# REMARKS ON THE GLOBAL SOLUTIONS OF 3-D NAVIER-STOKES SYSTEM WITH ONE SLOW VARIABLE

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ABSTRACT. By applying Wiegner' method in [15], we first prove the large time decay estimate for the global solutions of a 2.5 dimensional Navier-Stokes system, which is a sort of singular perturbed 2-D Navier-Stokes system in three space dimension. As an application of this decay estimate, we give a simplified proof for the global wellposedness result in [6] for 3-D Navier-Stokes system with one slow variable. Let us also mention that compared with the assumptions for the initial data in [6], here the assumptions in Theorem 1.3 are weaker.

*Keywords:* Incompressible Navier-Stokes Equations, slow variable, decay estimate, Littlewood-Paley Theory

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## 1. INTRODUCTION

The first part of this paper is devoted to the study of the following system

$$
\text{(NS2.5D)} \qquad \begin{cases} \partial_t u_\varepsilon^{\mathrm{h}} + \mathrm{div}_{\mathrm{h}}(u_\varepsilon^{\mathrm{h}} \otimes u_\varepsilon^{\mathrm{h}}) - \Delta_\varepsilon u_\varepsilon^{\mathrm{h}} = -\nabla_{\mathrm{h}} p_\varepsilon^{\mathrm{h}}, \qquad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\ \mathrm{div}_{\mathrm{h}}\, u_\varepsilon^{\mathrm{h}} = 0, \\ u_\varepsilon^{\mathrm{h}}|_{t=0} = u_0^{\mathrm{h}} \end{cases}
$$

where $u^h(t,x_{\mathrm{h}},z) = \big(u^1(t,x_{\mathrm{h}},z), u^2(t,x_{\mathrm{h}},z)\big)$ with $(x_{\mathrm{h}},z)$ in $\mathbb{R}^2_{\mathrm{h}} \times \mathbb{R}_{\mathrm{v}}$, $\nabla_{\mathrm{h}} = (\partial_1,\partial_2)$, $\Delta_{\mathrm{h}} = \partial_1^2 + \partial_2^2$ and $\Delta_\varepsilon = \Delta_{\mathrm{h}} + \varepsilon^2\partial_3^2$. Moreover, $\varepsilon$ is a (small) positive parameter. Let us first point out that in the case when $\varepsilon$ is equal to 0, the above system is simply the two dimensional incompressible Navier-Stokes system with an initial data depending on the real parameter $z$.

The motivation of studying this system (NS2.5D) comes from the study of global wellposedness of the three dimensionnal incompressible Navier-Stokes system (NS3D) which is

$$
\text{(NS3D)} \qquad \begin{cases} \partial_t u + \mathrm{div}(u \otimes u) - \Delta u = -\nabla p, \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^3 \\ \mathrm{div} u = 0, \\ u|_{t=0} = u_0 \end{cases}
$$

where the initial data are said to be "slowly varying" with respect to the vertical variable which means that they are of the form

$$
u_0(x) = (u_0^{\mathrm{h}}(x_{\mathrm{h}}, \varepsilon x_3), 0) \quad \text{with} \quad \mathrm{div}_h\, u_0^{\mathrm{h}}(\cdot, z) = 0.
$$

The interest of this type of initial data is that they are relevant tools to investigate the problem of global wellposedness for (NS3D). First of all, they provide a class of large initial data for the system (NS3D) which are globally wellposed and which do not have symmetries. Indeed, the following result holds (see [6], Theorem 1 and Proposition 1.1).

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**Theorem 1.1.** *Let $v_0^{\rm h}=(v_0^1,v_0^2)$ be a horizontal, smooth divergence free vector field on $\mathbb{R}^3$ (i.e. $v_0^{\rm h}$ is in $L^2(\mathbb{R}^3)$ as well as all its derivatives), belonging, as well as all its derivatives, to $L^2(\mathbb{R}_z;\dot H^{-1}(\mathbb{R}^2_{\rm h}))$. Then, there exists a positive $\varepsilon_0$ such that, if $\varepsilon\leq\varepsilon_0$, the initial data*

$$u_0^\varepsilon(x)=\bigl(v_0^{\rm h}(x_{\rm h},\varepsilon x_3),0\bigr)$$

*generates a unique, global solution $u^\varepsilon$ of (NS3D).*

*Moreover, if $v_0^{\rm h}(x_{\rm h},z)\stackrel{def}{=}\widetilde v_0(x_{\rm h})g(z)$ then if $\varepsilon$ is small enough,*

$$\|u_0^\varepsilon\|_{\dot B^{-1}_{\infty,\infty}(\mathbb{R}^3)}\geq\frac14\|\widetilde v_0^{\rm h}\|_{\dot B^{-1}_{\infty,\infty}(\mathbb{R}^2_{\rm h})}\|g\|_{L^\infty(\mathbb{R}_{\rm v})}\quad\text{where}\quad\|a\|_{\dot B^{-1}_{\infty,\infty}(\mathbb{R}^d)}\stackrel{def}{=}\sup_{t>0}t^{\frac12}\|e^{t\Delta}a\|_{L^\infty}.$$

This last inequality ensures that the above global wellposedness result is not a consequence of the Koch and Tataru theorem (see [12]) which claims that if the regular initial data $u_0$ of (NS3D) is sufficiently small in the norm of the space $\mathrm{BMO}^{-1}(\mathbb{R}^3)$, then it generates a global smooth solution. Here, let us simply recall that the space $\mathrm{BMO}^{-1}(\mathbb{R}^3)$ is continuously imbedded in the Besov space $\dot B^{-1}_{\infty,\infty}(\mathbb{R}^3)$.

Moreover, such slowly varying initial data allows to say something about the geometry of the set $\mathcal{G}$ of initial data in $\dot H^{\frac12}(\mathbb{R}^3)$ which generates global solution in the space $C(\mathbb{R}^+;\dot H^{\frac12}(\mathbb{R}^3))$. In [10], I. Gallagher, D. Iftimie and F. Planchon proved that this set is open and connected (see also [1] and [11] for the same property in more sophisticated spaces). Using slowly varying perturbations, I. Gallagher and the two authors proved in [8] that through any point of $\mathcal{G}$, there are uncountable lines of arbitrary length in the space $\dot B^{-1}_{\infty,\infty}(\mathbb{R}^3)$, and thus in the Sobolev space $\dot H^{\frac12}(\mathbb{R}^3)$, which is continuously imbedded in the space $\dot B^{-1}_{\infty,\infty}(\mathbb{R}^3)$. An interpretation in terms of support of the Fourier transform of the initial data is presented in [7].

Such initial data appears also in the study of the problem concerning the openness to the set $\mathcal{G}$ for weak topology (see [3] and [2]).

The way Theorem 1.1 is proved in [6] is as follows. Let us consider $u^{\rm h}(t,x_{\rm h},z)$ the (global) solution of the 2D incompressible Navier-Stokes system

$$(\mathrm{NS2D}_3)\begin{cases}\partial_t u^{\rm h}+u^{\rm h}\cdot\nabla_h u^{\rm h}-\Delta_h u^{\rm h}=-\nabla_h p^{\rm h}\quad\text{in }\mathbb{R}^+\times\mathbb{R}^2\\ \mathrm{div}_h\,u^{\rm h}=0\\ u^{\rm h}|_{t=0}=u_0^{\rm h}(\cdot,z).\end{cases}$$

This system is globally wellposed for any $z$ in $\mathbb{R}$, and the solution is smooth in (two dimensional) space, and in time. Let us define the approximate solution

$$\widetilde u^\varepsilon_{\rm app}(t,x)=(u^{\rm h}(t,x_h,\varepsilon x_3),0)\quad\text{and}\quad\widetilde p^\varepsilon_{\rm app}(t,x)=p^{\rm h}(t,x_h,\varepsilon x_3).\tag{1.1}$$

and let us search the solution of (NS3D) as

$$u^\varepsilon=\widetilde u^\varepsilon_{\rm app}+R^\varepsilon.$$

Classical computations leads to

$$\begin{gathered}\partial_t R^\varepsilon+R^\varepsilon\cdot\nabla R^\varepsilon-\Delta R^\varepsilon+\widetilde u^\varepsilon_{\rm app}\cdot\nabla R^\varepsilon+R^\varepsilon\cdot\nabla\widetilde u^\varepsilon_{\rm app}=\widetilde F^\varepsilon-\nabla q^\varepsilon\quad\text{with}\\ F^\varepsilon\stackrel{\rm def}{=}\bigl(\varepsilon^2\partial_z^2u^{\rm h},\varepsilon\partial_z p^{\rm h}\bigr)(t,x_h,\varepsilon x_3).\end{gathered}\tag{1.2}$$

It is easy to observe that, if we have good uniform estimates on $\widetilde u^\varepsilon_{\rm app}$ and that $\widetilde F_\varepsilon$ tends to 0 when $\varepsilon$ tends to 0 in a space like $L^2(\mathbb{R}^+;\dot H^{-\frac12}(\mathbb{R}^3))$, then the global wellposedness is proved (see [6] for the details). Here, the fact that $\varepsilon^2(\partial_z^2u^{\rm h})(t,x_{\rm h},\varepsilon x_3)$ appears as an error term is conceptually not satisfactory because it is a term coming from the viscosity and thus it is supposed to produce decay or regularity and yet here it is a source of some technical difficulty.

The idea here is to substitute ($\mathrm{NS2D}_3$) by (NS2.5D). We have to prove global wellposedness for (NS 2.5D) with regular initial data, which is not difficult, and also the space time estimate in $L^p$, which should of course be independent of the parameter $\varepsilon$. This is the new point of this paper. The precise statement is the following.

**Theorem 1.2.** *Let $u_0^{\rm h}$ and $\nabla_{\rm h} u_0^{\rm h}$ be in $L^2(\mathbb{R}^3)\cap L^\infty(\mathbb{R}_{\rm v};L^2(\mathbb{R}^2_{\rm h}))$. Then $u_0^{\rm h}$ generates a unique global solution to (NS2.5D) in the space $L^\infty_{\rm loc}(L^4(\mathbb{R}^3))$.*

*Moreover, if in addition $u_0^{\rm h}$ belongs to $L^\infty(\mathbb{R}_{\rm v};\dot H^{-\delta}(\mathbb{R}^2_{\rm h}))$ for some $\delta$ in $]0,1[$, then we have*

$$\int_0^\infty \|\nabla_{\rm h} u^{\rm h}(t)\|^2_{L^\infty_{\rm v}(L^2(\mathbb{R}^2_{\rm h}))}\,dt \leq A_\delta(u_0^{\rm h}) \quad \textit{with}$$

$$A_\delta(u_0^{\rm h}) \overset{def}{=} C_\delta\Bigl(\frac{\|\nabla_{\rm h} u_0^{\rm h}\|^2_{L^\infty_{\rm v}(L^2_{\rm h})}\|u_0^{\rm h}\|^{\frac{2}{\delta}}_{L^\infty_{\rm v}(\dot B^{-\delta}_{2,\infty}(\mathbb{R}^2_{\rm h}))}}{\|u_0^{\rm h}\|^{\frac{2}{\delta}}_{L^\infty_{\rm v}(L^2_{\rm h})}} + \|u_0^{\rm h}\|^2_{L^\infty_{\rm v}(L^2_{\rm h})}\Bigr) \times \exp\bigl(C_\delta \|u_0^{\rm h}\|^2_{L^\infty_{\rm v}(L^2_{\rm h})}(1+\|u_0^{\rm h}\|^2_{L^\infty_{\rm v}(L^2_{\rm h})})\bigr). \tag{1.3}$$

**Remark 1.1.** *Following the procedure in Section 3 and under the assumptions of Theorem 1.2, we can prove more precise large time decay estimates for $u^{\rm h}(t)$ as follows*

$$\|u^{\rm h}(t)\|^2_{L^\infty_{\rm v}(L^2_{\rm h})} \leq Ce^{CC_0^2}\langle t\rangle^{-\delta} \quad \textit{with} \quad C_0 \overset{def}{=} \|u_0^{\rm h}\|_{L^\infty_{\rm v}(L^2_{\rm h}\cap \dot H^{-\delta}_{\rm h})}\bigl(1+\|u_0^{\rm h}\|_{L^\infty_{\rm v}(L^2_{\rm h})}\bigr) \quad \textit{and}$$

$$\|\nabla_{\rm h} u^{\rm h}(t)\|^2_{L^\infty_{\rm v}(L^2_{\rm h})} \leq CC_1\langle t\rangle^{-(1+\delta)} \quad \textit{with} \quad C_1 \overset{def}{=} \bigl(1+\|\nabla_{\rm h} u_0^{\rm h}\|_{L^2}\bigr)e^{CC_0^2}.$$

*For a concise presentation, we shall not present the details here.*

The idea of the proof of this theorem is first to perform energy estimate in $L^2(\mathbb{R}^3)$ which is very basic and far from being enough here. Then we perform energy estimate in the horizontal variables only for the vector field $u^{\rm h}$ and its vorticity $\omega^{\rm h} \overset{\rm def}{=} \partial_1 u^2 - \partial_2 u^1$. The maximum principle for the heat equation provides global wellposedness of (NS2.5D). This is the purpose of the second section.

Unfortunately, this global wellposedness results does not yield any uniform bound for the solution with respect to $\varepsilon$ in any $L^p$ norm for time. Thus we have no uniform global stability of such global solutions in the sense that we want global stability of the global solutions of (NS2.5D) with the size of perturbation being independent of the parameter $\varepsilon$.

In the third section, we introduce Wiegner's method in the context of (NS2.5D). This method has been introduced by M. Wiegner in [15] in order to prove the large time decay estimate on the $L^2$ norm of a solution to the incompressible Navier-Stokes system in the whole space. For some developments and variations about this questions of decay of the $L^2$ norm in the whole space, see the works [4], [13] and [14].

In the forth section, we apply Theorem 1.2 to prove the following result.

**Theorem 1.3.** *Let $u_0^{\rm h}$ be in $H^1(\mathbb{R}^3)\cap L^\infty(\mathbb{R}_{\rm v};\dot H^{-\delta}(\mathbb{R}^2_{\rm h}))\cap L^\infty(\mathbb{R}_{\rm v};H^1(\mathbb{R}^2_{\rm h}))$ for some $\delta\in]0,1[$. Let us assume also $u_0^{\rm h}$ and $\partial_z u_0^{\rm h}$ belongs to $L^2(\mathbb{R}_{\rm v};\dot H^{-\frac12}\cap \dot H^{\frac12}(\mathbb{R}^2_{\rm h}))$. Then for $\varepsilon\leq\varepsilon_0$ depending only on the above norms, the initial data*

$$u_{0,\varepsilon}(x_{\rm h},x_3) = \bigl(u_0^{\rm h}(x_{\rm h},\varepsilon x_3),0\bigr)$$

*generates a global solution to (NS3D) in the space $C_b(\mathbb{R}^+;\dot H^{\frac12}(\mathbb{R}^3))\cap L^2(\mathbb{R}^+;\dot H^{\frac32}(\mathbb{R}^3))$.*

The idea of the proof of this theorem is to search a solution of (NS3D) as

$$u^\varepsilon = u^\varepsilon_{\rm app} + R^\varepsilon \quad\text{with}\quad u^\varepsilon_{\rm app}(t,x_{\rm h},x_3) \stackrel{\rm def}{=} (u^{\rm h}_\varepsilon(t,x_{\rm h},\varepsilon x_3),0)$$

where $(u^{\rm h}_\varepsilon, p^{\rm h}_\varepsilon)$ is the solution of (NS2.5D) with initial data $u^{\rm h}_0$. Classical computations leads to

$$\begin{aligned}&\partial_t R^\varepsilon + R^\varepsilon\cdot\nabla R^\varepsilon - \Delta R^\varepsilon + u^\varepsilon_{\rm app}\cdot\nabla R^\varepsilon + R^\varepsilon\cdot\nabla u^\varepsilon_{\rm app} = F^\varepsilon - \nabla q^\varepsilon \quad\text{with}\\ &\qquad\qquad F^\varepsilon \stackrel{\rm def}{=} \bigl(0,\varepsilon\partial_z p^{\rm h}_\varepsilon\bigr)(t,x_h,\varepsilon x_3).\end{aligned}\tag{1.4}$$

The external force $F^\varepsilon$ in (1.4) is much easier to be dealt with than the external force $\widetilde{F}^\varepsilon$ in (1.2).

## 2. Global wellposedness of (NS2.5D) and maximum principle

Let us first observe that the general theory of parabolic system implies that, for any positive $\varepsilon$, a unique maximal solution $u^{\rm h}_\varepsilon$ to (NS2.5D) exists in $C([0,T^\star_\varepsilon[;H^1(\mathbb{R}^3))$ and that

$$\text{if}\ \ T^\star_\varepsilon<\infty \implies \forall p>3\,,\ \lim_{t\to T^\star_\varepsilon}\|u^{\rm h}_\varepsilon\|_{L^p(\mathbb{R}^3)} = \infty.\tag{2.1}$$

All the forthcoming computations will be valid for $t$ less than $T^\star_\varepsilon$. For simplicity, we shall drop out the subscript $\varepsilon$ in what follows.

Multiplying (NS2.5D) by $u^{\rm h}$ and then integrating the resulting equation over $\mathbb{R}^2_{\rm h}$ gives

$$\frac12\Bigl(\frac{d}{dt}\|u^{\rm h}(t,\cdot,z)\|^2_{L^2_{\rm h}} - \varepsilon^2\partial_3^2\|u^{\rm h}(t,\cdot,z)\|^2_{L^2_{\rm h}}\Bigr) + \|\nabla_\varepsilon u^{\rm h}(t,\cdot,z)\|^2_{L^2_{\rm h}} = 0,\tag{2.2}$$

where $\nabla_\varepsilon = (\nabla_{\rm h},\varepsilon\partial_3)$. By integration (2.2) for both time and the vertical variable $z$, we get

$$\frac12\|u^{\rm h}(t)\|^2_{L^2(\mathbb{R}^3)} + \int_0^t \|\nabla_\varepsilon u^{\rm h}(t')\|^2_{L^2(\mathbb{R}^3)} = \frac12\|u^{\rm h}_0\|^2_{L^2(\mathbb{R}^3)}.\tag{2.3}$$

Moreover, from (2.2), we infer

$$\frac{d}{dt}\|u^{\rm h}(t,\cdot,z)\|^2_{L^2_{\rm h}} - \varepsilon^2\partial_3^2\|u^{\rm h}(t,\cdot,z)\|^2_{L^2_{\rm h}} \leq 0.$$

The fact that the heat flow is a contraction in $L^p$ space implies that

$$\forall p\in[2,\infty]^2\,,\ \|u^{\rm h}(t)\|_{L^p_{\rm v}(L^2_{\rm h})} \leq \|u^{\rm h}_0\|_{L^p_{\rm v}(L^2_{\rm h})}.\tag{2.4}$$

There is no evidence that Equality (2.2) provides an estimate of

$$\sup_z \int_0^t \|\nabla_h u^{\rm h}(t,\cdot,z)\|^2_{L^2_h}\,dt'$$

which is independent of $\varepsilon$ for small $\varepsilon$. Of course it is the case when $\varepsilon=0$. This shows that the system (NS2.5D) is really a singular perturbation problem.

Because the nonlinear term in (NS2.5D) is a two dimensional one, we have the following well-known equation on the vorticity $\omega^{\rm h} = \partial_1 u^2 - \partial_2 u^1$:

$$\partial_t\omega^{\rm h} + u^{\rm h}\cdot\nabla_{\rm h}\omega^{\rm h} - \Delta_\varepsilon\omega^{\rm h} = 0.\tag{2.5}$$

Arguing in the same way as the above, we get

$$\frac12\Bigl(\frac{d}{dt}\|\omega^{\rm h}(t,\cdot,z)\|^2_{L^2_{\rm h}} - \varepsilon^2\partial_3^2\|\omega^{\rm h}(t,\cdot,z)\|^2_{L^2_{\rm h}}\Bigr) + \|\nabla_\varepsilon\omega^{\rm h}(t,\cdot,z)\|^2_{L^2_{\rm h}} = 0\tag{2.6}$$

which leads us to

$$\forall p\in[2,\infty]^2\,,\quad \|\omega^{\rm h}(t)\|_{L^p_{\rm v}(L^2_{\rm h})} \leq \|\omega^{\rm h}_0\|_{L^p_{\rm v}(L^2_{\rm h})}\tag{2.7}$$

Now let us see that Inequalities (2.4) and (2.7) prevent the solution of (NS2.5D) from blowing up. Indeed, by interpolation between these two inequalities, we obtain

$$\forall p\in ]2,\infty[\,,\ \|u^{\rm h}(t)\|_{L^p(\mathbb{R}^3)}\leq C_p\leq \|u_0^{\rm h}\|^{\frac 2 p}_{L^p_{\rm v}(L^2_{\rm h})}\|\omega_0^{\rm h}\|^{1-\frac 2 p}_{L^p_{\rm v}(L^2_{\rm h})}.$$

Assertion (2.1) ensures that $T^\star_\varepsilon$ is infinite.

## 3. Singular perturbation of M. Wiegner' method

Let us first recall Wiegner' method in [15]. It consits in truncating the frequency space with an appropriate time dependent function. Given a positive function $g$ on $\mathbb{R}^+$, we define, for a $L^2$ function $a$ on $\mathbb{R}^2_{\rm h}$

$$a_{\flat,g}(t)\stackrel{\rm def}{=}\mathcal{F}^{-1}_{\mathbb{R}^2_{\rm h}}\bigl(\mathbb{1}_{S(t)}(\xi_{\rm h})\widehat a(\xi_{\rm h})\bigr)\quad\hbox{with}\quad S(t)\stackrel{\rm def}{=}\{\xi_{\rm h}\in\mathbb{R}^2_{\rm h}\,/\ |\xi_{\rm h}|\leq g(t)\}.$$

The key lemma here is the following.

**Lemma 3.1.** *Let $U$ be a regular function on $\mathbb{R}^+\times\mathbb{R}^2_{\rm h}\times\mathbb{R}_{\rm v}$ such that*

$$\partial_t\|U(t,\cdot,z)\|^2_{L^2_{\rm h}}-\varepsilon^2\partial_z^2\|U(t,\cdot,z)\|^2_{L^2_{\rm h}}+2\|\nabla_{\rm h}U(t,\cdot,z)\|^2_{L^2_{\rm h}}\leq 0. \tag{3.1}$$

*Then for any positive function $g$ on $\mathbb{R}^+$, we have*

$$\begin{aligned}\|U(t)\|^2_{L^\infty_{\rm v}(L^2_{\rm h})}\exp\Bigl(2\int_0^t g^2(t')dt'\Bigr)&\leq\|U(0)\|^2_{L^\infty_{\rm v}(L^2_{\rm h})}\\&\quad+C\int_0^t\|U_{\flat,g}(t')\|^2_{L^\infty_{\rm v}(L^2_{\rm h})}g^2(t')\exp\Bigl(2\int_0^{t'}g^2(t'')dt''\Bigr)dt'\,.\end{aligned}$$

*Proof.* Let us write that

$$\begin{aligned}\|\nabla_{\rm h}U(t,\cdot,z)\|^2_{L^2_{\rm h}}&=(2\pi)^{-2}\int_{\mathbb{R}^2_{\rm h}}|\xi_{\rm h}|^2|\widehat U(t,\xi_{\rm h},z)|^2d\xi_{\rm h}\\&\geq(2\pi)^{-2}\int_{\{\xi_{\rm h}\in\mathbb{R}^2_{\rm h}\,/\ |\xi_{\rm h}|\geq g(t)\}}|\xi_{\rm h}|^2|\widehat U(t,\xi_{\rm h},z)|^2d\xi_{\rm h}\\&\geq(2\pi)^{-2}g^2(t)\int_{\mathbb{R}^2_{\rm h}}|\widehat U(t,\xi_{\rm h},z)|^2d\xi_{\rm h}\\&\qquad\qquad-(2\pi)^{-2}g^2(t)\int_{\mathbb{R}^2_{\rm h}}|\widehat U_{\flat,g}(t,\xi_{\rm h},z)|^2d\xi_{\rm h}\,.\end{aligned}$$

Plugging this inequality in Hypothesis (3.1) gives

$$\partial_t\|U(t,\cdot,z)\|^2_{L^2_{\rm h}}-\varepsilon^2\partial_z^2\|U(t,\cdot,z)\|^2_{L^2_{\rm h}}+2g^2(t)\|U(t,\cdot,z)\|^2_{L^2_{\rm h}}\leq 2g^2(t)\|U_{\flat,g}(t,\cdot,z)\|^2_{L^2_{\rm h}}.$$

The multiplication by $\exp\Bigl(2\int_0^t g^2(t')dt'\Bigr)$ gives rise to

$$\begin{aligned}\partial_t\biggl(\|U(t,\cdot,z)\|^2_{L^2_{\rm h}}\exp\Bigl(2\int_0^t g^2(t')dt'\Bigr)\biggr)-\varepsilon^2\partial_z^2\biggl(\|U(t,\cdot,z)\|^2_{L^2_{\rm h}}\exp\Bigl(2\int_0^t g^2(t')dt'\Bigr)\biggr)\\ \leq 2\|U_{\flat,g}(t)\|^2_{L^\infty_{\rm v}(L^2_{\rm h})}\,g^2(t)\exp\Bigl(2\int_0^t g^2(t')dt'\Bigr).\end{aligned}$$

The maximum principle implies the lemma. □

In order to apply this lemma with $U=u^{\rm h}$ and $U=\omega^{\rm h}$, we need some control about low frequency part of $u^{\rm h}$ and $\omega^{\rm h}$.

**Lemma 3.2.** *If $u^{\rm h}$ is a regular solution of (NS2.5D), then we have, for any positive function $g$,*

$$
\begin{aligned}
\|u^{\rm h}_{\flat,g}(t)\|_{L^\infty_{\rm v}(L^2_{\rm h})} &\leq \|e^{t\Delta_{\rm h}}u^{\rm h}_0\|_{L^\infty_{\rm v}(L^2_{\rm h})} + Cg^2(t)\int_0^t \|u^{\rm h}(t')\|^2_{L^\infty_{\rm v}(L^2_{\rm h})}\,dt' \quad \textit{and}\\
\|\omega^{\rm h}_{\flat,g}(t)\|_{L^\infty_{\rm v}(L^2_{\rm h})} &\leq g(t)\|e^{t\Delta_{\rm h}}u^{\rm h}_0\|_{L^\infty_{\rm v}(L^2_{\rm h})} + Cg^2(t)\int_0^t \|u^{\rm h}(t')\|_{L^\infty_{\rm v}(L^2_{\rm h})}\|\omega^{\rm h}(t')\|_{L^\infty_{\rm v}(L^2_{\rm h})}\,dt'.
\end{aligned}
$$

*Proof.* It is in fact a lemma about the heat equation with vanishing diffusion in one direction. Let us consider $a$ and $f$ such that $\partial_t a - \Delta_\varepsilon a = f$. By definition of $\Delta_\varepsilon$, Duhamel's formula writes, after a Fourier transform with respect to the horizontal variables,

$$
\begin{aligned}
\widehat{a}_{\flat,g}(t,\xi_{\rm h},z) = {}& \mathbb{1}_{S(t)}(\xi_{\rm h})e^{-t|\xi_{\rm h}|^2}\frac{1}{(4\pi\varepsilon^2 t)^{\frac12}}\int_{\mathbb{R}_{\rm v}} e^{-\frac{|z-z'|^2}{4\varepsilon^2 t}}\widehat{a}(0,\xi_{\rm h},z')dz'\\
&+\int_0^t\int_{\mathbb{R}_{\rm v}} e^{-(t-t')|\xi_{\rm h}|^2}\frac{1}{(4\pi\varepsilon^2(t-t'))^{\frac12}}e^{-\frac{|z-z'|^2}{4\varepsilon^2(t-t')}}\mathbb{1}_{S(t)}(\xi_{\rm h})\widehat{f}(t',\xi_{\rm h},z')\,dz'dt'.
\end{aligned}
$$

As the norm of an integral is less than or equal to the integral of the norm, we get, for any $(t,z)$ in $\mathbb{R}^+\times\mathbb{R}_{\rm v}$,

$$
\begin{aligned}
\|a_{\flat,g}(t,\cdot,z)\|_{L^2_{\rm h}} \leq {}& \frac{1}{(4\pi\varepsilon^2 t)^{\frac12}}\int_{\mathbb{R}_{\rm v}} e^{-\frac{|z-z'|^2}{4\varepsilon^2 t}}\big\|\big(e^{t\Delta_{\rm h}}a(0,\cdot,z')\big)_{\flat,g}\big\|_{L^2_{\rm h}}\,dz'\\
&+\int_0^t\int_{\mathbb{R}_{\rm v}}\frac{1}{(4\pi\varepsilon^2(t-t'))^{\frac12}}e^{-\frac{|z-z'|^2}{4\varepsilon^2(t-t')}}\|\mathbb{1}_{S(t)}(D_{\rm h})f(t',\cdot,z')\|_{L^2_{\rm h}}\,dz'dt',
\end{aligned}
$$

where $\mathbb{1}_{S(t)}(D_{\rm h})$ denotes the Fourier multiplier with

$$
\mathbb{1}_{S(t)}(D_{\rm h})g(x_{\rm h}) = \mathcal{F}^{-1}\big(\mathbb{1}_{S(t)}(\xi_{\rm h})\widehat{g}(\xi_{\rm h})\big)(x_{\rm h}).
$$

Taking the $L^\infty$ norm with respect to the variable $z$ gives

$$
\|a_{\flat,g}(t)\|_{L^\infty_{\rm v}(L^2_{\rm h})} \leq \big\|\big(e^{t\Delta_{\rm h}}a(0)\big)_{\flat,g}\big\|_{L^\infty_{\rm v}(L^2_{\rm h})} + \int_0^t \|\mathbb{1}_{S(t)}(D_{\rm h})f(t')\|_{L^\infty_{\rm v}(L^2_{\rm h})}\,dt'. \tag{3.2}
$$

Using Berstein inequality in the horizontal variables gives

$$
\begin{aligned}
\|\big(e^{t\Delta_{\rm h}}\omega^{\rm h}_0\big)_{\flat,g}\|_{L^\infty_{\rm v}(L^2_{\rm h})} &\lesssim g(t)\|e^{t\Delta_{\rm h}}u^{\rm h}_0\|_{L^\infty_{\rm v}(L^2_{\rm h})}\\
\|\mathbb{1}_{S(t)}(D)\,{\rm div}_{\rm h}(u^{\rm h}(t')\otimes u^{\rm h}(t'))\|_{L^\infty_{\rm v}(L^2_{\rm h})} &\lesssim g^2(t)\|u^{\rm h}(t')\|^2_{L^\infty_{\rm v}(L^2_{\rm h})} \quad \text{and}\\
\big\|\mathbb{1}_{S(t)}(D)\,{\rm div}_{\rm h}(\omega^{\rm h}(t')u^{\rm h}(t'))\big\|_{L^\infty_{\rm v}(L^2_{\rm h})} &\lesssim g^2(t)\|u^{\rm h}(t')\|_{L^\infty_{\rm v}(L^2_{\rm h})}\|\omega^{\rm h}(t')\|_{L^\infty_{\rm v}(L^2_{\rm h})}.
\end{aligned}
$$

This together with the fact that

$$
u^{\rm h}(t) = e^{t\Delta_\varepsilon}u^{\rm h}_0 - \int_0^t e^{(t-t')\Delta_\varepsilon}\mathbb{P}^{\rm h}\,{\rm div}_{\rm h}(u^{\rm h}\otimes u^{\rm h})(t')\,dt'
$$

and (2.5) implies the required inequalities, where $\mathbb{P}^{\rm h} = I - \nabla_{\rm h}\Delta_{\rm h}^{-1}\,{\rm div}_{\rm h}$ denotes the Leray projection operator in two space dimension. □

Lemmas 3.1 and 3.2 can be summarized in the following corollary.

**Corollary 3.1.** *If $u^{\rm h}$ is a regular solution of (NS2.5D), then we have*

$$\|u^{\rm h}(t)\|^2_{L^\infty_{\rm v}(L^2_{\rm h})}\exp\Bigl(2\int_0^t g^2(t')dt'\Bigr)\leq\|u^{\rm h}_0\|^2_{L^\infty_{\rm v}(L^2_{\rm h})}+C\int_0^t\biggl(\|e^{t'\Delta_{\rm h}}u_0^{\rm h}\|^2_{L^\infty_{\rm v}(L^2_{\rm h})}$$
$$+g^4(t')\Bigl(\int_0^{t'}\|u^{\rm h}(t'')\|^2_{L^\infty_{\rm v}(L^2_{\rm h})}dt''\Bigr)^2\biggr)g^2(t')\exp\Bigl(2\int_0^{t'}g^2(t'')dt''\Bigr)dt'\quad\text{and}$$
$$\|\omega^{\rm h}(t)\|^2_{L^\infty_{\rm v}(L^2_{\rm h})}\exp\Bigl(2\int_0^t g^2(t')dt'\Bigr)\leq\|\nabla_{\rm h}u^{\rm h}_0\|^2_{L^\infty_{\rm v}(L^2_{\rm h})}+C\int_0^t\biggl(g^2(t')\|e^{t'\Delta_{\rm h}}u_0^{\rm h}\|^2_{L^\infty_{\rm v}(L^2_{\rm h})}$$
$$+g^4(t')\Bigl(\int_0^{t'}\|u^{\rm h}(t'')\|_{L^\infty_{\rm v}(L^2_{\rm h})}\|\omega^{\rm h}(t'')\|_{L^\infty_{\rm v}(L^2_{\rm h})}dt''\Bigr)\biggr)g^2(t')\exp\Bigl(2\int_0^{t'}g^2(t'')dt''\Bigr)dt'\,.$$

Now let us turn to the proof of Theorem 1.2.

*Proof of Theorem 1.2.* Following Wiegner's method in [15], the first step consists in the application of this corollary to get a very rough decay estimate on $\|u^{\rm h}(t)\|_{L^\infty_{\rm v}(L^2_{\rm h})}$. Let us choose the function $g$ such that

$$2g^2(t)=\frac{3}{T}\Bigl(e+\frac{t}{T}\Bigr)^{-1}\log^{-1}\Bigl(e+\frac{t}{T}\Bigr)\quad\text{which gives}\quad\exp\Bigl(2\int_0^t g^2(t')dt'\Bigr)=\log^3\Bigl(e+\frac{t}{T}\Bigr)\cdot$$

Here the positive real number $T$ is a scaling parameter which will be chosen later on. The first inequality of Corollary 3.1 together with Estimate (2.4) with $p=\infty$ gives

$$\|u^{\rm h}(t)\|^2_{L^\infty_{\rm v}(L^2_{\rm h})}\log^3\Bigl(e+\frac{t}{T}\Bigr)\leq\|u_0^{\rm h}\|^2_{L^\infty_{\rm v}(L^2_{\rm h})}$$
$$+C\int_0^t\Bigl(e+\frac{t'}{T}\Bigr)^{-1}\log^2\Bigl(e+\frac{t'}{T}\Bigr)\|e^{t'\Delta_{\rm h}}u_0^{\rm h}\|^2_{L^\infty_{\rm v}(L^2_{\rm h})}\frac{dt'}{T}$$
$$+C\|u_0^{\rm h}\|^4_{L^\infty_{\rm v}(L^2_{\rm h})}\int_0^t\Bigl(e+\frac{t'}{T}\Bigr)^{-3}\Bigl(\frac{t'}{T}\Bigr)^2\frac{dt'}{T}\,\cdot$$

For any positive $\delta$ less than 1, we have

$$\|e^{t\Delta_{\rm h}}u_0^{\rm h}\|^2_{L^\infty_{\rm v}(L^2_{\rm h})}\leq\frac{1}{t^\delta}\sup_{t>0}t^\delta\|e^{t\Delta_{\rm h}}u_0^{\rm h}\|^2_{L^\infty_{\rm v}(L^2_{\rm h})}.$$

Let us observe that

$$\sup_{t>0}t^\delta\|e^{t\Delta_{\rm h}}u_0^{\rm h}\|^2_{L^\infty_{\rm v}(L^2_{\rm h})}=\sup_{z\in\mathbb{R}_{\rm v}}\sup_{t>0}t^\delta\|e^{t\Delta_{\rm h}}u_0^{\rm h}(\cdot,z)\|^2_{L^2(\mathbb{R}^2_{\rm h})}=\|u_0^{\rm h}\|_{L^\infty_{\rm v}(\dot B^{-\delta}_{2,\infty}(\mathbb{R}^2_{\rm h}))}. \tag{3.3}$$

For $\delta$ positive and less than 1, the function $r\mapsto(e+r)^{-1}r^{-\delta}\log^2(e+r)$ is integrable on $\mathbb{R}^+$. Thus we get that

$$\begin{aligned}\|u^{\rm h}(t)\|^2_{L^\infty_{\rm v}(L^2_{\rm h})}&\leq C_\delta(u_0,T)\log^{-2}\Bigl(e+\frac{t}{T}\Bigr)\quad\text{with}\\ C_\delta(u_0,T)&\stackrel{\rm def}{=}C\|u_0^{\rm h}\|^2_{L^\infty_{\rm v}(L^2_{\rm h})}\bigl(1+\|u_0^{\rm h}\|^2_{L^\infty_{\rm v}(L^2_{\rm h})}\bigr)+\frac{C_\delta}{T^\delta}\|u_0^{\rm h}\|^2_{L^\infty_{\rm v}(\dot B^{-\delta}_{2,\infty}(\mathbb{R}^2_{\rm h}))}.\end{aligned}$$

Thus we have

$$\begin{gathered}T\geq T_\delta(u_0^{\rm h})\Longrightarrow\|u^{\rm h}(t)\|^2_{L^\infty_{\rm v}(L^2_{\rm h})}\leq C_\delta(u_0^{\rm h})\log^{-2}\Bigl(e+\frac{t}{T}\Bigr)\quad\text{with}\\ C(u_0^{\rm h})\stackrel{\rm def}{=}\|u_0^{\rm h}\|^2_{L^\infty_{\rm v}(L^2_{\rm h})}\bigl(1+\|u_0^{\rm h}\|^2_{L^\infty_{\rm v}(L^2_{\rm h})}\bigr)\quad\text{and}\quad T_\delta(u_0^{\rm h})\stackrel{\rm def}{=}C_\delta^{\frac1\delta}\biggl(\frac{\|u_0^{\rm h}\|_{L^\infty_{\rm v}(\dot B^{-\delta}_{2,\infty}(\mathbb{R}^2_{\rm h}))}}{\|u_0^{\rm h}\|_{L^\infty_{\rm v}(L^2_{\rm h})}}\biggr)^{\frac2\delta}.\end{gathered}\tag{3.4}$$

Now let us apply the second inequality of Corollary 3.1 with the function $g$ defined by

$$2g^2(t)=\frac{1}{T}\Bigl(1+\frac{\delta}{2}\Bigr)\Bigl(e+\frac{t}{T}\Bigr)^{-1}\quad\text{which gives}\quad \exp\Bigl(2\int_0^t g^2(t')dt'\Bigr)=e^{-1}\Bigl(e+\frac{t}{T}\Bigr)^{1+\frac{\delta}{2}}.$$

If we define

$$\Omega_\delta(t)\stackrel{\rm def}{=}\sup_{t'\leq t}\Bigl(e+\frac{t'}{T}\Bigr)^{1+\frac{\delta}{2}}\|\omega^{\rm h}(t')\|^2_{L^\infty_{\rm v}(L^2_{\rm h})}\,,$$

this gives

$$\begin{aligned}\Omega_\delta(t)\leq e\|\nabla_{\rm h}u_0^{\rm h}\|^2_{L^\infty_{\rm v}(L^2_{\rm h})}&+\frac{C}{T}\int_0^t\Bigl(e+\frac{t'}{T}\Bigr)^{-1+\frac{\delta}{2}}\|e^{t'\Delta_{\rm h}}u_0^{\rm h}\|^2_{L^\infty_{\rm v}(L^2_{\rm h})}\frac{dt'}{T}\\ &+C\int_0^t\Bigl(e+\frac{t'}{T}\Bigr)^{-2+\frac{\delta}{2}}\Bigl(\int_0^{t'}\|u^{\rm h}(t'')\|_{L^\infty_{\rm v}(L^2_{\rm h})}\|\omega(t'')\|_{L^\infty_{\rm v}(L^2_{\rm h})}\frac{dt''}{T}\Bigr)^2\frac{dt'}{T}\,\cdot\end{aligned}$$

Using (3.3) we get that

$$\begin{aligned}\int_0^t\Bigl(e+\frac{t'}{T}\Bigr)^{-1+\frac{\delta}{2}}\|e^{t'\Delta_{\rm h}}u_0^{\rm h}\|^2_{L^\infty_{\rm v}(L^2_{\rm h})}\frac{dt'}{T}&=\int_0^t\Bigl(e+\frac{t'}{T}\Bigr)^{-1+\frac{\delta}{2}}\Bigl(\frac{t'}{T}\Bigr)^{-\delta}\Bigl(\frac{t'}{T}\Bigr)^{\delta}\|e^{t'\Delta_{\rm h}}u_0^{\rm h}\|^2_{L^\infty_{\rm v}(L^2_{\rm h})}\frac{dt'}{T}\\ &\leq C_\delta\frac{1}{T^\delta}\|u_0^{\rm h}\|^2_{L^\infty_{\rm v}(\dot B^{-\delta}_{2,\infty}(\mathbb{R}^2_{\rm h}))}.\end{aligned}$$

Then Estimate (3.4) and the definition of $\Omega_\delta$ implies that, if $T\geq T_\delta(u_0^{\rm h})$,

$$\begin{aligned}\Omega_\delta(t)\leq e\|\nabla_{\rm h}u_0^{\rm h}\|^2_{L^\infty_{\rm v}(L^2_{\rm h})}&+C_\delta\frac{1}{T^{1+\delta}}\|u_0^{\rm h}\|^2_{L^\infty_{\rm v}(\dot B^{-\delta}_{2,\infty}(\mathbb{R}^2_{\rm h}))}\\ +C(u_0^{\rm h})\int_0^t\Bigl(e+\frac{t'}{T}\Bigr)^{-2+\frac{\delta}{2}}&\Bigl(\int_0^{t'}\Bigl(e+\frac{t''}{T}\Bigr)^{-\frac12-\frac{\delta}{4}}\log^{-1}\Bigl(e+\frac{t''}{T}\Bigr)\frac{dt''}{T}\Bigr)^2\Omega_\delta(t')\frac{dt'}{T}\,\cdot\end{aligned}$$

Since $\delta$ is less than 1, Cauchy Schwarz inequality with the measure $\Bigl(e+\dfrac{t''}{T}\Bigr)^{-\frac12-\frac{\delta}{4}}\dfrac{dt''}{T}$ gives,

$$\begin{aligned}\mathcal{I}(t')&\stackrel{\rm def}{=}\Bigl(\int_0^{t'}\Bigl(e+\frac{t''}{T}\Bigr)^{-\frac12-\frac{\delta}{4}}\log^{-1}\Bigl(e+\frac{t''}{T}\Bigr)\frac{dt''}{T}\Bigr)^2\\ &\leq\Bigl(\int_0^{t'}\Bigl(e+\frac{t''}{T}\Bigr)^{-\frac12-\frac{\delta}{4}}\frac{dt''}{T}\Bigr)\Bigl(\int_0^{t'}\Bigl(e+\frac{t''}{T}\Bigr)^{-\frac12-\frac{\delta}{4}}\log^{-2}\Bigl(e+\frac{t''}{T}\Bigr)\frac{dt''}{T}\Bigr)\\ &\leq C_\delta\Bigl(e+\frac{t'}{T}\Bigr)^{1-\frac{\delta}{2}}\log^{-2}\Bigl(e+\frac{t'}{T}\Bigr)\cdot\end{aligned}$$

Thus we obtain

$$\begin{aligned}\Omega_\delta(t)\leq e\|\nabla_{\rm h}u_0^{\rm h}\|^2_{L^\infty_{\rm v}(L^2_{\rm h})}&+C_\delta\frac{1}{T^{1+\delta}}\|u_0^{\rm h}\|^2_{L^\infty_{\rm v}(\dot B^{-\delta}_{2,\infty}(\mathbb{R}^2_{\rm h}))}\\ &+C(u_0^{\rm h})\int_0^t\Omega_\delta(t')\Bigl(e+\frac{t'}{T}\Bigr)^{-1}\log^{-2}\Bigl(e+\frac{t'}{T}\Bigr)\frac{dt'}{T}\,\cdot\end{aligned}$$

Gronwall lemma implies that

$$\Omega_\delta(t)\leq\Bigl(e\|\nabla_{\rm h}u_0^{\rm h}\|^2_{L^\infty_{\rm v}(L^2_{\rm h})}+C_\delta\frac{1}{T^{1+\delta}}\|u_0^{\rm h}\|^2_{L^\infty_{\rm v}(\dot B^{-\delta}_{2,\infty}(\mathbb{R}^2_{\rm h}))}\Bigr)\exp\bigl(C(u_0^{\rm h})\bigr)\,\cdot$$

By integration this gives, if $T \geq T_\delta(u_0^{\rm h})$,

$$\begin{aligned}\int_0^\infty \|\nabla_{\rm h} u^{\rm h}(t)\|^2_{L^\infty_{\rm v}(L^2_{\rm h})} dt &\leq C\int_0^\infty \Omega_\delta(t')\Big(e+\frac{t}{T}\Big)^{-1-\frac{\delta}{2}} dt\\ &\leq C\Big(\|\nabla_{\rm h} u_0^{\rm h}\|^2_{L^\infty_{\rm v}(L^2_{\rm h})} T + C_\delta \frac{1}{T^\delta}\|u_0^{\rm h}\|^2_{L^\infty_{\rm v}(\dot B^{-\delta}_{2,\infty}(\mathbb{R}^2_{\rm h}))}\Big)\exp\big(C_\delta(u_0^{\rm h})\big)\,.\end{aligned}$$

Selecting $T = T_\delta(u_0^{\rm h})$ in the above inequality concludes the proof of Theorem 1.2. $\square$

## 4. The global wellposedness of (NS3D) for slowly varying initial data

This section follows essentially the lines of [6] up to the fact that the external force due to the error term is simpler and that we deal with less regular solutions. Let us recall the procedure. We consider

$$u^\varepsilon_{\rm app}(t,x) = (u^{\rm h}(t,x_h,\varepsilon x_3),0) \quad \text{and} \quad p^\varepsilon_{\rm app}(t,x) = p^{\rm h}(t,x_h,\varepsilon x_3). \tag{4.1}$$

where $(u^{\rm h},p^{\rm h})$ is the solution of (NS2.5D) with initial data $u_0^{\rm h}$. Let us search the solution of (NS3D) as

$$u^\varepsilon = u^\varepsilon_{\rm app} + R^\varepsilon.$$

Classical computations leads to

$$\left\{\begin{array}{l}\partial_t R^\varepsilon + R^\varepsilon\cdot\nabla R^\varepsilon - \Delta R^\varepsilon + u^\varepsilon_{\rm app}\cdot\nabla R^\varepsilon + R^\varepsilon\cdot\nabla u^\varepsilon_{\rm app} = F^\varepsilon - \nabla q^\varepsilon \quad \text{with}\\ \operatorname{div} R^\varepsilon = 0 \quad \text{and} \quad F^\varepsilon \overset{\rm def}{=} \big(0,\varepsilon\partial_z p^{\rm h}\big)(t,x_h,\varepsilon x_3),\\ R^\varepsilon|_{t=0} = 0\,.\end{array}\right. \tag{4.2}$$

The first step of the proof is to prove a global existence lemma for a perturbed Navier-Stokes system with small external force. Mover precisely, we have the following lemma.

**Lemma 4.1.** *Let us consider a divergence free vector field $v$ such that*

$$\mathcal{N}_v \overset{def}{=} \int_0^\infty N(v(t))dt \quad \textit{with} \quad N(w) \overset{def}{=} \|w\|^2_{L^\infty(\mathbb{R}^3)} + \|\nabla_h w\|^2_{L^\infty_{\rm v}(L^2_{\rm h})} + \|\partial_3 w\|^2_{L^2_{\rm v}(\dot H^{\frac12}(\mathbb{R}^2_{\rm h}))}$$

*is finite and an external force $F$ in $L^2(\mathbb{R}^+;\dot H^{-\frac12}(\mathbb{R}^3))$. We consider the system*

$$(\mathrm{NS})_v\left\{\begin{array}{l}\partial_t w + w\cdot\nabla w - \Delta w + v\cdot\nabla w + w\cdot\nabla v = -\nabla p + F, \qquad (t,x)\in\mathbb{R}^+\times\mathbb{R}^3,\\ \qquad\qquad \operatorname{div} w = 0 \quad \textit{and} \quad w_{|t=0} = 0.\end{array}\right.$$

*Then a positive constant $C_0$ exists such that if*

$$\|F\|^2_{L^2(\mathbb{R}^+;\dot H^{-\frac12}(\mathbb{R}^3))} \leq \frac{1}{C_0}\exp\big(C_0\mathcal{N}_v\big)\,,$$

*the system $(\mathrm{NS})_v$ has a unique global solution $w$ in the space*

$$L^\infty(\mathbb{R}^+;\dot H^{\frac12}(\mathbb{R}^3))\cap L^2(\mathbb{R}^+;\dot H^{\frac32}(\mathbb{R}^3))\,.$$

*Proof.* The fact that the system $(\mathrm{NS})_v$ is locally wellposed follows from a classical Fujita and Kato theory ([9]). In order to prove global existence part of Lemma 4.1, it suffices to control

$$\|w(t)\|^2_{\dot H^{\frac12}(\mathbb{R}^3)} + \int_0^t \|\nabla w(t')\|^2_{\dot H^{\frac12}(\mathbb{R}^3)} dt'\,.$$

Performing a $\dot H^{\frac12}$ energy estimate for $(NS)_v$, we get

$$\begin{aligned}\frac12\frac{d}{dt}\|w(t)\|^2_{\dot H^{\frac12}(\mathbb{R}^3)} + \|\nabla w(t)\|^2_{\dot H^{\frac12}(\mathbb{R}^3)} &= -(w\cdot\nabla w|w)_{\dot H^{\frac12}(\mathbb{R}^3)}\\ &\quad - (v\cdot\nabla w|w)_{\dot H^{\frac12}(\mathbb{R}^3)} - (w\cdot\nabla v|w)_{\dot H^{\frac12}(\mathbb{R}^3)} - (F|w)_{\dot H^{\frac12}(\mathbb{R}^3)}\end{aligned} \tag{4.3}$$

Law of product in Sobolev spaces implies that $\|w\cdot\nabla w\|_{\dot H^{-\frac12}(\mathbb{R}^3)}\leq \|w\|_{\dot H^{\frac12}(\mathbb{R}^3)}\|\nabla w\|_{\dot H^{\frac12}(\mathbb{R}^3)}$. This gives

$$\big|(w\cdot\nabla w|w)_{\dot H^{\frac12}(\mathbb{R}^3)}\big|\leq C\|w\|_{\dot H^{\frac12}(\mathbb{R}^3)}\|\nabla w\|^2_{\dot H^{\frac12}(\mathbb{R}^3)}. \tag{4.4}$$

Using that

$$(a|b)_{\dot H^{\frac12}(\mathbb{R}^3)}\leq \|a\|_{L^2}\|\nabla b\|_{L^2}\,, \tag{4.5}$$

we infer that

$$\big|(v\cdot\nabla w|w)_{\dot H^{\frac12}(\mathbb{R}^3)}\big|\leq \|v\cdot\nabla w\|_{L^2(\mathbb{R}^3)}\|\nabla w\|_{L^2(\mathbb{R}^3)}\leq \|v\|_{L^\infty(\mathbb{R}^3)}\|\nabla w\|^2_{L^2(\mathbb{R}^3)}.$$

Interpolation inequality between Sobolev spaces ensures

$$\big|(v\cdot\nabla w|w)_{\dot H^{\frac12}(\mathbb{R}^3)}\big|\leq C\|v\|_{L^\infty(\mathbb{R}^3)}\|w\|_{\dot H^{\frac12}(\mathbb{R}^3)}\|\nabla w\|_{\dot H^{\frac12}(\mathbb{R}^3)}.$$

Then convexity inequality implies that

$$\big|(v\cdot\nabla w|w)_{\dot H^{\frac12}(\mathbb{R}^3)}\big|\leq \frac{1}{10}\|\nabla w\|^2_{\dot H^{\frac12}(\mathbb{R}^3)}+C\|v\|^2_{L^\infty(\mathbb{R}^3)}\|w\|^2_{\dot H^{\frac12}(\mathbb{R}^3)}. \tag{4.6}$$

The term $(w\cdot\nabla v|w)_{\dot H^{\frac12}(\mathbb{R}^3)}$ is a little bit more delicate. In order to treat it, we must take into account some anisotropy. Using (4.5), we get

$$\big|(w^{\rm h}\cdot\nabla_{\rm h}v|w)_{\dot H^{\frac12}(\mathbb{R}^3)}\big|\leq \|w^{\rm h}\cdot\nabla_{\rm h}v\|_{L^2(\mathbb{R}^3)}\|\nabla w\|_{L^2(\mathbb{R}^3)}.$$

Interpolation theory implies that

$$\|a\|_{L^\infty(\mathbb{R}^2_{\rm h})}\lesssim \|a\|^{\frac12}_{\dot H^{\frac12}(\mathbb{R}^2_{\rm h})}\|\nabla_{\rm h}a\|^{\frac12}_{\dot H^{\frac12}(\mathbb{R}^2_{\rm h})}. \tag{4.7}$$

We deduce that for any $x_3$ in $\mathbb{R}_{\rm v}$, we have

$$\|w^{\rm h}(\cdot,x_3)\cdot\nabla_{\rm h}v(\cdot,x_3)\|_{L^2(\mathbb{R}^2_{\rm h})}\lesssim \|w(\cdot,x_3)\|^{\frac12}_{\dot H^{\frac12}(\mathbb{R}^2_{\rm h})}\|\nabla w(\cdot,x_3)\|^{\frac12}_{\dot H^{\frac12}(\mathbb{R}^2_{\rm h})}\|\nabla_{\rm h}v\|_{L^\infty_{\rm v}(L^2_{\rm h})}.$$

As obviously $\|a\|_{L^2(\mathbb{R}_{\rm v};\dot H^{\frac12}(\mathbb{R}^2_{\rm h}))}$ is less than or equal to $\|a\|_{\dot H^{\frac12}(\mathbb{R}^3)}$, we deduce that

$$\big|(w^{\rm h}\cdot\nabla_{\rm h}v|w)_{\dot H^{\frac12}(\mathbb{R}^3)}\big|\lesssim \|w\|^{\frac12}_{\dot H^{\frac12}(\mathbb{R}^3)}\|\nabla w\|^{\frac12}_{\dot H^{\frac12}(\mathbb{R}^3)}\|\nabla_{\rm h}v\|_{L^\infty_{\rm v}(L^2_{\rm h})}\|\nabla w\|_{L^2(\mathbb{R}^3)}.$$

Interpolation inequality between Sobolev spaces and convexity inequality yields

$$\big|(w^{\rm h}\cdot\nabla_{\rm h}v|w)_{\dot H^{\frac12}(\mathbb{R}^3)}\big|\leq \frac{1}{10}\|\nabla w\|^2_{\dot H^{\frac12}(\mathbb{R}^3)}+C\|\nabla_{\rm h}v\|^2_{L^\infty_{\rm v}(L^2_{\rm h})}\|w\|^2_{\dot H^{\frac12}(\mathbb{R}^3)}. \tag{4.8}$$

Now let us examine the term which involves vertical derivative. Using again (4.5), we are reduced to estimate $\|w^3\partial_3 v\|_{L^2(\mathbb{R}^3)}$. Using law of product of Sobolev spaces in $\mathbb{R}^2_{\rm h}$, we get, for any $x_3$ in $\mathbb{R}_{\rm v}$,

$$\|w^3(\cdot,x_3)\partial_3 v(\cdot,x_3)\|_{L^2_{\rm h}}\leq C\|w(\cdot,x_3)\|_{\dot H^{\frac12}(\mathbb{R}^2_{\rm h})}\|\partial_3(\cdot,x_3)\|_{\dot H^{\frac12}(\mathbb{R}^2_{\rm h})}.$$

Observing that if $\mathcal{H}$ is a Hilbert space and $a$ is regular function from $\mathbb{R}_{\rm v}$into $\mathcal{H}$, we can write

$$\|a(x_3)\|^2_{\mathcal{H}}=2\int_{-\infty}^{x_3}\big(\partial_{y_3}a(y_3)\big|a(y_3)\big)_{\mathcal{H}}dy_3.$$

Then using Cauchy-Schwarz inequality, we get

$$\|a\|^2_{L^\infty(\mathbb{R};\mathcal{H})}\leq 2\|a\|_{L^2(\mathbb{R};\mathcal{H})}\|\partial_3 a\|_{L^2(\mathbb{R};\mathcal{H})}. \tag{4.9}$$

We infer that

$$\|w\|_{L^\infty_{\rm v}(\dot H^{\frac12}(\mathbb{R}^2_{\rm h}))} \leq \sqrt{2}\,\|w\|^{\frac12}_{L^2(\mathbb{R}_{\rm v};\dot H^{\frac12}(\mathbb{R}^2_{\rm h}))}\|\partial_3 w\|^{\frac12}_{L^2(\mathbb{R}_{\rm v};\dot H^{\frac12}(\mathbb{R}^2_{\rm h}))} \leq \sqrt{2}\,\|w\|^{\frac12}_{\dot H^{\frac12}(\mathbb{R}^3))}\|\partial_3 w\|^{\frac12}_{\dot H^{\frac12}(\mathbb{R}^3)}.$$

Thus

$$\|w^3\partial_3 v\|_{L^2(\mathbb{R}^3)} \leq C\|w\|^{\frac12}_{\dot H^{\frac12}(\mathbb{R}^3))}\|\partial_3 w\|^{\frac12}_{\dot H^{\frac12}(\mathbb{R}^3)}\|\partial_3 v\|_{L^2_{\rm v}(\dot H^{\frac12}(\mathbb{R}^2_{\rm h}))}.$$

After interpolation and convexity inequality, we infer that

$$\begin{aligned}\big|(w^3\cdot\partial_3 v|w)_{\dot H^{\frac12}(\mathbb{R}^3)}\big| \leq& \|w^3\partial_3 v\|_{L^2(\mathbb{R}^3)}\|\nabla w\|_{L^2(\mathbb{R}^3)}\\ \leq& \frac{1}{10}\|\nabla w\|^2_{\dot H^{\frac12}(\mathbb{R}^3)} + C\|\partial_3 v\|^2_{L^2_{\rm v}(\dot H^{\frac12}(\mathbb{R}^2_{\rm h}))}\|w\|^2_{\dot H^{\frac12}(\mathbb{R}^3)}.\end{aligned}\tag{4.10}$$

As we have

$$(F|w)_{\dot H^{\frac12}(\mathbb{R}^3)} \leq \frac{1}{10}\|\nabla w\|^2_{\dot H^{\frac12}(\mathbb{R}^3)} + 10\|F\|^2_{\dot H^{-\frac12}(\mathbb{R}^3)},$$

we infer from (4.3), (4.4), (4.6), (4.8) and (4.10) that

$$\begin{aligned}\frac12\frac{d}{dt}\|w(t)\|^2_{\dot H^{\frac12}(\mathbb{R}^3)} + \frac35\|\nabla w(t)\|^2_{\dot H^{\frac12}(\mathbb{R}^3)} \leq& C\|w(t)\|_{\dot H^{\frac12}(\mathbb{R}^3)}\|\nabla w(t)\|^2_{\dot H^{\frac12}(\mathbb{R}^3)}\\ &+ CN(v(t))\|w(t)\|^2_{\dot H^{\frac12}(\mathbb{R}^3)} + 10\|F\|^2_{\dot H^{\frac12}(\mathbb{R}^3)}.\end{aligned}$$

Defining $w_\lambda(t) \stackrel{\rm def}{=} \exp\Big(-\lambda\displaystyle\int_0^t N(v(t'))dt'\Big)$. Then for $\lambda \geq C$, we deduce that

$$\frac12\frac{d}{dt}\|w_\lambda(t)\|^2_{\dot H^{\frac12}(\mathbb{R}^3)} + \frac35\|\nabla w_\lambda(t)\|^2_{\dot H^{\frac12}(\mathbb{R}^3)} \leq C\|w(t)\|_{\dot H^{\frac12}(\mathbb{R}^3)}\|\nabla w_\lambda(t)\|^2_{\dot H^{\frac12}(\mathbb{R}^3)} + 10\|F\|^2_{\dot H^{\frac12}(\mathbb{R}^3)}.$$

Thus as long as

$$C\|w(t)\|_{\dot H^{\frac12}(\mathbb{R}^3)} \leq \frac{1}{10}\,,\tag{4.11}$$

we have that

$$\frac{d}{dt}\|w_\lambda(t)\|^2_{\dot H^{\frac12}(\mathbb{R}^3)} + \|\nabla w_\lambda(t)\|^2_{\dot H^{\frac12}(\mathbb{R}^3)} \leq 20\|F\|^2_{\dot H^{\frac12}(\mathbb{R}^3)}.$$

So that whenever the Condition (4.11) is satisfied, we have

$$\|w(t)\|^2_{\dot H^{\frac12}(\mathbb{R}^3)} + \int_0^t \|\nabla w(t')\|^2_{\dot H^{\frac12}(\mathbb{R}^3)}dt' \leq 20\|F\|^2_{L^2(\mathbb{R}^+;\dot H^{\frac12}(\mathbb{R}^3))}\exp\big(\lambda\mathcal{N}_v\big)\,.$$

Hence by a very classical continuation argument, we get that, if

$$20\|F\|^2_{L^2(\mathbb{R}^+;\dot H^{\frac12}(\mathbb{R}^3))}\exp\big(\lambda\mathcal{N}_v\big) \leq \frac{1}{121C^2}\,,$$

then Condition (4.11) is always satisfied and the norms $L^\infty(\mathbb{R}^+;\dot H^{\frac12}(\mathbb{R}^3))$ and $L^2(\mathbb{R}^+;\dot H^{\frac32}(\mathbb{R}^3))$ of the solution are controlled and the lemma is proved. □

Now let us compute $N(u^\varepsilon_{\rm app}(t))$. In view of (4.1) and because of the vertical scaling, we have

$$N(u^{\rm h}_{\varepsilon,{\rm app}}(t)) = \|u^{\rm h}(t)\|^2_{L^\infty(\mathbb{R}^3)} + \|\nabla_{\rm h}u^{\rm h}(t)\|^2_{L^\infty_{\rm v}(L^2_{\rm h})} + \varepsilon\|\partial_z u^{\rm h}(t)\|^2_{L^2_{\rm v}(\dot H^{\frac12}(\mathbb{R}^2_{\rm h}))}.\tag{4.12}$$

In order to control $N(u^\varepsilon_{\rm app}(t))$, we need the following propagation lemma.

**Lemma 4.2.** *Let $u^{\rm h}$ be a regular solution of (NS2.5D). Then for any $s$ between $-1$ and $1$, we have*

$$\|u^{\rm h}(t)\|^2_{L^\infty_{\rm v}(\dot H^s(\mathbb{R}^2_{\rm h}))} \leq \|u^{\rm h}_0\|^2_{L^\infty_{\rm v}(\dot H^s(\mathbb{R}^2_{\rm h}))} \exp\biggl(C_s\int_0^t \|\nabla_{\rm h}u^{\rm h}(t')\|^2_{L^\infty_{\rm v}(L^2_{\rm h})}dt'\biggr),$$

$$\|u^{\rm h}(t)\|^2_{L^2_{\rm v}(\dot H^s(\mathbb{R}^2_{\rm h}))} + \int_0^t \|\nabla_{\rm h}u^{\rm h}(t')\|^2_{L^2_{\rm v}(\dot H^s(\mathbb{R}^2_{\rm h}))}dt'$$
$$\leq \|u^{\rm h}_0\|^2_{L^2_{\rm v}(\dot H^s(\mathbb{R}^2_{\rm h}))} \exp\biggl(C_s\int_0^t \|\nabla_{\rm h}u^{\rm h}(t')\|^2_{L^\infty_{\rm v}(L^2_{\rm h})}dt'\biggr),$$

*and*

$$\|\partial_z u^{\rm h}(t)\|^2_{L^2_{\rm v}(\dot H^s(\mathbb{R}^2_{\rm h}))} + \int_0^t \|\nabla_{\rm h}\partial_z u^{\rm h}(t')\|^2_{L^2_{\rm v}(\dot H^s(\mathbb{R}^2_{\rm h}))}dt'$$
$$\leq \|\partial_z u^{\rm h}_0\|^2_{L^2_{\rm v}(\dot H^s(\mathbb{R}^2_{\rm h}))} \exp\biggl(C_s\int_0^t \|\nabla_{\rm h}u^{\rm h}(t')\|^2_{L^\infty_{\rm v}(L^2_{\rm h})}dt'\biggr).$$

*Proof.* Let us perform a $\dot H^s$ energy estimate in the horizontal variables for (NS2.5D). This gives, for any $z$ in $\mathbb{R}_{\rm v}$,

$$\frac12\frac{d}{dt}\|u^{\rm h}(t,\cdot,z)\|^2_{\dot H^s(\mathbb{R}^2_{\rm h})} + \|\nabla_\varepsilon u^{\rm h}(t,\cdot,z)\|^2_{\dot H^s(\mathbb{R}^2_{\rm h})} - \frac{\varepsilon^2}{2}\partial_z^2\|u^{\rm h}(t,\cdot,z)\|^2_{\dot H^s(\mathbb{R}^2_{\rm h})}$$
$$= -\bigl(u^{\rm h}(t,\cdot,z)\cdot\nabla_{\rm h}u^{\rm h}(t,\cdot,z)\big|u^{\rm h}(t,\cdot,z)\bigr)_{\dot H^s(\mathbb{R}^2_{\rm h})}.$$

Lemma 1.1 of [5] implies that

$$\bigl|\bigl(u^{\rm h}(t,\cdot,z)\cdot\nabla_{\rm h}u^{\rm h}(t,\cdot,z)\big|u^{\rm h}(t,\cdot,z)\bigr)_{\dot H^s(\mathbb{R}^2_{\rm h})}\bigr|$$
$$\lesssim \|\nabla_{\rm h}u^{\rm h}(t,\cdot,z)\|_{\dot H^s(\mathbb{R}^2_{\rm h})}\|\nabla_{\rm h}u^{\rm h}(t,\cdot,z)\|_{L^2(\mathbb{R}^2_{\rm h})}\|u^{\rm h}(t,\cdot,z)\|_{\dot H^s(\mathbb{R}^2_{\rm h})}.$$

Convexity inequality implies that

$$\bigl|\bigl(u^{\rm h}(t,\cdot,z)\cdot\nabla_{\rm h}u^{\rm h}(t,\cdot,z)\big|u^{\rm h}(t,\cdot,z)\bigr)_{\dot H^s(\mathbb{R}^2_{\rm h})}\bigr|$$
$$\leq \frac12\|\nabla_{\rm h}u^{\rm h}(t,\cdot,z)\|^2_{\dot H^s(\mathbb{R}^2_{\rm h})} + C\|\nabla_{\rm h}u^{\rm h}(t,\cdot,z)\|^2_{L^2(\mathbb{R}^2_{\rm h})}\|u^{\rm h}(t,\cdot,z)\|^2_{\dot H^s(\mathbb{R}^2_{\rm h})}.$$

Defining

$$u^{\rm h}_\lambda(t,\cdot,z) \stackrel{\rm def}{=} \exp\biggl(-\lambda\int_0^t \|\nabla_{\rm h}u^{\rm h}(t')\|_{L^\infty_{\rm v}(L^2_{\rm h})}dt'\biggr)u^{\rm h}(t,\cdot,z).$$

Then for $\lambda \geq C$, we can write

$$\frac{d}{dt}\|u^{\rm h}_\lambda(t,\cdot,z)\|^2_{\dot H^s(\mathbb{R}^2_{\rm h})} + \|\nabla_\varepsilon u^{\rm h}_\lambda(t,\cdot,z)\|^2_{\dot H^s(\mathbb{R}^2_{\rm h})} - \frac{\varepsilon^2}{2}\partial_z^2\|u^{\rm h}_\lambda(t,\cdot,z)\|^2_{\dot H^s(\mathbb{R}^2_{\rm h})} \leq 0.$$

We get the first inequality of the lemma by maximal principle, and the second one by integration in $z$ and in time.

In order to prove the third inequality, we take $\partial_z$ to the system (NS2.5D) and then perform a $\dot H^s$ energy estimate in the horizontal variables for the resulting equation. This gives, for any $z$ in $\mathbb{R}_{\rm v}$,

$$\frac12\frac{d}{dt}\|\partial_z u^{\rm h}(t,\cdot,z)\|^2_{\dot H^s(\mathbb{R}^2_{\rm h})} + \|\nabla_\varepsilon\partial_z u^{\rm h}(t,\cdot,z)\|^2_{\dot H^s(\mathbb{R}^2_{\rm h})} - \frac{\varepsilon^2}{2}\partial_z^2\|\partial_z u^{\rm h}(t,\cdot,z)\|^2_{\dot H^s(\mathbb{R}^2_{\rm h})}$$
$$= -\bigl(u^{\rm h}(t,\cdot,z)\cdot\nabla_{\rm h}\partial_z u^{\rm h}(t,\cdot,z)\big|\partial_z u^{\rm h}(t,\cdot,z)\bigr)_{\dot H^s(\mathbb{R}^2_{\rm h})} \tag{4.13}$$
$$-\bigl(\partial_z u^{\rm h}(t,\cdot,z)\cdot\nabla_{\rm h}u^{\rm h}(t,\cdot,z)\big|\partial_z u^{\rm h}(t,\cdot,z)\bigr)_{\dot H^s(\mathbb{R}^2_{\rm h})}.$$

Again Lemma 1.1 of [5] implies that

$$
\begin{aligned}
&\big|\big(u^{\rm h}(t,\cdot,z)\cdot\nabla_{\rm h}\partial_z u^{\rm h}(t,\cdot,z)\big|\partial_z u^{\rm h}(t,\cdot,z)\big)_{\dot H^s(\mathbb{R}^2_{\rm h})}\big| \\
&\qquad\lesssim \|\nabla_{\rm h}u^{\rm h}(t,\cdot,z)\|_{L^2(\mathbb{R}^2_{\rm h})}\|\partial_z\nabla_{\rm h}u^{\rm h}(t,\cdot,z)\|_{\dot H^s(\mathbb{R}^2_{\rm h})}\|\partial_z u^{\rm h}(t,\cdot,z)\|_{\dot H^s(\mathbb{R}^2_{\rm h})}\,.
\end{aligned}
\tag{4.14}
$$

If $s=0$, we use Sobolev embeddings and interpolation theory to write

$$
\begin{aligned}
\big|\big(\partial_z u^{\rm h}(t,\cdot,z)\cdot\nabla_{\rm h}u^{\rm h}(t,\cdot,z)\big|\partial_z u^{\rm h}(t,\cdot,z)\big)_{L^2(\mathbb{R}^2_{\rm h})}\big| &\leq \|\nabla_{\rm h}u^{\rm h}(t,\cdot,z)\|_{L^2(\mathbb{R}^2_{\rm h})}\|\partial_z u^{\rm h}(t,\cdot,z)\|^2_{L^4(\mathbb{R}^2_{\rm h})}\\
&\lesssim \|\nabla_{\rm h}u^{\rm h}(t,\cdot,z)\|_{L^2(\mathbb{R}^2_{\rm h})}\|\partial_z u^{\rm h}(t,\cdot,z)\|_{L^2(\mathbb{R}^2_{\rm h})}\\
&\qquad\qquad\times\|\nabla_{\rm h}\partial_z u^{\rm h}(t,\cdot,z)\|_{L^2(\mathbb{R}^2_{\rm h})}.
\end{aligned}
$$

If $s$ is different from 0, laws of product in Sobolev space imply that

$$
\|\partial_z u^{\rm h}(t,\cdot,z)\cdot\nabla_{\rm h}u^{\rm h}(t,\cdot,z)\|_{\dot H^{s-1}(\mathbb{R}^2_{\rm h})}\lesssim\|\partial_z u^{\rm h}(t,\cdot,z)\|_{\dot H^s(\mathbb{R}^2_{\rm h})}\|\nabla_{\rm h}u^{\rm h}(t,\cdot,z)\|_{L^2(\mathbb{R}^2_{\rm h})}
$$

if $s$ belongs to $]0,1[$. If $s$ is in $]-1,0[$, we have

$$
\|\partial_z u^{\rm h}(t,\cdot,z)\cdot\nabla_{\rm h}u^{\rm h}(t,\cdot,z)\|_{\dot H^{s}(\mathbb{R}^2_{\rm h})}\lesssim\|\partial_z\nabla_{\rm h} u^{\rm h}(t,\cdot,z)\|_{\dot H^s(\mathbb{R}^2_{\rm h})}\|\nabla_{\rm h}u^{\rm h}(t,\cdot,z)\|_{L^2(\mathbb{R}^2_{\rm h})}
$$

Plugging these estimates and (4.14) into (4.13) and using convexity inequality, we obtain

$$
\begin{aligned}
&\frac{d}{dt}\|\partial_z u^{\rm h}(t,\cdot,z)\|^2_{\dot H^s(\mathbb{R}^2_{\rm h})}+\|\nabla_\varepsilon\partial_z u^{\rm h}(t,\cdot,z)\|^2_{\dot H^s(\mathbb{R}^2_{\rm h})}-\frac{\varepsilon^2}{2}\partial_z^2\|\partial_z u^{\rm h}(t,\cdot,z)\|^2_{\dot H^s(\mathbb{R}^2_{\rm h})}\\
&\qquad\qquad\leq C\|\nabla_{\rm h}u^{\rm h}(t,\cdot,z)\|^2_{L^2(\mathbb{R}^2_{\rm h})}\|\partial_z u^{\rm h}(t,\cdot,z)\|^2_{\dot H^s(\mathbb{R}^2_{\rm h})}.
\end{aligned}
$$

Arguing as in the proof of the second inequality allows to conclude the proof of the lemma. □

We can deduce from this lemma the following corollary.

**Corollary 4.1.** *Let $u^{\rm h}$ be a regular solution of (NS2.5D). Then under the assumptions of Theorem 1.3, one has, for $A_\delta(u_0^{\rm h})$ given by* (1.3),

$$
\begin{aligned}
&\mathcal{N}_{u^\varepsilon_{\rm app}}\lesssim A_\delta(u_0^{\rm h})+U_0\exp\big(CA_\delta(u_0^{\rm h})\big)\quad\text{with}\\
&U_0\stackrel{\rm def}{=}\varepsilon\|\partial_z u_0^{\rm h}\|^2_{L^2_{\rm v}(\dot H^{-\frac12}(\mathbb{R}^2_{\rm h}))}+\|u_0^{\rm h}\|^{\frac12}_{L^2_{\rm v}(\dot H^{-\frac12}(\mathbb{R}^2_{\rm h}))}\|\partial_z u_0^{\rm h}\|^{\frac12}_{L^2_{\rm v}(\dot H^{-\frac12}(\mathbb{R}^2_{\rm h}))}\\
&\qquad\qquad\times\|u_0^{\rm h}\|^{\frac12}_{L^2_{\rm v}(\dot H^{\frac12}(\mathbb{R}^2_{\rm h}))}\|\partial_z u_0^{\rm h}\|^{\frac12}_{L^2_{\rm v}(\dot H^{\frac12}(\mathbb{R}^2_{\rm h}))}.
\end{aligned}
\tag{4.15}
$$

*Proof.* Lemma 4.2 implies that

$$
\int_0^\infty\|\partial_z u^{\rm h}(t)\|^2_{L^2_{\rm v}(\dot H^{\frac12}(\mathbb{R}^2_{\rm h}))}dt\leq\|\partial_z u_0^{\rm h}\|^2_{L^2_{\rm v}(\dot H^{-\frac12}(\mathbb{R}^2_{\rm h}))}\exp\biggl(C\int_0^\infty\|\nabla_{\rm h}u^{\rm h}(t,\cdot)\|^2_{L^\infty_{\rm v}(L^2_{\rm h})}dt\biggr).
\tag{4.16}
$$

As we have

$$
\|u^{\rm h}(t,\cdot,z)\|^2_{L^\infty_{\rm h}}\lesssim\|u^{\rm h}(t,\cdot,z)\|_{\dot H^{\frac12}(\mathbb{R}^2_{\rm h})}\|\nabla_{\rm h}u^{\rm h}(t,\cdot,z)\|_{\dot H^{\frac12}(\mathbb{R}^2_{\rm h})}.
$$

Using (4.9), we infer that

$$
\begin{aligned}
\|u^{\rm h}(t)\|^2_{L^\infty(\mathbb{R}^3)}&\lesssim\|u^{\rm h}(t)\|^{\frac12}_{L^2(\mathbb{R}_{\rm v};\dot H^{\frac12}(\mathbb{R}^2_{\rm h}))}\|\partial_z u^{\rm h}(t)\|^{\frac12}_{L^2(\mathbb{R}_{\rm v};\dot H^{\frac12}(\mathbb{R}^2_{\rm h}))}\\
&\qquad\times\|\nabla_{\rm h}u^{\rm h}(t)\|^{\frac12}_{L^2(\mathbb{R}_{\rm v};\dot H^{\frac12}(\mathbb{R}^2_{\rm h}))}\|\partial_z\nabla_{\rm h}u^{\rm h}(t)\|^{\frac12}_{L^2(\mathbb{R}_{\rm v};\dot H^{\frac12}(\mathbb{R}^2_{\rm h}))}.
\end{aligned}
$$

Lemma 4.2 implies that

$$\int_0^\infty \|u^{\rm h}(t)\|^2_{L^\infty(\mathbb{R}^3)}dt \lesssim \|u^{\rm h}_0\|^{\frac12}_{L^2_{\rm v}(\dot H^{-\frac12}(\mathbb{R}^2_{\rm h}))}\|\partial_z u^{\rm h}_0\|^{\frac12}_{L^2_{\rm v}(\dot H^{-\frac12}(\mathbb{R}^2_{\rm h}))}$$
$$\times \|u^{\rm h}_0\|^{\frac12}_{L^2_{\rm v}(\dot H^{\frac12}(\mathbb{R}^2_{\rm h}))}\|\partial_z \nabla_{\rm h} u^{\rm h}_0\|^{\frac12}_{L^2_{\rm v}(\dot H^{\frac12}(\mathbb{R}^2_{\rm h}))}\exp\Bigl(C\int_0^\infty \|\nabla_{\rm h}u^{\rm h}(t,\cdot)\|^2_{L^\infty_{\rm v}(L^2_{\rm h})}dt\Bigr).$$

Together with (4.12) and Theorem 1.2, this ensures (4.15). □

Finally let us present the proof of Theorem 1.3.

*Proof of Theorem 1.3.* By virtue of Lemma 4.1 and Corollary 4.1, in order to conclude the proof of the Theorem, it amounts to prove that

$$\|F_\varepsilon\|_{L^2(\mathbb{R}^+;\dot H^{-\frac12}(\mathbb{R}^3))} \lesssim C(u^{\rm h}_0)\,\varepsilon^{\frac12}. \tag{4.17}$$

Toward this, we get, by first applying the operator $\mathrm{div}_{\rm h}$ to the equation (NS2.5D) and then taking $\partial_z$ to the resulting equation, that

$$\partial_z p^{\rm h}(t,\cdot,,z) = 2\sum_{1\leq j,k\leq 2}(-\Delta_{\rm h})^{-1}\partial_j\partial_k\bigl(u^{{\rm h},j}(t,\cdot,z)\partial_z u^{{\rm h},k}(t,\cdot,z)\bigr).$$

Law of product for Sobolev spaces for the horizontal variables implies that

$$\bigl\|u^{{\rm h},j}(t,\cdot,z)\partial_z u^{{\rm h},k}(t,\cdot,z)\bigr\|_{\dot H^{-\frac12}(\mathbb{R}^2_{\rm h})} \lesssim \|u^{\rm h}(t,\cdot,z)\|_{L^2_{\rm h}}\bigl\|\partial_z u^{\rm h}(t,\cdot,z)\bigr\|_{\dot H^{\frac12}(\mathbb{R}^2_{\rm h})}.$$

As $(-\Delta_{\rm h})^{-1}\partial_j\partial_k$ is a bounded Fourier multiplier, we get that

$$\|\partial_z p^{\rm h}(t)\|_{L^2_{\rm v}(\dot H^{-\frac12}(\mathbb{R}^2_{\rm h}))} \lesssim \|u^{\rm h}(t)\|_{L^\infty_{\rm v}(L^2_{\rm h})}\bigl\|\partial_z u^{\rm h}(t)\bigr\|_{L^2_{\rm v}(\dot H^{\frac12}(\mathbb{R}^2_{\rm h}))}.$$

Changing variable $z$ into $\varepsilon x_3$ gives

$$\|F_\varepsilon(t)\|_{L^2_{\rm v}(\dot H^{-\frac12}(\mathbb{R}^2_{\rm h}))} \lesssim \varepsilon^{\frac12}\|\partial_z p^{\rm h}(t)\|_{L^2_{\rm v}(\dot H^{-\frac12}(\mathbb{R}^2_{\rm h}))} \lesssim \varepsilon^{\frac12}\|u^{\rm h}(t)\|_{L^\infty_{\rm v}(L^2_{\rm h})}\bigl\|\partial_z u^{\rm h}(t)\bigr\|_{L^2_{\rm v}(\dot H^{\frac12}(\mathbb{R}^2_{\rm h}))}.$$

Hence we obtain

$$\begin{aligned}\|F_\varepsilon\|_{L^2(\mathbb{R}^+;\dot H^{-\frac12}(\mathbb{R}^3))} &\lesssim \|F_\varepsilon\|_{L^2(\mathbb{R}^+;L^2_{\rm v}(\dot H^{-\frac12}(\mathbb{R}^2_{\rm h})))}\\ &\lesssim \varepsilon^{\frac12}\|u^{\rm h}\|_{L^\infty_t(L^\infty_{\rm v}(L^2_{\rm h}))}\bigl\|\partial_z u^{\rm h}\bigr\|_{L^2(\mathbb{R}^+;L^2_{\rm v}(\dot H^{\frac12}(\mathbb{R}^2_{\rm h})))}.\end{aligned}$$

Lemma 4.2 allows to conclude the proof of (4.17) and thus of Theorem 1.3. □

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