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ON THE THEORETICAL JUSTIFICATION OF POCKLINGTON'S EQUATION

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Pocklington's model consists in a one-dimensional integral equation relating the current at the surface of a straight finite wire to the tangential trace of an incident electromagnetic field. It is a simplification of the more usual single layer potential equation posed on a two-dimensional surface. We are interested in estimating the error between the solution of the exact integral equation and the solution of Pocklington's model. We address this problem for the model case of acoustics in a smooth geometry using results of asymptotic analysis.

Keywords: Asymptotic models; integral equation; Helmholtz; matched asymptotics; thin wire; ellipsoidal coordinates.

AMS Subject Classification: 35C20, 35C15

1. Introduction

We consider the question of the scattering of an electromagnetic wave by a perfectly conducting thin wire, assuming that the thickness of this wire is much smaller than the average wavelength of the incident field. This question appears in a wide range of applications, especially for the simulation of waves in media including antennas. In 1897, for the model case of a finite straight wire, Pocklington[16] proposed a simplified one-dimensional integral equation relating the current at the surface of the wire to the tangential trace of the incident field. A more recent derivation of this model can be found in Refs. 1 and 8. This equation has been derived using formal manipulations, assuming that the current is constant across any section of the wire. At least for the model problem of a finite, straight and cylindrical wire, this simplified equation has been proved to be well posed[9,18] and many results have been established for the regularity of the solution to this equation.[19,2]

For the simulation of wave propagation, it is possible to take into account thin wires using an integral equation approach. In practice, the wires are coupled to the rest of the medium via Pocklington's equation. This is a reason why the numerical resolution of this equation has also been extensively studied. A review of the methods used for solving Pocklington's equation is proposed in Ref. 5.

On the basis of heuristic developments, there is no doubt that Pocklington's equation is a valid approximation. But the error due to the model has to be quantified. The purpose of this paper is to address such a study for the model problem of acoustics. Our geometrical setting is the free space containing only one single thin wire and no other obstacle. It is possible to consider more complex situations with several other obstacles or heterogeneities, but this will imply only minor changes in the study we want to present here. For the rest of this paper we will be interested in the following acoustic scattering problem:

$$(\mathcal{P}^\varepsilon)\quad \begin{cases} \text{Find } u^\varepsilon \in H^2_{\text{loc}}(\mathbb{R}^3\backslash\,\overline{\Omega}_\varepsilon) \text{ such that} \\ \Delta u^\varepsilon + k^2 u^\varepsilon = f \text{ on } \mathbb{R}^3\backslash\,\overline{\Omega}_\varepsilon, \\ u^\varepsilon = 0 \text{ on } \Gamma^\varepsilon, \\ u^\varepsilon \text{ outgoing radiating,} \end{cases}$$

where Γ^ε refers to the surface of the wire, Ω_ε is the interior of the wire, and the datum $f \in L^2(\mathbb{R}^3)$ is assumed to have a compact support located in the exterior of the wire such that $\text{supp} f \cap \Omega_\varepsilon = \emptyset$. In this problem, $k \in \mathbb{R}_+\backslash\{0\}$ is the wave number. The parameter ε represents the thickness of the wire. It is assumed that $k\varepsilon \ll 1$. Since we suppose that k is fixed, this means that $\varepsilon \to 0$.

Derivation of Pocklington's equation. We shall now formally derive an acoustic version of Pocklington's equation, in the same manner as in the literature cited above. We consider a wire described by the surface $\Gamma^\varepsilon = \{\mathbf{x} \in \mathbb{R}^3 \,|\, x^2 + y^2 = \varepsilon^2, |z| < 1\} \cup \{\mathbf{x} \in \mathbb{R}^3 \,|\, x^2 + y^2 \leq \varepsilon^2, |z| = 1\}$. This surface is represented in Fig. 1 and corresponds to the shape traditionally studied in literature, i.e. Refs. 12, 18, 19, 9, 8, 2 and 5 all consider this type of wire. We suppose that it is embedded in free space. Consider the incident field u^0 defined as the solution of the scattering problem with no wire,

$$u^0(\mathbf{x}) = -\int_{\mathbb{R}^3} f(\mathbf{x}')\frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{4\pi|\mathbf{x}-\mathbf{x}'|}\; d\mathbf{x}'.$$

Problem $(\mathcal{P}^\varepsilon)$ can be solved using an integral equation. According to classical results of integral representation, u^ε can be represented by a single layer potential integral (see Theorem 2.1 in Ref. 4)

$$u^\varepsilon(\mathbf{x}) - u^0(\mathbf{x}) = -\int_{\Gamma^\varepsilon} p^\varepsilon(\mathbf{x}')\frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{4\pi|\mathbf{x}-\mathbf{x}'|} d\sigma(\mathbf{x}') \quad \text{with } p^\varepsilon = \frac{\partial u^\varepsilon}{\partial n}\bigg|_{\Gamma^\varepsilon}.$$

Here n is the unit normal vector to Γ^ε directed into the exterior of Ω_ε, and $d\sigma$ is the surface measure on Γ^ε. Provided that k does not belong to a countable set of resonant wave numbers, it is then a classical result (see Theorems 2.1 and 3.1 in Ref. 4) that p^ε

is the unique function in $H^{-1/2}(\Gamma^\varepsilon)$ satisfying

$$\int_{\Gamma^\varepsilon} p^\varepsilon(\mathbf{x}') \frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{4\pi|\mathbf{x}-\mathbf{x}'|} d\sigma(\mathbf{x}') = u^0(\mathbf{x}), \quad \forall\, \mathbf{x} \in \Gamma^\varepsilon. \tag{1.1}$$

Now come some heuristic argument. Denote $I = \{\mathbf{x} \in \mathbb{R}^3 \,|\, x = y = 0,\ |z| \le 1\}$. Because u^0 is a smooth function in a neighbourhood of I and ε is very small, for all $\mathbf{x} = (x, y, z) \in \Gamma^\varepsilon$, $u^0(\mathbf{x}) \simeq u^0(0, 0, z)$; this means that we approximate $u^0|_{\Gamma^\varepsilon}$ by a one-dimensional function, so we simply write $u^0(z)$ instead of $u^0(\mathbf{x})$. Then Pocklington's model assumes the existence of a one-dimensional function denoted $\mathfrak{p}^\varepsilon$ (that we normalize using a factor $1/2\pi\varepsilon$) that is very close to p^ε,

$$p^\varepsilon(\mathbf{x}') \simeq \frac{\mathfrak{p}^\varepsilon(z')}{2\pi\varepsilon} \quad \text{and} \quad u^0(\mathbf{x}) \simeq u^0(z) \quad \text{for } \mathbf{x}, \mathbf{x}' \in \Gamma^\varepsilon. \tag{1.2}$$

We will give more details about the precise definition of this new function $\mathfrak{p}^\varepsilon$ later in this paper. Pocklington's model also neglects the terms corresponding to $z = \pm 1$ in the integral in (1.1). With these approximations the integral equation can be rewritten as

$$\int_{-1}^{+1} \left(\frac{1}{2\pi} \int_0^{2\pi} \frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{4\pi|\mathbf{x}-\mathbf{x}'|} d\theta' \right) \mathfrak{p}^\varepsilon(z') dz' = u^0(z), \quad \forall\, z \in [-1, +1]. \tag{1.3}$$

Now we have to deal with an integral kernel averaged on each section of the wire. Note that the right-hand side in the above equation is independent of θ whereas, at first glance, the left-hand side can depend on θ via $\mathbf{x}$. However, $|\mathbf{x} - \mathbf{x}'| = ((z - z')^2 + 4\varepsilon^2 \sin^2(\frac{\theta-\theta'}{2}))^{1/2}$. If we use the change of variable $\varphi = \theta' - \theta$, Eq. (1.3) becomes, $\forall\, z \in [-1, +1]$

$$\int_{-1}^{+1} \left(\frac{1}{2\pi} \int_0^{2\pi} \frac{e^{ik\sqrt{(z-z')^2 + \varepsilon^2\, 4\sin^2(\varphi/2)}}}{4\pi\sqrt{(z-z')^2 + \varepsilon^2\, 4\sin^2(\varphi/2)}} d\varphi \right) \mathfrak{p}^\varepsilon(z') \ \ dz' = u^0(z). \tag{1.4}$$

We see that actually the left-hand side does not depend on θ. This last integral equation is indeed one-dimensional and we call it the acoustic version of Pocklington's equation.

Purpose and outline of this work. To our knowledge, there exists no result on the relative error that one commits when solving (1.4) instead of (1.1). We wish to tackle this latest question. Our basic idea consists in using asymptotic analysis considering the thickness ε as a small parameter going to 0. However, the shape corresponding to Fig. 1 (and traditionally considered in literature) is not very convenient for asymptotic analysis. So we consider a wire with slightly different shape, with rounded tips, such as in Fig. 2. Concerning asymptotics, a first strategy would consist in studying directly the integral equation (1.1) for $\varepsilon \to 0$. However, such a direct analysis seems difficult so we decided to use the asymptotic analysis available for the volumic formulation of this problem, and then use it to study the integral equation. Thus, we look at the behaviour of u^ε as $\varepsilon \to 0$, deriving the first terms of its asymptotic

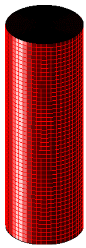


Fig. 1. A thin cylindrical wire.

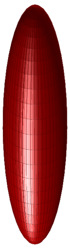


Fig. 2. A thin ellipsoid.

expansion with respect to ε. For this type of geometry, several authors have studied the solution u^ε in the electrostatic case ($k = 0$). It has been partially tackled by Rogier,[17] but the asymptotic analysis for the Laplace problem with a symmetry of revolution has been mainly solved by Fedoryuk.[7,6] Maslennikova[11] and Zhdanova[21] have directly followed Fedoryuk's ideas and extended his results in the case of coercive operator $\Delta - k^2$ with Neumann and Dirichlet boundary condition, and particular source terms. Maz'ya, Nazarov and Plamenevskii[13] have provided the same asymptotic analysis as Fedoryuk's one in the case of a wire described in cylindrical coordinates by $\Gamma^\varepsilon = \{(\mathbf{x}_\perp, z) \in \mathbb{R}^3 \,|\, \mathbf{x}_\perp \in \varepsilon\sqrt{1-z^2}\,\mathcal{C}\}$ where $\mathcal{C}$ is a given smooth closed curve in $\mathbb{R}^2$. Compared to these previous studies, what appears original in the present work is the use of a fictitious domain type formulation from which we can deduce some estimate on the field u^ε and also some estimate on the jump of its normal derivative p^ε on Γ^ε, a suitable norm for $H^{-1/2}(\Gamma^\varepsilon)$ being previously defined. To the best of our knowledge, no estimate on p^ε is available in the present literature.

As a first step, in Sec. 2 we present in detail the geometry of our problem and introduce ellipsoidal coordinates, which is a coordinate system well suited to the (rounded) tips of the wire. Then in Sec. 3 we introduce a volumic variational formulation close to the fictitious domain one. In addition to the unknown u^ε, it involves p^ε as unknown Lagrange multiplier. We study the stability of this problem with respect to ε, which requires showing that inf$-$sup conditions are satisfied uniformly with respect to ε. For this purpose we will take special care in choosing a norm for $H^{-1/2}(\Gamma^\varepsilon)$ (more precisely its dependency with respect to ε). Such a choice will provide a "stable lifting" of traces on Γ^ε which is sufficient to obtain uniform inf$-$sup conditions, and stability of the problem. In Sec. 4, we sketch the formal construction of $\tilde{u}^\varepsilon$, the first term of the asymptotic expansion of u^ε. The details of this construction can be found in Ref. 3. Defining $\tilde{p}^\varepsilon$ as the jump of the normal derivative of $\tilde{u}^\varepsilon$, we show that the couple $(\tilde{u}^\varepsilon, \tilde{p}^\varepsilon)$ is solution to the same problem as $(u^\varepsilon, p^\varepsilon)$ but with a small perturbation of the source. Using a fictitious domain formulation for dealing with asymptotics leads in particular to an estimate on p^ε which is one of the new ingredients of this paper. In Sec. 5, we study the properties of $\tilde{p}^\varepsilon$ and introduce an averaging operator denoted μ^ε and its transpose ${}^t\mu^\varepsilon$. The Hilbertian structure related to this transposition will be made clear in Sec. 5. Then we propose to replace $\mathcal{P}^\varepsilon$ by a new formulation $\mathbf{P}^\varepsilon$. This new formulation is in some sense simpler than the former one because it involves a one-dimensional space of Lagrange multipliers defined via μ^ε, instead of $H^{-1/2}(\Gamma^\varepsilon)$ that is two-dimensional. The solution of $\mathbf{P}^\varepsilon$ is denoted $(\mathfrak{u}^\varepsilon, \mathfrak{p}^\varepsilon)$. Using again the results of asymptotic analysis, we obtain estimates for the relative errors

$$\frac{\|u^\varepsilon - \mathfrak{u}^\varepsilon\|_{H^1(B_R)}}{\|u^\varepsilon\|_{H^1(B_R)}} \leq \kappa |\ln \varepsilon| \sqrt{\varepsilon} \quad \text{and} \quad \frac{|\, p^\varepsilon - {}^t\mu^\varepsilon[\mathfrak{p}^\varepsilon]|_{-1/2,\Gamma^\varepsilon}}{|\, p^\varepsilon|_{-1/2,\Gamma^\varepsilon}} \leq \kappa |\ln \varepsilon|^2 \sqrt{\varepsilon}$$

in suitable norms. It appears that we can associate to this new formulation an integral equation that has the same form as Eq. (1.4). This gives a precise description of the link between the exact integral equation and Pocklington's equation.

2. Geometry of the Problem and Adapted Coordinates

In this section we introduce several coordinate systems, and associated notations. One of them is the ellipsoidal system. We suppose the boundary Γ^ε to be described by an equation in this system. We give illustrations that justify the use of these coordinates and show why it is well suited for wire shapes.

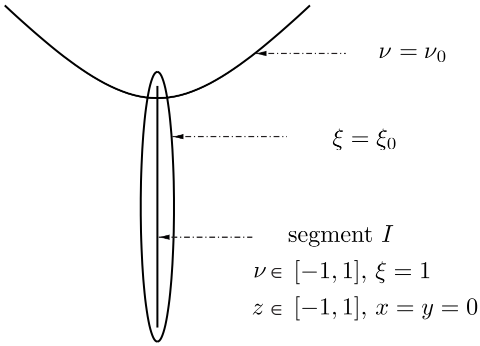
Ellipsoidal coordinates. In addition to Cartesian coordinates, we use a special coordinate system usually called prolate spheroidal; it is a particular case of ellipsoidal coordinates. We choose to call it ellipsoidal, and denote it by (ξ, ν, φ). It is given by the following correspondence with Cartesian coordinates

$$\begin{aligned} x &= \sqrt{(\xi^2 - 1)(1 - \nu^2)} \cos\varphi, & \xi &\in [1; +\infty[, \\ y &= \sqrt{(\xi^2 - 1)(1 - \nu^2)} \sin\varphi, & \nu &\in [-1; +1], \\ z &= \xi\,\nu, & \varphi &\in [0; 2\pi[. \end{aligned} \tag{2.1}$$

Take the set $U_{\mathrm{el}} = \mathbb{R}^3 \backslash \{\mathbf{x}(x, y, z) \in \mathbb{R}^3 \,|\, x = y = 0\}$, then the function $\phi_{\mathrm{el}} : U_{\mathrm{el}} \to\,]1, +\infty[\times] - 1, +1[\times(\mathbb{R}/2\pi\mathbb{Z})$ that associates to each point its ellipsoidal coordinates is a C^∞-diffeomorphism. In order to encourage a better intuition of these coordinates, we shall make some remarks. First of all, note that the set described by the equation $\xi = 1$ is simply the segment

$$I = \{\mathbf{x}(x, y, z) \in \mathbb{R}^3 \,|\, x = y = 0,\ |z| \leq 1\}.$$

We call it the origin segment. It will represent the limit shape of the shrinking wires we will consider. Let us now describe the surfaces defined by $\xi = \xi_0$ for a given ξ_0. We can eliminate the coordinates ν and φ in Eq. (2.1) and obtain a relation between Cartesian coordinates, $\frac{z^2}{\xi_0^2} + \frac{x^2+y^2}{\xi_0^2-1} = 1$. This is the equation of an ellipsoid with symmetry of revolution around the axis (oz). If we are interested in the surfaces defined by the equation $\nu = \nu_0$, then we are led to the equation in Cartesian coordinates, $\frac{z^2}{\nu_0^2} - \frac{x^2+y^2}{1-\nu_0^2} = 1$, which is the equation of a hyperboloid with symmetry of revolution around the axis (oz). The figure below provides a transversal view ($\varphi =$ constant) of these type of surface.



Definition of the wire. Using this special coordinate system we can define exactly the thin wire we wish to consider. We write Γ^ε for the boundary of this wire and define the wire as the set enclosed inside Γ^ε. We define Γ^ε by giving its equation in ellipsoidal coordinates,

$$(\Gamma^\varepsilon): \ \xi^2 - 1 = \varepsilon^2 \Phi^2(\nu, \varphi), \quad \nu \in [-1,1], \varphi \in [0, 2\pi[. \tag{2.2}$$

With this type of equation, the wire is close to a perturbed thin ellipsoid. Indeed if we take $\Phi = 1$, then Γ^ε is simply a thin ellipsoid. Here we suppose that Φ is a function defined on $[-1,1] \times \mathbb{R}/2\pi\mathbb{Z}$ satisfying three hypothesis,

- **A1:** There exists Φ_0 such that $\Phi_0 < \Phi(\nu;\varphi)$, $\forall \nu \in [-1;+1]$, $\forall \varphi \in [0;2\pi[$,
- **A2:** There exists $\nu_0 \in \,]0;1[$ such that $\Phi(\nu;\varphi) = \Phi(\nu)$ if $|\nu| > \nu_0$,
- **A3:** Φ is C^∞ as a function defined on $[-1,1] \times \mathbb{R}/2\pi\mathbb{Z}$.

Remarks. Hypothesis A1 is not really restrictive, because this type of condition is satisfied if the wire is thick. Hypothesis A2 is a bit more restrictive, because it imposes a symmetry of revolution in a neighbourhood of the tips. In most of the geometries considered for antennas, this type of condition is satisfied. Hypothesis A3 supposes that our wire is smooth. Consequently, it excludes the usual geometry (see Fig. 1) considered for antennas. For the sake of comparison we describe this usual geometry in ellipsoidal coordinates. Again consider the set $\{\mathbf{x} \in \mathbb{R}^3 \,|\, x^2 + y^2 = \varepsilon^2,\, z^2 < 1 + \varepsilon^2\} \cup \{\mathbf{x} \in \mathbb{R}^3 \,|\, x^2 + y^2 \leq \varepsilon^2,\ z^2 = 1 + \varepsilon^2\}$ of the type of Fig. 1. We have assumed that $z^2 \leq 1 + \varepsilon^2$ instead of $z^2 \leq 1$ but this is a minor difference. This surface can be described in ellipsoidal coordinates using the equation,

$$\xi^2 = 1 + \frac{\varepsilon^2}{1-\nu^2} \mathbf{1}_{|\nu| < \nu_0(\varepsilon)} + \frac{\varepsilon^2 + 1 - \nu^2}{\nu^2} \mathbf{1}_{|\nu| > \nu_0(\varepsilon)},$$
$$\text{with } \nu_0(\varepsilon) = \left((\sqrt{1+\varepsilon^2} - \varepsilon)\sqrt{1+\varepsilon^2} \right)^{1/2} \underset{\varepsilon \to 0}{=} 1 - \frac{\varepsilon}{2} + O(\varepsilon^2).$$

We see that this equation does not have the same form as (2.2), and the parametrisation is not smooth.

We also introduce a chart on the wire. Denote $O_+^{\Gamma^\varepsilon}$ (resp. $O_-^{\Gamma^\varepsilon}$) the upper (resp. lower) pole on Γ^ε. Then we define $\phi_{\Gamma^\varepsilon} : \Gamma^\varepsilon \backslash \{O_-^{\Gamma^\varepsilon}, O_+^{\Gamma^\varepsilon}\} \to \,]-1,+1[\, \times \mathbb{R}/2\pi\mathbb{Z}$ as the function associating to any point $\mathbf{x} \in \Gamma^\varepsilon$ its ellipsoidal coordinates (ν, φ). This is a C^∞-diffeomorphism. We remind that Ω_ε refers to the interior domain delimited by Γ^ε.

Adapted parametrisation of the unit sphere. One of the difficulties when dealing with objects related to Γ^ε lies in their dependency with respect to ε. To bypass this problem, we will transform Γ^ε into a fixed surface namely the unit sphere S^2. Indeed, these two surfaces are diffeomorphic. As a consequence, we need to introduce coordinates (ν, φ) for S^2. We use the same notation ν and φ for ellipsoidal coordinates; this will be clarified in the next paragraph. These coordinates are defined by the following correspondence with Cartesian coordinates,

$$\begin{aligned} x &= \sqrt{1-\nu^2}\cos\varphi, \quad \nu \in [-1;+1], \\ y &= \sqrt{1-\nu^2}\sin\varphi, \quad \varphi \in [0;2\pi], \\ z &= \nu. \end{aligned}$$

Denoting $O_+^{S^2}$ (resp. $O_-^{S^2}$) the upper (resp. lower) pole on S^2, we also need to introduce $\phi_{S^2} : S^2 \backslash \{O_-^{S^2}, O_+^{S^2}\} \to \,]-1,1[\, \times (\mathbb{R}/2\pi\mathbb{Z})$ the diffeomorphism that associates to any point $\mathbf{x} \in S^2 \backslash \{O_-^{S^2}, O_+^{S^2}\}$ its coordinates on S^2 according to the correspondence given above.

Note that using charts for transporting Γ^ε on S^2 is natural. Indeed we have the simple relation $\Gamma^\varepsilon = \phi_{\Gamma^\varepsilon}^{-1} \circ \phi_{S^2}(S^2)$. This is the reason why we identify the coordinates on Γ^ε and on S^2, thus making the map $\phi_{\Gamma^\varepsilon}^{-1} \circ \phi_{S^2}$ implicit. Sometimes however we will write this map explicitly. It is possible to state a regularity result implying that Γ^ε and S^2 are C^∞-diffeomorphic. We do not prove it because it relies on usual differential calculus. However the proof can be found in Ref. 3.

Lemma 2.1. *The mapping* $\Psi \triangleq \phi_{\Gamma^\varepsilon}^{-1} \circ \phi_{S^2} : S^2 \backslash \{O_-^{S^2}, O_+^{S^2}\} \to \Gamma^\varepsilon \backslash \{O_-^{\Gamma^\varepsilon}, O_+^{\Gamma^\varepsilon}\}$ *can be extended into a* C^∞*-diffeomorphism from* S^2 *to* Γ^ε*. Moreover,* Ψ *induces a continuous isomorphism* $\Psi_* : H^s(\Gamma^\varepsilon) \to H^s(S^2)$, $s \in \mathbb{R}$ *such that* $\Psi_* v = v \circ \Psi$, $\forall v \in C^\infty(\Gamma^\varepsilon)$.

3. Setting of the Problem

We will now write in full details the formulation we consider in order to solve the problem of scattering by a wire. As we mentioned before, this is a fictitious domain type formulation that involves p^ε the jump of the normal derivative of u^ε on Γ^ε. It is classical to prove that, for a given ε, this problem is well posed. More unusual is to prove that such a mixed formulation is stable when ε goes to 0. By stability we mean that for bounded and fixed data, independent of ε, the solution $(u^\varepsilon, p^\varepsilon)$ is bounded when $\varepsilon \to 0$ for a suitable norm that we shall define. This problem will be set on $B_R = B(0,R)$, where $\Gamma_R = \partial B_R$ denotes the fictitious boundary on which we shall impose an outgoing radiation condition with R large enough for B_R to contain the support of f.

Functional spaces. First define the usual space $\mathcal{H} = H^1(B_R)$, a scalar product associated to this space $(u, v)^2_{\mathcal{H}} = \int_{B_R} \nabla u \cdot \nabla \overline{v} + \int_{B_R} u\overline{v}$ and the corresponding norm $\|u\|^2_{\mathcal{H}} = (u, u)_{\mathcal{H}}$. We also introduce $\mathcal{H}^\varepsilon_0 = \{v \in \mathcal{H} \,|\, v_{|\Gamma^\varepsilon} = 0\}$. In addition, we consider the usual space of traces $H^{1/2}(\Gamma^\varepsilon) = \{v_{|\Gamma^\varepsilon} \text{ such that } v \in \mathcal{H}\}$. We have to choose a norm for this space and at the same time stay aware of the fact that $H^{1/2}(\Gamma^\varepsilon)$ depends on ε. A strategy consists in getting back to a space independent of ε. We use the correspondence between Γ^ε and S^2 that we introduced in the preceding paragraph. Lemma 2.1 provides an explicit characterisation of $H^{1/2}(\Gamma^\varepsilon)$, namely $u \in H^{1/2}(\Gamma^\varepsilon)$ if and only if $\Psi_* u \in H^{1/2}(S^2)$. On the basis of this characterisation we propose a norm on $H^{1/2}(\Gamma^\varepsilon)$,

$$|u|_{1/2,\Gamma^\varepsilon} \triangleq \|\Psi_* u\|_{H^{1/2}(S^2)}.$$

This can be rewritten using the characterisation of $H^{1/2}(S^2)$ based on the spherical harmonics Y_l^m

$$|u|^2_{1/2,\Gamma^\varepsilon} = \sum_{l\geq 0, |m|\leq l} (1+l^2)^{1/2} |\langle \Psi_* u,\, Y_l^m \rangle_{L^2(S^2)}|^2. \tag{3.1}$$

More details about spherical harmonics can be found in Ref. 14. Here we use the L^2 product on the sphere, $\langle u, v\rangle_{L^2(S^2)} = \int_{S^2} u\overline{v}$. This leads to a dual characterisation of $H^{-1/2}(\Gamma^\varepsilon)$ and to the corresponding norm

$$|u|_{-1/2,\Gamma^\varepsilon} = \sup_{v \in H^{1/2}(\Gamma^\varepsilon)} \frac{\int_{\Gamma^\varepsilon} u\overline{v}}{|v|_{1/2,\Gamma^\varepsilon}}. \tag{3.2}$$

Formulation of the problem. Then consider a sesquilinear form associated with the Helmholtz problem (with outgoing radiation condition) on B_R,

$$a(u, v) = \int_{B_R} \nabla u \cdot \nabla \overline{v} - k^2 \int_{B_R} u\overline{v} + \int_{\Gamma_R} \overline{v} T_R u, \quad u, v \in \mathcal{H}.$$

Here T_R refers to the usual Dirichlet-to-Neumann map used to impose an outgoing radiation condition. It is defined explicitly using the spherical harmonics $(Y_l^m)_{l,m}$ (on Γ_R),

$$\int_{\Gamma_R} \overline{v} T_R u = -\sum_{n=0}^{+\infty} \sum_{m=-n}^{+n} \frac{k}{R^2} \frac{h_n^{(1)\prime}(kR)}{h_n^{(1)}(kR)} \left(\int_{\Gamma_R} u \overline{Y}_n^m \right) \overline{\left(\int_{\Gamma_R} v \overline{Y}_n^m \right)}.$$

In this expression, the functions $h_n^{(1)}$ are the spherical Hankel functions also described in Ref. 14. We remind the reader of a classical positivity result concerning the operator T_R, namely

$$\Re e \left\{ \int_{\Gamma_R} \overline{v} T_R v \right\} \geq 0, \quad \forall\, v \in H^{1/2}(\Gamma_R) \tag{3.3}$$

see Theorem 2.6.4 in Ref. 15. Suppose given a function $f \in L^2(B_R)$ whose support does not intersect the wire Γ^ε, i.e. there exists $\varepsilon > 0$ such that $\mathrm{supp} f \cap \Omega_\varepsilon = \emptyset$. We will make particular use of the following reformulation of problem $(\mathcal{P}^\varepsilon)$ introduced at the beginning of this paper,

$$(\mathcal{P}^\varepsilon)\begin{cases}\text{Find}(u^\varepsilon,p^\varepsilon)\in\mathcal{H}\times H^{-1/2}(\Gamma^\varepsilon)\text{ such that,}\\[1ex] a(u^\varepsilon,v)+\displaystyle\int_{\Gamma^\varepsilon}p^\varepsilon\overline{v}=-\int_{B_R}f\overline{v}, & \forall\, v\in\mathcal{H},\\[1ex] \displaystyle\int_{\Gamma^\varepsilon}u^\varepsilon q=0, & \forall\, q\in H^{-1/2}(\Gamma^\varepsilon).\end{cases}$$

It is strictly equivalent to the traditional variational formulation $u^\varepsilon\in\mathcal{H}_0^\varepsilon$ and $a(u^\varepsilon,v)=-\int_{B_R}f\overline{v},\forall\, v\in\mathcal{H}_0^\varepsilon$. We impose a homogeneous Dirichlet boundary condition on Γ^ε using Lagrange multipliers chosen in $H^{-1/2}(\Gamma^\varepsilon)$. Note that $u^\varepsilon=0$ in Ω_ε, and that p^ε is the jump of its normal derivative on Γ^ε,

$$p^\varepsilon=\frac{\partial}{\partial n}(u^\varepsilon_{|B_R\setminus\overline{\Omega}_\varepsilon})-\frac{\partial}{\partial n}(u^\varepsilon_{|\Omega_\varepsilon}),$$

where n is the unit normal vector to Γ^ε directed into the exterior of Ω_ε. For ε fixed, problem $(\mathcal{P}^\varepsilon)$ is well posed: this is a classical result. Moreover, there is continuous dependency of the solution with respect to the data. This means that there exist $\kappa_1^\varepsilon,\kappa_2^\varepsilon>0$ (independent of f) such that $\|u^\varepsilon\|_{\mathcal{H}}\leq\kappa_1^\varepsilon\|f\|_{L^2(B_R)}$ and $|p^\varepsilon|_{-1/2,\Gamma^\varepsilon}\leq\kappa_2^\varepsilon\|f\|_{L^2(B_R)}$. However, we do not know *a priori* the behaviour of these constants κ_i^ε when $\varepsilon\to0$. This is an important issue that we will clarify through the rest of this section.

Stability result. In a first part we are interested in an intermediate homogeneous problem, which enables to "isolate" the bilinear form $a(\,,\,)$. Let us define $\mathcal{H}'$ the dual space of $\mathcal{H}$ and $\|\ \|_{\mathcal{H}'}$ the dual norm of $\|\ \|_{\mathcal{H}}$ with definition similar to (3.2) replacing $H^{1/2}(\Gamma^\varepsilon)$ by $\mathcal{H}$. Take an arbitrary element of this space, potentially dependent of ε, $f^\varepsilon\in\mathcal{H}'$ and consider the problem

$$\text{Find } v_0^\varepsilon\in\mathcal{H}_0^\varepsilon\text{ such that } a(v_0^\varepsilon,w)=\langle f^\varepsilon,w\rangle_{\mathcal{H}',\mathcal{H}},\ \forall\, w\in\mathcal{H}_0^\varepsilon. \tag{3.4}$$

For this problem we establish a stability result which contains the proof of a uniform inf−sup condition on the bilinear form a. This is also a classical proof.

Lemma 3.1. *For any $\varepsilon\in\ ]0,1[$ and for any $f^\varepsilon\in\mathcal{H}'$, problem (3.4) admits a unique solution $v_0^\varepsilon\in\mathcal{H}_0^\varepsilon$. Moreover, there exist $\kappa_1,\kappa_2>0$ independent of ε such that*

$$\kappa_1\|v_0^\varepsilon\|_{\mathcal{H}}\leq\|f^\varepsilon\|_{\mathcal{H}'}\leq\kappa_2\|v_0^\varepsilon\|_{\mathcal{H}}\quad\forall\,\varepsilon\in\ ]0,1[.$$

Proof. We begin with a simple observation. There obviously exists a constant $\kappa_2>0$ independent of ε such that

$$\sup_{w\in\mathcal{H}_0^\varepsilon}\frac{a(v,w)}{\|w\|_{\mathcal{H}}}\leq\kappa_2\|v\|_{\mathcal{H}},\quad\forall\, v\in\mathcal{H}_0^\varepsilon,\quad\forall\,\varepsilon\in\ ]0,1[. \tag{3.5}$$

Then we prove the uniform inf−sup condition on a, i.e. there exists a constant $\kappa_1 > 0$ independent of ε such that

$$\kappa_1 \|v\|_{\mathcal{H}} \leq \sup_{w \in \mathcal{H}_0^\varepsilon} \frac{a(v,w)}{\|w\|_{\mathcal{H}}}, \quad \forall\, v \in \mathcal{H}_0^\varepsilon, \quad \forall \varepsilon \in \,]0,1[. \tag{3.6}$$

We proceed by contradiction. Suppose there exist sequences (ε_n) and $w_n \in \mathcal{H}_0^{\varepsilon_n}$ such that $\lim_{n\to+\infty} \varepsilon_n = 0, \|w_n\|_{\mathcal{H}} = 1$ and $\lim_{n\to+\infty} \sup_{w\in\mathcal{H}_0^{\varepsilon_n}} a(w_n,w)/\|w\|_{\mathcal{H}} = 0$. Extracting subsequences if necessary, we can assume that (w_n) is strongly convergent in $L^2(B_R)$ and weakly convergent in $\mathcal{H}$ toward a function $w_0 \in \mathcal{H}$. Then take an arbitrary $w \in \mathcal{H}_* = \{v \in \mathcal{H} \,|\, v = 0$ in a neighbourhood of $I\}$. $w \in \mathcal{H}_0^\varepsilon$ for ε small enough, so $a(w_n, w) \to 0$ when $n \to +\infty$. We conclude that $a(w_0, w) = 0$, for any $w \in \mathcal{H}_*$. As a consequence of the fact that the trace on a line is not a H^1-continuous operator in three dimensions, $\mathcal{H}_*$ is dense in $\mathcal{H}$ (see also Lemma A.8 in Ref. 3) so we obtain that $a(w_0, w) = 0$, for any $w \in \mathcal{H}$. Since $a(\,,)$ classically satisfies inf−sup conditions on $\mathcal{H} \times \mathcal{H}$, this implies that $w_0 = 0$. We have just shown that $\|w_n\|_{L^2(B_R)} \to 0$. We now prove the same for $\|\nabla w_n\|_{L^2(B_R)}$. Taking into account the positivity property (3.3) for the Dirichlet-to-Neumann operator T_R, we can write

$$\begin{aligned} \|\nabla w_n\|^2_{L^2(B_R)} &\leq \|\nabla w_n\|^2_{L^2(B_R)} + \Re e\left\{ \int_{\Gamma_R} \overline{w_n} T_R\, w_n \right\} \\ &\leq \Re e\{a(w_n, w_n)\} + k^2 \|w_n\|^2_{L^2(B_R)} \underset{n\to+\infty}{\longrightarrow} 0. \end{aligned}$$

We have proved that $\|\nabla w_n\|_{L^2(B_R)} \to 0$. This and the same result on the L^2 norm provide $\|w_n\|_{\mathcal{H}} \to 0$. This is impossible since we have assumed that $\|w_n\|_{\mathcal{H}} = 1$ for any n. This gives a contradiction which yields the inf−sup condition (3.6). The property (3.6) provides unicity of the solution of problem (3.4). Then, according to Fredholm alternative (applied to $a(\,,)$) unicity implies existence of the solution to this problem. Finally, given a $f^\varepsilon \in \mathcal{H}'$, use inequalities (3.5) and (3.6) to obtain $\kappa_1 \|v_0^\varepsilon\|_{\mathcal{H}_0^\varepsilon} \leq \|f^\varepsilon\|_{\mathcal{H}'} \leq \kappa_2 \|v_0^\varepsilon\|_{\mathcal{H}_0^\varepsilon}$. ☐

This first result can be applied in particular to u^ε. Indeed $\int_{\Gamma^\varepsilon} q\overline{u}^\varepsilon = 0, \forall\, q \in H^{-1/2}(\Gamma^\varepsilon)$ so $u^\varepsilon \in \mathcal{H}_0^\varepsilon$ and, as a consequence of $\mathcal{P}^\varepsilon$, it has to satisfy (3.4) with $\langle f^\varepsilon, v\rangle_{\mathcal{H}',\mathcal{H}} = -\int_{B_R} f\overline{v}$. Hence $\|u^\varepsilon\|_{\mathcal{H}} \leq \kappa \|f\|_{L^2(B_R)}$ for a suitable $\kappa > 0$ independent of ε and for any $\varepsilon \in \,]0,1[$. In order to prove a similar result of continuous dependency of p^ε with respect to f, we need to construct a lifting operator bounded uniformly with respect to ε. For this purpose we need a particular result of geometry. Here again we do not give the proof of this result (it can be found in Lemma 2.2 in Ref. 3), because it relies on basic differential calculus.

Lemma 3.2. *Suppose given two manifolds* Γ_i^ε, $i = 0, 1$ *parametrised in ellipsoidal coordinates by* $(\Gamma_i^\varepsilon) : \xi^2 = 1 + \varepsilon^2 \Phi_i(\nu, \varphi)$, *where* Φ_i *satisfy Assumptions* A1, A2 *and*

A3. *There exists a C^∞-diffeomorphism $\mathcal{G}$ defined in ellipsoidal coordinates in a neighbourhood of I by*

$$\phi_{\text{el}} \circ \mathcal{G} \circ \phi_{\text{el}}^{-1}(\xi,\nu,\varphi) = \left(\sqrt{1 + \left(\frac{\Phi_1(\nu,\varphi)}{\Phi_0(\nu,\varphi)}\right)^2 (\xi^2-1)},\, \nu,\, \varphi \right),$$

such that $\mathcal{G}(\Gamma_0^\varepsilon) = \Gamma_1^\varepsilon$ and $\mathcal{G}$ induces a continuous map $\mathcal{G}_* : \mathcal{H} \to \mathcal{H}$ defined by $\mathcal{G}_* u = u \circ \mathcal{G}$.

Note that $\mathcal{G}$ also induces a bijective isometry $\mathcal{G}_* : H^{1/2}(\Gamma_1^\varepsilon) \to H^{1/2}(\Gamma_0^\varepsilon)$. With this result we are able to provide a lifting operator.

Lemma 3.3. *There exists $\kappa > 0$ independent of ε and $\mathcal{R}^\varepsilon : H^{1/2}(\Gamma^\varepsilon) \to \mathcal{H}$ such that $\mathcal{R}^\varepsilon(u)_{|\Gamma^\varepsilon} = u$ and $\|\mathcal{R}^\varepsilon(u)\|_{\mathcal{H}} \le \kappa |u|_{1/2,\Gamma^\varepsilon}$ for any $u \in H^{1/2}(\Gamma^\varepsilon)$ and any $\varepsilon \in \,]0,1[$.*

Proof. According to Lemma 3.2 ($\mathcal{G}$ is independent of ε) it is sufficient to prove this result in the case where the boundary of the wire is given by $(\Gamma^\varepsilon) : \xi^2 = 1 + \varepsilon^2$. Choose a function $u \in H^{1/2}(\Gamma^\varepsilon)$ and denote $\hat{u}(l,m) = \langle\, \Psi_* u,\, Y_l^m \rangle_{L^2(S^2)}$. According to Eq. (3.1),

$$|u|^2_{1/2,\Gamma^\varepsilon} = \sum_{l,m} (1+l^2)^{1/2} |\hat{u}(l,m)|^2.$$

We introduce an auxiliary function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ defined by $\alpha(r) = r$ if $0 \le r \le 1$ and $\alpha(r) = 1/r$ if $1 \le r$. Note that $\alpha(1) = 1$. We define $\mathcal{R}^\varepsilon(u)$ in ellipsoidal coordinates by

$$\mathcal{R}^\varepsilon(u)(\xi,\nu,\varphi) = \sum_{l=0}^{+\infty} \sum_{m=-l}^{+l} \hat{u}(l,m)\, Y_l^m \circ \phi_{S^2}^{-1}(\nu,\varphi) \left(\alpha\left(\frac{\xi^2-1}{\varepsilon^2} \right) \right)^{l/2}. \tag{3.7}$$

Let us denote $\mathcal{R}^\varepsilon_{l,m}$ the terms in the sum above. Clearly this sum converges in the sense of $L^2(B_R)$ since $|\alpha| \le 1$ and $u \in L^2(\Gamma^\varepsilon)$. Now we verify that $\mathcal{R}^\varepsilon(u) \in \mathcal{H}$ by proving that the sum (3.7) converges in $\mathcal{H}$ and bound straightforwardly $\|\nabla \mathcal{R}^\varepsilon_{l,m}\|_{L^2(B_R)}$. Note that the L^2 norm of the gradient of a function v over B_R is given by

$$\begin{aligned} \|\nabla v\|^2_{L^2(B_R)} &= \int_{B_R} (\xi^2-1) \left| \frac{\partial v}{\partial \xi} \right|^2 + \frac{1}{\xi^2-1} \left| \frac{\partial v}{\partial \varphi} \right|^2 d\xi\, d\nu\, d\varphi \\ &\quad + \int_{B_R} (1-\nu^2) \left| \frac{\partial v}{\partial \nu} \right|^2 + \frac{1}{1-\nu^2} \left| \frac{\partial v}{\partial \varphi} \right|^2 d\xi\, d\nu\, d\varphi = T_\xi(v) + T_\nu(v). \end{aligned}$$

The norm $\|\nabla v\|^2_{L^2(B_R)}$ thus splits into two terms that we successively estimate. Take a $\xi_* > 1$ such that for any $\mathbf{x} \in B_R$ with ellipsoidal coordinates (ξ,ν,φ), we have $\xi < \xi_*$.

As a consequence, each term of the sum in (3.7) can be bounded as follows

$$T_\xi(\mathcal{R}^\varepsilon_{l,m}) \leq |\hat{u}(l,m)|^2 \int_{\xi=1}^{\xi_*} (\xi^2-1) \left| \frac{\partial}{\partial \xi}\left(\alpha\left(\frac{\xi^2-1}{\varepsilon^2}\right)\right)^{l/2}\right|^2 + \frac{l^2}{\xi^2-1}\left|\alpha\left(\frac{\xi^2-1}{\varepsilon^2}\right)\right|^l d\xi$$

$$\leq |\hat{u}(l,m)|^2 \int_{\xi=1}^{\xi_*} l^2 \, \frac{\xi_*^2+1}{\xi^2-1}\left|\alpha\left(\frac{\xi^2-1}{\varepsilon^2}\right)\right|^l d\xi.$$

In the preceding estimate we have used the fact that $|r\alpha'(r)| = |\alpha(r)|$. We use the change of variable $\rho = \sqrt{\xi^2-1}/\varepsilon$ in order to bound the integral term for $l \geq 1$, and then find a constant $\kappa > 0$ independent of ε and l such that

$$\int_{\xi=1}^{\xi_*} \frac{1}{\xi^2-1}\left|\alpha\left(\frac{\xi^2-1}{\varepsilon^2}\right)\right|^l d\xi = \int_{\rho=0}^{\sqrt{\xi_*^2-1}/\varepsilon} \frac{|\alpha(\rho^2)|^l}{\sqrt{1+\varepsilon^2\rho^2}} \frac{d\rho}{\rho} \leq \frac{\kappa}{l+1}. \tag{3.8}$$

As a result there exists another constant $\kappa > 0$ independent of ε such that $T_\xi(\mathcal{R}^\varepsilon_{l,m}) \leq \kappa(1+l^2)^{1/2}|\hat{u}(l,m)|^2$ and, since $\sum_{l,m}(1+l^2)^{1/2}|\hat{u}(l,m)|^2$ converges, so does $\sum_{l,m} T_\xi(\mathcal{R}^\varepsilon_{l,m})$. In order to bound the terms $T_\nu(\mathcal{R}^\varepsilon_{l,m})$ we first note that

$$T_\nu(\mathcal{R}^\varepsilon_{l,m}) \leq |\hat{u}(l,m)|^2 \, \|\nabla_{S^2} Y_l^m\|^2_{L^2(S^2)} \int_{\xi=1}^{\xi_*} \left|\alpha\left(\frac{\xi^2-1}{\varepsilon^2}\right)\right|^l d\xi.$$

Let us recall the well-known property of spherical harmonics $\|\nabla_{S^2} Y_l^m\|^2_{L^2(S^2)} = l(1+l)\|Y_l^m\|^2_{L^2(S^2)} = l(1+l)$ (see proof of Lemma 9.15 in Ref. 14). In order to bound the integral term for $l \geq 1$, we simply use (3.8) which yields the existence of $\kappa > 0$ independent of ε such that $T_\nu(\mathcal{R}^\varepsilon_{l,m}) \leq \kappa(1+l^2)^{1/2}|\hat{u}(l,m)|^2$, so $\sum_{l,m} T_\nu(\mathcal{R}^\varepsilon_{l,m})$ converges. Gathering the estimates we have obtained on $T_\xi(\mathcal{R}^\varepsilon_{l,m})$ and $T_\nu(\mathcal{R}^\varepsilon_{l,m})$ and summing over $l \in \mathbb{N}$ and $m \in [-l,l]$, we are led to the existence of $\kappa > 0$ independent of ε such that $\|\mathcal{R}^\varepsilon(u)\|_{\mathcal{H}} \leq \kappa |u|_{1/2,\Gamma^\varepsilon}$, which is the desired result. □

We can now prove a full stability estimate. It will be proved for problems similar to $\mathcal{P}^\varepsilon$. Given $f^\varepsilon \in \mathcal{H}'$ and $g^\varepsilon \in H^{1/2}(\Gamma^\varepsilon)$, consider the problem

$$\begin{cases} \text{Find } (v^\varepsilon, q^\varepsilon) \in \mathcal{H} \times H^{-1/2}(\Gamma^\varepsilon) \text{ such that,} \\ a(v^\varepsilon, v) + \displaystyle\int_{\Gamma^\varepsilon} q^\varepsilon \overline{v} = \langle f^\varepsilon, v\rangle_{\mathcal{H}',\mathcal{H}}, \quad \forall v \in \mathcal{H}, \\ \displaystyle\int_{\Gamma^\varepsilon} q\,\overline{v}^\varepsilon = \langle q, g^\varepsilon\rangle_{H^{-1/2}(\Gamma^\varepsilon),H^{1/2}(\Gamma^\varepsilon)}, \quad \forall q \in H^{-1/2}(\Gamma^\varepsilon). \end{cases} \tag{3.9}$$

We prove that there is uniform boundedness of the solution with respect to ε, f^ε and g^ε. In particular, this result can be applied to the problem $\mathcal{P}^\varepsilon$ where $\langle f^\varepsilon, v\rangle = \int_{B_R} f\overline{v}$ and $g^\varepsilon = 0$. Applied to $\mathcal{P}^\varepsilon$, it yields the existence of a unique solution $(u^\varepsilon, p^\varepsilon)$ that remains bounded when $\varepsilon \to 0$.

Theorem 3.1. *For any* $\varepsilon \in \,]0,1[$ *and for any* $f^\varepsilon \in \mathcal{H}', g^\varepsilon \in H^{1/2}(\Gamma^\varepsilon)$, *the problem* (3.9) *admits a unique solution* $(v^\varepsilon; q^\varepsilon) \in \mathcal{H} \times H^{-1/2}(\Gamma^\varepsilon)$. *Moreover, there exist two constants* $\kappa_1, \kappa_2 > 0$ *independent of* ε *such that* $\forall\, \varepsilon \in \,]0,1[$

$$\|v^\varepsilon\|_{\mathcal{H}} \leq \kappa_1 \|f^\varepsilon\|_{\mathcal{H}'} + \kappa_2 |g^\varepsilon|_{1/2,\Gamma^\varepsilon} \quad \textit{and} \quad |q^\varepsilon|_{-1/2,\Gamma^\varepsilon} \leq \kappa_1 \|f^\varepsilon\|_{\mathcal{H}'} + \kappa_2 |g^\varepsilon|_{1/2,\Gamma^\varepsilon}.$$

Proof. Existence and unicity of the solution $(v^\varepsilon, q^\varepsilon)$ to problem (3.9) is classical and we do not prove it here. Now take $w^\varepsilon = v^\varepsilon - \mathcal{R}^\varepsilon(g^\varepsilon)$ and note that $w^\varepsilon \in \mathcal{H}_0^\varepsilon$. It also satisfies $a(w^\varepsilon, v) = \langle f^\varepsilon, v\rangle_{\mathcal{H}',\mathcal{H}} - a(\mathcal{R}^\varepsilon(g^\varepsilon), v)$, $\forall\, v \in \mathcal{H}_0^\varepsilon$. Applying Lemmas 3.1 and 3.3, we see that there exists a $\kappa > 0$ independent of ε such that $\|w^\varepsilon\|_{\mathcal{H}} \leq \kappa\|f^\varepsilon\|_{\mathcal{H}'} + \kappa|g^\varepsilon|_{1/2,\Gamma^\varepsilon}$, and this implies a similar result on v^ε namely $\|v^\varepsilon\|_{\mathcal{H}} \leq \kappa\|f^\varepsilon\|_{\mathcal{H}'} + \kappa|g^\varepsilon|_{1/2,\Gamma^\varepsilon}$. Now using directly problem (3.9), we see that for any $v \in H^{1/2}(\Gamma^\varepsilon)$ we have $|\int_{\Gamma^\varepsilon} q^\varepsilon \bar{v}| \leq |a(v^\varepsilon, \mathcal{R}^\varepsilon(v))| + |\langle f^\varepsilon, \mathcal{R}^\varepsilon(v)\rangle_{\mathcal{H}',\mathcal{H}}| \leq \kappa\|v^\varepsilon\|_{\mathcal{H}}\|\mathcal{R}^\varepsilon(v)\|_{\mathcal{H}} + \|f^\varepsilon\|_{\mathcal{H}'}\|\mathcal{R}^\varepsilon(v)\|_{\mathcal{H}} \leq \kappa(\|f^\varepsilon\|_{\mathcal{H}'} + |g^\varepsilon|_{1/2,\Gamma^\varepsilon})\|\mathcal{R}^\varepsilon(v)\|_{\mathcal{H}}$. Applying once again Lemma 3.3, there exists a $\kappa > 0$ independent of ε such that $|\int_{\Gamma^\varepsilon} q^\varepsilon \bar{v}| \leq \kappa(\|f^\varepsilon\|_{\mathcal{H}'} + |g^\varepsilon|_{1/2,\Gamma^\varepsilon})$ $|v|_{1/2,\Gamma^\varepsilon}$. Since $v \in H^{1/2}(\Gamma^\varepsilon)$ is chosen arbitrarily this yields $|q^\varepsilon|_{-1/2,\Gamma^\varepsilon} \leq \kappa(\|f^\varepsilon\|_{\mathcal{H}'} + |g^\varepsilon|_{1/2,\Gamma^\varepsilon})$. □

As was mentioned before, Theorem 3.1 proves that $(u^\varepsilon, p^\varepsilon)$ remains bounded when $\varepsilon \to 0$. However, we would like to obtain more precision about the behaviour of $(u^\varepsilon, p^\varepsilon)$ when $\varepsilon \to 0$. This will be the subject of the next section.

Integral equation. A single layer integral equation is classically associated to problem $\mathcal{P}^\varepsilon$. Indeed consider the integral kernel

$$G(\mathbf{x}, \mathbf{x}') = -\frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{4\pi|\mathbf{x}-\mathbf{x}'|}$$

which is the outgoing Green kernel in free space for the Helmholtz equation, and define $u^0 \in \mathcal{H}$ as the unique function satisfying

$$a(u^0, v) = -\int_{B_R} f\bar{v}, \quad \forall\, v \in \mathcal{H}.$$

Provided that k is not a resonant wave number for the Helmholtz equation posed in Ω_ε then p^ε is the unique solution to the integral equation (see Theorems 2.1 and 3.1 in Ref. 4)

$$\int_{\Gamma^\varepsilon} G(\mathbf{x}, \mathbf{x}')\, p^\varepsilon(\mathbf{x}')\, d\sigma(\mathbf{x}') = -u^0(\mathbf{x}), \quad \forall\, \mathbf{x} \in \Gamma^\varepsilon. \tag{3.10}$$

This equation can be rewritten in a variational form, $\forall\, q \in H^{-1/2}(\Gamma^\varepsilon)$

$$\int_{\Gamma^\varepsilon \times \Gamma^\varepsilon} G(\mathbf{x}, \mathbf{x}')\, p^\varepsilon(\mathbf{x}')q(\mathbf{x})\ d\sigma(\mathbf{x}')d\sigma(\mathbf{x}) = -\int_{\Gamma^\varepsilon} u^0(\mathbf{x})q(\mathbf{x})\ d\sigma(\mathbf{x}). \tag{3.11}$$

Note that Theorem 3.1 can also be used to provide a bound for the norm of the inverse of the single layer operator associated with Eq. (3.10). Indeed take a $g \in H^{1/2}(\Gamma^\varepsilon)$, and consider $q^\varepsilon \in H^{-1/2}(\Gamma^\varepsilon)$ as the unique solution of (3.10) with

g instead of u^0 as right-hand side. Define $v^\varepsilon \in \mathcal{H}$ by $v^\varepsilon(\mathbf{x}) = \int_{\Gamma^\varepsilon} G(\mathbf{x}, \mathbf{x}') \cdot q^\varepsilon(\mathbf{x}') d\sigma(\mathbf{x}'),\ \ \mathbf{x} \in B_R$. Then $(v^\varepsilon, q^\varepsilon)$ is the unique solution to problem (3.9) with $f^\varepsilon = 0$ and $g^\varepsilon = g$. Applying Theorem 3.1 we find the existence of a $\kappa > 0$ independent of ε such that $|q^\varepsilon|_{-1/2,\Gamma^\varepsilon} \leq \kappa |g|_{1/2,\Gamma^\varepsilon}$.

4. First Term of an Asymptotic Expansion

Now we introduce results of asymptotic analysis that will help us to know more about $(u^\varepsilon, p^\varepsilon)$. In Sec. 4.5, we propose an approximate field $(\tilde{u}^\varepsilon, \tilde{p}^\varepsilon)$ with an expression as explicit as possible and very close to $(u^\varepsilon, p^\varepsilon)$ in the sense that (cf. Theorem 4.1) there exists $\kappa > 0$ independent of ε such that

$$\|u^\varepsilon - \tilde{u}^\varepsilon\|_{\mathcal{H}} + |p^\varepsilon - \tilde{p}^\varepsilon|_{1/2,\Gamma^\varepsilon} \leq \kappa |\ln \varepsilon| \sqrt{\varepsilon} \quad \forall \varepsilon \in\,]0, 1[.$$

Moreover, since $(\tilde{u}^\varepsilon, \tilde{p}^\varepsilon)$ is explicitly described, it will be easy to collect information that we shall apply to $(u^\varepsilon, p^\varepsilon)$ according to estimates we have just given. The construction of the couple $(\tilde{u}^\varepsilon, \tilde{p}^\varepsilon)$ could be the subject of a whole study. We will not give a complete proof of Theorem 4.1, we refer the reader to the proof of Theorem 7.1 in Ref. 3. We only want to use this result, and the purpose of the present section is to give intuitive ideas on how to prove it. Besides note that it is in good agreement with the results of Refs. 7, 6, 11 and 21, although the technique these authors used is closer to multiscale expansions, which leads to slightly sharper error estimates in L^∞-norm. Concerning the comparison between matched asymptotics and multiscale expansions see Ref. 20.

4.1. *General presentation of the method*

Let us begin with a description of the method we use for the definition of the approximate field. It is called *matched asymptotic expansions.* For the present problem, it consists in four main steps. Steps 1, 2 and 3 are formal in the sense that we do not worry about error estimates. This will be the subject of Step 4.

- **Step 1.** In this step, we are interested in the behaviour of the field u^ε in a region *far* from the wire Γ^ε. What we mean by *far* shall be precised by the definition of a geometric region $\mathcal{Z}_{\mathrm{f}}^\varepsilon$. Then, on the basis of formal considerations we will postulate a form for the approximate field in this region. However, there will remain an indeterminacy in the definition of the approximate field $\tilde{u}^\varepsilon$ in this region, represented by an unknown function $a^\varepsilon \in L^2(I)$ that is to be defined in Step 3.
- **Step 2.** In this second step, we are interested in the behaviour of the field u^ε in a *small* region around the wire Γ^ε. Here again we shall precise the word *small* and define a neighbourhood $\mathcal{Z}_{\mathrm{n}}^\varepsilon$ of the wire. In order to study u^ε in a normalised geometry, we use a change of coordinates that depends on ε. Using formal calculus, we are led to solve a Laplace problem in order to define the approximate field $\tilde{u}^\varepsilon$ in this region. This construction will leave an indeterminacy in the construction of $\tilde{u}^\varepsilon$ represented by an unknown function $b^\varepsilon \in L^2(I)$ that is to be defined in Step 3.

- **Step 3.** In this step, we look for necessary conditions for the constructions of Steps 1 and 2 to coincide in the intermediate region $\mathcal{Z}_{\mathrm{n}}^{\varepsilon} \cap \mathcal{Z}_{\mathrm{f}}^{\varepsilon}$. We are thus led to Eqs. (4.8) and (4.9) involving a^{ε} and b^{ε}. One of these equations is a one-dimensional integral equation. These equations determine a^{ε} and b^{ε}, and conclude the formal construction of $\tilde{u}^{\varepsilon}$. The function $\tilde{p}^{\varepsilon}$ will be defined by

$$\tilde{p}^{\varepsilon} = \frac{\partial}{\partial n}(\tilde{u}^{\varepsilon}_{|B_R\setminus\overline{\Omega}_{\varepsilon}}) - \frac{\partial}{\partial n}(\tilde{u}^{\varepsilon}_{|\Omega_{\varepsilon}}).$$

- **Step 4.** In this step we bound the error $\|u^{\varepsilon} - \tilde{u}^{\varepsilon}\|_{\mathcal{H}} + |p^{\varepsilon} - \tilde{p}^{\varepsilon}|_{1/2,\Gamma^{\varepsilon}}$. The proof is based on the idea that the approximate field satisfies the same problem as $(u^{\varepsilon}, p^{\varepsilon})$ with a small perturbation in the source term. In this step we will use the stability result (3.1).

We just want to detail Steps 1, 2 and 3. We admit the results of Step 4, and refer to Ref. 3 for the precise and rigorous asymptotic analysis.

4.2. *Step* 1: *The far field*

We are first interested in the behaviour of u^{ε} far from the wire. Let us define the far field region $\mathcal{Z}_{\mathrm{f}}^{\varepsilon} \subset B_R$ by

$$\mathcal{Z}_{\mathrm{f}}^{\varepsilon} = \{\mathbf{x}(\xi,\nu,\theta) \in B_R \,|\, \xi^2 - 1 > \varepsilon\}.$$

We say that a point $\mathbf{x}$ is far from the wire Γ^{ε} when $\mathbf{x} \in \mathcal{Z}_{\mathrm{f}}^{\varepsilon}$. Note that when ε goes to *0* the region $\mathcal{Z}_{\mathrm{f}}^{\varepsilon}$ gets closer to the set $B_R\setminus I$. We want to propose an approximate field in this region. On the basis of formal considerations, it appears reasonable to think that when ε goes to *0* the wire disappears, so that the field u^{ε} tends to the solution to the same problem with no wire. We recall that u^0 has been defined as the unique element of $\mathcal{H}$ satisfying

$$a(u^0, v) = -\int_{B_R} f\bar{v}, \quad \forall\, v \in \mathcal{H}. \tag{4.1}$$

This is the same problem as $\mathcal{P}^{\varepsilon}$ but without the obstacle Γ^{ε}. We call u^0 the incident field. We can reasonably conjecture that $\|u^{\varepsilon} - u^0\|_{\mathcal{H}} \to 0$ when $\varepsilon \to 0$, so the incident field appears like the order 0 term in an asymptotic expansion of u^{ε} with respect to ε. However, we want a sharper description of u^{ε}, because u^0 realises an approximation of u^{ε} that does not take into account the presence of the wire.

In order to define a higher-order term denoted u_1^{ε}, we look for a problem as independent of ε as possible. Note that $u^{\varepsilon} - u^0$ satisfies a homogeneous Helmholtz equation in the far field region $\mathcal{Z}_{\mathrm{f}}^{\varepsilon}$, so we impose on u_1^{ε} to satisfy the homogeneous Helmholtz equation in $B_R\setminus I$. Since $u^{\varepsilon} - u^0$ satisfy the outgoing radiation condition, we also impose this condition on u_1^{ε}. This implies that u_1^{ε} has to be singular in the neighbourhood of *I*.

Indeed suppose for a moment that u_1^{ε} is a H^1 function in a neighbourhood of *I*. Then it would be a H^1 solution of a homogeneous Helmholtz equation in a smooth domain, with homogeneous Dirichlet boundary condition on Γ^ε and outgoing radiation condition. From this we would conclude that $u_1^\varepsilon = 0$ whereas we strongly suspect that $u^\varepsilon - u^0 \neq 0$.

For this reason we assume that u_1^ε is singular in a neighbourhood of *I*, but we also conjecture that its singularity is of the weakest possible kind. We conjecture that u_1^ε is a field radiated by a lineic repartition of source points located on *I*, and define it by

$$u_1^\varepsilon(\mathbf{x}) = -\int_I a^\varepsilon(\mathbf{x}')G(\mathbf{x},\mathbf{x}')d\mathbf{x}', \quad \mathbf{x} \in B_R\backslash I. \tag{4.2}$$

In this expression we have not defined a^ε yet. We postpone it to Step 3, and only assume (which admittedly is a purely empiric decision) that $a^\varepsilon \in L^2(I)$. This leaves an indeterminacy on the definition of u_1^ε, which will be cured applying the matching principle in Step 3. So we conjecture that

$$u^\varepsilon = u^0 + u_1^\varepsilon + O(|\ln\varepsilon|\sqrt{\varepsilon}) \text{ in } H^1(\omega), \quad \forall\, \omega \subset \overline{\omega} \subset B_R\backslash I.$$

Finally we present the first-order terms of the radial expansion of u_1^ε in the neighbourhood of the origin segment *I*. Here appears clearly one of the advantages of using ellipsoidal coordinates: this allows a simple description of this expansion. Indeed if (ξ,ν,φ) are the ellipsoidal coordinates of $\mathbf{x}$ then,

$$\begin{aligned} u_1^\varepsilon(\mathbf{x}) = {} & \frac{a^\varepsilon(\nu)}{4\pi}\ln\left(\frac{4}{\xi^2-1}\right) + \int_{-1}^{+1}\frac{a^\varepsilon(z)-a^\varepsilon(\nu)}{4\pi|z-\nu|}\,dz \\ & + \int_{-1}^{+1} a^\varepsilon(z)\frac{e^{ik|\nu-z|}-1}{4\pi|\nu-z|}\,dz + \underset{\xi\to 1_+}{O}(\sqrt{\xi^2-1}). \end{aligned} \tag{4.3}$$

We only wrote this radial expansion (with respect to ξ^2-1) up to the constant terms. With this formula we see that the singularity of u_1^ε is logarithmic.

4.3. *Step 2: The near field*

Now we look for an approximation of u^ε in a region located close to the wire. We call this region the near field region and define it by

$$\mathcal{Z}_{\mathrm{n}}^\varepsilon = \{\mathbf{x}(\xi,\nu,\theta) \in B_R \,|\, \xi^2 - 1 < 2\varepsilon\}.$$

This region becomes smaller and smaller as ε goes to 0, so we use a change of coordinates in order to get back to a normalised domain. We introduce a new radial coordinate defined by

$$\rho = \frac{\sqrt{\xi^2-1}}{\varepsilon} \in [0,+\infty[. \tag{4.4}$$

This change of coordinate looks complicated, but it appears natural when one notice that the equation of the wire with this new coordinate becomes $(\Gamma^\varepsilon) : \rho = \Phi(\nu,\varphi)$, which is quite simple. Also interesting is the expression of the Laplace operator with this new coordinate,

$$(\rho^2 + 1 - \nu^2)\Delta = \frac{1}{\varepsilon^2}\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2}\right)$$
$$+\rho\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + \rho\frac{\partial}{\partial\rho} + \frac{\partial}{\partial\nu}(1-\nu^2)\frac{\partial}{\partial\nu} + \frac{1}{1-\nu^2}\frac{\partial^2}{\partial\varphi^2}.$$

We see that the leading part (in the sense that it is the one associated with the lowest power of ε) is $\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2}$. It looks like a two-dimensional Laplace operator expressed with polar coordinates. This raises two remarks. First ρ and φ are not polar coordinates here. However, using a geometric mapping, it might be possible to reinterpret (ρ, φ) as polar coordinates. The other remark is that this principal part does not contain any operator related to ν, so that it cannot lead to a well-posed three-dimensional problem with traditional boundary conditions.

We will now propose an approximate field denoted U_1^ε. We will express it using the coordinates (ρ, ν, φ). First in the near field region the function u^ε satisfies a homogeneous Helmholtz equation, hence U_1^ε should satisfy a homogeneous problem associated with the principal part of the Helmholtz operator $\Delta + k^2$ expressed in coordinates (ρ, ν, φ) (its principal part is the same as for the Laplace operator):

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial U_1^\varepsilon}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2 U_1^\varepsilon}{\partial\varphi^2} = 0. \tag{4.5}$$

We do not impose that U_1^ε satisfy a Helmholtz equation because this would not lead to a problem independent of ε. In order to deal with a standard problem we look at this equation for each $\nu \in [-1, +1]$. Thus we introduce a function $\widehat{U}^\varepsilon_{1,\nu}(\mathbf{x}_\perp) = U_1^\varepsilon(\mathbf{x}_\perp, \nu)$ indexed by the coordinate ν. As a consequence, we are led to a description of the geometry in "normalised slices". Indeed we consider two-dimensional domains $\omega_{\mathrm{n}}(\nu) = \{\mathbf{x}_\perp(\rho, \varphi) \in \mathbb{R}^2 \,|\, \rho > \Phi(\varphi, \nu)\}$ (the slices) that are indexed by ν. Here $\mathbf{x}_\perp$ represents a point in $\mathbb{R}^2$ with polar coordinates (ρ, φ). Then we reinterpret Eq. (4.5) as a collection (indexed by ν) of homogeneous Laplace problems where (ρ, φ) are polar coordinates. Since u^ε satisfies a homogeneous Dirichlet boundary condition on Γ^ε we impose the following Dirichlet boundary conditions for each $\widehat{U}^\varepsilon_{1,\nu}$,

$$\widehat{U}^\varepsilon_{1,\nu} = 0 \quad \text{on } \partial\omega_{\mathrm{n}}(\nu), \quad \forall\, \nu \in [-1, 1].$$

Now if we assume that $\widehat{U}^\varepsilon_{1,\nu}$ (for each ν) is bounded at infinity (the standard condition at infinity for Laplace problems), this implies that $\widehat{U}^\varepsilon_{1,\nu} = 0$ and, for the same reason as in the section on the far field, this is not satisfying. As a consequence we impose on $\widehat{U}^\varepsilon_{1,\nu}$ to have a growing behaviour at infinity. Since the singularity of the far field is logarithmic, we choose a logarithmic growth for the near field, in order to compensate the singularity of the far field, when applying the matching principle in Step 3. In order to define the near field, we introduce a reference solution of such a

Laplace problem with logarithmic growth at infinity with the following lemma. The proof of it is quite classical (see for example Sec. I.3 in Part 2 of Ref. 17).

Lemma 4.1. *For any* $\nu \in [-1,1]$ *there exists a unique couple* $(\widehat{V}_\nu, \widehat{V}_{0,\nu}) \in H^1_{\mathrm{loc}}(\omega_{\mathrm{n}}(\nu)) \times \mathbb{C}$ *satisfying*

$$\begin{cases} \Delta_\perp \widehat{V}_\nu = 0 & \text{in } \omega_{\mathrm{n}}(\nu), \\ \widehat{V}_\nu = 0 & \text{on } \partial\omega_{\mathrm{n}}(\nu), \\ \lim\limits_{\rho\to+\infty} \sup\limits_{\varphi\in[0,2\pi]} |\,\widehat{V}_\nu(\rho,\varphi) - \widehat{V}_{0,\nu} - \ln\rho\,| = 0. \end{cases}$$

Here we use the notation $\Delta_\perp$ in order to refer to the two-dimensional Laplace operator. The function $\widehat{V}_\nu$ will be useful for the definition of an averaging operator in Sec. 5.1, and the definition of a reduced space of Lagrange multipliers. The constant $\widehat{V}_{0,\nu}$ appearing in this lemma is called the capacity of the obstacle $\omega_{\mathrm{n}}(\nu)$; it will also appear in the equations of the matching principle (Step 3). Besides, note that for $|\nu| > \nu_0$, $\widehat{V}_\nu(\rho,\varphi) = \ln(\rho/\Phi(\nu))$. Taking into account this remark and Assumption A3 on Φ, it is easy to prove that $\widehat{V}_\nu(\rho,\varphi)$ is C^∞ as a function of the three variables. Because of the unicity part of Lemma 4.1 we choose $\widehat{U}^\varepsilon_{1,\nu}$ to be proportional to $\widehat{V}_\nu$. We define it by $\widehat{U}^\varepsilon_{1,\nu} = b^\varepsilon(\nu)\widehat{V}_\nu$, where $b^\varepsilon(\nu)$ is one-dimensional function that is to be defined in Step 3 when applying the matching principle. We define

$$U^\varepsilon_1(\rho,\nu,\varphi) = \widehat{U}^\varepsilon_{1,\nu}(\rho,\varphi) = b^\varepsilon(\nu)\widehat{V}_\nu(\rho,\varphi), \quad \nu \in [-1,1]. \tag{4.6}$$

From now on we will consider that $U^\varepsilon_1 = 0$ in Ω_ε so that $U^\varepsilon_1 \in \mathcal{H}$ since it satisfies a homogeneous Dirichlet boundary condition on Γ^ε. Finally, applying classical separation of variables techniques in order to derive the expansion of $\widehat{V}_{0,\nu}$ when $\rho \to \infty$, we see that

$$U^\varepsilon_1(\rho,\nu,\varphi) = b^\varepsilon(\nu)\ln(\rho) + b^\varepsilon(\nu)\widehat{V}_{0,\nu} + \underset{\rho\to\infty}{O}(1/\rho). \tag{4.7}$$

4.4. *Step 3: Matching principle*

So far there has remained an indeterminacy in the definition of the far field (because a^ε is still not defined) and the near field (because b^ε is still not defined). In order to remove this indeterminacy, we impose

$$u^0(\mathbf{x}) + u^\varepsilon_1(\mathbf{x}) - U^\varepsilon_1(\mathbf{x}) \underset{\varepsilon\to0}{\to} 0, \quad \forall\,\mathbf{x} \in \mathcal{Z}^\varepsilon_{\mathrm{f}} \cap \mathcal{Z}^\varepsilon_{\mathrm{n}}.$$

More precisely we identify the first two terms of their radial expansion. For a point $\mathbf{x} \in \mathcal{Z}^\varepsilon_{\mathrm{f}} \cap \mathcal{Z}^\varepsilon_{\mathrm{n}}$ with ellipsoidal coordinates (ξ,ν,φ) and modified coordinates (ρ,ν,φ) we have $\xi \to 1$ and $\rho \to +\infty$, so we use the radial expansion of $u^0 + u^\varepsilon_1$ in the neighbourhood of $I\,(\xi \to 1)$ and use the radial expansion of U^ε_1 in the neighbourhood of infinity $(\rho \to +\infty)$. Denote $\tilde{u}^0(\nu) = u^0\,(\xi = 1,\nu,\varphi)$, indeed it does not depend on φ.

Taking into acount Expansion (4.3) and (4.7), we are led to

$$\tilde{u}^0(\nu) + \frac{a^\varepsilon(\nu)}{4\pi}\ln\left(\frac{4}{\xi^2-1}\right) + \int_{-1}^{+1}\frac{a^\varepsilon(z)-a^\varepsilon(\nu)}{4\pi|z-\nu|}\,dz + \int_{-1}^{+1} a^\varepsilon(z)\frac{e^{ik|\nu-z|}-1}{4\pi|\nu-z|}\,dz$$
$$-\frac{b^\varepsilon(\nu)}{2}\ln\left(\frac{\xi^2-1}{\varepsilon^2}\right) - b^\varepsilon(\nu)\,\widehat{V}_{0,\nu} = 0$$

which yields a pair of equations involving only a^ε and b^ε. The first one is obtained identifying the coefficients of $\ln(\xi^2-1)$,

$$\frac{a^\varepsilon}{4\pi} + \frac{b^\varepsilon}{2} = 0. \tag{4.8}$$

The second equation is obtained identifying the terms independent of ξ. We can get rid of the first equation replacing b^ε by $-a^\varepsilon/2\pi$ in the second equation. The second equation takes the form,

$$\text{Find } a^\varepsilon \in L^2(I) \text{ such that } \left(\ln\frac{1}{\varepsilon^2}\,\mathrm{Id} + A + B\right)a^\varepsilon = -4\pi\,\tilde{u}^0, \tag{4.9}$$

where

$$\begin{cases} (Au)(\nu) = \displaystyle\int_{-1}^{+1}\frac{u(z)-u(\nu)}{|z-\nu|}\,dz, \\ B = B_1 + B_2 \quad \text{with}, \\ \qquad (B_1u)(\nu) = (\ln(4) + 2\,\widehat{V}_{0,\nu})\,u(\nu), \\ \qquad (B_2u)(\nu) = \displaystyle\int_{-1}^{+1} u(z)\frac{e^{ik|\nu-z|}-1}{|\nu-z|}\,dz. \end{cases}$$

We call (4.9) the matching equation. It is a one-dimensional integral equation composed with bounded linear operators acting from $L^2(I) \to L^2(I)$ except for A. Indeed one can prove that B_1 and B_2 are bounded operators whereas A is unbounded as an operator from $L^2(I) \to L^2(I)$. The next lemma provides an explicit diagonalisation result for A which is a direct way to describe $\ln(\frac{1}{\varepsilon^2})\,\mathrm{Id} + A$ (for a proof of this lemma see Theorem 1.1 in Ref. 7).

Lemma 4.2. *Consider $(P_n)_{n\in\mathbb{N}}$ the Legendre polynomials and for each n let $W_n = -\sum_{k=1}^{n}\frac{1}{k}$. The operator A is diagonalised by the Legendre polynomials*

$$AP_n = 2W_n\,P_n, \quad \forall\, n \in \mathbb{N}.$$

We remind the reader that the set of Legendre polynomials P_n is a Hilbertian orthogonal basis for $L^2(I)$, see Chap. 4 in Ref. 10 for a detailed description of these polynomials. With this result we know exactly the spectrum of A, $\sigma(A) = \{2W_n\}_{n\in\mathbb{N}}$. If we write $\lambda_n = \exp(W_n)$, then $\lambda_n \sim e^{-\gamma}/n$ when $n \to +\infty$ and $\ln\frac{1}{\varepsilon^2}\,\mathrm{Id} + A$ admits $\ln(\lambda_n/\varepsilon)^2$ as eigenvalues. This is a problem if one wants to prove the well-posedness of Eq. (4.9). A solution consists in regularising $\ln\frac{1}{\varepsilon^2}\,\mathrm{Id} + A$. Another consequence of Lemma 4.2 is that a convenient functional setting should be adapted to Legendre polynomials. This is why we introduce particular one-dimensional spaces. For any $r \in \mathbb{R}$ let us denote

$$E^r(I) = \left\{ v \in L^2(I) \,|\, \|v\|^2_{E^r(I)} = \sum_{l=0}^{+\infty} (1+l^2)^{r+1/2} \,|\langle v, P_l\rangle_{L^2(I)}|^2 < +\infty \right\}.$$

The spaces $E^{1/2}(I)$ and $E^{-1/2}(I)$ will be of particular interest in the next section. Later we will establish a link between $\| \ \|_{E^{1/2}(I)}$ and $| \ |_{1/2,\Gamma^\varepsilon}$ defined by (3.1). Standard Hilbertian theory shows that the following identity holds

$$\|v\|_{E^{-1/2}(I)} = \sup_{w \in E^{1/2}(I)} \frac{\langle v, w\rangle_{L^2(I)}}{\|w\|_{E^{1/2}(I)}} \quad \forall\, v \in E^{-1/2}(I).$$

The following lemma provides a solution to a regularised version of Eq. (4.9), and express estimates in terms of the norms we have introduced.

Lemma 4.3. *Given $\tilde{u}^0 \in C^\infty(I)$, $\tilde{u}^0 \neq 0$, there exists two functions $\tau^\varepsilon(\tilde{u}^0)$ and a^ε belonging to $\cap_{n\in\mathbb{N}} E^n(I)$ such that*

$$\left(\ln\frac{1}{\varepsilon^2}\,\mathrm{Id} + A + B\right) a^\varepsilon = -4\pi\,\tilde{u}^0 + \tau^\varepsilon(\tilde{u}^0),$$

with estimates on a^ε and τ^ε : for any $r, s > 0$ there exist κ_1, κ_2 and $\kappa_3 > 0$ independent of ε such that,

$$\frac{\kappa_1}{|\ln\varepsilon|} \leq \|a^\varepsilon\|_{E^r(I)} \leq \frac{\kappa_2}{|\ln\varepsilon|} \quad \textit{and} \quad \|\tau^\varepsilon(\tilde{u}^0)\|_{E^r(I)} \leq \kappa_3 \varepsilon^s \quad \forall\, \varepsilon \in\,]0,1[.$$

For technical reasons, we have to assume that $\tilde{u}^0 \neq 0$ which is a bit restrictive. We do not give any proof for this lemma and refer to Lemma 5.4 in Ref. 3 (see also Lemma 2.2 in Ref. 7). This lemma definitely concludes the construction of the first term of an asymptotic expansion of u^ε. Indeed with Lemma 4.3 the function a^ε is well defined, and this leads at the same time to the definition of $b^\varepsilon = -a^\varepsilon/2\pi$.

4.5. *Step 4: Approximate field and error estimates*

There only remains to propose a relevant approximate field using the far field and the near field. First of all we define the far field u_1^ε and the near field U_1^ε according to Eqs. (4.2) and (4.6). Then we proceed in a very classical way considering a C^∞ cutoff function $\chi : \mathbb{R} \to [0,1]$ that we choose to be decreasing and such that

$$\chi(x) = \begin{cases} 0 & \text{for } x \geq 1 \\ 1 & \text{for } x \leq 0 \end{cases} \quad \text{and set } \chi^\varepsilon(\mathbf{x}) = \chi\left(\frac{\xi^2 - 1 - \varepsilon}{\varepsilon}\right).$$

Note that $\chi^\varepsilon = 1$ in $\mathcal{Z}_\mathrm{n}^\varepsilon \backslash \mathcal{Z}_\mathrm{n}^\varepsilon \cap \mathcal{Z}_\mathrm{f}^\varepsilon$, and $\chi^\varepsilon = 0$ in the far field zone $\mathcal{Z}_\mathrm{f}^\varepsilon \backslash \mathcal{Z}_\mathrm{n}^\varepsilon \cap \mathcal{Z}_\mathrm{f}^\varepsilon$. We simply define the approximate field $\tilde{u}^\varepsilon$ and $\tilde{p}^\varepsilon$ by

$$\begin{aligned} &\tilde{u}^\varepsilon(\mathbf{x}) = (1-\chi^\varepsilon)(u^0(\mathbf{x}) + u_1^\varepsilon(\mathbf{x})) + \chi^\varepsilon U_1^\varepsilon(\mathbf{x}), \quad \mathbf{x} \in B_R \\ &\text{and} \quad \tilde{p}^\varepsilon = \frac{\partial}{\partial n}(\tilde{u}^\varepsilon_{|B_R\backslash\overline{\Omega}_\varepsilon}) - \frac{\partial}{\partial n}(\tilde{u}^\varepsilon_{|\Omega_\varepsilon}) = \frac{\partial U_1^\varepsilon}{\partial n}. \end{aligned} \tag{4.10}$$

Taking into account our preceding remarks on the cutoff function, we see that $\tilde{u}^\varepsilon = u^0 + u_1^\varepsilon$ in $\mathcal{Z}_\mathrm{f}^\varepsilon \backslash \mathcal{Z}_\mathrm{n}^\varepsilon \cap \mathcal{Z}_\mathrm{f}^\varepsilon$ and $\tilde{u}^\varepsilon = U_1^\varepsilon$ in $\mathcal{Z}_\mathrm{n}^\varepsilon \backslash \mathcal{Z}_\mathrm{n}^\varepsilon \cap \mathcal{Z}_\mathrm{f}^\varepsilon$. Note that the couple $(\tilde{u}^\varepsilon, \tilde{p}^\varepsilon)$ satisfies a perturbed version of problem $\mathcal{P}^\varepsilon$. Indeed there exists $\tilde{f}^\varepsilon \in \mathcal{H}'$ such that

$$\begin{cases} a(\tilde{u}^\varepsilon, v) + \displaystyle\int_{\Gamma^\varepsilon} \tilde{p}^\varepsilon \overline{v} = -\int_{B_R} f\overline{v} + \langle \tilde{f}^\varepsilon, v\rangle_{\mathcal{H}',\mathcal{H}}, & \forall\, v \in \mathcal{H}, \\ \displaystyle\int_{\Gamma^\varepsilon} \tilde{u}^\varepsilon q = 0, & \forall\, q \in H^{-1/2}(\Gamma^\varepsilon). \end{cases}$$

According to Theorem 3.1, it is sufficient to establish an estimate for $\|\tilde{f}^\varepsilon\|_{\mathcal{H}'}$ in order to bound the errors $\|u^\varepsilon - \tilde{u}^\varepsilon\|_{\mathcal{H}}$ and $|p^\varepsilon - \tilde{p}^\varepsilon|_{-1/2,\Gamma^\varepsilon}$. Moreover one aim of the construction of Steps 1, 2 and 3 has been to minimise $\|\tilde{f}^\varepsilon\|_{\mathcal{H}'}$. Indeed this quantity can be subdivided into two main parts. One part comes from the fact that the near field does not exactly satisfies an Helmholtz equation (in the near field region it rather satisfies a transverse Laplace equation). The other part comes from the difference between the far field and the near field in the matching zone $\mathcal{Z}_\mathrm{f}^\varepsilon \cap \mathcal{Z}_\mathrm{n}^\varepsilon$ (but the difference is very small because the matching equation (4.9) is nearly satisfied). The preceding considerations enable to bound $\|\tilde{f}^\varepsilon\|_{\mathcal{H}'}$ as follows.

Proposition 4.1. *There exists $\kappa > 0$ independent of ε such that $\|\tilde{f}^\varepsilon\|_{\mathcal{H}'} \leq \kappa|\ln\varepsilon|\sqrt{\varepsilon}$, $\forall\,\varepsilon \in\ ]0,1[$.*

For the proof of this proposition, we refer the reader to Theorem 7.1 in Ref. 3. Combining the variational formulation of problem $\mathcal{P}^\varepsilon$ given in Sec. 3 and the stability result of Theorem 3.1, Proposition 4.1 leads straightforwardly to the following theorem.

Theorem 4.1. *There exists $\kappa > 0$ independent of ε such that the following error estimate holds between $(u^\varepsilon, p^\varepsilon)$ solution of $(\mathcal{P}^\varepsilon)$ and $(\tilde{u}^\varepsilon, \tilde{p}^\varepsilon)$ defined by (4.10),*

$$\|u^\varepsilon - \tilde{u}^\varepsilon\|_{\mathcal{H}} \leq \kappa\sqrt{\varepsilon}\,|\ln\varepsilon| \quad \textit{and} \quad |p^\varepsilon - \tilde{p}^\varepsilon|_{-1/2,\Gamma^\varepsilon} \leq \kappa\sqrt{\varepsilon}\,|\ln\varepsilon| \quad \forall\,\varepsilon \in\ ]0,1[.$$

Of course the difficult part for the justification of this theorem is Proposition 4.1. As we said before we do not give any proof for this result because it is quite long and it is not the central subject of the present paper. The estimate is optimal for $\|u^\varepsilon - \tilde{u}^\varepsilon\|_{\mathcal{H}}$, but for the second estimate we believe (no proof is available at present) that there exists $\kappa > 0$ such that $|p^\varepsilon - \tilde{p}^\varepsilon|_{-1/2,\Gamma^\varepsilon} \leq \kappa\,\varepsilon$. Theorem 4.1 clearly establishes that $\tilde{u}^\varepsilon$ is the first term of an asymptotic expansion of u^ε with respect to ε. Indeed Cauchy−Schwarz inequality applied to (4.2) yields

$$\|u_1^\varepsilon\|_{L^2(B_R)}^2 \leq \|a^\varepsilon\|_{L^2(I)}^2 \int_{B_R}\int_I |G(\mathbf{x},\mathbf{x}')|^2 d\mathbf{x}d\mathbf{x}'.$$

Taking into account Lemma 4.3, we are led to the existence of $\kappa > 0$ independent of ε such that

$$\|u_1^\varepsilon\|_{L^2(B_R)} \leq \frac{\kappa}{|\ln\varepsilon|} \quad \forall\,\varepsilon \in\ ]0,1[.$$

Choose an arbitrary open set $\mathcal{O} \subset B_R$, $\mathcal{O} \neq \emptyset$ and such that $\overline{\mathcal{O}} \cap I = \emptyset$. As a consequence of the preceding remarks, there exists ε_0 (independent of ε) such that

$$\|\tilde{u}^\varepsilon\|_{\mathcal{H}} > \|\tilde{u}^\varepsilon\|_{L^2(\mathcal{O})} > \|u^0\|_{L^2(\mathcal{O})} - \|u_1^\varepsilon\|_{L^2(B_R)} > \|u^0\|_{L^2(\mathcal{O})}/2 \quad \forall\, \varepsilon \in\ ]0, \varepsilon_0[.$$

Since $\|u^\varepsilon\|_{\mathcal{H}} = \|\tilde{u}^\varepsilon\|_{\mathcal{H}} + O(\varepsilon^{1/3})$ according to Theorem 4.1, this means that the error bound given in Theorem 4.1 is actually relative: there exists $\varepsilon_0, \kappa_0 > 0$ independent of ε such that

$$\frac{\|u^\varepsilon - \tilde{u}^\varepsilon\|_{\mathcal{H}}}{\|u^\varepsilon\|_{\mathcal{H}}} \leq \kappa_0 \sqrt{\varepsilon}\, |\ln \varepsilon|, \quad \forall\, \varepsilon \in\ ]0, \varepsilon_0[.$$

From this point of view, the situation does not appear so clear for p^ε, because we have no information about $|p^\varepsilon|_{-1/2,\Gamma^\varepsilon}$ or $|\tilde{p}^\varepsilon|_{-1/2,\Gamma^\varepsilon}$, but we know only about $|p^\varepsilon - \tilde{p}^\varepsilon|_{-1/2,\Gamma^\varepsilon}$. This will be clarified in Sec. 5.2 by finding the behaviour of $|p^\varepsilon|_{-1/2,\Gamma^\varepsilon}$.

5. One-Dimensional Wire Model

In this section we introduce a new simplified problem. Its solution, denoted $(\mathfrak{u}^\varepsilon, \mathfrak{p}^\varepsilon)$, is to be a good approximation for $(u^\varepsilon, p^\varepsilon)$. The idea consists in taking a weaker condition on Γ^ε than homogeneous Dirichlet condition. We will impose an "averaged Dirichlet condition": we will only impose that the mean value of the solution on each section of Γ^ε has to be 0. In order to impose such a condition we first introduce an averaging operator, then we define the new simplified problem and finally we provide error estimates for the difference between $(u^\varepsilon, p^\varepsilon)$ and $(\mathfrak{u}^\varepsilon, \mathfrak{p}^\varepsilon)$.

5.1. *The averaging operator*

In order to define such an operator, we first need to introduce some notation. We introduce the density $\gamma^\varepsilon(\nu, \varphi)$ of the surface measure on Γ^ε defined by the relation $d\sigma = \gamma^\varepsilon(\nu, \varphi)\, d\nu\, d\varphi$. Moreover we define a function V^ε by $V^\varepsilon \circ \phi_{\mathrm{el}}^{-1}(\xi, \nu, \varphi) = \widehat{V}_\nu(\sqrt{\xi^2 - 1}/\varepsilon, \varphi)$ and remind the reader of the relation $\sqrt{\xi^2 - 1}/\varepsilon = \rho$. We need also a technical lemma proved in Appendix.

Lemma 5.1. *Let n be the unit normal vector to Γ^ε directed into the exterior of Ω_ε. There exists $\kappa > 0$ independent of ε such that*

$$\left| \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial V^\varepsilon}{\partial n} \gamma^\varepsilon\, d\varphi - 1 \right| < \kappa\, \varepsilon \quad \forall\, \varepsilon \in\ ]0, 1[.$$

As a consequence of this result we see that for ε small enough $\gamma_\mu^\varepsilon(\nu) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial V^\varepsilon}{\partial n} \gamma^\varepsilon(\nu, \varphi) d\varphi \neq 0$. In the Appendix we prove a stronger result stating that all the derivatives of γ_μ^ε remain bounded as $\varepsilon \to 0$. Note that $\gamma_\mu^\varepsilon \in C^\infty(I)$ since $\widehat{V}_\nu(\rho, \varphi)$ is smooth, as was underlined at the end of Sec. 4.3. Now take a $v \in C^\infty(\Gamma^\varepsilon)$ and define its image by the averaging operator

$$\mu^\varepsilon[v](\nu) = \frac{\int_0^{2\pi} v \circ \phi_{\Gamma^\varepsilon}^{-1}(\nu, \varphi)\, \frac{\partial V^\varepsilon}{\partial n} \circ \phi_{\Gamma^\varepsilon}^{-1}(\nu, \varphi)\, \gamma^\varepsilon(\nu, \varphi)\, d\varphi}{\int_0^{2\pi} \frac{\partial V^\varepsilon}{\partial n} \circ \phi_{\Gamma^\varepsilon}^{-1}(\nu, \varphi)\, \gamma^\varepsilon(\nu, \varphi)\, d\varphi}, \quad \nu \in I, \forall\, \varepsilon \in\ ]0, 1[.$$

It is well defined for ε small enough and belongs to $L^2(I)$. Moreover, if we take a $v \in C^\infty(\Gamma^\varepsilon)$ such that $\frac{\partial v}{\partial \varphi} = 0$ (v depends only on the variable ν) then we have $\mu^\varepsilon[v] = v$. In order to illustrate this operator, let us take an example and assume for a moment that Γ^ε is described by the equation $\xi^2 = 1 + \varepsilon^2$ which corresponds to a thin ellipsoid. In this situation, tedious but basic differential calculus shows that $\frac{\partial V^\varepsilon}{\partial n}\gamma^\varepsilon = \sqrt{1+\varepsilon^2}$, so that $\mu^\varepsilon[v](\nu) = \frac{1}{2\pi}\int_0^{2\pi} v \circ \phi_{\Gamma^\varepsilon}^{-1}(\nu,\varphi)d\varphi$.

Back to the general case, using Cauchy−Schwarz inequality, one can check easily that μ^ε maps continuously $H^{1/2}(\Gamma^\varepsilon)$ into $L^2(I)$. Denote its image $\mathcal{M}^\varepsilon = \operatorname{Im}\mu^\varepsilon$. The space $\mathcal{M}^\varepsilon$ is one-dimensional, and we wish to use it as a reduced space of Lagrange multipliers. Besides, we have already introduced one-dimensional spaces $E^r(I)$, so a natural question consists in determining whether there exists a relationship between $\mathcal{M}^\varepsilon$ and $E^r(I)$. This is the subject of the next lemma.

Lemma 5.2.

$$\mathcal{M}^\varepsilon = E^{1/2}(I).$$

Proof. We first show that $\mathcal{M}^\varepsilon \subset E^{1/2}(I)$ and for this purpose we will use the correspondence between Γ^ε and S^2 that we introduced in Lemma 2.1. The function $\gamma^\varepsilon \frac{\partial V^\varepsilon}{\partial n}$ is of class C^∞ on Γ^ε so that if $v \in H^{1/2}(\Gamma^\varepsilon)$, then $v\gamma^\varepsilon \frac{\partial V}{\partial n} \in H^{1/2}(\Gamma^\varepsilon)$. Moreover, classical results of decomposition on spherical harmonics tell us that if $w \in H^{1/2}(S^2)$ then the function $\nu \mapsto \int_0^{2\pi} w \circ \phi_{\Gamma^\varepsilon}^{-1}(\nu,\varphi)\, d\varphi$ belongs to $E^{1/2}(I)$. Finally recall that in Lemma 2.1 a continuous isomorphism $\Psi_* : H^{1/2}(\Gamma^\varepsilon) \to H^{1/2}(S^2)$ was introduced, and for $v \in H^{1/2}(\Gamma^\varepsilon)$, $\mu^\varepsilon[v](\nu) = \int_0^{2\pi}(\Psi_*(v\,\frac{\partial V}{\partial n})) \circ \phi_{S^2}^{-1}\,\gamma^\varepsilon\, d\varphi$, so according to the preceding remarks $\mu^\varepsilon[v] \in E^{1/2}(I)$. This gives the intermediate result,

$$\mathcal{M}^\varepsilon \subset E^{1/2}(I).$$

On the other hand, take a function $v \in E^{1/2}(I)$. Then using characterisation via spherical harmonics, we see that $v \circ \phi_{S^2} \in H^{1/2}(S^2)$ hence, using Lemma 2.1, $v \circ \phi_{\Gamma^\varepsilon} \in H^{1/2}(\Gamma^\varepsilon)$. Note that $v \circ \phi_{\Gamma^\varepsilon}$ does not depend on φ so $v = \mu^\varepsilon[v \circ \phi_{\Gamma^\varepsilon}] \in \mathcal{M}^\varepsilon$. From this we finally obtain the opposite inclusion, $E^{1/2}(I) \subset \mathcal{M}^\varepsilon$. □

In the proof of the preceding lemma, we saw that for a $v \in E^{1/2}(I)$ it is possible to define

$$\sigma^\varepsilon[v] = v \circ \phi_{\Gamma^\varepsilon} \in H^{1/2}(\Gamma^\varepsilon) \quad \text{such that } \mu^\varepsilon \circ \sigma^\varepsilon[v] = v.$$

In other words, $\sigma^\varepsilon : E^{1/2}(I) \to H^{1/2}(\Gamma^\varepsilon)$ and $\mu^\varepsilon \circ \sigma^\varepsilon = \mathrm{Id}_{E^{1/2}(I)}$. As a consequence $\sigma^\varepsilon \circ \mu^\varepsilon : H^{1/2}(\Gamma^\varepsilon) \to H^{1/2}(\Gamma^\varepsilon)$ is a projection. Moreover, the identification of Lemma 5.2 leads to $\|\ \|_{E^{1/2}(I)}$ as a natural norm for $\mathcal{M}^\varepsilon$. We will also consider the transpose map ${}^t\mu^\varepsilon$ defined by

$$\langle u, \mu^\varepsilon[v]\rangle_{E^{-1/2},E^{1/2}} = \langle\, {}^t\mu^\varepsilon[u], v\rangle_{H^{-1/2},H^{1/2}}$$

and ${}^t\sigma^\varepsilon$ defined by $\langle u, \sigma^\varepsilon[v]\rangle_{H^{-1/2},H^{1/2}} = \langle\, {}^t\sigma^\varepsilon[u], v\rangle_{E^{-1/2},E^{1/2}}$. Here again we give an example in a simple case. Suppose anew that $(\Gamma^\varepsilon) : \xi^2 = 1+\varepsilon^2$. In this case we have ${}^t\mu^\varepsilon[v](\nu,\varphi) = \frac{v(\nu)}{2\pi\varepsilon\sqrt{\varepsilon^2+1-\nu^2}}$. The next result shows that the norm we chose for $H^{1/2}(\Gamma^\varepsilon)$ is convenient for dealing with such operators. Indeed μ^ε and σ^ε are bounded uniformly with respect to ε when using $|\ |_{1/2,\Gamma^\varepsilon}$.

Lemma 5.3. *There exists $\kappa > 0$ independent of ε such that for any $\varepsilon \in\ ]0,1[$*

(i) $\forall\, v \in H^{1/2}(\Gamma^\varepsilon) \qquad \|\mu^\varepsilon[v]\|_{E^{1/2}(I)} \leq\ \kappa|v|_{1/2,\Gamma^\varepsilon}$,

(ii) $\forall\, v \in E^{1/2}(I) \qquad \|v\|_{E^{1/2}(I)} = |\sigma^\varepsilon[v]|_{1/2,\Gamma^\varepsilon}$,

(iii) $\forall\, q \in H^{-1/2}(\Gamma^\varepsilon) \qquad \|{}^t\sigma^\varepsilon[q]\|_{E^{-1/2}(I)} \leq\ \kappa|q|_{-1/2,\Gamma^\varepsilon}$,

(iv) $\forall\, q \in E^{-1/2}(I) \qquad \|q\|_{E^{-1/2}(I)} \leq |{}^t\mu^\varepsilon[q]|_{-1/2,\Gamma^\varepsilon} \leq \kappa\|q\|_{E^{-1/2}(I)}$.

Proof. Recall that we defined $\gamma^\varepsilon_\mu(\nu) = \frac{1}{2\pi}\int_0^{2\pi}\frac{\partial V^\varepsilon}{\partial n}\gamma^\varepsilon\, d\varphi$. In order to prove the first point (i), we start by writing the explicit expression

$$\|\mu^\varepsilon[u]\|^2_{E^{1/2}(I)} = \sum_{n=0}^{+\infty}(1+n^2)\left(\int_{-1}^{+1}\!\!\int_0^{2\pi}\left(u\,\frac{\partial V}{\partial n}\,\frac{\gamma^\varepsilon}{\gamma^\varepsilon_\mu}\right)\circ\phi^{-1}_{\Gamma^\varepsilon}(\nu,\varphi)\,P_n(\nu)\,d\varphi\,d\nu\right)^2.$$

Note that this expression looks like a norm on the sphere S^2. Indeed,

$$\begin{aligned}&\left\|\Psi_*\left(u\,\frac{\partial V}{\partial n}\gamma^\varepsilon\right)\right\|^2_{H^{1/2}(S^2)}\\ &\quad= \sum_{n=0}^{+\infty}\sum_{m=-n}^{n}(1+n^2)\left(\int_{-1}^{+1}\!\!\int_0^{2\pi}\left(u\,\frac{\partial V}{\partial n}\gamma^\varepsilon\right)\circ\phi^{-1}_{\Gamma^\varepsilon}(\nu,\varphi)\,P_n(\nu)\,d\varphi\,d\nu\right)^2.\end{aligned}$$

Since $\Psi_*(\frac{\partial V^\varepsilon}{\partial n}) \in C^\infty(S^2)$, and $\gamma^\varepsilon_\mu \geq \pi$ for ε small enough and γ^ε_μ admits uniformly bounded derivatives, basic interpolation results show that there exists a $\kappa > 0$ independent of ε such that

$$\left\|\Psi_*\left(u\,\frac{\partial V}{\partial n}\gamma^\varepsilon\right)\right\|^2_{H^{1/2}(S^2)} \leq \kappa\|\Psi_* u\|^2_{H^{1/2}(S^2)} = \kappa|u|^2_{1/2,\Gamma^\varepsilon}.$$

Equality (ii) is a direct consequence of the definition of $|\ |_{1/2,\Gamma^\varepsilon}$ and σ^ε. Equality (iii) is a consequence of

$$\begin{aligned}\|{}^t\sigma^\varepsilon[p]\|_{E^{-1/2}(I)} &= \sup_{v\in E^{1/2}(I)}\frac{\langle\, {}^t\sigma^\varepsilon[p], v\rangle_{E^{-1/2},E^{1/2}}}{\|v\|_{E^{1/2}(I)}} = \sup_{v\in E^{1/2}(I)}\frac{\langle p, \sigma^\varepsilon[v]\rangle_{H^{-1/2},H^{1/2}}}{|\sigma^\varepsilon[v]|_{1/2,\Gamma^\varepsilon}}\\ &\leq \sup_{v\in H^{1/2}(\Gamma^\varepsilon)}\frac{\langle p, v\rangle_{H^{-1/2},H^{1/2}}}{|v|_{1/2,\Gamma^\varepsilon}} = |p|_{-1/2,\Gamma^\varepsilon}.\end{aligned}$$

The proof of equality (iv) uses (i). First we prove directly,

$$\begin{aligned}\|p\|_{E^{-1/2}(I)} &= \sup_{v\in E^{1/2}(I)} \frac{\langle p, v\rangle_{E^{-1/2},E^{1/2}}}{\|v\|_{E^{1/2}(I)}} = \sup_{v\in E^{1/2}(I)} \frac{\langle p, \mu^\varepsilon \circ \sigma^\varepsilon[v]\rangle_{E^{-1/2},E^{1/2}}}{|\sigma^\varepsilon[v]|_{1/2,\Gamma^\varepsilon}}\\
&= \sup_{v\in E^{1/2}(I)} \frac{\langle {}^t\mu^\varepsilon[p], \sigma^\varepsilon[v]\rangle_{H^{-1/2},H^{1/2}}}{|\sigma^\varepsilon[v]|_{1/2,\Gamma^\varepsilon}}\\
&\le \sup_{v\in H^{1/2}(\Gamma^\varepsilon)} \frac{\langle {}^t\mu^\varepsilon[p], v\rangle_{H^{-1/2},H^{1/2}}}{|v|_{1/2,\Gamma^\varepsilon}} = |{}^t\mu^\varepsilon[p]|_{-1/2,\Gamma^\varepsilon}.\end{aligned}$$

We end the proof of (iv) using (i) and the fact that μ^ε is onto,

$$\begin{aligned}|{}^t\mu^\varepsilon[p]|_{-1/2,\Gamma^\varepsilon} &= \sup_{v\in H^{1/2}(\Gamma^\varepsilon)} \frac{\langle {}^t\mu^\varepsilon[p], v\rangle_{H^{-1/2},H^{1/2}}}{|v|_{1/2,\Gamma^\varepsilon}}\\
&= \sup_{v\in H^{1/2}(\Gamma^\varepsilon)} \frac{\langle p, \mu^\varepsilon[v]\rangle_{E^{-1/2},E^{1/2}}}{\|\mu^\varepsilon[v]\|_{E^{1/2}(I)}} \frac{\|\mu^\varepsilon[v]\|_{E^{1/2}(I)}}{|v|_{1/2,\Gamma^\varepsilon}}\\
&\le \kappa \sup_{v\in E^{1/2}(I)} \frac{\langle p, v\rangle_{E^{-1/2},E^{1/2}}}{\|v\|_{E^{1/2}(I)}} = \|p\|_{E^{-1/2}(I)}.\end{aligned}$$

□

5.2. *Error estimate for the jump of the normal derivative*

Using the averaging operator μ^ε and the associated bounds given by Lemma 5.3, it is now possible to prove that the error estimate of Theorem 4.1 leads to a relative error bound. Indeed, V^ε satisfies a homogeneous Dirichlet boundary condition on Γ^ε and is equal to 0 in Ω_ε, so we have

$$\tilde{p}^\varepsilon = \left[\frac{\partial \tilde{u}^\varepsilon}{\partial n}\right]\Bigg|_{\Gamma^\varepsilon} = \left[\frac{\partial b^\varepsilon}{\partial n}\right]\Bigg|_{\Gamma^\varepsilon} V^\varepsilon|_{\Gamma^\varepsilon} + b^\varepsilon|_{\Gamma^\varepsilon} \left[\frac{\partial V^\varepsilon}{\partial n}\right]\Bigg|_{\Gamma^\varepsilon} = b^\varepsilon \frac{\partial V^\varepsilon}{\partial n}\Bigg|_{\Gamma^\varepsilon}. \tag{5.1}$$

As a consequence, for $v \in H^{1/2}(\Gamma^\varepsilon)$, we see that $\langle \tilde{p}^\varepsilon, v\rangle = \int_{-1}^1 \int_0^{2\pi} v b^\varepsilon \frac{\partial V^\varepsilon}{\partial n} \gamma^\varepsilon d\varphi d\nu = \int_{-1}^1 b^\varepsilon \gamma_\mu^\varepsilon \, \mu^\varepsilon[v] \, d\nu$ which implies

$$\tilde{p}^\varepsilon = {}^t\mu^\varepsilon[b^\varepsilon \gamma_\mu^\varepsilon].$$

Using result (iv) of Lemma 5.3 and the equation $b^\varepsilon = -a^\varepsilon/2\pi$ (see Sec. 4.4) we obtain the estimates $\|a^\varepsilon \gamma_\mu^\varepsilon/2\pi\|_{E^{-1/2}(I)} \le |\tilde{p}^\varepsilon|_{-1/2,\Gamma^\varepsilon} \le \kappa \|a^\varepsilon \gamma_\mu^\varepsilon/2\pi\|_{E^{-1/2}(I)}$. Since all the derivatives of γ_μ^ε remain bounded as $\varepsilon \to 0$, there exist $\kappa_1, \kappa_2 > 0$ independent of ε such that $\kappa_1 \|a^\varepsilon\|_{E^{-1/2}(I)} \le \|a^\varepsilon \gamma_\mu^\varepsilon/2\pi\|_{E^{-1/2}(I)} \le \kappa_2 \|a^\varepsilon\|_{E^{-1/2}(I)}$. Finally Lemma 4.3 provides estimates on a^ε, so we conclude that, if $\tilde{u}^0 \ne 0$, there exist $\kappa_1, \kappa_2 > 0$ independent of ε such that

$$\frac{\kappa_1}{|\ln \varepsilon|} \le |\tilde{p}^\varepsilon|_{1/2,\Gamma^\varepsilon} \le \frac{\kappa_2}{|\ln \varepsilon|} \quad \forall\, \varepsilon \in \ ]0,1[, \tag{5.2}$$

and according to Theorem 4.1, we can conclude the same for p^ε. Moreover, this estimate justifies that $\tilde{p}^\varepsilon$ is the first term of an asymptotic expansion for p^ε.

5.3. *Simplified problem*

Now we introduce our simplified problem. It is based on a simple observation of Eq. (5.1). Note that $\tilde{p}^\varepsilon = {}^t\mu^\varepsilon[b^\varepsilon\gamma^\varepsilon_\mu] \in \mathrm{Im}\,{}^t\mu^\varepsilon$ so, taking into account (5.2) and the estimate of Theorem 4.1, it is natural to think that p^ε "nearly belongs" to $\mathrm{Im}\,{}^t\mu^\varepsilon$. Thus we take $(\mathcal{M}^\varepsilon)' = E^{-1/2}(I)$ as the space of Lagrange multipliers of a new problem,

$$(\mathbf{P}^\varepsilon)\begin{cases} \text{Find } (\mathfrak{u}^\varepsilon, \mathfrak{p}^\varepsilon) \in \mathcal{H} \times E^{-1/2}(I) \text{ such that,} \\ a(\mathfrak{u}^\varepsilon, v) + \displaystyle\int_{\Gamma^\varepsilon} {}^t\mu^\varepsilon[\mathfrak{p}^\varepsilon]\,\overline{v} = -\int_{B_R} f\overline{v}, \quad \forall\, v \in \mathcal{H}, \\ \displaystyle\int_{\Gamma^\varepsilon} {}^t\mu^\varepsilon[q]\,\overline{\mathfrak{u}}^\varepsilon = 0, \quad \forall\, q \in E^{-1/2}(I). \end{cases}$$

This time the space of Lagrange multipliers is one-dimensional. Moreover, the homogeneous Dirichlet boundary condition on Γ^ε has been replaced by the second equation of problem $\mathbf{P}^\varepsilon$ simply equivalent to $\mu^\varepsilon[\mathfrak{u}^\varepsilon] = 0$, which can be interpreted as an averaged Dirichlet condition weaker that the original Dirichlet condition.

Well-posedness and stability. This simplified problem is nevertheless still well-posed because we have reduced the space of Lagrange multipliers. Indeed consider the space $\mathcal{H}_\mu = \{v \in \mathcal{H} \mid \mu^\varepsilon[v_{|\Gamma^\varepsilon}] = 0\}$. Then it is easy to adapt the proof of Lemma 3.1 (with $\mathcal{H}^\varepsilon_0$ replaced by $\mathcal{H}^\varepsilon_\mu$) in order to obtain

Lemma 5.4. *There exists $\varepsilon_0 > 0$ independent of ε such that for any $f^\varepsilon \in \mathcal{H}'$ and for any $\varepsilon \in\,]0, \varepsilon_0[$ the problem*

$$\text{Find } v^\varepsilon_\mu \in \mathcal{H}^\varepsilon_\mu \text{ such that } a(v^\varepsilon_\mu, w) = \langle f^\varepsilon, w\rangle_{\mathcal{H}',\mathcal{H}} \quad \forall\, w \in \mathcal{H}^\varepsilon_\mu, \tag{5.3}$$

admits a unique solution $v^\varepsilon_\mu \in \mathcal{H}^\varepsilon_\mu$. Moreover, there exist $\kappa_1, \kappa_2 > 0$ independent of ε such that

$$\kappa_1\|v^\varepsilon_\mu\|_{\mathcal{H}} \le \|f^\varepsilon\|_{\mathcal{H}'} \le \kappa_2\|v^\varepsilon_\mu\|_{\mathcal{H}} \quad \forall\, \varepsilon \in\,]0, \varepsilon_0[.$$

Moreover, taking $f^\varepsilon \in \mathcal{H}'$ and $g^\varepsilon \in E^{1/2}(I)$, consider the general problem

$$\begin{cases} \text{Find } (v^\varepsilon, q^\varepsilon) \in \mathcal{H} \times E^{-1/2}(I) \text{ such that,} \\ a(v^\varepsilon, v) + \displaystyle\int_{\Gamma^\varepsilon} {}^t\mu^\varepsilon[q^\varepsilon]\overline{v} = \langle f^\varepsilon, v\rangle_{\mathcal{H}',\mathcal{H}}, \quad \forall\, v \in \mathcal{H}, \\ \displaystyle\int_{\Gamma^\varepsilon} {}^t\mu^\varepsilon[q]\,\overline{v}^\varepsilon = \langle q, g^\varepsilon\rangle_{E^{-1/2},E^{1/2}}, \quad \forall\, q \in E^{-1/2}(I). \end{cases} \tag{5.4}$$

Then using the lifting operator of Lemma 3.3 and (iv) of Lemma 5.3, it is again easy to adapt the proof of Lemma 3.1 in order to obtain the following (existence, unicity and stability) result for the simplified problem $\mathbf{P}^\varepsilon$.

Theorem 5.1. *There exists $\varepsilon_0 > 0$ independent of ε such that for any $f^\varepsilon \in \mathcal{H}'$, any $g^\varepsilon \in E^{1/2}(I)$ and $\forall\, \varepsilon \in\,]0, \varepsilon_0[$, problem (5.4) admits a unique solution $(v^\varepsilon, q^\varepsilon) \in \mathcal{H} \times E^{-1/2}(\Gamma^\varepsilon)$. Moreover, there exist two constants $\kappa_1, \kappa_2 > 0$ independent of ε*

such that

$$\|v^\varepsilon\|_{\mathcal{H}} \leq \kappa_1 \|f^\varepsilon\|_{\mathcal{H}'} + \kappa_2 \|g^\varepsilon\|_{E^{1/2}(I)} \quad \textit{and}$$
$$\|q^\varepsilon\|_{E^{-1/2}(I)} \leq \kappa_1 \|f^\varepsilon\|_{\mathcal{H}'} + \kappa_2 \|g^\varepsilon\|_{E^{1/2}(I)} \quad \forall \varepsilon \in \,]0, \varepsilon_0[.$$

As in Sec. 3, this result can be applied to problem $\mathbf{P}^\varepsilon$ in order to prove that $(\mathfrak{u}^\varepsilon, \mathfrak{p}^\varepsilon)$ remains bounded when $\varepsilon \to 0$. This result will also be useful for asymptotic considerations in the last paragraph.

Associated integral equation. Now we look for a single layer type integral equation equivalent to problem $\mathbf{P}^\varepsilon$. Because we have introduced an averaging operator inside the formulation, this new integral equation should not involve the coordinate φ, and thus leads to a Pocklington equation corresponding to our geometry. According to the definition of the incident field u^0 given by (4.1), one can see that $a(\mathfrak{u}^\varepsilon - u^0, v) = -\int_{\Gamma^\varepsilon} {}^t\mu^\varepsilon[\mathfrak{p}^\varepsilon]\bar{v},\ \forall\, v \in \mathcal{H}$ and this implies

$${}^t\mu^\varepsilon[\mathfrak{p}^\varepsilon] = \frac{\partial}{\partial n}(\mathfrak{u}^\varepsilon|_{B_R \setminus \overline{\Omega}_\varepsilon}) - \frac{\partial}{\partial n}(\mathfrak{u}^\varepsilon|_{\Omega_\varepsilon})$$

and $(\Delta + k^2)(\mathfrak{u}^\varepsilon - u^0) = 0$ in $B_R \backslash \Gamma^\varepsilon$ in the distributional sense. Moreover, $\mathfrak{u}^\varepsilon - u^0 \in \mathcal{H}$, so $\mathfrak{u}^\varepsilon - u^0$ can be represented by a single layer integral, $\mathfrak{u}^\varepsilon(\mathbf{x}) - u^0(\mathbf{x}) = \int_{\Gamma^\varepsilon} {}^t\mu^\varepsilon(\mathfrak{p}^\varepsilon)(\mathbf{x}')G(\mathbf{x}, \mathbf{x}')d\mathbf{x}'$, $\mathbf{x} \in B_R$ and this equality holds in particular on Γ^ε in the sense of $H^{1/2}(\Gamma^\varepsilon)$. Take an arbitrary $q \in E^{-1/2}(I)$ and multiply this equation by ${}^t\mu^\varepsilon(q)$ and integrate over Γ^ε. Taking into account the fact that $\mu^\varepsilon[\mathfrak{u}^\varepsilon] = 0$, we are led by $\forall\, q \in E^{-1/2}(I)$ to

$$\int_{\Gamma^\varepsilon \times \Gamma^\varepsilon} G(\mathbf{x}, \mathbf{x}')\, {}^t\mu^\varepsilon[q](\mathbf{x})\, {}^t\mu^\varepsilon[\mathfrak{p}^\varepsilon](\mathbf{x}') d\sigma(\mathbf{x}) d\sigma(\mathbf{x}') = -\int_{-1}^{+1} q(\nu)\mu^\varepsilon[u^0](\nu)\, d\nu. \quad (5.5)$$

This equation is very close to Eq. (3.11), except for the presence of averaging operators μ^ε. Actually it is one-dimensional, contrary to (3.11). Indeed applying the definition of transposition in Eq. (5.5), we define $\mathfrak{G}^\varepsilon(\nu, \nu')$ the outgoing Green kernel to which is applied twice the averaging operator (one for each variable), i.e. $\mathfrak{G}^\varepsilon(\nu, \nu') = (\mu^\varepsilon_{\mathbf{x}} \times \mu^\varepsilon_{\mathbf{x}'})[G(\cdot, \cdot)](\nu, \nu')$. Thus the preceding integral equation rewrites

$$\int_{I \times I} \mathfrak{G}^\varepsilon(\nu, \nu')\, q(\nu) \mathfrak{p}^\varepsilon(\nu')\, d\nu d\nu' = -\int_I q(\nu)\mu^\varepsilon[u^0](\nu)\, d\nu, \quad \forall\, q \in E^{-1/2}(I). \quad (5.6)$$

It is also possible to rewrite it in a non-variational manner,

$$\int_I \mathfrak{G}^\varepsilon(\nu, \nu')\, \mathfrak{p}^\varepsilon(\nu')\, d\nu' = -\mu^\varepsilon[u^0](\nu), \quad \forall\, \nu \in I. \quad (5.7)$$

Relation with Pocklington's equation. Since Eqs. (5.5)–(5.7) are one-dimensional, comparing them to Eqs. (3.10) and (3.11), they appear as "ellipsoidal" versions of Pocklington's equation (1.3). Indeed consider the particular case where $\Phi(\nu, \varphi) = 1$ so that the equation of Γ^ε is $\xi^2 = 1 + \varepsilon^2$. In this case $d\sigma(\mathbf{x}) = \gamma^\varepsilon(\nu, \varphi) d\nu d\varphi = \varepsilon\sqrt{\varepsilon^2 + 1 - \nu^2}\, d\nu d\varphi$, $\gamma^\varepsilon_\mu = 2\pi\sqrt{1 + \varepsilon^2}$ and $\frac{\partial V^\varepsilon}{\partial n} \gamma^\varepsilon d\varphi = \sqrt{1 + \varepsilon^2} d\varphi$. Thus Eq. (5.7) can be rewritten as

$$\int_{-1}^{+1} \mathfrak{p}^{\varepsilon}(\nu')\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{4\pi|\mathbf{x}-\mathbf{x}'|}\,d\varphi d\varphi' d\nu' = \frac{1}{2\pi}\int_0^{2\pi} u^0(\mathbf{x})d\varphi, \quad \mathbf{x}\in\Gamma^{\varepsilon}.$$

Note also that if $\mathbf{x},\mathbf{x}'\in\Gamma^{\varepsilon}$ then $|\mathbf{x}-\mathbf{x}'|^2 = \varepsilon^2((1-\nu^2)+(1-\nu'^2)-2\sqrt{1-\nu^2}\cdot\sqrt{1-\nu'^2}\cos(\varphi-\varphi'))+(1+\varepsilon^2)(\nu-\nu')^2$ so that the left-hand side of the preceding equation actually does not depend on φ, and Eq. (5.7) finally takes the same form as (1.3),

$$\int_{-1}^{+1}\underbrace{\left(\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{4\pi|\mathbf{x}-\mathbf{x}'|}\,d\varphi\right)}_{=\,-\mathfrak{G}^{\varepsilon}(\nu,\nu')}\mathfrak{p}^{\varepsilon}(\nu')\,d\nu' = \underbrace{\frac{1}{2\pi}\int_0^{2\pi}u^0(\mathbf{x})d\varphi}_{\mu^{\varepsilon}[u^0](\nu)}, \quad \mathbf{x}\in\Gamma^{\varepsilon}.$$

5.4. *Asymptotic estimates*

In the same spirit as for the results that were presented in Sec. 4, we will now provide some results of asymptotic analysis for the solution of the simplified problem $\mathbf{P}^{\varepsilon}$. We will see in this section that the first term of the expansion of $(\mathfrak{u}^{\varepsilon},\mathfrak{p}^{\varepsilon})$ is simply given by $(\tilde{u}^{\varepsilon},\tilde{p}^{\varepsilon})$. This implies in particular that, up to the first order in their respective expansions, $(u^{\varepsilon},p^{\varepsilon})$ and $(\mathfrak{u}^{\varepsilon},\mathfrak{p}^{\varepsilon})$ coincide. We will then be able to appropriately bound their difference. This will justify replacing the exact problem by the simplified one, or equivalently replacing the exact integral equation (3.11) by the one-dimensional equation (5.6).

First of all, note as in Sec. 5.2 that $\tilde{p}^{\varepsilon} = {}^t\mu^{\varepsilon}[b^{\varepsilon}\gamma_{\mu}^{\varepsilon}]$. As a consequence, according to the definition of $\tilde{f}^{\varepsilon}$ given in Sec. 4.5, the couple $(\tilde{u}^{\varepsilon},b^{\varepsilon}\gamma_{\mu}^{\varepsilon})\in\mathcal{H}\times E^{-1/2}(I)$ and satisfies

$$\begin{cases} a(\tilde{u}^{\varepsilon},v)+\displaystyle\int_{\Gamma^{\varepsilon}}{}^t\mu^{\varepsilon}[b^{\varepsilon}\gamma_{\mu}^{\varepsilon}]\bar{v} = -\int_{B_R}f\bar{v}+\langle\tilde{f}^{\varepsilon},v\rangle_{\mathcal{H}',\mathcal{H}}, & \forall\, v\in\mathcal{H},\\ \displaystyle\int_{\Gamma^{\varepsilon}}{}^t\mu^{\varepsilon}[q]\,\bar{\tilde{u}}^{\varepsilon} = 0, & \forall\, q\in E^{-1/2}(I).\end{cases}$$

Indeed if $q\in E^{-1/2}(I)$ then ${}^t\mu^{\varepsilon}[q]\in H^{-1/2}(\Gamma^{\varepsilon})$. Comparing the preceding equations and problem $\mathbf{P}^{\varepsilon}$, we can use the stability result of Theorem 5.1 and obtain straightforwardly that there exist $\kappa_0,\varepsilon_0>0$ independent of ε such that

$$\|\mathfrak{u}^{\varepsilon}-\tilde{u}^{\varepsilon}\|_{\mathcal{H}}+\|\mathfrak{p}^{\varepsilon}-b^{\varepsilon}\gamma_{\mu}^{\varepsilon}\|_{E^{-1/2}(I)}\le\kappa\|\tilde{f}^{\varepsilon}\|_{\mathcal{H}'}\quad\forall\,\varepsilon\in\,]0,\varepsilon_0[.$$

Then we can bound the right-hand side of this inequality according to Proposition 4.1, which yields $\|\mathfrak{u}^{\varepsilon}-\tilde{u}^{\varepsilon}\|_{\mathcal{H}}+\|\mathfrak{p}^{\varepsilon}-b^{\varepsilon}\gamma_{\mu}^{\varepsilon}\|_{E^{-1/2}(I)}\le\kappa\sqrt{\varepsilon}|\ln\varepsilon|$. We conclude by using Schwarz inequality and the results of Lemma 5.3.

$$\begin{aligned}\|u^{\varepsilon}-\mathfrak{u}^{\varepsilon}\|_{\mathcal{H}}+|p^{\varepsilon}-{}^t\mu^{\varepsilon}[\mathfrak{p}^{\varepsilon}]|_{-1/2,\Gamma^{\varepsilon}} &\le \|u^{\varepsilon}-\tilde{u}^{\varepsilon}\|_{\mathcal{H}}+|p^{\varepsilon}-\tilde{p}^{\varepsilon}|_{-1/2,\Gamma^{\varepsilon}}\\ &\quad+\|\mathfrak{u}^{\varepsilon}-\tilde{u}^{\varepsilon}\|_{\mathcal{H}}+|{}^t\mu^{\varepsilon}[b^{\varepsilon}\gamma_{\mu}^{\varepsilon}]-{}^t\mu^{\varepsilon}[\mathfrak{p}^{\varepsilon}]|_{-1/2,\Gamma^{\varepsilon}}\\ &\le\kappa\sqrt{\varepsilon}\,|\ln\varepsilon|+\kappa'\sqrt{\varepsilon}\,|\ln\varepsilon|.\end{aligned}$$

We can also obtain relative bounds, combining the results of Theorem 4.1 and Eq. (5.2). This yields the final result of this paper, the one that validates Pocklington's model in the case of an ellipsoidal geometry.

Theorem 5.2. *If* p^ε *is the solution to the two-dimensional integral equation* (3.11), *if* $\mathfrak{p}^\varepsilon$ *is the solution to the one-dimensional equation* (5.6) *and if* $u^0|_I \neq 0$ *then there exist* $\kappa_0, \varepsilon_0 > 0$ *independent of* ε *such that*

$$\frac{|p^\varepsilon - {}^t\mu^\varepsilon[\mathfrak{p}^\varepsilon]|_{-1/2,\Gamma^\varepsilon}}{|p^\varepsilon|_{-1/2,\Gamma^\varepsilon}} \leq \kappa_0 \sqrt{\varepsilon}\,|\ln \varepsilon|^2 \quad \forall \varepsilon \in \,]0, \varepsilon_0[.$$

Note that the constant κ_0 involved in this result does depend on u^0 (defined by Eq. (4.1)).

Appendix A

Here we give a proof of Lemma 5.1 used for the definition of the averaging operator. We will also use the proof of this lemma for obtaining a smoothness result on γ_μ^ε. In what follows, if (ρ, ν, φ) are the coordinates defined by formulas (2.1) and (4.4), and $\mathbf{e}_\mathrm{x}$, $\mathbf{e}_\mathrm{y}$, $\mathbf{e}_\mathrm{z}$ are the Cartesian basis of $\mathbb{R}^3$, the vector $\mathbf{e}_\rho$ is defined by $\mathbf{e}_\rho = \frac{\partial x}{\partial \rho}\,\mathbf{e}_\mathrm{x} + \frac{\partial y}{\partial \rho}\,\mathbf{e}_\mathrm{y} + \frac{\partial z}{\partial \rho}\,\mathbf{e}_\mathrm{z}$ and the vectors $\mathbf{e}_\varphi$ and $\mathbf{e}_\nu$ are defined in the same manner. Similarly, if (ρ, φ) refer to the polar coordinates in $\mathbb{R}^2$, then we define $\widehat{\mathbf{e}}_\rho = \frac{\partial x}{\partial \rho}\,\mathbf{e}_\mathrm{x} + \frac{\partial y}{\partial \rho}\,\mathbf{e}_\mathrm{y}$ and $\widehat{\mathbf{e}}_\varphi$ is defined in the same manner.

Lemma 5.1. *There exists a* $\kappa > 0$ *independent of* ε *such that*

$$\left| \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial V^\varepsilon}{\partial n} \gamma^\varepsilon \, d\varphi - 1 \right| < \kappa\varepsilon.$$

Proof. First of all, we remind the reader that in the coordinates (ρ, ν, φ), the equation of the wire is $(\Gamma^\varepsilon) : \rho = \Phi(\nu, \varphi)$. Using classical differential calculus it can be checked that if the outward normal vector on Γ^ε is given by $n = n_\rho\,\mathbf{e}_\rho + n_\nu\,\mathbf{e}_\nu + n_\varphi\,\mathbf{e}_\varphi$, then we have the simple relation

$$\gamma^\varepsilon n_\rho = \Phi(1 + \varepsilon^2\Phi^2)^{1/2} = \Phi + \frac{\varepsilon^2\Phi^3}{1 + \sqrt{1 + \varepsilon^2\Phi^2}}.$$

Similarly, if one considers the coordinates (ρ, φ) as polar coordinates in $\omega_\mathrm{n}(\nu)$, denote $\widehat{\gamma}_\nu(\varphi)$ the density of the measure on $\partial\omega_\mathrm{n}(\nu)$ and $\widehat{n} = \widehat{n}_{\nu,\rho}\,\widehat{\mathbf{e}}_\rho + \widehat{n}_{\nu,\varphi}\,\widehat{\mathbf{e}}_\varphi$ the outward normal to $\partial\omega_\mathrm{n}(\nu)$, then $\widehat{\gamma}_\nu\,\widehat{n}_{\nu,\rho} = \Phi$. As a consequence, there exists $\kappa > 0$ independent of ε such that $|\gamma^\varepsilon n_\rho - \widehat{\gamma}_\nu\,\widehat{n}_{\nu,\rho}| < \kappa\varepsilon^2$. Using the regularity of V^ε (with respect to the three variables ρ, ν and φ), it can be proved that there exists $\kappa > 0$ such that

$$\left| \frac{n_\varphi}{n_\rho} \frac{\partial V^\varepsilon}{\partial \varphi} + \frac{n_\nu}{n_\rho} \frac{\partial V^\varepsilon}{\partial \nu} - \frac{\widehat{n}_{\nu,\varphi}}{\widehat{n}_{\nu,\rho}} \frac{\partial \widehat{V}_\nu}{\partial \varphi} \right| \leq \kappa\varepsilon.$$

Note that $\gamma^\varepsilon \frac{\partial V^\varepsilon}{\partial n} = \gamma^\varepsilon n_\rho (\frac{\partial V^\varepsilon}{\partial \rho} + \frac{n_\varphi}{n_\rho} \frac{\partial V^\varepsilon}{\partial \varphi} + \frac{n_\nu}{n_\rho} \frac{\partial V^\varepsilon}{\partial \nu})$ and $\widehat{\gamma}_\nu \frac{\partial \widehat{V}_\nu}{\partial \widehat{n}_\nu} = \widehat{\gamma}_\nu \widehat{n}_{\nu,\rho} (\frac{\partial \widehat{V}_\nu}{\partial \rho} + \frac{\widehat{n}_{\nu,\varphi}}{\widehat{n}_{\nu,\rho}} \frac{\partial \widehat{V}_\nu}{\partial \varphi})$ so we obtain from the preceding estimates that

$$\left| \int_0^{2\pi} \gamma^\varepsilon \frac{\partial V^\varepsilon}{\partial n}\, d\varphi - \int_0^{2\pi} \widehat{\gamma}_\nu \frac{\partial \widehat{V}_\nu}{\partial \widehat{n}_\nu}\, d\varphi \right| \leq \kappa\varepsilon.$$

There only remains to note that, according to classical theory of Laplace problems in two dimensions, $\int_0^{2\pi} \widehat{\gamma}_\nu \frac{\partial \widehat{V}_\nu}{\partial \widehat{n}_\nu}\, d\varphi = 2\pi$. This concludes the proof. □

Lemma A.1. *For any* $n \in \mathbb{N}$, *there exists* $\kappa_n > 0$ *independent of* ε *such that* $\|\frac{\partial^n \gamma_\mu^\varepsilon}{\partial \nu^n}\|_{L^\infty(I)} \leq \kappa_n, \quad \forall\, \varepsilon > 0.$

Proof. The proof for this result lies on calculus and observation. Let us repeat the expression γ_μ^ε,

$$\gamma_\mu^\varepsilon(\nu) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial V^\varepsilon}{\partial n} \gamma^\varepsilon(\varphi, \nu) d\varphi.$$

Just decompose the expression located inside the integral. Taking into account the remarks about $n_\rho \gamma^\varepsilon$ in the preceding lemma, this yields

$$\frac{\partial V^\varepsilon}{\partial n} \gamma^\varepsilon = \left[\frac{\partial V^\varepsilon}{\partial \rho} + \frac{n_\varphi}{n_\rho} \frac{\partial V^\varepsilon}{\partial \varphi} + \frac{n_\nu}{n_\rho} \frac{\partial V^\varepsilon}{\partial \nu} \right] \left[\Phi + \frac{\varepsilon^2 \Phi^3}{1 + \sqrt{1 + \varepsilon^2 \Phi^2}} \right].$$

Clearly the term on the right in the product has all its derivatives uniformly bounded as $\varepsilon \to 0$. So there only remains to deal with the first factor. First note that according to the definition of V^ε we have $\frac{\partial V^\varepsilon}{\partial \rho} = \frac{\partial}{\partial \rho} (\widehat{V}_\nu(\rho, \varphi))$, and similar equalities hold for the other derivatives. As a consequence the only terms that depend on ε are $\frac{n_\varphi}{n_\rho}$ and $\frac{n_\nu}{n_\rho}$. These two are given by the explicit expressions,

$$\frac{n_\varphi}{n_\rho} = -\frac{\varepsilon^2 \Phi^2 + 1 - \nu^2}{\Phi^2 (1 + \varepsilon^2 \Phi^2)(1 - \nu^2)} \frac{\partial \Phi}{\partial \varphi} \quad \text{and} \quad \frac{n_\nu}{n_\rho} = -\varepsilon^2 \frac{1 - \nu^2}{1 + \varepsilon^2 \Phi^2} \frac{\partial \Phi}{\partial \nu}.$$

These expressions clearly show uniform boundedness of all the derivatives, except maybe for n_φ / n_ρ because of the factor $1/(1 - \nu^2)$. Since $\partial \Phi / \partial \varphi = 0$ for $|\nu| > \nu_0$ this actually does not raise any problem. □

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