense of distributions on $\mathbb{R}^d\setminus\overline{\Gamma}$). A priori, and this is a crucial observation, we have $\mathbf{p}\neq\nabla u$ in the sense of distributions on $\mathbb{R}^d$. We shall equip this space with the scalar product

$$(u,v)_{\mathrm{H}^1(\mathbb{R}^d\setminus\overline{\Gamma})} := \int\limits_{\mathbb{R}^d\setminus\overline{\Gamma}} u\overline{v}\,d\mathbf{x} + \int\limits_{\mathbb{R}^d\setminus\overline{\Gamma}} (\nabla u|_{\mathbb{R}^d\setminus\overline{\Gamma}})\cdot(\nabla\overline{v}|_{\mathbb{R}^d\setminus\overline{\Gamma}})\,d\mathbf{x}$$
$$\forall u,v\in\mathrm{H}^1(\mathbb{R}^d\setminus\overline{\Gamma}). \tag{4.1}$$

With this scalar product, it is routine calculus to check that $\mathrm{H}^1(\mathbb{R}^d\setminus\overline{\Gamma})$ is a Hilbert space. We equip this space with the norm defined by

$$\|u\|^2_{\mathrm{H}^1(\mathbb{R}^d\setminus\overline{\Gamma})} := \|u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} + \|\mathbf{p}\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$$
$$\text{where}\quad \mathbf{p}=\nabla u|_{\mathbb{R}^d\setminus\overline{\Gamma}}.$$

The space $\mathrm{H}^1(\mathbb{R}^d\setminus\overline{\Gamma})$ *strictly* contains $\mathrm{H}^1(\mathbb{R}^d)$ as a non-trivial closed subspace. Indeed the elements of $\mathrm{H}^1(\mathbb{R}^d\setminus\overline{\Gamma})$ may "jump" across Γ (a precise definition of "jumps" will be provided in Sect. 6.2) whereas this is not possible for elements of $\mathrm{H}^1(\mathbb{R}^d)$. In the sequel we shall also denote

$$\mathrm{H}^1_{\mathrm{loc}}(\mathbb{R}^d\setminus\overline{\Gamma}) = \big\{u\in\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^d)\mid\varphi u\in\mathrm{H}^1(\mathbb{R}^d\setminus\overline{\Gamma})\quad\forall\varphi\in\mathscr{D}(\mathbb{R}^d)\big\},$$

equipping this space with its classical Frechet topology induced by the semi-norms $\|\ \|_{\mathrm{H}^1(\mathrm{K})}$ for all compact sets $\mathrm{K}\subset\mathbb{R}^d$, see [19, Chap. 1].

In addition, we consider similar definitions for $\mathrm{H}(\mathrm{div},\mathbb{R}^d\setminus\overline{\Gamma})$ and $\mathrm{H}_{\mathrm{loc}}$ $(\mathrm{div},\,\mathbb{R}^d\setminus\overline{\Gamma})$. For the scalar product, the operator div replaces the operator

∇. Once again $\mathrm{H}(\mathrm{div}, \mathbb{R}^d \setminus \overline{\Gamma})$ contains $\mathrm{H}(\mathrm{div}, \mathbb{R}^d)$ as a strict non-trivial closed subspace.

It is clear how to generalize the previous definitions to the case of functions defined over $\Omega \setminus \overline{\Gamma}$ (instead of $\mathbb{R}^d \setminus \overline{\Gamma}$) where Ω is any bounded Lipschitz open set containing $\overline{\Gamma}$.

Proposition 4.1 (Rellich embedding theorem). *Consider any bounded Lipschitz open set* $\Omega \subset \mathbb{R}^d$ *such that* $\overline{\Gamma} \subset \Omega$. *Then* $\mathrm{H}^1(\Omega \setminus \overline{\Gamma})$ *is compactly embedded into* $\mathrm{L}^2(\Omega)$.

Proof. Take a sequence $u_n \in \mathrm{H}^1(\Omega \setminus \overline{\Gamma}), n \geq 0$ such that $(\|u_n\|_{\mathrm{H}^1(\mathbb{R}^d \setminus \overline{\Gamma})})_{n\geq 0}$ is bounded. Consider an open neighbourhood ω_0 of Γ such that $\overline{\Gamma} \subset \omega_0 \subset \overline{\omega}_0 \subset \Omega$. Consider another open set $\omega_1 \subset \mathbb{R}^d$ such that $\overline{\Gamma} \cap \overline{\omega}_1 = \emptyset$ and $\mathbb{R}^d \subset \omega_0 \cup \omega_1$. Take two smooth cut-off functions ψ_0, ψ_1 that form a partition of unity subordinated to $\omega_0 \cup \omega_1$. Clearly $\psi_1 u_n \in \mathrm{H}^1(\Omega)$ for all n so, extracting a subsequence if necessary, it may be assumed that $(\psi_1 u_n)_{n\geq 0}$ converges in $\mathrm{L}^2(\Omega)$.

There remains to show that, up to some extraction, the sequence $(\psi_0 u_n)_{n\geq 0}$ converges in $\mathrm{L}^2(\Omega)$. Since ψ_0 vanishes in the neighbourhood of $\partial\Omega$, $\psi_0 u_n$ can be considered as defined over all of $\mathbb{R}^d$. Like in Definition 2.3, there exists a Lipschitz partition $\mathbb{R}^d = \cup_{j=0}^n \overline{\Omega}_j$ such that $\Gamma \subset \cup_{j=0}^n \partial\Omega_j$. For each $j = 0 \ldots n$, we have $\psi_0 u_n|_{\Omega_j} \in \mathrm{H}^1(\Omega_j)$ and the sequence $(\psi_0 u_n|_{\Omega_j})_{n\geq 0}$ admits a subsequence that converges in $\mathrm{L}^2(\Omega_j)$. Since $(\Omega_j)_{j=0\ldots n}$ is a *finite* family, this concludes the proof. □

We also need to introduce spaces of functions that vanish in the neighborhood of Γ (such functions, in particular, do not jump across Γ).

Definition 4.2. We define $\mathrm{H}^1_{0,\Gamma}(\mathbb{R}^d)$ to be the closure in $\mathrm{H}^1(\mathbb{R}^d \setminus \overline{\Gamma})$ of the set $\mathscr{D}(\mathbb{R}^d \setminus \overline{\Gamma})$. We also define $\mathrm{H}_{0,\Gamma}(\mathrm{div}, \mathbb{R}^d)$ as the closure in $\mathrm{H}(\mathrm{div}, \mathbb{R}^d \setminus \overline{\Gamma})$ of the set $\mathscr{D}(\mathbb{R}^d \setminus \overline{\Gamma})^d$.

We do not rely on any trace operator for defining $\mathrm{H}^1_{0,\Gamma}(\mathbb{R}^d)$ and $\mathrm{H}_{0,\Gamma}(\mathrm{div}, \mathbb{R}^d)$. Indeed Γ is not (a priori) a Lipschitz manifold, hence the trace operator has not been properly defined yet. By construction $\mathrm{H}^1_{0,\Gamma}(\mathbb{R}^d)$ and $\mathrm{H}_{0,\Gamma}(\mathrm{div}, \mathbb{R}^d)$ are closed subspaces of $\mathrm{H}^1(\mathbb{R}^d \setminus \overline{\Gamma})$ and $\mathrm{H}(\mathrm{div}, \mathbb{R}^d \setminus \overline{\Gamma})$ respectively.

5. Multi-Trace Spaces

In the sequel, we shall introduce several types of trace spaces. The first one is a counterpart of the trace spaces we already introduced in a previous article dedicated to boundary integral formulations for the scattering by multi-subdomain objects, see [6]. With these spaces, traces at the boundary of the screen may admit different values depending on which side of the screen is considered. Taking the cue from (3.2) and (3.4), to construct such traces, we use quotient spaces. We set

$$\begin{aligned}\mathbb{H}^{+\frac{1}{2}}(\Gamma) &:= \mathrm{H}^1(\mathbb{R}^d \setminus \overline{\Gamma})/\mathrm{H}^1_{0,\Gamma}(\mathbb{R}^d),\\ \mathbb{H}^{-\frac{1}{2}}(\Gamma) &:= \mathrm{H}(\mathrm{div}, \mathbb{R}^d \setminus \overline{\Gamma})/\mathrm{H}_{0,\Gamma}(\mathrm{div}, \mathbb{R}^d).\end{aligned} \tag{5.1}$$

The spaces $\mathbb{H}^{1/2}(\Gamma)$ and $\mathbb{H}^{-1/2}(\Gamma)$ will be called *Dirichlet and Neumann multi-trace spaces*, respectively. Their elements will be tagged by $\dot{}$, for instance $\dot{u}, \dot{p}$, and they are equipped with the usual quotient space norms $\| \;\|_{\mathbb{H}^{\pm 1/2}(\Gamma)}$.

In Definition (5.1), it is very important to keep in mind that $\mathrm{H}^1(\mathbb{R}^d \setminus \overline{\Gamma}) \neq \mathrm{H}^1(\mathbb{R}^d)$. In the sequel, we introduce "trace like" operators as the canonical surjections

$$\pi_{\mathrm{D}} : \mathrm{H}^1(\mathbb{R}^d \setminus \overline{\Gamma}) \to \mathbb{H}^{+\frac{1}{2}}(\Gamma) \qquad \text{and} \qquad \pi_{\mathrm{N}} : \mathrm{H}(\mathrm{div}, \mathbb{R}^d \setminus \overline{\Gamma}) \to \mathbb{H}^{-\frac{1}{2}}(\Gamma).$$

For two elements $u, v \in \mathrm{H}^1(\mathbb{R}^d \setminus \overline{\Gamma})$ such that u and v coincide on a bounded neighbourhood of $\overline{\Gamma}$, we have $\pi_{\mathrm{D}}(u) = \pi_{\mathrm{D}}(v)$. This allows to extend π_{D} as a continuous map from $\mathrm{H}^1_{\mathrm{loc}}(\mathbb{R}^d \setminus \overline{\Gamma})$ to $\mathbb{H}^{1/2}(\Gamma)$. Similarly π_{N} can be extended as a continuous map from $\mathrm{H}_{\mathrm{loc}}(\mathrm{div}, \mathbb{R}^d \setminus \overline{\Gamma})$ to $\mathbb{H}^{-1/2}(\Gamma)$.

5.1. Duality Pairing

Note that Green's Formula in $\mathbb{R}^d$ does not hold for elements of $\mathrm{H}^1(\mathbb{R}^d \setminus \overline{\Gamma})$ and $\mathrm{H}(\mathrm{div}, \mathbb{R}^d \setminus \overline{\Gamma})$. Indeed, pick any $u \in \mathrm{H}^1(\mathbb{R}^d \setminus \overline{\Gamma})$ and any $\mathbf{p} \in \mathrm{H}(\mathrm{div}, \mathbb{R}^d \setminus \overline{\Gamma})$, which, in general, will yield $\int_{\mathbb{R}^d \setminus \overline{\Gamma}} \mathbf{p} \cdot \nabla u + u \,\mathrm{div}(\mathbf{p})\, d\mathbf{x} \neq 0$. However, note that

$$\int_{\mathbb{R}^d \setminus \overline{\Gamma}} (\mathbf{p} + \mathbf{q}) \cdot \nabla(u + v) + (u + v)\,\mathrm{div}(\mathbf{p} + \mathbf{q})\, d\mathbf{x} = \int_{\mathbb{R}^d \setminus \overline{\Gamma}} \mathbf{p} \cdot \nabla u + u\,\mathrm{div}(\mathbf{p})\, d\mathbf{x}$$
$$\forall v \in \mathrm{H}^1_{0,\Gamma}(\mathbb{R}^d), \;\; \forall \mathbf{q} \in \mathrm{H}_{0,\Gamma}(\mathrm{div}, \mathbb{R}^d).$$

This suggests a bilinear pairing between $\mathbb{H}^{1/2}(\Gamma)$ and $\mathbb{H}^{-1/2}(\Gamma)$. Indeed, for $\dot{u} \in \mathbb{H}^{1/2}(\Gamma)$ and $\dot{p} \in \mathbb{H}^{-1/2}(\Gamma)$, choose $u \in \mathrm{H}^1(\mathbb{R}^d \setminus \overline{\Gamma})$ and $\mathbf{p} \in \mathrm{H}(\mathrm{div}, \mathbb{R}^d \setminus \overline{\Gamma})$ such that $\pi_{\mathrm{D}}(u) = \dot{u}$ and $\pi_{\mathrm{N}}(\mathbf{p}) = \dot{p}$, and set

$$\int_{[\Gamma]} \dot{u}\, \dot{p}\, d\sigma := \int_{\mathbb{R}^d \setminus \overline{\Gamma}} \mathbf{p} \cdot \nabla u + u\,\mathrm{div}(\mathbf{p})\, d\mathbf{x}. \tag{5.2}$$

Please be aware that the integral in the left hand side above should not be read as an integral with respect to the Lebesgue measure on Γ. It is merely a notational convention hinting at the relationship of (5.2) with Green's Formula. Similarly to $\mathrm{H}^{\pm 1/2}(\partial\Omega)$ in the case of a smooth boundary, see Sect. 3, $\mathbb{H}^{\pm 1/2}(\Gamma)$ are dual to each other via this pairing.

Proposition 5.1. *The pairing* $\ll , \gg : \mathbb{H}^{+1/2}(\Gamma) \times \mathbb{H}^{-1/2}(\Gamma) \to \mathbb{C}$ *defined by*

$$\ll \dot{v}, \dot{q} \gg = \int_{[\Gamma]} \dot{q}\, \dot{v}\, d\sigma \qquad \forall \dot{v} \in \mathbb{H}^{\frac{1}{2}}(\Gamma), \;\; \forall \dot{q} \in \mathbb{H}^{-\frac{1}{2}}(\Gamma), \tag{5.3}$$

induces an isometric duality *between* $\mathbb{H}^{+1/2}(\Gamma)$ *and* $\mathbb{H}^{-1/2}(\Gamma)$.

Proof. Pick any $\dot{u} \in \mathbb{H}^{+1/2}(\Gamma)$ and write $\ll \dot{u}, \cdot \gg$ for the linear form $\dot{p} \mapsto \ll \dot{u}, \dot{p} \gg$ on $\mathbb{H}^{-1/2}(\Gamma)$. We find

$$\| \ll \dot{u}, \cdot \gg \|_{\mathbb{H}^{-1/2}(\Gamma)} = \sup_{\substack{\dot{p}\in\mathbb{H}^{-\frac{1}{2}}(\Gamma)\\ \dot{q}\neq 0}} \frac{| \ll \dot{u}, \dot{p} \gg |}{\|\dot{p}\|_{\mathbb{H}^{-\frac{1}{2}}(\Gamma)}}$$

$$= \sup_{\substack{\mathbf{p}\in \mathrm{H}(\mathrm{div},\mathbb{R}^d\setminus\overline{\Gamma})\\ \mathbf{p}\neq 0}} \frac{\int_{\mathbb{R}^d\setminus\overline{\Gamma}} \mathbf{p}\cdot\nabla u + u\,\mathrm{div}(\mathbf{p})\,d\mathbf{x}}{\|\mathbf{p}\|_{\mathrm{H}(\mathrm{div},\mathbb{R}^d\setminus\overline{\Gamma})}} \leq \|u\|_{\mathrm{H}^1(\mathbb{R}^d\setminus\overline{\Gamma})}\,.$$

The above inequality holds for any $u \in \mathrm{H}^1(\mathbb{R}^d \setminus \overline{\Gamma})$ such that $\pi_{\mathrm{D}}(u) = \dot{u}$. Hence, $\ll \dot{u}, \cdot \gg \in (\mathbb{H}^{-1/2}(\Gamma))'$. Let $u \in \mathrm{H}^1(\mathbb{R}^d \setminus \Gamma)$ be the minimal norm representative of $\dot{u}$, which fulfills

$$\int_{\mathbb{R}^d\setminus\overline{\Gamma}} \nabla u\cdot\nabla v + u\,v\,d\mathbf{x} = 0 \quad \forall v \in \mathrm{H}^1_{0,\Gamma}(\mathbb{R}^d). \tag{5.4}$$

This implies $\|u\|_{\mathrm{H}^1(\mathbb{R}^d\setminus\Gamma)} = \|\dot{u}\|_{\mathbb{H}^{1/2}(\Gamma)}$. Set $\mathbf{p} := \nabla u$. Since (5.4) means that $-\Delta u + u = 0$ in $\mathbb{R}^d \setminus \overline{\Gamma}$, we infer $\mathbf{p} \in \mathrm{H}(\mathrm{div}, \mathbb{R}^d \setminus \overline{\Gamma})$, $\mathrm{div}(\mathbf{p}) = u$, and, finally, $\|\mathbf{p}\|_{\mathrm{H}(\mathrm{div},\mathbb{R}^d\setminus\overline{\Gamma})} = \|u\|_{\mathrm{H}^1(\mathbb{R}^d\setminus\overline{\Gamma})}$. As a consequence

$$\| \ll \dot{u}, \cdot \gg \|_{\mathbb{H}^{-1/2}(\Gamma)} \geq \frac{\int_{\mathbb{R}^d\setminus\overline{\Gamma}} \mathbf{p}\cdot\nabla u + u\,\mathrm{div}(\mathbf{p})\,d\mathbf{x}}{\|\mathbf{p}\|_{\mathrm{H}(\mathrm{div},\mathbb{R}^d\setminus\overline{\Gamma})}} \geq \|u\|_{\mathrm{H}^1(\mathbb{R}^d\setminus\overline{\Gamma})}\,.$$

We conclude that $\dot{u} \mapsto \ll \dot{u}, \cdot \gg$ is an isometry $\mathbb{H}^{+1/2}(\Gamma) \mapsto (\mathbb{H}^{-1/2}(\Gamma))'$. By similar arguments, one establishes that also $\dot{p} \mapsto \ll \cdot, \dot{p} \gg$ spawns an isometry $\mathbb{H}^{-1/2}(\Gamma) \mapsto (\mathbb{H}^{+1/2}(\Gamma))'$, which concludes the proof. □

As an immediate consequence of the duality between $\mathbb{H}^{\frac{1}{2}}(\Gamma)$ and $\mathbb{H}^{-\frac{1}{2}}(\Gamma)$ we obtain that $\mathrm{H}^1(\mathbb{R}^d \setminus \overline{\Gamma})$ and $\mathrm{H}_{0,\Gamma}(\mathrm{div}, \mathbb{R}^d)$ are polar to each other, as well as $\mathrm{H}^1_{0,\Gamma}(\mathbb{R}^d)$ and $\mathrm{H}(\mathrm{div}, \mathbb{R}^d \setminus \overline{\Gamma})$.

Corollary 5.2. *Let* $u \in \mathrm{H}^1(\mathbb{R}^d \setminus \overline{\Gamma})$ *and* $\mathbf{p} \in \mathrm{H}(\mathrm{div}, \mathbb{R}^d \setminus \overline{\Gamma})$. *We have the following characterizations,*

$$u \in \mathrm{H}^1_{0,\Gamma}(\mathbb{R}^d) \iff \int_{\mathbb{R}^d\setminus\overline{\Gamma}} \mathbf{q}\cdot\nabla u + u\,\mathrm{div}(\mathbf{q})\,d\mathbf{x} = 0 \quad \forall \mathbf{q} \in \mathrm{H}(\mathrm{div}, \mathbb{R}^d \setminus \overline{\Gamma})$$

$$\mathbf{p} \in \mathrm{H}_{0,\Gamma}(\mathrm{div}, \mathbb{R}^d) \iff \int_{\mathbb{R}^d\setminus\overline{\Gamma}} \mathbf{p}\cdot\nabla v + v\,\mathrm{div}(\mathbf{p})\,d\mathbf{x} = 0 \quad \forall v \in \mathrm{H}^1(\mathbb{R}^d \setminus \overline{\Gamma})$$

5.2. Interpretation of Multi-Traces in Terms of Functions

In this paragraph we describe as explicitly as possible the spaces $\mathbb{H}^{\pm 1/2}(\Gamma)$ for particular situations, relating these spaces to the more standard functional framework recalled in Sect. 3.

The skeleton of a Lipschitz partition. We first illustrate the concepts introduced at the beginning of Sect. 5 by applying them to the particular case where $\Gamma = \cup_{j=0}^{n} \partial\Omega_j$ for some Lipschitz partition $(\Omega_j)_{j=0\ldots n}$ of $\mathbb{R}^d$, see Definition 2.2. In this situation, depicted in Fig. 5, simple localization provides an isometric isomorphism

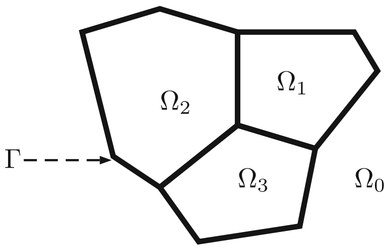

FIGURE 5. Multi-screens obtained from sub-domain boundaries

$$\mathrm{Loc} : \mathrm{H}^1(\mathbb{R}^d \setminus \Gamma) \to \mathrm{H}^1(\Omega_0) \times \cdots \times \mathrm{H}^1(\Omega_n).$$

Writing $\mathrm{Ext}_j : \mathrm{H}^{\frac{1}{2}}(\partial\Omega_j) \to \mathrm{H}^1(\Omega_j)$ for some right inverse of the point trace operator, obviously

$$\pi_D \circ \mathrm{Loc}^{-1} \circ (\mathrm{Ext}_0 \times \cdots \times \mathrm{Ext}_n) : \mathrm{H}^{\frac{1}{2}}(\partial\Omega_1) \times \cdots \times \mathrm{H}^{\frac{1}{2}}(\partial\Omega_n) \to \mathbb{H}^{+\frac{1}{2}}(\Gamma). \tag{5.5}$$

is a well-defined *isometric isomorphism*. A similar isomorphism can be obtained for Neumann traces. Casually speaking, this permits us to identify

$$\begin{aligned} \mathbb{H}^{+\frac{1}{2}}(\Gamma) &\cong \mathrm{H}^{+\frac{1}{2}}(\partial\Omega_0) \times \cdots \times \mathrm{H}^{+\frac{1}{2}}(\partial\Omega_n), \\ \mathbb{H}^{-\frac{1}{2}}(\Gamma) &\cong \mathrm{H}^{-\frac{1}{2}}(\partial\Omega_0) \times \cdots \times \mathrm{H}^{-\frac{1}{2}}(\partial\Omega_n). \end{aligned} \tag{5.6}$$

There is a clear interpretation of (5.2) in this case: take $u \in \mathrm{H}^1(\mathbb{R}^d \setminus \overline{\Gamma})$ and $\mathbf{p} \in \mathrm{H}(\mathrm{div}, \mathbb{R}^d \setminus \overline{\Gamma})$. Let $u_j = u|_{\Omega_j}$ and $\mathbf{p}_j = \mathbf{p}|_{\Omega_j}$, and set $v_j = u_j|_{\partial\Omega_j} \in \mathrm{H}^{1/2}(\partial\Omega_j)$ and $q_j = \mathbf{n}_j \cdot \mathbf{p}_j|_{\partial\Omega_j} \in \mathrm{H}^{-1/2}(\partial\Omega_j)$ where $\mathbf{n}_j$ is the normal to $\partial\Omega_j$ directed toward the exterior of Ω_j. Identity (5.2) then reads

$$\begin{aligned} \ll \dot{u}, \dot{p} \gg = \int_{[\Gamma]} \dot{u}\, \dot{p}\, d\sigma &= \int_{\mathbb{R}^d \setminus \overline{\Gamma}} \mathbf{p} \cdot \nabla u + u\, \mathrm{div}(\mathbf{p})\, d\mathbf{x} \\ &= \sum_{j=0}^{n} \int_{\Omega_j} \mathbf{p}_j \cdot \nabla u_j + u_j\, \mathrm{div}(\mathbf{p}_j)\, d\mathbf{x} = \sum_{j=0}^{n} \int_{\partial\Omega_j} v_j\, p_j\, d\sigma \end{aligned} \tag{5.7}$$

Identity (5.7) is consistent with the usual Green formula.

Standard Lipschitz screens. Next, we illustrate the concepts of this section by applying them to another special situation shown in Fig. 6. Let Ω_1 be a bounded Lipschitz domain, and let $\Gamma \subset \partial\Omega_1$ be a Lipschitz screen in the sense of Definition 2.1. Let us denote $\Omega_2 = \mathbb{R}^d \setminus \overline{\Omega}_1$. We have $\mathrm{H}^1(\mathbb{R}^d \setminus \overline{\Gamma}) \subset \mathrm{H}^1(\mathbb{R}^d \setminus \partial\Omega_1)$ which induces a natural injection

$$\mathbb{H}^{\frac{1}{2}}(\Gamma) = \mathrm{H}^1(\mathbb{R}^d \setminus \overline{\Gamma})/\mathrm{H}^1_{0,\Gamma}(\mathbb{R}^d) \quad \hookrightarrow \quad \mathrm{H}^1(\mathbb{R}^d \setminus \partial\Omega_1)/\mathrm{H}^1_{0,\Gamma}(\mathbb{R}^d).$$

From the natural identification $\mathrm{H}^1(\mathbb{R}^d \setminus \partial\Omega_1) \cong \mathrm{H}^1(\Omega_1) \times \mathrm{H}^1(\Omega_2)$ that associates u with $(u|_{\Omega_1}, u|_{\Omega_2})$ we obtain an isomorphism that we may express as

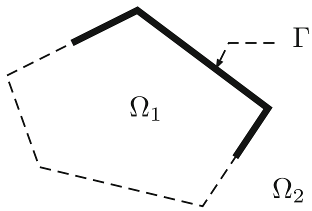

FIGURE 6. Lipschitz screen contained in the boundary of a domain

$$\mathrm{H}^1(\mathbb{R}^d \setminus \partial\Omega_1)/\mathrm{H}^1_{0,\Gamma}(\mathbb{R}^d) \cong \Big[\mathrm{H}^1(\Omega_1)/\mathrm{H}^1_{0,\Gamma}(\Omega_1)\Big] \times \Big[\mathrm{H}^1(\Omega_2)/\mathrm{H}^1_{0,\Gamma}(\Omega_2)\Big]$$
$$\cong \mathrm{H}^{\frac{1}{2}}(\Gamma) \times \mathrm{H}^{\frac{1}{2}}(\Gamma).$$

Here, in the spirit of (3.2), we have linked $\mathrm{H}^{\frac{1}{2}}(\Gamma)$ with quotient spaces. From this discussion we can conclude an injection

$$\mathbb{H}^{+\frac{1}{2}}(\Gamma) \quad\hookrightarrow\quad \mathrm{H}^{\frac{1}{2}}(\Gamma) \times \mathrm{H}^{\frac{1}{2}}(\Gamma). \tag{5.8}$$

Now let us show how the injection (5.8) can be constructed in detail. Consider an element $\dot{u} \in \mathbb{H}^{1/2}(\Gamma)$. Take any $u \in \mathrm{H}^1(\mathbb{R}^d \setminus \overline{\Gamma})$ such that $\pi_{\mathrm{D}}(u) = \dot{u}$, and make the following identification

$$\dot{u} \quad\leftrightarrow\quad (u_1|_\Gamma, u_2|_\Gamma) \qquad \text{where} \qquad u_j = u|_{\Omega_j},\ j = 1,2.$$

In this construction, the traces u_1, u_2 actually satisfy a compatibility condition. Indeed consider a function $\tilde{u} \in \mathrm{H}^1(\mathbb{R}^d)$ such that $\tilde{u}|_{\Omega_2} = u_2$ (which exists thanks to Sobolev extension theorems), and set $\tilde{u}_1 = \tilde{u}|_{\Omega_1}$. Observe that $\tilde{u}_1|_\Gamma = u_2|_\Gamma$, and we have

$$u_1 - \tilde{u}_1 \in \mathrm{H}^1_{0,\partial\Omega\setminus\overline{\Gamma}}(\Omega) \quad\Rightarrow\quad u_1|_\Gamma - \tilde{u}_1|_\Gamma = u_1|_\Gamma - u_2|_\Gamma \ \in \widetilde{\mathrm{H}}^{\frac{1}{2}}(\Gamma). \tag{5.9}$$

A thorough inspection of the above arguments shows that (5.9) is a necessary and sufficient condition to ensure that there exists $u \in \mathrm{H}^1(\mathbb{R}^d \setminus \overline{\Gamma})$ such that $u_1|_\Gamma, u_2|_\Gamma$ are the traces of $u|_{\Omega_1}$ and $u|_{\Omega_2}$ on Γ. Thus, localization to Ω_1 and Ω_2 together with local traces yield an isomorphism

$$\mathbb{H}^{+\frac{1}{2}}(\Gamma) \cong \big\{(v_1, v_2) \in \mathrm{H}^{\frac{1}{2}}(\Gamma)^2 \ \ |\ \ v_1 - v_2 \in \widetilde{\mathrm{H}}^{\frac{1}{2}}(\Gamma)\big\}. \tag{5.10}$$

Similarly we can prove

$$\mathbb{H}^{-\frac{1}{2}}(\Gamma) \cong \big\{(q_1, q_2) \in \mathrm{H}^{-\frac{1}{2}}(\Gamma)^2 \ \ |\ \ q_1 + q_2 \in \widetilde{\mathrm{H}}^{-\frac{1}{2}}(\Gamma)\big\}. \tag{5.11}$$

The "+" sign coming into play in Definition (5.11) is related to the change in the normal direction depending on wich side of Γ is involved, see Fig. 1.

In addition, it is possible to give an explicit expression of the duality pairing $\ll\ ,\ \gg$ defined in (5.2)–(5.3) by means of Identifications (5.10) and (5.11). Indeed consider any $\dot{u} \in \mathbb{H}^{1/2}(\Gamma)$ and $\dot{p} \in \mathbb{H}^{-1/2}(\Gamma)$. Pick $u \in \mathrm{H}^1(\mathbb{R}^d \setminus \overline{\Gamma})$ and $\mathbf{p} \in \mathrm{H}(\mathrm{div}, \mathbb{R}^d \setminus \overline{\Gamma})$ such that $\pi_{\mathrm{D}}(u) = \dot{u}$ and $\pi_{\mathrm{N}}(\mathbf{p}) = \dot{p}$. For $j = 1,2$, set $u_j = u|_{\Omega_j}$ and $\mathbf{p}_j = \mathbf{p}|_{\Omega_j}$, and let $\mathbf{n}_j$ refer to the normal vector to Ω_j

directed toward the exterior of Ω_j. Statement (5.10) and (5.11) is based on the following identifications

$$\dot{u} \leftrightarrow (v_1, v_2) \quad \text{and} \quad \dot{p} \leftrightarrow (q_1, q_2),$$
$$\text{where} \quad \begin{cases} v_1 = u_1|_\Gamma, \quad v_2 = u_2|_\Gamma, \\ q_1 = \mathbf{n}_1 \cdot \mathbf{p}_1|_\Gamma, \quad q_2 = \mathbf{n}_2 \cdot \mathbf{p}_2|_\Gamma. \end{cases}$$

According to Green's formula, we have

$$\begin{aligned} \int_{[\Gamma]} \dot{u}\,\dot{p}\,d\sigma &= \int_{\mathbb{R}^d\setminus\overline{\Gamma}} \mathbf{p}\cdot\nabla u + u\,\mathrm{div}(\mathbf{p})\,d\mathbf{x} \\ &= \sum_{j=1,2}\int_{\Omega_j} \mathbf{p}\cdot\nabla u + u\,\mathrm{div}(\mathbf{p})\,d\mathbf{x} \\ &= \sum_{j=1,2}\int_{\partial\Omega_j} q_j\,v_j\,d\sigma. \end{aligned}$$

The boundary terms in the identity above can be simplified further. Indeed, since $u \in \mathrm{H}^1(\mathbb{R}^d\setminus\overline{\Gamma})$ we have $v_1 = v_2$ on $\partial\Omega_1\setminus\Gamma$. Similarly, since $\mathbf{p} \in \mathrm{H}(\mathrm{div}, \mathbb{R}^d\setminus\overline{\Gamma})$ and $\mathbf{n}_1 = -\mathbf{n}_2$, we have $q_1 = -q_2$ on $\partial\Omega_1\setminus\Gamma$, which leads to cancellation of terms off Γ:

$$\ll \dot{v}, \dot{q} \gg = \int_{[\Gamma]} \dot{u}\,\dot{p}\,d\sigma = \sum_{j=1,2}\int_{\partial\Omega_j} q_j\,v_j\,d\sigma = \int_{\Gamma} v_1\,q_1 + v_2\,q_2\,d\sigma. \quad (5.12)$$

6. Single-Trace and Jump Spaces

We return to the general case of an arbitrary multi-screen $\Gamma \subset \mathbb{R}^d$ according to Definition 2.1. As regards the multi-trace spaces $\mathbb{H}^{\pm 1/2}(\Gamma)$ they contain multi-valued functions: the sides of each panel of Γ could be regarded as distinct surfaces, and traces on both sides do not necessarily match, see Fig. 1. Now we are going to single out subspaces of $\mathbb{H}^{\pm 1/2}(\Gamma)$ that may be considered as standard trace spaces of single-valued functions.

6.1. Single-Trace Spaces

We can obtain particular subspaces of $\mathbb{H}^{\pm 1/2}(\Gamma)$ by simply replacing "$\mathbb{R}^d\setminus\overline{\Gamma}$" by "$\mathbb{R}^d$" in (5.1).

Definition 6.1 (*Single-trace spaces*). We introduce *single-trace spaces* as the quotient spaces

$$\begin{aligned} \mathrm{H}^{+\frac{1}{2}}([\Gamma]) &= \mathrm{H}^1(\mathbb{R}^d)/\mathrm{H}^1_{0,\Gamma}(\mathbb{R}^d) \\ \mathrm{H}^{-\frac{1}{2}}([\Gamma]) &= \mathrm{H}(\mathrm{div}, \mathbb{R}^d)/\mathrm{H}_{0,\Gamma}(\mathrm{div}, \mathbb{R}^d). \end{aligned} \quad (6.1)$$

They owe their name to the intuitive point of view that the elements of the single-trace spaces can be understood as multi-traces whose values on both sides of the screen either agree (in the case of $\mathrm{H}^{1/2}([\Gamma])$) or have opposite sign (in the case of $\mathrm{H}^{-1/2}([\Gamma])$).

Corollary 6.2. *The space* $\mathrm{H}^{+1/2}([\Gamma])$ *(resp.* $\mathrm{H}^{-1/2}([\Gamma])$*) is a closed subspace of* $\mathbb{H}^{+1/2}(\Gamma)$ *(resp.* $\mathbb{H}^{-1/2}(\Gamma)$*)*

Proof. The assertion is immediate, since $\mathrm{H}^1(\mathbb{R}^d)$ (resp. $\mathrm{H}(\mathrm{div},\mathbb{R}^d)$) is closed in $\mathrm{H}^1(\mathbb{R}^d\setminus\overline{\Gamma})$ (resp. $\mathbb{H}(\mathrm{div},\mathbb{R}^d\setminus\overline{\Gamma})$). □

There is no duality relationship between $\mathrm{H}^{+\frac{1}{2}}([\Gamma])\subset\mathbb{H}^{+1/2}(\Gamma)$ and $\mathrm{H}^{-\frac{1}{2}}([\Gamma])\subset\mathbb{H}^{-1/2}(\Gamma)$ with respect to the pairing $\ll\,,\,\gg$ between the multi-trace spaces. On the contrary, both spaces are polar to each other, which provides a weak characterization:

Proposition 6.3. *For* $\dot{u}\in\mathbb{H}^{+\frac{1}{2}}(\Gamma)$ *and* $\dot{p}\in\mathbb{H}^{-\frac{1}{2}}(\Gamma)$ *holds true*

$$\begin{aligned}\dot{u}\in\mathrm{H}^{+\frac{1}{2}}([\Gamma]) &\iff \int_{[\Gamma]}\dot{u}\,\dot{q}\,d\sigma=0\quad\forall\dot{q}\in\mathrm{H}^{-\frac{1}{2}}([\Gamma]),\\ \dot{p}\in\mathrm{H}^{-\frac{1}{2}}([\Gamma]) &\iff \int_{[\Gamma]}\dot{v}\,\dot{p}\,d\sigma=0\quad\forall\dot{v}\in\mathrm{H}^{+\frac{1}{2}}([\Gamma]).\end{aligned}\tag{6.2}$$

Proof. We will show only the first assertion, since the proof for the second is very similar. First, take any $u\in\mathrm{H}^1(\mathbb{R}^d)$ such that $\pi_{\mathrm{D}}(u)=\dot{u}$. Then for any $\dot{q}\in\mathrm{H}^{-\frac{1}{2}}([\Gamma])$, considering $\mathbf{q}\in\mathrm{H}(\mathrm{div},\mathbb{R}^d)$ such that $\pi_{\mathrm{N}}(\mathbf{q})=\dot{q}$, the standard Green formula over all $\mathbb{R}^d$ yields

$$\int_{[\Gamma]}\dot{u}\,\dot{q}\,d\sigma=\int_{\mathbb{R}^d\setminus\overline{\Gamma}}\mathbf{q}\cdot\nabla u+u\,\mathrm{div}(\mathbf{q})d\mathbf{x}=\int_{\mathbb{R}^d}\mathbf{q}\cdot\nabla u+u\,\mathrm{div}(\mathbf{q})d\mathbf{x}=0.$$

Now consider $\dot{u}\in\mathbb{H}^{1/2}(\Gamma)$ such that the condition in the right hand side of (6.2) holds. Take a $u\in\mathrm{H}^1(\mathbb{R}^d\setminus\overline{\Gamma})$ such that $\pi_{\mathrm{D}}(u)=\dot{u}$. We need to show that $u\in\mathrm{H}^1(\mathbb{R}^d)$. We already know that there exists some $\mathbf{p}\in\mathrm{L}^2(\mathbb{R}^d)^d$ such that $\int_{\mathbb{R}^d}\mathbf{p}\cdot\mathbf{q}+u\,\mathrm{div}(\mathbf{q})\,d\mathbf{x}=0$ for any $\mathbf{q}\in\mathscr{D}(\mathbb{R}^d\setminus\overline{\Gamma})^d$. Take any $\mathbf{q}\in\mathscr{D}(\mathbb{R}^d)^d$, so that $\pi_{\mathrm{N}}(\mathbf{q})=\dot{q}\in\mathrm{H}^{-1/2}([\Gamma])$. Applying Definition (5.2), we obtain

$$\int_{\mathbb{R}^d}\mathbf{p}\cdot\mathbf{q}+u\,\mathrm{div}(\mathbf{q})\,d\mathbf{x}=\int_{[\Gamma]}\dot{u}\,\dot{q}\,d\sigma=0\quad\forall\mathbf{q}\in\mathscr{D}(\mathbb{R}^d).$$

This proves that $\mathbf{p}=\nabla u$ in the sense of distributions over $\mathbb{R}^d$ (and not just $\mathbb{R}^d\setminus\overline{\Gamma}$), so that $u\in\mathrm{H}^1(\mathbb{R}^d)$. This concludes the proof. □

6.2. Jump Spaces

Another type of trace space may be obtained by considering the duals to single-trace spaces.

Definition 6.4 (*Jump spaces*). We introduce *jump spaces* as the dual spaces

$$\widetilde{\mathrm{H}}^{+\frac{1}{2}}([\Gamma]):=\big(\mathrm{H}^{-\frac{1}{2}}([\Gamma])\big)'\quad\text{and}\quad\widetilde{\mathrm{H}}^{-\frac{1}{2}}([\Gamma]):=\big(\mathrm{H}^{+\frac{1}{2}}([\Gamma])\big)'.$$

We endow theses spaces with their natural dual norms

$$\|\varphi\|_{\widetilde{\mathrm{H}}^{\pm\frac{1}{2}}([\Gamma])}:=\sup_{\substack{\dot{q}\in\mathrm{H}^{\mp\frac{1}{2}}([\Gamma])\\ \dot{q}\neq0}}\frac{|\langle\,\varphi,\dot{q}\,\rangle|}{\|\dot{q}\|_{\mathbb{H}^{\mp\frac{1}{2}}(\Gamma)}}.$$

Clearly, any element of $\mathbb{H}^{1/2}(\Gamma)$ (resp. $\mathbb{H}^{-1/2}(\Gamma)$) induces an element $\widetilde{\mathrm{H}}^{-1/2}([\Gamma])$ (resp. $\widetilde{\mathrm{H}}^{1/2}([\Gamma])$) via the duality pairing (5.2).

Definition 6.5 (*Jump operators*). We define continuous *jump operators* $[\ ]: \mathbb{H}^{+1/2}(\Gamma) \to \widetilde{\mathrm{H}}^{1/2}([\Gamma])$ and $[\ ]: \mathbb{H}^{-1/2}(\Gamma) \to \widetilde{\mathrm{H}}^{-1/2}([\Gamma])$ as follows: For any $\dot{u} \in \mathbb{H}^{+1/2}(\Gamma)$ (resp. any $\dot{p} \in \mathbb{H}^{-1/2}(\Gamma)$), let $[\dot{u}]$ (resp. $[\dot{p}]$) be the unique element of $\widetilde{\mathrm{H}}^{1/2}([\Gamma])$ (resp. $\widetilde{\mathrm{H}}^{-1/2}([\Gamma])$) satisfying

$$\begin{aligned} \langle [\dot{u}], \dot{q}\rangle &:= \int_{[\Gamma]} \dot{u}\, \dot{q}\, d\sigma \quad \forall \dot{q} \in \mathrm{H}^{-\frac{1}{2}}([\Gamma]), \\ \langle \dot{v}, [\dot{p}]\rangle &:= \int_{[\Gamma]} \dot{v}\dot{p}\, d\sigma \quad \forall \dot{v} \in \mathrm{H}^{+\frac{1}{2}}([\Gamma]), \end{aligned} \tag{6.3}$$

where $\langle\ ,\ \rangle$ denotes the duality pairing either between $\mathrm{H}^{1/2}([\Gamma])$ and $\widetilde{\mathrm{H}}^{-1/2}([\Gamma])$, or between $\widetilde{\mathrm{H}}^{1/2}([\Gamma])$ and $\mathrm{H}^{-1/2}([\Gamma])$.

An immediate consequence of Proposition 6.3 is a characterization of the kernels of the jump operators that matches the intuition that "single-valued traces do not jump".

Corollary 6.6 (Kernels of jump operators). *For* $v \in \mathbb{H}^{1/2}(\Gamma)$ *and* $q \in \mathbb{H}^{-1/2}([\Gamma])$ *holds true*

$$v \in \mathrm{H}^{1/2}([\Gamma]) \Leftrightarrow [v] = 0,$$

$$q \in \mathrm{H}^{-1/2}([\Gamma]) \Leftrightarrow [q] = 0.$$

Proposition 6.7 (Range of jump operators). *The jump operators* $[\ ]: \mathbb{H}^{1/2}(\Gamma) \to \widetilde{\mathrm{H}}^{1/2}([\Gamma])$ *and* $[\ ]: \mathbb{H}^{-1/2}(\Gamma) \to \widetilde{\mathrm{H}}^{-1/2}([\Gamma])$ *from Definition* 6.5 *are surjective.*

Proof. The statement follows from Proposition 5.1, the Hahn-Banach Theorem (see [19, Thm 3.3] for example), and Corollary 6.6. ☐

We end this section by pointing out an alternative description of jump spaces provided by the next proposition. The proof is a direct consequence of Corollary 6.6 and Problem 9, §3.8 in [21].

Proposition 6.8 (Quotient space characterization of jump spaces). *The jump operators induce* isometric isomorphisms

$$\widetilde{\mathrm{H}}^{1/2}([\Gamma]) \cong \mathbb{H}^{1/2}(\Gamma)/\mathrm{H}^{1/2}([\Gamma]) \quad \textit{and}$$
$$\widetilde{\mathrm{H}}^{-1/2}([\Gamma]) \cong \mathbb{H}^{-1/2}(\Gamma)/\mathrm{H}^{-1/2}([\Gamma]).$$

6.3. Interpretation of Single-Traces in Terms of Functions

In this paragraph we will try to describe as explicitly as possible the spaces $\mathrm{H}^{\pm 1/2}([\Gamma])$ and $\widetilde{\mathrm{H}}^{\pm 1/2}([\Gamma])$ for the two special situations that we considered in Sect. 5.2.

The skeleton of a Lipschitz partition. First, we focus on the situation where $\Gamma = \cup_{j=0}^{n} \partial\Omega_j$ for some Lipschitz partition $(\Omega_j)_{j=0...n}$ of $\mathbb{R}^d$, see Fig. 5. Pick $\dot{u} \in \mathrm{H}^{1/2}([\Gamma])$. As explained in Sect. 5.2, considering any $u \in \mathrm{H}^1(\mathbb{R}^d)$ such that $\pi_{\mathrm{D}}(u) = \dot{u}$, and setting $u_j = u|_{\Omega_j}$, we can make the identification

$$\begin{aligned} \dot{u} \leftrightarrow (v_0, \ldots, v_n) \in \mathrm{H}^{\frac{1}{2}}(\partial\Omega_0) \times \cdots \times \mathrm{H}^{\frac{1}{2}}(\partial\Omega_n), \\ \text{where} \quad v_j = u_j|_{\partial\Omega_j}, \quad j = 0 \ldots n. \end{aligned} \tag{6.4}$$

The condition $\dot{u} \in \mathrm{H}^{1/2}([\Gamma])$ amounts to $v_j = v_k$ on $\partial\Omega_j \cap \partial\Omega_k$ $\forall j, k = 0 \ldots n$, and (6.4) gives rise to an isomorphism

$$\mathrm{H}^{\frac{1}{2}}([\Gamma]) \cong \Big\{ (v_j)_{0\leq j\leq n} \in \prod_{j=0}^{n} \mathrm{H}^{\frac{1}{2}}(\partial\Omega_j) \mid v_j - v_k = 0 \quad \text{on } \partial\Omega_j \cap \partial\Omega_k \ \ \forall j, k \Big\}.$$

Similarly, we find an isomorphism

$$\mathrm{H}^{-\frac{1}{2}}([\Gamma]) \cong \Big\{ (q_j)_{0\leq j\leq n} \in \prod_{j=0}^{n} \mathrm{H}^{-\frac{1}{2}}(\partial\Omega_j) \mid q_j + q_k = 0 \quad \text{on } \partial\Omega_j \cap \partial\Omega_k \ \ \forall j, k \Big\}$$

The spaces $\mathrm{H}^{\pm 1/2}([\Gamma])$ have been considered in [5,6] (where they were noted $\mathbb{X}^{\pm 1/2}(\Gamma)$), and Proposition 6.3 above is a generalization of Proposition 2.1 in [5].

It seems to us that it is not possible to develop any explicit description of $\widetilde{\mathrm{H}}^{\pm 1/2}([\Gamma])$ for the case where Γ is the skeleton of some Lipschitz partition except if, in this partition, each interface separates at most two subdomains. The latter case is covered in the next paragraph.

Standard Lipschitz screens. Now we consider the case where $\Gamma \subset \partial\Omega$ is a Lipschitz screen in the sense of Definition 2.1, where Ω is a bounded Lipschitz open set, as in Fig. 6. Pick $\dot{u} \in \mathrm{H}^{1/2}([\Gamma])$ and set $\Omega_1 = \Omega$ and $\Omega_2 = \mathbb{R}^d \setminus \overline{\Omega}$. In accordance with the discussion in Sect. 5.2, for any $u \in \mathrm{H}^1(\mathbb{R}^d \setminus \overline{\Gamma})$ we have the identification

$$\begin{aligned} \dot{u} \ \leftrightarrow \ (v_1, v_2) \in \mathrm{H}^{\frac{1}{2}}(\Gamma) \times \mathrm{H}^{\frac{1}{2}}(\Gamma) \\ \text{where} \quad v_j = u_j|_{\Gamma} \quad \text{and} \quad u_j = u|_{\Omega_j}, \quad j = 1, 2. \end{aligned}$$

Since $\dot{u} \in \mathrm{H}^{1/2}([\Gamma])$ we actually have $u \in \mathrm{H}^1(\mathbb{R}^d)$ which implies $v_1 = v_2$. This leads to the conclusion that (compare with (5.10))

$$\begin{aligned} \mathrm{H}^{\frac{1}{2}}([\Gamma]) \cong \Big\{ (v_1, v_2) \in \mathrm{H}^{\frac{1}{2}}(\Gamma) \times \mathrm{H}^{\frac{1}{2}}(\Gamma) \mid v_1 - v_2 = 0 \text{ on } \Gamma \Big\}, \\ \text{i.e.}\ \mathrm{H}^{\frac{1}{2}}([\Gamma]) \cong \phi_+\Big(\mathrm{H}^{\frac{1}{2}}(\Gamma)\Big) \ \cong \ \mathrm{H}^{\frac{1}{2}}(\Gamma), \quad \text{where} \quad \phi_+(x) := (x, x). \end{aligned} \tag{6.5}$$

Similar results hold for the Neumann single-trace space. A slight adaptation of the above arguments shows that

$$\begin{aligned} \mathrm{H}^{-\frac{1}{2}}([\Gamma]) \ \cong \ \Big\{ \ (q_1, q_2) \in \mathrm{H}^{-\frac{1}{2}}(\Gamma) \times \mathrm{H}^{-\frac{1}{2}}(\Gamma) \mid q_1 + q_2 = 0 \text{ on } \Gamma \ \Big\}, \\ \text{i.e.}\ \ \mathrm{H}^{-\frac{1}{2}}([\Gamma]) \ \cong \ \phi_-\Big(\mathrm{H}^{-\frac{1}{2}}(\Gamma)\Big) \ \cong \ \mathrm{H}^{-\frac{1}{2}}(\Gamma), \\ \text{where} \ \ \phi_-(x) := (x, -x). \end{aligned} \tag{6.6}$$

Remark 6.9. This discussion confirms the agreement of the new functional framework we have introduced with standard Sobolev trace spaces on surfaces and screens. Such a simple and explicit description does not seem to be possible for more complicated screens that are multi-screens but not standard Lipschitz screens. In this sense, our new functional framework is a genuine generalization of standard Sobolev trace spaces.

Let us now look for some explicit description of $\widetilde{\mathrm{H}}^{\pm 1/2}([\Gamma])$, still considering the case where $\Gamma \subset \partial\Omega_1$ is a standard Lipshitz screen. We make use of the isomorphism

$$\iota : \mathbb{H}^{1/2}(\Gamma) \to \Big\{(v_1, v_2) \in \mathrm{H}^{\frac{1}{2}}(\Gamma) \times \mathrm{H}^{\frac{1}{2}}(\Gamma) \quad | \quad v_1 - v_2 \in \widetilde{\mathrm{H}}^{\frac{1}{2}}(\Gamma)\Big\}$$

that underlies (5.10). Pick any $v \in \widetilde{\mathrm{H}}^{1/2}(\Gamma)$, see (3.1). Following the discussion of Sect. 5.2, if $(v_1, v_2) = (v, -v) = \phi_-(v)$, we have $v_1 - v_2 = 2v \in \widetilde{\mathrm{H}}^{1/2}(\Gamma)$, so that $\iota^{-1}(\phi_-(v)) \in \mathbb{H}^{1/2}(\Gamma)$ in the sense of (5.10), and the linear mapping

$$[\iota^{-1}\phi_-] := [\ \ ] \circ \iota^{-1} \circ \phi_- : \ \ \widetilde{\mathrm{H}}^{\frac{1}{2}}(\Gamma) \to \widetilde{\mathrm{H}}^{\frac{1}{2}}([\Gamma]) \tag{6.7}$$

is well defined and continuous.

Theorem 6.10 (Isomorphism connecting $\widetilde{\mathrm{H}}^{\frac{1}{2}}(\Gamma)$ and $\widetilde{\mathrm{H}}^{\frac{1}{2}}([\Gamma])$). *In the special situation of a Lipschitz screen* Γ *the mapping from* (6.7) *is an isomorphism.*

Proof. (i) Injectivity: Assume that $[\iota^{-1}\phi_-(v)] = 0$ for some $v \in \widetilde{\mathrm{H}}^{\frac{1}{2}}(\Gamma)$. Above in (6.6) we have seen that any element $\dot{q} \in \mathrm{H}^{-1/2}([\Gamma])$ takes the form $\dot{q} = \theta^{-1}(\phi_-(q)) = \theta^{-1}(q, -q)$ for some $q \in \mathrm{H}^{-1/2}(\Gamma)$, where θ designates the isomorphism underlying (6.6). As a consequence of (5.12), $[\iota^{-1}(\phi_-(v))] = 0$ implies

$$\begin{aligned} 0 = \langle\, [\iota^{-1}\phi_-(v)], \theta^{-1}(\phi_-(q))\,\rangle &= \int_{[\Gamma]} \iota^{-1}(\phi_-(v))\, \theta^{-1}(\phi_-(q))\, d\sigma \\ &= 2\int_{\Gamma} v\, q\, d\sigma \quad \forall q \in \mathrm{H}^{-1/2}(\Gamma). \end{aligned} \tag{6.8}$$

Since $\widetilde{\mathrm{H}}^{1/2}(\Gamma) = \mathrm{H}^{-1/2}(\Gamma)'$, Identity (6.8) implies that $v = 0$.

(ii) Surjectivity: Pick some $\varphi \in \widetilde{\mathrm{H}}^{1/2}([\Gamma])$. According to the Hahn-Banach Theorem (see [19, Thm 3.3]) and Proposition 5.1, there exists $\dot{v} \in \mathbb{H}^{1/2}(\Gamma)$ such that $\|\dot{v}\|_{\mathbb{H}^{1/2}(\Gamma)} = \|\varphi\|_{\widetilde{\mathrm{H}}^{1/2}([\Gamma])}$ and

$$\langle\, \varphi, \dot{q}\,\rangle = \int_{[\Gamma]} \dot{v}\, \dot{q}\, d\sigma \qquad \forall \dot{q} \in \mathrm{H}^{-1/2}([\Gamma]).$$

Moreover, by (5.10) there exists $v_1, v_2 \in \mathrm{H}^{1/2}(\Gamma)$ such that $v_1 - v_2 \in \widetilde{\mathrm{H}}^{1/2}(\Gamma)$, and $\dot{v} = \iota^{-1}(v_1, v_2)$. Any $\dot{q} \in \mathrm{H}^{-1/2}([\Gamma])$ can be written as $\dot{q} = \theta^{-1}(\phi_-(q)) = \theta^{-1}(q, -q)$ in the sense of (6.6) for some $q \in \mathrm{H}^{-1/2}(\Gamma)$. Setting $v = \frac{1}{2}(v_1 - v_2)$, we have

$$\begin{aligned}\langle\, \varphi, \dot{q}\,\rangle &= \int_{[\Gamma]} \dot{v}\,\dot{q}\,d\sigma = \int_{\Gamma} v_1\, q \;-\; v_2\, q\; d\sigma \\ &= \int_{\Gamma} v\, q + (-v)\,(-q)\; d\sigma = \int_{[\Gamma]} \iota^{-1}(\phi_-(v))\,\dot{q}\,d\sigma\end{aligned}$$

Since $\dot{q} \in \mathrm{H}^{-1/2}([\Gamma])$ is arbitrary, this proves that $\varphi = [\iota^{-1}(\phi_-(v))]$, and bears out the surjectivity of the map (6.7). □

To summarize, the mapping (6.7) induces an isomorphism

$$\widetilde{\mathrm{H}}^{\frac{1}{2}}([\Gamma]) \cong \widetilde{\mathrm{H}}^{\frac{1}{2}}(\Gamma).$$

We can also find an analogous isomorphism

$$\widetilde{\mathrm{H}}^{-\frac{1}{2}}([\Gamma]) \cong \widetilde{\mathrm{H}}^{-\frac{1}{2}}(\Gamma).$$

Let us end this paragraph by mentioning that the inclusion "$\widetilde{\mathrm{H}}^{1/2}([\Gamma]) \subset \mathrm{H}^{1/2}([\Gamma])$" does *not* hold. This inclusion has to be replaced with some injection relation, as can be readily seen from

$$\widetilde{\mathrm{H}}^{1/2}([\Gamma]) \cong \widetilde{\mathrm{H}}^{1/2}(\Gamma) \;\subset\; \mathrm{H}^{1/2}(\Gamma) \cong \mathrm{H}^{1/2}([\Gamma]).$$

7. Boundary Value Problems

Once again, we come back to general multi-screens, and continue the construction of our framework, introducing concepts better adapted to boundary value problems set in the exterior of such objects. We first need to introduce generalizations of usual trace operators.

7.1. Dirichlet and Neumann Trace Operators

Let $\mathrm{H}^1(\Delta, \mathbb{R}^d \setminus \overline{\Gamma}) = \{u \in \mathrm{H}^1(\mathbb{R}^d \setminus \overline{\Gamma}) \mid \nabla u \in \mathrm{H}(\mathrm{div}, \mathbb{R}^d \setminus \overline{\Gamma})\ \}$ and denote $\mathrm{H}^1_{\mathrm{loc}}(\Delta, \mathbb{R}^d \setminus \overline{\Gamma}) = \{u \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^d) \mid \varphi\, u \in \mathrm{H}^1(\Delta, \mathbb{R}^d \setminus \overline{\Gamma})\ \ \forall \varphi \in \mathscr{D}(\mathbb{R}^d \setminus \overline{\Gamma})\ \}$. For any element of this space we can define its Dirichlet and Neumann traces on Γ in the following manner

$$\gamma_{\mathrm{D}}(u) = \pi_{\mathrm{D}}(u) \qquad \text{and} \qquad \gamma_{\mathrm{N}}(u) = \pi_{\mathrm{N}}(\nabla u). \tag{7.1}$$

Clearly $\gamma_{\mathrm{D}} : \mathrm{H}^1_{\mathrm{loc}}(\Delta, \mathbb{R}^d \setminus \overline{\Gamma}) \to \mathbb{H}^{1/2}(\Gamma)$ and $\gamma_{\mathrm{N}} : \mathrm{H}^1_{\mathrm{loc}}(\Delta, \mathbb{R}^d \setminus \overline{\Gamma}) \to \mathbb{H}^{-1/2}(\Gamma)$ are continuous maps. Besides, if $u \in \mathrm{H}^1_{\mathrm{loc}}(\Delta, \mathbb{R}^d \setminus \overline{\Gamma})$ and $v \in \mathrm{H}^1_{\mathrm{loc}}(\Delta, \mathbb{R}^d \setminus \overline{\Gamma})$ coincide in a neighbourhood of Γ, then $\gamma_{\mathrm{D}}(u) = \gamma_{\mathrm{D}}(v)$ and $\gamma_{\mathrm{N}}(u) = \gamma_{\mathrm{N}}(v)$.

Lemma 7.1. *The trace operators $\gamma_{\mathrm{D}}, \gamma_{\mathrm{N}}$ both admit a continuous right-inverse.*

Proof. For any $\dot{u} \in \mathbb{H}^{1/2}(\Gamma)$, define $\mathrm{S}_{\mathrm{D}}(\dot{u})$ as the unique element of $\mathrm{H}^1(\mathbb{R}^d \setminus \overline{\Gamma})$ satisfying $\pi_{\mathrm{D}}\big(\,\mathrm{S}_{\mathrm{D}}(\dot{u})\,\big) = \dot{u}$ and $\|\mathrm{S}_{\mathrm{D}}(\dot{u})\|_{\mathrm{H}^1(\mathbb{R}^d\setminus\overline{\Gamma})} = \|\dot{u}\|_{\mathbb{H}^{1/2}(\Gamma)}$. As pointed out in the proof of Proposition 5.1, we have $-\Delta \mathrm{S}_{\mathrm{D}}(\dot{u}) + \mathrm{S}_{\mathrm{D}}(\dot{u}) = 0$ in $\mathbb{R}^d \setminus \overline{\Gamma}$. As a consequence, $\mathrm{S}_{\mathrm{D}} : \mathbb{H}^{1/2}(\Gamma) \to \mathrm{H}^1_{\mathrm{loc}}(\Delta, \mathbb{R}^d \setminus \overline{\Gamma})$ is a continuous right-inverse for γ_{D}.

Similarly, for $\dot{p} \in \mathbb{H}^{-1/2}(\Gamma)$, define $\mathrm{S}_{\mathrm{N}}(\dot{p})$ as the unique element of $\mathrm{H}(\mathrm{div}, \mathbb{R}^d \setminus \overline{\Gamma})$ satisfying $\pi_{\mathrm{N}}\big(\,\mathrm{S}_{\mathrm{N}}(\dot{p})\,\big) = \dot{p}$ and $\|\mathrm{S}_{\mathrm{N}}(\dot{p})\|_{\mathrm{H}(\mathrm{div},\mathbb{R}^d\setminus\overline{\Gamma})} = \|\dot{p}\|_{\mathbb{H}^{-1/2}(\Gamma)}$. We have $-\nabla \mathrm{div}\big(\,\mathrm{S}_{\mathrm{N}}(\dot{p})\,\big) + \mathrm{S}_{\mathrm{N}}(\dot{p}) = 0$ in $\mathbb{R}^d \setminus \overline{\Gamma}$, and we see $v := \mathrm{div}\big(\,\mathrm{S}_{\mathrm{N}}(\dot{p})\,\big) \in$

$\mathrm{H}^1_{\mathrm{loc}}(\Delta,\mathbb{R}^d\setminus\overline{\Gamma})$. Obviously, $\dot{p} = \gamma_{\mathrm{N}}(v)$, so that $\mathrm{div}\big(\mathrm{S}_{\mathrm{N}}(\cdot)\big) : \mathbb{H}^{-1/2}(\Gamma) \to \mathrm{H}^1_{\mathrm{loc}}(\Delta,\mathbb{R}^d\setminus\overline{\Gamma})$ is a continuous right-inverse for γ_{N}. $\square$

Note that we may also consider the operator $[\gamma_{\mathrm{D}}] : \mathrm{H}^1_{\mathrm{loc}}(\Delta,\mathbb{R}^d\setminus\overline{\Gamma}) \to \widetilde{\mathrm{H}}^{1/2}([\Gamma])$ as well as $[\gamma_{\mathrm{N}}] : \mathrm{H}^1_{\mathrm{loc}}(\Delta,\mathbb{R}^d\setminus\overline{\Gamma}) \to \widetilde{\mathrm{H}}^{-1/2}([\Gamma])$ obtained by composing the Dirichlet and Neumann traces with the jump operators described at § 6.2. An interesting identity is obtained by applying twice Formula (5.2). This yields a generalization of the second Green Formula,

$$\int_{\mathbb{R}^d} u\Delta v - v\Delta u \, d\mathbf{x} = \int_{[\Gamma]} \gamma_{\mathrm{D}}(u)\gamma_{\mathrm{N}}(v) - \gamma_{\mathrm{D}}(v)\gamma_{\mathrm{N}}(u) d\sigma$$

$$\forall u, v \in \mathrm{H}^1(\Delta,\mathbb{R}^d\setminus\overline{\Gamma}). \tag{7.2}$$

It is possible to consider boundary value problems with Dirichlet or Neumann condition on Γ prescribed by means of γ_{D} and γ_{N}.

Proposition 7.2 (Exterior Dirichlet problem). *Suppose that Γ is multi-screen in the sense of Definition 2.3, and that $\mathbb{R}^d\setminus\overline{\Gamma}$ is connected. Take $g \in \mathbb{H}^{1/2}(\Gamma)$ and $\kappa \in \mathbb{R}_+\setminus\{0\}$. Then there exists a unique $u \in \mathrm{H}^1_{\mathrm{loc}}(\Delta,\mathbb{R}^d\setminus\overline{\Gamma})$ satisfying the following equations*

$$-\Delta u - \kappa^2 u = 0 \quad \textit{in } \mathbb{R}^d\setminus\overline{\Gamma} \qquad \gamma_{\mathrm{D}}(u) = g \quad \textit{and} \quad u \textit{ is outgoing.} \tag{7.3}$$

Moreover, if we denote $\mathrm{S} : \mathbb{H}^{1/2}(\Gamma) \to \mathrm{H}^1_{\mathrm{loc}}(\Delta,\mathbb{R}^d\setminus\overline{\Gamma})$ the operator mapping any $g \in \mathbb{H}^{1/2}(\Gamma)$ to the unique solution to (7.3), then S is continuous.

Proof. Let Ω be an open ball with radius large enough to guarantee $\overline{\Gamma} \subset \Omega$. Let $\mathrm{T} : \mathrm{H}^{1/2}(\partial\Omega) \to \mathrm{H}^{-1/2}(\partial\Omega)$ be the exterior Steklov-Poincaré map associated to the homogeneous Helmholtz equation in $\mathbb{R}^d\setminus\overline{\Omega}$. Let $u_g \in \mathrm{H}^1(\mathbb{R}^d\setminus\overline{\Gamma})$ satisfy $\gamma_{\mathrm{D}}(u_g) = g$. This element u_g can be chosen so as to guarantee that $g \mapsto u_g$ is continuous from $\mathbb{H}^{1/2}(\Gamma)$ to $\mathrm{H}^1(\mathbb{R}^d\setminus\overline{\Gamma})$ according to Lemma 7.1. Using Green's formula, Problem (7.3) can be reformulated as

Find $u \in \mathrm{H}^1_{0,\Gamma}(\Omega)$ such that

$$(\mathrm{A}u, v)_{\mathrm{H}^1(\Omega^d\setminus\overline{\Gamma})} = -(\mathrm{A}u_g, v)_{\mathrm{H}^1(\Omega^d\setminus\overline{\Gamma})} \quad \forall v \in \mathrm{H}^1_{0,\Gamma}(\Omega)$$

$$\text{where} \quad (\mathrm{A}u, v)_{\mathrm{H}^1(\Omega^d\setminus\overline{\Gamma})} = \int_{\Omega\setminus\overline{\Gamma}} \nabla u\cdot\nabla\overline{v} - \kappa^2 u\,\overline{v}\, d\mathbf{x} + \int_{\partial\Omega} \overline{v}\mathrm{T}u\, d\sigma$$

To prove the desired result, it suffices to show that A is a continuous isomorphism. Using the compactness result of Proposition (4.1), one can check by means of classical arguments that A is a Fredholm operator with index 0. As a consequence, proving that A is a continuous isomorphism boils down to showing that, when $g = 0$, the only solution to (7.3) is $u = 0$.

Now assume that $u \in \mathrm{H}^1_{\mathrm{loc}}(\mathbb{R}^d\setminus\overline{\Gamma})$ satisfies (7.3) with $g = 0$. For any $\rho > 0$, let $\mathcal{B}_\rho$ refer to the ball centered at 0 with radius ρ, and denote $\mathbf{n}_\rho$ the unit normal vector to $\partial\mathcal{B}_\rho$ directed toward the exterior of $\mathcal{B}_\rho$. Applying (5.2) in $\mathcal{B}_\rho\setminus\overline{\Gamma}$ with $\mathbf{p} = \nabla\overline{u}$, and taking into account that $\gamma_{\mathrm{D}}(u) = 0$, we obtain

$$\int_{\partial\mathcal{B}_\rho} u\,\mathbf{n}_\rho\cdot\nabla\overline{u}\,d\sigma_\rho = \int_{\mathcal{B}_\rho\setminus\overline{\Gamma}} |\nabla u|^2 - \kappa^2|u|^2 d\mathbf{x}$$

where $d\sigma_\rho$ refers to the surface Lebesgue-measure on $\partial\mathcal{B}_\rho$. Since the right hand side in the identity above is real, and according to Sommerfeld's radiation condition, we have

$$\kappa\int_{\partial\mathcal{B}_\rho} |u|^2\,d\sigma_\rho = \Im m\left\{\int_{\partial\mathcal{B}_\rho} u\,(i\kappa\,\overline{u} - \mathbf{n}_\rho\cdot\nabla\overline{u})\,d\sigma_\rho\right\}$$
$$\Longrightarrow\quad \lim_{\rho\to\infty}\|u\|^2_{\mathrm{L}^2(\partial\mathcal{B}_\rho)} \leq \frac{1}{\kappa^2}\lim_{\rho\to\infty}\|\,i\kappa u - \mathbf{n}_\rho\cdot\nabla u\,\|^2_{\mathrm{L}^2(\partial\mathcal{B}_\rho)} = 0.$$

It follows by Rellich's Theorem (see e.g. Müller [17]) that u vanishes on a neighbourhood of infinity. Using an analytic continuation theorem, since $\mathbb{R}^d\setminus\overline{\Gamma}$ is connected, we obtain that $u = 0$ in $\mathbb{R}^d\setminus\overline{\Gamma}$. □

7.2. Density Results

Generalizing results by Costabel [8], in this subsection we will prove density theorem that will be useful for the study of boundary integral operators in the next section.

Proposition 7.3. *Consider the continuous operator* $\gamma : \mathrm{H}^1_{\mathrm{loc}}(\Delta,\mathbb{R}^d\setminus\overline{\Gamma}) \to \mathbb{H}^{+1/2}(\Gamma)\times\mathbb{H}^{-1/2}(\Gamma)$ *defined by* $\gamma(\varphi) = (\gamma_{\mathrm{D}}(\varphi),\gamma_{\mathrm{N}}(\varphi))$ *for all* $\varphi\in\mathrm{H}^1_{\mathrm{loc}}(\Delta,\mathbb{R}^d\setminus\overline{\Gamma})$. *The range of* γ *is dense in* $\mathbb{H}^{+1/2}(\Gamma)\times\mathbb{H}^{-1/2}(\Gamma)$.

Proof. Note that $(\gamma_{\mathrm{D}},\gamma_{\mathrm{N}})$ induce a map from $\mathrm{H}^1(\Delta,\mathbb{R}^d\setminus\overline{\Gamma})\times\mathrm{H}^1(\Delta,\mathbb{R}^d\setminus\overline{\Gamma})$ to $\mathbb{H}^{1/2}(\Gamma)\times\mathbb{H}^{-1/2}(\Gamma)$ that is continuous. Consider the pairing defined by

$$\big((u,p),(v,q)\big)\ \mapsto \int_{[\Gamma]} uq\,d\sigma - \int_{[\Gamma]} vp\,d\sigma \quad \forall u,v\in\mathbb{H}^{\frac{1}{2}}(\Gamma)\ \ \forall p,q\in\mathbb{H}^{-\frac{1}{2}}(\Gamma). \tag{7.4}$$

According to Proposition 5.1, the space $\mathbb{H}^{1/2}(\Gamma)\times\mathbb{H}^{-1/2}(\Gamma)$ is dual to itself under the pairing (7.4). Hence, according to Hahn-Banach's Theorem, it suffices to show that

$$\int_{[\Gamma]} u\gamma_{\mathrm{N}}(v)d\sigma = \int_{[\Gamma]} p\,\gamma_{\mathrm{D}}(v)d\sigma \quad \forall v\in\mathrm{H}^1(\Delta,\mathbb{R}^d\setminus\overline{\Gamma})$$
$$\Longrightarrow\quad u = 0,\ p = 0. \tag{7.5}$$

Take $(u,p)\in\mathbb{H}^{1/2}(\Gamma)\times\mathbb{H}^{-1/2}(\Gamma)$ satisfying the condition in the left-hand side of (7.5). For any $f\in\mathrm{L}^2(\mathbb{R}^d)$ with compact support, denote $\mathcal{S}(f)$ the unique element of $\mathrm{H}^1(\Delta,\mathbb{R}^d\setminus\overline{\Gamma})$ satisfying the equations

$$-\Delta\mathcal{S}(f) + \mathcal{S}(f) = f \quad \text{in } \mathbb{R}^d\setminus\overline{\Gamma}, \qquad \gamma_{\mathrm{D}}\big(\mathcal{S}(f)\big) = 0 \ \text{ on } \ \Gamma.$$

Using Proposition 7.2, it is straightforward to check that $\mathcal{S}(f)$ is properly defined. Denote also $\mathrm{S}_{\mathrm{D}}(u)$ the unique element of $\mathrm{H}^1(\mathbb{R}^d\setminus\overline{\Gamma})$ satisfying $\pi_{\mathrm{D}}\big(\mathrm{S}_{\mathrm{D}}(u)\big) = \gamma_{\mathrm{D}}\big(\mathrm{S}_{\mathrm{D}}(u)\big) = u$ and $\|\mathrm{S}_{\mathrm{D}}(u)\|_{\mathrm{H}^1(\mathbb{R}^d\setminus\overline{\Gamma})} = \|u\|_{\mathrm{H}^{1/2}(\Gamma)}$. As was pointed out in the proof of Proposition 5.1, we have $-\Delta \mathrm{S}_{\mathrm{D}}(u)+\mathrm{S}_{\mathrm{D}}(u) = 0$ in $\mathbb{R}^d \setminus \overline{\Gamma}$. Hence

$$\begin{aligned}
0 &= \int_{[\Gamma]} p\,\gamma_{\mathrm{D}}\big(\mathcal{S}(f)\big)\,d\sigma = \int_{[\Gamma]} \gamma_{\mathrm{D}}\big(\mathrm{S}_{\mathrm{D}}(u)\big)\,\gamma_{\mathrm{N}}\big(\mathcal{S}(f)\big)\,d\sigma \\
&= \int_{[\Gamma]} \gamma_{\mathrm{D}}\big(\mathrm{S}_{\mathrm{D}}(u)\big)\,\gamma_{\mathrm{N}}\big(\mathcal{S}(f)\big) - \gamma_{\mathrm{N}}\big(\mathrm{S}_{\mathrm{D}}(u)\big)\,\gamma_{\mathrm{D}}\big(\mathcal{S}(f)\big)\,d\sigma \\
&= \int_{\mathbb{R}^d\setminus\overline{\Gamma}} \mathrm{S}_{\mathrm{D}}(u)\,\Delta\mathcal{S}(f) - \mathcal{S}(f)\Delta\mathrm{S}_{\mathrm{D}}(u)\,d\sigma = -\int_{\mathbb{R}^d\setminus\overline{\Gamma}} f\,\mathrm{S}_{\mathrm{D}}(u)\,d\mathbf{x}
\end{aligned}$$

Since this holds for any $f \in \mathrm{L}^2(\mathbb{R}^d)$ with compact support, this implies that $\mathrm{S}_{\mathrm{D}}(u) = 0$. Hence $u = \gamma_{\mathrm{D}}(\mathrm{S}_{\mathrm{D}}(u)) = 0$. As a consequence $\int_{[\Gamma]} p\,\gamma_{\mathrm{D}}(v)d\sigma = 0$ for any $v \in \mathrm{H}^1(\Delta, \mathbb{R}^d \setminus \overline{\Gamma})$, and since $\gamma_{\mathrm{D}} : \mathrm{H}^1(\Delta, \mathbb{R}^d \setminus \overline{\Gamma}) \to \mathbb{H}^{1/2}(\Gamma)$ is onto, this finally implies $p = 0$. □

One may wonder if a result comparable to the previous proposition holds for single-trace spaces. The answer is positive but, to prove it, we first need an intermediary result.

Lemma 7.4. *Assume that $\Gamma = \cup_{j=0\dots n}\partial\Omega_j$ is the skeleton of some Lipschitz partition $(\Omega_j)_{j=0\dots n}$ of $\mathbb{R}^d$. Consider the operator $\gamma : \mathrm{H}^1_{\mathrm{loc}}(\Delta, \mathbb{R}^d \setminus \overline{\Gamma}) \to \mathbb{H}^{+1/2}(\Gamma) \times \mathbb{H}^{-1/2}(\Gamma)$. The range of γ restricted to $\mathrm{H}^1_{\mathrm{loc}}(\Delta, \mathbb{R}^d)$ is dense in $\mathrm{H}^{+1/2}([\Gamma]) \times \mathrm{H}^{-1/2}([\Gamma])$.*

Proof. First of all, according to Proposition 5.1, and the Hahn-Banach theorem, it suffices to show that if $(\dot{u}, \dot{p}) \in \mathbb{H}^{1/2}(\Gamma) \times \mathbb{H}^{-1/2}(\Gamma)$ satisfies $\int_{[\Gamma]} \dot{u}\,\gamma_{\mathrm{N}}(\varphi) - \dot{p}\,\gamma_{\mathrm{D}}(\varphi)\,d\sigma = 0\ \forall\varphi \in \mathrm{H}^1(\Delta, \mathbb{R}^d)$, then

$$\int_{[\Gamma]} \dot{u}\,\dot{q} - \dot{p}\,\dot{v}\,d\sigma = 0 \quad \forall(\dot{v}, \dot{q}) \in \mathrm{H}^{1/2}([\Gamma]) \times \mathrm{H}^{-1/2}([\Gamma]) \tag{7.6}$$

which is equivalent to $(\dot{u}, \dot{p}) \in \mathrm{H}^{1/2}([\Gamma]) \times \mathrm{H}^{-1/2}([\Gamma])$ according to Proposition 6.3. Hence, let us consider such a pair $(\dot{u}, \dot{p}) \in \mathbb{H}^{1/2}(\Gamma) \times \mathbb{H}^{-1/2}(\Gamma)$.

For each Ω_j, let us denote $\gamma^j_{\mathrm{D}}(\varphi) = \varphi|_{\partial\Omega_j}$ and $\gamma^j_{\mathrm{N}}(\varphi) = \mathbf{n}_j \cdot \nabla\varphi|_{\partial\Omega_j}$, $\forall\varphi \in \mathrm{H}^1(\Delta, \overline{\Omega}_j)$, where the traces are taken from the interior of Ω_j, and $\mathbf{n}_j$ refers to the normal vector to $\partial\Omega_j$ directed toward the exterior of Ω_j. According to (6.1), the space $\mathrm{H}^{+1/2}([\Gamma]) \times \mathrm{H}^{-1/2}([\Gamma])$ can be identified with the space

$$\mathbb{X}(\Gamma) = \Big\{ \Big(\gamma^j_{\mathrm{D}}(v), \gamma^j_{\mathrm{N}}(q)\Big)_{j=0}^{n} \;|\; (v, q) \in \mathrm{H}^1(\Delta, \mathbb{R}^d)^2 \Big\}.$$

Setting $\gamma^j(\varphi) = (\gamma^j_{\mathrm{D}}(\varphi), \gamma^j_{\mathrm{N}}(\varphi))$, we also consider the space $\mathcal{C}(\Gamma) = \{(\gamma^j(\varphi))_{j=0}^n \mid \varphi \in \mathrm{H}^1(\Delta, \mathbb{R}^d \setminus \overline{\Gamma})$ and $-\Delta\varphi + \varphi = 0$ in $\Omega_j, j = 0 \dots n\}$. Then according to [6, Prop. 6.1], we have $\mathbb{X}(\Gamma) \oplus \mathcal{C}(\Gamma) = \mathbb{H}^{1/2}(\Gamma) \times \mathbb{H}^{-1/2}(\Gamma)$. As a consequence there exists $u, p \in \mathrm{H}^1(\Delta, \mathbb{R}^d)$ and functions $\psi_j \in \mathrm{H}^1(\Delta, \overline{\Omega}_j)$ with $\Delta\psi_j = \psi_j$, such that $\dot{u} = (\gamma^j_{\mathrm{D}}(u) + \gamma^j_{\mathrm{D}}(\psi_j))_{j=0\dots n}$ and $\dot{p} = (\gamma^j_{\mathrm{N}}(p) + \gamma^j_{\mathrm{N}}(\psi_j))_{j=0\dots n}$. To finish the proof, it suffices to show that $\psi_j = 0, j = 0 \dots n$.

According to [5, Prop. 2.1] (that admits Proposition 6.3 as a generalization), we have

$$
\begin{aligned}
0 &= \int_{[\Gamma]} \dot{u}\gamma_{\mathrm{N}}(\varphi) - \dot{p}\gamma_{\mathrm{D}}(\varphi)\, d\sigma \\
&= \sum_{j=0}^{n} \int_{\partial\Omega_j} \gamma_{\mathrm{D}}(u+\psi_j)\gamma_{\mathrm{N}}(\varphi) - \gamma_{\mathrm{N}}(p+\psi_j)\gamma_{\mathrm{D}}(\varphi)\, d\sigma \\
&= \sum_{j=0}^{n} \int_{\partial\Omega_j} \gamma_{\mathrm{D}}(\psi_j)\gamma_{\mathrm{N}}(\varphi) - \gamma_{\mathrm{N}}(\psi_j)\gamma_{\mathrm{D}}(\varphi)\, d\sigma = \sum_{j=0}^{n} \int_{\Omega_j} \psi_j \Delta\varphi - \varphi\Delta\psi_j \, d\mathbf{x} \\
&= \sum_{j=0}^{n} \int_{\Omega_j} \psi_j(\Delta\varphi - \varphi)\, d\mathbf{x} = \int_{\mathbb{R}^d} \psi(\Delta\varphi - \varphi)\, d\mathbf{x} \qquad \forall \varphi \in \mathrm{H}^1(\Delta, \mathbb{R}^d)
\end{aligned}
$$

where we define $\psi \in \mathrm{L}^2(\mathbb{R}^d)$ by $\psi|_{\Omega_j} = \psi_j$. Now, for any $f \in \mathscr{D}(\mathbb{R}^d)$, there exists $\varphi \in \mathrm{H}^1(\Delta, \mathbb{R}^d)$ such that $-\Delta\varphi + \varphi = f$ in $\mathbb{R}^d$. From this we deduce that $\int_{\mathbb{R}^d} \psi f d\mathbf{x} = 0$ for all $f \in \mathscr{D}(\mathbb{R}^d)$, which implies that $\psi = 0$. □

Proposition 7.5. *In the case where* Γ *is any multi-screen (not necessarily the skeleton of some Lipschitz partition), consider the continuous operator* $\gamma : \mathrm{H}^1_{\mathrm{loc}}(\Delta, \mathbb{R}^d \setminus \overline{\Gamma}) \to \mathbb{H}^{+1/2}([\Gamma]) \times \mathbb{H}^{-1/2}([\Gamma])$. *The range of* γ *restricted to* $\mathrm{H}^1_{\mathrm{loc}}(\Delta, \mathbb{R}^d)$ *is dense in the space* $\mathrm{H}^{+1/2}([\Gamma]) \times \mathrm{H}^{-1/2}([\Gamma])$.

Proof. Take a $\dot{u} \in \mathrm{H}^{1/2}([\Gamma])$ and $\dot{p} \in \mathrm{H}^{-1/2}([\Gamma])$ such that $\dot{u} = \pi_{\mathrm{N}}(u)$ and $\dot{p} = \pi_{\mathrm{N}}(\mathbf{p})$ for some $u \in \mathrm{H}^1_\star(\mathbb{R}^d) = \{v \in \mathrm{H}^1(\mathbb{R}^d) \mid v = 0 \text{ in a neighbourhood of } \partial\Gamma\}$ and some $\mathbf{p} \in \mathrm{H}_\star(\mathrm{div}, \mathbb{R}^d) = \{\mathbf{s} \in \mathrm{H}(\mathrm{div}, \mathbb{R}^d) \mid \mathbf{s} = 0 \text{ in a neighbourhood of } \partial\Gamma\}$. Take a Lipschitz partition $\mathbb{R}^d = \cup_{k=0}^K \overline{\Omega}_k$ like in Definition 2.3, and set $\Sigma = \cup_{k=0}^K \partial\Omega_k$. Since u and $\mathbf{p}$ vanish in a neighbourhood of $\partial\Gamma$, it may be assumed that u and $\mathbf{p}$ vanish on $\Sigma \setminus \overline{\Gamma}$, using some adapted cut-off function if necessary.

Since $\Gamma \subset \Sigma$, using extension by 0, the traces $\dot{u}$ and $\dot{p}$ can be considered as single-traces on Σ i.e. $\dot{u} \in \mathrm{H}^{1/2}([\Sigma])$ and $\dot{p} \in \mathrm{H}^{-1/2}([\Sigma])$. Let us denote $\gamma_{\mathrm{D}}^{\Sigma}, \gamma_{\mathrm{N}}^{\Sigma}$ the trace operators on Σ, as defined by (7.1) but considering Σ instead of Γ. According to Proposition 7.4, there exists a sequence $\xi_n \in \mathrm{H}^1_{\mathrm{loc}}(\Delta, \mathbb{R}^d)$ such that

$$
\lim_{n\to+\infty} \Big(\|\dot{u} - \gamma_{\mathrm{D}}^{\Sigma}(\xi_n)\|^2_{\mathbb{H}^{+\frac{1}{2}}(\Sigma)} + \|\dot{p} - \gamma_{\mathrm{N}}^{\Sigma}(\xi_n)\|^2_{\mathbb{H}^{-\frac{1}{2}}(\Sigma)} \Big) = 0. \tag{7.7}
$$

Using a cut-off function if necessary, we can assume that $\mathrm{supp}(\xi_n) \cap (\Sigma \setminus \overline{\Gamma}) = \emptyset$, so that the traces of ξ_n on Σ and Γ coincide. As a consequence, (7.7) actually holds with Σ replaced by Γ, and $\gamma_{\mathrm{D}}^{\Sigma}, \gamma_{\mathrm{N}}^{\Sigma}$ replaced by $\gamma_{\mathrm{D}}, \gamma_{\mathrm{N}}$. This concludes the proof for the case where $\dot{u} \in \pi_{\mathrm{D}}\big(\mathrm{H}^1_\star(\mathbb{R}^d)\big)$ and $\dot{p} \in \pi_{\mathrm{N}}\big(\mathrm{H}_\star(\mathrm{div}, \mathbb{R}^d)\big)$. It only remains to observe that $\mathrm{H}^1_\star(\mathbb{R}^d)$ and $\mathrm{H}_\star(\mathrm{div}, \mathbb{R}^d)$ are dense in $\mathrm{H}^1(\mathbb{R}^d)$ and $\mathrm{H}(\mathrm{div}, \mathbb{R}^d)$ according to Proposition 8.11 below. So the proof is complete. □

8. Potential Operators

As we have an adapted functional framework at hand, we can now build potential operators for scattering by multi-screens. We will adapt proofs contained in [8], relying on the trace spaces and operators that we introduced before.

In the sequel, we will study boundary integral formulations to scalar wave propagation problems around a screen Γ. To simplify our presentation, in the remaining of this document, we make the following assumption

Assumption: $\Gamma \subset \mathbb{R}^d$ is a Lipschitz multi-screen such that $\mathbb{R}^d \setminus \overline{\Gamma}$ is connected

Note that this assumption rules out the case where Γ would be the skeleton fo some Lipschitz partition of $\mathbb{R}^d$.

The forthcoming analysis could be carried out without this connectedness assumption. However this hypothesis will help making the presentation clearer. Moreover, the results that we present below could be generalized to any strongly elliptic partial differential operator, following a presentation similar to [16]. However we focus on Helmholtz equation for the sake of simplicity.

Let $\mathscr{G}_\kappa(\mathbf{x})$ refer to the outgoing Green kernel for the Helmholtz operator, i.e. it satisfies $(-\Delta - \kappa^2)\mathscr{G}_\kappa = \delta_0$ in $\mathbb{R}^d$ in the sense of distributions. Consider some $\mathbf{x} \in \mathbb{R}^d \setminus \overline{\Gamma}$, and observe that the function $\mathscr{G}_{\kappa,\mathbf{x}} : \mathbf{y} \mapsto \mathscr{G}_\kappa(\mathbf{x} - \mathbf{y})$ is C^∞ in the neighbourhood of Γ. Thus, using a cut-off function if necessary (so as to remove the singularity of $\mathscr{G}_{\kappa,\mathbf{x}}(\mathbf{y})$ at $\mathbf{y} = \mathbf{x}$) we may consider the following operators, named respectively single layer and double layer potential,

$$\begin{aligned} \mathrm{SL}_\kappa(\dot{q})(\mathbf{x}) &:= \int_{[\Gamma]} \gamma_{\mathrm{D}}(\mathscr{G}_{\kappa,\mathbf{x}})\, \dot{q}\, d\sigma \qquad \forall \dot{q} \in \mathbb{H}^{-\frac{1}{2}}(\Gamma) \\ \mathrm{DL}_\kappa(\dot{v})(\mathbf{x}) &:= -\int_{[\Gamma]} \gamma_{\mathrm{N}}(\mathscr{G}_{\kappa,\mathbf{x}})\, \dot{v}\, d\sigma \quad \forall \dot{v} \in \mathbb{H}^{+\frac{1}{2}}(\Gamma). \end{aligned} \tag{8.1}$$

Clearly $\mathrm{SL}_\kappa : \mathbb{H}^{-1/2}(\Gamma) \to \mathrm{C}^\infty(\mathbb{R}^d \setminus \overline{\Gamma})$ and $\mathrm{DL}_\kappa : \mathbb{H}^{1/2}(\Gamma) \to \mathrm{C}^\infty(\mathbb{R}^d \setminus \overline{\Gamma})$ since, if $U, V \subset \mathbb{R}^d$ are two bounded open sets such that $\overline{\Gamma} \subset V$ and $U \cap V = \emptyset$, the function $\mathbf{x} \mapsto \mathscr{G}_{\kappa,\mathbf{x}}$, $\mathbf{x} \in U$, is a smooth function valued in $\mathrm{H}^1(\Delta, V \setminus \overline{\Gamma})$.

8.1. Representation Formula

Following [8, 16], we may write the expression of the potential operators (8.1) in a manner that is more convenient for calculus in the sense of distributions. Denote $\mathscr{G}_\kappa * : \mathrm{C}^\infty(\mathbb{R}^d)' \to \mathscr{D}(\mathbb{R}^d)'$ the operation of convolution (in the sense of distributions if necessary) with kernel $\mathscr{G}_\kappa$. Let $\gamma_{\mathrm{D}}' : \mathbb{H}^{-1/2}(\Gamma) \to \mathrm{H}^1_{\mathrm{loc}}(\mathbb{R}^d \setminus \overline{\Gamma})'$ and $\gamma_{\mathrm{N}}' : \mathbb{H}^{+1/2}(\Gamma) \to \mathrm{H}^1_{\mathrm{loc}}(\Delta, \mathbb{R}^d \setminus \overline{\Gamma})'$ refer to the adjoints of γ_{D} and γ_{N}. Then we have

$$\mathrm{SL}_\kappa = \mathscr{G}_\kappa * \gamma_{\mathrm{D}}' \quad \text{and} \quad \mathrm{DL}_\kappa = -\mathscr{G}_\kappa * \gamma_{\mathrm{N}}'. \tag{8.2}$$

Take a function $u \in \mathrm{H}^1(\mathbb{R}^d \setminus \overline{\Gamma})$ and assume in addition that $\mathrm{supp}(u)$ is bounded. Consider identity (7.2). Choosing v in $\mathscr{D}(\mathbb{R}^d)$, we can interpret this identity in the sense of distributions, using the adjoint of the trace operators, which yields

$$(\Delta u)|_{\mathbb{R}^d} = (\Delta u)|_{\mathbb{R}^d\setminus\overline{\Gamma}} \; + \; \gamma_{\mathrm{N}}' \cdot \gamma_{\mathrm{D}}(u) - \gamma_{\mathrm{D}}' \cdot \gamma_{\mathrm{N}}(u)$$

where $\gamma_{\mathrm{N}}' \cdot \gamma_{\mathrm{D}}(u)$ and $\gamma_{\mathrm{D}}' \cdot \gamma_{\mathrm{N}}(u)$ are distributions supported in $\overline{\Gamma}$. Now, since $\mathrm{supp}(u)$ is bounded, we can convolve the previous identity with the Green kernel, which yields the following result.

Lemma 8.1. *For any $u \in \mathrm{H}^1(\Delta, \mathbb{R}^d \setminus \overline{\Gamma})$ with bounded support, if $f = -\Delta u - \kappa^2 u$ in the sense of distributions in $\mathbb{R}^d \setminus \overline{\Gamma}$, we have the following formula*

$$u = \mathscr{G}_\kappa * f + \mathrm{SL}_\kappa \cdot \gamma_{\mathrm{N}}(u) + \mathrm{DL}_\kappa \cdot \gamma_{\mathrm{D}}(u) \quad \textit{in } \; \mathbb{R}^d \setminus \overline{\Gamma}. \tag{8.3}$$

Identity (8.3) is a representation formula, in the parlance of boundary integral equations. Although we have established it in the case where $\mathrm{supp}(u)$ is bounded, it actually also holds in the case where u is outgoing radiating.

Proposition 8.2. *Assume that $u \in \mathrm{H}^1_{\mathrm{loc}}(\Delta, \mathbb{R}^d \setminus \overline{\Gamma})$ satisfies Sommerfeld's radiation condition. Define $f \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^d)$ by $f = -\Delta u - \kappa^2 u$ in the sense of distributions in $\mathbb{R}^d \setminus \overline{\Gamma}$, and suppose in addition that f has bounded support. Then formula (8.3) still holds.*

Proof. Consider a C^∞ cut-off function $\chi : \mathbb{R}^d \to \mathbb{R}_+$ such that $\chi(\mathbf{x}) = 1$ for $\mathbf{x} \in \mathrm{supp}(f)$, and χ is compactly supported. The function $u\chi$ has compact support so we can apply Lemma 8.1,

$$\begin{aligned} \chi u = {} & \mathscr{G}_\kappa * f + \mathrm{SL}_\kappa \cdot \gamma_{\mathrm{N}}(u) + \mathrm{DL}_\kappa \cdot \gamma_{\mathrm{D}}(u) \\ & - \mathscr{G}_\kappa * (u\Delta\chi + 2\,\nabla u \cdot \nabla\chi) \end{aligned} \tag{8.4}$$

Set $\psi := 1 - \chi$ and observe that ψu satisfies Sommerfeld's radiation condition. In addition, since $\Delta u + \kappa^2 u = 0$ in $\mathrm{supp}(\psi)$, and $\nabla\psi = -\nabla\chi$, we have $(-\Delta - \kappa^2)(\psi u) = g$ in $\mathbb{R}^d$ where $g = u\Delta\chi + 2\nabla u \cdot \nabla\chi$ has compact support. As a consequence we can apply standard representation theorems based on Newton potential [20, Thm 3.1.4] to conclude that

$$\psi u = u - \chi u = \mathscr{G}_\kappa * g = \mathscr{G}_\kappa * (u\Delta\chi + 2\,\nabla u \cdot \nabla\chi) \quad \text{in } \; \mathbb{R}^d. \tag{8.5}$$

Combining (8.4) and (8.5) leads directly to an expression of u, which concludes the proof. ☐

Proposition 8.2 extends [16, Thm 6.10] to problems set in domains containing multi-screens. Now let us study the continuity properties of the potential operators SL_κ and DL_κ.

Proposition 8.3 (Continuity of single layer potential). *The potential operator SL_κ continuously maps $\mathbb{H}^{-1/2}(\Gamma)$ into $\mathrm{H}^1_{\mathrm{loc}}(\Delta, \mathbb{R}^d \setminus \overline{\Gamma}) \cap \mathrm{H}^1_{\mathrm{loc}}(\mathbb{R}^d)$.*

Proof. First of all, since $\mathrm{H}^1_{\mathrm{loc}}(\mathbb{R}^d) \subset \mathrm{H}^1_{\mathrm{loc}}(\mathbb{R}^d \setminus \overline{\Gamma})$, the space $\mathrm{H}^1_{\mathrm{loc}}(\mathbb{R}^d \setminus \overline{\Gamma})'$ is continuously embedded into $\mathrm{H}^1_{\mathrm{loc}}(\mathbb{R}^d)'$. Hence $\gamma_{\mathrm{D}}' : \mathbb{H}^{-1/2}(\Gamma) \to \mathrm{H}^1_{\mathrm{loc}}(\mathbb{R}^d)'$ is continuous. Besides $\mathscr{G}_\kappa *$ is a pseudodifferential operator of order -2 on $\mathbb{R}^d$, mapping $\mathrm{H}^1_{\mathrm{loc}}(\mathbb{R}^d)' \to \mathrm{H}^1_{\mathrm{loc}}(\mathbb{R}^d)$ continuously. Finally, observe that $\Delta \mathrm{SL}_\kappa(p) +$

$\kappa^2\mathrm{SL}_\kappa(p) = 0$ in $\mathbb{R}^d \setminus \overline{\Gamma}$, in the sense of distributions, for any $p \in \mathbb{H}^{-1/2}(\Gamma)$. Hence if $f = (\Delta \mathrm{SL}_\kappa(p))|_{\mathbb{R}^d\setminus\overline{\Gamma}}$, then

$$\|f\|_{\mathrm{L}^2(K)} \leq \kappa^2 \|\mathrm{SL}_\kappa(p)\|_{\mathrm{L}^2(K)} \leq C_K \|p\|_{\mathbb{H}^{-1/2}(\Gamma)}$$

for any compact subset $K \subset \mathbb{R}^d$ and some $C_K > 0$ independent of p. This concludes the proof. □

Proposition 8.4 (Continuity of double layer potential). *The potential operator* DL_κ *continuously maps* $\mathbb{H}^{+1/2}(\Gamma)$ *into* $\mathrm{H}^1_{\mathrm{loc}}(\Delta, \mathbb{R}^d \setminus \overline{\Gamma})$.

Proof. First of all, consider $\mathrm{S} : \mathbb{H}^{1/2}(\Gamma) \to \mathrm{H}^1_{\mathrm{loc}}(\mathbb{R}^d \setminus \overline{\Gamma})$ as the solution operator such that for any $g \in \mathbb{H}^{1/2}(\Gamma)$ the function $\mathrm{S}(g)$ is the unique solution to Problem (7.3). In particular we have $\gamma_{\mathrm{D}} \cdot \mathrm{S}(v) = v$ for any $v \in \mathbb{H}^{1/2}(\Gamma)$. Since $\mathrm{S}(v)$ is a solution to the homogeneous Helmholtz equation in $\mathbb{R}^d \setminus \overline{\Gamma}$, we can apply identity (8.3) which yields

$$\mathrm{DL}_\kappa(v) = \mathrm{S}(v) - \mathrm{SL}_\kappa \cdot \gamma_{\mathrm{N}} \cdot \mathrm{S}(v) \qquad \forall v \in \mathbb{H}^{\frac{1}{2}}(\Gamma).$$

The continuity result that we want to prove is then a clear consequence of the continuity of $\mathrm{S}, \mathrm{SL}_\kappa$ and γ_{N}, see Propositions 8.3 and 7.2. □

8.2. Jump Relations

As predictible, functions of the form $\mathrm{DL}_\kappa(v)$ do *not* belong to $\mathrm{H}^1_{\mathrm{loc}}(\mathbb{R}^d)$. Their Neumann traces, though, admit no jump across the screen Γ. The following result summarizes the behaviour of both the single layer and double layer potentials across the screen Γ.

Proposition 8.5 (Jump relations).

$$[\gamma_{\mathrm{D}}] \cdot \mathrm{DL}_\kappa(\dot{u}) = [\dot{u}], \quad [\gamma_{\mathrm{N}}] \cdot \mathrm{DL}_\kappa(\dot{u}) = 0 \qquad \forall \dot{u} \in \mathbb{H}^{+\frac{1}{2}}(\Gamma),$$

$$[\gamma_{\mathrm{D}}] \cdot \mathrm{SL}_\kappa(\dot{p}) = 0, \qquad [\gamma_{\mathrm{N}}] \cdot \mathrm{SL}_\kappa(\dot{p}) = [\dot{p}] \quad \forall \dot{p} \in \mathbb{H}^{-\frac{1}{2}}(\Gamma).$$

Proof. We will focus on the proof of the identities concerning the double layer potential. The identities concerning the single layer potential may be proved in a similar manner. Consider any $\dot{u} \in \mathbb{H}^{1/2}(\Gamma)$, set $\psi(\mathbf{x}) = \mathrm{DL}_\kappa(\dot{u})(\mathbf{x})$. According to Relation (8.2), we have $\Delta\psi + \kappa^2\psi = \gamma'_{\mathrm{N}}(\dot{u})$ in the sense of distributions over $\mathbb{R}^d$. As a consequence we have

$$\int_{\mathbb{R}^d} \psi\,(\Delta\varphi + \kappa^2\varphi) d\mathbf{x} = -\langle \gamma'_{\mathrm{N}}(\dot{u}), \varphi\rangle = \int_{[\Gamma]} \dot{u}\,\gamma_{\mathrm{N}}(\varphi)\, d\mathbf{x} \qquad \forall \varphi \in \mathscr{D}(\mathbb{R}^d). \quad (8.6)$$

where $\langle\ ,\ \rangle$ must be understood as the duality pairing between $\mathscr{D}(\mathbb{R}^d)$ and $\mathscr{D}(\mathbb{R}^d)'$. On the other hand consider the integral in the left hand side above, and apply the generalized 2nd Green Formula (7.2). Since $\Delta\psi + \kappa^2\psi = 0$ in $\mathbb{R}^d \setminus \overline{\Gamma}$, this yields

$$\begin{aligned}\int_{\mathbb{R}^d} \psi\,(\Delta\varphi + \kappa^2\varphi) d\mathbf{x} &= \int_{\mathbb{R}^d\setminus\overline{\Gamma}} \psi\,(\Delta\varphi + \kappa^2\varphi) d\mathbf{x} \\ &= \int_{[\Gamma]} \gamma_{\mathrm{D}}(\psi)\gamma_{\mathrm{N}}(\varphi) - \gamma_{\mathrm{N}}(\psi)\gamma_{\mathrm{D}}(\varphi) d\sigma \qquad (8.7)\end{aligned}$$

and this has to hold for any $\varphi \in \mathscr{D}(\mathbb{R}^d)$ as well. Now take the difference between Eqs. (8.6) and (8.7), and observe that $\gamma_{\mathrm{N}}(\varphi) \in \mathrm{H}^{-1/2}([\Gamma])$ whenever $\varphi \in \mathscr{D}(\mathbb{R}^d)$. This yields

$$\int_{[\Gamma]} \gamma_{\mathrm{N}}(\psi)\gamma_{\mathrm{D}}(\varphi)\,d\sigma - \int_{[\Gamma]} \big(\gamma_{\mathrm{D}}(\psi) - \dot{u}\big)\gamma_{\mathrm{N}}(\varphi)\,d\sigma = 0 \qquad \forall\varphi \in \mathscr{D}(\mathbb{R}^d). \quad (8.8)$$

Using the density of $\mathscr{D}(\mathbb{R}^d)$ in $\mathrm{H}^1_{\mathrm{loc}}(\Delta, \mathbb{R}^d)$, as well as Proposition 7.5, we see that (8.8) implies that $\int_{[\Gamma]} \gamma_{\mathrm{N}}(\psi)\dot{v}d\sigma = 0$ for all $\dot{v} \in \mathrm{H}^{+1/2}([\Gamma])$, and $\int_{[\Gamma]}(\gamma_{\mathrm{D}}(\psi) - \dot{u})\dot{q}\,d\sigma = 0$ for all $\dot{q} \in \mathrm{H}^{-1/2}([\Gamma])$. According to Proposition 6.3, and the definition of the jump operators given in Sect. 6.2, this concludes the proof. □

In spite of a clear parallel, there is also a remarkable difference between Proposition 8.5 above, and the usual jump relations, e.g., from Lemma 4.1 in [8]. Indeed, in the right hand sides of the identities of Proposition 8.5, what appears is $[\dot{u}]$ and $[\dot{p}]$, and not just $\dot{u}$ and $\dot{p}$. This is a specific feature of screen's geometries. In the present case, the operators $\gamma_{\mathrm{D}}\cdot\mathrm{SL}_\kappa : \mathbb{H}^{-1/2}(\Gamma) \to \mathbb{H}^{+1/2}(\Gamma)$ and $\gamma_{\mathrm{N}} \cdot \mathrm{DL}_\kappa : \mathbb{H}^{+1/2}(\Gamma) \to \mathbb{H}^{-1/2}(\Gamma)$ are not onto. As exhibited by the next result, they are not injective neither.

Lemma 8.6 (Kernels of potentials). *We have* $\mathrm{SL}_\kappa(\dot{p}) = 0\ \forall\dot{p} \in \mathrm{H}^{-1/2}([\Gamma])$ *and* $\mathrm{DL}_\kappa(\dot{u}) = 0\ \forall\dot{u} \in \mathrm{H}^{+1/2}([\Gamma])$.

Proof. We prove the result only for the single layer potential, since for the double layer potential, the proof is very similar. For any $\dot{p} \in \mathrm{H}^{-1/2}([\Gamma])$, set $\psi = \mathrm{SL}_\kappa(\dot{p})$. The function ψ belongs to $\mathrm{H}^1_{\mathrm{loc}}(\Delta, \mathbb{R}^d\setminus\overline{\Gamma})$, and since $[\gamma_{\mathrm{D}}(\psi)] = 0$ and $[\gamma_{\mathrm{N}}(\psi)] = [\dot{p}] = 0$ according to Proposition 8.5, we deduce that $\gamma_{\mathrm{D}}(\psi) \in \mathrm{H}^{1/2}([\Gamma])$ and $\gamma_{\mathrm{N}}(\psi) \in \mathrm{H}^{-1/2}([\Gamma])$, so that $\psi \in \mathrm{H}^1_{\mathrm{loc}}(\Delta, \mathbb{R}^d)$. Since $\Delta\psi+\kappa^2\psi = 0$ in the sense of distributions in $\mathbb{R}^d\setminus\overline{\Gamma}$, we deduce that actually $\Delta\psi+\kappa^2\psi = 0$ in $\mathbb{R}^d$. To summarize, $\Delta\psi + \kappa^2\psi = 0$ in $\mathbb{R}^d$ and ψ is outgoing, which implies that $\psi = 0$. □

This lemma combined with Proposition 6.8 shows that SL_κ induces a continuous map from $\widetilde{\mathrm{H}}^{-1/2}([\Gamma])$ to $\mathrm{H}^1_{\mathrm{loc}}(\Delta, \mathbb{R}^d \setminus \overline{\Gamma})$. Similarly DL_κ induces a continuous map from $\widetilde{\mathrm{H}}^{+1/2}([\Gamma])$ to $\mathrm{H}^1_{\mathrm{loc}}(\Delta, \mathbb{R}^d \setminus \overline{\Gamma})$. For both induced maps, we keep the same notations $\mathrm{SL}_\kappa, \mathrm{DL}_\kappa$ so that

$$\mathrm{SL}_\kappa : \widetilde{\mathrm{H}}^{-\frac{1}{2}}([\Gamma]) \to \mathrm{H}^1_{\mathrm{loc}}(\Delta, \mathbb{R}^d \setminus \overline{\Gamma}) \quad \text{and} \quad \mathrm{DL}_\kappa : \widetilde{\mathrm{H}}^{+\frac{1}{2}}([\Gamma]) \to \mathrm{H}^1_{\mathrm{loc}}(\Delta, \mathbb{R}^d \setminus \overline{\Gamma})$$

are continuous operators. We will now examine the invertibility property of the integral operators $\gamma_{\mathrm{D}} \cdot \mathrm{SL}_\kappa$ and $\gamma_{\mathrm{N}} \cdot \mathrm{DL}_\kappa$.

Proposition 8.7. *Assume that* $\kappa = \imath$ *(imaginary unit). There exists a constant* $C > 0$ *such that*

$$\Re e\left\{\int_{[\Gamma]} q\,\gamma_{\mathrm{D}} \cdot \mathrm{SL}_\imath(\,\overline{q}\,)\,d\sigma\right\} \geq C\,\|q\|^2_{\widetilde{\mathrm{H}}^{-\frac{1}{2}}([\Gamma])} \qquad \forall q \in \widetilde{\mathrm{H}}^{-\frac{1}{2}}([\Gamma]),$$

$$\Re e \left\{ \int_{[\Gamma]} v\, \gamma_{\mathrm{N}} \cdot \mathrm{DL}_{\imath}(\,\overline{v}\,)\, d\sigma \right\} \geq C\, \|v\|^2_{\widetilde{\mathrm{H}}^{\frac{1}{2}}([\Gamma])} \qquad \forall v \in \widetilde{\mathrm{H}}^{\frac{1}{2}}([\Gamma]).$$

Proof. Once again we only prove the statement for $\mathrm{SL}_{\imath}$ since the statement concerning $\mathrm{DL}_{\imath}$ is very similar. Take any $q \in \widetilde{\mathrm{H}}^{-1/2}([\Gamma])$ and denote $\psi = \mathrm{SL}_{\kappa}(q)$ so that $-\Delta\psi + \psi = 0$ in the sense of distributions in $\mathbb{R}^d \setminus \overline{\Gamma}$, and $[\gamma_{\mathrm{N}}(\psi)] = q$. By definition of the jump operator introduced in Sect. 6.2, we have

$$\begin{aligned} \int_{[\Gamma]} q\, \gamma_{\mathrm{D}} \cdot \mathrm{SL}_{\imath}(\overline{q})\, d\sigma &= \int_{[\Gamma]} \gamma_{\mathrm{N}}(\psi)\gamma_{\mathrm{D}}(\overline{\psi})\, d\sigma = \int_{\mathbb{R}^d\setminus\overline{\Gamma}} |\nabla\psi|^2 + \overline{\psi}\Delta\psi\, d\mathbf{x} \\ &= \int_{\mathbb{R}^d\setminus\overline{\Gamma}} |\nabla\psi|^2 + |\psi|^2\, d\mathbf{x} \geq \frac{1}{2}\, \|\psi\|^2_{\mathrm{H}^1(\Delta,\mathbb{R}^d\setminus\overline{\Gamma})} \end{aligned}$$

In the calculus above we used the generalized Green formula (7.2), as well as the fact that $\Delta\psi = \psi$. Now since $[\gamma_{\mathrm{N}}(\psi)] = q$ and since $\gamma_{\mathrm{N}} : \mathrm{H}^1(\Delta, \mathbb{R}^d \setminus \overline{\Gamma}) \to \mathbb{H}^{-1/2}(\Gamma)$ and $[\ \ ] : \mathbb{H}^{-1/2}(\Gamma) \to \widetilde{\mathrm{H}}^{-1/2}([\Gamma])$ are continuous, we deduce that there exists $C > 0$, independent of q such that

$$\|q\|_{\widetilde{\mathrm{H}}^{-\frac{1}{2}}([\Gamma])} \leq C\, \|\psi\|_{\mathrm{H}^1(\Delta,\mathbb{R}^d\setminus\overline{\Gamma})},$$

which concludes the proof. □

Proposition 8.8 (Coercivity of boundary integral operators). *For any wave number* $\kappa \in \mathbb{C} \setminus \{0\}$ *such that* $\Im m\{\kappa\} \geq 0$*, define the operators* $\mathrm{V} : \widetilde{\mathrm{H}}^{-1/2}([\Gamma]) \to \mathrm{H}^{+1/2}([\Gamma])$ *and* $\mathrm{W} : \widetilde{\mathrm{H}}^{1/2}(\Gamma) \to \mathrm{H}^{-1/2}([\Gamma])$ *by*

$$\mathrm{V} = \gamma_{\mathrm{D}} \cdot \mathrm{SL}_{\kappa} \qquad \textit{and} \qquad \mathrm{W} = \gamma_{\mathrm{N}} \cdot \mathrm{DL}_{\kappa}.$$

Then there exists compact operators $\mathrm{K}_{\mathrm{V}} : \widetilde{\mathrm{H}}^{-1/2}([\Gamma]) \to \mathrm{H}^{1/2}([\Gamma])$ *and* $\mathrm{K}_{\mathrm{W}} : \widetilde{\mathrm{H}}^{1/2}([\Gamma]) \to \mathrm{H}^{-1/2}([\Gamma])$ *such that the following generalized Gårding identities are satisfied*

$$\Re e \left\{ \int_{[\Gamma]} q\, (\mathrm{V} + \mathrm{K}_{\mathrm{V}})\, \overline{q}\, d\sigma \right\} \geq C\, \|q\|^2_{\widetilde{\mathrm{H}}^{-\frac{1}{2}}([\Gamma])} \qquad \forall q \in \widetilde{\mathrm{H}}^{-\frac{1}{2}}([\Gamma]),$$

$$\Re e \left\{ \int_{[\Gamma]} v\, (\mathrm{W} + \mathrm{K}_{\mathrm{W}})\, \overline{v}\, d\sigma \right\} \geq C\, \|v\|^2_{\widetilde{\mathrm{H}}^{\frac{1}{2}}([\Gamma])} \qquad \forall v \in \widetilde{\mathrm{H}}^{\frac{1}{2}}([\Gamma]).$$

Proof. Denote by $\mathscr{G}_{\imath}$ and $\mathrm{SL}_{\imath}, \mathrm{DL}_{\imath}$ the outgoing Green kernel and the single and double layer potentials associated to the value $\imath$ for the wave number, so that Proposition 8.7 applies to $\mathrm{SL}_{\imath}$ and $\mathrm{DL}_{\imath}$. Besides, following Remark 3.1.3 in [20], the operator $(\mathscr{G}_{\imath} - \mathscr{G}_{\kappa})*$ is pseudo-differential operator of order -4 mapping $\mathrm{H}^1_{\mathrm{loc}}(\mathbb{R}^d)'$ to $\mathrm{H}^3_{\mathrm{loc}}(\mathbb{R}^d)$ which implies that both $\mathrm{K}_{\mathrm{V}} := \gamma_{\mathrm{D}} \cdot (\mathrm{SL}_{\imath} - \mathrm{SL}_{\kappa})$ and $\mathrm{K}_{\mathrm{W}} := \gamma_{\mathrm{N}} \cdot (\mathrm{DL}_{\imath} - \mathrm{DL}_{\kappa})$ are compact as operators mapping respectively $\widetilde{\mathrm{H}}^{-1/2}([\Gamma])$ to $\mathbb{H}^{1/2}(\Gamma)$ and $\widetilde{\mathrm{H}}^{1/2}([\Gamma])$ to $\mathbb{H}^{-1/2}(\Gamma)$. We finally obtain coercivity of both $\mathrm{V} + \mathrm{K}_{\mathrm{V}}$ and $\mathrm{W} + \mathrm{K}_{\mathrm{W}}$ by application of Proposition 8.7. □

The previous result implies that both $\mathrm{V} : \widetilde{\mathrm{H}}^{-1/2}([\Gamma]) \to \mathrm{H}^{1/2}([\Gamma])$ and $\mathrm{W} : \widetilde{\mathrm{H}}^{1/2}([\Gamma]) \to \mathrm{H}^{-1/2}([\Gamma])$ are Fredholm operators with index 0. As one may expect by analogy with a more standard problem, they are actually isomorphisms.

Proposition 8.9. *The operators* $\mathrm{V} : \widetilde{\mathrm{H}}^{-1/2}([\Gamma]) \to \mathrm{H}^{1/2}([\Gamma])$ *and* $\mathrm{W} : \widetilde{\mathrm{H}}^{1/2}([\Gamma]) \to \mathrm{H}^{-1/2}([\Gamma])$ *are isomorphisms.*

Proof. According to Fredholm alternative, all we need to prove is that these operators are one-to-one. We prove this only for V, since the proof for W is analogous. Consider any $\dot{q} \in \widetilde{\mathrm{H}}^{-1/2}([\Gamma])$ such that $\mathrm{V}(\dot{q}) = 0$. Take any $\dot{p} \in \mathbb{H}^{-1/2}(\Gamma)$ such that $[\dot{p}] = \dot{q}$. Injectivity will be proved if we show that $\dot{p} \in \mathrm{H}^{-1/2}([\Gamma])$ which is equivalent to $[\dot{p}] = \dot{q} = 0$. Set $\psi = \mathrm{SL}_\kappa(\dot{p})$. Then $\gamma_{\mathrm{D}}(\psi) = \mathrm{V}(\dot{p}) = \mathrm{V}(\dot{q}) = 0$ and ψ is an outgoing solution to the homogeneous Helmholtz equation in $\mathbb{R}^d \setminus \overline{\Gamma}$. Hence according to Proposition 7.2, $\psi = 0$ i.e. $\mathrm{SL}_\kappa(\dot{p}) = 0$. We conclude with the jump formula $[\dot{p}] = [\gamma_{\mathrm{N}}] \cdot \mathrm{SL}_\kappa(\dot{p}) = 0$ provided by Proposition 8.5. □

Acknowledgements

The authors would like to thank the reviewers for suggesting simplifications in the proof of Proposition 8.2.

Appendix

Quotient spaces. As this is a concept constantly used across this article, in the first part of this appendix we recall elementary results concerning quotient spaces and their norms. For a full justification of these results, we refer to [19, chapter 1 & 4],

Assume here that $(\mathrm{H}, \|\ \|_{\mathrm{H}})$ is some Banach space, and that X is a closed sub-space of H. Then we define the quotient space H/X as the set

$$\mathrm{H}/\mathrm{X} := \{\, x + \mathrm{X} \mid x \in \mathrm{H} \,\}.$$

The quotient space H/X is the set of equivalence classes associated to the equivalence relation $x \sim y \iff x - y \in \mathrm{X}$. The addition and multiplication by scalars induce natural counterparts in H/X, so that H/X inherits a structure of vector space from H. We equip this space with the norm

$$\|\, \dot{y} \,\|_{\mathrm{H}/\mathrm{X}} = \inf_{x \in \mathrm{X}} \|y + x\|_{\mathrm{H}} \quad \text{for any } y \in \dot{y}. \tag{8.9}$$

Recall that if $(\mathrm{H}, \|\ \|_{\mathrm{H}})$ is a Banach space, then H/X equipped with $\|\ \|_{\mathrm{H}/\mathrm{X}}$ is a Banach space as well. Finally, we would like to remind the reader that the canonical surjection $\pi : \mathrm{H} \to \mathrm{H}/\mathrm{X}$ is an open mapping.

Observe that, for the topology induced by (8.9), a set $U \subset \mathrm{H}/\mathrm{X}$ is open if and only if $\pi^{-1}(U)$ is an open set of H. Using this obervation, it is easy to prove the following result.

Lemma 8.10. *Let* $(\mathrm{H}, \|\ \|)$ *and* $(\mathrm{Y}, \|\ \|_{\mathrm{Y}})$ *be two Banach spaces, and assume that* X *is a closed subspace of* H*. Consider a continuous linear map* $\Theta : \mathrm{H} \to$

Y. *If* $\mathrm{X} \subset \mathrm{Ker}(\Theta)$*, then* Θ *induces a continuous linear map* $\theta : \mathrm{H}/\mathrm{X} \to \mathrm{Y}$ *that is uniquely determined by the identity* $\Theta = \theta \circ \pi$*.*

Density result. In this part of the appendix, we recall a density result proved in [9, Lemma 2.4]. We need this result when exploiting the local structure of screens. We provide a proof for the sake of completeness.

Proposition 8.11. *For* H *being one of the spaces* $\mathrm{H}^1(\mathbb{R}^d)$ *or* $\mathrm{H}(\mathrm{div}, \mathbb{R}^d)$*, denote* $\mathrm{H}_\star$ *the space of* $v \in \mathrm{H}$ *that vanish in a neighbourhood of* $\partial\Gamma$*. Then* $\mathrm{H}_\star$ *is dense in* H*.*

Proof. We prove this result for $\mathrm{H} = \mathrm{H}^1(\mathbb{R}^d)$. The case where $\mathrm{H} = \mathrm{H}(\mathrm{div}, \mathbb{R}^d)$ follows the same lines. Since $\mathrm{C}^\infty(\mathbb{R}^d)$ is dense in $\mathrm{H}^1(\mathbb{R}^d)$, it suffices to show that any $u \in \mathrm{C}^\infty(\mathbb{R}^d)$ is the limit of some sequence $u_1, u_2, u_3, \dots$ of $\mathrm{H}^1_\star(\mathbb{R}^d)$.

According to Definition 2.3, there exists a Lipschitz partition $(\Omega_j)_{j=0\dots Q}$ such that $\overline{\Gamma} \cap \partial\Omega_j = \overline{\Gamma}_j$ where Γ_j is a Lipschitz screen in the sense of Definition 2.1. Set $\Sigma_j = \partial\Gamma_j$, and observe that $\partial\Gamma \subset \cup_{j=0}^{Q}\Sigma_j$. Considering a partition of unity, the proof can be reduced to the case where u is supported in some ball B centered at a point $\mathbf{x} \in \partial\Gamma$. Considering a smaller radius for B if necessary, one may consider that each Σ_j can be described like in (2.1). This implies in particular that there exist Lipschitz diffeomorphisms $\Psi_j : \mathrm{B} \to \Psi_j(\mathrm{B}) \subset \mathbb{R}^d$ such that

$$\Psi_j(\mathrm{B} \cap \Sigma_j) \subset \widehat{\Sigma} := \{\ (0,0,z) \mid z \in \mathbb{R}\ \}.$$

Assume first that we have constructed functions $\tau_{j,k} \in \mathrm{H}^1(\mathrm{B}) \cap \mathrm{L}^\infty(\mathrm{B})$ such that $\tau_{j,k} = 0$ in some neighbourhood of $\mathrm{B} \cap \Sigma_j$ and $\lim_{k\to\infty} \|1 - \tau_{j,k}\|_{\mathrm{H}^1(\mathrm{B})} = 0$. Setting

$$\tau_k(\mathbf{x}) = \tau_{0,k}(\mathbf{x})\tau_{1,k}(\mathbf{x}) \dots \tau_{1,Q}(\mathbf{x})$$

we obtain $\tau_k \in \mathrm{H}^1(\mathrm{B}) \cap \mathrm{L}^\infty(\mathrm{B})$ such that $\tau_k = 0$ in a neighbourhood of $\mathrm{B} \cap (\cup_{j=0}^{Q}\Sigma_j) \supset \mathrm{B} \cap \partial\Gamma$, and such that $\lim_{k\to\infty} \|1 - \tau_k\|_{\mathrm{H}^1(\mathrm{B})} = 0$. Set $u_k(\mathbf{x}) := \tau_k(\mathbf{x})u(\mathbf{x})$. Since $\mathrm{supp}(u) \subset \mathrm{B}$ and $u \in \mathrm{L}^\infty(\mathrm{B})$ and $\nabla u \in \mathrm{L}^\infty(\mathrm{B})$, we obtain that $u_k \in \mathrm{H}^1_\star(\mathbb{R}^d)$ and

$$\|u_k - u\|_{\mathrm{H}^1(\mathbb{R}^d)} \leq 2\ \|1 - \tau_k\|_{\mathrm{H}^1(\mathrm{B})}\ \big(\ \|u\|_{\mathrm{L}^\infty(\mathrm{B})} + \|\nabla u\|_{\mathrm{L}^\infty(\mathrm{B})}\ \big) \underset{k\to\infty}{\longrightarrow} 0.$$

To conclude the proof, there only remains to construct the cut-off functions $\tau_{j,k}(\mathbf{x})$. Consider a subset $\Sigma \subset \mathrm{B}$ such that $\Psi(\mathrm{B} \cap \Sigma) \subset \widehat{\Sigma} := \{\ (0,0,z) \mid z \in \mathbb{R}\ \}$ for some Lipschitz diffeomorphism $\Psi : \mathrm{B} \to \widehat{\mathrm{B}} = \Psi(\mathrm{B})$. We consider $\widehat{\tau}_k \in \mathrm{H}^1_{\mathrm{loc}}(\mathbb{R}^d) \cap \mathrm{L}^\infty(\mathbb{R}^d)$ defined by

$$\widehat{\tau}_k(r,\theta,z) = \begin{cases} 0 & \text{if } r \leq 1/k \\ \ln(kr)/\ln(k) & \text{if } 1/k \leq r \leq 1 \\ 1 & \text{if } 1 \leq r \end{cases}$$

Straightforward calculus yields $\lim_{k\to\infty} \|1 - \tau_k\|_{\mathrm{H}^1(\widehat{\mathrm{B}})} = 0$. Now we can define $\tau_k(\mathbf{x}) = \widehat{\tau}_k(\Psi(\mathbf{x}))$. Clearly $\tau_k \in \mathrm{L}^\infty(\mathrm{B})$. According to Theorem 3.23, Chapter 3 in [16], we also have $\tau_k \in \mathrm{H}^1(\mathrm{B})$, and

$$\|1 - \tau_k\|^2_{\mathrm{H}^1(\mathrm{B})} \leq \Big(1 + \|\mathrm{D}\Psi\|^2_{\mathrm{L}^\infty(\mathrm{B})}\Big)\, \big\|\mathrm{Jac}(\Psi)^{-1}\big\|^2_{\mathrm{L}^\infty(\mathrm{B})}\, \|1 - \widehat{\tau}_k\|^2_{\mathrm{H}^1(\widehat{\mathrm{B}})} \underset{k\to\infty}{\longrightarrow} 0$$

with $\mathrm{Jac}(\Psi)(\mathbf{x}) := \det(\Psi(\mathbf{x}))$. Finally it is clear, according to this construction, that $\tau_k = 0$ in a neighbourhood of $\mathrm{B} \cap \Sigma$. This concludes the proof. $\square$

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Received: January 28, 2013.
Revised: August 2, 2013.