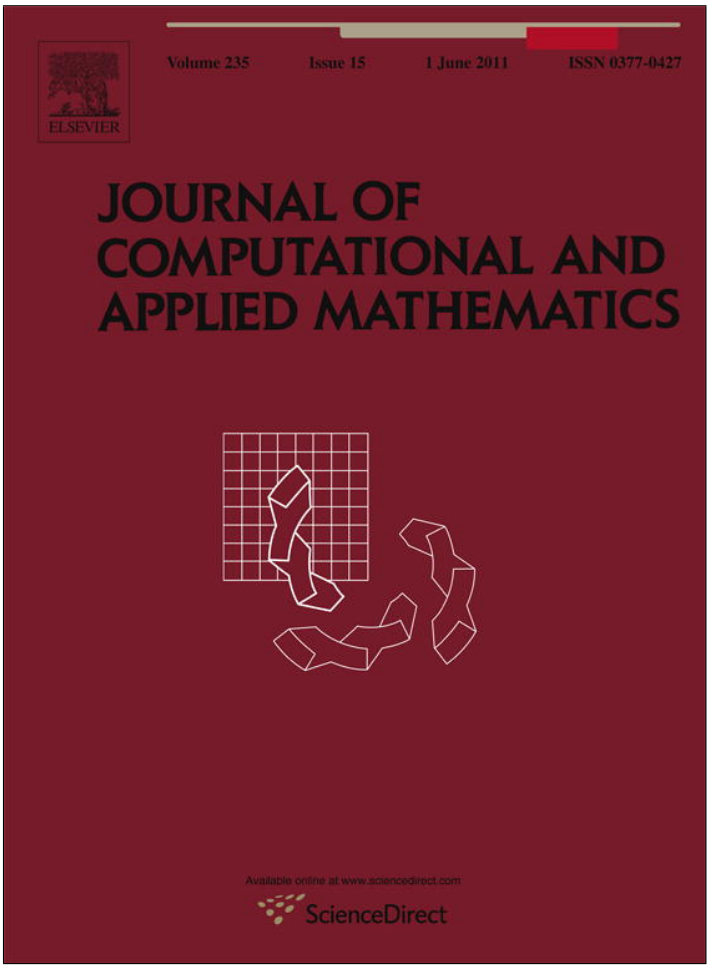

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Asymptotic and numerical analysis for Holland and Simpson's thin wire formalism

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ABSTRACT

In the context of simulation of electromagnetic propagation, the thin wire formalism of Holland and Simpson allows one to deal with scattering by perfectly conducting thin wires by coupling a standard FDTD method with a discrete 1D wave equation ruling the current inside the wires. This method can be very accurate, but it involves a fitting parameter that requires careful calibration.

We propose a consistency analysis and derive a formula for the calibration of this parameter in the case of a simplified 2D analogue of the method of Holland and Simpson. Our proof relies on the observation that this method is actually a hidden version of the singular function method well known in the context of elliptic equations in domains with a singular boundary.

0. Introduction

Taking into account perfectly conducting thin wires in electromagnetic wave propagation is a usual issue when dealing with a volumic method of simulation. In such a context, a thin wire should be considered as a diffracting object with thickness ε much smaller than the wavelength λ. In a typical concrete situation, it is desirable to treat wires as geometrical details that would require only a minor perturbation of the numerical scheme. This point of view discards the use of mesh refinement as a solution for treating wires. Indeed the mesh should be generated in accordance with the wavelength, which raises the problem of taking into account the influence of scatterers whose size is smaller than the average step h of a mesh cell. To briefly sum up,

$$\varepsilon \ll \lambda \quad \text{and} \quad \mathrm{h} \simeq \frac{\lambda}{10} \quad \text{so } \varepsilon \ll \mathrm{h}.$$

In this situation, a standard method (such as FDTD) cannot be expected to be accurate. A solution proposed by Holland and Simpson for FDTD [1,2] consists in adding a term coupling the electromagnetic field with a 1D equation ruling the current at the surface of the wires. This method is widely used at present for it does not require a drastic modification of the classical FDTD scheme and provides accurate results at least in the case of ideal geometric configurations. Moreover, to the best of our knowledge, it is the only volumic method for wires offering such advantages. Many variants have been proposed in the engineer literature, see [3–6] and the references therein. However, Holland and Simpson's method contains a fitting parameter that requires a calibration, which raises many practical difficulties. To the best of our knowledge, no satisfying systematic procedure of calibration has been proposed so far.

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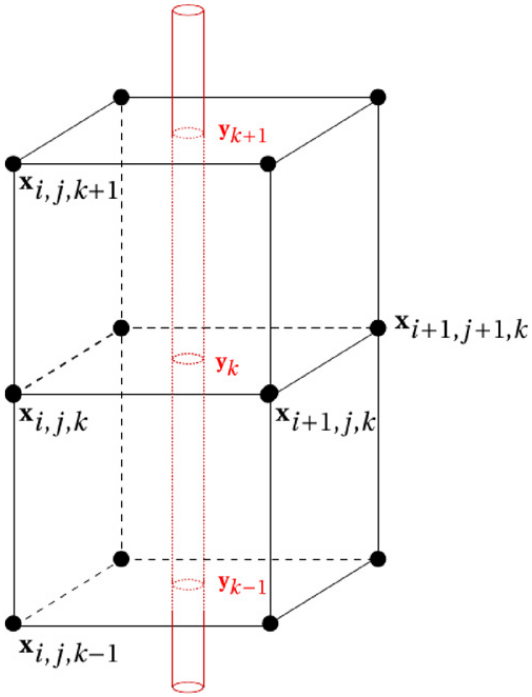


Fig. 1. Mesh cells adjacent to the wire.

This method remains obscure as regards mathematical aspects. Holland and Simpson themselves proposed directly a discrete formulation of their method without providing any underlying continuous setting. In [7], Collino and Millot proposed a continuous variational formulation that leads to the method of Holland and Simpson for a well chosen discretization. This variational setting also naturally leads to variants, each one corresponding to a particular standard discretization (finite elements, discontinuous Galerkin,…etc.). However there still does not exist any theoretical numerical analysis of Holland and Simpson's method. In particular why is this method so accurate remains an opened question.

This article aims at clarifying the formalism of Holland and Simpson from a theoretical point of view. The main idea of this article can be formulated in simple words: this method is nothing else but a hidden form of the singular function method well known in the literature concerned with the solution to elliptic problems in domains with edges and corners on their boundary, see for example [8–10] and references therein. The other main point of the present article is the proof of consistency for Holland–Simpson's scheme for a simple model problem. We would like to point out that the present article is not aiming at determining whether the method of Holland and Simpson is more efficient than the singular function method, although we shall formulate comments concerning this issue in Section 5.

The outline of this article is as follows. In the first section we present the thin wire formalism of Holland and Simpson for the full Maxwell's equations following a presentation very close to [1,2]. Then we discuss, from a numerical analysis point of view, what is problematic in the derivation of this method. Finally we restrict our study to a problem invariant under translation along the axis of the wire with harmonic dependency in time. Whereas the first section is only supposed to rely on rather formal arguments, the rest of the paper claims for mathematical rigor. In the second section we state a well posed problem corresponding to the last physical model of the first section: this is an Helmholtz equation with homogeneous Dirichlet condition on the boundary of an obstacle whose size is small compared to the wavelength. In the third section we introduce simplifications of this problem and justify them using results of asymptotic analysis proved in [11]. In the fourth section we present both a standard finite element discretization of the simplified problem and a variant of the corresponding Holland and Simpson's scheme. In the last section, that contains the main results of this paper, we show that Holland and Simpson's method is related to the Augmented Galerkin method tackled in [11]. This leads to error estimates for the method of Holland and Simpson scheme and to a theoretical expression for its fitting parameter.

1. The original derivation of Holland and Simpson

This section is devoted to the presentation of the thin wire formalism of Holland and Simpson. For the derivation of this numerical scheme, we will stick to the original papers [1,2]. We only introduce minor simplifications that will have no incidence but the clarification of the presentation. Unlike the rest of this article, the present section is based on formal and intuitive arguments.

The method of Holland and Simpson was designed for solving Maxwell's equations in a situation where a Cartesian mesh has been generated in order to apply a Yee scheme, and a straight wire aligned with one of the axis of the mesh must be taken into account. It is assumed that the thickness of this wire is much smaller than the step of the mesh. For the sake

of simplicity, we will suppose that the wire crosses mesh cells in their center, as represented in Fig. 1. The approach of Holland and Simson consists in writing the usual Yee equations except for the nodes adjacent to the wire where slight modifications are introduced. To derive the coupling equations inside an adjacent mesh cell, Holland and Simpson study the electromagnetic scattering by an infinite perfectly conducting thin straight wire with constant circular cross-section of thickness ε in a 3D free space. Since they are only interested in what happens inside a single mesh cell containing the wire, in this model the wire is supposed to be "alone" i.e. there is no other diffracting obstacle.

The whole surrounding space is an homogeneous material characterized by a constant electric permittivity and a constant magnetic permeability. For the sake of simplicity, we will suppose that these constants are equal to 1, however this is just a matter of convention. In what follows, we denote r, θ, z the cylindrical coordinates, taking $r = 0$ as the equation of the median line of the wire. We decompose the vector fields according to the cylindrical coordinates $\mathbf{F} = \mathrm{F}_r\mathbf{e}_r + \mathrm{F}_\theta\mathbf{e}_\theta + \mathrm{F}_z\mathbf{e}_z$ where $\mathbf{e}_r = (\cos\theta, \sin\theta, 0)$, $\mathbf{e}_\theta = (-\sin\theta, \cos\theta, 0)$ and $\mathbf{e}_z = (0, 0, 1)$ are unit vectors.

1.0.1. Exact physical model

Holland and Simpson consider Maxwell's equations in the neighborhood of the wire. We note $(\mathbf{E}, \mathbf{H})$ the electromagnetic field, and suppose that it is generated by some source currents $\mathbf{J}$ circulating far away from the wire. These equations are

$$\mathbf{curl}\,\mathbf{E} + \partial_t\mathbf{H} = \mathbf{0} \quad \text{and} \quad \mathbf{curl}\,\mathbf{H} - \partial_t\mathbf{E} = -(\mathbf{H}\times\mathbf{n}_\varepsilon)\delta_{\Gamma_\varepsilon} + \mathbf{J}. \tag{1}$$

Here $\mathbf{n}_\varepsilon$ is the normal vector on the surface of the wire Γ_ε directed toward the exterior of the wire, $\delta_{\Gamma_\varepsilon}$ is the Dirac distribution associated with Γ_ε, and $\mathbf{H}\times\mathbf{n}_\varepsilon$ refers to the exterior trace of $\mathbf{H}\times\mathbf{n}_\varepsilon$ on Γ_ε (the interior trace is $\mathbf{0}$ since the wire is a perfect conductor). Moreover the exterior trace of the electric field on the boundary of the wire satisfies the perfect conductor condition

$$\mathbf{E}\times\mathbf{n}_\varepsilon = \mathbf{0} \quad \text{for } r = \varepsilon. \tag{2}$$

1.1. Analytical part of the model

Although it was not explicitly stated as such in [1,2], the first ingredient in the method of Holland and Simpson consists in a "thin wire approximation" i.e. the wire is so thin compared to the average wavelength that we can neglect the azimuthal dependences at the surface of the wire and write

$$(\mathbf{H}\times n_{\Gamma_\varepsilon})(\varepsilon,\theta,z) \simeq \left\{\frac{-1}{2\pi}\int_0^{2\pi}\mathrm{H}_\theta(\varepsilon,\alpha,z)\mathrm{d}\alpha\right\}\mathbf{e}_z \quad \text{and} \quad \mathrm{E}_r(\varepsilon,\theta,z) \simeq \frac{1}{2\pi}\int_0^{2\pi}\mathrm{E}_r(\varepsilon,\alpha,z)\mathrm{d}\alpha. \tag{3}$$

Holland and Simpson's model consists in deriving an equation coupling the electromagnetic field inside the mesh cells containing the wire with currents and charges at the surface of the wire, so we introduce the following definition for the density of current $\mathrm{I}(z)$ and the density of charge $\mathrm{Q}(z)$ per unit length at the surface of the wire,

$$\frac{\mathrm{I}(z)}{2\pi\varepsilon} \coloneqq \frac{1}{2\pi}\int_0^{2\pi}\mathrm{H}_\theta(\varepsilon,\alpha,z)\mathrm{d}\alpha \quad \text{and} \quad \frac{\mathrm{Q}(z)}{2\pi\varepsilon} \coloneqq \frac{1}{2\pi}\int_0^{2\pi}\mathrm{E}_r(\varepsilon,\alpha,z)\mathrm{d}\alpha.$$

Note that I and Q depend only on z and t (but we do not write the t-dependency for the sake of brevity). Taking into account the thin wire approximation (3), the electromagnetic field satisfies the following equations in the vicinity of the wire (it is assumed that $\mathbf{J} = \mathbf{0}$ close to the wire)

$$\mathbf{curl}\,\mathbf{E} + \partial_t\mathbf{H} = \mathbf{0} \quad \text{and} \quad \mathbf{curl}\,\mathbf{H} - \partial_t\mathbf{E} = \frac{\mathrm{I}(z)}{2\pi\varepsilon}\delta_{\Gamma_\varepsilon}\mathbf{e}_z. \tag{4}$$

The equation on $(\mathbf{E}, \mathbf{H})$ corresponding to the θ-component of $\mathbf{rot}\,\mathbf{E}$ then writes

$$\partial_z\mathrm{E}_r - \partial_r\mathrm{E}_z + \partial_t\mathrm{H}_\theta = 0. \tag{5}$$

The next step toward a coupling equation consists in integrating Eq. (5) between $r = \varepsilon$ and $r = \rho$ and taking into account that $\mathrm{E}_z(r = \varepsilon, \theta, z) = 0$ (according to (2) since the cross-section of the wire is constant), which leads to

$$\mathrm{E}_z(\rho,\theta,z) = \partial_t\left\{\int_\varepsilon^\rho \mathrm{H}_\theta(r,\theta,z)\mathrm{d}r\right\} + \partial_z\left\{\int_\varepsilon^\rho \mathrm{E}_r(r,\theta,z)\mathrm{d}r\right\}. \tag{6}$$

Let us derive some analytical expressions for each of the terms contained between braces. Here we are only interested in the points $\mathbf{x}(\rho,\theta,z)$ located inside the mesh cell adjacent to the wire. We introduce the quasi-static approximation $\partial_t\mathbf{E} \simeq 0$. This is intuitively justified by the fact that $\rho < \mathrm{h} \ll \lambda$ where λ is the average wavelength of the field we want to compute. As a consequence inside a cell adjacent to the wire we have $\mathrm{div}\,\mathbf{E} = 0 \Rightarrow r^{-1}\partial_r(r\mathrm{E}_r) + \partial_\theta\mathrm{E}_\theta + \partial_z\mathrm{E}_z = 0$ and $\mathbf{curl}\,\mathbf{H} \simeq \mathbf{0} \Rightarrow r^{-1}\partial_r(r\,\mathrm{H}_\theta) - \partial_\theta\mathrm{H}_r = 0$. These equations hold outside the boundary of the wire. We integrate them over

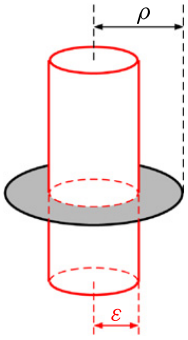


Fig. 2. Surface of integration.

the corona represented in Fig. 2 (between $r = \varepsilon$ and $r = \rho$). Since the terms containing ∂_θ vanish, we obtain the following identities

$$\rho \int_0^{2\pi} \mathrm{H}_\theta(\rho,\theta,z)\mathrm{d}\theta = \mathrm{I}(z) \quad \text{and} \quad \rho \int_0^{2\pi} \mathrm{E}_r(\rho,\theta,z)\mathrm{d}\theta + \underbrace{\int_0^{2\pi}\int_\varepsilon^\rho \partial_z \mathrm{E}_z(r,\theta,z) r\mathrm{d}r\,\mathrm{d}\theta}_{\simeq O(\rho^2)} = \mathrm{Q}(z).$$

According to the assumption $\rho < \mathrm{h} \ll \lambda$, we can neglect the term containing $\partial_z \mathrm{E}_z$ that is $O(\rho^2)$ and write $\int_0^{2\pi} \mathrm{H}_\theta \mathrm{d}\theta \simeq 2\pi \mathrm{H}_\theta$ and $\int_0^{2\pi} \mathrm{E}_r \mathrm{d}\theta \simeq 2\pi \mathrm{E}_r$ (no dependency on θ). As a consequence we end up with the following explicit expressions

$$\mathrm{H}_\theta(\rho,\theta,z) \simeq \frac{1}{2\pi}\int_0^{2\pi} \mathrm{H}_\theta(\rho,\alpha,z)\mathrm{d}\alpha \simeq \frac{\mathrm{I}(z)}{2\pi\rho}$$
$$\mathrm{E}_r(\rho,\theta,z) \simeq \frac{1}{2\pi}\int_0^{2\pi} \mathrm{E}_r(\rho,\alpha,z)\mathrm{d}\alpha \simeq \frac{\mathrm{Q}(z)}{2\pi\rho}.$$

Plugging these expressions into Eq. (6) leads to a relation involving E_z, I and Q inside mesh cells adjacent to the wire. We complete it with the conservation of charge inside the wire and then obtain

$$\mathrm{E}_z(r,\theta,z,t) = \frac{\ln(r/\varepsilon)}{2\pi}\Big(\partial_t \mathrm{I}(z,t) + \partial_z \mathrm{Q}(z,t)\Big) \quad \text{and} \quad \partial_z \mathrm{I}(z,t) + \partial_t \mathrm{Q}(z,t) = 0. \tag{7}$$

1.2. Semi-discretization of the model

We consider a volumic Cartesian mesh with cubic cells and, for the wire, a line mesh whose nodes correspond to the middle of the faces intersecting the wire, see Fig. 1. We wish to propose a discretization of Eqs. (4) and (7). In the context of finite differences it is not clear how to proceed because the nodes of the Cartesian mesh do not coincide with the nodes on the wire. As a consequence, in Eqs. (4) no discrete meaning can be naturally assigned to "$\mathrm{I}(z)\delta_{\Gamma_\varepsilon}$". Similarly in Eqs. (7) no discrete meaning can be naturally assigned to "E_z".
Here is what Holland and Simpson propose for the discretization of (4). We introduce $\mathrm{Q_h} = (\mathrm{Q_h^k})$ the array containing the numerical values of the charge at node $\mathbf{y}_k$ on the wire. We also consider $(\mathrm{I_h^{k+1/2}})$ the array containing the numerical values of the current on the segment $[\mathbf{y}_k, \mathbf{y}_{k+1}]$. In [1,2] the authors define a volumic current $\delta_\mathrm{h} \cdot \mathrm{I_h} = ((\delta_\mathrm{h} \cdot \mathrm{I_h})^{i,j,k+1/2})$ as follows

$$\begin{aligned} &\big(\delta_\mathrm{h} \cdot \mathrm{I_h}\big)^{i,j,k+\frac{1}{2}} := \frac{\mathrm{I_h^{k+\frac{1}{2}}}}{4} \quad \text{if } \mathbf{x}_{i,j,k} \text{ belongs to the face containing the node } \mathbf{y}_k, \\ &\big(\delta_\mathrm{h} \cdot \mathrm{I_h}\big)^{i,j,k+\frac{1}{2}} := 0 \quad \text{in the other cases.} \end{aligned} \tag{8}$$

Let $\mathbf{E_h}$, $\mathbf{H_h}$ and $\mathbf{J_h}$ be the arrays containing the numerical values of the electric and magnetic fields and the current corresponding to the nodes $\mathbf{x}_{i,j,k}$ of the Cartesian mesh. Recall that, according to the Yee scheme, half indices must be taken into account in the discretization process, see [12]. For example the z-component of $\mathbf{E_h}$ will be noted $(\mathrm{E}_{z,\mathrm{h}}^{i,j,k+1/2})$. For the discretization of Eqs. (7), Holland and Simpson introduce an average value of the discrete electric field

$$\left\langle \mathrm{E}_{z,\mathrm{h}}^{k+\frac{1}{2}} \right\rangle := \frac{1}{4} \sum_{\mathbf{x}_{i,j,k} \in \mathrm{f}_k} \mathrm{E}_z^{i,j,k+\frac{1}{2}} \quad \text{where } \mathrm{f}_k \text{ is the face containing the node } \mathbf{y}_k. \tag{9}$$

Let $\mathbf{curl}_h\,\mathbf{E}_h$ and $\mathbf{curl}_h\mathbf{H}_h$ be the discrete counterparts of $\mathbf{curl}\,\mathbf{E}$ and $\mathbf{curl}\,\mathbf{H}$ corresponding to the classical finite difference discretization. Taking into account notations (8) and (9) the volumic equations of the formalism of Holland and Simpson write

$$\mathbf{curl}_h\mathbf{E}_h + \partial_t\mathbf{H}_h = \mathbf{0} \quad \text{and} \quad \mathbf{curl}_h\mathbf{H}_h - \partial_t\mathbf{E}_h = \frac{\delta_h \cdot \mathrm{I}_h}{h^2}\mathbf{e}_z + \mathbf{J}_h. \tag{10}$$

The equations of Holland and Simpson's model at a node $\mathbf{y}_k$ of the mesh on the wire are obtained by considering some kind of average of (7) on each face f_k containing the node $\mathbf{y}_k$,

$$\left\langle \mathrm{E}_{z,h}^{k+\frac{1}{2}} \right\rangle = \mathrm{L}_{\varepsilon,h}\left(\partial_t \mathrm{I}_h^{k+\frac{1}{2}} + \frac{\mathrm{Q}_h^{k+1} - \mathrm{Q}_h^k}{h}\right) \quad \text{and} \quad \frac{\mathrm{I}_h^{k+\frac{1}{2}} - \mathrm{I}_h^{k-\frac{1}{2}}}{h} + \partial_t \mathrm{Q}_h^k = 0. \tag{11}$$

In this equation $\mathrm{L}_{\varepsilon,h}$ is a constant called line inductance. Integrating the first equation of (7) over one face of the mesh (area h^2), Holland and Simpson propose the following expression for this parameter

$$\mathrm{L}_{\varepsilon,h} = \frac{\int_\varepsilon^{\mathrm{R}} \ln(r/\varepsilon) r\,\mathrm{d}r}{\int_0^{\mathrm{R}} r\,\mathrm{d}r} \quad \text{where } \mathrm{R} \text{ is defined by } \pi\,\mathrm{R}^2 = h^2. \tag{12}$$

1.3. Comments on the thin wire formalism of Holland and Simpson

The system of Eqs. (10)–(11) is what we call the (semi-discrete version of) Holland and Simpson's method. In [1,2] this method was presented with a full discretization of Maxwell's equations. The authors only considered straight wires with a constant circular cross-section, and assumed that the median line of the wires were aligned with one of the axis of the Cartesian mesh. Holland and Simpson compared the results provided by this method with the results obtained with a method of moments. This experiment showed that Holland and Simpson's method dramatically lacks accuracy, but the authors observed that their method becomes much more precise if $\mathrm{L}_{\varepsilon,h}$ is chosen with a slightly different value. They determined this new value according to experimental considerations.

The thin wire formalism of Holland and Simpson received many generalizations. In [3] this method was adapted to the case of wire bundles i.e. several straight wires very close to each others. In [13,14] the method of Holland and Simpson was adapted for the framework of TLM and FEM discretization. In [4,15] the authors tested several types of interpolation for (8) and (9), and used them to propose different variants of Holland and Simpson's formalism adapted to wires arbitrarily oriented with respect to the mesh. It must be pointed out that these articles only provide experimental facts, without any theoretical error analysis.

The presentation we have just given of Holland and Simpson's method remains unclear on several points. First of all there are several approximations in the analytical part of the derivation such as quasi-staticity. Even if they can be well understood from the point of view of physics, their mathematical justification is not obvious. Another point concerns the choice of the value for the line inductance $\mathrm{L}_{\varepsilon,h}$. How come that the theoretical value (12) is not the one Holland and Simpson choose in the end? Indeed in practice $\mathrm{L}_{\varepsilon,h}$ is a fitting parameter of the method that must be chosen according to a calibration process. Even in simple situations such as a wire arbitrarily oriented with respect to the mesh, this calibration turns out to be delicate. Unfortunately the accuracy of Holland and Simpson's method is highly sensitive to any variation of $\mathrm{L}_{\varepsilon,h}$.

In [7] Collino and Millot proposed an approach based on a fictitious domain formulation that provides a continuous variational setting underlying Eqs. (10) and (11). This variational setting leads naturally to definitions similar to (8) and (9) in the case of a finite difference discretization. This variational setting also provides natural generalizations for any other type of discretization that can be deduced from a variational framework: finite elements, discontinuous Galerkin, etc…Additionally, Collino and Millot tackled a question on which the rest of the literature remained silent (except [1,2]): is there a systematic way for choosing $\mathrm{L}_{\varepsilon,h}$ which would not require any calibration? The answer is yes for an infinite wire parallel to one line of the FDTD mesh; in this case, the problem can be solved by hand, and one can show that there is one value for $\mathrm{L}_{\varepsilon,h}$ that ensures consistency of Holland and Simpson's scheme. This value takes the form

$$\mathrm{L}_{\varepsilon,h} = \frac{1}{2\pi}\ln\left(\frac{h}{\varepsilon}\right) + C_h$$

where C_h is a pure constant that varies only with the relative position of the wire with respect to the neighboring nodes of the mesh. Besides, the formula shows that the erroneous value of $\mathrm{L}_{\varepsilon,h}$ in (12) (that was obtained by averaging the relation between current and field on the face containing the wire) is due to the error between the continuous Green's functions and its discrete counterpart in the vicinity of the wire.

1.4. Model problem for the rest of the article

The main purpose of this article is to provide a theoretical approach of the method developed by Holland and Simpson. We propose not to tackle this method in the general 3D case, but we rather consider a reduced model. Getting back to the physical context presented at the beginning of this section, we impose two additional simplifying hypothesis:

- every fields are invariant under translation along the z-axis,
- every fields admit harmonic time dependency.

Let us present an intuitive derivation of this model problem, plugging these two simplifying assumptions inside (1) and (2). We also formulate the formalism of Holland and Simpson adapted to this simpler new context. We decompose the vector fields in the following manner $\mathbf{F} = \mathbf{F}_\perp + \mathrm{F}_z\,\mathbf{e}_z$ with $\mathbf{e}_z \cdot \mathbf{F}_\perp = 0$. Taking into account the translation invariance along z, Maxwell's equations (4) split into two decoupled systems of equations: one set of equations involving E_z, $\mathbf{H}_\perp$ (the TE mode) and another one involving H_z, $\mathbf{E}_\perp$ (the TH mode). We will only focus on the TE mode whose equations write

$$\mathbf{curl}_\perp \mathrm{E}_z + \partial_t \mathbf{H}_\perp = \mathbf{0}, \qquad \mathrm{curl}_\perp \mathbf{H}_\perp - \partial_t \mathrm{E}_z = \mathrm{H}_\theta|_{\Gamma_\varepsilon}\delta_{\Gamma_\varepsilon} + \mathrm{J}_z \quad \text{in } \mathbb{R}^3 \quad \text{and} \quad \mathrm{E}_z(r=\varepsilon,\theta) = 0$$

with $\mathbf{curl}_\perp \mathrm{E}_z := (\partial_y \mathrm{E}_z, -\partial_x \mathrm{E}_z)$ and $\mathrm{curl}_\perp \mathbf{H}_\perp := \partial_x \mathrm{H}_y - \partial_y \mathrm{H}_x$. Moreover $\mathrm{H}_\theta|_{\Gamma_\varepsilon}$ refers to the exterior trace of H_θ on Γ_ε. We plug the second equation into the first, and end up with an Helmholtz equation on E_z involving also $\partial_t \mathrm{I}$ together with a Dirichlet boundary condition on the wire,

$$-\Delta_\perp \mathrm{E}_z + \partial_t^2 \mathrm{E}_z + (\partial_t \mathrm{H}_\theta)|_{\Gamma_\varepsilon}\delta_{\Gamma_\varepsilon} = -\partial_t \mathrm{J}_z \quad \text{in } \mathbb{R}^3 \quad \text{and} \quad \mathrm{E}_z(r=\varepsilon,\theta) = 0 \tag{13}$$

with $\Delta_\perp \mathrm{E}_z := \partial_x^2 \mathrm{E}_z + \partial_y^2 \mathrm{E}_z$. This problem obviously reduces to a 2D Helmholtz problem in the exterior of a small obstacle representing a cross-section of the wire. Indeed, just define u^ε, p^ε and f by

$$\mathrm{E}_z = u^\varepsilon(x,y)\mathrm{e}^{-\mathrm{i}\omega t}, \qquad (\partial_t \mathrm{H}_\theta)|_{\Gamma_\varepsilon} = \frac{p^\varepsilon}{2\pi\varepsilon}\mathrm{e}^{-\mathrm{i}\omega t} \quad \text{and} \quad \partial_t \mathrm{J}_z = f(x,y)\mathrm{e}^{-\mathrm{i}\omega t},$$

with u^ε, p^ε and f independent of time. With these new notations Eqs. (13) lead to our model problem (the next section contains a more precise and rigorous formulation) that we consider as a 2D time harmonic counterpart of (1) and (2)

$$-\Delta_\perp u^\varepsilon - \omega^2 u^\varepsilon + \frac{\delta_{\Gamma_\varepsilon}}{2\pi\varepsilon} p^\varepsilon = f \quad \text{in } \mathbb{R}^2 \quad \text{and} \quad u^\varepsilon(r=\varepsilon,\theta) = 0. \tag{14}$$

2. Mathematical setting of the model problem

Our purpose for the rest of the article will consist in stating a rigorous formulation for Problem (14) and study the consistency of the corresponding Holland and Simpson's model. Note that finite differences is actually a particular case of finite element discretization. Since the theoretical framework of finite elements appeared to be more convenient for our analysis, we will give proofs for general finite element schemes. As a consequence, the analysis we will give is valid for any type of mesh that enters the finite element framework, not only for Cartesian meshes.

2.1. Geometry and functional setting

First of all we choose a mathematical formulation close to (14) and call it "the exact problem". In our 2D problem the propagation medium is made of an homogeneous material containing a small obstacle with boundary Γ_ε representing a cross-section of the wire. We suppose that Γ_ε is simply the circle of center 0 and radius ε

$$(\Gamma_\varepsilon) : r = \varepsilon. \tag{15}$$

The computational domain is assumed to be a disk, with boundary $\Gamma := \partial D(0,\varrho)$ where $\varrho > 0$, containing this small obstacle,

$$\Omega_\varepsilon := \left\{\mathbf{x} \in \mathbb{R}^2 \text{ s.t. } \varepsilon < |\mathbf{x}| < \varrho\right\} \quad \text{and} \quad \Omega := D(0,\varrho).$$

Although this geometry is very simple, the analysis we present can be easily adapted to the case where Ω contains "fixed" (not depending on ε) obstacles with arbitrary shape in addition to Γ_ε.

Given a set $\omega \subset \mathbb{R}^2$, $L^2(\omega)$ will refer to the set of measurable functions v such that $\|v\|_{0,\omega}^2 := \int_\omega |v|^2 < \infty$ and $H^p(\omega)$ will be the Sobolev space of order p over ω equipped with the norm $\|v\|_{p,\omega}^2 := \sum_{k=0}^p \|\nabla^k v\|_{0,\omega}^2$, see [16] for example. In the sequel, $H^{1/2}(\Gamma)$ will refer to the space of traces on Γ of elements of $H^1(\Omega)$, equipped with the norm $\|v\|_{1/2,\Gamma}^2 = \inf\{\|w\|_{1,\Omega} \mid w_{|\Gamma} = v\}$, and $H^{-1/2}(\Gamma)$ will refer to its topological dual. We also consider similar definitions for $H^{1/2}(\Gamma_\varepsilon)$ and $H^{-1/2}(\Gamma_\varepsilon)$.

2.2. Formulation for the exact problem

We consider a source function $f \in C^\infty(\Omega)$ such that there exists an open neighborhood ω of $\mathbf{0}$ satisfying $\mathrm{supp} f \cap \omega = \emptyset$. The smoothness assumption on f allows a simplification in our presentation but does not restrict seriously our results. The equation that we consider inside Ω are given by (14). They can be rewritten as

$$\Delta u^\varepsilon + \omega^2 u^\varepsilon = -f \quad \text{in } \Omega_\varepsilon \quad \text{and} \quad u^\varepsilon = 0 \quad \text{in } \Omega \setminus \overline{\Omega}_\varepsilon \quad \text{with } \frac{p^\varepsilon}{2\pi\varepsilon} := \frac{\partial}{\partial n_\varepsilon}\left(u^\varepsilon|_{\Omega_\varepsilon}\right). \tag{16}$$

Here $\mathbf{n}_\varepsilon$ is the unit vector normal to Γ_ε directed in the interior of Ω_ε. We also have to impose a condition on the outer boundary of the domain. We shall consider some generic boundary condition that take the form

$$\partial_r u^\varepsilon + T_\Gamma u^\varepsilon = 0 \quad \text{on } \Gamma = \partial\Omega \tag{17}$$

where T_Γ is a continuous linear operator mapping $H^{1/2}(\Gamma)$ into $H^{-1/2}(\Gamma)$. For the forthcoming analysis, we need to assume that T_Γ fulfills two additional conditions,

$$\begin{aligned} &\text{Condition 1: There exists a unique } u_0 \in H^1(\Omega) \text{ such that} \\ &\Delta u_0 + \omega^2 u_0 = -f \quad \text{in } \Omega \quad \text{and} \quad \partial_r u_0 + T_\Gamma u_0 = 0 \quad \text{on } \Gamma. \\ &\text{Condition 2: } \Re e\left\{\int_\Gamma \overline{v} T_\Gamma v \, \mathrm{d}\sigma\right\} \geqslant 0 \quad \forall v \in H^{1/2}(\Gamma). \end{aligned} \tag{18}$$

Such conditions are satisfied in many usual wave propagation problems. For example, a possible choice for T_Γ is the Dirichlet-to-Neumann operator for the 2D Helmholtz equation, see for example [17]. Such a choice corresponds to a Sommerfeld radiation condition which is a rather canonical condition.

In the remaining of the present article, we shall make use of results established in [11]. Although the later article considers Problem (16)–(17) where T_Γ is a 2D Dirichlet-to-Neumann operator, a careful examination shows that, in this article, the properties of T_Γ just come into play in Lemma 3.3 of [11], where only conditions (18) are required. As a consequence, the results of [11] remain valid, only assuming that Conditions (18) are satisfied.

Now we derive a variational formulation for (16). Since u^ε was extended by 0 inside Γ_ε, it belongs to the following subspace of $H^1(\Omega)$ that is closed with respect to $\| \ \|_{1,\Omega}$

$$\mathbb{V}_0^\varepsilon := \left\{ v \in H^1(\Omega) \mid v = 0 \text{ in } \Omega \setminus \overline{\Omega}_\varepsilon \right\}.$$

Take the Helmholtz equation of (16), multiply it by a test function $v \in \mathbb{V}_0^\varepsilon$ and then integrate over Ω_ε. Applying a Green formula over Ω_ε and taking into account Eq. (17), we obtain that u^ε satisfies

$$\begin{aligned} &u^\varepsilon \in \mathbb{V}_0^\varepsilon \quad \text{and} \quad a(u^\varepsilon, v) = \int_\Omega f\overline{v} \, \mathrm{d}\mathbf{x} \quad \forall v \in \mathbb{V}_0^\varepsilon \\ &\text{with} \quad a(v, w) := \int_\Omega \nabla v \nabla \overline{w} \, \mathrm{d}\mathbf{x} - \omega^2 \int_\Omega v\overline{w} \, \mathrm{d}\mathbf{x} + \int_\Gamma \overline{w} T_\Gamma v \mathrm{d}\sigma \quad \forall v, w \in H^1(\Omega). \end{aligned} \tag{19}$$

Note that, according to the continuity properties of T_Γ, the bilinear form $a(\,,\,)$ is continuous on $H^1(\Omega) \times H^1(\Omega)$. This bilinear form also satisfies inf – sup conditions on $\mathbb{V}_0^\varepsilon$, so that (19) is well posed, see [11].

3. Derivation of a simplified model

In this section we propose a mathematical derivation of the continuous setting of Collino and Millot adapted to the case of our model problem. It consists in a new problem that is a simplified version of (16). In a further section we will obtain Holland and Simpson's scheme by applying a particular discretization to this simplified continuous model.

3.1. Simplified problem

In this new problem, the only difference with (19) lies in the boundary condition imposed on Γ_ε. The idea consists in weakening the exact Dirichlet condition. The new condition is suggested by the results of [18–20]. We introduce another closed subspace of $H^1(\Omega)$ that takes into account the new boundary condition on Γ_ε. We define

$$\mathbb{V}_\mu^\varepsilon := \left\{ v \in H^1(\Omega) \Big| \int_{\Gamma_\varepsilon} v \mathrm{d}\sigma = \varepsilon \int_0^{2\pi} v(r = \varepsilon, \theta) \mathrm{d}\theta = 0 \right\}.$$

Note that $\mathbb{V}_0^\varepsilon \subset \mathbb{V}_\mu^\varepsilon$. The new condition is an averaged version of the exact Dirichlet condition. The new variational problem we consider is given by

$$\text{Find } \tilde{u}^\varepsilon \in \mathbb{V}_\mu^\varepsilon \quad \text{such that } a(\tilde{u}^\varepsilon, v) = \int_\Omega f\overline{v} \mathrm{d}\mathbf{x} \quad \forall v \in \mathbb{V}_\mu^\varepsilon. \tag{20}$$

This formulation was introduced and studied in [21,11] where this problem is proved to be well posed because required inf – sup conditions are verified for the bilinear form $a(,)$ restricted to $\mathbb{V}^\varepsilon_\mu$. The following result is contained in Lemmas 1.1 and 1.3 in [11].

Lemma 3.1. *For $V = H^1(\Omega)$, $\mathbb{V}^\varepsilon_0$, $\mathbb{V}^\varepsilon_\mu$ there exists $\kappa_0, \varepsilon_0 > 0$ independent of $\varepsilon > 0$ such that*

$$\inf_{u\in V}\sup_{v\in V}\frac{|a(u,v)|}{\|u\|_{1,\Omega}\|v\|_{1,\Omega}} > \kappa_0 \quad \text{and} \quad \inf_{u\in V}\sup_{v\in V}\frac{|a(v,u)|}{\|u\|_{1,\Omega}\|v\|_{1,\Omega}} > \kappa_0 \quad \forall \varepsilon \in]0,\varepsilon_0[.$$

These inf – sup conditions are uniform with respect to ε. Note that $\tilde{u}^\varepsilon$ does not vanish *a priori* in $\Omega \setminus \overline{\Omega}_\varepsilon$, nor on Γ_ε. Just as for the exact problem, choosing suitable test functions in (20) and applying Green's formula leads to

$$\Delta\tilde{u}^\varepsilon + \omega^2\tilde{u}^\varepsilon = -f \quad \text{in } \Omega_\varepsilon, \qquad \Delta\tilde{u}^\varepsilon + \omega^2\tilde{u}^\varepsilon = 0 \quad \text{in } \Omega\setminus\overline{\Omega}_\varepsilon \quad \text{and} \quad \partial_r\tilde{u}^\varepsilon + T_\Gamma\tilde{u}^\varepsilon = 0 \quad \text{on } \Gamma. \tag{21}$$

The next result, which is Theorem 2.1 of [11], shows that Problem (20) is indeed an approximate model: solving it instead of problem (19) only implies an error in $O(|\ln\varepsilon|\sqrt{\varepsilon})$.

Theorem 3.1. *There exists $\kappa_0, \varepsilon_0 > 0$ independent of ε such that*

$$\|u^\varepsilon - \tilde{u}^\varepsilon\|_{1,\Omega} < \kappa_0|\ln\varepsilon|\sqrt{\varepsilon} \quad \forall\varepsilon\in]0,\varepsilon_0[.$$

3.2. Fictitious domain formulation

In practice, we wish to use a mesh that has been generated independently of the small obstacle. This is the reason why we do not want to take the boundary condition on Γ_ε into account via the variational space. So we consider another formulation of the fictitious domain type (see [22,23] for example) in which the boundary condition appears as an additional equation. The new formulation reads

$$\text{Find } (\tilde{u}^\varepsilon,\tilde{p}^\varepsilon)\in H^1(\Omega)\times\mathbb{C} \quad \text{such that} \quad \begin{cases} a(\tilde{u}^\varepsilon,v) + \tilde{p}^\varepsilon b^\varepsilon(\overline{v}) = \displaystyle\int_\Omega f\overline{v}\,\mathrm{d}\mathbf{x} \quad \forall v\in H^1(\Omega), \\ b^\varepsilon(\tilde{u}^\varepsilon) = 0 \end{cases} \tag{22}$$

$$\text{with} \quad b^\varepsilon(v) := \frac{1}{2\pi\varepsilon}\int_{\Gamma_\varepsilon} v\,\mathrm{d}\sigma.$$

Although the linear form b^ε is bounded, this boundedness is non-uniform with respect to ε. However, according to [11] Lemma 3.1, there exists $\kappa > 0$ independent of ε such that

$$|b^\varepsilon(v)| \leqslant \kappa\sqrt{|\ln\varepsilon|}\|v\|_{1,\Omega} \quad \forall v\in H^1(\Omega),\ \forall\varepsilon\in]0,1[. \tag{23}$$

Formulation (22) is strictly equivalent to (20). Indeed in [21,11] we proved that $(\tilde{u}^\varepsilon,\tilde{p}^\varepsilon)$ is solution to (22) if and only if $\tilde{u}^\varepsilon$ is solution to (20) and

$$\frac{\tilde{p}^\varepsilon}{2\pi\varepsilon} = \frac{\partial}{\partial n_\varepsilon}\big(\tilde{u}^\varepsilon|_{\Omega_\varepsilon}\big) - \frac{\partial}{\partial n_\varepsilon}\big(\tilde{u}^\varepsilon|_{\Omega\setminus\Omega_\varepsilon}\big).$$

4. Standard numerical scheme and the method of Holland and Simpson

We consider a family of triangulations $(\mathcal{T}_\mathrm{h})_{\mathrm{h}\in]0,1[}$ over Ω, made up of triangles or quadrangles, where h is a mesh parameter supposed to go to 0 and representing the size of the cells

$$\mathrm{h} = \max\big\{\mathrm{diam}(K)\mid K\in\mathcal{T}_\mathrm{h}\big\} \to 0. \tag{24}$$

For the definition of a triangulation see [24, Section 2.1]. Since Ω is a circle, it cannot be exactly covered by any triangulation, so we denote $\Omega_\mathrm{h} = \cup_{K\in\mathcal{T}_\mathrm{h}} K$. We assume:

A1 : The family of triangulations$\mathcal{T}_\mathrm{h}$, h > 0is regular (see [24]).

For each h we consider a collection $(K,\Theta_K,\Sigma_K)_{K\in\mathcal{T}_\mathrm{h}}$ of Lagrange finite elements. According to the definition of a regular family of triangulations, there exists a reference finite element $(\hat{K},\hat{\Theta},\hat{\Sigma})$ such that for any $\mathrm{h} > 0$ and any $K\in\mathcal{T}_\mathrm{h}$, there exists a bijection $F_K : \mathbb{R}^2\to\mathbb{R}^2$ such that $F_K\in\mathbb{P}^2_k$ in the case of triangles and $F_K\in\mathbb{Q}^2_k$ in the case of quadrangles for a certain $k\in\mathbb{N}$ (the order of the method), and $K = F_K(\hat{K})$, $\Sigma_K = F_K(\hat{\Sigma})$ and $\Theta_K = \{\hat{v}\circ F_K^{-1}\mid \hat{v}\in\hat{\Theta}\}$. The set Σ_K of degrees

of freedom in the element K will be identified with a set of points in K. In our numerical experiments $\hat{\Theta}$ will be a space of polynomials. So we assume that our discretization satisfies

A2 : $\hat{\Theta} = \mathbb{P}_k$ or $\mathbb{Q}_k$ with $k =$ order of the method.

Let the approximation space that we call "standard" be the set of all continuous functions piecewise polynomial with order k on each element of the mesh namely

$$\mathbb{V}^{\mathrm{h}} := \left\{ v \in C^0(\Omega_{\mathrm{h}}) \mid v_{|K} \in \Theta_K,\ \forall K \in \mathcal{T}_{\mathrm{h}} \right\}. \tag{25}$$

Finally we introduce the usual continuous interpolation operator $\mathrm{P}_{\mathrm{h}} : C^0(\Omega) \to \mathbb{V}^{\mathrm{h}}$ that is uniquely defined by $\mathrm{P}_{\mathrm{h}} v \in \mathbb{V}^{\mathrm{h}}$ and $(\mathrm{P}_{\mathrm{h}} v)(\mathbf{x}) = v(\mathbf{x})$, $\forall \mathbf{x} \in \cup_{K \in \mathcal{T}_{\mathrm{h}}} \Sigma_K$ and $\forall v \in C^0(\Omega)$.

4.0.1. Remark about the curved boundary of the domain

Rigorously we have $\Omega_{\mathrm{h}} \neq \Omega$. Indeed Ω_{h} is only an approximation of Ω, and this generates a numerical error. However the code we used includes high order isoparametric finite elements at the boundary, so that this numerical error was negligible compared to other sources of error. As a consequence, in the remainder of this paper, we neglect the difference between Ω and Ω_{h}, and simply write Ω.

4.0.2. Assumption on the thickness of the small obstacle

In what follows we will study a problem depending on the small parameters i.e. both ε and h. The results that we present are specific to the situation where $\varepsilon \ll \mathrm{h}$, so we introduce a third assumption that somehow translates this in mathematical terms.

A3 : There exists $\alpha > 0$ independent of h such that $\varepsilon = \mathrm{h}^{1+\alpha}, \quad \forall \mathrm{h} \in]0, 1[$.

This assumption implies that, in what follows, functions of ε, h may actually be considered as functions of h only.

4.1. Standard numerical scheme

A first numerical scheme for solving (22) can be straightforwardly derived using the approximation space $\mathbb{V}^{\mathrm{h}}$. Applying standard discretization we obtain

$$\text{Find } (u_{\mathrm{h}}^{\varepsilon}, p_{\mathrm{h}}^{\varepsilon}) \in \mathbb{V}^{\mathrm{h}} \times \mathbb{C} \quad \text{such that } \begin{cases} a(u_{\mathrm{h}}^{\varepsilon}, v) + p_{\mathrm{h}}^{\varepsilon} b^{\varepsilon}(\overline{v}) = \displaystyle\int_{\Omega} f \overline{v}\, \mathrm{d}\mathbf{x} & \forall v \in \mathbb{V}^{\mathrm{h}}, \\ b^{\varepsilon}(u_{\mathrm{h}}^{\varepsilon}) = 0. \end{cases} \tag{26}$$

One advantage of the above numerical scheme is that it allows not to adapt the mesh to the presence of the small obstacle. Moreover (26) is well posed, since inf – sup conditions uniform with respect to ε and h are satisfied (see Lemmas 3.2 and 3.3 in [11]).

Lemma 4.1. *Let* $\mathbb{V}_{\mu}^{\varepsilon,\mathrm{h}} := \{ v \in \mathbb{V}^{\mathrm{h}} \mid b^{\varepsilon}(v) = 0 \}$. *For* $V = \mathbb{V}^{\mathrm{h}}$ *or* $\mathbb{V}_{\mu}^{\varepsilon,\mathrm{h}}$ *there exists* $\kappa_0, \varepsilon_0 > 0$ *independent of* ε *such that*

$$\inf_{u \in V} \sup_{v \in V} \frac{|a(u, v)|}{\|u\|_{1,\Omega} \|v\|_{1,\Omega}} > \kappa_0 \quad \textit{and} \quad \inf_{u \in V} \sup_{v \in V} \frac{|a(v, u)|}{\|u\|_{1,\Omega} \|v\|_{1,\Omega}} > \kappa_0 \quad \forall \varepsilon \in]0, \varepsilon_0[.$$

Such uniform conditions are necessary in order to obtain interesting error estimates. In spite of this result, it was pointed out in [11] that a standard scheme is definitely not sufficiently accurate. This is due to the presence of a logarithmic term in the asymptotic expansion of u^{ε} that a standard approximation space such as $\mathbb{V}^{\mathrm{h}}$ cannot reproduce.

4.2. The method of Holland and Simpson

This lack of accuracy with a standard method is a motivation for introducing the approach of Holland and Simpson that consists in adding a regularization term coupling Lagrange multipliers. For a given parameter $\mathrm{L}_{\varepsilon,\mathrm{h}}$ such a method then reads

$$\text{Find } (u_{\mathrm{h}}^{\varepsilon}, p_{\mathrm{h}}^{\varepsilon}) \in \mathbb{V}^{\mathrm{h}} \times \mathbb{C} \quad \text{such that } \begin{cases} a(u_{\mathrm{h}}^{\varepsilon}, v) + p_{\mathrm{h}}^{\varepsilon} b^{\varepsilon}(\overline{v}) = \displaystyle\int_{\Omega} f \overline{v}\, \mathrm{d}\mathbf{x}, & \forall v \in \mathbb{V}^{\mathrm{h}}, \\ b^{\varepsilon}(u_{\mathrm{h}}^{\varepsilon}) - \mathrm{L}_{\varepsilon,\mathrm{h}} p_{\mathrm{h}}^{\varepsilon} = 0. \end{cases} \tag{27}$$

This formulation takes the same form as the numerical scheme of Holland and Simpson for Problem (14) (see Appendix A.1 for more details). We shall discuss later on the value that must be allocated to $\mathrm{L}_{\varepsilon,\mathrm{h}}$ for $u_{\mathrm{h}}^{\varepsilon}$ to be a good approximation of $\tilde{u}^{\varepsilon}$. A direct advantage of Holland and Simpson's scheme is that it is "local": the influence of the small obstacle induces a

perturbation of the standard scheme only on the nodes belonging to an element containing the small obstacle. Note that, for $\mathrm{L}_{\varepsilon,\mathrm{h}} \neq 0$, $(u^\varepsilon_\mathrm{h}, p^\varepsilon_\mathrm{h})$ is a solution to Problem (27) if and only if

$$p^\varepsilon_\mathrm{h} = \frac{1}{\mathrm{L}_{\varepsilon,\mathrm{h}}} b^\varepsilon(u^\varepsilon_\mathrm{h}) \quad \text{and} \quad a(u^\varepsilon_\mathrm{h}, v) + \frac{1}{\mathrm{L}_{\varepsilon,\mathrm{h}}} b^\varepsilon(u^\varepsilon_\mathrm{h}) b^\varepsilon(\overline{v}) = \int_\Omega f\overline{v}\mathrm{d}\mathbf{x} \quad \forall v \in \mathbb{V}^\mathrm{h}. \tag{28}$$

Using this remark, we prove uniform inf – sup conditions on the sesquilinear form associated to (28) under some conditions on $\mathrm{L}_{\varepsilon,\mathrm{h}}$, which then implies well posedness of Problem (27).

Lemma 4.2. *Let* $\mathbb{C}_+ := \{\lambda \in \mathbb{C} \mid \Re e\{\lambda\} > 0\}$. *There exists* $\mathrm{h}_0, \varepsilon_0, \kappa_0 > 0$ *independent of* ε, h *and* L *such that*

$$\inf_{u\in\mathbb{V}^\mathrm{h}} \sup_{v\in\mathbb{V}^\mathrm{h}} \frac{|a(u,v) + \mathrm{L}^{-1} b^\varepsilon(u) b^\varepsilon(\overline{v})|}{\|u\|_{1,\Omega}\|v\|_{1,\Omega}} > \kappa_0 \quad \textit{and}$$

$$\inf_{v\in\mathbb{V}^\mathrm{h}} \sup_{u\in\mathbb{V}^\mathrm{h}} \frac{|a(u,v) + \mathrm{L}^{-1} b^\varepsilon(u) b^\varepsilon(\overline{v})|}{\|u\|_{1,\Omega}\|v\|_{1,\Omega}} > \kappa_0 \quad \forall \varepsilon \in]0, \varepsilon_0[,\ \forall \mathrm{h} \in]0, \mathrm{h}_0[,\ \forall \mathrm{L} \in \mathbb{C}_+.$$

Proof. The method for proving both of these two inf – sup conditions is nearly the same, so we prove only the first one. Proceed by contradiction and suppose that there are sequences $\mathrm{h}_n \in \mathbb{R}_+, \varepsilon_n \in \mathbb{R}_+, \mathrm{L}_n \in \mathbb{C}_+$ and $u_n \in \mathbb{V}^{\mathrm{h}_n}$ such that $\mathrm{h}_n + \varepsilon_n \to 0$, $\|u_n\|_{1,\Omega} = 1$ and

$$\sup_{v\in\mathbb{V}^\mathrm{h}} \frac{|a(u_n,v) + \mathrm{L}_n^{-1} b^{\varepsilon_n}(u_n) b^{\varepsilon_n}(\overline{v})|}{\|v\|_{1,\Omega}} \xrightarrow[n\to\infty]{} 0. \tag{29}$$

Since (u_n) is bounded in $H^1(\Omega)$, we can consider (extracting a subsequence if necessary) that (u_n) converges weakly in $H^1(\Omega)$ and strongly in $L^2(\Omega)$ toward a limit $u_\infty \in H^1(\Omega)$. Take an arbitrary $v \in C^\infty(\Omega) \cap H^1_\star(\Omega)$ where $H^1_\star(\Omega) := \{v \in H^1(\Omega) \mid v = 0$ in a neighborhood of 0 $\}$ satisfying $\|v\|_{1,\Omega} = 1$. Note that there exists $N \in \mathbb{N}$ independent of n (although it depends on v) such that

$$\mathrm{P}_{\mathrm{h}_n} v \in H^1_\star(\Omega) \quad \text{and} \quad \|\mathrm{P}_{\mathrm{h}_n} v\|_{1,\Omega} \geqslant \frac{1}{2} \quad \forall n \geqslant N.$$

Taking N larger if necessary, we thus have $b^{\varepsilon_n}(v) = b^{\varepsilon_n}(\mathrm{P}_{\mathrm{h}_n} v) = 0$ for $n \geqslant N$. Besides note that $\|\mathrm{P}_{\mathrm{h}_n} v - v\|_{1,\Omega} \to 0$ since $v \in C^\infty(\Omega)$. Applying weak convergence in $H^1(\Omega)$, Eq. (29) leads to

$$a(u_\infty, v) = \lim_{n\to\infty} \left\{ a(u_n, \mathrm{P}_{\mathrm{h}_n} v) + \mathrm{L}_n^{-1} b^{\varepsilon_n}(u_n) b^{\varepsilon_n}(\overline{\mathrm{P}_{\mathrm{h}_n} v}) + a(u_n, \mathrm{P}_{\mathrm{h}_n} v - v) \right\} = 0.$$

We chose v arbitrarily, so we have $a(u_\infty, v) = 0,\ \forall v \in C^\infty(\Omega) \cap H^1_\star(\Omega)$, and this implies $a(u_\infty, v) = 0,\ \forall v \in H^1(\Omega)$ according to the density of $C^\infty(\Omega) \cap H^1_\star(\Omega)$ in $H^1(\Omega)$ (see Proposition 5.14 in [25]). Since $a(,)$ satisfies inf – sup conditions on $H^1(\Omega)$ according to Lemma 3.1, this provides $u_\infty = 0$. In conclusion we obtained $\|u_n\|_{0,\Omega} \to 0$. Now note that

$$\Re e\left\{ \int_\Gamma \overline{v} T_\Gamma v \,\mathrm{d}\sigma \right\} \geqslant 0 \quad \text{and} \quad \Re e\left\{ \mathrm{L}_n^{-1}\, b^\varepsilon(v) b^\varepsilon(\overline{v}) \right\} \geqslant 0 \quad \forall v \in H^1(\Omega). \tag{30}$$

As a consequence we have

$$\begin{aligned} \|\nabla u_n\|^2_{0,\Omega} &\leqslant \|\nabla u_n\|^2_{0,\Omega} + \Re e\left\{ \int_\Gamma \overline{u}_n T_\Gamma u_n \,\mathrm{d}\sigma \right\} + \Re e\left\{ \mathrm{L}_n^{-1}\, b^\varepsilon(u_n) b^\varepsilon(\overline{u}_n) \right\} \\ &\leqslant |a(u_n, u_n) + \mathrm{L}_n^{-1} b^{\varepsilon_n}(u_n) b^{\varepsilon_n}(\overline{u}_n)| + \|u_n\|^2_{0,\Omega} \xrightarrow[n\to\infty]{} 0. \end{aligned}$$

In conclusion we obtained that $\|u_n\|^2_{1,\Omega} = \|u_n\|^2_{0,\Omega} + \|\nabla u_n\|^2_{0,\Omega} \to 0$ whereas we supposed at the beginning that $\|u_n\|^2_{1,\Omega} = 1$. This provides a contradiction and concludes the proof. □

4.3. Calibration and precision of Holland and Simpson's method

In this paragraph, we want to show the results of an experiment that lead to the following conclusion: for a given mesh, and a given small obstacle, there exists a value of $\mathrm{L}_{\varepsilon,\mathrm{h}}$ for which the numerical scheme (27) is accurate. Let us present the conditions of this experiment. First of all the domain of computation is $\Omega = D(0, 3)$. We considered the operator

$$T_\Gamma v = \lambda_\Gamma v \quad \text{with } \lambda_\Gamma := -\omega \frac{\mathrm{H}_0^{(1)\prime}(\omega\varrho)}{\mathrm{H}_0^{(1)}(\omega\varrho)}$$

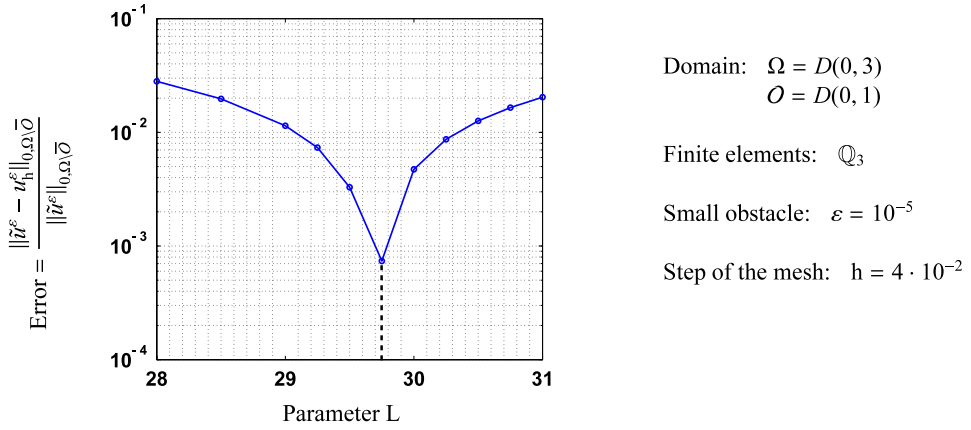


Fig. 3. Calibration of Holland–Simpson's method.

where $H_0^{(1)}(r)$ refers to the Hankel function of order 1 of the first kind (see Chapter 5 of [26] for a detailed definition). This choice of T_Γ matches the conditions (18). In particular we have $\Re e\{\lambda_\Gamma\} \geqslant 0$, see Formula (A.37) in [27] for example. We considered the following problem,

$$\text{Find } \tilde{u}^\varepsilon \in \mathbb{V}_\mu^\varepsilon \quad \text{such that } a(\tilde{u}^\varepsilon, v) = \int_\Gamma \left(\frac{\partial u_i}{\partial n_\Gamma} + \lambda_\Gamma u_i \right) \overline{v}\, d\sigma \quad \forall v \in \mathbb{V}_\mu^\varepsilon \tag{31}$$

with $u_i(\mathbf{x}) = -e^{i\omega r \cos\theta}$. Problem (31) has exactly the same form as (20). Admittedly, the right hand side is slightly different but, as can be easily checked, our analysis remains fully valid for a source term of this kind. The motivation for considering such a right hand side is that it makes implementation easier. The explicit expression of the solution to Problem (31) can be computed using the Jacobi–Anger formula, see [26] formula (5.10.8),

$$\tilde{u}^\varepsilon(\mathbf{x}) = u_i(\mathbf{x}) + J_0(\omega|\mathbf{x}|) \quad \text{if } |\mathbf{x}| < \varepsilon \quad \text{and} \quad \tilde{u}^\varepsilon(\mathbf{x}) = u_i(\mathbf{x}) + J_0(\omega\varepsilon)\frac{H_0^{(1)}(\omega|\mathbf{x}|)}{H_0^{(1)}(\omega\varepsilon)} \quad \text{if } |\mathbf{x}| > \varepsilon. \tag{32}$$

For our numerical experiment, we solved the following associated Holland–Simpson's scheme, for a fixed mesh and a fixed value of ε, and different values of the fitting parameter L: find $(u_h^\varepsilon, p_h^\varepsilon) \in \mathbb{V}^h \times \mathbb{C}$ such that

$$\begin{cases} \tilde{a}(u_h^\varepsilon, v) + p_h^\varepsilon b^\varepsilon(\overline{v}) = \displaystyle\int_\Gamma \left(\frac{\partial u_i}{\partial n_\Gamma} + \lambda_\Gamma u_i \right) \overline{v}\, d\sigma \quad \forall v \in \mathbb{V}^h, \\ b^\varepsilon(u_h^\varepsilon) - L \cdot p_h^\varepsilon = 0. \end{cases} \tag{33}$$

We considered a frequency $\omega = 2\pi$, a quadrangular mesh with average step $h = 0.04$ and a small obstacle with thickness $\varepsilon = 10^{-5}$, and we used $\mathbb{Q}_3$ finite elements. In Fig. 3 we represent the relative error that we obtained when varying the parameter L. Note that the error is measured in an opened set that excludes a neighborhood of the small obstacle $\mathcal{O} = D(0, 1)$.

With this experiment it appears that there exists one single value of L for which the discrete formulation (33) yields an approximate solution with good accuracy. We call calibration a procedure consisting in determining this critical value.

4.3.1. Calibration in practice

Obviously the preceding remarks raise the question of how to determine the critical value of Holland–Simpson's parameter when the analytical solution of the problem under study is not known. In practice, one refers to tables containing the critical values of the parameter for simple model situations where the solution of the problem is known, and choose among these model problems the one that is closest to one's concrete problem. However this approach remains often unsatisfactory and there are many simple situations where calibrating becomes a tricky problem.

5. Consistency of Holland and Simpson's scheme

In this section we propose to derive again Holland–Simpson's method in a different manner, exhibiting a link between this method and the augmented Galerkin scheme studied in [11] that is very close the singular function method. As the consistency of the augmented Galerkin scheme is already established (Theorem 3.1 in [11]), this will lead to a proof of consistency for the method of Holland and Simpson, and to a theoretical value for Holland–Simpson's parameter.

5.1. Recall on the Augmented Galerkin approach

The Augmented Galerkin scheme consists in adding a singular function in the approximation space so that it becomes possible to approximate the logarithmic behavior appearing in the expansion of $\tilde{u}^\varepsilon$ when $\varepsilon \to 0$. The augmented Galerkin scheme is a discrete formulation similar to (26) but with the following volumic approximation space

$$\mathbb{V}^{\mathrm{h}}_{\mathbf{e}} := \mathbb{V}^{\mathrm{h}} \oplus \mathrm{span}\{\Psi^\varepsilon\} \quad \text{with } \Psi^\varepsilon(\mathbf{x}) := \ln\left(\frac{|\mathbf{x}|}{\varepsilon}\right)\chi(\mathbf{x})\mathbf{1}_{\mathrm{r}>\varepsilon}. \tag{34}$$

In this definition, $\chi \in C^\infty(\overline{\mathbb{R}}_+)$ is a decreasing cut-off function that satisfies $\chi : \mathbb{R}_+ \to [0, 1]$ and $\chi = 1$ in a neighborhood of $\mathbf{0}$, and $\chi = 0$ in a neighborhood of Γ. The augmented Galerkin formulation then writes

$$\text{Find } (\tilde{u}^\varepsilon_{\mathrm{h}}, \tilde{p}^\varepsilon_{\mathrm{h}}) \in \mathbb{V}^{\mathrm{h}}_{\mathbf{e}} \times \mathbb{C} \quad \text{such that } \begin{cases} a(\tilde{u}^\varepsilon_{\mathrm{h}}, v) + \tilde{p}^\varepsilon_{\mathrm{h}}\, b^\varepsilon(\overline{v}) = \displaystyle\int_\Omega f\overline{v}\,\mathrm{d}\mathbf{x} \quad \forall v \in \mathbb{V}^{\mathrm{h}}_{\mathbf{e}}, \\ b^\varepsilon(\tilde{u}^\varepsilon_{\mathrm{h}}) = 0. \end{cases} \tag{35}$$

The linear forms $a()$ and $b^\varepsilon()$ satisfy inf – sup conditions that are uniform with respect to ε and h, see [11] Lemmas 3.2 and 3.3. This implies that Formulation (35) is well posed. We denote its solution $(\tilde{u}^\varepsilon_{\mathrm{h}}, \tilde{p}^\varepsilon_{\mathrm{h}})$ in order to distinguish it from $(u^\varepsilon_{\mathrm{h}}, p^\varepsilon_{\mathrm{h}})$ that refers to the solution to (27). Uniform inf – sup conditions also imply that $(\tilde{u}^\varepsilon_{\mathrm{h}}, \tilde{p}^\varepsilon_{\mathrm{h}})$ is uniformly bounded with respect to ε, h i.e. there exists $\kappa > 0$ independent of ε, h such that

$$\|\tilde{u}^\varepsilon_{\mathrm{h}}\|_{1,\Omega} + |\tilde{p}^\varepsilon_{\mathrm{h}}| \leqslant \kappa \|f\|_{0,\Omega} \quad \forall \varepsilon]0, 1[,\ \forall \mathrm{h} \in]0, 1[. \tag{36}$$

In [11] Theorem 3.1, we proved that the formulation (35) satisfies the following "quasi-locking free" consistency result.

Theorem 5.1. *There exists $\varepsilon_0, \kappa_0 > 0$ independent of ε such that*

$$\|\tilde{u}^\varepsilon - \tilde{u}^\varepsilon_{\mathrm{h}}\|_{1,\Omega} + |\tilde{p}^\varepsilon - \tilde{p}^\varepsilon_{\mathrm{h}}| \leqslant \kappa_0 |\ln \varepsilon|^{\frac{3}{2}} \left(\sqrt{\varepsilon} + \mathrm{h}\right) \quad \forall \varepsilon \in]0, \varepsilon_0[,\ \forall \mathrm{h} \in]0, 1[.$$

5.2. Study of the additional shape function

In order to propose another derivation of the formalism of Holland and Simpson using (35), we need to study the impact of the additional shape function in the augmented Galerkin method. This boils down to determining how well Ψ^ε can be approximated by the elements of $\mathbb{V}^{\mathrm{h}}$.

5.2.1. Elliptic projections

Let us introduce two additional operators $\Pi_{\mathrm{h}}, \Pi^*_{\mathrm{h}} : H^1(\Omega) \to \mathbb{V}^{\mathrm{h}}$ defined as follows

$$\begin{aligned} \forall u \in H^1(\Omega) \quad & \Pi_{\mathrm{h}}(u) \in \mathbb{V}^{\mathrm{h}} \quad \text{such that } a\big(\Pi_{\mathrm{h}}(u), v\big) = a(u, v)\ \forall v \in \mathbb{V}^{\mathrm{h}}, \\ & \Pi^*_{\mathrm{h}}(u) \in \mathbb{V}^{\mathrm{h}} \quad \text{such that } a\big(v, \Pi^*_{\mathrm{h}}(u)\big) = a(v, u)\ \forall v \in \mathbb{V}^{\mathrm{h}}. \end{aligned} \tag{37}$$

According to the inf – sup conditions provided by Lemma 4.1, the problems defining Π_{h} and Π^*_{h} are well posed so that these operators are well defined. For a smooth function u, one could reasonably expect that $\Pi_{\mathrm{h}}(u)$ and $\Pi^*_{\mathrm{h}}(u)$ represent somehow optimal approximations of u by elements of $\mathbb{V}^{\mathrm{h}}$. Indeed there exist $\kappa > 0$ independent of h such that

$$\inf_{v\in\mathbb{V}^{\mathrm{h}}} \|u - v\|_{1,\Omega} \leqslant \|u - \Pi_{\mathrm{h}}(u)\|_{1,\Omega} \leqslant \kappa \inf_{v\in\mathbb{V}^{\mathrm{h}}} \|u - v\|_{1,\Omega} \quad \forall u \in H^1(\Omega),\ \forall \mathrm{h} \in]0, 1[\tag{38}$$

and a similar statement holds for Π^*_{h}. We also introduce a notation for the error related to these projections,

$$\mathrm{D}_{\mathrm{h}}(u) := u - \Pi_{\mathrm{h}}(u) \quad \text{and} \quad \mathrm{D}^*_{\mathrm{h}}(u) := u - \Pi^*_{\mathrm{h}}(u) \quad \forall u \in H^1(\Omega).$$

As a consequence of classical results on finite element methods (Ω is smooth and we use isoparametric elements with sufficiently high order), see [24] for example, for any $p \in \mathbb{N}$ there exists $\kappa_p > 0$ independent of h such that

$$\|\mathrm{D}_{\mathrm{h}}(u)\|_{1,\Omega} + \|\mathrm{D}^*_{\mathrm{h}}(u)\|_{1,\Omega} \leqslant \kappa_p \mathrm{h}^p \|u\|_{p+1,\Omega} \quad \forall u \in H^{p+1}(\Omega),\ \forall \mathrm{h} \in]0, 1[. \tag{39}$$

5.2.2. Global approximation by standard shape functions

Whereas $\Pi_{\mathrm{h}}(u)$ and $\Pi^*_{\mathrm{h}}(u)$ are good approximations for u with respect to $\|\ \|_{1,\Omega}$ as soon as u is smooth, the following proposition shows that a similar result does not hold for Ψ^ε. A proof can be found in Appendix.

Proposition 5.1. *Under assumptions* **A1–A3** *there exists* $\kappa > 0$ *independent of* h *(and consequently also independent of* ε*) such that*

$$\kappa < \|\mathrm{D_h}(\Psi^\varepsilon)\|_{1,\Omega} \quad \textit{and} \quad \kappa < \|\mathrm{D_h^*}(\Psi^\varepsilon)\|_{1,\Omega} \quad \forall \mathrm{h} \in]0,\,1[.$$

According to (38), Proposition 5.1 shows that it is impossible to approximate properly Ψ^ε in the H^1 norm with standard shape functions. Besides an easy and explicit calculus (recall that $\varepsilon = \mathrm{h}^{1+\alpha}$) yields $\kappa_0 > 0$ independent of h such that

$$\|\Psi^\varepsilon\|_{1,\Omega} \leqslant \kappa_0 |\ln \mathrm{h}| \quad \forall \mathrm{h} \in]0,\,1[.$$

Since there exists $\kappa_1 > 0$ independent of ε and h such that $\|\Pi_\mathrm{h}^*(v)\|_{1,\Omega} + \|\Pi_\mathrm{h}(v)\|_{1,\Omega} \leqslant \kappa_1 \|v\|_{1,\Omega}$ (see definition (37) and Lemma 4.1), this leads to $\kappa_2 > 0$ independent of h (and thus independent of ε) such that

$$\|\mathrm{D_h^*}(\Psi^\varepsilon)\|_{1,\Omega} + \|\mathrm{D_h}(\Psi^\varepsilon)\|_{1,\Omega} \leqslant \kappa_2 |\ln \mathrm{h}| \quad \forall \mathrm{h} \in]0,\,1[.$$

As expected though, the next result shows that possible estimates in the L^2 norm are sharper.

Lemma 5.1. *Under assumptions* **A1–A3** *there exists* $\kappa > 0$ *independent of* h *(and consequently also independent of* ε*) such that*

$$\|\mathrm{D_h}(\Psi^\varepsilon)\|_{0,\Omega} < \kappa \mathrm{h} \|\mathrm{D_h}(\Psi^\varepsilon)\|_{1,\Omega} \quad \textit{and} \quad \|\mathrm{D_h^*}(\Psi^\varepsilon)\|_{0,\Omega} < \kappa \mathrm{h} \|\mathrm{D_h^*}(\Psi^\varepsilon)\|_{1,\Omega} \quad \forall \mathrm{h} \in]0,\,1[.$$

Proof. We only give the proof for the estimate on $\|\mathrm{D_h}(\Psi^\varepsilon)\|_{0,\Omega}$ since the proof for $\|\mathrm{D_h^*}(\Psi^\varepsilon)\|_{0,\Omega}$ is very similar. For a given $v \in L^2(\Omega)$ denote $g(v)$ the unique element of $H^2(\Omega)$ verifying $a(w, g(v)) = \int_\Omega w\,\overline{v}\,\mathrm{d}\mathbf{x}$, $\forall w \in H^1(\Omega)$. Such a function exists according to the inf – sup conditions of Lemma 4.1. Write the characterization of $\|\ \|_{0,\Omega}$ based on the scalar product

$$\|\mathrm{D_h}(\Psi^\varepsilon)\|_{0,\Omega} = \sup_{v \in L^2(\Omega)} \frac{\int_\Omega \mathrm{D_h}(\Psi^\varepsilon)\overline{v}\,\mathrm{d}\mathbf{x}}{\|v\|_{0,\Omega}} = \sup_{v \in L^2(\Omega)} \frac{a\big(\mathrm{D_h}(\Psi^\varepsilon), g(v)\big)}{\|v\|_{0,\Omega}} = \sup_{v \in L^2(\Omega)} \frac{a\big(\mathrm{D_h}(\Psi^\varepsilon), \mathrm{D_h}(g(v))\big)}{\|v\|_{0,\Omega}}.$$

Note that $g(v)$ continuously depends on v: $\exists \kappa_0 > 0$ such that $\|g(v)\|_{2,\Omega} \leqslant \kappa_0 \|v\|_{0,\Omega}$, $\forall v \in L^2(\Omega)$. Since $a(,\,)$ is continuous and (39) holds, $\exists \kappa_1, \kappa_2 > 0$ independent of ε, h such that

$$\|\mathrm{D_h}(\Psi^\varepsilon)\|_{0,\Omega} \leqslant \kappa_1 \|\mathrm{D_h}(\Psi^\varepsilon)\|_{1,\Omega} \sup_{v \in L^2(\Omega)} \{\|\mathrm{D_h}(g(v))\|_{1,\Omega}/\|v\|_{0,\Omega}\} \leqslant \kappa_2\, \mathrm{h} \|\mathrm{D_h}(\Psi^\varepsilon)\|_{1,\Omega}. \quad \square$$

Note that, according to the definition of Π_h, we have $a(\mathrm{D_h}(\Psi^\varepsilon), \mathrm{D_h^*}(\Psi^\varepsilon)) = a(\mathrm{D_h}(\Psi^\varepsilon), \mathrm{D_h}(\Psi^\varepsilon))$. Besides, according to inequality (30), Proposition 5.1 and Lemma 5.1, there exists $\kappa_0, \kappa_1 > 0$ independent of ε, h such that $\forall \varepsilon, \mathrm{h} \in]0,\,1[$ we have

$$\Re e\Big\{a\big(\mathrm{D_h}(\Psi^\varepsilon), \mathrm{D_h^*}(\Psi^\varepsilon)\big)\Big\} \geqslant \|\mathrm{D_h}(\Psi^\varepsilon)\|_{1,\Omega}^2 - \omega^2 \|\mathrm{D_h}(\Psi^\varepsilon)\|_{0,\Omega}^2 \geqslant \kappa_1\big(1 - \kappa_0(1+\omega^2)\mathrm{h}^2\big). \tag{40}$$

5.2.3. Local approximation by standard shape functions

The singularity of Ψ^ε is the reason why a standard approximation space cannot properly reproduce it. However it is possible to state sharper results when we restrict our study to a subset of Ω that excludes a fixed neighborhood of 0.

Lemma 5.2. *Let* k *be the order of finite elements used for discretization. Let* $\mathcal{O}$ *be an open set verifying* $\mathcal{O} \subset \overline{\mathcal{O}} \subset \Omega \setminus \{0\}$. *If the assumptions* **A1–A3** *are satisfied, there exist constants* $\kappa_0 > 0$ *independent of* ε *and* h *such that*

$$\|\mathrm{D_h}(\Psi^\varepsilon)\|_{1,\mathcal{O}} \leqslant \kappa_0 |\ln \varepsilon| \mathrm{h}^k \quad \textit{and} \quad \|\mathrm{D_h^*}(\Psi^\varepsilon)\|_{1,\mathcal{O}} \leqslant \kappa_0 |\ln \varepsilon| \mathrm{h}^k \quad \forall \varepsilon, \mathrm{h} \in]0,\,1[.$$

Proof. Once again we only prove the estimate related to $\mathrm{D_h}(\Psi^\varepsilon)$, since very similar arguments would provide the result for $\mathrm{D_h^*}(\Psi^\varepsilon)$. To prove this, we make use of the Nitsche and Schatz theorem [28] for which we state a simplified version in Appendix. Consider another open set $\mathcal{O}'$ satisfying $\mathcal{O} \subset \mathcal{O}' \subset \overline{\mathcal{O}'} \subset \Omega \setminus \{0\}$. Let us apply Nitsche and Schatz theorem to Ψ^ε in $\mathcal{O}$: there exist constant $\kappa_0 > 0$ independent of ε, h such that

$$\|\mathrm{D_h}(\Psi^\varepsilon)\|_{1,\mathcal{O}} \leqslant \kappa_0\big(\mathrm{h}^k \|\Psi^\varepsilon\|_{k+1,\mathcal{O}'} + \|\mathrm{D_h}(\Psi^\varepsilon)\|_{-k,\mathcal{O}'}\big) \quad \forall \varepsilon, \mathrm{h} \in]0,\,1[. \tag{41}$$

First of all, since $\mathcal{O}'$ excludes a fixed neighborhood of 0, there clearly exists a constant $\kappa > 0$ independent of ε, h such that $\|\Psi^\varepsilon\|_{k+1,\mathcal{O}'} \leqslant \kappa |\ln \varepsilon|$. Let us consider $H_0^k(\mathcal{O}')$ defined as the closure of $\mathcal{D}(\mathcal{O}')$ in $H^k(\mathcal{O}')$, and equipped with the norm $\|\ \|_{k,\mathcal{O}'}$. By definition $H^{-k}(\mathcal{O}')$ is the dual space of $H_0^k(\mathcal{O}')$ which is a Banach space equipped with the norm

$$\|w\|_{-k,\mathcal{O}'} := \sup_{v \in H_0^k(\mathcal{O}')} \frac{\int_\Omega w\overline{v}\,\mathrm{d}\mathbf{x}}{\|v\|_{k,\mathcal{O}'}}. \tag{42}$$

Like in the proof of the preceding lemma, for $v \in L^2(\Omega)$ let us define $g(v)$ as the unique function in $H^2(\Omega)$ satisfying $a(w, g(v)) = \int_\Omega w\overline{v}\,\mathrm{d}\mathbf{x}$, $\forall w \in H^1(\Omega)$. According to the continuity of $a(,)$ there exists a constant $\kappa > 0$ independent of ε, h such that $\forall v \in H^k(\Omega)$ we have

$$\left|\int_\Omega \mathrm{D_h}(\Psi^\varepsilon)\overline{v}\,\mathrm{d}\mathbf{x}\right| = |a\big(\mathrm{D_h}(\Psi^\varepsilon), g(v)\big)| = |a\big(\mathrm{D_h}(\Psi^\varepsilon), \mathrm{D_h}(g(v))\big)| \leqslant \kappa\|\mathrm{D_h}(\Psi^\varepsilon)\|_{1,\Omega}\|\mathrm{D_h}(g(v))\|_{1,\Omega}.$$

Note that $g(v) \in H^{k+2}(\Omega)$ when $v \in H^k(\Omega)$ and there exists $\kappa' > 0$ such that $\|g(v)\|_{k+2,\Omega} \leqslant \kappa'\|v\|_{k,\Omega}$. Using (39), we obtain the existence of a constant $\kappa'' > 0$ independent of ε, h such that, for any h $\in]0, 1[$ and any $v \in H^k(\Omega)$, we have

$$\left|\int_\Omega \mathrm{D_h}(\Psi^\varepsilon)\overline{v}\,\mathrm{d}\mathbf{x}\right| \leqslant \kappa''\mathrm{h}^{k+1}\|\mathrm{D_h}(\Psi^\varepsilon)\|_{1,\Omega}\|g(v)\|_{k+2,\Omega} \leqslant \kappa'\kappa''\mathrm{h}^{k+1}\|\mathrm{D_h}(\Psi^\varepsilon)\|_{1,\Omega}\|v\|_{k,\Omega}.$$

Choose $v \in H_0^k(\mathcal{O}')$. Extending v by 0 in $\Omega \setminus \overline{\mathcal{O}'}$, we can consider that $v \in H^k(\Omega)$ with $\|v\|_{k,\mathcal{O}'} = \|v\|_{k,\Omega}$. Then using the preceding inequality, we obtain the existence of $\kappa > 0$ such that

$$\left|\int_\Omega \mathrm{D_h}(\Psi^\varepsilon)\overline{v}\,\mathrm{d}\mathbf{x}\right| \leqslant \kappa\mathrm{h}^{k+1}\|\mathrm{D_h}(\Psi^\varepsilon)\|_{1,\Omega}\|v\|_{k,\mathcal{O}'} \quad \forall v \in H_0^k(\mathcal{O}')\ \forall \mathrm{h} \in]0, 1[. \tag{43}$$

According to inf – sup conditions applied to (37), there exists a constant $\kappa > 0$ independent of ε, h such that $\|\mathrm{D_h}(\Psi^\varepsilon)\|_{1,\Omega} \leqslant \kappa\|\Psi^\varepsilon\|_{1,\Omega}$. Besides $\|\Psi^\varepsilon\|_{1,\Omega} \leqslant 2\sqrt{|\Omega|}|\ln\varepsilon|$ for any $\varepsilon \in]0, 1[$. Plugging (43) into (42) and then into (41) we obtain the desired result. □

The conclusion coming out from this section is that the additional shape function is well approximated by elements of $\mathbb{V}^\mathrm{h}$ in any open set that excludes a fixed neighborhood of the origin, but its behavior near 0 cannot be reproduced by the standard approximation space. So $\mathbb{V}^\mathrm{h}_\mathbf{e}$ provides wider approximation properties as ε(h) and h go to 0.

5.3. Reformulation of the Augmented Galerkin method

The presence of the additional shape function in Formulation (35) is not so comfortable a situation. For example, computing accurately the integrals involving Ψ^ε can be problematic because of its singular behavior. As a consequence we propose to rewrite Formulation (35) so as to get rid of terms related to the additional shape function Ψ^ε by decomposing $\tilde{u}^\varepsilon_\mathrm{h}$ the solution to (35) in a particular manner. Consider

$$\mathbb{V}^\mathrm{h}_\mathbf{e} = \mathbb{V}^\mathrm{h} \oplus \mathrm{span}\big\{\mathrm{D_h}(\Psi^\varepsilon)\big\} = \mathbb{V}^\mathrm{h} \oplus \mathrm{span}\big\{\mathrm{D^*_h}(\Psi^\varepsilon)\big\}. \tag{44}$$

The following decomposition of $\tilde{u}^\varepsilon_\mathrm{h}$ uniquely defines $\hat{u}^\varepsilon_\mathrm{h}$ and $\hat{p}^\varepsilon_\mathrm{h}$ as follows

$$\tilde{u}^\varepsilon_\mathrm{h} = \hat{u}^\varepsilon_\mathrm{h} + \hat{p}^\varepsilon_\mathrm{h}\cdot\mathrm{D_h}(\Psi^\varepsilon), \quad \hat{u}^\varepsilon_\mathrm{h} \in \mathbb{V}^\mathrm{h} \text{ and } \hat{p}^\varepsilon_\mathrm{h} \in \mathbb{C}.$$

Plugging this decomposition into Formulation (35), and decomposing test functions according to the second direct sum of (44), we obtain: $(\hat{u}^\varepsilon_\mathrm{h}, \hat{p}^\varepsilon_\mathrm{h}, \tilde{p}^\varepsilon_\mathrm{h}) \in \mathbb{V}^\mathrm{h} \times \mathbb{C}^2$ and

$$\begin{cases} a(\hat{u}^\varepsilon_\mathrm{h}, v) + \tilde{p}^\varepsilon_\mathrm{h} b^\varepsilon(\overline{v}) = \displaystyle\int_\Omega f\overline{v}\,\mathrm{d}\mathbf{x} \quad \forall v \in \mathbb{V}^\mathrm{h}, \\ \hat{p}^\varepsilon_\mathrm{h} a\big(\mathrm{D_h}(\Psi^\varepsilon), \mathrm{D^*_h}(\Psi^\varepsilon)\big) + \tilde{p}^\varepsilon_\mathrm{h}\, b^\varepsilon(\overline{\mathrm{D^*_h}(\Psi^\varepsilon)}) = \displaystyle\int_\Omega f\,\overline{\mathrm{D^*_h}(\Psi^\varepsilon)}\,\mathrm{d}\mathbf{x}, \\ b^\varepsilon(\hat{u}^\varepsilon_\mathrm{h}) + \hat{p}^\varepsilon_\mathrm{h} b^\varepsilon\big(\mathrm{D_h}(\Psi^\varepsilon)\big) = 0. \end{cases} \tag{45}$$

Now we get rid of the unknown $\hat{p}^\varepsilon_\mathrm{h}$ by a simple algebraic manipulation: we express $\hat{p}^\varepsilon_\mathrm{h}$ with $\tilde{p}^\varepsilon_\mathrm{h}$ using the second equation. We obtain

$$\hat{p}^\varepsilon_\mathrm{h} = -\frac{b^\varepsilon\big(\overline{\mathrm{D^*_h}(\Psi^\varepsilon)}\big)}{a\big(\mathrm{D_h}(\Psi^\varepsilon), \mathrm{D^*_h}(\Psi^\varepsilon)\big)}\tilde{p}^\varepsilon_\mathrm{h} + \frac{\int_\Omega f\overline{\mathrm{D^*_h}(\Psi^\varepsilon)}\,\mathrm{d}\mathbf{x}}{a\big(\mathrm{D_h}(\Psi^\varepsilon), \mathrm{D^*_h}(\Psi^\varepsilon)\big)}.$$

The above identity assumes that $a\big(\mathrm{D_h}(\Psi^\varepsilon), \mathrm{D^*_h}(\Psi^\varepsilon)\big) \neq 0$ which is verified according to (40). Plugging this identity into the third equation of (45) we see that $(\hat{u}^\varepsilon_\mathrm{h}, \tilde{p}^\varepsilon_\mathrm{h}) \in \mathbb{V}^\mathrm{h} \times \mathbb{C}$ such that

$$\begin{cases} a(\hat{u}^\varepsilon_\mathrm{h}, v) + \tilde{p}^\varepsilon_\mathrm{h} b^\varepsilon(\overline{v}) = \displaystyle\int_\Omega f\overline{v}\,\mathrm{d}\mathbf{x} \quad \forall v \in \mathbb{V}^\mathrm{h}, \\ b^\varepsilon(\hat{u}^\varepsilon_\mathrm{h}) - \Lambda_{\varepsilon,\mathrm{h}}\tilde{p}^\varepsilon_\mathrm{h} = g_{\varepsilon,\mathrm{h}}. \end{cases} \tag{46}$$

We eliminated the terms related to the additional shape function, reducing the problem to a formulation where appear two numbers:

$$\Lambda_{\varepsilon,\mathrm{h}} = \frac{b^\varepsilon\big(\mathrm{D_h}(\Psi^\varepsilon)\big)b^\varepsilon\big(\overline{\mathrm{D}^*_{\mathrm{h}}(\Psi^\varepsilon)}\big)}{a\big(\mathrm{D_h}(\Psi^\varepsilon),\mathrm{D}^*_{\mathrm{h}}(\Psi^\varepsilon)\big)} \quad \text{and} \quad g_{\varepsilon,\mathrm{h}} = -\frac{b^\varepsilon\big(\mathrm{D_h}(\Psi^\varepsilon)\big)}{a\big(\mathrm{D_h}(\Psi^\varepsilon),\mathrm{D}^*_{\mathrm{h}}(\Psi^\varepsilon)\big)}\int_\Omega f\overline{\mathrm{D}^*_{\mathrm{h}}(\Psi^\varepsilon)}\,\mathrm{d}\mathbf{x}. \tag{47}$$

Observe that, according to Lemmas 5.2 and A.1, the number $\Lambda_{\varepsilon,\mathrm{h}}$ mainly depends on the value of $\mathrm{D}^*_{\mathrm{h}}(\Psi^\varepsilon)$ in the neighborhood of the small obstacle. Another important remark for what follows is that $\Lambda_{\varepsilon,\mathrm{h}}$ admits a positive real part for h small enough, according to the following lemma.

Lemma 5.3. *Under assumptions* **A1–A3**, *there exists* $\mathrm{h}_0 > 0$ *such that* $\Re e\{\Lambda_{\varepsilon,\mathrm{h}}\} > 0,\ \forall \mathrm{h} \in]0, \mathrm{h}_0[$.

Proof. Denote $\lambda_{\mathrm{h}} = a(\mathrm{D_h}(\Psi^\varepsilon), \mathrm{D}^*_{\mathrm{h}}(\Psi^\varepsilon))$. Lemma A.2 in Appendix shows that $\Lambda_{\varepsilon,\mathrm{h}} = \lambda_{\mathrm{h}} \cdot (1 + o(\mathrm{h}\ln\mathrm{h}))$ since $\varepsilon = \mathrm{h}^{1+\alpha}$ according to Assumption **A3**. Finally (40) shows that there exists $\mathrm{h}_0 > 0$ such that $\Re e\{\lambda_{\mathrm{h}}\} > 0\ \forall \mathrm{h} \in]0, \mathrm{h}_0[$. □

5.4. Reinterpretation of Holland and Simpson's scheme

Now we exhibit a relation between Holland–Simpson's method and the Augmented Galerkin approach. Note that Formulation (46) presents strong similarities with the method of Holland and Simpson (27): the equations take exactly the same form except that in (46) there appears an additional source term $g_{\varepsilon,\mathrm{h}}$ instead of 0 in (27). Actually this additional source term can be considered negligible.

Proposition 5.2. *Let* k *be the order of finite elements used for discretization. If the assumptions* **A1–A3** *are satisfied, there exists a constant* $\kappa_0 > 0$ *independent of* h *such that*

$$|g_{\varepsilon,\mathrm{h}}| \leqslant \kappa_0 |\ln \mathrm{h}| \mathrm{h}^k \quad \forall \mathrm{h} \in]0, 1[.$$

Proof. The number $g_{\varepsilon,\mathrm{h}}$ is simply the product of three terms that we estimate separately. Using Lemma A.2 in Appendix, we easily obtain upper bounds for the terms $b^\varepsilon(\mathrm{D_h}(\Psi^\varepsilon))$ and $a(\mathrm{D_h}(\Psi^\varepsilon), \mathrm{D}^*_{\mathrm{h}}(\Psi^\varepsilon))^{-1}$: there exists $\kappa_0 > 0$ independent of ε, h such that

$$\left|\frac{b^\varepsilon\big(\mathrm{D}^*_{\mathrm{h}}(\Psi^\varepsilon)\big)}{a\big(\mathrm{D_h}(\Psi^\varepsilon),\mathrm{D}^*_{\mathrm{h}}(\Psi^\varepsilon)\big)}\right| + \left|\frac{b^\varepsilon\big(\mathrm{D_h}(\Psi^\varepsilon)\big)}{a\big(\mathrm{D_h}(\Psi^\varepsilon),\mathrm{D}^*_{\mathrm{h}}(\Psi^\varepsilon)\big)}\right| \leqslant \kappa_0, \quad \forall \varepsilon, \mathrm{h} \in]0, 1[. \tag{48}$$

For the last term we use Lemma 5.2 noting that, according to the description of our model problem in Section 2.2, there exists an open neighborhood $\mathcal{O}$ of 0 such that $\mathrm{supp} f \cap \mathcal{O} = \emptyset$. According to Cauchy–Schwarz inequality there exists $\kappa_1 > 0$ independent of ε, h such that

$$\left|\int_\Omega f\overline{\mathrm{D}^*_{\mathrm{h}}(\Psi^\varepsilon)}\,\mathrm{d}\mathbf{x}\right| \leqslant \|f\|_{0,\Omega}\|\mathrm{D}^*_{\mathrm{h}}(\Psi^\varepsilon)\|_{1,\mathcal{O}} \leqslant \kappa_1\|f\|_{0,\Omega}|\ln \mathrm{h}|\mathrm{h}^k \quad \forall \varepsilon, \mathrm{h} \in]0, 1[. \quad \square \tag{49}$$

Note that, according to Lemma A.1 in Appendix, such an estimate would also hold for a right hand side of the same form as in (31).

Proposition 5.2 suggests to consider a simplified version of (46) where $g_{\varepsilon,\mathrm{h}}$ is replaced by 0, which would simply yield Formulation (27) where one has chosen $\mathrm{L}_{\varepsilon,\mathrm{h}} = \Lambda_{\varepsilon,\mathrm{h}}$. According to Lemma 5.3, there exists $\mathrm{h}_0 > 0$ such that Holland–Simpson's scheme obtained in this manner is well posed for all $\mathrm{h} \in]0, \mathrm{h}_0[$. The preceding remarks provide at once a proof for the consistency of Holland and Simpson's method (27) and a formula for its calibration.

Theorem 5.2. *Assume that* **A1–A3** *are satisfied. Let* $(u^\varepsilon_{\mathrm{h}}, p^\varepsilon_{\mathrm{h}}) \in \mathbb{V}^{\mathrm{h}} \times \mathbb{C}$ *be the solution to* (27) *taking* $\mathrm{L}_{\varepsilon,\mathrm{h}} = \Lambda_{\varepsilon,\mathrm{h}}$ *where* $\Lambda_{\varepsilon,\mathrm{h}}$ *is given by* (47). *Let* $(\tilde{u}^\varepsilon_{\mathrm{h}}, \tilde{p}^\varepsilon_{\mathrm{h}}) \in \mathbb{V}^{\mathrm{h}}_{\mathbf{e}} \times \mathbb{C}$ *be the unique solution to* (35). *Let* $\mathcal{O}$ *be any arbitrary open set verifying* $\overline{\mathcal{O}} \subset \Omega \setminus \{0\}$. *Then there exists* $\kappa_0 > 0$ *independent of* ε, h *such that*

$$\|u^\varepsilon_{\mathrm{h}} - \tilde{u}^\varepsilon_{\mathrm{h}}\|_{1,\mathcal{O}} + |p^\varepsilon_{\mathrm{h}} - \tilde{p}^\varepsilon_{\mathrm{h}}| \leqslant \kappa_0 \mathrm{h}^k |\ln \mathrm{h}|^2 \quad \forall \mathrm{h} \in]0, 1[.$$

Proof. We start by defining

$$v^\varepsilon_{\mathrm{h}} = u^\varepsilon_{\mathrm{h}} + \left\{-p^\varepsilon_{\mathrm{h}}\frac{b^\varepsilon\big(\mathrm{D}^*_{\mathrm{h}}(\Psi^\varepsilon)\big)}{a\big(\mathrm{D_h}(\Psi^\varepsilon),\mathrm{D}^*_{\mathrm{h}}(\Psi^\varepsilon)\big)} + \frac{\int_\Omega f\overline{\mathrm{D}^*_{\mathrm{h}}(\Psi^\varepsilon)}\,\mathrm{d}\mathbf{x}}{a\big(\mathrm{D_h}(\Psi^\varepsilon),\mathrm{D}^*_{\mathrm{h}}(\Psi^\varepsilon)\big)}\right\}\mathrm{D_h}(\Psi^\varepsilon).$$

Applying once again the calculus of Section 5.3 reversely, we observe that $(v^\varepsilon_{\mathrm{h}}, p^\varepsilon_{\mathrm{h}})$ satisfies a problem very close to (35), so that the difference $(v^\varepsilon_{\mathrm{h}} - \tilde{u}^\varepsilon_{\mathrm{h}}, p^\varepsilon_{\mathrm{h}} - \tilde{p}^\varepsilon_{\mathrm{h}})$ verifies

$$a(v^\varepsilon_{\mathrm{h}} - \tilde{u}^\varepsilon_{\mathrm{h}}, v) + (p^\varepsilon_{\mathrm{h}} - \tilde{p}^\varepsilon_{\mathrm{h}})b^\varepsilon(\overline{v}) = 0 \quad \forall v \in \mathbb{V}^{\mathrm{h}}_{\mathbf{e}} \quad \text{and} \quad b^\varepsilon(v^\varepsilon_{\mathrm{h}} - \tilde{p}^\varepsilon_{\mathrm{h}}) = -g_{\varepsilon,\mathrm{h}}.$$

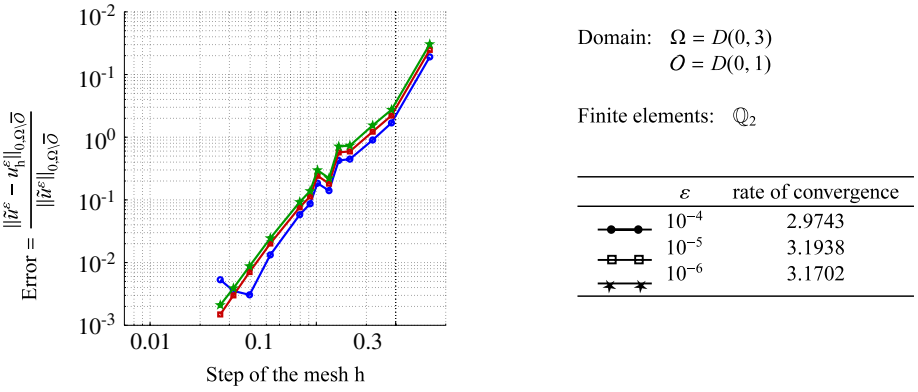


Fig. 4. Validation of the calibration formula.

The bilinear forms $a()$ and $b()$ satisfy inf – sup conditions with respect to $\mathbb{V}_{\mathbf{e}}^{\mathrm{h}}$ and $\mathbb{C}$ that are uniform with respect to ε and h, see Lemmas 3.2 and 3.3 in [11]. As a consequence, according to Proposition 5.2 and Theorem 11 of Section II.1 of [29], there exist $\kappa_2, \kappa_3 > 0$ independent of ε, h such that

$$\|v_{\mathrm{h}}^\varepsilon - \tilde{u}_{\mathrm{h}}^\varepsilon\|_{1,\Omega} + |p_{\mathrm{h}}^\varepsilon - \tilde{p}_{\mathrm{h}}^\varepsilon| \leqslant \kappa_2 |g_{\varepsilon,\mathrm{h}}| \leqslant \kappa_3 \mathrm{h}^k |\ln \mathrm{h}| \quad \forall \varepsilon, \mathrm{h} \in]0,1[. \tag{50}$$

To conclude, choose an open set $\mathcal{O} \subset \Omega$ such that $\mathcal{O} \subset \overline{\mathcal{O}} \subset \Omega \setminus \{0\}$. There remains to bound $\|u_{\mathrm{h}}^\varepsilon - \tilde{u}_{\mathrm{h}}^\varepsilon\|_{1,\mathcal{O}}$. According to (48) and (49) and Lemma 5.2, there exist $\kappa_4, \kappa_5 > 0$ such that

$$\begin{aligned}\|u_{\mathrm{h}}^\varepsilon - \tilde{u}_{\mathrm{h}}^\varepsilon\|_{1,\mathcal{O}} &\leqslant \kappa_4 \|v_{\mathrm{h}}^\varepsilon - \tilde{u}_{\mathrm{h}}^\varepsilon\|_{1,\mathcal{O}} + \kappa_4 \left| \frac{p_{\mathrm{h}}^\varepsilon b^\varepsilon\left(\mathrm{D}_{\mathrm{h}}^*(\Psi^\varepsilon)\right) - \int_\Omega f \overline{\mathrm{D}_{\mathrm{h}}^*(\Psi^\varepsilon)}\,\mathrm{d}\mathbf{x}}{a\left(\mathrm{D}_{\mathrm{h}}(\Psi^\varepsilon), \mathrm{D}_{\mathrm{h}}^*(\Psi^\varepsilon)\right)} \right| \|\mathrm{D}_{\mathrm{h}}(\Psi^\varepsilon)\|_{1,\mathcal{O}} \\ &\leqslant \kappa_5 |\ln \mathrm{h}| \mathrm{h}^{k+1} + \kappa_5 \left(1 + |p_{\mathrm{h}}^\varepsilon|\right) |\ln \mathrm{h}| \mathrm{h}^k.\end{aligned}$$

As a consequence the theorem will be proved if we show that $|p_{\mathrm{h}}^\varepsilon|$ is bounded independently of ε, h. According to (50), it is sufficient to prove that $|\tilde{p}_{\mathrm{h}}^\varepsilon|$ is bounded. There exist $\kappa_6 > 0$ such that $|\tilde{p}_{\mathrm{h}}^\varepsilon| \leqslant \kappa_6 \|f\|_{0,\Omega}$ $\forall \varepsilon \in]0,1[, \forall \mathrm{h} \in]0,1[$ according to (35) and (36). Since $\|f\|_{0,\Omega}$ is bounded independently of ε, h, this concludes the proof. □

5.5. Numerical validation

We present the results of a numerical experiment that confirm the conclusion of Theorem 5.2. We come back to the situation described in Section 4.3 and solve again Problem (33) with $\mathbb{Q}_2$ finite elements, for $\varepsilon = 10^{-4}, 10^{-5}, 10^{-6}$, each time representing the relative error as $\mathrm{h} \to 0$ (Fig. 4).

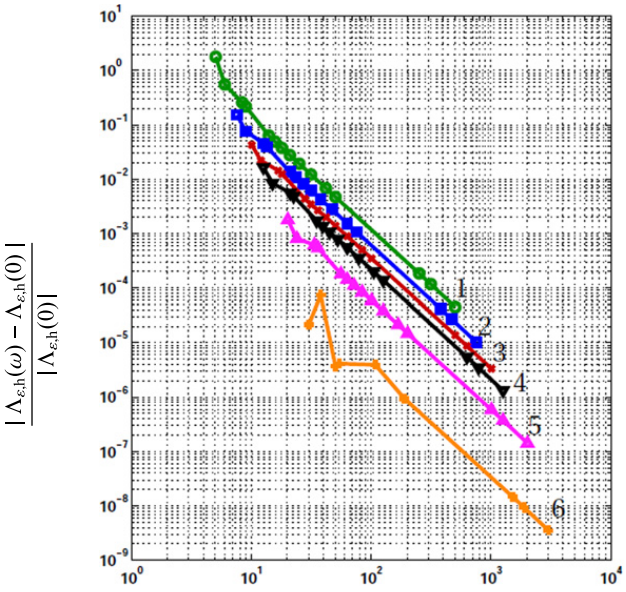
Holland and Simpson's method with our calibration formula turns out to be as precise as the Augmented Galerkin method (compare with the results presented in [11]): the rate of convergence is optimal (around 3 for a discretization with $\mathbb{Q}_2$) and the consistency properties deteriorate only slowly as $\varepsilon \to 0$. We observe a deterioration of Holland and Simpson's method at the end of the curve associated to $\varepsilon = 10^{-4}$. We believe that this is due to the approximation $\int_0^{2\pi} v(r = \varepsilon, \theta)\mathrm{d}\theta \simeq 2\pi v(0)$ that is (intuitively) valid only for $\mathrm{h} \gg \varepsilon$.

5.5.1. Remark: practical relevance of the calibration formula

In this situation where the Augmented Galerkin method is already at hand, is it worth using Holland and Simpson's scheme instead, dealing with a calibration process? Since Holland and Simpson's scheme is much easier to implement, it seems that the answer is yes under the condition that the calibration procedure that one has selected is fast and cheap. As regards the calibration issue of Holland and Simpson's method, although Formula (47) seems interesting from a theoretical point of view, this formula may seem much less relevant from a computational point of view. Indeed it suggests that, to apply Holland and Simpson's method, one should first compute $\mathrm{D}_{\mathrm{h}}(\Psi^\varepsilon)$ and then solve Problem (27). Besides, computing exactly $\mathrm{D}_{\mathrm{h}}(\Psi^\varepsilon)$ requires solving a Problem of the form (37) which seems to be a costly task. An interesting consequence of Formula (47) though, is that a fast procedure for computing $\mathrm{D}_{\mathrm{h}}(\Psi^\varepsilon)$ would lead to an efficient way to calibrate Holland–Simpson's scheme.

Actually it seems possible to compute approximately $\mathrm{D}_{\mathrm{h}}(\Psi^\varepsilon)$ without fully solving a problem of the form (37). Let us briefly *sketch* a possible approach for achieving such a computation. Take any subset $\Omega_0 \subset \Omega$ containing the small obstacle Γ_ε. According to Lemmas 5.2 and A.1, the function $\mathrm{D}_{\mathrm{h}}(\Psi^\varepsilon)$ has negligible amplitude far from the small obstacle i.e. $\|\mathrm{D}_{\mathrm{h}}(\Psi^\varepsilon)\|_{1,\Omega\setminus\overline{\Omega}_0} = O(\mathrm{h}^k |\ln \mathrm{h}|)$. This motivates the approximation

$$a(\mathrm{D}_{\mathrm{h}}(\Psi^\varepsilon), v_{\mathrm{h}}) \simeq a_0(\mathrm{D}_{\mathrm{h}}(\Psi^\varepsilon), v_{\mathrm{h}}) = \int_{\Omega_0} \nabla \mathrm{D}_{\mathrm{h}}(\Psi^\varepsilon) \cdot \nabla \overline{v}_{\mathrm{h}}\,\mathrm{d}\mathbf{x} - \omega^2 \int_{\Omega_0} \mathrm{D}_{\mathrm{h}}(\Psi^\varepsilon)\overline{v}_{\mathrm{h}}\mathrm{d}\mathbf{x} + \mathrm{i} \int_{\partial\Omega_0} \mathrm{D}_{\mathrm{h}}(\Psi^\varepsilon)\overline{v}_{\mathrm{h}}\mathrm{d}\sigma.$$



Domain: $\Omega = D(0,3)$
Finite elements: $\mathbb{Q}_2$
Radius of the obstacle: $\varepsilon = 10^{-6}$

Linear regression

$$\ln\Big(\frac{|\Lambda_{\varepsilon,h}(\omega)-\Lambda_{\varepsilon,h}(0)|}{|\Lambda_{\varepsilon,h}(0)|}\Big) \simeq \alpha_0 \ln\Big(\frac{4\pi}{\omega h}\Big) + \beta_0$$

curve	step h	$\Re e\, \Lambda_{\varepsilon,h}(\omega = 0)$	rate α_0	origin ordinate β_0
1	7.9372 1e-2	45.8579	-2.1569	3.1985
2	5.2872 1e-2	48.6451	-2.0430	2.0174
3	3.9638 1e-2	48.1400	-2.0236	1.3793
4	3.1703 1e-2	47.1408	-2.0182	0.8776
5	1.9807 1e-2	44.4313	-2.0131	-0.4394
6	1.3202 1e-2	41.9246	-1.9308	-3.9097

Fig. 5. Dependency of $\Lambda_{\varepsilon,h}$ with respect to frequency.

Since $a(D_h(\Psi^\varepsilon), v_h) = 0\ \forall v_h \in \mathbb{V}^h$ by definition, the observation above leads to the idea of choosing Ω_0 (much) smaller that Ω and compute $D_h(\Psi^\varepsilon) \simeq \Psi^\varepsilon - \mathcal{P}_h(\Psi^\varepsilon)$ where $\mathcal{P}_h(\Psi^\varepsilon)$ solves the reduced problem

$$a_0(\mathcal{P}_h(\Psi^\varepsilon), v_h) \simeq a_0(\Psi^\varepsilon, v_h) \quad \forall v_h \in \mathbb{V}^h.$$

For practical implementation, one may contemplate choosing Ω_0 as a union of triangles of the mesh (in the case of a triangular mesh). However, one should expect that the approximation $D_h(\Psi^\varepsilon) \simeq \Psi^\varepsilon - \mathcal{P}_h(\Psi^\varepsilon)$ becomes less relevant when the size of Ω_0 reaches the size of mesh cells.

5.6. Dependency of the calibration with respect to the frequency

The critical value of Holland–Simpson's parameter given by formula (47) a priori depends on h and ε. It is also natural to ask about its dependency with respect to the frequency ω. Although we were not able to provide satisfying theoretical results for this question, we briefly present a numerical study on this subject. Let us denote $\Lambda_{\varepsilon,h}(\omega)$ the number computed by means of Formula (47) at frequency ω. For the computation of $\Lambda_{\varepsilon,h}(\omega)$ we have considered Formulation (35) posed on the domain $\Omega = D(0,3)$ and a small obstacle with fixed radius $\varepsilon = 10^{-6}$. For the discretization we have considered $\mathbb{Q}_2$-finite elements with 6 different mesh whose characteristic step size is given in Fig. 5. The table also provides the characteristics of the line obtained by linear regression. These results suggest that there exists a constant κ independent of ω and h such that

$$\frac{|\Lambda_{\varepsilon,h}(\omega) - \Lambda_{\varepsilon,h}(0)|}{|\Lambda_{\varepsilon,h}(0)|} \leqslant \kappa\omega^2 h^2 \quad \forall \omega \in \mathbb{R}_+,\ \forall h \in]0,1[.$$

This conclusion confirms the analysis presented in [7] concerning the critical value of Holland–Simpson's parameter.

5.7. Conclusion

In this paper, we presented how the formalism of Holland and Simpson can be reinterpreted as a singular function method for an elementary problem. This leads to a well known, more comfortable framework for the analysis of this method. This approach led in particular to a proof of consistency, as well as a formula for the calibration of Holland–Simpson's parameter.

It does not seem clear which approach is preferable between Holland–Simpson's scheme and the Augmented Galerkin scheme. The Augmented Galerkin method is systematic, consistent, and fits a traditional Galerkin framework. However the Augmented Galerkin approach has also several important drawbacks: the matrix associated to this method is not sparse due to the interaction between the additional shape function and standard shape functions. Besides, its conditioning is not as comfortable as the conditioning of a standard finite element scheme. In addition, in higher-dimensional situations, the quadrature of terms involving the additional shape function can be problematic and costly, see [21].

On the other hand, the method of Holland and Simpson can be very sharp and it differs from a standard scheme only at nodes adjacent to the wires, which makes its implementation far easier compared to the Augmented Galerkin approach. Besides Holland and Simpson's method is usually better conditioned than the Augmented Galerkin approach. However it requires a calibration process. Whether or not calibration can be conducted efficiently appears as the key consideration in order to choose either the Augmented Galerkin approach or Holland and Simpson's scheme. Last but not least, it would be highly desirable to devise a fast method for computing the parameter $\Lambda_{\varepsilon,\mathrm{h}}$ given by Formula (47).

Obviously a possible improvement of this work would consist in the extension to scattering by a real "3D wire". Some development and numerical experiments in this direction can be found in [21]. Besides the numerical results of Section 5.6 suggest that our approach could be adapted for a time dependent problem, since the critical value of Holland–Simpson's parameter seems to be frequency independent at first order. This will be the subject of a forthcoming work.

Appendix

A.1. Holland and Simpson's formalism in the case of our model problem

In this part of the appendix, we want to show that applying Holland and Simpson's approach with finite differences to (14) yields a numerical scheme of the same form as (27). Since (13) is invariant under translation along z, an initial 3D Cartesian discretization grid for this problem can be seen a 2D Cartesian grid with vertices denoted $\mathbf{x}_{i,j}$. We decompose $\mathbf{E}_\mathrm{h}$ in the same manner as in Section 1.4, $\mathbf{E}_\mathrm{h} = \mathbf{E}_{\perp,\mathrm{h}} + \mathrm{E}_{z,\mathrm{h}}\mathbf{e}_z$ with $\mathbf{e}_z \cdot \mathbf{E}_{\perp,\mathrm{h}} = 0$. We apply to (10) the same algebraic manipulations as to (4), splitting it into two decoupled discrete sets of equations: discrete TE and TM systems. The discrete TE system reduces to an equation involving only $\mathrm{E}_{z,\mathrm{h}}$ and I_h namely

$$-\Delta_\mathrm{h}\mathrm{E}_{z,\mathrm{h}} + \partial_t^2\mathrm{E}_{z,\mathrm{h}} + \frac{\delta_\mathrm{h}\cdot\partial_t\mathrm{I}_\mathrm{h}}{\mathrm{h}^2} = -\partial_t\mathrm{J}_{z,\mathrm{h}} \quad \text{with} \quad \Delta_\mathrm{h}\mathrm{E}_{z,\mathrm{h}} := \frac{1}{\mathrm{h}^2}\left\{\mathrm{E}_{z,\mathrm{h}}^{i+1,j} + \mathrm{E}_{z,\mathrm{h}}^{i-1,j} + \mathrm{E}_{z,\mathrm{h}}^{i,j+1} + \mathrm{E}_{z,\mathrm{h}}^{i,j-1} - 4\mathrm{E}_{z,\mathrm{h}}^{i,j}\right\}.$$

Note that since we supposed invariance of the problem under translation along the z-axis, $\mathrm{E}_{z,\mathrm{h}}$ and $\partial_t\mathrm{I}_\mathrm{h}$ do not depend on the third index k anymore. Besides in the present case we have the following definition: $(\delta_\mathrm{h}\cdot\partial_t\mathrm{I}_\mathrm{h})^{i,j} := \frac{1}{4}\partial_t\mathrm{I}_\mathrm{h}$ if $\mathbf{x}_{i,j}$ belongs to the face containing $\mathbf{0}$, and $(\delta_\mathrm{h}\cdot\partial_t\mathrm{I}_\mathrm{h})^{i,j} := 0$ otherwise. Similarly the average $\langle\mathrm{E}_{z,\mathrm{h}}\rangle$ introduced with (11) for the method of Holland and Simpson becomes a simple number (with harmonic dependence in time though). Eq. (11) then simply writes

$$\langle\mathrm{E}_{z,\mathrm{h}}\rangle - \mathrm{L}_{\varepsilon,\mathrm{h}}\partial_t\mathrm{I}_\mathrm{h} = 0.$$

Once again we rewrite these equations in 2D by setting $\mathrm{E}_{z,\mathrm{h}}^{i,j} = u_\mathrm{h}^{i,j}\mathrm{e}^{-\mathrm{i}\omega t}$, $\partial_t\mathrm{I}_\mathrm{h} = p_\mathrm{h}\mathrm{e}^{-\mathrm{i}\omega t}$ and $\partial_t\mathrm{J}_{z,\mathrm{h}}^{i,j} = f_\mathrm{h}^{i,j}\mathrm{e}^{-\mathrm{i}\omega t}$. Holland–Simpson's method corresponding to our model problem finally writes

$$-\Delta_\mathrm{h}u_\mathrm{h} - \omega^2 u_\mathrm{h} + \frac{\delta_\mathrm{h}\cdot p_\mathrm{h}}{\mathrm{h}^2} = f \quad \text{and} \quad \langle u_\mathrm{h}\rangle - \mathrm{L}_{\varepsilon,\mathrm{h}}p_\mathrm{h} = 0. \tag{51}$$

The equations of (27) appear as variational counterparts of such equations. In particular, in (27), the term $b^\varepsilon(u_\mathrm{h})$ should be interpreted as the term $\langle u_\mathrm{h}\rangle$ of (51).

A.2. Technical results

In this part of the appendix we recall or establish several technical results. First, we recall a theorem from Nitsche and Schatz, detailed and proved in [28], that provides interior a priori estimates for the solution to a finite element scheme. Here we give a simplified reformulation because we do not need this result in full generality.

Theorem A.1 (*Nitsche and Schatz*). *Let $a()$ be the bilinear form defined in (19) and $\mathbb{V}^\mathrm{h}$ be the space defined by (25). Take two open sets $\mathcal{O}_0 \subset\subset \mathcal{O}_1 \subset\subset \Omega$ and a number $p \in \mathbb{R}_+$. Finally let k be the order of the finite elements in $\mathbb{V}^\mathrm{h}$. In these conditions there exist constants $\kappa_0, \mathrm{h}_0 > 0$ independent of h such that for any couple $(u, u_\mathrm{h}) \in H^{k+1}(\mathcal{O}_1)\times\mathbb{V}^\mathrm{h}$ verifying*

$$a(u - u_\mathrm{h}, v_\mathrm{h}) = 0 \quad \forall v_\mathrm{h} \in \mathbb{V}^\mathrm{h} \text{ such that } \mathrm{supp}(v_\mathrm{h}) \subset \mathcal{O}_1$$

we have

$$\|u-u_{\mathrm{h}}\|_{0,\mathcal{O}_0} \leqslant \kappa_0\Big(\mathrm{h}^q\|u\|_{q,\mathcal{O}_1}+\|u-u_{\mathrm{h}}\|_{-p,\mathcal{O}_0}\Big)$$
$$\|u-u_{\mathrm{h}}\|_{1,\mathcal{O}_0} \leqslant \kappa_0\Big(\mathrm{h}^{q-1}\|u\|_{q,\mathcal{O}_1}+\|u-u_{\mathrm{h}}\|_{-p,\mathcal{O}_0}\Big) \quad \forall \mathrm{h}\in]0,\mathrm{h}_0[,\ \forall q=1\ldots k.$$

Here is a proposition that generalizes the results of Lemma 5.2 providing local estimates in energy norm up to the outer boundary of the domain.

Lemma A.1. *Let k be the order of finite elements used for discretization. Take $r_0>0$ such that $\overline{D}_{r_0}\subset\Omega$, where D_r is the disk of center 0 of radius r. If the assumptions* **A1**–**A3** *are satisfied, there exist constants $\kappa_0>0$ independent of ε and* h *such that*

$$\|\mathrm{D_h}(\Psi^\varepsilon)\|_{1,\Omega\setminus\overline{D}_{r_0}}\leqslant\kappa_0|\ln\varepsilon|\mathrm{h}^k \quad \textit{and} \quad \|\mathrm{D_h^*}(\Psi^\varepsilon)\|_{1,\Omega\setminus\overline{D}_{r_0}}\leqslant\kappa_0|\ln\varepsilon|\mathrm{h}^k \quad \forall\varepsilon,\mathrm{h}\in]0,1[.$$

Proof. We only prove the estimate related to $\mathrm{D_h}(\Psi^\varepsilon)$, since very similar arguments would provide the result for $\mathrm{D_h^*}(\Psi^\varepsilon)$. Recall that according to Section 5.1, we have Ψ^ε in a neighborhood of Γ. Using Lemma 5.2 and a suitable cut-off function if necessary, we may assume that $r_0>0$ is chosen large enough to guaranty that $\Psi^\varepsilon=0$ on $\Omega\setminus\overline{D}_{r_0/3}$. Let us set $\varphi_{\mathrm{h}}^\varepsilon=\Pi_{\mathrm{h}}(\Psi^\varepsilon)$, and introduce a C^∞ cut-off function $\hat\chi:\Omega\to[0,1]$ such that $\hat\chi(\mathbf{x})=0$ if $|\mathbf{x}|<r_0/3$ and $\hat\chi(\mathbf{x})=1$ if $|\mathbf{x}|>r_0/2$. Observe that $\Pi_{\mathrm{h}}(\hat\chi\varphi_{\mathrm{h}}^\varepsilon)$ coincide with $\varphi_{\mathrm{h}}^\varepsilon$ in $\Omega\setminus\overline{D}_{r_0}$. As a consequence, using the uniform inf–sup condition given by Lemma 4.1 with $V=\mathbb{V}^{\mathrm{h}}$, we have

$$\|\mathrm{D_h}(\Psi^\varepsilon)\|_{1,\Omega\setminus\overline{D}_{r_0}}\leq\|\Pi_{\mathrm{h}}(\hat\chi\varphi_{\mathrm{h}}^\varepsilon)\|_{1,\Omega}\leq\sup_{v_{\mathrm{h}}\in\mathbb{V}^{\mathrm{h}}}\frac{a\big(\Pi_{\mathrm{h}}(\hat\chi\varphi_{\mathrm{h}}^\varepsilon),v_{\mathrm{h}}\big)}{\|v_{\mathrm{h}}\|_{1,\Omega}}. \tag{52}$$

We provide an upper bound $a\big(\Pi_{\mathrm{h}}(\hat\chi\varphi_{\mathrm{h}}^\varepsilon),v_{\mathrm{h}}\big)$. Denote $\mathcal{O}=D_{r_0/2}\setminus\overline{D}_{r_0/3}$. Observe that $\mathrm{supp}\big(\Pi_{\mathrm{h}}(\hat\chi v_{\mathrm{h}})-\hat\chi v_{\mathrm{h}}\big)\subset\mathcal{O}$ for any $v_{\mathrm{h}}\in\mathbb{V}^{\mathrm{h}}$. Besides $\mathrm{supp}(\nabla\hat\chi)\subset\mathcal{O}$. As a consequence, since $\varphi_{\mathrm{h}}^\varepsilon\in\mathbb{V}^{\mathrm{h}}$, we see that there exists a constant $\kappa>0$ independent of ε, h such that

$$\begin{aligned}\big|a\big(\Pi_{\mathrm{h}}(\hat\chi\varphi_{\mathrm{h}}^\varepsilon),v_{\mathrm{h}}\big)\big| &\leq \big|a\big((\mathrm{Id}-\Pi_{\mathrm{h}})(\hat\chi\varphi_{\mathrm{h}}^\varepsilon),v_{\mathrm{h}}\big)\big|+\big|a\big(\varphi_{\mathrm{h}}^\varepsilon,(\mathrm{Id}-\Pi_{\mathrm{h}})(\hat\chi v_{\mathrm{h}})\big)\big|+\big|a(\hat\chi\varphi_{\mathrm{h}}^\varepsilon,v_{\mathrm{h}})-a(\varphi_{\mathrm{h}}^\varepsilon,\hat\chi v_{\mathrm{h}})\big|\\ &\leq \kappa\|\varphi_{\mathrm{h}}^\varepsilon\|_{1,\mathcal{O}}\|v_{\mathrm{h}}\|_{1,\Omega}.\end{aligned} \tag{53}$$

Since $\varphi_{\mathrm{h}}^\varepsilon=\mathrm{D_h}(\Psi^\varepsilon)$ in $\mathcal{O}$, there exists a constant $\kappa'>0$ such that $\|\varphi_{\mathrm{h}}^\varepsilon\|_{1,\mathcal{O}}\leq\kappa'\mathrm{h}^k|\ln\varepsilon|$ according to Lemma 5.2. Plugging this into estimate (53) and then in (52), we obtain the desired result. □

Here is also a technical proposition used in Section 5.2 that shows that the additional shape function Ψ^ε is not well approximated by functions from $\mathbb{V}^{\mathrm{h}}$.

Proposition 5.1. *Under assumptions* **A1**–**A3** *there exists $\kappa>0$ independent of* h *(and consequently also independent of ε) such that*

$$\kappa<\|\mathrm{D_h}(\Psi^\varepsilon)\|_{1,\Omega} \quad \textit{and} \quad \kappa<\|\mathrm{D_h^*}(\Psi^\varepsilon)\|_{1,\Omega} \quad \forall\mathrm{h}\in]0,1[.$$

Proof. During this proof $k\in\mathbb{N}$ will refer to the order of our finite element method. We give only the proof in the case of $\mathbb{P}_k$ finite elements, but similar arguments would hold with $\mathbb{Q}_k$ finite elements.
According to the definition of h given by (24), for any $\mathrm{h}\in]0,1[$ there exists $K\in\mathcal{T}_{\mathrm{h}}$ such that $K\subset\{\mathbf{x}\in\Omega\ \text{s.t.}\ \mathrm{h}<|\mathbf{x}|<4\mathrm{h}\}$. For each h, let us choose one such K that we denote $K(\mathrm{h})$. In order to show the desired result it is sufficient to show that there exists $\kappa_0>0$ independent of h such that

$$\int_{K(\mathrm{h})}|\nabla\mathrm{D_h}(\Psi^\varepsilon)|^2\mathrm{d}\mathbf{x}>\kappa_0 \quad \forall\mathrm{h}\in]0,1[.$$

At the beginning of Section 4 we introduced for each $K\in\mathcal{T}_{\mathrm{h}}$ a bijection $F_K:\hat K\to K$ where $\hat K$ is a reference triangle. This application is of the form $F_K=(F_K^x,F_K^y)\in\mathbb{P}_k\times\mathbb{P}_k$. In the sequel we denote $\hat{\mathbf{x}}$ an arbitrary point of $\hat K$. By definition, there always holds $\mathrm{h}<|F_K(\hat{\mathbf{x}})|<4\mathrm{h}$. Using the change of variables induced by F_K and the explicit expression of Ψ^ε given by (34), we see that it suffices to show the existence of $\kappa_1>0$ independent of h such that

$$\inf_{Q\in\mathbb{P}_k}\int_{\hat K}\left|\frac{F^\alpha_{K(\mathrm{h})}(\hat{\mathbf{x}})}{|F_{K(\mathrm{h})}(\hat{\mathbf{x}})|^2}-Q(\hat{\mathbf{x}})\right|^2|\mathrm{Jac}(F_{K(\mathrm{h})})|\,\mathrm{d}\hat{\mathbf{x}}>\kappa_1 \quad \forall\mathrm{h}\in]0,1[,\ \alpha=x,y. \tag{54}$$

We will show such a uniform inequality only for $\alpha=x$, as the proof is nearly the same for $\alpha=y$. In the above inequality $\mathrm{Jac}(F_{K(\mathrm{h})})$ refers to the determinant of the Jacobian matrix associated to F_K. For proving (54), we proceed by contradiction.

Assume that there exists a sequence $h_n \to 0$ (with associated triangle rewritten $K_n := K(h_n)$) and a sequence of polynomials $Q_n \in \mathbb{P}_k$ with maximum degree k such that

$$\int_{\hat{K}} \left| \frac{F^x_{K_n}(\hat{\mathbf{x}})}{|F_{K_n}(\hat{\mathbf{x}})|^2} - \frac{1}{h_n} Q_n(\hat{\mathbf{x}}) \right|^2 |\mathrm{Jac}(F_{K_n})| \, d\hat{\mathbf{x}} \xrightarrow[n\to\infty]{} 0. \tag{55}$$

We know from [30] that there exists $\kappa_2 > 0$ independent of h such that $|\mathrm{Jac}(F_{K(h)})| > \kappa\, h^2$ for any $h \in]0, 1[$ which implies

$$\int_{\hat{K}} \left| \frac{F^x_{K_n}(\hat{\mathbf{x}})}{|F_{K_n}(\hat{\mathbf{x}})|^2} - \frac{1}{h_n} Q_n(\hat{\mathbf{x}}) \right|^2 |\mathrm{Jac}(F_{K_n})| \, d\hat{\mathbf{x}} > \kappa_2 \int_{\hat{K}} \left| \frac{h_n F^x_{K_n}(\hat{\mathbf{x}})}{|F_{K_n}(\hat{\mathbf{x}})|^2} - Q_n(\hat{\mathbf{x}}) \right|^2 d\hat{\mathbf{x}}.$$

Since $\sup_{\hat{K}} |F_{K_n}/h_n| < 4$ we conclude that the sequence F_{K_n}/h_n is bounded in $\mathbb{P}_k \times \mathbb{P}_k$. Since this space is of finite dimension, we can assume that F_{K_n}/h_n converges toward a limit $F_\infty \in \mathbb{P}_k \times \mathbb{P}_k$, extracting a subsequence if necessary. By continuity we have $1 < F_\infty(\hat{\mathbf{x}}) < 4, \ \forall \hat{\mathbf{x}} \in \hat{K}$, which implies in particular that F_∞ does not vanish on $\hat{K}$. In the same manner, $\|Q_n\|_{0,\hat{K}}$ stays bounded as $n \to \infty$, indeed we have

$$\|Q_n\|_{0,\hat{K}} \leqslant 2 \int_{\hat{K}} \left| \frac{h_n F^x_{K_n}(\hat{\mathbf{x}})}{|F_{K_n}(\hat{\mathbf{x}})|^2} - Q_n(\hat{\mathbf{x}}) \right|^2 d\hat{\mathbf{x}} + 2 \int_{\hat{K}} \left| \frac{h_n F^x_{K_n}(\hat{\mathbf{x}})}{|F_{K_n}(\hat{\mathbf{x}})|^2} \right|^2 d\hat{\mathbf{x}}.$$

Since $Q_n \in \mathbb{P}_k, \ \forall n \in \mathbb{N}$ and $\mathbb{P}_k$ has finite dimension we can also assume, extracting a subsequence if necessary, that Q_n converges toward a limit $Q_\infty \in \mathbb{P}_k$. Then according to (55) we must have

$$\int_{\hat{K}} \left| \frac{F^x_\infty(\hat{\mathbf{x}})}{|F_\infty(\hat{\mathbf{x}})|^2} - Q_\infty(\hat{\mathbf{x}}) \right|^2 d\hat{\mathbf{x}} = 0 \Rightarrow \frac{F^x_\infty}{|F_\infty|^2} = Q_\infty.$$

Suppose first that Q_∞ is not a constant. This leads to a contradiction since

$$\frac{|F^x_\infty(\hat{\mathbf{x}})|}{|F_\infty(\hat{\mathbf{x}})|^2} \xrightarrow[|\hat{\mathbf{x}}|\to\infty]{} 0 \quad \text{and} \quad |Q_\infty(\hat{\mathbf{x}})| \xrightarrow[|\hat{\mathbf{x}}|\to\infty]{} \infty.$$

Assume now that Q_∞ is constant, then this implies that F_∞ is a constant itself, and this leads to another contradiction since, as $n \to \infty$, we have $|\mathrm{Jac}(F_\infty)| > \kappa_2 > 0$. In conclusion we necessarily obtain a contradiction, which concludes the proof. □

Here is also a technical result that we need in order to study the sign of the theoretical formula that we provide for the critical value of Holland–Simpson's parameter.

Lemma A.2. *Assume that* **A1**–**A3** *are satisfied. There exists a constant* $\kappa_0 > 0$ *independent of* h *such that for any* $h \in]0, 1[$

$$\left| \frac{b^\varepsilon\big(\mathrm{D_h}(\Psi^\varepsilon)\big)}{a\big(\mathrm{D_h}(\Psi^\varepsilon), \mathrm{D^*_h}(\Psi^\varepsilon)\big)} - 1 \right| \leqslant \kappa_0 h |\ln h| \quad \textit{and} \quad \left| \frac{b^\varepsilon\big(\mathrm{D^*_h}(\Psi^\varepsilon)\big)}{a\big(\mathrm{D_h}(\Psi^\varepsilon), \mathrm{D^*_h}(\Psi^\varepsilon)\big)} - 1 \right| \leqslant \kappa_0 h |\ln h|.$$

Proof. We only prove the first estimate, since the proof for the second estimate follows nearly the same lines. Observe that we have $a(\mathrm{D_h}(\Psi^\varepsilon), \mathrm{D^*_h}(\Psi^\varepsilon)) = a(\mathrm{D_h}(\Psi^\varepsilon), \Psi^\varepsilon)$. Applying a Green formula over Ω_ε yields

$$2\pi b^\varepsilon\big(\mathrm{D_h}(\Psi^\varepsilon)\big) = a\big(\mathrm{D_h}(\Psi^\varepsilon), \mathrm{D_h}(\Psi^\varepsilon)\big) + 2\int_\Omega \mathrm{D_h}(\Psi^\varepsilon) \nabla\chi \cdot \nabla \ln|\mathbf{x}| d\mathbf{x} + \int_\Omega \ln\left(\frac{|\mathbf{x}|}{\varepsilon}\right)\big(\Delta\chi + \omega^2\chi\big)\mathrm{D_h}(\Psi^\varepsilon) d\mathbf{x}.$$

Since $|a(\mathrm{D_h}(\Psi^\varepsilon), \mathrm{D^*_h}(\Psi^\varepsilon))|$ is bounded from below according to (40), there only remain to properly estimate the last two terms in the right hand side above. First, note that there exists a constant $\kappa > 0$ such that $|\ln(|\mathbf{x}|/\varepsilon)| \leqslant \kappa |\ln\varepsilon|$ for all $\mathbf{x} \in \Omega$. Then, the conclusion is obtained by using Lemmas 5.1 and 5.2 and the fact $\mathrm{supp}\chi$ excludes a fixed neighborhood of 0. □

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