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Augmented Galerkin schemes for the numerical solution of scattering by small obstacles

X. Claeys · F. Collino

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Abstract We are interested in the problem of a bidimensional acoustic wave propagation in a medium including a small obstacle with homogeneous Dirichlet boundary condition. We present and analyse a numerical scheme suitable for finite elements that does not suffer from numerical locking, and takes the presence of the small obstacle into account. It is based on the fictitious domain method combined with matched asymptotic expansions.

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0 Introduction

The general context of this work is the simulation of wave propagation in media containing small obstacles, i.e. objects with a diameter much smaller than the average wavelength. In order to perform such a computation, a first approach consists in applying standard techniques (finite elements for instance) everywhere in the domain of computation. Small obstacles require mesh refinement in order to be taken into account: time and memory used for these computations increase significantly because of these details, which implies bad computational efficiency.

X. Claeys (✉)
INRIA Paris-Rocquencourt, Domaine de Voluceau, B.P. 105, 78153 Le Chesnay, France
e-mail: xavier.claeys@inria.fr

F. Collino
CERFACS, 42 avenue G. Coriolis, 31057 Toulouse, France
e-mail: collino@cerfacs.fr

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In this work we propose a way to avoid such mesh refinement. There is little in the literature concerning the study, from a numerical analysis point of view, of the effect of the presence of small obstacles on the solution of an elliptic problem. In [4] the authors study the effect of small disks with Dirichlet or mixed boundary conditions on the resolution of a Laplace problem. In [11] the authors are interested in the effect of a small perturbation of the boundary on the resolution of an electrostatic problem. Both of these methods require special treatment of the small obstacle via a precomputation. But our approach is pragmatic: we would like to take into account small obstacles without any preliminary treatment, realizing only one computation. To our knowledge, there exists no volumic numerical method proposing such an alternative in the presence of small obstacles. We choose a simplified context and consider a 2-D geometry where the Helmholtz equation has to be solved (see [7] for extensions).

Our method consists, on the basis of an asymptotic analysis, in augmenting the approximation space with an additional shape function containing a singularity adapted to a small obstacle. This method is similar to singular complement methods (or singular function methods) used in electromagnetics in order to obtain consistency when computational domains contain reentrant corners. There is a vast literature on this subject (see [1,2,5,12,14,21] and references therein). The justification of the singular complement method relies on asymptotic analysis in the neighborhood of an edge or a corner. Concerning corners and edges there is also a very extensive literature, let us just quote [19,24] or [13].

The first particularity of the present work lies in the application of such a method to a problem whose geometry depends on a small parameter. Therefore the numerical scheme we describe also depends on this parameter. Consequently we refer to preexisting results of asymptotic analysis around a small obstacle (see [16,20] or [23] as well as to our preliminary work [8]). A second particular feature of our method is the use of a fictitious domain formulation. This enables us to avoid adapting the mesh to the small obstacle. Fictitious domain methods have been presented and studied in [10,18] and references therein. In the fictitious domain formulation, the boundary condition is taken into account using Lagrange multipliers. In the present work, using asymptotics, the space of Lagrange multipliers is reduced, thus lowering computational cost and making implementation easier.

Our geometrical setting is as follows. Let $\Omega = D(0,\varrho)$ be the disk of radius $\varrho > 0$ with center 0 and let $\Gamma = \partial\Omega$. We will use this boundary as a fictitious line for imposing a radiation condition in order to restrict our scattering problem to the bounded domain Ω. In the remaining of this paper (r,θ) will refer to polar coordinates. Let us consider a small scatterer with boundary Γ_ε defined by the equation

$$(\Gamma_\varepsilon): \quad r = \varepsilon\,\gamma(\theta) \quad \forall\theta \in [0, 2\pi).$$

In this definition $\varepsilon \in (0,1)$ is a parameter that must be thought of as going to 0, which simply introduces in our analysis the assumption of a "small obstacle". We assume that γ is a 2π-periodic $\mathscr{C}^\infty$ function such that $\gamma(\theta) > \gamma_* > 0, \forall\theta \in [0,2\pi)$, $\gamma_* \in \mathbb{R}$. In order to simplify future calculation, we also assume that $\gamma(\theta) < 1$. We denote the exterior of the small obstacle by

$$\Omega_\varepsilon := \{\mathbf{x}(r,\theta) \in \Omega \mid \varepsilon\gamma(\theta) < r < \varrho\}.$$

This paper is made up of three sections. In the first section we present two different models for time harmonic diffraction by the small obstacle with boundary Γ_ε. The first model is the Helmholtz equation

$$\Delta u^\varepsilon + k^2 u^\varepsilon = -f \quad \text{in} \quad \Omega_\varepsilon, \quad u^\varepsilon = 0 \quad \text{on } \Gamma_\varepsilon.$$

We also impose the Sommerfeld radiation condition $\lim_{r\to\infty}\sqrt{r}(\partial_r u^\varepsilon + iku^\varepsilon) = 0$ reformulated as a condition on Γ via a Dirichlet-to-Neumann operator. The second model consists in the same equations with a modified condition imposed on Γ_ε that is an averaged version of the exact Dirichlet condition. In the second section, we prove that the solutions of the first and second models are asymptotically close to each other (Theorem 2.1). Finally in the third section we propose a numerical scheme for the computation of the solution to the second model. This numerical scheme has the interesting property of keeping good accuracy even when $\varepsilon \to 0$: it is "quasi locking-free". We prove a consistency estimate (Theorem 3.1), and present numerical results.

Remark on the notations All through this article, $\kappa, \kappa_0, \kappa_1 \ldots$ and so on, will always refer to strictly positive constants that does not depend on the small parameter ε.

1 Description of two models

We first introduce notations related to standard functional spaces. Given a set $\omega \subset \mathbb{R}^2$, $L^2(\omega)$ will refer to the set of measurable functions v such that $\|v\|_{0,\omega}^2 := \int_\omega |v|^2 < \infty$ and $H^p(\omega)$ will be the Sobolev space of order p over ω equipped with the norm $\|v\|_{p,\omega}^2 := \sum_{k=0}^p \|\nabla^k v\|_{0,\omega}^2$ (see [27] for example). We will also refer to the following space

$$\mathbb{V}_0^\varepsilon := \left\{ v \in H^1(\Omega) \mid v = 0 \text{ in } \Omega \setminus \overline{\Omega}_\varepsilon \right\}. \tag{1.1}$$

In the sequel, $H^{1/2}(\Gamma)$ will refer to the space of traces on Γ of elements of $H^1(\Omega)$, equipped with the norm $\|v\|_{1/2,\Gamma}^2 = \inf\{\|w\|_{1,\Omega} \mid w_{|\Gamma} = v\}$, and $H^{-1/2}(\Gamma)$ will refer to its topological dual. We also consider similar definitions for $H^{1/2}(\Gamma_\varepsilon)$ and $H^{-1/2}(\Gamma_\varepsilon)$.

1.1 The exact problem

Our initial problem is as follows. We consider a source function $f \in \mathscr{C}^\infty(\Omega)$ such that there exists an open neighborhood ω of 0 satisfying $\operatorname{supp} f \cap \omega = \emptyset$. The smoothness assumption on f allows a simplification in our presentation but does not represent a serious restriction. Our study can be easily adapted to the case $f \in H^k(\Omega), k \in \mathbb{N}$. We wish to solve the Helmholtz equation $\Delta u^\varepsilon + k^2 u^\varepsilon = -f$ in Ω_ε with homogeneous Dirichlet boundary condition on Γ_ε and outgoing radiation condition. A standard variational formulation for this problem is given by

$$\text{Find } u^\varepsilon \in \mathbb{V}_0^\varepsilon \text{ such that } \quad a(u^\varepsilon, v) = \int_\Omega f\overline{v} \quad \forall v \in \mathbb{V}_0^\varepsilon$$

$$\text{with } \quad a(v,w) := \int_\Omega \nabla v \nabla \overline{w} - k^2 \int_\Omega v\overline{w} + \int_\Gamma \overline{w} T_\Gamma v \quad \forall v, w \in H^1(\Omega). \quad (1.2)$$

In this definition T_Γ refers to the Dirichlet-to-Neumann operator for the 2D-Helmholtz equation. For details concerning non-reflecting boundary conditions (see [17]). This operator admits an analytical expression that can be decomposed using Fourier expansion and the Hankel functions $H_n^{(1)}$,

$$T_\Gamma v \; := \; -\sum_{n\in\mathbb{Z}} v_n \, k \, \frac{H_n^{(1)'}(k\,\varrho)}{H_n^{(1)}(k\,\varrho)} e^{in\theta} \quad \text{where} \quad v_n \; = \; \frac{1}{2\pi}\int_0^{2\pi} v(r,\alpha)\, e^{-in\alpha} d\alpha.$$

For a definition of $H_n^{(1)}$ (see [25]). It is classical that T_Γ is a continuous operator from $H^{1/2}(\Gamma)$ into $H^{-1/2}(\Gamma)$ so that $(v, w) \mapsto \int_\Omega \overline{w}\, T_\Gamma v$ is a continuous sesquilinear form on $H^1(\Omega) \times H^1(\Omega)$. As a consequence $a(\;,\;)$ is also continuous on $H^1(\Omega) \times H^1(\Omega)$. The variational Formulation (1.2) is well posed (see [27] Lemma 9.9 and Theorem 9.11). Let n_Γ be the unit vector outward normal on Γ. Choose suitable test functions in (1.2) and apply Green's formula to obtain the following equations

$$\begin{aligned} &\Delta u^\varepsilon + k^2 u^\varepsilon = -f \text{ in } \Omega_\varepsilon \quad \Delta u^\varepsilon + k^2 u^\varepsilon = 0 \text{ in } \Omega \setminus \overline{\Omega_\varepsilon} \quad \text{and} \\ &\frac{\partial u^\varepsilon}{\partial n_\Gamma} + T_\Gamma u^\varepsilon = 0 \text{ on } \Gamma. \end{aligned} \quad (1.3)$$

According to Remark 1.6 Section II.1 in [3], the well posedness of Problem (1.2) is equivalent to the fact that $a(\;,\;)$ satisfies two inf–sup conditions. These inf–sup conditions are actually satisfied uniformly with respect to ε. The proof for this is quite similar to the proof of Proposition 3.3 and simpler. Thus it is left to the reader.

Lemma 1.1 (Stability of the exact problem) *There exists $\kappa > 0$ such that*

$$\begin{aligned} &\inf_{u\in\mathbb{V}_0^\varepsilon} \sup_{v\in\mathbb{V}_0^\varepsilon} \frac{|\,a(u,v)\,|}{\|u\|_{1,\Omega}\,\|v\|_{1,\Omega}} > \kappa \\ &\textit{and} \quad \inf_{u\in\mathbb{V}_0^\varepsilon} \sup_{v\in\mathbb{V}_0^\varepsilon} \frac{|\,a(v,u)\,|}{\|u\|_{1,\Omega}\,\|v\|_{1,\Omega}} > \kappa \quad \forall \varepsilon \in (0,1). \end{aligned}$$

1.2 An intermediate electrostatic problem

We wish to present a second problem close to the former one. To describe it, we introduce a function appearing in known results of asymptotic analysis around a small obstacle, such as those contained in [8] or [20,26]. Consider two "normalized versions" of Ω_ε and Γ_ε, defined in polar coordinates by

$$\Omega_{\mathrm{N}} := \left\{\mathbf{x}(r,\theta) \in \mathbb{R}^2 \mid \gamma(\theta) < r\right\} \quad \text{and} \quad \Gamma_{\mathrm{N}} := \left\{\mathbf{x}(r,\theta) \in \mathbb{R}^2 \mid r = \gamma(\theta)\right\}.$$

We define $\mathfrak{S}$ as the solution to the following problem

$$\begin{aligned} &\mathfrak{S} \in \ln|\mathbf{x}| + \mathbb{W}_{\log}(\Omega_{\mathrm{N}}) \quad \text{such that} \quad \Delta\mathfrak{S} = 0 \quad \text{in } \Omega_{\mathrm{N}} \quad \text{and} \quad \mathfrak{S} = 0 \quad \text{on } \Gamma_{\mathrm{N}} \\ &\text{with} \quad \mathbb{W}_{\log}(\Omega_{\mathrm{N}}) := \left\{ v \in H^1_{\mathrm{loc}}(\Omega_{\mathrm{N}}) \;\middle|\; \int_{\Omega_{\mathrm{N}}} \frac{|v|^2\, d\mathbf{x}}{|\mathbf{x}|^2 \ln^2(1+|\mathbf{x}|)} + \int_{\Omega_{\mathrm{N}}} |\nabla v|^2\, d\mathbf{x} < +\infty \right\}. \end{aligned} \tag{1.4}$$

Recall that $H^1_{\mathrm{loc}}(\overline{\Omega}_{\mathrm{N}})$ is the space of all functions v defined on Ω_{N} such that $v\varphi \in H^1(\Omega_{\mathrm{N}})$ for all $\varphi \in \mathscr{C}^\infty(\overline{\Omega}_{\mathrm{N}})$ with compact support. That (1.4) is well posed can be straightforwardly deduced from standard results on the Laplace equation in $\mathbb{R}^2$ (see [15] chap. 2 for instance). At infinity, $\mathfrak{S}$ admits the following behaviour: there exists a constant $c_\gamma \in \mathbb{R}$ such that

$$\sup_{\theta\in[0,2\pi)} |\mathfrak{S}(r,\theta) - \ln r - c_\gamma| + \sup_{\theta\in[0,2\pi)} |r\partial_r\mathfrak{S} - 1| + \sup_{\theta\in[0,2\pi)} |\partial_\theta\mathfrak{S}| = \underset{r\to\infty}{O}\left(\frac{1}{r}\right), \tag{1.5}$$

where $O(\;)$ refers to the usual Landau notation. Equation (1.5) is a direct consequence of separation of variables applied to Problem (1.4). The function $\mathfrak{S}$ satisfies also the following property.

Lemma 1.2 *Let $n_{\Gamma_{\mathrm{N}}}$ be the normal vector to Γ_{N} directed into the interior of Ω_{N}, then* $\displaystyle\int_{\Gamma_{\mathrm{N}}} \frac{\partial\mathfrak{S}}{\partial n_{\Gamma_{\mathrm{N}}}} = 2\pi$.

Proof Let $\mathfrak{S}(r,\theta) = \ln r + \sum_{p\in\mathbb{Z}} \mathfrak{S}_p e^{ip\theta} r^{-|p|}$ be the decomposition of $\mathfrak{S}$ derived by separation of variables (so $c_\gamma = \mathfrak{S}_0$). Now consider some $r_0 > 1$ so that $\mathbb{R}^2 \setminus \Omega_{\mathrm{N}} \subset D(0,r_0)$. Then apply a Green's formula after integrating the Laplace equation satisfied by $\mathfrak{S}$ over $\Omega_{\mathrm{N}} \cap D(0,r_0)$, which leads to

$$0 = \int_0^{2\pi}\int_{\gamma(\theta)}^{r_0} \Delta\mathfrak{S}\; r dr d\theta = \int_0^{2\pi} \frac{\partial\mathfrak{S}}{\partial r}(r_0,\theta) r_0 d\theta - \int_{\Gamma_{\mathrm{N}}} \frac{\partial\mathfrak{S}}{\partial n_{\Gamma_{\mathrm{N}}}}$$

$$\text{hence} \int_{\Gamma_{\mathrm{N}}} \frac{\partial\mathfrak{S}}{\partial n_{\Gamma_{\mathrm{N}}}} = r_0 \int_0^{2\pi} \frac{\partial}{\partial r}\left(\ln r + \sum_{p\in\mathbb{Z}} \mathfrak{S}_p e^{ip\theta} r^{-|p|}\right)_{|r=r_0} d\theta.$$

According to standard elliptic regularity results (see Theorem 4.16 in [27]), the expansion converges in $H^n(\omega)$ for any $n \in \mathbb{N}$ and any $\omega \subset \Omega_{\mathrm{N}}$. So we can move the sum out of the integral which gives

$$\int_{\Gamma_{\mathrm{N}}} \frac{\partial \mathfrak{S}}{\partial n_{\Gamma_{\mathrm{N}}}} = r_0 \int_0^{2\pi} \frac{\partial}{\partial r} (\ln r)_{|r=r_0} \, d\theta + \sum_{p\in\mathbb{Z}} \mathfrak{S}_p r_0 \int_0^{2\pi} \frac{\partial}{\partial r} \left(r^{-|p|}\right)_{|r=r_0} e^{ip\theta} \, d\theta = 2\pi.$$

□

1.3 A simplified problem

We modify Problem (1.2) by weakening the Dirichlet boundary condition. Let n_{Γ_ε} be the outward normal vector to Γ_ε. The new variational problem we consider is given by

$$\text{Find } \tilde{u}^\varepsilon \in \mathbb{V}^\varepsilon_\mu \text{ such that } \quad a(\tilde{u}^\varepsilon, v) = \int_\Omega f\, \overline{v} \quad \forall v \in \mathbb{V}^\varepsilon_\mu$$

$$\text{where } \quad \mathbb{V}^\varepsilon_\mu := \left\{ v \in H^1(\Omega) \Big| \int_{\Gamma_\varepsilon} \frac{\partial\, \mathfrak{S}^\varepsilon}{\partial n_{\Gamma_\varepsilon}} \overline{v} d\sigma = 0 \right\} \quad \text{with} \quad \mathfrak{S}^\varepsilon(\mathbf{x}) := \mathfrak{S}\left(\frac{\mathbf{x}}{\varepsilon}\right). \tag{1.6}$$

The sesquilinear form $a(\ ,\ )$ verify inf–sup conditions also when it is restricted to $\mathbb{V}^\varepsilon_\mu$ as stated in the next lemma. Its proof is quite similar to the proof of Proposition 3.3 below and actually a bit simpler. Thus it will not be given here.

Lemma 1.3 (Stability of the simplified problem) *There exists* $\kappa_0, \varepsilon_0 > 0$ *such that*

$$\inf_{u\in\mathbb{V}^\varepsilon_\mu} \sup_{v\in\mathbb{V}^\varepsilon_\mu} \frac{|a(u, v)|}{\|u\|_{1,\Omega}\, \|v\|_{1,\Omega}} > \kappa_0$$

$$\textit{and} \quad \inf_{u\in\mathbb{V}^\varepsilon_\mu} \sup_{v\in\mathbb{V}^\varepsilon_\mu} \frac{|a(v, u)|}{\|u\|_{1,\Omega}\, \|v\|_{1,\Omega}} > \kappa_0 \quad \forall \varepsilon \in (0, \varepsilon_0).$$

These inf–sup conditions straightforwardly imply that the variational formulation (1.6) is well posed for any $\varepsilon \in (0, \varepsilon_0)$. As a consequence $\tilde{u}^\varepsilon$ is well defined for any $\varepsilon \in (0, \varepsilon_0)$. Note that $\tilde{u}^\varepsilon$ does not vanish a priori in $\Omega \setminus \overline{\Omega}_\varepsilon$, nor on Γ_ε. Just as for the exact problem, choosing suitable test functions in (1.6) and applying Green's formula leads to

$$\Delta\tilde{u}^\varepsilon + k^2\tilde{u}^\varepsilon = -f \text{ in } \Omega_\varepsilon, \quad \Delta\tilde{u}^\varepsilon + k^2\tilde{u}^\varepsilon = 0 \text{ in } \Omega \setminus \overline{\Omega}_\varepsilon$$

$$\text{and} \quad \frac{\partial \tilde{u}^\varepsilon}{\partial n_\Gamma} + T_\Gamma \tilde{u}^\varepsilon = 0 \text{ on } \Gamma. \tag{1.7}$$

2 Asymptotic analysis

In order to demonstrate the relevance of substituting Problem (1.6) for Problem (1.2), we will show that $\|u^\varepsilon - \tilde{u}^\varepsilon\|_{1,\Omega} \to 0$ as $\varepsilon \to 0$. To the best of our knowledge, the following asymptotic treatment of $\tilde{u}^\varepsilon$ is new.

2.1 Definition of the first term of the matched expansion

We will prove that u^ε and $\tilde{u}^\varepsilon$ are both asymptotically close to a common intermediate function $u^\varepsilon_\chi \in H^1(\Omega)$ that we define now. Consider the limit field u^0 defined as the unique solution to the following variational problem:

$$u^0 \in H^1(\Omega) \quad \text{and} \quad a(u^0, v) = \int_\Omega f\overline{v} \quad \forall v \in H^1(\Omega). \tag{2.1}$$

Problem (2.1) is well posed, and $u^0 \in H^2(\Omega) \subset \mathscr{C}^0(\Omega)$ according to elliptic regularity theorems (see [27]). Consider now a non-increasing $\mathscr{C}^\infty$-function $\chi : \mathbb{R} \to [0, 1]$ that satisfies $\chi(r) = 1$ when $r < 1$ and $\chi(r) = 0$ when $r > 2$. We define the intermediate function u^ε_χ by

$$\begin{aligned} &u^\varepsilon_\chi(\mathbf{x}) := \left(1 - \chi^\varepsilon(\mathbf{x})\right) u^\varepsilon_0(\mathbf{x}) + \chi^\varepsilon(\mathbf{x}) U^\varepsilon_0(\mathbf{x}) \\ &\text{with} \quad u^\varepsilon_0(\mathbf{x}) := u^0(\mathbf{x}) + \frac{-u^0(0)}{\eta(\varepsilon)} \frac{\pi}{2i} H^{(1)}_0(k|\mathbf{x}|), \quad U^\varepsilon_0(\mathbf{x}) := \frac{-u^0(0)}{\eta(\varepsilon)} \mathfrak{S}(\mathbf{x}/\varepsilon)\, \mathbf{1}_{\Omega_\varepsilon} \\ &\qquad \eta(\varepsilon) := \ln\left(\frac{k\,\varepsilon}{2}\right) + \gamma_{\mathbf{e}} - i\frac{\pi}{2} - c_\gamma \quad \text{and} \quad \chi^\varepsilon(\mathbf{x}) := \chi\left(\frac{|\mathbf{x}|}{\sqrt{\varepsilon}}\right). \end{aligned} \tag{2.2}$$

In this definition $\gamma_{\mathbf{e}} = 0.5772156\ldots$ is the Euler–Mascheroni constant. According to formulas (5.2.2), (5.5.3) and (5.6.1) in [25] we have

$$\begin{aligned} &\frac{\pi}{2i} H^{(1)}_0(kr) = \ln\left(\frac{kr}{2}\right) + \gamma_{\mathbf{e}} - i\frac{\pi}{2} + \underset{r\to 0}{O}(r) \quad \text{and} \\ &r\partial_r\left(\frac{\pi}{2i} H^{(1)}_0(kr)\right) = 1 + \underset{r\to 0}{O}(r). \end{aligned} \tag{2.3}$$

Note that $u^\varepsilon_0 \in H^2_{\text{loc}}(\Omega \setminus \{0\})$ and the following equations hold in the distributional sense:

$$\Delta u^\varepsilon_0 + k^2 u^\varepsilon_0 = -f \quad \text{in } \Omega \setminus \{0\} \quad \text{and} \quad \frac{\partial u^\varepsilon_0}{\partial n_\Gamma} + T_\Gamma u^\varepsilon_0 = 0 \quad \text{on } \Gamma, \quad \forall \varepsilon \in (0, 1). \tag{2.4}$$

Note also that $\mathfrak{S} = 0$ on Γ_{N} so $\mathfrak{S}(\mathbf{x}/\varepsilon) = 0$ when $\mathbf{x} \in \Gamma_\varepsilon$. Thus $u^\varepsilon_\chi \in \mathbb{V}^\varepsilon_0$ according to Definition (2.2) and in particular $u^\varepsilon_\chi \in \mathbb{V}^\varepsilon_\mu$. In order to distinguish regions "far" and "close" to the obstacle, we set $Z^\varepsilon_f := \{\mathbf{x} \in \Omega \mid |\mathbf{x}| > \sqrt{\varepsilon}\}$, $Z^\varepsilon_n := \{\mathbf{x} \in \Omega \mid |\mathbf{x}| < 2\sqrt{\varepsilon}\}$ and $Z^\varepsilon_t := Z^\varepsilon_f \cap Z^\varepsilon_n$.

2.2 Asymptotic stability

To obtain an asymptotic estimate for $u^\varepsilon - \tilde{u}^\varepsilon$ we proceed in two steps. The first step consists in bounding $\|u^\varepsilon - \tilde{u}^\varepsilon\|_{1,\Omega}$ by a quantity depending only on u^ε_χ.

Proposition 2.1 *There exist* $\kappa_0, \varepsilon_0 > 0$ *such that* $\|u^\varepsilon - \tilde{u}^\varepsilon\|_{1,\Omega} \leq \kappa_0|\ln\varepsilon|\ \|r(f + \Delta u^\varepsilon_\chi + k^2 u^\varepsilon_\chi)\|_{0,\Omega_\varepsilon}$, $\forall\varepsilon \in (0,\varepsilon_0)$.

Proof We start by applying the triangle inequality, $\|u^\varepsilon - \tilde{u}^\varepsilon\|_{1,\Omega} \leq \|u^\varepsilon - u^\varepsilon_\chi\|_{1,\Omega} + \|u^\varepsilon_\chi - \tilde{u}^\varepsilon\|_{1,\Omega}$ and bound each of the two resulting terms. For the first term, since $u^\varepsilon_\chi \in \mathbb{V}^\varepsilon_0$, Lemma 1.1 provides $\kappa_0, \varepsilon_0 > 0$ such that

$$\|u^\varepsilon - u^\varepsilon_\chi\|_{1,\Omega} \ < \ \kappa_0 \sup_{v\in\mathbb{V}^\varepsilon_0} \frac{|a(u^\varepsilon - u^\varepsilon_\chi, v)|}{\|v\|_{1,\Omega}} \quad \forall\varepsilon \in (0,\varepsilon_0).$$

Since $v = 0$ in $\Omega\setminus\overline{\Omega}_\varepsilon$ when $v \in \mathbb{V}^\varepsilon_0$, we simply apply a Green's formula in Ω_ε in order to rearrange the expression of $a(u^\varepsilon - u^\varepsilon_\chi, v)$. According to (1.3) and (2.4), $\forall v \in \mathbb{V}^\varepsilon_0$, $\forall\varepsilon \in (0,1)$

$$\begin{aligned}|a(u^\varepsilon - u^\varepsilon_\chi, v)| &= \left| \int_{\Omega_\varepsilon} \left(f + \Delta u^\varepsilon_\chi + k^2 u^\varepsilon_\chi\right)\overline{v} \right| \\ &\leq \ \|r(f + \Delta u^\varepsilon_\chi + k^2 u^\varepsilon_\chi)\|_{0,\Omega_\varepsilon} \ \left\| \frac{v}{r} \right\|_{0,\Omega_\varepsilon}.\end{aligned}$$

There only remains to use Lemma A.1. Since $\|v\|_{1,\Omega_\varepsilon} \leq \|v\|_{1,\Omega}$, $\forall v \in H^1(\Omega)$, we obtain $\kappa_1, \varepsilon_1 > 0$ such that

$$\|u^\varepsilon - u^\varepsilon_\chi\|_{1,\Omega} \ \leq \ \kappa_1|\ln\varepsilon|\ \|\,r\,(f + \Delta u^\varepsilon_\chi + k^2 u^\varepsilon_\chi)\|_{0,\Omega_\varepsilon} \quad \forall\varepsilon \in (0,\varepsilon_1). \tag{2.5}$$

The bound for the term related to $\tilde{u}^\varepsilon$ is derived in a similar manner. Taking into account Eqs. (1.7), we apply a Green's formula on both $\Omega\setminus\overline{\Omega}_\varepsilon$ and Ω_ε, choosing this time test functions in $\mathbb{V}^\varepsilon_\mu$. Applying Cauchy–Schwartz inequality and Lemma A.1, we obtain the existence of $\kappa_3, \varepsilon_3 > 0$ such that

$$\begin{aligned}|\,a(\tilde{u}^\varepsilon - u^\varepsilon_\chi, v)\,| \ \leq \ \kappa_3|\ln\varepsilon|\ \|r(f + \Delta u^\varepsilon_\chi + k^2 u^\varepsilon_\chi)\|_{0,\Omega_\varepsilon}\ \|v\|_{1,\Omega} \\ \forall v \in \mathbb{V}^\varepsilon_\mu,\ \forall\varepsilon \in (0,\varepsilon_3).\end{aligned} \tag{2.6}$$

Finally apply Lemma 1.3 taking into account Inequality (2.6) and the fact that $u^\varepsilon_\chi \in \mathbb{V}^\varepsilon_\mu$. This leads to the existence of $\kappa_4, \varepsilon_4 > 0$ such that

$$\|\tilde{u}^{\varepsilon} - u^{\varepsilon}_{\chi}\|_{1,\Omega} \;<\; \kappa_4 \,|\ln\varepsilon|\, \|r(f + \Delta u^{\varepsilon}_{\chi} + k^2 u^{\varepsilon}_{\chi})\|_{0,\Omega_\varepsilon} \quad \forall\varepsilon \in (0, \varepsilon_4). \qquad (2.7)$$

To conclude the proof it suffices to combine Eqs. (2.5) and (2.7). □

2.3 Asymptotic consistency

The second step of the asymptotic analysis consists in proving that $\|r(f + \Delta u^{\varepsilon}_{\chi} + k^2 u^{\varepsilon}_{\chi})\|_{0,\Omega_\varepsilon}$ is small as $\varepsilon \to 0$. Since there exists $\varepsilon_0 > 0$ such that $f = (1-\chi^{\varepsilon})f$, $\forall\varepsilon \in (0, \varepsilon_0)$ and $\operatorname{supp} f$ excludes a fixed neighborhood of 0, according to the definition of u^{ε}_{χ} given by (2.2), we have

$$f + \Delta u^{\varepsilon}_{\chi} + k^2 u^{\varepsilon}_{\chi} = (1-\chi^{\varepsilon})\,(f + \Delta u^{\varepsilon}_0 + k^2 u^{\varepsilon}_0) + \chi^{\varepsilon}\,(\Delta U^{\varepsilon}_0 + k^2 U^{\varepsilon}_0)$$
$$+[\Delta;\chi^{\varepsilon}]\left(U^{\varepsilon}_0 - u^{\varepsilon}_0\right) \quad \text{in } \Omega_\varepsilon$$

for all $\varepsilon \in (0, 1)$ in the distributional sense. In this expression we used $[\Delta;\chi^{\varepsilon}] := \Delta\chi^{\varepsilon} - \chi^{\varepsilon}\Delta$. According to (2.4), some parts in the right hand side vanish: $(1-\chi^{\varepsilon})\,(f + \Delta u^{\varepsilon}_0 + k^2 u^{\varepsilon}_0) = 0$ and $\chi^{\varepsilon}\Delta U^{\varepsilon}_0 = 0$, $\forall\varepsilon \in (0, 1)$. Taking into account the definition of χ^{ε} given in (2.2), we have

$$r^2\,[\Delta;\chi^{\varepsilon}](U^{\varepsilon}_0 - u^{\varepsilon}_0) = \left((r\partial_r)^2\chi^{\varepsilon}\right)\,(U^{\varepsilon}_0 - u^{\varepsilon}_0)$$
$$+2\,\left(r\partial_r\chi^{\varepsilon}\right)\,r\partial_r(U^{\varepsilon}_0 - u^{\varepsilon}_0) \qquad (2.8)$$

Given a set $\omega \subset \mathbb{R}$ or $\mathbb{R}^2$ and a function $v \in \mathscr{C}^0(\overline{\omega})$, denote $|v|_{\infty,\omega} := \sup_{\omega}|v|$. Recall that $|\mathbf{x}| < 2\sqrt{\varepsilon}$ when $\mathbf{x} \in \operatorname{supp}\chi^{\varepsilon}$ and $|\mathbf{x}| > \sqrt{\varepsilon}$ when $\mathbf{x} \in \operatorname{supp}\{r\partial_r\chi^{\varepsilon}\}$ or when $\mathbf{x} \in \operatorname{supp}\{(r\partial_r)^2\chi^{\varepsilon}\}$. Joining this remark to Identity (2.8) leads to the estimate

$$\|r(f + \Delta u^{\varepsilon}_{\chi} + k^2 u^{\varepsilon}_{\chi})\|_{0,\Omega_\varepsilon} \le 2\sqrt{\varepsilon}\,|\chi^{\varepsilon}|_{\infty,\Omega}\,k^2\|U^{\varepsilon}_0\|_{0,Z^{\varepsilon}_n}$$
$$+\frac{1}{\sqrt{\varepsilon}}\,|(r\partial_r)^2\chi^{\varepsilon}|_{\infty,\Omega}\,\|u^{\varepsilon}_0 - U^{\varepsilon}_0\|_{0,Z^{\varepsilon}_t}$$
$$+\frac{2}{\sqrt{\varepsilon}}\,|r\partial_r\chi^{\varepsilon}|_{\infty,\Omega}\|r\partial_r\left(u^{\varepsilon}_0 - U^{\varepsilon}_0\right)\|_{0,Z^{\varepsilon}_t}. \qquad (2.9)$$

Estimates for the cut-off function We provide a bound for each term in the right hand side of the previous inequality. Denote χ' and χ'' the first and second derivative of χ. Then we have

$$\left(r\partial_r\chi^{\varepsilon}\right)(r) = \left(\frac{r}{\sqrt{\varepsilon}}\right)\,\chi'\left(\frac{r}{\sqrt{\varepsilon}}\right) \quad \text{and}$$
$$\left((r\partial_r)^2\chi^{\varepsilon}\right)(r) = \left(\frac{r}{\sqrt{\varepsilon}}\right)\,\chi'\left(\frac{r}{\sqrt{\varepsilon}}\right) + \left(\frac{r}{\sqrt{\varepsilon}}\right)^2\,\chi''\left(\frac{r}{\sqrt{\varepsilon}}\right),$$

Since $\chi^{\varepsilon}(\mathbf{x}) \neq 0$ only if $|\mathbf{x}| < 2\sqrt{\varepsilon}$, we obtain $|(r\partial_r)^q\chi^{\varepsilon}|_{\infty,\Omega} \le |\chi|_{\infty,\mathbb{R}} + 2|\chi'|_{\infty,\mathbb{R}} + 4|\chi''|_{\infty,\mathbb{R}}$ for $q = 0, 1, 2$. Plugging this inequality into (2.9) and using rough estimates, for any $\varepsilon \in (0, 1)$ we have

$$\begin{aligned}\|r(f + \Delta u^\varepsilon_\chi + k^2 u^\varepsilon_\chi)\|_{0,\Omega_\varepsilon} \leq\ & \frac{\kappa(\chi)}{\sqrt{\varepsilon}} \\ & \left(\varepsilon\ \|U^\varepsilon_0\|_{0,Z^\varepsilon_n} + \|u^\varepsilon_0 - U^\varepsilon_0\|_{0,Z^\varepsilon_t} + \|r\partial_r \left(u^\varepsilon_0 - U^\varepsilon_0\right)\|_{0,Z^\varepsilon_t}\right) \\ & \text{with}\ \ \kappa(\chi) := 2\left(1 + k^2\right)\ \left(|\chi|_{\infty,\mathbb{R}} + 2|\chi'|_{\infty,\mathbb{R}} + 4|\chi''|_{\infty,\mathbb{R}}\right). \end{aligned} \tag{2.10}$$

Estimate of the term related to the near field We first give an upper bound for the term $\|U^\varepsilon_0\|_{0,Z^\varepsilon_n}$, making the change of coordinate $r = \varepsilon R$. Using Eq. (1.5), we obtain the existence of $\kappa > 0$ such that for any $\varepsilon \in (0, 1)$

$$\begin{aligned}\|U^\varepsilon_0\|^2_{0,Z^\varepsilon_n} &= \left|\frac{u^0(0)}{\eta(\varepsilon)}\right|^2 \int_0^{2\pi}\int_{\varepsilon\gamma(\theta)}^{2\sqrt{\varepsilon}} |\mathfrak{S}(r/\varepsilon, \theta)|^2 r\,dr\,d\theta \\ &\leq 2\varepsilon^2 \left|\frac{u^0(0)}{\eta(\varepsilon)}\right|^2 \int_0^{2\pi}\int_{\gamma(\theta)}^{2/\sqrt{\varepsilon}} \left(|\mathfrak{S}(R, \theta) - \ln R|^2 + |\ln R|^2\right) R\,dR\,d\theta \leq \kappa\varepsilon.\end{aligned} \tag{2.11}$$

Estimate of the first term related to the transition zone Now we bound $\|u^\varepsilon_0 - U^\varepsilon_0\|_{0,Z^\varepsilon_t}$, using the following identity easily deduced from (2.2)

$$\begin{aligned}u^\varepsilon_0(\mathbf{x}) - U^\varepsilon_0(\mathbf{x}) &= u^0(\mathbf{x}) - u^0(0) \\ &+ \frac{-u^0(0)}{\eta(\varepsilon)}\left[\frac{\pi}{2i} H^{(1)}_0(k|\mathbf{x}|) - \ln\left(\frac{k|\mathbf{x}|}{2}\right) - \gamma_{\mathbf{e}} + i\frac{\pi}{2}\right] \\ &+ \frac{u^0(0)}{\eta(\varepsilon)}\left[\mathfrak{S}\left(\frac{\mathbf{x}}{\varepsilon}\right) - \ln\left(\frac{|\mathbf{x}|}{\varepsilon}\right) - c_\gamma\right] \quad \forall\varepsilon \in (0, 1).\end{aligned} \tag{2.12}$$

Then we give an estimate for each of the three terms in the right hand side above. For $u^0(\mathbf{x}) - u^0(0)$ we simply use a Taylor expansion of u^0 in the neighborhood of 0. Since $\sqrt{\varepsilon} \leq |\mathbf{x}| \leq 2\sqrt{\varepsilon}$ when $\mathbf{x} \in Z^\varepsilon_t$, this provides $\kappa > 0$ such that

$$\|u^0(\mathbf{x}) - u^0(0)\|_{0,Z^\varepsilon_t} \leq \kappa\varepsilon \quad \forall\varepsilon \in (0, 1). \tag{2.13}$$

We deal now with the second term of (2.12). According to Eq. (2.3), there exists $\kappa > 0$ such that

$$\left\| H^{(1)}_0(k|\mathbf{x}|) - \ln\left(\frac{k|\mathbf{x}|}{2}\right) - \gamma_{\mathbf{e}} + i\frac{\pi}{2}\right\|^2_{0,Z^\varepsilon_t} \leq \kappa \int_{\sqrt{\varepsilon}}^{2\sqrt{\varepsilon}} r^3 dr = \frac{15}{4}\kappa\varepsilon^2 \quad \forall\varepsilon \in (0, 1). \tag{2.14}$$

Finally we deal with the third term in (2.12). First we apply the change of coordinate $r = \varepsilon R$, and then we take into account Asymptotics (1.5). This leads to a constant $\kappa > 0$ such that for any $\varepsilon \in (0, 1)$ we have

$$\left\| \mathfrak{S}\left(\frac{\mathbf{x}}{\varepsilon}\right) - \ln\left(\frac{|\mathbf{x}|}{\varepsilon}\right) - c_\gamma \right\|^2_{0,Z_t^\varepsilon} = \varepsilon^2 \int_0^{2\pi} \int_{1/\sqrt{\varepsilon}}^{2/\sqrt{\varepsilon}} |\mathfrak{S}(R,\theta) - \ln R - c_\gamma|^2 R\, dR\, d\theta \leq 2\pi\kappa\varepsilon^2. \tag{2.15}$$

There only remains to combine Eqs. (2.12)–(2.15). This provides $\kappa > 0$ (depending on γ, f but not on ε) such that

$$\|u_0^\varepsilon - U_0^\varepsilon\|_{0,Z_t^\varepsilon} \leq \kappa\varepsilon \quad \forall\varepsilon \in (0, 1). \tag{2.16}$$

Estimate of the second term related to the transition zone The estimate for the last term in the right hand side of (2.10) follows the same lines as for the second term, so we only give some indications about its derivation. First we use Decomposition (2.12). Then, using Asymptotics (1.5) and (2.3), we obtain estimates similar to (2.13), (2.14) and (2.15) but involving the operator $r\partial_r$. This does not raise any problem since this operator is homogeneous (if $r = \varepsilon R$ then $r\partial_r = R\partial_R$). In the end we obtain $\kappa > 0$ such that

$$\|r\partial_r\left(u_0^\varepsilon - U_0^\varepsilon\right)\|_{0,Z_t^\varepsilon} \leq \kappa\varepsilon \quad \forall\varepsilon \in (0, 1). \tag{2.17}$$

Final consistency result Taking Eqs. (2.11), (2.16) and (2.17) and plugging them into (2.10), we obtain the existence of $\kappa > 0$ such that $\|r(f + \Delta u_\chi^\varepsilon + k^2 u_\chi^\varepsilon)\|_{0,\Omega_\varepsilon} \leq \kappa\sqrt{\varepsilon}$, $\forall\varepsilon \in (0, 1)$. Combining this inequality with Proposition 2.1 yields the following theorem.

Theorem 2.1 (Validation of the simplified model) *There exist* $\kappa_0, \varepsilon_0 > 0$ *such that* $\| u^\varepsilon - \tilde{u}^\varepsilon\|_{1,\Omega} < \kappa_0 |\ln\varepsilon|\sqrt{\varepsilon}$, $\forall\varepsilon \in (0, \varepsilon_0)$.

3 Augmented Galerkin scheme

Theorem 2.1 suggests solving Problem (1.6) instead of Problem (1.2) for the computation of the field diffracted by the small obstacle, the associated boundary condition being weaker. In this section we present a numerical scheme for computing the solution to this new problem. We start by introducing an equivalent formulation of Problem (1.6) which is more convenient for numerical computation.

3.1 A fictitious domain formulation for the simplified problem

In practice, we wish to use a mesh that has been generated independently of the small obstacle. This is the reason why we do not want to take the boundary condition on Γ_ε

into account via the variational space, but rather as an additional constraint imposed by means of Lagrange multipliers, where the Lagrange multiplier space is chosen as

$$\mathbb{L}(\Gamma_\varepsilon) := \left\{ q = \hat{q}\,\frac{\partial \mathfrak{S}^\varepsilon}{\partial n_{\Gamma_\varepsilon}} \Big| \hat{q} \in \mathbb{C} \right\} \quad \text{and} \quad |\, q\,|_{\mathbb{L}(\Gamma_\varepsilon)} := |\hat{q}| = \left| \frac{1}{2\pi} \int_{\Gamma_\varepsilon} q \right|.$$

Although the norm $|\ \ |_{\mathbb{L}(\Gamma_\varepsilon)}$ is equivalent to any other norm on $\mathbb{L}(\Gamma_\varepsilon)$ that would be more standard, we will use $|\ \ |_{\mathbb{L}(\Gamma_\varepsilon)}$ because it takes into account that $\dim \mathbb{L}(\Gamma_\varepsilon) = 1$. Note that

$$q \in \mathbb{L}(\Gamma_\varepsilon) \iff \int_{\Gamma_\varepsilon} q\overline{v} = 0 \quad \forall v \in \mathbb{V}_\mu^\varepsilon. \tag{3.1}$$

The boundary condition on Γ_ε is imposed weakly by testing $\tilde{u}^\varepsilon_{|\Gamma_\varepsilon}$ on each element of $\mathbb{L}(\Gamma_\varepsilon)$. The new formulation reads as follows

$$\text{Find } (\tilde{u}^\varepsilon, \tilde{p}^\varepsilon) \in H^1(\Omega) \times \mathbb{L}(\Gamma_\varepsilon) \text{ such that } \begin{cases} a(\tilde{u}^\varepsilon, v) + \displaystyle\int_{\Gamma_\varepsilon} \tilde{p}^\varepsilon \overline{v} = \int_\Omega f\overline{v} \ \forall v \in H^1(\Omega), \\ \displaystyle\int_{\Gamma_\varepsilon} q\overline{\tilde{u}}^\varepsilon = 0 \qquad\qquad \forall q \in \mathbb{L}(\Gamma_\varepsilon). \end{cases} \tag{3.2}$$

3.2 Well posedness of the fictitious domain formulation

Formulation (3.2) is equivalent to Formulation (1.6) in the following sense. Suppose first that $(\tilde{u}^\varepsilon, \tilde{p}^\varepsilon)$ is a solution to (3.2). In this case, it is straightforward that $\tilde{u}^\varepsilon \in \mathbb{V}_\mu^\varepsilon$ because of the second equation, and the first equation directly implies that $\tilde{u}^\varepsilon$ satisfies $a(\tilde{u}^\varepsilon, v) = \int_\Omega f\overline{v}$ for any $v \in \mathbb{V}_\mu^\varepsilon$. On the contrary suppose that $\tilde{u}^\varepsilon$ satisfies (1.6) and let

$$\tilde{p}^\varepsilon = \frac{\partial}{\partial n_{\Gamma_\varepsilon}} \left(\tilde{u}^\varepsilon_{|\Omega_\varepsilon} \right) - \frac{\partial}{\partial n_{\Gamma_\varepsilon}} \left(\tilde{u}^\varepsilon_{|\Omega \setminus \overline{\Omega}_\varepsilon} \right). \tag{3.3}$$

According to the definition of $\mathbb{V}_\mu^\varepsilon$, it is clear that $\tilde{u}^\varepsilon$ satisfies the second equation of (3.2). Let us show that $\tilde{p}^\varepsilon \in \mathbb{L}(\Gamma_\varepsilon)$ using (3.1). Choose an arbitrary $v \in H^1(\Omega)$ and apply a Green's formula to the expression for $a(\tilde{u}^\varepsilon, v)$ separately in Ω_ε and in $\Omega \setminus \overline{\Omega}_\varepsilon$,

$$\begin{aligned} a(\tilde{u}^\varepsilon, v) = &- \int_{\Omega_\varepsilon} \overline{v}(\Delta \tilde{u}^\varepsilon + k^2 \tilde{u}^\varepsilon) - \int_{\Omega \setminus \overline{\Omega}_\varepsilon} \overline{v}(\Delta \tilde{u}^\varepsilon + k^2 \tilde{u}^\varepsilon) \\ &+ \int_\Gamma \left(\frac{\partial \tilde{u}^\varepsilon}{\partial n_\Gamma} + T_\Gamma \tilde{u}^\varepsilon \right) \overline{v} - \int_{\Gamma_\varepsilon} \tilde{p}^\varepsilon \overline{v}, \quad \forall v \in H^1(\Omega). \end{aligned}$$

Taking into account (1.7) and (1.6), we are left with $a(\tilde{u}^\varepsilon, v) + \int_{\Gamma_\varepsilon} \tilde{p}^\varepsilon \overline{v} = \int_\Omega f\overline{v}$, $\forall v \in H^1(\Omega)$ and $\int_{\Gamma_\varepsilon} \tilde{p}^\varepsilon \overline{v} = 0$, $\forall v \in \mathbb{V}^\varepsilon_\mu$. This means that $\tilde{p}^\varepsilon \in \mathbb{L}(\Gamma_\varepsilon)$. This also yields the first equation of (3.2).

Another proof of well posedness for the simplified problem The well-posedness of (3.2) is a consequence of its equivalence with (1.6). Since (3.2) admits a mixed formulation, an alternative manner of obtaining well posedness would consist in applying Brezzi and Fortin's theory: according to Theorem 1.1 of Section II.1 in [3], the well posedness of (3.2) is equivalent to the continuity of sesquilinear and antilinear forms and three inf–sup conditions. It is clear that $a(\;,\;)$ is continuous independently of ε. Moreover the sesquilinear form $(q, v) \mapsto \int_{\Gamma_\varepsilon} q\overline{v}$ is also continuous but we have to know how this continuity depends on ε.

Lemma 3.1 (Quasi-uniform continuity for the simplified problem)

There exists $\beta_0^+ > 0$ *s.t.*
$$\sup_{q\in\mathbb{L}(\Gamma_\varepsilon)} \sup_{v\in H^1(\Omega)} \frac{\left|\displaystyle\int_{\Gamma_\varepsilon} q\overline{v}\right|}{\|v\|_{1,\Omega} | q |_{\mathbb{L}(\Gamma_\varepsilon)}} < \beta_0^+ \sqrt{|\ln\varepsilon|} \quad \forall\varepsilon\in(0,1).$$

Proof Using a density argument, it is sufficient to prove this result when the superior bound is taken over $v \in \mathscr{C}^\infty(\Omega)$ with compact support included in Ω. Choose arbitrary $q \in \mathbb{L}(\Gamma_\varepsilon)$ and $v \in \mathscr{C}^\infty(\Omega)$ such that supp$v \subset \Omega$. From the definition of $\mathbb{L}(\Gamma_\varepsilon)$ we deduce that there exists $\kappa(\gamma) > 0$ such that

$$\left|\int_{\Gamma_\varepsilon} q\overline{v}\right| < \kappa(\gamma) \left|\frac{\partial\mathfrak{S}}{\partial n_{\Gamma_\mathrm{N}}}\right|_{\infty,\Gamma_\mathrm{N}} | q |_{\mathbb{L}(\Gamma_\varepsilon)} \int_0^{2\pi} |v(\varepsilon\gamma(\theta), \theta)| d\theta \quad \forall\varepsilon \in (0, 1),$$

where $\kappa(\gamma) := \sup_{\theta\in[0,2\pi)} \left\{\sqrt{|\gamma(\theta)|^2 + |\partial_\theta\gamma(\theta)|^2}\right\}$ and $\left|\frac{\partial\mathfrak{S}}{\partial n_{\Gamma_\mathrm{N}}}\right|_{\infty,\Gamma_\mathrm{N}} = \varepsilon \left|\frac{\partial\mathfrak{S}^\varepsilon}{\partial n_{\Gamma_\varepsilon}}\right|_{\infty,\Gamma_\varepsilon}$.

The term related to $\mathfrak{S}$ is independent of ε. Since $v(\varrho, \theta) = 0$, $\forall\theta \in [0, 2\pi)$, we have

$$|v(\varepsilon\gamma(\theta), \theta)| \le \int_{\varepsilon\gamma(\theta)}^{\varrho} \left|\frac{\partial v}{\partial r}(r, \theta)\right| dr < \left(\int_{\varepsilon\gamma(\theta)}^{\varrho} \frac{dr}{r}\right)^{1/2} \left(\int_{\varepsilon\gamma(\theta)}^{\varrho} \left|\frac{\partial v}{\partial r}\right|^2 r dr\right)^{1/2}.$$

Therefore there exists $\kappa_0 > 0$ such that $\int_0^{2\pi} |v(\varepsilon\gamma(\theta), \theta)| d\theta < \kappa_0 \sqrt{|\ln\varepsilon|} \|v\|_{1,\Omega}$ $\forall\varepsilon \in (0, 1)$ since $\gamma(\theta) > \gamma_* > 0$ for $\theta \in [0, 2\pi)$ and γ is smooth by assumption. As v and q are arbitrary, this concludes the proof. □

Now we have to take care of the inf–sup conditions. Two of them are provided by Lemma 1.3. The next lemma shows that the last one is satisfied uniformly with respect to ε. This shows in another manner that Problem (3.2) is well posed.

Lemma 3.2 (Uniform inf–sup condition for the simplified problem)

$$\textit{There exists } \beta_0^- > 0 \textit{ s.t.} \quad \inf_{q\in\mathbb{L}(\Gamma_\varepsilon)} \sup_{v\in H^1(\Omega)} \frac{\left|\int_{\Gamma_\varepsilon} q\overline{v}\right|}{\|v\|_{1,\Omega}|q|_{\mathbb{L}(\Gamma_\varepsilon)}}$$

$$> \inf_{q\in\mathbb{L}(\Gamma_\varepsilon)} \frac{\left|\int_{\Gamma_\varepsilon} q\right|}{\sqrt{|\Omega|}|q|_{\mathbb{L}(\Gamma_\varepsilon)}} > \beta_0^- \quad \forall\varepsilon \in (0,1).$$

Proof To prove the desired result, take an arbitrary element $q \in \mathbb{L}(\Gamma_\varepsilon)$. Define $\mathbf{1}_\Omega$ as the constant function equal to 1 on the whole domain Ω, and note that $\|\mathbf{1}_\Omega\|_{1,\Omega} = \sqrt{|\Omega|}$ where $|\Omega|$ refers to the area of Ω. We choose $v = \mathbf{1}_\Omega$ as test function. Applying the change of coordinate $r = \varepsilon R$ and Lemma 1.2, we get

$$\int_{\Gamma_\varepsilon} q\mathbf{1}_\Omega = \hat{q}\int_{\Gamma_\varepsilon} \frac{\partial\mathfrak{S}^\varepsilon}{\partial n_{\Gamma_\varepsilon}} = \hat{q}\int_{\Gamma_N} \frac{\partial\mathfrak{S}}{\partial n_{\Gamma_N}}$$

$$= \frac{2\pi}{\sqrt{|\Omega|}}\|\mathbf{1}_\Omega\|_{1,\Omega}\hat{q} \;\Rightarrow\; \sup_{v\in H^1(\Omega)} \frac{\left|\int_{\Gamma_\varepsilon} q\overline{v}\right|}{\|v\|_{1,\Omega}} > \frac{2\pi}{\sqrt{|\Omega|}}|\hat{q}|.$$

Since by definition $|\hat{q}| = |\, q\,|_{\mathbb{L}(\Gamma_\varepsilon)}$ and q is arbitrary, we have proved the desired result with $\beta_0^- = 2\pi/\sqrt{|\Omega|}$. □

3.3 Discretization of the simplified problem

We describe here a possible discretization for Problem (3.2). For a detailed presentation of the finite element method we refer the reader to [6]. We consider a family of triangulations $(\mathcal{T}_h)_{h\in(0,1)}$ over Ω, made up of triangles or rectangles, where h is a mesh parameter supposed to go to 0, $h = \max\{\mathrm{diam}(K)|K \in \mathcal{T}_h\} \to 0$. For the definition of a triangulation (see [6] section 2.1). Since Ω is a circle, it cannot be completely covered by any triangulation, so we denote $\Omega_h = \cup_{K\in\mathcal{T}_h} K$. We assume:

H1: The family of triangulations $\mathcal{T}_h, h > 0$ is regular.

For the definition of a regular family of triangulations (see [6] section 3.2). For the "standard" approximation space denoted $\mathbb{V}^h$ we take

H2: $\mathbb{V}^h := \mathbb{P}_k$ or $\mathbb{Q}_k$-finite element space on $\mathcal{T}_h$.

Finally let $\Pi_h : \mathscr{C}^0(\Omega) \to \mathbb{V}^h$ be the continuous projection, called the Lagrange interpolation operator defined as follows. Identifying the set Σ_h of degrees of freedom with a set of points in Ω_h,

$$\forall v \in \mathscr{C}^0(\Omega), \quad \Pi_h v \in \mathbb{V}^h \quad \text{and} \quad \Pi_h v(\mathbf{x}) = v(\mathbf{x}) \quad \forall \mathbf{x} \in \Sigma_h.$$

A remark about the curved boundary of the domain Rigorously we have $\Omega_h \neq \Omega$. Indeed Ω_h is only an approximation of Ω, and this generates a numerical error. However the code we used includes high order isoparametric finite elements at the boundary, so that this numerical error was negligible compared to other sources of error. As a consequence, in the remainder of this paper, we neglect the difference between Ω and Ω_h, and simply write Ω.

An augmented approximation space We wish to consider a scatterer smaller than the step of the mesh, and it is clear that a standard method would fail in reproducing phenomenons with a characteristic length below the step of the mesh (see section 3 in [9] for more details). As a consequence, the space $\mathbb{V}^h$ will not be our actual approximation space. We consider an additional shape function that will be able to reproduce the behaviour of $\tilde{u}^\varepsilon$ in the neighborhood of 0 when $\varepsilon \to 0$. Consider a cut-off function $\chi_{\mathbf{e}}$ that satisfies the following properties

$$\begin{aligned}&\chi_{\mathbf{e}} : \mathbb{R}_+ \to [0, 1], \quad \chi_{\mathbf{e}} \in \mathscr{C}^\infty(\overline{\mathbb{R}}_+), \quad \chi_{\mathbf{e}} \text{ is non-increasing },\\ &\chi_{\mathbf{e}} = 1 \text{ in a neighborhood of } 0.\end{aligned}$$

This cut-off function is arbitrarily chosen following practical considerations. Note that $\chi_{\mathbf{e}}$ does not depend on ε. The additional shape function and our actual approximation space are then defined by

$$\Psi^\varepsilon_{\mathbf{e}} := \chi_{\mathbf{e}} \mathbf{1}_{\Omega_\varepsilon} \mathfrak{S}^\varepsilon \quad \text{and} \quad \mathbb{V}^h_{\mathbf{e}} := \mathbb{V}^h \oplus \operatorname{span}\{\Psi^\varepsilon_{\mathbf{e}}\} \tag{3.4}$$

where $\mathbf{1}_{\Omega_\varepsilon}$ is equal to 1 on Ω_ε and equal to 0 on $\Omega \setminus \overline{\Omega}_\varepsilon$. The discrete formulation we consider is

$$\text{Find } (u^\varepsilon_h, p^\varepsilon_h) \in \mathbb{V}^h_{\mathbf{e}} \times \mathbb{L}(\Gamma_\varepsilon) \text{ such that } \begin{cases} a(u^\varepsilon_h, v) + \displaystyle\int_{\Gamma_\varepsilon} p^\varepsilon_h \overline{v} = \int_\Omega f \overline{v} & \forall v \in \mathbb{V}^h_{\mathbf{e}}, \\ \displaystyle\int_{\Gamma_\varepsilon} q \overline{u}^\varepsilon_h = 0 & \forall q \in \mathbb{L}(\Gamma_\varepsilon). \end{cases} \tag{3.5}$$

This formulation is well posed. Indeed $\mathbb{V}^h_{\mathbf{e}} \subset H^1(\Omega)$, so it is clear that continuity properties still hold in this discrete case: $\exists \alpha_0^+, \beta_0^+ > 0$ such that $\forall \varepsilon \in (0, 1)$, $\forall u, v \in \mathbb{V}^h_{\mathbf{e}}$ and $\forall q \in \mathbb{L}(\Gamma_\varepsilon)$

$$|a(u, v)| \leq \alpha_0^+ \|u\|_{1,\Omega} \|v\|_{1,\Omega} \quad \text{and} \quad \left| \int_{\Gamma_\varepsilon} q \overline{v} \right| \leq \beta_0^+ \sqrt{|\ln \varepsilon|} \|v\|_{1,\Omega} |\, q\,|_{\mathbb{L}(\Gamma_\varepsilon)}. \tag{3.6}$$

We also have to ensure that inf–sup conditions are satisfied uniformly with respect to h and ε. According to Lemma 3.2, the uniform inf–sup condition related to $(q, v) \mapsto$

$\int_{\Gamma_\varepsilon} q\overline{v}$, $q \in \mathbb{L}(\Gamma_\varepsilon)$, $v \in \mathbb{V}^h_{\mathbf{e}}$ is verified, since $\mathbf{1}_\Omega \in \mathbb{V}^h_{\mathbf{e}}$. Concerning $a(\ ,\ )$ we have the following lemma.

Lemma 3.3 (Stability of the discrete problem) *Let* $\mathbb{V}^{\varepsilon,h}_\mu := \{v \in \mathbb{V}^h_{\mathbf{e}} | \int_{\Gamma_\varepsilon} q\overline{v} = 0, \forall q \in \mathbb{L}(\Gamma_\varepsilon)\}$. *Then there exists* $\alpha_0^- > 0$ *such that*

$$\inf_{u\in\mathbb{V}^{\varepsilon,h}_\mu} \sup_{v\in\mathbb{V}^{\varepsilon,h}_\mu} \frac{|a(u,v)|}{\|u\|_{1,\Omega}\|v\|_{1,\Omega}} > \alpha_0^- \quad \textit{and}$$
$$\inf_{u\in\mathbb{V}^{\varepsilon,h}_\mu} \sup_{v\in\mathbb{V}^{\varepsilon,h}_\mu} \frac{|a(v,u)|}{\|u\|_{1,\Omega}\|v\|_{1,\Omega}} > \alpha_0^- \quad \forall \varepsilon, h \in (0,1).$$

Proof We only prove the first inf–sup condition since the second one can be proved in the same manner. Proceed by contradiction and suppose that there exists two sequences (ε_n) and (h_n) such that $\varepsilon_n + h_n \to 0$ as $n \to \infty$, and a sequence (u_n) such that for any n,

$$u_n \in \mathbb{V}^{\varepsilon_n,h_n}_\mu, \quad \|u_n\|_{1,\Omega} = 1 \quad \text{and} \quad \sup_{v\in\mathbb{V}^{\varepsilon,h}_\mu} \frac{|a(u,v)|}{\|v\|_{1,\Omega}} \underset{n\to\infty}{\longrightarrow} 0.$$

Since (u_n) is bounded in $H^1(\Omega)$, it is possible to extract a subsequence weakly converging in $H^1(\Omega)$ and strongly converging in $L^2(\Omega)$ toward $u \in H^1(\Omega)$. Take an arbitrary $v \in \mathscr{C}^\infty_\star(\Omega)$ where $\mathscr{C}^\infty_\star(\Omega) := \{v \in \mathscr{C}^\infty(\Omega)\text{s.t}v = 0 \text{ in a neighborhood of } 0\}$. For n large enough, $\Pi_{h_n} v \in \mathbb{V}^{\varepsilon_n,h_n}_\mu$, so that

$$a(u_n, v) = a(u_n, \Pi_{h_n} v) + a(u_n, v - \Pi_{h_n} v) \underset{n\to\infty}{\longrightarrow} 0 \quad \text{and} \quad a(u_n, v) \underset{n\to\infty}{\longrightarrow} a(u, v).$$

Since v has been chosen arbitrarily, and according to the density of $\mathscr{C}^\infty_\star(\Omega)$ in $H^1(\Omega)$ (see Proposition 5.14 in [28]), we conclude that $a(u, v) = 0$, $\forall v \in H^1(\Omega)$. It is classical that this property implies $u = 0$. Thus $\|u_n\|_{0,\Omega} \to 0$ when $n \to \infty$. Recall that

$$\Re e \left\{ \int_\Gamma \overline{v} T_\Gamma v \right\} \geq 0, \quad \forall v \in H^{1/2}(\Gamma). \tag{3.7}$$

This is a consequence of the Wronskian formula for Bessel functions and Nicholson's formula, see for example Equations (2.15) and (2.30) in [22]. As a consequence we have

$$\|\nabla u_n\|^2_{0,\Omega} \leq \|\nabla u_n\|^2_{0,\Omega} + \Re e \left\{ \int_\Gamma \overline{u}_n T_\Gamma u_n \right\} = \Re e\{a(u_n, u_n)\} + k^2 \|u_n\|^2_{0,\Omega}.$$

Since $a(u_n, u_n) \to 0$ when $n \to \infty$, we obtain that $\|u_n\|_{1,\Omega} \to 0$ when $n \to \infty$. This leads to a contradiction since $\|u_n\|_{1,\Omega} = 1$, which yields the desired result. □

According to Lemma 3.2, Proposition 3.3 and Inequalities (3.6), we can apply Theorem 1.1 in Section II.1 of [3] that gives existence and uniqueness of the solution to Problem (3.5).

3.4 Consistency of the augmented Galerkin scheme

Brezzi and Fortin's theory also provides an estimate for $\|\tilde{u}^\varepsilon - u_h^\varepsilon\|_{1,\Omega}$ in terms of interpolation error. According to Propositions 2.4 and 2.5 in Section II.2 of [3], there exist $\varepsilon_0, \kappa_1, \kappa_2 > 0$ such that

$$\begin{aligned}
\|\tilde{u}^\varepsilon - u_h^\varepsilon\|_{1,\Omega} &\leq \left(1 + \frac{\alpha_0^+}{\alpha_0^-}\right)\left(1 + \frac{\beta_0^+\sqrt{|\ln\varepsilon|}}{\beta_0^-}\right) \inf_{v_h \in \mathbb{V}_{\mathbf{e}}^h} \|\tilde{u}^\varepsilon - v_h\|_{1,\Omega} \\
&\leq \kappa_1\sqrt{|\ln\varepsilon|} \inf_{v_h \in \mathbb{V}_{\mathbf{e}}^h} \|\tilde{u}^\varepsilon - v_h\|_{1,\Omega} \\
|\,\tilde{p}^\varepsilon - p_h^\varepsilon\,|_{\mathbb{L}(\Gamma_\varepsilon)} &\leq \frac{\alpha_0^+}{\beta_0^-}\|\tilde{u}^\varepsilon - u_h^\varepsilon\|_{1,\Omega} \\
&\leq \kappa_2\sqrt{|\ln\varepsilon|} \inf_{v_h \in \mathbb{V}_{\mathbf{e}}^h} \|\tilde{u}^\varepsilon - v_h\|_{1,\Omega} \qquad \forall \varepsilon, h \in (0, \varepsilon_0).
\end{aligned} \tag{3.8}$$

The numbers α_0^-, α_0^+, β_0^- and β_0^+ appearing in the inequalities above are those involved in Lemma 3.2, Proposition 3.3 and Inequalities (3.6). In order to prove a consistency result for the numerical scheme associated with Formulation (3.5), it is sufficient to prove that $\inf_{v_h \in \mathbb{V}_{\mathbf{e}}^h} \|\tilde{u}^\varepsilon - v_h\|_{1,\Omega}$ goes to 0 as $\varepsilon + h \to 0$.

Proposition 3.1 (Consistency of the augmented Galerkin scheme) *There exists* κ_0, $\varepsilon_0 > 0$ *such that* $\inf_{v_h \in \mathbb{V}_{\mathbf{e}}^h} \|\tilde{u}^\varepsilon - v_h\|_{1,\Omega} \leq \kappa_0 |\ln\varepsilon| \left(\sqrt{\varepsilon} + h\right) \quad \forall \varepsilon \in (0, \varepsilon_0), \quad \forall h \in (0, 1)$.

Proof We use the first term of the matched expansion for which we gave a quasi-explicit expression with Eq. (2.2). According to Theorem 2.1, there exist $\kappa_1, \varepsilon_1 > 0$ such that $\|\tilde{u}^\varepsilon - u_\chi^\varepsilon\|_{1,\Omega} < \kappa_1 |\ln\varepsilon|\sqrt{\varepsilon}$, $\forall \varepsilon \in (0, \varepsilon_1)$. By means of a triangular inequality, this leads to

$$\begin{aligned}
\inf_{v_h \in \mathbb{V}_{\mathbf{e}}^h} \|\tilde{u}^\varepsilon - v_h\|_{1,\Omega} &\leq \|\tilde{u}^\varepsilon - u_\chi^\varepsilon\|_{1,\Omega} + \inf_{v_h \in \mathbb{V}_{\mathbf{e}}^h} \|u_\chi^\varepsilon - v_h\|_{1,\Omega} \\
&\leq \kappa_1 |\ln\varepsilon|\sqrt{\varepsilon} + \inf_{v_h \in \mathbb{V}^h} \left\| u_\chi^\varepsilon + \frac{u^0(0)}{\eta(\varepsilon)} \Psi_{\mathbf{e}}^\varepsilon - v_h \right\|_{1,\Omega}.
\end{aligned}$$

Getting back to Eqs. (2.2) and (3.4), we have

$$\begin{aligned}
u_\chi^\varepsilon(\mathbf{x}) + \frac{u^0(0)}{\eta(\varepsilon)}\Psi_{\mathbf{e}}^\varepsilon(\mathbf{x}) = \quad &(1 - \chi_{\mathbf{e}}(\mathbf{x}))\left(u^0(\mathbf{x}) - \frac{u^0(0)}{\eta(\varepsilon)}\frac{\pi}{2i}H_0^{(1)}(k|\mathbf{x}|)\right) \\
&+ \left(\chi_{\mathbf{e}}(\mathbf{x}) - \chi^\varepsilon(\mathbf{x})\right)\left(u^0(\mathbf{x}) - \frac{u^0(0)}{\eta(\varepsilon)}\frac{\pi}{2i}H_0^{(1)}(k|\mathbf{x}|) + \frac{u^0(0)}{\eta(\varepsilon)}\mathfrak{S}^\varepsilon\right)
\end{aligned}$$

Taking into account the expression of $\eta(\varepsilon)$ given in (2.2), the behaviour of $H_0^{(1)}(k|\mathbf{x}|)$ in the neighborhood of 0 given by (2.3), and the behaviour of $\mathfrak{S}(\mathbf{x})$ in the neighborhood of infinity given by (1.5), we further decompose the right hand side of the preceding identity

$$\begin{aligned}
u^{\varepsilon}_{\chi}(\mathbf{x}) + \frac{u^0(0)}{\eta(\varepsilon)}\Psi^{\varepsilon}_{\mathbf{e}}(\mathbf{x}) &= (1-\chi_{\mathbf{e}}(\mathbf{x}))\left(u^0(\mathbf{x}) - \frac{u^0(0)}{\eta(\varepsilon)}\frac{\pi}{2i}H_0^{(1)}(k|\mathbf{x}|)\right)\\
&+ \big(\chi_{\mathbf{e}}(\mathbf{x}) - \chi^{\varepsilon}(\mathbf{x})\big)\left(u^0(\mathbf{x}) - u^0(0)\right)\\
&+ \big(\chi_{\mathbf{e}}(\mathbf{x}) - \chi^{\varepsilon}(\mathbf{x})\big)\frac{-u^0(0)}{\eta(\varepsilon)}\left(\frac{\pi}{2i}H_0^{(1)}(k|\mathbf{x}|) - \ln\left(\frac{k|\mathbf{x}|}{2}\right) - \gamma_{\mathbf{e}} + i\frac{\pi}{2}\right)\\
&+ \big(\chi_{\mathbf{e}}(\mathbf{x}) - \chi^{\varepsilon}(\mathbf{x})\big)\frac{u^0(0)}{\eta(\varepsilon)}\left(\mathfrak{S}(\mathbf{x}/\varepsilon) - \ln\left(\frac{|\mathbf{x}|}{\varepsilon}\right) - c_{\gamma}\right)\\
&:= v_1^{\varepsilon} + v_2^{\varepsilon} + v_3^{\varepsilon} + v_4^{\varepsilon}.
\end{aligned}$$

In order to conclude the proof, it is sufficient to provide an estimate for $\inf_{v_h\in\mathbb{V}^h}\|v_i^{\varepsilon} - v_h\|_{1,\Omega}$, $i = 1,\ldots,4$. To begin with, it is clear that $\|v_1^{\varepsilon}\|_{2,\Omega}$ is bounded independently of ε whence $\kappa_1' > 0$ such that

$$\begin{aligned}
\inf_{v_h\in\mathbb{V}^h}\|v_1^{\varepsilon} - v_h\|_{1,\Omega} &= \inf_{v_h\in\mathbb{V}^h}\left\|(1-\chi_{\mathbf{e}}(\mathbf{x}))\left(u^0(\mathbf{x}) - \frac{u^0(0)}{\eta(\varepsilon)}\frac{\pi}{2i}H_0^{(1)}(k|\mathbf{x}|)\right) - v_h\right\|_{1,\Omega}\\
&\le \kappa_1' h \quad \forall\varepsilon, h\in(0,1).
\end{aligned}$$

In this estimate we used a classical result of interpolation of regular functions by a $\mathbb{P}_k$ (or $\mathbb{Q}_k$) finite element space on a regular triangulation, see Theorem 4.4.6 in [6] for example. The derivation of such an estimate for v_2^{ε}, v_3^{ε} and v_4^{ε} is longer, so we prove it in appendix with Lemmas A.2–A.4. To conclude, we gather the estimates corresponding to each v_i^{ε}, which yields $\kappa_5, \varepsilon_5 > 0$ such that

$$\inf_{v_h\in\mathbb{V}^h_{\mathbf{e}}}\|\tilde{u}^{\varepsilon} - v_h\|_{1,\Omega} \le \kappa_5|\ln\varepsilon|\left(\sqrt{\varepsilon} + h\right) \quad \forall\varepsilon\in(0,\varepsilon_5), \quad \forall h\in(0,1).$$

□

Using Inequalities (3.8) and Proposition 3.1, we finally obtain a global consistency result.

Theorem 3.1 (Final consistency result) *There exists* $\kappa_0, \varepsilon_0 > 0$ *such that*

$$\|\tilde{u}^{\varepsilon} - u_h^{\varepsilon}\|_{1,\Omega} + |\,\tilde{p}^{\varepsilon} - p_h^{\varepsilon}\,|_{\mathbb{L}(\Gamma_\varepsilon)} \le \kappa_0|\ln\varepsilon|^{3/2}\left(\sqrt{\varepsilon} + h\right) \quad \forall\varepsilon\in(0,\varepsilon_0), \quad \forall h\in(0,1).$$

Since $\tilde{u}^{\varepsilon}$ is close to u^{ε} according to Theorem 2.1, this theorem leads straightforwardly to an estimate of $\|u^{\varepsilon} - u_h^{\varepsilon}\|_{1,\Omega}$. The result of the preceding corollary proves that the numerical scheme associated with (3.5) is quasi locking-free in the sense that it looses approximation efficiency only slowly as $\varepsilon \to 0$.

There is admittedly a consistency error proportional to $|\ln \varepsilon|^{\frac{3}{2}}\sqrt{\varepsilon}$ and independent of h, but this error can be considered negligible if $\sqrt{\varepsilon}/h$ is small. This corresponds to the situation we wish to tackle: the scatterer is small with respect to the step of the mesh.

3.5 Numerical results

In this section we present the results of a numerical experiment that put into evidence the consistency of the numerical scheme associated with (3.5). For the domain of computation, we have considered that $\Omega = D(0,3)$. The obstacle is a small disc, $\Omega_\varepsilon = D(0,3) \setminus \overline{D}(0,\varepsilon)$ and $(\Gamma_\varepsilon) : r = \varepsilon$. The source term is slightly different from f introduced with (1.2): we consider an incident wave $u_{\mathrm{i}}(\mathbf{x}) = -\exp(ikr\cos\theta)$. We are interested in solving

$$\text{Find } u^\varepsilon \in \mathbb{V}_0^\varepsilon \quad \text{such that} \quad a(u^\varepsilon, v) = \int_\Gamma \left(\frac{\partial u_{\mathrm{i}}}{\partial n_\Gamma} + T_\Gamma u_{\mathrm{i}}\right)\overline{v} \quad \forall v \in \mathbb{V}_0^\varepsilon \tag{3.9}$$

It is very easy to adapt Sects. 1–3 for problems including this type of source term, and to propose a simplified problem similar to (3.9) whose solution will be denoted $\tilde{u}^\varepsilon$ as well.

Since this particular geometry is simple, we are able to derive an analytical expression for u^ε and $\tilde{u}^\varepsilon$ by means of Bessel functions J_p and Hankel functions $H_p^{(1)}$ (see Chapter 5 of [25]). First of all the trace of the incident wave on Γ_ε can be derived using the Jacobi–Anger formula $u_{\mathrm{i}}(\mathbf{x}) = -\sum_{p\in\mathbb{Z}} i^{|p|}\mathrm{J}_{|p|}(k\varepsilon)e^{ip\theta}$ $\forall \mathbf{x} \in \Gamma_\varepsilon$ (see Identity (5.12.2) in [25]). As a result, the explicit expressions of u^ε and $\tilde{u}^\varepsilon$ are given by

$$u^\varepsilon - u_{\mathrm{i}} = \begin{cases} \displaystyle\sum_{p\in\mathbb{Z}} i^{|p|}\mathrm{J}_{|p|}(kr)e^{ip\theta} & \text{if } r \le \varepsilon \\ \displaystyle\sum_{p\in\mathbb{Z}} i^{|p|}\mathrm{J}_{|p|}(k\varepsilon)\frac{H_{|p|}^{(1)}(kr)}{H_{|p|}^{(1)}(k\varepsilon)}e^{ip\theta} & \text{if } r \ge \varepsilon \end{cases},$$

$$\tilde{u}^\varepsilon - u_{\mathrm{i}} = \begin{cases} \mathrm{J}_0(kr) & \text{if } r \le \varepsilon \\ \mathrm{J}_0(k\varepsilon)\dfrac{H_0^{(1)}(kr)}{H_0^{(1)}(k\varepsilon)} & \text{if } r \ge \varepsilon. \end{cases} \tag{3.10}$$

The pair $(\tilde{u}^\varepsilon, \tilde{p}^\varepsilon)$ can be characterized as the solution to a formulation similar to (3.2) in which appears the DtN operator T_Γ, as well as the unique solution to the following formulation

Find $(\tilde{u}^\varepsilon, \tilde{p}^\varepsilon) \in H^1(\Omega) \times \mathbb{L}(\Gamma_\varepsilon)$ such that

$$\begin{cases} \displaystyle\int_\Omega \nabla\tilde{u}^\varepsilon \cdot \nabla\overline{v} - k^2 \int_\Omega \tilde{u}^\varepsilon\overline{v} + \lambda_\Gamma \int_\Gamma \tilde{u}^\varepsilon\overline{v} + \int_{\Gamma_\varepsilon} \tilde{p}^\varepsilon\overline{v} = \int_\Gamma \left(\frac{\partial u_{\mathrm{i}}}{\partial n_\Gamma} + \lambda_\Gamma u_{\mathrm{i}}\right)\overline{v} & \forall v \in H^1(\Omega) \\ \displaystyle\int_{\Gamma_\varepsilon} q\overline{\tilde{u}}^\varepsilon = 0 & \forall q \in \mathbb{L}(\Gamma_\varepsilon). \end{cases} \tag{3.11}$$

Instead of a Sommerfeld condition, we impose the Robin type condition

$$\frac{\partial\tilde{u}^\varepsilon}{\partial n_\Gamma} + \lambda_\Gamma\tilde{u}^\varepsilon = \frac{\partial u_{\mathrm{i}}}{\partial n_\Gamma} + \lambda_\Gamma u_{\mathrm{i}} \quad \text{on } \Gamma \quad \text{with} \quad \lambda_\Gamma := -k\frac{H_0^{(1)\prime}(k\varrho)}{H_0^{(1)}(k\varrho)}. \tag{3.12}$$

We chose to discretize Formulation (3.11), rather than a formulation similar to (3.2), because the Robin type condition (3.12) is much easier to implement. The associated discrete formulation is the following,

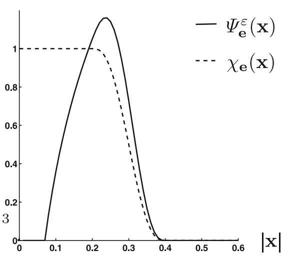
Find $(u_h^\varepsilon, p_h^\varepsilon) \in \mathbb{V}_{\mathbf{e}}^h \times \mathbb{L}(\Gamma_\varepsilon)$ such that

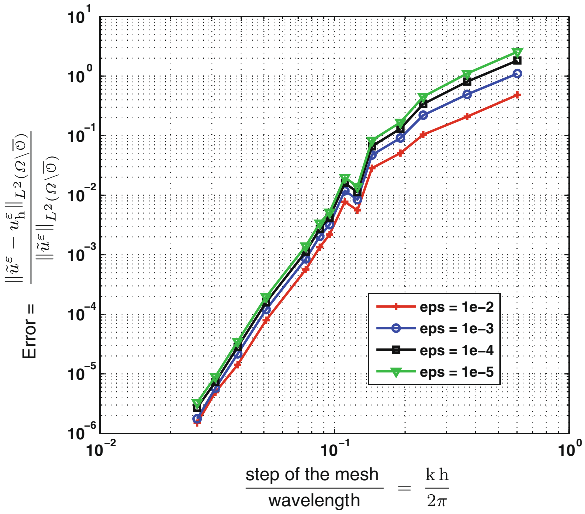
$$\begin{cases} \displaystyle\int_\Omega \nabla u_h^\varepsilon \cdot \nabla\overline{v} - k^2 \int_\Omega \tilde{u}^\varepsilon\overline{v} + \lambda_\Gamma \int_\Gamma u_h^\varepsilon\overline{v} + \int_{\Gamma_\varepsilon} p_h^\varepsilon\overline{v} = \int_\Gamma \left(\frac{\partial u_{\mathrm{i}}}{\partial n_\Gamma} + \lambda_\Gamma u_{\mathrm{i}}\right)\overline{v} & \forall v \in \mathbb{V}_{\mathbf{e}}^h \\ \displaystyle\int_{\Gamma_\varepsilon} q\overline{u}_h^\varepsilon = 0 & \forall q \in \mathbb{L}(\Gamma_\varepsilon). \end{cases} \tag{3.13}$$

In the present case, the additional shape function takes the form $\Psi_{\mathbf{e}}^\varepsilon(\mathbf{x}) = \chi_{\mathbf{e}}(\mathbf{x}) \ln(|\mathbf{x}|/\varepsilon)\mathbf{1}_{\Omega_\varepsilon}$. This formulation has been implemented in the code MONTJOIE developed at project POEMS by Marc Duruflé. A complete documentation describing this code is available at the address http://www.math.u-bordeaux1.fr/~durufle/montjoie. The wave number considered was $k = 2\pi$. We used order 3 quadrangular finite elements. For the cut-off function, we considered the following,

$$\chi_{\mathrm{e}}(\mathbf{x}) = \begin{cases} 1 & \text{if } |\mathbf{x}| \leqslant \frac{\varrho_0}{3} \\ Q(|\mathbf{x}|) & \text{if } \frac{\varrho_0}{3} \leqslant |\mathbf{x}| \leqslant 2\frac{\varrho_0}{3} \\ 0 & \text{if } |\mathbf{x}| \geqslant 2\frac{\varrho_0}{3} \end{cases}$$

$$\text{with} \quad \begin{cases} \varrho_0 = 0.6 \\ Q(r) = 6\cdot\left(2-\frac{3r}{\varrho_0}\right)^5 - 15\cdot\left(2-\frac{3r}{\varrho_0}\right)^4 + 10\cdot\left(2-\frac{3r}{\varrho_0}\right)^3 \end{cases}$$





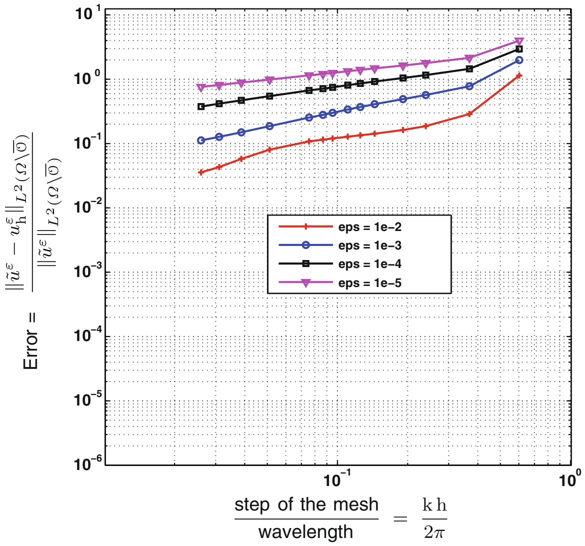
Domain: $\Omega = D(0,3)$

$\mathcal{O} = D(0,1)$

Finite elements: $\mathbb{Q}_3$

ε	rates of convergence
10^{-2}	3.8883
10^{-3}	4.1136
10^{-4}	4.2392
10^{-5}	4.3103

Fig. 1 Relative error for the augmented Galerkin scheme



Domain: $\Omega = D(0,3)$

$\mathcal{O} = D(0,1)$

Finite elements: $\mathbb{Q}_3$

ε	rates of convergence
10^{-2}	0.9051
10^{-3}	0.8697
10^{-4}	0.6163
10^{-5}	0.5074

Fig. 2 Relative error for a standard Galerkin scheme

On the first figure above, we have represented the graph of the additional shape function $\Psi_{\mathbf{e}}^{\varepsilon}$ and the cut-off function $\chi_{\mathbf{e}}$ versus r for $\varepsilon = 0.06$. Such a definition ensures that $\chi_{\mathbf{e}} \in \mathscr{C}^2(\Omega)$. In Fig.1, we represent the relative errors obtained with this experiment as $h \to 0$ for four values of ε.

Note that the graph goes up very slowly as $\varepsilon \to 0$. Moreover the rates of convergence are in accordance with the order 3 of the finite element method. Let us compare with Fig. 2 that represents the results when solving Formulation (3.13) replacing $\mathbb{V}_{\mathbf{e}}^{h}$

by $\mathbb{V}^h$.The numerical scheme becomes inefficient when there is no additional shape function: if $\varepsilon \geq 10^{-3}$ the relative error is at least equal to 10%.

In the context of this experiment, the exact expression of the Lagrange multiplier $\tilde{p}^\varepsilon$ is derived using (3.3) and (3.10),

$$\tilde{p}^\varepsilon = k\frac{\mathrm{J}_0(k\varepsilon)H_0^{(1)\prime}(k\varepsilon) - \mathrm{J}_0'(k\varepsilon)H_0^{(1)}(k\varepsilon)}{H_0^{(1)}(k\varepsilon)} = \frac{2i}{\pi\varepsilon H_0^{(1)}(k\varepsilon)}.$$

Since the graphs of the errors for Lagrange multipliers are very similar to those of Figs. 1 and 2, we only give the rates of convergence. The conclusions are the same.

Error	Standard Galerkin scheme		Augmented Galerkin scheme	
=	ε	Rates of convergence	ε	Rates of convergence
$\frac{\lvert \tilde{p}^\varepsilon - p_h^\varepsilon \rvert_{\mathbb{L}(\Gamma_\varepsilon)}}{\lvert \tilde{p}^\varepsilon \rvert_{\mathbb{L}(\Gamma_\varepsilon)}}$	10^{-2}	0.7373	10^{-2}	4.5777
	10^{-3}	0.7352	10^{-3}	4.6909
	10^{-4}	0.5139	10^{-4}	4.7637
	10^{-5}	0.4067	10^{-5}	4.7968

Whereas Theorem 3.1 only states a convergence in $O(h + \sqrt{\varepsilon})$, the results appearing in Fig. 1 suggest that the convergence of the augmented Galerkin scheme (3.5) is optimal, i.e. convergence in $O(h^k)$ with order k finite elements. This is an interesting point that we are not able to theoretically explain at present.

Appendix

Lemma A.1 *There exists* $\kappa > 0$ *such that* $\|v/r\|_{0,\Omega_\varepsilon} < \kappa|\ln\varepsilon|\|v\|_{1,\Omega_\varepsilon}, \forall v \in H^1(\Omega_\varepsilon)$, $\forall\varepsilon \in (0, 1)$.

Proof Since $\mathscr{C}^\infty(\overline{\Omega}_\varepsilon)$ is dense in $H^1(\Omega_\varepsilon)$, it is sufficient to find a $\kappa > 0$ independent of ε such that the desired estimate holds for any $v \in \mathscr{C}^\infty(\overline{\Omega}_\varepsilon)$. Take an arbitrary $v \in \mathscr{C}^\infty(\overline{\Omega}_\varepsilon)$. Since Γ is defined by $r = \varrho$, applying Cauchy–Schwartz inequality, we obtain

$$v(r,\theta) = v(\varrho,\theta) + \int_r^\varrho \frac{\partial v}{\partial r}(t,\theta)\,dt \quad \Rightarrow \quad |v(r,\theta)|^2 \leq 2|v(\varrho,\theta)|^2 + 2\left(\int_r^\varrho \frac{d\tau}{\tau}\right)\int_r^\varrho \left|\frac{\partial v}{\partial r}(t,\theta)\right|^2 t dt$$

By assumption there exists $\gamma_* \in \mathbb{R}_+$ independent of ε such that $\gamma(\theta) > \gamma_* > 0,\ \forall\theta \in [0, 2\pi[$. Integrating over $\theta \in [0, 2\pi[$ and $r \in [\varepsilon\gamma(\theta), \varrho]$, we obtain

$$\left\| \frac{v}{r} \right\|_{0,\Omega_\varepsilon}^2 \leq \frac{2}{\varrho} \left(\int_{\varepsilon\gamma_*}^{\varrho} \frac{ds}{s} \right) \|v\|_{0,\Gamma}^2 + 2 \left(\int_{\varepsilon\gamma_*}^{\varrho} \ln(\frac{\varrho}{s}) \frac{ds}{s} \right) \|v\|_{1,\Omega_\varepsilon}^2$$
$$\leq \frac{2}{\varrho} \ln \left(\frac{\varrho}{\varepsilon\gamma_*} \right) \|v\|_{0,\Gamma}^2 + 2 \ln^2 \left(\frac{\varrho}{\varepsilon\gamma_*} \right) \|v\|_{1,\Omega_\varepsilon}^2$$

Finally there exists $\kappa(\varrho) > 0$ that depends only on ϱ (and not on ε nor on v) such that $\|w\|_{0,\Gamma}^2 \leq \kappa(\varrho) \|w\|_{1,\Omega_\varepsilon}^2$ $\forall w \in H^1(\Omega_\varepsilon),\ \forall \varepsilon \in (0, 1)$. This leads to the estimate

$$\left\| \frac{v}{r} \right\|_{0,\Omega_\varepsilon}^2 \leq \left[\frac{2\kappa(\varrho)}{\varrho} \ln \left(\frac{\varrho}{\varepsilon\gamma_*} \right) + 2 \ln^2 \left(\frac{\varrho}{\varepsilon\gamma_*} \right) \right] \|v\|_{1,\Omega_\varepsilon}^2$$

□

Lemma A.2 *If $\mathbb{V}_0^\varepsilon$ is the standard approximation space associated with finite elements of order k, then there exists $\kappa > 0$ such that*

$$\inf_{v_h \in \mathbb{V}^h} \| \left(\chi_{\mathbf{e}}(\mathbf{x}) - \chi^\varepsilon(\mathbf{x})\right) \left(u^0(\mathbf{x}) - u^0(0)\right) - v_h \|_{1,\Omega}$$
$$\leq \kappa(\sqrt{\varepsilon} + h^k) \quad \forall \varepsilon \in (0, 1), \quad \forall h \in (0, 1).$$

Proof First of all, note that there exists $v^0 \in \mathscr{C}^\infty(\Omega)^2$ such that $u^0(\mathbf{x}) = u^0(0) + \mathbf{x} \cdot v^0(\mathbf{x}),\ \forall \mathbf{x} \in \Omega$. Following the definition of χ^ε given in (2.2), if $\mathbf{x} \in \operatorname{supp}\chi^\varepsilon$ then $|\mathbf{x}| < 2\sqrt{\varepsilon}$. According to very classical results of approximation by finite elements (see theorem 4.4.6 in [6] for example), there exist $\kappa, \kappa' > 0$ and h such that

$$\inf_{v_h \in \mathbb{V}^h} \| \left(\chi_{\mathbf{e}}(\mathbf{x}) - \chi^\varepsilon(\mathbf{x})\right) \left(u^0(\mathbf{x}) - u^0(0)\right) - v_h \|_{1,\Omega} \leq \underbrace{\|\chi^\varepsilon(\mathbf{x}) \mathbf{x} \cdot v^0(\mathbf{x})\|_{1,\Omega}}_{\leq \kappa\sqrt{\varepsilon}}$$
$$+ \underbrace{\inf_{v_h \in \mathbb{V}^h} \|\chi_{\mathbf{e}}(\mathbf{x}) \mathbf{x} \cdot v^0(\mathbf{x}) - v_h\|_{1,\Omega}}_{\leq \kappa' h^k \|\chi_{\mathbf{e}}(\mathbf{x}) \mathbf{x} \cdot v^0(\mathbf{x})\|_{k+1,\Omega}} \quad \forall \varepsilon \in (0, 1), \quad \forall h \in (0, 1).$$

□

Lemma A.3 *If $\mathbb{V}_0^\varepsilon$ is the standard approximation space associated with finite elements of order $k \geq 1$, then there exists $\kappa > 0$ such that $\forall \varepsilon \in (0, 1)$, $\forall h \in (0, 1)$ we have*

$$\inf_{v_h \in \mathbb{V}^h} \left\| \left(\chi_{\mathbf{e}}(\mathbf{x}) - \chi^\varepsilon(\mathbf{x})\right) \left(\frac{\pi}{2i}\ H_0^{(1)}(k|\mathbf{x}|) - \ln \left(\frac{k|\mathbf{x}|}{2} \right) - \gamma_{\mathbf{e}} + i \frac{\pi}{2} \right) - v_h \right\|_{1,\Omega}$$
$$\leq \kappa(\sqrt{\varepsilon} + h).$$

Proof For the proof of this lemma, we adopt essentially the same strategy as for the preceding lemma. According to the definition of the Hankel function given in [25], there exists a function v^0 such that

$$|\mathbf{x}|v^0 \in H^2(\Omega),\ \ v^0 \in L^\infty(\Omega),\ \ \nabla v^0 \in L^\infty(\Omega) \quad \text{and}$$
$$\frac{\pi}{2i}\ H_0^{(1)}(k|\mathbf{x}|) - \ln\left(\frac{k|\mathbf{x}|}{2}\right) - \gamma_{\mathbf{e}} + i\frac{\pi}{2} = |\mathbf{x}|v^0(\mathbf{x})$$

Recall that, according to the definition of χ^ε given in (2.2), if $\mathbf{x} \in \mathrm{supp}\chi^\varepsilon$ then $|\mathbf{x}| < 2\sqrt{\varepsilon}$. As a consequence there exists $\kappa, \kappa' > 0$ such that

$$\inf_{v_h \in \mathbb{V}^h} \left\| \left(\chi_{\mathbf{e}}(\mathbf{x}) - \chi^\varepsilon(\mathbf{x})\right)\left(\frac{\pi}{2i}\ H_0^{(1)}(k|\mathbf{x}|) - \ln\left(\frac{k|\mathbf{x}|}{2}\right) - \gamma_{\mathbf{e}} + i\frac{\pi}{2}\right) - v_h \right\|_{1,\Omega}$$
$$\leq \underbrace{\|\chi^\varepsilon(\mathbf{x})|\mathbf{x}|v^0(\mathbf{x})\|_{1,\Omega}}_{\leq \kappa\sqrt{\varepsilon}} + \underbrace{\inf_{v_h \in \mathbb{V}^h} \|\chi_{\mathbf{e}}(\mathbf{x})|\mathbf{x}|v^0(\mathbf{x}) - v_h\|_{1,\Omega}}_{\leq \kappa' h \|\chi_{\mathbf{e}}(\mathbf{x})|\mathbf{x}|v^0(\mathbf{x})\|_{2,\Omega}} \quad \forall \varepsilon \in (0,1),$$
$$\forall h \in (0,1).$$

□

Lemma A.4 *There exists* $\kappa > 0$ *such that* $\left\|\left(\chi_{\mathbf{e}} - \chi^\varepsilon\right)\left(\mathfrak{S}^\varepsilon - \ln(r/\varepsilon) - c_\gamma\right)\right\|_{1,\Omega} \leq \kappa\sqrt{\varepsilon}|\ln\varepsilon|$, $\forall\varepsilon \in (0,1)$.

Proof Since the norm $\|\|_{1,\Omega}$ contains two terms, i.e. $\|v\|_{1,\Omega}^2 := \|v\|_{0,\Omega}^2 + \|\nabla v\|_{0,\Omega}^2$, we split the proof in two parts corresponding to those two terms. For this proof, let us introduce the set $O_\varepsilon := \{\mathbf{x} \in \Omega | \sqrt{\varepsilon} < |\mathbf{x}| < \varrho\}$. Note that $\mathrm{supp}(\chi_{\mathbf{e}} - \chi^\varepsilon) \subset O_\varepsilon$ and also that $|\chi_{\mathbf{e}} - \chi^\varepsilon|_{\infty,\Omega} \leq 1, \forall\varepsilon \in (0,1)$. Applying the change of coordinate $r = \varepsilon R$ we obtain

$$\begin{aligned} &\|\left(\chi_{\mathbf{e}} - \chi^\varepsilon\right)\left(\mathfrak{S}^\varepsilon - \ln(r/\varepsilon) - c_\gamma\right)\|_{0,\Omega}^2 \leq \|\mathfrak{S}^\varepsilon - \ln(r/\varepsilon) - c_\gamma\|_{0,O_\varepsilon}^2 \\ &\leq \varepsilon^2 \int_{1/\sqrt{\varepsilon}}^{\varrho/\varepsilon}\int_0^{2\pi} |\mathfrak{S}(R,\theta) - \ln(R) - c_\gamma|^2 R\, dR\, d\theta \end{aligned} \tag{3.14}$$

Recall that ϱ refers to the radius of Γ. According to Asymptotics (1.5), there exists $\kappa > 0$ independent of R, θ, ε such that $|\mathfrak{S}(\mathbf{x}) - \ln|\mathbf{x}| - c_\gamma|^2 < \kappa/|\mathbf{x}|^2$ for all $\mathbf{x} \in \Omega_{\mathrm{N}}$. Putting this inequality inside (3.14) leads to

$$\begin{aligned} &\int_{1/\sqrt{\varepsilon}}^{\varrho/\varepsilon}\int_0^{2\pi} |\mathfrak{S}(R,\theta) - \ln(R) - c_\gamma|^2 R\, dR\, d\theta \\ &\leq \kappa \int_{1/\sqrt{\varepsilon}}^{\varrho/\varepsilon}\int_0^{2\pi} \frac{dR\, d\theta}{R} = 2\pi\kappa\ln\left(\frac{\varrho}{\sqrt{\varepsilon}}\right), \quad \forall\varepsilon \in (0,1). \end{aligned} \tag{3.15}$$

Gathering (3.14) and (3.15) leads to the first part of the desired estimate: there exists $\kappa > 0$ such that

$$\|(\chi_{\mathbf{e}} - \chi^{\varepsilon})(\mathfrak{S}^{\varepsilon} - \ln(r/\varepsilon) - c_{\gamma})\|_{0,\Omega} \leq \kappa\varepsilon\sqrt{|\ln\varepsilon|}, \quad \forall\varepsilon \in (0,1). \tag{3.16}$$

Applying triangular inequality, we have

$$\begin{aligned}
&\|\nabla(\chi_{\mathbf{e}} - \chi^{\varepsilon})(\mathfrak{S}^{\varepsilon} - \ln(r/\varepsilon) - c_{\gamma})\|_{0,\Omega} \\
&\quad \leq |\nabla(\chi_{\mathbf{e}} - \chi^{\varepsilon})|_{\infty,\Omega}\|\mathfrak{S}^{\varepsilon} - \ln(r/\varepsilon) - c_{\gamma}\|_{0,O_{\varepsilon}} \\
&\qquad + |\chi_{\mathbf{e}} - \chi^{\varepsilon}|_{\infty,\Omega}\|\nabla\left(\mathfrak{S}^{\varepsilon} - \ln(r/\varepsilon) - c_{\gamma}\right)\|_{0,O_{\varepsilon}}.
\end{aligned} \tag{3.17}$$

An elementary calculus shows that there exists $\kappa > 0$ such that $|\nabla(\chi_{\mathbf{e}} - \chi^{\varepsilon})|^2_{\infty,\Omega} < \kappa_1/\sqrt{\varepsilon}$, $\forall\varepsilon \in (0,1)$. Since we have also $|\chi_{\mathbf{e}} - \chi^{\varepsilon}|^2_{\infty,\Omega} \leq 1$, this yields the existence of $\kappa_1 > 0$ independent of ε such that

$$\begin{aligned}
&\|\nabla(\chi_{\mathbf{e}} - \chi^{\varepsilon})(\mathfrak{S}^{\varepsilon} - \ln(r/\varepsilon) - c_{\gamma})\|_{0,\Omega} \\
&\quad \leq \frac{\kappa_1}{\sqrt{\varepsilon}}\|\mathfrak{S}^{\varepsilon} - \ln(r/\varepsilon) - c_{\gamma}\|_{0,O_{\varepsilon}} + \|\nabla\left(\mathfrak{S}^{\varepsilon} - \ln(r/\varepsilon) - c_{\gamma}\right)\|_{0,O_{\varepsilon}}
\end{aligned} \tag{3.18}$$

According to (3.14) and (3.15), it is clear that there exists $\kappa_2 > 0$ independent of ε such that $\|\mathfrak{S}^{\varepsilon} - \ln(r/\varepsilon) - c_{\gamma}\|_{0,O_{\varepsilon}} \leq \kappa_2\varepsilon\sqrt{|\ln\varepsilon|}$. To deal with the second term of (3.18) we apply the change of coordinate $r = \varepsilon R$ and take into account Asymptotics (1.5) which yields $\kappa_3 > 0$ independent of ε such that $\forall\varepsilon \in (0,1)$

$$\begin{aligned}
\|\nabla\left(\mathfrak{S}^{\varepsilon} - \ln(r/\varepsilon) - c_{\gamma}\right)\|^2_{0,O_{\varepsilon}} &\leq \int_{1/\sqrt{\varepsilon}}^{\varrho/\sqrt{\varepsilon}}\int_0^{2\pi}|R\partial_R\mathfrak{S} - 1|^2 + |\partial_\theta\mathfrak{S}|^2\frac{dR\,d\theta}{R} \\
&\leq 2\pi\kappa_3\int_{1/\sqrt{\varepsilon}}^{\varrho/\sqrt{\varepsilon}}\frac{dR}{R^3} \leq \pi\kappa_3\varepsilon
\end{aligned} \tag{3.19}$$

Plugging (3.18) and (3.19) into (3.17) leads to the existence of $\kappa_5 > 0$ such that

$$\|\nabla(\chi_{\mathbf{e}} - \chi^{\varepsilon})(\mathfrak{S}^{\varepsilon} - \ln(r/\varepsilon) - c_{\gamma})\|_{0,\Omega} \leq \kappa_5\sqrt{\varepsilon|\ln\varepsilon|} \quad \forall\varepsilon \in (0,1). \tag{3.20}$$

In order to conclude the proof, it is sufficient to join estimates (3.16) and (3.20). □

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