ON THE CRITICAL ONE COMPONENT REGULARITY FOR 3-D NAVIER-STOKES SYSTEM

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Abstract. Given an initial data v_0 with vorticity $\Omega_0 = \nabla \times v_0$ in $L^{\frac{3}{2}}$, (which implies that v_0 belongs to the Sobolev space $H^{\frac{1}{2}}$), we prove that the solution v given by the classical Fujita-Kato theorem blows up in a finite time $T^\star$ only if, for any p in $]4,6[$ and any unit vector e in $\mathbb{R}^3$, there holds $\int_0^{T^\star} \|v(t)\cdot e\|^p_{H^{\frac{1}{2}+\frac{2}{p}}}\,dt = \infty$. We remark that all these quantities are scaling invariant under the scaling transformation of Navier-Stokes system.

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1. Introduction

In the present work, we investigate necessary conditions for the breakdown of the regularity of regular solutions to the following 3-D homogeneous incompressible Navier-Stokes system

$$(NS)\qquad \begin{cases} \partial_t v + \mathrm{div}(v\otimes v) - \Delta v + \nabla \Pi = 0, & (t,x)\in \mathbb{R}^+\times\mathbb{R}^3,\\ \mathrm{div}\, v = 0,\\ v|_{t=0} = v_0, \end{cases}$$

where $v = (v^1, v^2, v^3)$ stands for the velocity of the fluid and Π for the pressure. Let us first recall some fundamental results proved by J. Leray in his seminal paper [19].

Theorem 1.1. *Let us consider an initial data v_0 which belongs to the inhomogeneous Sobolev space $H^1_{\mathrm{in}}(\mathbb{R}^3)$. There exists a (unique) maximal positive time of existence $T^\star$ such that a unique solution v of (NS) exists on $[0,T^\star[\times\mathbb{R}^3$, which is continuous with value in $H^1_{\mathrm{in}}(\mathbb{R}^3)$ and the gradient of which belongs to $L^2_{\mathrm{loc}}([0,T^\star[;H^1_{\mathrm{in}}(\mathbb{R}^3))$. Moreover, if $\|v_0\|_{L^2}\|\nabla v_0\|_{L^2}$ is small enough, then $T^\star$ is infinite. If $T^\star$ is finite, we have, for any q greater than 3,*

$$\forall\ t < T^\star\,,\ \ \|v(t)\|_{L^q} \geq \frac{C_q}{(T^\star - t)^{\frac{1}{2}\left(1-\frac{3}{q}\right)}}\,\cdotp$$

Let us also mention that in [19], J. Leray proved also the existence (but not the uniqueness) of global weak (turbulent in J. Leray's terminology) solutions of (NS) with initial data only in $L^2(\mathbb{R}^3)$. In the present paper, we only deal with solutions which are regular to be unique.

In [19], J. Leray emphasized two basic facts about the homogeneous incompressible Navier-Stokes system: the L^2 energy estimate and the scaling invariance.

Because the vector field v is divergence free, the energy estimate formally reads

$$\frac{1}{2}\frac{d}{dt}\|v(t)\|^2_{L^2} + \|\nabla v(t)\|^2_{L^2} = 0.$$

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After time integration, this gives

$$\frac{1}{2}\|v(t)\|_{L^2}^2+\int_0^t\|\nabla v(t')\|_{L^2}^2dt'=\frac{1}{2}\|v_0\|_{L^2}^2. \tag{1.1}$$

This estimate is the cornerstone of the proof of the existence of global turbulent solution to (NS) done by J. Leray in [19]. The energy estimate relies (formally) on the fact that if v is a divergence free vector field, $(v\cdot\nabla f|f)_{L^2}=0$ and that $(\nabla p|v)_{L^2}=0$. In the present work, we shall use the more general fact that for any divergence free vector field v and any function a, we have

$$\int_{\mathbb{R}^3}v(x)\cdot\nabla a(x)|a(x)|^{p-2}a(x)\,dx=0\qquad\text{for any}\ \ p\in]1,\infty[.$$

This will lead to the L^p type energy estimate.

The scaling invariance is the fact that if v is a solution of (NS) on $[0,T]\times\mathbb{R}^3$ associated with an initial data v_0, then $\lambda v(\lambda^2t,\lambda x)$ is also a solution of (NS) on $[0,\lambda^{-2}T]\times\mathbb{R}^3$ associated with the initial data $\lambda v_0(\lambda x)$. The importance of this point can be illustrated by this sentence coming from [19] *" . . . les équations aux dimensions permettent de prévoir a priori presque toutes les inégalités que nous écrirons . . . "*[1] The scaling property is also the foundation of the Kato theory which gives a general method to solve (locally or globally) the incompressible Navier-Stokes equation in critical spaces i.e. spaces with the norms of which are invariant under the scaling. In the present work, we only use such scaling invariant spaces. Let us exhibit some examples of scaling invariant norms. For $p\geq 2$, the norms of

$$L^p_t(H^{\frac12+\frac2p})\quad\text{and}\quad L^p_t(L_x^{3+\frac{6}{p-2}}).$$

are scaling invariant norms. The spaces $H^{\frac12}$ are L^3 are scaling invariant spaces for the initial data v_0 . Let us point out that in the case when the space dimension is two, the energy norm which appears in Relation (1.1) is scaling invariant. This allows to prove that in the two dimensional case, turbulent solutions are unique and regular.

The first result of local (and global for small initial data) wellposedness of (NS) in a scaling invariant space was proved by H. Fujita and T. Kato in 1964 (see [14]) for initial data in the homogenenous Sobolev space $H^{\frac12}$. More precisely, we have the following statement.

Theorem 1.2. *Let us consider an initial data v_0 in the homogeneous Sobolev space $H^{\frac12}(\mathbb{R}^3)$. There exists a (unique) maximal positive time of existence $T^\star$ such that a unique solution v of (NS) exists on $[0,T^\star[\times\mathbb{R}^3$ which is continuous in time with value in $H^{\frac12}(\mathbb{R}^3)$ and belongs to $L^2_{\rm loc}([0,T^\star[;H^{\frac32}(\mathbb{R}^3))$. Moreover, if the quantity $\|v_0\|_{H^{\frac12}}$ is small enough, then $T^\star$ is infinite. If $T^\star$ is finite, we have, for any q greater than 3,*

$$\forall\ t<T^\star,\ \|v(t)\|_{L^q}\geq C_q\frac{1}{(T^\star-t)^{\frac12\left(1-\frac3q\right)}}\,\cdot$$

Let us point out that the above necessary condition for blow up implies that

$$T^\star<\infty\Longrightarrow\int_0^{T^\star}\|v(t)\|_{L^q}^pdt=\infty\quad\text{with}\quad\frac2p+\frac3q=1\quad\text{and}\quad p<\infty. \tag{1.2}$$

Let us mention that it is possible to prove this theorem without using the energy estimate and this theorem is true for a large class of systems which have the same scaling as the incompressible Navier-Stokes system.

[1] This can be translated by *"The scaling allows to guess almost all the inequalities written in this paper"*

Using results related to the energy estimate, L. Iskauriaza, G. A. Serëgin and V. Sverak proved in 2003 the end point case of (1.2) when p is infinite (see [12]). This remarkable result has been extended to Besov space with negative index (see [10]). Let us also mention a blow up criteria proposed by Beirão da Veiga [3], which states that if the maximal time $T^\star$ of existence of a regular solution v to (NS) is finite, then we have

$$\int_0^{T^\star} \|\nabla v(t)\|_{L^q}^p dt = \infty \quad \text{with} \quad \frac{2}{p}+\frac{3}{q}=2 \quad \text{for} \quad q \geq \frac{3}{2}\cdot \tag{1.3}$$

Let us observe that because of the fact that homogeneous bounded Fourier multipliers maps L^p into L^p, this criteria is equivalent, for q is finite, to

$$\int_0^{T^\star} \|\Omega(t)\|_{L^q}^p dt = \infty \quad \text{where} \quad \Omega \stackrel{\text{def}}{=} \nabla \times v. \tag{1.4}$$

In this case when q is infinite, this criteria is the classical Beale-Kato-Majda theorem (see [2]) which is in fact a result about Euler equation and where the viscosity plays no role.

In the present paper, we want to establish necessary conditions for breakdown of regularity of solutions to (NS) given by Theorem 1.2 in term of the scaling invariant norms of one component of the velocity field. Because we shall use the $L^{\frac{3}{2}}$ norm of the vorticity, we work with solution given by the following theorem, which are a little bit more regular than that given by Theorem 1.2.

Theorem 1.3. *Let us consider an initial data v_0 with vorticity $\Omega_0 = \nabla \times v_0$ in $L^{\frac{3}{2}}$. Then a unique maximal solution v of (NS) exists in the space $C([0,T^*[;H^{\frac{1}{2}}) \cap L^2_{\rm loc}([0,T^\star[;H^{\frac{3}{2}})$ for some positive time T^*, and the vorticity $\Omega = \nabla \times v$ is continuous on $[0,T^*[$ with value in $L^{\frac{3}{2}}$ and Ω satisfies*

$$|\nabla\Omega|\,|\Omega|^{-\frac{1}{4}} \in L^2_{\rm loc}([0,T^\star[;L^2).$$

This theorem is classical. For the reader's convenience, we prove it in the third section where we insist on the importance of $L^{\frac{3}{2}}$ energy estimate for the vorticity.

The main theorem of this paper is the following.

Theorem 1.4. *We consider a maximal solution v of (NS) given by Theorem 1.3. Let p be in $]4,6[$, e a unit vector of $\mathbb{R}^3$, and $v_e \stackrel{\text{def}}{=} v\cdot e$. Then if $T^\star < \infty$, we have*

$$\int_0^{T^\star} \|v_e(t)\|^p_{H^{\frac{1}{2}+\frac{2}{p}}}\, dt = \infty. \tag{1.5}$$

Let us remark that the quantity $\displaystyle\int_0^{T} \|v_e(t)\|^p_{H^{\frac{1}{2}+\frac{2}{p}}}\, dt$ is scaling invariant. Moreover, it gives a necessary blow up condition which involves only a scaling invariant norm to one component of the velocity. Or equivalently, it claims that if the maximal time of existence $T^\star$ is finite, v blows up in any direction and thus is in some sense isotropic.

The first result in that direction is obtained in a pionnier work by J. Neustupa and P. Penel (see [20]) but the norm involded was not scaling invariant. A lot of works (see [5, 6, 16, 17, 21, 23, 24, 25, 26]) generalized established conditions of the type

$$\int_0^{T^\star} \|v^3(t,\cdot)\|_{L^q}^p dt = \infty \quad \text{or} \quad \int_0^{T^\star} \|\partial_j v^3(t,\cdot)\|_{L^q}^p dt = \infty$$

with relations on p and q which make these quantities not scaling invariant.

Let us mention that I. Kukavica and M. Ziane proved in [18] further that

$$T^\star < \infty \Longrightarrow \int_0^{T^\star} \|\partial_3 v(t,\cdot)\|_{L^q}^p dt = \infty \quad \text{with} \quad \frac{2}{p} + \frac{3}{q} = 2 \quad \text{and} \quad q \in [9/4, 3]$$

The restriction "p less than 6" is probably technical. As we shall see, it comes from the domain of validity of law of product for some Sobolev or Besov spaces. It is not clear how to overcome this difficulty if the remain in the frame of solution which have only the critical regularity. Nevertheless, if we assume that the solution is more regular, for instance continuous in H^1, it is probably possible to prove the theorem for all finite p with much more technical difficulties. On the other hand, the case when $p = \infty$ seems out of reach. Indeed in the isotropic case, if if the maximal time of existence $T^\star$ of a solution u is finite, then

$$\limsup_{t \to T^\star} \|u(t)\|_{H^{\frac{1}{2}}} = \infty$$

(see [12] or [15] for the proof). The proof uses the fact that if the initial data is small then the solution is global. The equivalent of this result in our framework would be that, if $\|v_0 \cdot e\|_{H^{\frac{1}{2}}}$ is small for some unit vector e, then the solution is globally regular. Such a result, if it is true, seems out of reach for the time being.

The proof of Theorem 1.4 uses a result which claims that the control (in term of Besov spaces of negative indices) can be different for each component of the jacobian matrix Dv. In order to state the theorem, let us recall the definition of some class of Besov spaces.

Definition 1.1. *If σ is a positive real number, we define the space $B^{-\sigma}_{\infty,\infty}$ as the space of tempered distributions f such that*

$$\|f\|_{B^{-\sigma}_{\infty,\infty}} \stackrel{def}{=} \sup_{t>0} t^{\frac{\sigma}{2}} \|e^{t\Delta} f\|_{L^\infty} < \infty.$$

For p in $]1,\infty[$, we shall use the notation $\mathcal{B}_p \stackrel{def}{=} B^{-2+\frac{2}{p}}_{\infty,\infty}$.

These spaces are in some sense the largest ones which have a fixed scaling. Indeed, let us consider any Banach space E which can be continuously embedded into the space of tempered distribution $\mathcal{S}'(\mathbb{R}^3)$ such that

$$\forall (\lambda, \overrightarrow{a}) \in]0,\infty[\times \mathbb{R}^3\,,\ \lambda^\sigma \|f(\lambda \cdot + \overrightarrow{a})\|_E \sim \|f\|_E.$$

The first hypothesis on E implies that a constant C exists such that

$$\langle f, e^{-|\cdot|^2} \rangle \leq C\|f\|_E.$$

The scaling hypothesis on E implies, after a change of variables in the left-hand side of the above inequality, that

$$\forall t \in]0,\infty[\,,\ t^{\frac{\sigma}{2}} \|e^{t\Delta} f\|_{L^\infty} \leq C\|f\|_E.$$

As an example, let us apply the above inequality with the Sobolev space $E = H^{-\frac{1}{2}+\frac{2}{p}}$. This gives immediatly that

$$\|\partial_\ell a\|_{\mathcal{B}_p} \lesssim \|\partial_\ell a\|_{H^{-\frac{1}{2}+\frac{2}{p}}} \lesssim \|a\|_{H^{\frac{1}{2}+\frac{2}{p}}}. \tag{1.6}$$

Then the following theorem can be understood as an end point blow up theorem for the incompressible Navier-Stokes equation.

Theorem 1.5. *Let v be a solution of (NS) in the space $C([0,T^\star[;H^{\frac12})\cap L^2_{\rm loc}([0,T^\star[;H^{\frac32})$. If $T^\star$ is the maximal time of existence and $T^*<\infty$, then for any $(p_{k,\ell})$ in $]1,\infty[^9$, one has*

$$\sum_{1\leq k,\ell\leq 3}\int_0^{T^\star}\|\partial_\ell v^k(t)\|^{p_{k,\ell}}_{\mathcal{B}_{p_{k,\ell}}}\,dt=\infty.$$

It is easy to observe that

$$\|\partial_\ell v\|_{\mathcal{B}_p}\lesssim\|v\|_{L^q}\quad\text{with}\quad\frac2p+\frac3q=1,\,p>2,\quad\text{and}\quad L^q\subset\mathcal{B}_p\quad\text{with}\quad\frac2p+\frac3q=2\cdot\tag{1.7}$$

In particular, Theorem 1.5 implies blow up criteria (1.2) and (1.3). It generalizes also the result by D. Fang and C. Qian (see [13]) who proved sort of combined version of blow up criteria (1.2) and (1.3), like for instance critical Lebesgue norms of horizontal components of the vorticity and of derivative to the third component of the velocity.

2. Ideas and structure of the proof

First of all, let us remark that it makes no restriction to assume that the unit vector e is the vertical vector $(0,0,1)$. The first idea of the present work consists in writing the incompressible homogeneous Navier-Stokes system in terms of two unknowns:

- the third component of the vorticity Ω, which we denote by
$$\omega=\partial_1v^2-\partial_2v^1$$
and which can be understood as the 2D vorticity for the vector field $v^{\rm h}\stackrel{\rm def}{=}(v^1,v^2)$,
- the quantity ∂_3v^3 which is $-\,{\rm div}_h\,v^{\rm h}=-\partial_1v^1-\partial_2v^2$ because v is divergence free.

Immediate computations gives

$$(\widetilde{NS})\quad\begin{cases}\partial_t\omega+v\cdot\nabla\omega-\Delta\omega=\partial_3v^3\omega+\partial_2v^3\partial_3v^1-\partial_1v^3\partial_3v^2\\ \partial_t\partial_3v^3+v\cdot\nabla\partial_3v^3-\Delta\partial_3v^3+\partial_3v\cdot\nabla v^3=\partial_3^2\Delta^{-1}\Big(\displaystyle\sum_{\ell,m=1}^3\partial_\ell v^m\partial_m v^\ell\Big).\end{cases}$$

Keeping in mind that we control v^3 in the norm $L^p_T\big(H^{\frac12+\frac2p}\big)$ with p greater than 4, which implies that the order of regularity in space variables is less than 1. Let us analyze this system. We first introduce the notations

$$\nabla_{\rm h}^\perp=(-\partial_2,\partial_1),\quad\Delta_{\rm h}=\partial_1^2+\partial_2^2,\quad v^{\rm h}_{\rm curl}\stackrel{\rm def}{=}\nabla_{\rm h}^\perp\Delta_{\rm h}^{-1}\omega\quad\text{and}\quad v^{\rm h}_{\rm div}\stackrel{\rm def}{=}-\nabla_{\rm h}\Delta_{\rm h}^{-1}\partial_3v^3.\tag{2.1}$$

Let us observe that thanks to the divergence free condition on v, we have

$${\rm div}_{\rm h}\,v^{\rm h}_{\rm div}=\nabla_{\rm h}\Delta_{\rm h}^{-1}\,{\rm div}_{\rm h}\,v^{\rm h}.$$

Then we have, using the Biot-Savart's law in the horizontal variables

$$v^{\rm h}=v^{\rm h}_{\rm curl}+v^{\rm h}_{\rm div}.\tag{2.2}$$

Thus the righthand side term of the equation on ω in $(\widetilde{NS})$ contains terms which are linear in ω, namely

$$\partial_3v^3\omega+\partial_2v^3\partial_3v^1_{\rm curl}-\partial_1v^3\partial_3v^2_{\rm curl},$$

and a term that appears as a forcing term, namely

$$\partial_2v^3\partial_3v^1_{\rm div}-\partial_1v^3\partial_3v^2_{\rm div}.$$

The only quadratic term in ω is $v^{\rm h}_{\rm curl}\cdot\nabla_{\rm h}\omega$. A way to get rid of it is to use an energy type estimate and the divergence free condition on v. As we want to work only with scaling invariant norms, the only way is to perform a $L^{\frac{3}{2}}$ energy estimate in the equation on ω. This is possible thanks to the following lemma.

Lemma 2.1. *Let p be in $]1,2[$ and a_0 a function in L^p. Let us consider a function f in $L^1_{\rm loc}(\mathbb{R}^+;L^p)$ and v a divergence free vector field in $L^2_{\rm loc}(\mathbb{R}^+;L^\infty)$. If a solves*

$$(T_v)\qquad \left\{\begin{array}{c}\partial_t a-\Delta a+v\cdot\nabla a=f\\ a_{|t=0}=a_0\end{array}\right.$$

then a is such that $|a|^{\frac{p}{2}}$ belongs to $L^\infty_{\rm loc}(\mathbb{R}^+;L^2)\cap L^2_{\rm loc}(\mathbb{R}^+;H^1)$ and

$$\begin{aligned}\frac1p\int_{\mathbb{R}^3}|a(t,x)|^pdx+(p-1)\int_0^t\int_{\mathbb{R}^3}|\nabla a(t',x)|^2|a(t',x)|^{p-2}dx\,dt'\\ =\frac1p\int_{\mathbb{R}^3}|a_0(x)|^pdx+\int_0^t\int_{\mathbb{R}^3}f(t',x)a(t',x)|a(t',x)|^{p-2}dx\,dt'.\end{aligned}\tag{2.3}$$

Then it seems reasonable to control ω using some norm on v^3. Unfortunately, we need more regularity on v^3 than the $H^{\frac12+\frac2p}$ regularity. As shown by the forthcoming Proposition 2.1, we need higher order regularity on v^3. Indeed, the application of the above lemma leads in particular to the control of

$$\int_{\mathbb{R}^3}\bigl(\partial_2v^3\partial_3v^1_{\rm div}-\partial_1v^3\partial_3v^2_{\rm div}\bigr)\omega|\omega|^{-\frac12}dx.$$

It is clear that we need more regularity on ∂_3v^3 than v^3 belongs to $H^{\frac12+\frac2p}$. This leads to investigate the second equation of $(\widetilde{NS})$, which is

$$\partial_t\partial_3v^3+v\cdot\nabla\partial_3v^3-\Delta\partial_3v^3+\partial_3v\cdot\nabla v^3=\partial_3^2\Delta^{-1}\Bigl(\sum_{\ell,m=1}^3\partial_\ell v^m\partial_m v^\ell\Bigr).$$

The main feature of this equation is that it contains only one quadratic term with respect to ω, namely the term

$$\partial_3^2\Delta^{-1}\Bigl(\sum_{\ell,m=1}^2\partial_\ell v^m_{\rm curl}\partial_m v^\ell_{\rm curl}\Bigr)$$

A way to get rid of this term is to perform an energy estimate on ∂_3v^3, namely an estimate on

$$\|\partial_3v^3(t)\|_{\mathcal{H}}$$

for an adapted Hilbert space $\mathcal{H}$. Indeed, we hope that if we control v^3, we can control terms of the type

$$\bigl(\partial_3^2\Delta^{-1}(\partial_\ell v^m_{\rm curl}\partial_m v^\ell_{\rm curl})\big|\partial_3v^3\bigr)_{\mathcal{H}}$$

with quadratic terms in ω and thus it fits with $\|\partial_3v^3\|^2_{\mathcal{H}}$ and we can hope to close the estimate. Again here, the scaling helps us for the choice of the Hilbert space $\mathcal{H}$. The scaling of $\mathcal{H}$ must be the scaling of $H^{-\frac12}$. Moreover, because of the operator $\nabla_{\rm h}\Delta_{\rm h}^{-1}$, it is natural to measure horizontal derivatives and vertical derivatives differently. This leads to the following definition.

Definition 2.1. *For (s,s') in $\mathbb{R}^2$, $H^{s,s'}$ denotes the space of tempered distribution a such that*

$$\|a\|^2_{H^{s,s'}}\stackrel{\rm def}{=}\int_{\mathbb{R}^3}|\xi_{\rm h}|^{2s}|\xi_3|^{2s'}|\widehat{a}(\xi)|^2d\xi<\infty\quad\textit{with}\quad\xi_{\rm h}=(\xi_1,\xi_2).$$

For θ in $]0,1/2[$, we denote $\mathcal{H}_\theta\stackrel{\rm def}{=}H^{-\frac12+\theta,-\theta}$.

Let us first remark that

$$
(2.4)\qquad \begin{aligned} (s,s')\in[0,\infty[^2 &\Longrightarrow \|a\|_{H^{s,s'}}\leq \|a\|_{H^{s+s'}} \quad\text{and}\\ (s,s')\in]-\infty,0]^2 &\Longrightarrow \|a\|_{H^{s+s'}}\leq \|a\|_{H^{s,s'}}. \end{aligned}
$$

We want to emphasize the fact that anisotropy in the regularity is highly related to the divergence free condition. Indeed, let us consider a divergence free vector field $w=(w^{\rm h},w^3)$ in $H^{\frac12}$ and let us estimate $\|\partial_3 w^3\|_{\mathcal{H}_\theta}$. By definition of the $\mathcal{H}_\theta$ norm, we have

$$
\|\partial_3 w^3\|_{\mathcal{H}_\theta}^2 = A_L + A_H \quad\text{with}\quad A_L \stackrel{\rm def}{=} \int_{|\xi_{\rm h}|\leq|\xi_3|} |\xi_{\rm h}|^{-1+2\theta}|\xi_3|^{-2\theta}|\mathcal{F}(\partial_3 w^3)(\xi)|^2 d\xi.
$$

In the case when $|\xi_{\rm h}|\geq|\xi_3|$, we write that

$$
A_H \leq \int_{\mathbb{R}^3} |\xi_3|\,|\widehat{w}^3(\xi)|^2 d\xi \leq \|w^3\|_{H^{\frac12}}^2.
$$

In the case when $|\xi_{\rm h}|\leq|\xi_3|$, we use divergence free condition and write that

$$
\begin{aligned} A_L &\leq \int_{|\xi_{\rm h}|\leq|\xi_3|} |\xi_{\rm h}|^{-1}|\mathcal{F}(\mathrm{div}_{\rm h}\, w^{\rm h})(\xi)|^2 d\xi\\ &\leq \int_{\mathbb{R}^3}|\xi_{\rm h}|\,|\widehat{w}^{\rm h}(\xi)|^2 d\xi = \|w^{\rm h}\|_{H^{\frac12}}^2. \end{aligned}
$$

Thus for any divergence free vector field w in $H^{\frac12}$, we have

$$
(2.5)\qquad \|\partial_3 w^3\|_{\mathcal{H}_\theta} \leq C\|w\|_{H^{\frac12}}.
$$

The first step of the proof of Theorem 1.4 is the following proposition:

Proposition 2.1. *Let us consider a solution v of (NS) given by Theorem 1.3. Then for p in $]4,6[$, a constant C exists such that for any $t<T^\star$*

$$
(2.6)\qquad \begin{aligned} &\frac23\big\|\,\omega_{\frac34}(t)\big\|_{L^2}^2 + \frac59\int_0^t \big\|\nabla\omega_{\frac34}(t')\big\|_{L^2}^2\,dt' \leq \bigg(\frac23\|\,|\omega_0|^{\frac34}\|_{L^2}^2\\ &\qquad + \Big(\int_0^t \|\partial_3^2 v^3(t')\|_{\mathcal{H}_\theta}^2\,dt'\Big)^{\frac34}\bigg)\exp\Big(C\int_0^t \|v^3(t')\|^p_{H^{\frac12+\frac2p}}\,dt'\Big). \end{aligned}
$$

Here and in all that follows, for scalar function a and for α in the interval $]0,1[$, we always denote

$$
(2.7)\qquad a_\alpha \stackrel{\rm def}{=} \frac{a}{|a|}|a|^\alpha,
$$

so that in particular $\omega_{\frac34}=\omega|\omega|^{-\frac14}$ and $\omega_{\frac12}=\omega|\omega|^{-\frac12}$.

Next we want to control $\|\partial_3^2 v^3\|_{L^2_t(\mathcal{H}_\theta)}$. As already explained, a way to get rid of the only quadratic term in ω, namely

$$
\partial_3^2\Delta^{-1}\Big(\sum_{\ell,m=1}^2 \partial_\ell v^m_{\rm curl}\partial_m v^\ell_{\rm curl}\Big).
$$

is to perform an energy estimate for the norm $\mathcal{H}_\theta$.

Proposition 2.2. *Let us consider a solution v of (NS) given by Theorem 1.3. For any p in $]4,6[$ and θ in $\left]\frac{1}{2}-\frac{2}{p},\frac{1}{6}\right[$, a constant C exists such that for any $t<T^\star$, we have*

$$\begin{aligned}\|\partial_3 v^3(t)\|_{\mathcal{H}_\theta}^2+\int_0^t\|\nabla\partial_3 v^3(t')\|_{\mathcal{H}_\theta}^2\,dt' \leq C\exp\Bigl(C\int_0^t\|v^3(t')\|_{H^{\frac{1}{2}+\frac{2}{p}}}^p\,dt'\Bigr)\\ \times\biggl(\|\Omega_0\|_{L^{\frac{3}{2}}}^2+\int_0^t\Bigl(\|v^3(t')\|_{H^{\frac{1}{2}+\frac{2}{p}}}\bigl\|\omega_{\frac{3}{4}}(t')\bigr\|_{L^2}^{2\left(\frac{1}{3}+\frac{1}{p}\right)}\bigl\|\nabla\omega_{\frac{3}{4}}(t')\bigr\|_{L^2}^{2\left(1-\frac{1}{p}\right)}\\ +\|v^3(t')\|_{H^{\frac{1}{2}+\frac{2}{p}}}^2\bigl\|\omega_{\frac{3}{4}}(t')\bigr\|_{L^2}^{2\left(\frac{1}{3}+\frac{2}{p}\right)}\bigl\|\nabla\omega_{\frac{3}{4}}(t')\bigr\|_{L^2}^{2\left(1-\frac{2}{p}\right)}\Bigr)\,dt'\biggr).\end{aligned}\tag{2.8}$$

As aforementioned observation, the non-linear terms of the equation on $\partial_3 v^3$ contains quadratic terms in ω. In spite of that, the terms in ω and in $\partial_3 v^3$ have the same homogeneity in (2.8). Let us point out that this is also the case in the estimate of Proposition 2.1. This will allow us to close the estimates using Gronwall type arguments. More precisely, we have the following proposition.

Proposition 2.3. *Let v be the unique solution of (NS) given by Theorem 1.3. Then for any p in $]4,6[$ and θ in $\left]\frac{1}{2}-\frac{2}{p},\frac{1}{6}\right[$, a constant C exists such that, for any $t<T^*$, we have*

$$\begin{aligned}\bigl\|\omega_{\frac{3}{4}}(t)\bigr\|_{L^2}^{2\frac{p+3}{3}}+\bigl\|\nabla\omega_{\frac{3}{4}}\bigr\|_{L^2_t(L^2)}^{2\frac{p+3}{3}} &\leq C\|\Omega_0\|_{L^{\frac{3}{2}}}^{\frac{p+3}{2}}\mathcal{E}(t)\quad\textit{and}\\ \|\partial_3 v^3(t)\|_{\mathcal{H}_\theta}^2+\|\nabla\partial_3 v^3(t')\|_{L^2_t(\mathcal{H}_\theta)}^2 &\leq \|\Omega_0\|_{L^{\frac{3}{2}}}^2\mathcal{E}(t)\quad\textit{with}\\ \mathcal{E}(t) &\stackrel{\text{def}}{=} \exp\Bigl(C\exp\Bigl(C\int_0^t\|v^3(t)\|_{H^{\frac{1}{2}+\frac{2}{p}}}^p\,dt'\Bigr)\Bigr).\end{aligned}$$

The proof of this proposition from Propositions 2.1 and 2.2 is the purpose of Section 7. It consists in plugging the estimate of Proposition 2.2 into the one of Proposition 2.1 and making careful use of Hölder and convexity inequalities.

Now in order to conclude the proof of Theorem 1.4, we need to prove that the control of

$$\|\omega\|_{L^\infty_t(L^{\frac{3}{2}})},\quad\int_0^t\|\nabla\omega_{\frac{3}{4}}(t')\|_{L^2}^2\,dt',\ \int_0^t\|\partial_3^2 v^3(t')\|_{\mathcal{H}_\theta}^2\,dt'\quad\text{and}\quad\int_0^t\|v^3(t')\|_{H^{\frac{1}{2}+\frac{2}{p}}}^p\,dt'$$

allows to apply Theorem 1.5.

The paper is organized as follows. In the third section, we first prove Lemma 2.1 and explain how this estimate applied to the vorticity equation allows to prove the local existence of a solution to (NS) which satisfies the smoothing effect "$\nabla|\Omega|^{\frac{3}{4}}$ belongs to $L^2_{\rm loc}([0,T^\star[;L^2(\mathbb{R}^3)$".

Because of the term $\omega|\omega|^{-\frac{1}{2}}$ which appear when the apply the $L^{\frac{3}{2}}$ energy estimate, it is not possible to remain in the framework of Sobolev spaces but we have to deal with anisotropic spaces. In the fourth section, we present the anisotropic Littlewood-Paley theory and some properties of anisotropic Besov spaces, in particular laws of products which come from paraproduct decomposition in both horizontal and vertical variable and which play a key role in the proof of Propositions 2.1 and 2.2.

The fifth section is devoted to the proof of Proposition 2.1. The main point is the study of expression of the type

$$\int_{\mathbb{R}^3}\partial_{\rm h}\Delta_{\rm h}^{-1}f\partial_{\rm h}a\,\omega|\omega|^{-\frac{1}{2}}\,dx\,.$$

The first step is the description of the regularity of $\omega|\omega|^{-\frac{1}{2}}$ is term of Besov space knowing that ω belongs to $L^{\frac{3}{2}}$ and $\nabla\omega^{\frac{3}{4}}$ belongs to L^2 which is essentially made with forthcoming Lemma 5.1. Then the use of anisotropic Sobolev type embeddings and law of products. This is a place where the technical restriction "p less than 6" appears.

The sixth section is devoted to the proof of Proposition 2.2. The main point is the control of trilinear terms of the type

$$\big(\partial_3^2\Delta^{-1}(\partial_\ell v\partial_m v^\ell)\big|\partial_3 v^3\big)_{\mathcal{H}_\theta}$$

and especially the terms

$$\big(\partial_3^2\Delta^{-1}(\partial_\ell v^m_{\rm curl}\partial_m v^\ell_{\rm curl})\big|\partial_3 v^3\big)_{\mathcal{H}_\theta}$$

The main tool is the law of products for anisotropic Besov spaces. Again the technical restriction "p less than 6" appears.

In the seventh section, we explain how to deduce Proposition 2.3 from Propositions 2.1 and 2.2. The proof relies on a mixing between Gronwall type arguments and Hölder inequality.

In the eighth section, we prove first prove Theorem 1.5. Let us point out that the proof uses the particular structure of the incompressible Navier-Stokes system in the case when some index p less than or equal to 2. Indeed the skew-symmetry of the operator $v\cdot\nabla$ plays a key role. After this, we conclude the proof of Theorem 1.4 using Biot-Savart's law in the horizontal variable and Sobolev type inequalities in the spirit of Inequality (1.6).

Before going on, let us introduce some notations that will be used in all that follows. For $a\lesssim b$, we mean that there is a uniform constant C, which may be different on different lines, such that $a\leq Cb$. We denote by $(a|b)_{L^2}$ the $L^2(\mathbb{R}^3)$ inner product of a and b. For X a Banach space and I an interval of $\mathbb{R}$, we denote by $C(I;\,X)$ the set of continuous functions on I with values in X. For q in $[1,+\infty]$, the notation $L^q(I;\,X)$ stands for the set of measurable functions on I with values in X, such that $t\longmapsto\|f(t)\|_X$ belongs to $L^q(I)$. Finally, we denote by $L^p_T(L^q_{\rm h}(L^r_{\rm v}))$ the space $L^p([0,T];L^q(\mathbb{R}_{x_{\rm h}};L^r(\mathbb{R}_{x_3})))$ with $x_{\rm h}=(x_1,x_2)$, $\nabla_{\rm h}=(\partial_{x_1},\partial_{x_2})$ and $\Delta_{\rm h}=\partial^2_{x_1}+\partial^2_{x_2}$.

3. The local wellposedness of (NS) for vorticity in $L^{\frac{3}{2}}$ revisited

The first step of the proof is the proof of Lemma 2.1.

Proof of Lemma 2.1. Note that for p in $]1,2[$, $\nabla a=\nabla a|a|^{\frac{p-2}{2}}\times|a|^{\frac{2-p}{2}}$, which belongs to the space $L^2_{\rm loc}(\mathbb{R}^+;L^p)$ according to the energy inequality (2.3). Moreover, v belongs to the space $L^2_{\rm loc}(\mathbb{R}^+;L^\infty)$, so that $v\cdot\nabla a$ is in $L^1_{\rm loc}(\mathbb{R}^+;L^p)$, and hence arguing by density, we can assume that all the functions in (T_v) are smooth. As the function $r\mapsto r^p$ is C^1, we first write that

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\int_{\mathbb{R}^3}|a(t,x)|^pdx &= \int_{\mathbb{R}^3}\partial_t a\,a\,|a|^{p-2}dx\\ &= -\frac{1}{p}\int_{\mathbb{R}^3}v(t,x)\cdot\nabla|a|^p(t,x)dx+\int_{\mathbb{R}^3}\Delta a(t,x)a(t,x)|a(t,x)|^{p-2}dx\\ &\qquad+\int_{\mathbb{R}^3}f(t,x)a(t,x)|a(t,x)|^{p-2}dx.\end{aligned}$$

As v is assumed to be divergence free, we get

$$\frac{1}{p}\frac{d}{dt}\int_{\mathbb{R}^3}|a(t,x)|^pdx=\int_{\mathbb{R}^3}\Delta a(t,x)a(t,x)|a(t,x)|^{p-2}dx+\int_{\mathbb{R}^3}f(t,x)a(t,x)|a(t,x)|^{p-2}dx.$$

Integrating the above inequality over $[0,t]$ yields

$$\begin{aligned}\frac{1}{p}\int_{\mathbb{R}^3}|a(t,x)|^p dx = \frac{1}{p}\int_{\mathbb{R}^3}|a_0(x)|^p dx + \int_0^t\int_{\mathbb{R}^3}\Delta a(t',x)a(t',x)|a(t',x)|^{p-2}dxdt'\\ + \int_0^t\int_{\mathbb{R}^3} f(t',x)a(t',x)|a(t',x)|^{p-2}dx.\end{aligned}\tag{3.1}$$

In the case when $p\geq 2$, the function $r\mapsto r^{p-1}$ is C^1 and then an integration by parts implies that

$$\int_{\mathbb{R}^3}\Delta a(t,x)a(t,x)|a(t,x)|^{p-2}dx = -(p-1)\int_{\mathbb{R}^3}|\nabla a(t,x)|^2|a(t,x)|^{p-2}dx.$$

In the case when p is less than 2, some regularization has to be made. Indeed, even for smooth function, the fact that $|a|^{\frac{p}{2}}$ belongs to H^1 is not obvious. As a is supposed to be smooth, in particular, we have that a is bounded and $\Delta a\,|a|^{p-1}$ belongs to $L^\infty_{\rm loc}(\mathbb{R}^+,L^1)$. Thus, using Lebesgue's theorem, we infer that

$$\begin{aligned}\lim_{\delta\to 0}\int_0^t\int_{\mathbb{R}^3}\Delta a(t',x)a(t',x)\bigl(|a(t',x)|+\delta\bigr)^{p-2}\,dx\,dt'\\ = \int_0^t\int_{\mathbb{R}^3}\Delta a(t',x)a(t',x)|a(t',x)|^{p-2}\,dx\,dt'.\end{aligned}\tag{3.2}$$

As the function $r\mapsto (r+\delta)^{p-2}$ is smooth for any positive δ, we obtain

$$\begin{aligned}-\int_{\mathbb{R}^3}\Delta a(t',x)a(t',x)\bigl(|a(t',x)|+\delta\bigr)^{p-2}\,dx = \int_{\mathbb{R}^3}|\nabla a(t',x)|^2\bigl(|a(t',x)|+\delta\bigr)^{p-2}\,dx\\ + (p-2)\int_{\mathbb{R}^3}\nabla a(t',x)\cdot(\nabla|a|)(t',x)a(t',x)\bigl(|a(t',x)|+\delta\bigr)^{p-3}dx.\end{aligned}$$

It is well-known that

$$\nabla|a| = \nabla a\,\frac{a}{|a|}\,\cdot$$

Thus we get by time integration that

$$\begin{aligned}\int_0^t\int_{\mathbb{R}^3}\Delta a(t',x)a(t',x)\bigl(|a(t',x)|+\delta\bigr)^{p-2}dx = \int_0^t\int_{\mathbb{R}^3}|\nabla a(t',x)|^2\bigl(|a(t',x)|+\delta\bigr)^{p-2}\,dx\,dt'\\ + (p-2)\int_0^t\int_{\mathbb{R}^3}|\nabla a(t',x)|^2|a(t',x)|\bigl(|a(t',x)|+\delta\bigr)^{p-3}\,dx\,dt'.\end{aligned}$$

For the term in the right-hand side of the above inequality, thanks to (3.2) and to the monotonic convergence theorem, we get that $|\nabla a|^2|a|^{p-2}$ belongs to $L^1_{\rm loc}(\mathbb{R}^+;L^1)$ and that

$$-\int_0^t\int_{\mathbb{R}^3}\Delta a(t',x)a(t',x)|a(t',x)|^{p-2}\,dx\,dt' = (p-1)\int_0^t\int_{\mathbb{R}^3}|\nabla a(t',x)|^2|a(t',x)|^{p-2}\,dx\,dt'.$$

Resuming the above estimate into (3.1) leads to (2.3). This proves the lemma. □

We remark that we shall use Lemma 2.1 in the case when $p=3/2$. Indeed, by virtue of (2.7), one has

$$|\nabla a_{\frac34}| = \bigl|\nabla|a|^{\frac34}\bigr| = \frac{3}{4}|\nabla a|\,|a|^{-\frac14}.$$

Then (2.3) applied for $p=\frac32$ gives rise to

$$\frac{2}{3}\bigl\|a_{\frac34}(t)\bigr\|_{L^2}^2 + \frac{8}{9}\int_0^t\bigl\|\nabla a_{\frac34}(t')\bigr\|_{L^2}^2dt' = \frac{2}{3}\bigl\||a_0|^{\frac34}\bigr\|_{L^2}^2 + \int_0^t\int_{\mathbb{R}^3}f(t',x)a_{\frac12}(t',x)\,dx\,dt'.\tag{3.3}$$

Let us turn to the proof of Theorem 1.3.

Proof of Theorem 1.3. We use the equation on the vorticity By virtue of (NS), the vorticity $\Omega=\nabla\times v$ satisfies the equation

$$(NSV)\quad \begin{cases} \partial_t\Omega-\Delta\Omega+v\cdot\nabla\Omega-\Omega\cdot\nabla v=0\\ \Omega_{|t=0}=\Omega_0. \end{cases}$$

Biot-Sarvart's law claims that $v_0=-\nabla\Delta^{-1}\times\Omega_0$. This implies that $\|v_0\|_{H^{\frac12}}\lesssim\|\Omega_0\|_{H^{-\frac12}}$. Using the dual Sobolev embedding $\|f\|_{H^{-\frac12}}\lesssim\|f\|_{L^{\frac32}}$, we deduce that v_0 belongs to $H^{\frac12}$. Then applying Fujita-Kato theory [14] ensures that (NS) has a unique solution v on $[0,T^*[$ in the space $C([0,T^*[;H^{\frac12})\cap L^2_{\rm loc}([0,T^\star[;H^{\frac32})$. Moreover, it follows from Proposition B.1 of [9] that v belongs to $L^2_{\rm loc}([0,T^*[;L^\infty)$. Then to apply Lemma 2.1 for (NSV) with the external force $f=\Omega\cdot\nabla v$, we only need to estimate this term. Indeed as the solution v belongs to $L^2_{\rm loc}([0,T^\star[;H^{\frac32})$, we use Sobolev inequality to get

$$\begin{aligned}\Big|\int_0^t\int_{\mathbb{R}^3}\Omega\cdot\nabla v\,\Omega|\Omega|^{-\frac12}\,dx\,dt'\Big| &\leq \int_0^t\|\Omega\cdot\nabla v(t')\|_{L^{\frac32}}\|\Omega(t')\|^{\frac12}_{L^{\frac32}}\,dt'\\ &\leq \int_0^t\|\Omega(t')\|_{L^3}\|\nabla v(t')\|_{L^3}\|\Omega(t')\|^{\frac12}_{L^{\frac32}}\,dt'\\ &\leq C\int_0^t\|\nabla v(t')\|^2_{H^{\frac12}}\|\Omega(t')\|^{\frac12}_{L^{\frac32}}\,dt'.\end{aligned}$$

By virtue of (NSV), by applying Lemma 2.1 and using the convexity inequality

$$ab\leq\frac23a^{\frac32}+\frac13b^3\quad\text{with}\quad a=\|\nabla v(t')\|^{\frac43}_{H^{\frac12}}\quad\text{and}\quad b=\|\nabla v(t')\|^{\frac23}_{H^{\frac12}}\|\Omega(t')\|^{\frac12}_{L^{\frac32}},$$

we infer that

$$\begin{aligned}&\frac23\int_{\mathbb{R}^3}|\Omega(t,x)|^{\frac32}\,dx+\frac12\int_0^t\int_{\mathbb{R}^3}|\nabla\Omega(t',x)|^2|\Omega(t',x)|^{-\frac12}\,dx\,dt'\\ &\qquad\leq\frac23\int_{\mathbb{R}^3}|\Omega_0(x)|^{\frac32}\,dx+\int_0^t\|\nabla v(t')\|^2_{H^{\frac12}}\,dt'+C\int_0^t\|\nabla v(t')\|^2_{H^{\frac12}}\|\Omega(t')\|^{\frac32}_{L^{\frac32}}\,dt'.\end{aligned}$$

Applying Gronwall Lemma gives rise to

$$\begin{aligned}&\frac23\int_{\mathbb{R}^3}|\Omega(t,x)|^{\frac32}\,dx+\frac12\int_0^t\int_{\mathbb{R}^3}|\nabla\Omega(t',x)|^2|\Omega(t',x)|^{-\frac12}\,dx\,dt'\\ &\qquad\leq\Big(\frac23\int_{\mathbb{R}^3}|\Omega_0(x)|^{\frac32}\,dx+\int_0^t\|\nabla v(t')\|^2_{H^{\frac12}}\,dt'\Big)\exp\Big(C\int_0^t\|\nabla v(t')\|^2_{H^{\frac12}}\,dt'\Big).\end{aligned}$$

Thus Theorem 1.3 is proved. □

As a conclusion of this section, let us establish some Sobolev type inequalities which involves the regularities of $a_{\frac34}$ and $\nabla a_{\frac34}$ in L^2.

Lemma 3.1. *We have*

$$\|\nabla a\|_{L^{\frac32}}\lesssim\big\|\nabla a_{\frac34}\big\|_{L^2}\big\|a_{\frac34}\big\|^{\frac13}_{L^2}. \tag{3.4}$$

Moreover, for s in $\big[-1/2\,,5/6\big]$, we have

$$\|a\|_{H^s}\leq C\|a_{\frac34}\|^{\frac56-s}_{L^2}\|\nabla a_{\frac34}\|^{\frac12+s}_{L^2}. \tag{3.5}$$

Proof. Notice that due to (2.7), $|\nabla a| = \dfrac{4}{3}|\nabla a_{\frac{3}{4}}|\,|a|^{\frac{1}{4}}$, then we get (3.4) by using Hölder inequality. The dual Sobolev inequality claims that

$$\|a\|_{H^{-\frac{1}{2}}} \leq C\|a\|_{L^{\frac{3}{2}}} = C\|a_{\frac{3}{4}}\|_{L^2}^{\frac{4}{3}}. \tag{3.6}$$

Moreover, using again that $|\nabla a| = \frac{4}{3}|\nabla a_{\frac{3}{4}}|\,|a|^{\frac{1}{4}}$, Hölder inequality implies that

$$\begin{aligned} \|\nabla a\|_{L^{\frac{9}{5}}} &\leq \frac{4}{3}\|\nabla a_{\frac{3}{4}}\|_{L^2}\|\,|a|^{\frac{1}{4}}\|_{L^{18}} \\ &\leq \frac{4}{3}\|\nabla a_{\frac{3}{4}}\|_{L^2}\|a_{\frac{3}{4}}\|_{L^6}^{\frac{1}{3}}. \end{aligned}$$

Sobolew embedding of H^1 into L^6 then ensures that

$$\|\nabla a\|_{L^{\frac{9}{5}}} \leq C\|\nabla a_{\frac{3}{4}}\|_{L^2}^{\frac{4}{3}}. \tag{3.7}$$

Sobolev embedding of $W^{1,\frac{9}{5}}$ into $H^{\frac{5}{6}}$ leads to

$$\|a\|_{H^{\frac{5}{6}}} \leq C\|\nabla a_{\frac{3}{4}}\|_{L^2}^{\frac{4}{3}},$$

from which and (3.6), we concludes the proof of (3.5) and hence the lemma by using interpolation inequality between H^s Sobolev spaces. □

4. Some estimates related to Littlewood-Paley analysis

As we shall use the anisotropic Littlewood-Paley theory, we recall the functional space framework we are going to use in this section. As in [8], [11] and [22], the definitions of the spaces we are going to work with requires anisotropic dyadic decomposition of the Fourier variables. Let us recall from [1] that

$$\begin{aligned} &\Delta_k^{\mathrm{h}} a = \mathcal{F}^{-1}(\varphi(2^{-k}|\xi_{\mathrm{h}}|)\widehat{a}), && \Delta_\ell^{\mathrm{v}} a = \mathcal{F}^{-1}(\varphi(2^{-\ell}|\xi_3|)\widehat{a}),\\ &S_k^{\mathrm{h}} a = \mathcal{F}^{-1}(\chi(2^{-k}|\xi_{\mathrm{h}}|)\widehat{a}), && S_\ell^{\mathrm{v}} a = \mathcal{F}^{-1}(\chi(2^{-\ell}|\xi_3|)\widehat{a}) \quad \text{and}\\ &\Delta_j a = \mathcal{F}^{-1}(\varphi(2^{-j}|\xi|)\widehat{a}), && S_j a = \mathcal{F}^{-1}(\chi(2^{-j}|\xi|)\widehat{a}), \end{aligned} \tag{4.1}$$

where $\xi_{\mathrm{h}} = (\xi_1, \xi_2)$, $\mathcal{F}a$ and $\widehat{a}$ denote the Fourier transform of the distribution a, $\chi(\tau)$ and $\varphi(\tau)$ are smooth functions such that

$$\begin{aligned} &\mathrm{Supp}\ \varphi \subset \Big\{\tau \in \mathbb{R}\ /\ \ \frac{3}{4} \leq |\tau| \leq \frac{8}{3}\Big\} \quad \text{and} \quad \forall \tau > 0\,,\ \sum_{j\in\mathbb{Z}} \varphi(2^{-j}\tau) = 1,\\ &\mathrm{Supp}\ \chi \subset \Big\{\tau \in \mathbb{R}\ /\ \ |\tau| \leq \frac{4}{3}\Big\} \quad \text{and} \quad \chi(\tau) + \sum_{j\geq 0} \varphi(2^{-j}\tau) = 1. \end{aligned}$$

Definition 4.1. *Let (p, r) be in $[1, +\infty]^2$ and s in $\mathbb{R}$. Let us consider u in $\mathcal{S}_h'(\mathbb{R}^3)$, which means that u is in $\mathcal{S}'(\mathbb{R}^3)$ and satisfies $\lim\limits_{j\to-\infty}\|S_j u\|_{L^\infty} = 0$. We set*

$$\|u\|_{B^s_{p,r}} \stackrel{\text{def}}{=} \big\|\big(2^{js}\|\Delta_j u\|_{L^p}\big)_j\big\|_{\ell^r(\mathbb{Z})}.$$

- *For $s < \frac{3}{p}$ (or $s = \frac{3}{p}$ if $r = 1$), we define $B^s_{p,r}(\mathbb{R}^3) \stackrel{\text{def}}{=} \big\{u \in \mathcal{S}_h'(\mathbb{R}^3)\ \big|\ \|u\|_{B^s_{p,r}} < \infty\big\}$.*
- *If k is a positive integer and if $\frac{3}{p} + k \leq s < \frac{3}{p} + k + 1$ (or $s = \frac{3}{p} + k + 1$ if $r = 1$), then we define $B^s_{p,r}(\mathbb{R}^3)$ as the subset of distributions u in $\mathcal{S}_h'(\mathbb{R}^3)$ such that $\partial^\beta u$ belongs to $B^{s-k}_{p,r}(\mathbb{R}^3)$ whenever $|\beta| = k$.*

Let us remark that in the particular case when $p=r=2$, $B^s_{p,r}$ coincides with the classical homogeneous Sobolev spaces H^s. Moreover, in the case when $p=r=\infty$, it coincides with the spaces defined in Definition 2.1 (see for instance Theorem 2.34 on page 76 of [1]).

Similar to Definition 4.1, we can also define the homogeneous anisotropic Besov space.

Definition 4.2. *Let us define the space $\bigl(B^{s_1}_{p,q_1}\bigr)_{\rm h}\bigl(B^{s_2}_{p,q_2}\bigr)_{\rm v}$ as the space of distribution in $\mathcal{S}'_h$ such that*

$$\|u\|_{\bigl(B^{s_1}_{p,q_1}\bigr)_{\rm h}\bigl(B^{s_2}_{p,q_2}\bigr)_{\rm v}} \overset{\rm def}{=} \biggl(\sum_{k\in\mathbb{Z}} 2^{q_1 k s_1}\Bigl(\sum_{\ell\in\mathbb{Z}} 2^{q_2\ell s_2}\|\Delta_k^{\rm h}\Delta_\ell^{\rm v}u\|_{L^p}^{q_2}\Bigr)^{\frac{q_1}{q_2}}\biggr)^{\frac{1}{q_1}}$$

is finite.

We remark that when $p=q_1=q_2=2$, the anisotropic Besov space $\bigl(B^{s_1}_{p,q_1}\bigr)_{\rm h}\bigl(B^{s_2}_{p,q_2}\bigr)_{\rm v}$ coincides with the classical homogeneous anisotropic Sobolev space H^{s_1,s_2} and thus the space $\bigl(B^{-\frac12+\theta}_{2,2}\bigr)_{\rm h}\bigl(B^{-\theta}_{2,2}\bigr)_{\rm v}$ is the space $\mathcal{H}_\theta$ defined in Definition 2.1. Let us also remark that in the case when q_1 is different from q_2, the order of summation is important.

For the convenience of the readers, we recall the following anisotropic Bernstein type lemma from [11, 22]:

Lemma 4.1. *Let $\mathcal{B}_h$ (resp. $\mathcal{B}_v$) a ball of $\mathbb{R}^2_h$ (resp. $\mathbb{R}_v$), and $\mathcal{C}_h$ (resp. $\mathcal{C}_v$) a ring of $\mathbb{R}^2_h$ (resp. $\mathbb{R}_v$); let $1\le p_2\le p_1\le\infty$ and $1\le q_2\le q_1\le\infty$. Then there holds:*
If the support of $\widehat{a}$ is included in $2^k\mathcal{B}_h$, then

$$\|\partial^\alpha_{x_{\rm h}}a\|_{L^{p_1}_{\rm h}(L^{q_1}_{\rm v})}\lesssim 2^{k\bigl(|\alpha|+2\bigl(\frac{1}{p_2}-\frac{1}{p_1}\bigr)\bigr)}\|a\|_{L^{p_2}_{\rm h}(L^{q_1}_{\rm v})}.$$

If the support of $\widehat{a}$ is included in $2^\ell\mathcal{B}_v$, then

$$\|\partial_3^\beta a\|_{L^{p_1}_{\rm h}(L^{q_1}_{\rm v})}\lesssim 2^{\ell(\beta+(\frac{1}{q_2}-\frac{1}{q_1}))}\|a\|_{L^{p_1}_{\rm h}(L^{q_2}_{\rm v})}.$$

If the support of $\widehat{a}$ is included in $2^k\mathcal{C}_h$, then

$$\|a\|_{L^{p_1}_{\rm h}(L^{q_1}_{\rm v})}\lesssim 2^{-kN}\sup_{|\alpha|=N}\|\partial^\alpha_{x_{\rm h}}a\|_{L^{p_1}_{\rm h}(L^{q_1}_{\rm v})}.$$

If the support of $\widehat{a}$ is included in $2^\ell\mathcal{C}_v$, then

$$\|a\|_{L^{p_1}_{\rm h}(L^{q_1}_{\rm v})}\lesssim 2^{-\ell N}\|\partial^N_{x_3}a\|_{L^{p_1}_{\rm h}(L^{q_1}_{\rm v})}.$$

As a corollary of Lemma 4.1, we have the following inequality, if $1\le p_2\le p_1$,

$$\|a\|_{\bigl(B^{s_1-2(\frac{1}{p_2}-\frac{1}{p_1})}_{p_1,q_1}\bigr)_{\rm h}\bigl(B^{s_2-(\frac{1}{p_2}-\frac{1}{p_1})}_{p_1,q_2}\bigr)_{\rm v}}\lesssim\|a\|_{\bigl(B^{s_1}_{p_2,q_1}\bigr)_{\rm h}\bigl(B^{s_2}_{p_2,q_2}\bigr)_{\rm v}}. \tag{4.2}$$

To consider the product of a distribution in the isentropic Besov space with a distribution in the anisotropic Besov space, we need the following result which allows to embed isotropic Besov spaces into the anisotropic ones.

Lemma 4.2. *Let s be a positive real number and (p,q) in $[1,\infty]$ with $p\ge q$. Then one has*

$$\|a\|_{L^p_{\rm h}\bigl((B^s_{p,q})_{\rm v}\bigr)}\lesssim\|a\|_{B^s_{p,q}}.$$

Proof. Once noticed that, an integer N_0 exists such that, if j is less or equal to $\ell - N_0$ then the operator $\Delta_\ell^{\rm v}\Delta_j$ is identically 0, we can write that

$$\begin{aligned} 2^{\ell s}\|\Delta_\ell^{\rm v} a\|_{L^p} &\lesssim 2^{\ell s}\sum_{\ell\leq j+N_0}\|\Delta_\ell^{\rm v}\Delta_j a\|_{L^p}\\ &\lesssim \sum_{\ell\leq j+N_0} 2^{(\ell-j)s}2^{js}\|\Delta_j a\|_{L^p}. \end{aligned}$$

Because s is positive, Young inequality on $\mathbb{Z}$ implies that

$$\big\|\big(2^{\ell s}\|\Delta_\ell^{\rm v} a\|_{L^p}\big)_\ell\big\|_{\ell^q(\mathbb{Z})}\lesssim \|a\|_{B^s_{p,q}}.$$

Due to $p\geq q$, Minkowski inequality implies that

$$\begin{aligned} \|a\|_{L^p_{\rm h}\left((B^s_{p,q})_{\rm v}\right)} &= \big\|\big(2^{\ell s}\|\Delta_\ell^{\rm v} a(x_{\rm h},\cdot)\|_{L^p_{\rm v}}\big)_\ell\big\|_{\ell^q(\mathbb{Z})}\big\|_{L^p_{\rm h}}\\ &\lesssim \big\|\big(2^{\ell s}\|\Delta_\ell^{\rm v} a\|_{L^p}\big)_\ell\big\|_{\ell^q(\mathbb{Z})}\\ &\lesssim \|a\|_{B^s_{p,q}}. \end{aligned}$$

The lemma is proved. □

Lemma 4.3. *For any s positive and any θ in $]0,s[$, we have*

$$\|f\|_{(B^{s-\theta}_{p,q})_{\rm h}(B^\theta_{p,1})_{\rm v}}\lesssim \|f\|_{B^s_{p,q}}.$$

Proof. This lemma means exactly that

$$V_k \stackrel{\rm def}{=} \sum_{\ell\in\mathbb{Z}} 2^{\ell\theta}\|\Delta_k^{\rm h}\Delta_\ell^{\rm v} f\|_{L^p}\lesssim c_{k,q}2^{-k(s-\theta)}\|f\|_{B^s_{p,q}} \quad \text{with} \quad (c_{k,q})_k\in\ell^q(\mathbb{Z}). \tag{4.3}$$

We distinguish the case when ℓ is less or equal to k from the case when ℓ is greater than k. Using the fact that the operators $\Delta_\ell^{\rm v}$ are uniformly bounded on L^p, we write

$$\begin{aligned} 2^{k(s-\theta)}V_k &= 2^{k(s-\theta)}\sum_{\ell\leq k}2^{\ell\theta}\|\Delta_k^{\rm h}\Delta_\ell^{\rm v} f\|_{L^p}+2^{k(s-\theta)}\sum_{\ell>k}2^{\ell\theta}\|\Delta_k^{\rm h}\Delta_\ell^{\rm v} f\|_{L^p}\\ &\lesssim 2^{ks}\|\Delta_k^{\rm h} f\|_{L^p}+2^{k(s-\theta)}\sum_{\ell>k}2^{\ell\theta}\|\Delta_k^{\rm h}\Delta_\ell^{\rm v} f\|_{L^p}. \end{aligned} \tag{4.4}$$

In the case when ℓ is greater than k, the set $2^k\mathcal{C}_{\rm h}\times 2^\ell\mathcal{C}_{\rm v}$ is included a ring of the type $2^\ell\widetilde{\mathcal{C}}$. Thus, if $|j-\ell|$ is greater than some fixed integer N_0, then we have $\Delta_j\Delta_k^{\rm h}\Delta_\ell^{\rm v}\equiv 0$. This gives

$$\sum_{\ell>k}2^{\ell\theta}\|\Delta_k^{\rm h}\Delta_\ell^{\rm v} f\|_{L^p}\lesssim \sum_{\substack{|j-\ell|\leq N_0\\ \ell>k}} 2^{\ell\theta}\|\Delta_j\Delta_k^{\rm h}\Delta_\ell^{\rm v} f\|_{L^p}.$$

Then using again that the operators $\Delta_\ell^{\rm v}$ and $\Delta_k^{\rm h}$ are uniformly bounded on L^p, we infer that

$$\sum_{\ell>k}2^{\ell\theta}\|\Delta_k^{\rm h}\Delta_\ell^{\rm v} f\|_{L^p}\lesssim \sum_{j>k-N_0} 2^{-j(s-\theta)}2^{js}\|\Delta_j f\|_{L^p}.$$

Moreover, we have $\Delta_j\Delta_k^{\rm h}=0$ if $j\leq k-N_1$. We thus deduce from (4.4) that

$$2^{k(s-\theta)}V_k\lesssim \sum_{j\geq k-N_1} 2^{-(j-k)s}2^{js}\|\Delta_j f\|_{L^p}+\sum_{j\geq k-N_0} 2^{-(j-k)(s-\theta)}2^{js}\|\Delta_j f\|_{L^p}.$$

This gives (4.3) and thus the lemma. □

One of the main motivation of using anisotropic Besov space is the proof of the following proposition.

Proposition 4.1. *Let v be a divergence free vector field. Let us consider (α,θ) in $]0,1/2[^2$. Then we have*

$$\|v^{\rm h}\|_{\left(B^1_{2,1}\right)_{\rm h}\left(B^{\frac12-\alpha}_{2,1}\right)_{\rm v}} \lesssim \big\|\,\omega_{\frac34}\big\|_{L^2}^{\frac13+\alpha}\big\|\nabla\omega_{\frac34}\big\|_{L^2}^{1-\alpha}+\|\partial_3 v^3\|_{L^2}^{\alpha}\|\nabla\partial_3 v^3\|_{\mathcal{H}_\theta}^{1-\alpha}.$$

Proof. Using horizontal Biot-Savart law (2.1) and Lemma 4.1, we have

$$\|v^{\rm h}\|_{\left(B^1_{2,1}\right)_{\rm h}\left(B^{\frac12-\alpha}_{2,1}\right)_{\rm v}} \lesssim \|\omega\|_{\left(B^0_{2,1}\right)_{\rm h}\left(B^{\frac12-\alpha}_{2,1}\right)_{\rm v}}+\|\partial_3 v^3\|_{\left(B^0_{2,1}\right)_{\rm h}\left(B^{\frac12-\alpha}_{2,1}\right)_{\rm v}}. \tag{4.5}$$

Applying Lemmas 4.1 and Lemma 4.3 gives

$$\begin{aligned}\|\omega\|_{\left(B^0_{2,1}\right)_{\rm h}\left(B^{\frac12-\alpha}_{2,1}\right)_{\rm v}} &\lesssim \|\omega\|_{\left(B^{\frac19}_{\frac95,1}\right)_{\rm h}\left(B^{\frac59-\alpha}_{\frac95,1}\right)_{\rm v}}\\ &\lesssim \|\omega\|_{B^{\frac23-\alpha}_{\frac95,1}}.\end{aligned} \tag{4.6}$$

Now let us estimate $\|\omega\|_{B^s_{\frac95,1}}$ in terms of $\big\|\omega_{\frac34}\big\|_{L^2}$ and $\big\|\nabla\omega_{\frac34}\big\|_{L^2}$. For s in $]-1/3,1[$ and any positive integer N, which we shall choose hereafter, we write that

$$\begin{aligned}\|\omega\|_{B^s_{\frac95,1}} &= \sum_{j\le N}2^{js}\|\Delta_j\omega\|_{L^{\frac95}}+\sum_{j>N}2^{js}\|\Delta_j\omega\|_{L^{\frac95}}\\ &\lesssim \sum_{j\le N}2^{j(s+\frac13)}\|\Delta_j\omega\|_{L^{\frac32}}+\sum_{j>N}2^{j(s-1)}\|\Delta_j\nabla\omega\|_{L^{\frac95}}\\ &\lesssim 2^{N(s+\frac13)}\|\omega\|_{L^{\frac32}}+2^{N(s-1)}\|\nabla\omega\|_{L^{\frac95}}.\end{aligned}$$

Choosing $N=\left[\log_2\left(e+\left(\frac{\|\nabla\omega\|_{L^{\frac95}}}{\|\omega\|_{L^{\frac32}}}\right)^{\frac34}\right)\right]$ yields

$$\|\omega\|_{B^s_{\frac95,1}}\lesssim\|\omega\|_{L^{\frac32}}^{\frac34(1-s)}\|\nabla\omega\|_{L^{\frac95}}^{\frac34(s+\frac13)}.$$

Using (3.7), we infer that

$$\|\omega\|_{B^s_{\frac95,1}}\lesssim\big\|\omega_{\frac34}\big\|_{L^2}^{1-s}\big\|\nabla\omega_{\frac34}\big\|_{L^2}^{\frac13+s}.$$

Using this inequality with $s=\dfrac23-\alpha$ and (4.6) gives

$$\|\omega\|_{\left(B^0_{2,1}\right)_{\rm h}\left(B^{\frac12-\alpha}_{2,1}\right)_{\rm v}}\lesssim\big\|\,\omega_{\frac34}\big\|_{L^2}^{\frac13+\alpha}\big\|\nabla\omega_{\frac34}\big\|_{L^2}^{1-\alpha}. \tag{4.7}$$

Now let us prove the following lemma.

Lemma 4.4. *Let us consider (α,θ) in $]0,1/2[^2$.Then we have*

$$\|a\|_{\left(B^0_{2,1}\right)_{\rm h}\left(B^{\frac12-\alpha}_{2,1}\right)_{\rm v}}\lesssim\|a\|_{\mathcal{H}_\theta}^{\alpha}\|\nabla a\|_{\mathcal{H}_\theta}^{1-\alpha}.$$

Proof. By definition of $\|\cdot\|_{(B^0_{2,1})_{\rm h}(B^{\frac12-\alpha}_{2,1})_{\rm v}}$, we have

$$
\begin{aligned}
\|a\|_{(B^0_{2,1})_{\rm h}(B^{\frac12-\alpha}_{2,1})_{\rm v}} &= H_L(a)+V_L(a) \quad\text{with}\\
H_L(a) &\stackrel{\rm def}{=} \sum_{k\leq\ell}\|\Delta^{\rm h}_k\Delta^{\rm v}_\ell a\|_{L^2}2^{\ell\left(\frac12-\alpha\right)}\quad\text{and}\\
V_L(a) &\stackrel{\rm def}{=} \sum_{k>\ell}\|\Delta^{\rm h}_k\Delta^{\rm v}_\ell a\|_{L^2}2^{\ell\left(\frac12-\alpha\right)}
\end{aligned}
\tag{4.8}
$$

In order to estimate $H_L(a)$, we classically estimate differently high and low vertical frequencies which are here the dominant ones. Using Lemma 4.1, we write that for any N in $\mathbb{Z}$,

$$
H_L(a)\lesssim \sum_{k\leq\ell\leq N}\|\Delta^h_k\Delta^v_\ell a\|_{L^2}2^{\ell\left(\frac12-\alpha\right)}+\sum_{\substack{k\leq\ell\\ \ell>N}}\|\Delta^h_k\Delta^v_\ell\partial_3 a\|_{L^2}2^{-\ell\left(\frac12+\alpha\right)}.
$$

By definition of the norm of $\mathcal{H}_\theta$, we get

$$
H_L(a)\lesssim \|a\|_{\mathcal{H}_\theta}\sum_{k\leq\ell\leq N}2^{k\left(\frac12-\theta\right)}2^{\ell\left(\frac12-\alpha+\theta\right)}+\|\partial_3 a\|_{\mathcal{H}_\theta}\sum_{\substack{k\leq\ell\\ \ell>N}}2^{k\left(\frac12-\theta\right)}2^{-\ell\left(\frac12+\alpha-\theta\right)}
$$

The hypothesis on (α,θ) imply that

$$
\begin{aligned}
H_L(a) &\lesssim \|a\|_{\mathcal{H}_\theta}\sum_{\ell\leq N}2^{\ell(1-\alpha)}+\|\partial_3 a\|_{\mathcal{H}_\theta}\sum_{\ell>N}2^{-\alpha\ell}\\
&\lesssim \|a\|_{\mathcal{H}_\theta}2^{N(1-\alpha)}+\|\partial_3 a\|_{\mathcal{H}_\theta}2^{-N\alpha}
\end{aligned}
$$

Choosing N such that $2^N\sim\dfrac{\|\partial_3 a\|_{\mathcal{H}_\theta}}{\|a\|_{\mathcal{H}_\theta}}$ gives

$$
H_L(a)\lesssim \|a\|^\alpha_{\mathcal{H}_\theta}\|\partial_3 a\|^{1-\alpha}_{\mathcal{H}_\theta}. \tag{4.9}
$$

The term $V_L(a)$ is estimated along the same lines. In fact, we get, by using again Lemma 4.1, that

$$
\begin{aligned}
V_L(a) &\lesssim \sum_{\ell<k\leq N}\|\Delta^h_k\Delta^v_\ell a\|_{L^2}2^{\ell\left(\frac12-\alpha\right)}+\sum_{\substack{\ell<k\\ k>N}}\|\Delta^h_k\Delta^v_\ell\nabla_{\rm h} a\|_{L^2}2^{\ell\left(\frac12-\alpha\right)}2^{-k}\\
&\lesssim \|a\|_{\mathcal{H}_\theta}\sum_{\ell<k\leq N}2^{\ell\left(\frac12-\alpha+\theta\right)}2^{k\left(\frac12-\theta\right)}+\|\nabla_{\rm h} a\|_{\mathcal{H}_\theta}\sum_{\substack{\ell\leq k\\ k>N}}2^{\ell\left(\frac12-\alpha+\theta\right)}2^{-k\left(\frac12+\theta\right)}\\
&\lesssim \|a\|_{\mathcal{H}_\theta}2^{N(1-\alpha)}+\|\nabla_{\rm h} a\|_{\mathcal{H}_\theta}2^{-N\alpha}
\end{aligned}
$$

Choosing N such that $2^N\sim\dfrac{\|\nabla_{\rm h} a\|_{\mathcal{H}_\theta}}{\|a\|_{\mathcal{H}_\theta}}$ yields

$$
V_L(a)\lesssim \|a\|^\alpha_{\mathcal{H}_\theta}\|\nabla_{\rm h} a\|^{1-\alpha}_{\mathcal{H}_\theta}.
$$

Together with (4.8) and (4.9), this gives the lemma. □

The application of Lemma 4.4 together with (4.7) leads to Proposition 4.1. □

To study product laws between distributions in the anisotropic Besov spaces, we need to modify the isotropic para-differential decomposition of Bony [4] to the setting of anisotropic version. We first recall the isotropic para-differential decomposition from [4]: let a and b be in $\mathcal{S}'(\mathbb{R}^3)$,

$$
\begin{aligned}
&ab = T(a,b) + \bar{T}(a,b) + R(a,b) \quad \text{with} \\
&T(a,b) = \sum_{j\in\mathbb{Z}} S_{j-1}a\Delta_j b, \quad \bar{T}(a,b) = T(b,a), \quad \text{and} \\
&R(a,b) = \sum_{j\in\mathbb{Z}} \Delta_j a\tilde{\Delta}_j b, \quad \text{with} \quad \tilde{\Delta}_j b = \sum_{\ell=j-1}^{j+1} \Delta_\ell a.
\end{aligned}
\tag{4.10}
$$

As an application of the above basic facts on Littlewood-Paley theory, we present the following product laws in the anisotropic Besov spaces.

Lemma 4.5. *Let $q \geq 1$, $p_1 \geq p_2 \geq 1$ with $\frac{1}{p_1} + \frac{1}{p_2} \leq 1$, and $s_1 < \frac{2}{p_1}$, $s_2 < \frac{2}{p_2}$ (resp. $s_1 \leq \frac{2}{p_1}$, $s_2 \leq \frac{2}{p_2}$ if $q = 1$) with $s_1 + s_2 > 0$. Let $\sigma_1 < \frac{1}{p_1}$, $\sigma_2 < \frac{1}{p_2}$ (resp. $\sigma_1 \leq \frac{1}{p_1}$, $\sigma_2 \leq \frac{1}{p_2}$ if $q = 1$) with $\sigma_1 + \sigma_2 > 0$. Then for a in $\big(B^{s_1}_{p_1,q}\big)_{\rm h}\big(B^{\sigma_1}_{p_1,q}\big)_{\rm v}$ and b in $\big(B^{s_2}_{p_2,q}\big)_{\rm h}\big(B^{\sigma_2}_{p_2,q}\big)_{\rm v}$, the product ab belongs to $\big(B^{s_1+s_2-\frac{2}{p_2}}_{p_1,q}\big)_{\rm h}\big(B^{\sigma_1+\sigma_2-\frac{1}{p_2}}_{p_1,q}\big)_{\rm v}$, and*

$$
\|ab\|_{\big(B^{s_1+s_2-\frac{2}{p_2}}_{p_1,q}\big)_{\rm h}\big(B^{\sigma_1+\sigma_2-\frac{1}{p_2}}_{p_1,q}\big)_{\rm v}} \lesssim \|a\|_{\big(B^{s_1}_{p_1,q}\big)_{\rm h}\big(B^{\sigma_1}_{p_1,q}\big)_{\rm v}} \|b\|_{\big(B^{s_2}_{p_2,q}\big)_{\rm h}\big(B^{\sigma_2}_{p_2,q}\big)_{\rm v}}.
$$

The proof of the above Lemma is a standard application of Bony's decomposition (4.10) in both horizontal and vertical variables and Definition 4.2. We skip the details here.

5. Proof of the estimate for the horizontal vorticity

The purpose of this section to present the proof of Proposition 2.1. Let us recall the first equation of our reformulation $(\widetilde{NS})$ of the incompressible Navier-Stokes equation which is

$$
\partial_t\omega + v\cdot\nabla\omega - \Delta\omega = F \stackrel{\rm def}{=} \partial_3 v^3\omega + \partial_2 v^3\partial_3 v^1 - \partial_1 v^3 \partial_3 v^2.
$$

As already explained in the second section, we decompose F as a sum of three terms. Hence by virtue of (3.3), we obtain

$$
\begin{aligned}
&\frac{2}{3}\big\|\omega_{\frac{3}{4}}(t)\big\|_{L^2}^2 + \frac{8}{9}\int_0^t \big\|\nabla\omega_{\frac{3}{4}}(t')\big\|_{L^2}^2\,dt' = \frac{2}{3}\big\||\omega_0|^{\frac{3}{4}}\big\|_{L^2}^2 + \sum_{\ell=1}^{3} F_\ell(t) \quad \text{with} \\
&F_1(t) \stackrel{\rm def}{=} \int_0^t \int_{\mathbb{R}^3} \partial_3 v^3 |\omega|^{\frac{3}{2}}\,dx\,dt', \\
&F_2(t) \stackrel{\rm def}{=} \int_0^t \int_{\mathbb{R}^3} \big(\partial_2 v^3\partial_3 v^1_{\rm curl} - \partial_1 v^3 \partial_3 v^2_{\rm curl}\big)\omega_{\frac{1}{2}}\,dx\,dt' \quad \text{and} \\
&F_3(t) \stackrel{\rm def}{=} \int_0^t \int_{\mathbb{R}^3} \big(\partial_2 v^3\partial_3 v^1_{\rm div} - \partial_1 v^3 \partial_3 v^2_{\rm div}\big)\omega_{\frac{1}{2}}\,dx\,dt',
\end{aligned}
\tag{5.1}
$$

where $v^{\rm h}_{\rm curl}$ (resp. $v^{\rm h}_{\rm div}$) corresponds the horizontal divergence free (resp. curl free) part of the horizontal vector $v^{\rm h} = (v^1, v^2)$, which is given by (2.1), and where $\omega_{\frac{1}{2}} \stackrel{\rm def}{=} |\omega|^{-\frac{1}{2}}\,\omega$.

Let us start with the easiest term F_1. We first get, by using integration by parts, that

$$
|F_1(t)| \leq \frac{3}{2}\int_0^t \int_{\mathbb{R}^3} |v^3(t',x)|\,|\partial_3\omega(t',x)|\,|\omega(t',x)|^{\frac{1}{2}}\,dx\,dt'.
$$

Using that

$$\frac{p-2}{3p}+\frac{2}{3}+\frac{2}{3p}=1,$$

we apply Hölder inequality to get

$$|F_1(t)|\leq\frac{3}{2}\int_0^t\|v^3(t')\|_{L^{\frac{3p}{p-2}}}\|\partial_3\omega(t')\|_{L^{\frac{3}{2}}}\big\|\omega_{\frac{3}{4}}(t')\big\|_{L^p}^{\frac{2}{3}}\,dt'.$$

As p is in $]4,6[$, Sobolev embedding and interpolation inequality imply that

$$\big\|\omega_{\frac{3}{4}}(t')\big\|_{L^p}\lesssim\big\|\omega_{\frac{3}{4}}(t')\big\|_{H^{3(\frac{1}{2}-\frac{1}{p})}}\lesssim\big\|\omega_{\frac{3}{4}}(t')\big\|_{L^2}^{\frac{3}{p}-\frac{1}{2}}\big\|\nabla\omega_{\frac{3}{4}}(t')\big\|_{L^2}^{\frac{3}{2}-\frac{3}{p}}.$$

Using (3.4), this gives

$$|F_1(t)|\lesssim\int_0^t\|v^3(t')\|_{H^{\frac{1}{2}+\frac{2}{p}}}\big\|\partial_3\omega_{\frac{3}{4}}(t')\big\|_{L^2}\big\|\omega_{\frac{3}{4}}(t')\big\|_{L^2}^{\frac{1}{3}}\big\|\nabla\omega_{\frac{3}{4}}(t')\big\|_{L^2}^{1-\frac{2}{p}}\big\|\omega_{\frac{3}{4}}(t')\big\|_{L^2}^{\frac{2}{p}-\frac{1}{3}}\,dt'.$$

Applying convex inequality, we obtain

$$\begin{aligned}|F_1(t)|&\lesssim\int_0^t\|v^3(t')\|_{H^{\frac{1}{2}+\frac{2}{p}}}\big\|\omega_{\frac{3}{4}}(t')\big\|_{L^2}^{\frac{2}{p}}\big\|\nabla\omega_{\frac{3}{4}}(t')\big\|_{L^2}^{2(1-\frac{1}{p})}\,dt'\\&\leq\frac{1}{9}\int_0^t\big\|\nabla\omega_{\frac{3}{4}}(t')\big\|_{L^2}^2\,dt'+C\int_0^t\|v^3(t')\|_{H^{\frac{1}{2}+\frac{2}{p}}}^p\big\|\omega_{\frac{3}{4}}(t')\big\|_{L^2}^2\,dt'.\end{aligned}\tag{5.2}$$

The other two terms requires a refined way of the description of the regularity of $\omega_{\frac{1}{2}}$ and demands a detailed study of the anisotropic operator $\nabla_{\rm h}\Delta_{\rm h}^{-1}$ associated with the Biot-Savart's law in horizontal variables. Now we state the lemmas which allows us to treat the terms F_2 and F_3 in (5.1).

Lemma 5.1. *Let (s,α) be in $]0,1[^2$ and (p,q) in $[1,\infty]^2$. We consider a function G from $\mathbb{R}$ to $\mathbb{R}$ which is Hölderian of exponent α. Then for any a in the Besov space $B^s_{p,q}$, one has*

$$\|G(a)\|_{B^{\alpha s}_{\frac{p}{\alpha},\frac{q}{\alpha}}}\lesssim\|G\|_{C^\alpha}\big(\|a\|_{B^s_{p,q}}\big)^\alpha\quad\textit{with}\quad\|G\|_{C^\alpha}\stackrel{\rm def}{=}\sup_{r\neq r'}\frac{|G(r)-G(r')|}{|r-r'|^\alpha}\,\cdot$$

Proof. Because the indices s and α are between 0 and 1, we use the definition of Besov spaces coming from integral in the physical space (see for instance Theorem 2.36 of [1]). Indeed as

$$\big|G(a)-G(b)\big|\lesssim\|G\|_{C^\alpha}|a-b|^\alpha,$$

we infer that

$$\begin{aligned}\|G(a)-G(a(\cdot+y))\|_{L^{\frac{p}{\alpha}}}&=\Big(\int_{\mathbb{R}^d}\big|G(a(x))-G(a(x+y))\big|^{\frac{p}{\alpha}}dx\Big)^{\frac{\alpha}{p}}\\&\lesssim\|G\|_{C^\alpha}\Big(\int_{\mathbb{R}^d}|a(x)-a(x+y)|^pdx\Big)^{\frac{\alpha}{p}}\\&\lesssim\|G\|_{C^\alpha}\|a-a(\cdot+y)\|_{L^p}^\alpha.\end{aligned}$$

Then for any $q<\infty$, we write that

$$\begin{aligned}\Big(\int_{\mathbb{R}^d}\frac{\|G(a)-G(a(\cdot+y))\|_{L^{\frac{p}{\alpha}}}^{\frac{q}{\alpha}}}{|y|^{\alpha s\times\frac{q}{\alpha}}}\frac{dy}{|y|^d}\Big)^{\frac{\alpha}{q}}&\lesssim\|G\|_{C^\alpha}\Big(\int_{\mathbb{R}^d}\frac{\|a-a(\cdot+y)\|_{L^p}^q}{|y|^{sq}}\frac{dy}{|y|^d}\Big)^{\frac{\alpha}{q}}\\&\lesssim\|G\|_{C^\alpha}\big(\|a\|_{B^s_{p,q}}\big)^\alpha.\end{aligned}$$

The case for $q=\infty$ is identical. This completes the proof of the lemma. □

Lemma 5.2. *Let θ be in $]0,1/6[$, σ in $]3/4,1[$, and $s=\dfrac{1}{2}+1-\dfrac{2}{3}\sigma$. Then we have*

$$
\begin{aligned}
&\Big|\int_{\mathbb{R}^3}\partial_{\rm h}\Delta_{\rm h}^{-1}f\partial_{\rm h}a\,\omega_{\frac12}\,dx\Big|\lesssim\|f\|_{L^{\frac32}}\|a\|_{H^s}\big\|\omega_{\frac34}\big\|_{H^\sigma}^{\frac23}\quad\textit{and}\\
&\Big|\int_{\mathbb{R}^3}\partial_{\rm h}\Delta_{\rm h}^{-1}f\partial_{\rm h}a\,\omega_{\frac12}\,dx\Big|\lesssim\|f\|_{\mathcal{H}_\theta}\|a\|_{H^s}\big\|\omega_{\frac34}\big\|_{H^\sigma}^{\frac23},
\end{aligned}
\tag{5.3}
$$

for $\mathcal{H}_\theta$ given by Definition 2.1.

Proof. Let us observe that $\omega_{\frac12}=G(\omega_{\frac34})$ with $G(r)\overset{\rm def}{=}r|r|^{-\frac13}$. Using Lemma 5.1, we obtain

$$
\big\|\omega_{\frac12}\big\|_{B^{\frac23\sigma}_{3,3}}\lesssim\big\|\omega_{\frac34}\big\|_{H^\sigma}^{\frac23}.
\tag{5.4}
$$

Let us study the product $\partial_{\rm h}a\,\omega_{\frac12}$. Using Bony's decomposition (4.10) and the Leibnitz formula, we write

$$
\begin{aligned}
\partial_{\rm h}a\,\omega_{\frac12}&=T(\partial_{\rm h}a,\omega_{\frac12})+R(\partial_{\rm h}a,\omega_{\frac12})+T(\omega_{\frac12},\partial_{\rm h}a)\\
&=\partial_{\rm h}T(\omega_{\frac12},a)+A(a,\omega)\quad\text{with}\\
A(a,\omega)&\overset{\rm def}{=}T(\partial_{\rm h}a,\omega_{\frac12})+R(\partial_{\rm h}a,\omega_{\frac12})-T(\partial_{\rm h}\omega_{\frac12},a).
\end{aligned}
$$

We first get, by using Lemma 4.1, that

$$
\begin{aligned}
\|\Delta_jT(\omega_{\frac12},a)\|_{L^2}&\lesssim\sum_{|j-j'|\leq4}\|S_{j'-1}\omega_{\frac12}\|_{L^\infty}\|\Delta_{j'}a\|_{L^2}\\
&\lesssim c_{j,2}2^{-j(s+\frac23\sigma-1)}\|\omega_{\frac12}\|_{B^{\frac23\sigma}_{3,3}}\|a\|_{H^s},
\end{aligned}
$$

which together with (5.4) ensures that

$$
\big\|T(\omega_{\frac12},a)\big\|_{H^{\frac12}}\lesssim\|a\|_{H^s}\big\|\omega_{\frac34}\big\|_{H^\sigma}^{\frac23}.
$$

Using that the operator $\partial_{\rm h}^2\Delta_{\rm h}^{-1}$ is a bounded Fourier multiplier and the dual Sobolev embedding $L^{\frac32}\subset H^{-\frac12}$, we get that

$$
\begin{aligned}
\Big|\int_{\mathbb{R}^3}\partial_{\rm h}\Delta_{\rm h}^{-1}f\partial_{\rm h}T(\omega_{\frac12},a)\,dx\Big|&=\Big|\int_{\mathbb{R}^3}\partial_{\rm h}^2\Delta_{\rm h}^{-1}fT(\omega_{\frac12},a)\,dx\Big|\\
&\leq\|f\|_{H^{-\frac12}}\big\|T(\omega_{\frac12},a)\big\|_{H^{\frac12}}\\
&\lesssim\|f\|_{L^{\frac32}}\|a\|_{H^s}\big\|\omega_{\frac34}\big\|_{H^\sigma}^{\frac23}.
\end{aligned}
\tag{5.5}
$$

In the case of the anisotropic norm, recalling that $\mathcal{H}_\theta=H^{-\frac12+\theta,-\theta}$, and using Lemma 4.3, we write

$$
\begin{aligned}
\Big|\int_{\mathbb{R}^3}\partial_{\rm h}^2\Delta_{\rm h}^{-1}fT(\omega_{\frac12},a)\,dx\Big|&\leq\|f\|_{\mathcal{H}_\theta}\|T(\omega_{\frac12},a)\|_{H^{\frac12-\theta,\theta}}\\
&\leq\|f\|_{\mathcal{H}_\theta}\|T(\omega_{\frac12},a)\|_{H^{\frac12}}\\
&\leq\|f\|_{\mathcal{H}_\theta}\|a\|_{H^s}\big\|\omega_{\frac34}\big\|_{H^\sigma}^{\frac23}.
\end{aligned}
\tag{5.6}
$$

Now let us take into account the anisotropy induced by the operator $\partial_{\rm h}\Delta_{\rm h}^{-1}$. Hardy-Littlewood-Sobolev inequality implies that $\partial_{\rm h}\Delta_{\rm h}^{-1}f$ belongs to $L^{\frac32}_{\rm v}(L^6_{\rm h})$ if f is in $L^{\frac32}$. So that it amounts to prove that $A(a,\omega)$ belongs to $L^3_{\rm v}(L^{\frac65}_{\rm h})$, which is simply an anisotropic Sobolev type embedding. Because of $s<1$, we get, by using Lemma 4.1, that

$$\begin{aligned}\|\Delta_j T(\partial_{\rm h} a,\omega_{\frac12})\|_{L^{\frac65}} &\lesssim \sum_{|j'-j|\leq 4}\|S_{j'-1}\partial_{\rm h} a\|_{L^2}\|\Delta_{j'}\omega_{\frac12}\|_{L^3}\\ &\lesssim \sum_{|j'-j|\leq 4} c_{j',2}c_{j',3}2^{-j'(s+\frac23\sigma-1)}\|a\|_{H^s}\|\omega_{\frac12}\|_{B^{\frac23\sigma}_{3,3}}\\ &\lesssim c_{j,\frac65}2^{-\frac j2}\|a\|_{H^s}\big\|\omega_{\frac34}\big\|^{\frac23}_{H^\sigma}.\end{aligned}$$

Along the same line, it is easy to check that the other two terms in $A(a,\omega)$ satisfy the same estimate. This leads to

$$\|A(a,\omega)\|_{B^{\frac12}_{\frac65,\frac65}}\lesssim \|a\|_{H^s}\big\|\omega_{\frac34}\big\|^{\frac23}_{H^\sigma}. \tag{5.7}$$

While it follows from Lemma 4.2 that

$$B^{\frac12}_{\frac65,\frac65}\subset L^{\frac65}_{\rm h}\big((B^{\frac12}_{\frac65,\frac65})_{\rm v}\big).$$

Sobolev type embedding theorem (see for instance Theorem 2.40 of [1]) claims that

$$B^{\frac12}_{\frac65,\frac65}(\mathbb{R})\subset B^0_{3,2}(\mathbb{R})\subset L^3(\mathbb{R}).$$

As a consequence, by virtue of (5.7), we obtain

$$\begin{aligned}\Big|\int_{\mathbb{R}^3}\partial_{\rm h}\Delta_{\rm h}^{-1}f A(a,\omega)\,dx\Big| &\lesssim \|\partial_{\rm h}\Delta_{\rm h}^{-1}f\|_{L^{\frac32}_v(L^6_{\rm h})}\|A(a,\omega)\|_{L^3_{\rm v}(L^{\frac65}_h)}\\ &\lesssim \|f\|_{L^{\frac32}}\|A(a,\omega)\|_{L^{\frac65}_h(L^3_{\rm v})}\\ &\lesssim \|f\|_{L^{\frac32}}\|a\|_{H^s}\big\|\omega_{\frac34}\big\|^{\frac23}_{H^\sigma},\end{aligned}$$

which together with (5.5) leads to the first inequality of (5.3).

In order to prove the second inequality of (5.3), we observe that

$$\|\nabla_h\Delta_{\rm h}^{-1}f\|_{H^{\frac12+\theta,-\theta}}\lesssim\|f\|_{H^{-\frac12+\theta,-\theta}}=\|f\|_{\mathcal{H}_\theta}.$$

Thus thanks to (5.7), for θ given by the lemma, what we only need to prove now is that

$$B^{\frac12}_{\frac65,\frac65}\subset H^{-\frac12-\theta,\theta}. \tag{5.8}$$

As a matter of fact, using Lemma 4.3 and Lemma 4.1, we have, for any α in $\Big]0,\dfrac12\Big[$,

$$\begin{aligned}B^{\frac12}_{\frac65,\frac65} &\subset \big(B^{\frac12-\alpha}_{\frac65,\frac65}\big)_{\rm h}\big(B^{\alpha}_{\frac65,\frac65}\big)_{\rm v}\quad\text{and}\\ \big(B^{\frac12-\alpha}_{\frac65,\frac65}\big)_{\rm h}\big(B^{\alpha}_{\frac65,\frac65}\big)_{\rm v} &\subset \big(B^{\frac12-\alpha-2\left(\frac56-\frac12\right)}_{2,2}\big)_{\rm h}\big(B^{\alpha-\left(\frac56-\frac12\right)}_{2,2})\big)_{\rm v}\\ &\subset H^{-\alpha-\frac16,\alpha-\frac13}.\end{aligned}$$

Choosing $\alpha=\dfrac13+\theta$ gives (5.8) because θ is less than $\dfrac16$. This completes the proof of the lemma. $\square$

The estimate of $F_2(t)$ uses the Biot-Savart's law in the horizontal variables (namely (2.1)) and Lemma 5.2 with $f = \partial_3\omega$, $a = v^3$. This gives for any time $t < T^\star$ and σ in $]3/4, 1[$ that

$$I_\omega(t) \stackrel{\text{def}}{=} \Big|\int_{\mathbb{R}^3} \big(\partial_2 v^3(t,x)\partial_3 v^1_{\text{curl}}(t,x) - \partial_1 v^3(t,x)\partial_3 v^2_{\text{curl}}(t,x)\big)\omega_{\frac12}(t,x)\,dx\Big|$$
$$\lesssim \|\partial_3\omega(t)\|_{L^{\frac32}} \|v^3(t)\|_{H^{\frac32-\frac23\sigma}} \big\|\omega_{\frac34}(t)\big\|^{\frac23}_{H^\sigma}.$$

By virtue of (3.4) and of the interpolation inequalities between L^2 and H^1, we thus obtain

$$I_\omega(t) \lesssim \|v^3(t)\|_{H^{\frac12+2\left(\frac12-\frac\sigma3\right)}} \big\|\omega_{\frac34}(t)\big\|^{2\left(\frac12-\frac\sigma3\right)}_{L^2} \big\|\nabla\omega_{\frac34}(t)\big\|^{2\left(\frac12+\frac\sigma3\right)}_{L^2}.$$

Choosing $\sigma = 3\Big(\dfrac12 - \dfrac1p\Big)$, which is between 3/4 and 1 because p is between 4 and 6, gives

$$I_\omega(t) \lesssim \|v^3(t)\|_{H^{\frac12+\frac2p}} \big\|\omega_{\frac34}(t)\big\|^{\frac2p}_{L^2} \big\|\nabla\omega_{\frac34}(t)\big\|^{2\left(1-\frac1p\right)}_{L^2}.$$

Then by using convexity inequality and time integration, we get

$$|F_2(t)| \leq \frac19 \int_0^t \big\|\nabla\omega_{\frac34}(t')\big\|^2_{L^2} dt' + C\int_0^t \|v^3(t')\|^p_{H^{\frac12+\frac2p}} \big\|\omega_{\frac34}(t')\big\|^2_{L^2}\,dt'. \tag{5.9}$$

In order to estimate $F_3(t)$, we write that

$$F_3(t) = -\int_0^t \int_{\mathbb{R}^3} \Big(\partial_2 v^3(t',x)(\partial_1\Delta_{\rm h}^{-1}\partial_3^2 v^3)(t',x)$$
$$- \partial_1 v^3(t',x)(\partial_2\Delta_{\rm h}^{-1}\partial_3^2 v^3)(t',x)\Big)\,\omega_{\frac12}(t',x)dxdt'.$$

As $\dfrac2p = 1 - \dfrac{2\sigma}{3}$, thanks to interpolation inequality between Sobolev spaces, we get, by applying Lemma 5.2 with $f = \partial_3^2 v^3$ and $a = v^3$, that

$$\begin{aligned}
|F_3(t)| &\lesssim \int_0^t \|\partial_3^2 v^3(t')\|_{\mathcal{H}_\theta} \|v^3(t')\|_{H^{\frac32-\frac23\sigma}} \big\|\omega_{\frac34}(t')\big\|^{\frac23}_{H^\sigma}\,dt'\\
&\lesssim \int_0^t \|\partial_3^2 v^3(t')\|_{\mathcal{H}_\theta} \|v^3(t')\|_{H^{\frac12+\frac2p}} \big\|\omega_{\frac34}(t')\big\|^{\frac23(1-\sigma)}_{L^2} \big\|\nabla\omega_{\frac34}(t')\big\|^{\frac23\sigma}_{L^2}\,dt'\\
&\lesssim \int_0^t \|\partial_3^2 v^3(t')\|_{\mathcal{H}_\theta} \|v^3(t')\|^{\frac p6}_{H^{\frac12+\frac2p}}\\
&\qquad\qquad \times \Big(\|v^3(t')\|^p_{H^{\frac12+\frac2p}} \big\|\omega_{\frac34}(t')\big\|^2_{L^2}\Big)^{\frac1p-\frac16} \big\|\nabla\omega_{\frac34}(t')\big\|^{2\left(\frac12-\frac1p\right)}_{L^2}\,dt'.
\end{aligned}$$

As we have

$$\frac12 + \frac16 + \Big(\frac1p - \frac16\Big) + \Big(\frac12 - \frac1p\Big) = 1,$$

applying Hölder inequality ensures that

$$|F_3(t)| \lesssim \Big(\int_0^t \|\partial_3^2 v^3(t')\|^2_{\mathcal{H}_\theta}\,dt'\Big)^{\frac12} \Big(\int_0^t \|v^3(t')\|^p_{H^{\frac12+\frac2p}}\,dt'\Big)^{\frac16}$$
$$\times \Big(\int_0^t \|v^3(t')\|^p_{H^{\frac12+\frac2p}} \big\|\omega_{\frac34}(t')\big\|^2_{L^2}\,dt'\Big)^{\frac1p-\frac16} \Big(\int_0^t \big\|\nabla\omega_{\frac34}(t')\big\|^2_{L^2} dt'\Big)^{\frac12-\frac1p}.$$

Applying the convexity inequality

$$
\begin{aligned}
&a_1a_2a_3\leq \frac{1}{p_1}a^{p_1}+\frac{1}{p_2}a^{p_2}+\frac{1}{p_3}a^{p_3}\quad\text{with}\\
&a_1=C\Bigl(\int_0^t\|\partial_3^2v^3(t')\|_{\mathcal{H}_\theta}^2\,dt'\Bigr)^{\frac12}\Bigl(\int_0^t\|v^3(t')\|^p_{H^{\frac12+\frac2p}}\,dt'\Bigr)^{\frac16},\\
&a_2=\Bigl(9^{\frac{3(p-2)}{6-p}}\int_0^t\|v^3(t')\|^p_{H^{\frac12+\frac2p}}\bigl\|\omega_{\frac34}(t')\bigr\|_{L^2}^2\,dt'\Bigr)^{\frac1p-\frac16},\\
&a_3=\Bigl(\frac19\int_0^t\bigl\|\nabla\omega_{\frac34}(t')\bigr\|_{L^2}^2dt'\Bigr)^{\frac12-\frac1p}\quad\text{and}\\
&\frac{1}{p_1}=\frac23,\ \ \frac{1}{p_2}=\frac1p-\frac16\quad\text{and}\quad\frac{1}{p_3}=\frac12-\frac1p
\end{aligned}
$$

leads to

$$
\begin{aligned}
|F_3(t)|\leq\frac19\int_0^t\bigl\|\nabla\omega_{\frac34}(t')\bigr\|_{L^2}^2dt'+C\int_0^t\|v^3(t')\|^p_{H^{\frac12+\frac2p}}\bigl\|\omega_{\frac34}(t')\bigr\|_{L^2}^2\,dt'\\
+C\Bigl(\int_0^t\|v^3(t')\|^p_{H^{\frac12+\frac2p}}\,dt'\Bigr)^{\frac14}\Bigl(\int_0^t\|\partial_3^2v^3(t')\|_{\mathcal{H}_\theta}^2\,dt'\Bigr)^{\frac34}.
\end{aligned}
\tag{5.10}
$$

Conclusion of the proof to Proposition 2.1. Resuming the estimates (5.2), (5.9) and (5.10) into (5.1), we obtain

$$
\begin{aligned}
&\frac23\bigl\|\omega_{\frac34}(t)\bigr\|_{L^2}^2+\frac59\int_0^t\|\nabla\omega_{\frac34}(t')\|_{L^2}^2\,dt'\\
&\quad\leq\frac23\bigl\||\omega_0|^{\frac34}\bigr\|_{L^2}^2+C\Bigl(\int_0^t\|v^3(t')\|^p_{H^{\frac12+\frac2p}}\,dt'\Bigr)^{\frac14}\Bigl(\int_0^t\|\partial_3^2v^3(t')\|_{\mathcal{H}_\theta}^2\,dt'\Bigr)^{\frac34}\\
&\qquad\qquad+C\int_0^t\|v^3(t')\|^p_{H^{\frac12+\frac2p}}\bigl\|\omega_{\frac34}(t')\bigr\|_{L^2}^2\,dt'.
\end{aligned}
$$

Inequality (2.6) follows from Gronwall lemma once notice that $x^{\frac14}e^{Cx}\lesssim e^{C'x}$ for $C'>C$. □

6. Proof of the estimate for the second vertical derivatives of v^3

In this section, we shall present the proof of Proposition 2.2. Let $\mathcal{H}_\theta$ be given by Definition 2.1. We get, by taking the $\mathcal{H}_\theta$ inner product of the ∂_3v^3 equation of $(\widetilde{NS})$ with ∂_3v^3, that

$$
\begin{aligned}
&\frac12\frac{d}{dt}\|\partial_3v^3(t)\|_{\mathcal{H}_\theta}^2+\|\nabla\partial_3v^3(t)\|_{\mathcal{H}_\theta}^2=\sum_{n=1}^3\bigl(Q_n(v,v)\,|\,\partial_3v^3\bigr)_{\mathcal{H}_\theta}\quad\text{with}\\
&Q_1(v,v)\stackrel{\text{def}}{=}\bigl(-\mathrm{Id}+\partial_3^2\Delta^{-1}\bigr)(\partial_3v^3)^2+\partial_3^2\Delta^{-1}\Bigl(\sum_{\ell,m=1}^2\partial_\ell v^m\partial_mv^\ell\Bigr),\\
&Q_2(v,v)\stackrel{\text{def}}{=}\bigl(-\mathrm{Id}+2\partial_3^2\Delta^{-1}\bigr)\Bigl(\sum_{\ell=1}^2\partial_3v^\ell\partial_\ell v^3\Bigr)\quad\text{and}\\
&Q_3(v,v)\stackrel{\text{def}}{=}-v\cdot\nabla\partial_3v^3.
\end{aligned}
\tag{6.1}
$$

The estimate involving Q_1 relies on the the following lemma.

Lemma 6.1. *Let A be a bounded Fourier multiplier. If p and θ satisfy*

$$0<\theta<\frac{1}{2}-\frac{1}{p}\,, \tag{6.2}$$

then we have

$$\bigl|\bigl(A(D)(fg)\,|\,\partial_3 v^3\bigr)_{\mathcal{H}_\theta}\bigr|\lesssim \|f\|_{H^{\theta,\frac12-\theta-\frac1p}}\|g\|_{H^{\theta,\frac12-\theta-\frac1p}}\|v^3\|_{H^{\frac12+\frac2p}}\cdot$$

Proof. Let us first observe that, for any couple (α,β) in $\mathbb{R}^2$, we have, thanks to Cauchy-Schwartz inequality, that, for any real valued function a and b,

$$\begin{aligned}\bigl|(a|b)_{\mathcal{H}_\theta}\bigr| &= \Bigl|\int_{\mathbb{R}^3}|\xi_{\rm h}|^{-1+2\theta-\alpha}|\xi_3|^{-2\theta-\beta}\widehat{a}(\xi)|\xi_{\rm h}|^{\alpha}\,|\xi_3|^{\beta}\widehat{b}(-\xi)d\xi\Bigr|\\ &\leq \|a\|_{H^{-1+2\theta-\alpha,-2\theta-\beta}}\|b\|_{H^{\alpha,\beta}}.\end{aligned} \tag{6.3}$$

As $A(D)$ is a bounded Fourier multiplier, applying (6.3) with $\alpha=0$ and $\beta=-\dfrac{1}{2}+\dfrac{2}{p}$, we obtain

$$\bigl|\bigl(A(D)(fg)\,|\,\partial_3 v^3\bigr)_{\mathcal{H}_\theta}\bigr|\lesssim \|fg\|_{H^{-1+2\theta,\frac12-\frac2p-2\theta}}\|\partial_3 v^3\|_{H^{0,-\frac12+\frac2p}} \tag{6.4}$$

Because $H^{s,s'}=\bigl(B^s_{2,2}\bigr)_{\rm h}\bigl(B^{s'}_{2,2}\bigr)_{\rm v}$ and thanks to Condition (6.2), law of products of Lemma 4.5 implies in particular that

$$\|fg\|_{H^{-1+2\theta,\frac12-\frac2p-2\theta}}\lesssim\|f\|_{H^{\theta,\frac12-\theta-\frac1p}}\|g\|_{H^{\theta,\frac12-\theta-\frac1p}}\cdot$$

As $\|\partial_3 v^3\|_{H^{0,-\frac12+\frac2p}}\lesssim\|v^3\|_{H^{0,\frac12+\frac2p}}\leq\|v^3\|_{H^{\frac12+\frac2p}}$, the lemma is proved. □

Because both $\partial_3^2\Delta^{-1}$ and $\partial_{\rm h}^2\Delta_{\rm h}^{-1}$ are bounded Fourier multipliers, applying Lemma 6.1 with f and g of the form $\partial_{\rm h}v^{\rm h}_{\rm curl}$ or $\partial_{\rm h}v^{\rm h}_{\rm div}$ or with $f=g=\partial_3 v^3$ gives,

$$\bigl|\bigl(Q_1(v,v)\,|\,\partial_3 v^3\bigr)_{\mathcal{H}_\theta}\bigr|\lesssim\|v^3\|_{H^{\frac12+\frac2p}}\bigl(\|\omega\|^2_{H^{\theta,\frac12-\theta-\frac1p}}+\|\partial_3 v^3\|^2_{H^{\theta,\frac12-\theta-\frac1p}}\bigr).$$

Because of Condition (6.2), we get, by using Lemma 4.3 and Lemma 3.1, that

$$\|\omega\|_{H^{\theta,\frac12-\theta-\frac1p}}\leq\|\omega\|_{H^{\frac12-\frac1p}}\lesssim\bigl\|\,\omega_{\frac34}\bigr\|_{L^2}^{\frac{p+3}{3p}}\bigl\|\nabla\omega_{\frac34}\bigr\|_{L^2}^{1-\frac1p}.$$

While it follows from Definition 2.1 that

$$\begin{aligned}\|a\|^2_{H^{\theta,\frac12-\theta-\frac1p}} &= \int_{\mathbb{R}^3}|\xi_{\rm h}|^{2\theta}|\xi_3|^{1-2\theta-\frac2p}|\widehat{a}(\xi)|^2\,d\xi\\ &\leq \int_{\mathbb{R}^3}|\widehat{a}(\xi)|^{\frac2p}\bigl(|\xi||\widehat{a}(\xi)|\bigr)^{2\left(1-\frac1p\right)}|\xi_{\rm h}|^{2(-\frac12+\theta)}|\xi_3|^{-2\theta}\,d\xi.\end{aligned}$$

Applying Hölder's inequality with measure $|\xi_{\rm h}|^{2(-\frac12+\theta)}|\xi_3|^{-2\theta}\,d\xi$ yields

$$\|a\|_{H^{\theta,\frac12-\theta-\frac1p}}\leq\|a\|_{\mathcal{H}_\theta}^{\frac1p}\|\nabla a\|_{\mathcal{H}_\theta}^{1-\frac1p},$$

We then infer that

$$\bigl|\bigl(Q_1(v,v)\,|\,\partial_3 v^3\bigr)_{\mathcal{H}_\theta}\bigr|\lesssim\|v^3\|_{H^{\frac12+\frac2p}}\Bigl(\bigl\|\,\omega_{\frac34}\bigr\|_{L^2}^{\frac{2(p+3)}{3p}}\bigl\|\nabla\omega_{\frac34}\bigr\|_{L^2}^{2-\frac2p}+\|\partial_3 v^3\|_{\mathcal{H}_\theta}^{\frac2p}\|\nabla\partial_3 v^3\|_{\mathcal{H}_\theta}^{2-\frac2p}\Bigr).$$

Convexity inequality ensures

$$
\begin{aligned}
\bigl|\bigl(Q_1(v,v)\,|\,\partial_3 v^3\bigr)_{\mathcal{H}_\theta}\bigr| \leq \frac{1}{6}\|\nabla\partial_3 v^3\|_{\mathcal{H}_\theta}^2 + C\|v^3\|^p_{H^{\frac12+\frac2p}}\|\partial_3 v^3\|^2_{\mathcal{H}_\theta}\\
+ C\|v^3\|_{H^{\frac12+\frac2p}}\bigl\|\,\omega_{\frac34}\bigr\|_{L^2}^{\frac{2(p+3)}{3p}}\bigl\|\nabla\omega_{\frac34}\bigr\|_{L^2}^{2-\frac2p}.
\end{aligned}
\tag{6.5}
$$

The estimates of the two terms involving $Q_2(v,v)$ and $Q_3(v,v)$ rely on the following lemma.

Lemma 6.2. *Let A be a bounded Fourier multiplier. If p and θ satisfy Condition (6.2) and $\theta<\frac2p$. We have, for ℓ in $\{1,2\}$,*

$$
\begin{aligned}
\bigl|\bigl(A(D)(v^\ell\partial_\ell\partial_3 v^3)\,|\,\partial_3 v^3\bigr)_{\mathcal{H}_\theta}\bigr| &\lesssim \|v^3\|_{H^{\frac12+\frac2p}}\\
&\times\Bigl(\bigl\|\,\omega_{\frac34}\bigr\|_{L^2}^{\frac13+\frac2p}\bigl\|\nabla\omega_{\frac34}\bigr\|_{L^2}^{1-\frac2p}+\|\partial_3 v^3\|_{\mathcal{H}_\theta}^{\frac2p}\|\nabla\partial_3 v^3\|_{\mathcal{H}_\theta}^{1-\frac2p}\Bigr)\|\nabla\partial_3 v^3\|_{\mathcal{H}_\theta}.
\end{aligned}
$$

Proof. Using (6.4) and the law of product of Lemma 4.5 gives,

$$
\begin{aligned}
\bigl|\bigl(A(D)(v^\ell\partial_\ell\partial_3 v^3)\,|\,\partial_3 v^3\bigr)_{\mathcal{H}_\theta}\bigr| &\leq \|v^\ell\partial_\ell\partial_3 v^3\|_{H^{-1+2\theta,\frac12-\frac2p-2\theta}}\|\partial_3 v^3\|_{H^{0,-\frac12+\frac2p}}\\
&\lesssim \|v^\ell\|_{(B^1_{2,1})_{\rm h}(B^{\frac12-\frac2p}_{2,1})_{\rm v}}\|\partial_\ell\partial_3 v^3\|_{H^{-1+2\theta,\frac12-2\theta}}\|\partial_3 v^3\|_{H^{0,-\frac12+\frac2p}}\\
&\lesssim \|v^\ell\|_{(B^1_{2,1})_{\rm h}(B^{\frac12-\frac2p}_{2,1})_{\rm v}}\|\partial_3 v^3\|_{H^{2\theta,\frac12-2\theta}}\|v^3\|_{H^{\frac12+\frac2p}}.
\end{aligned}
$$

However, notice from Definition 2.1 that

$$
\begin{aligned}
\|\partial_3 v^3\|^2_{H^{2\theta,\frac12-2\theta}} &= \int_{\mathbb{R}^3}|\xi_{\rm h}|^{4\theta}|\xi_3|^{1-4\theta}|\widehat{\partial_3 v^3}(\xi)|^2\,d\xi\\
&\leq \int_{\mathbb{R}^3}|\xi_{\rm h}|^{-1+2\theta}|\xi_3|^{-2\theta}|\xi|^2|\widehat{\partial_3 v^3}(\xi)|^2\,d\xi = \|\nabla\partial_3 v^3\|^2_{\mathcal{H}_\theta}.
\end{aligned}
$$

We thus obtain

$$
\bigl|\bigl(A(D)(v^\ell\partial_\ell\partial_3 v^3)\,|\,\partial_3 v^3\bigr)_{\mathcal{H}_\theta}\bigr| \lesssim \|v^\ell\|_{(B^1_{2,1})_{\rm h}(B^{\frac12-\frac2p}_{2,1})_{\rm v}}\|\nabla\partial_3 v^3\|_{\mathcal{H}_\theta}\|v^3\|_{H^{\frac12+\frac2p}}.
$$

Then Proposition 4.1 leads to the result. □

In order to estimate $\bigl(Q_2(v,v)\,|\,\partial_3 v^3\bigr)_{\mathcal{H}_\theta}$, we write that

$$
\begin{aligned}
\bigl((\mathrm{Id}+2\partial_3^2\Delta^{-1})(\partial_3 v^\ell\partial_\ell v^3)\,|\,\partial_3 v^3\bigr)_{\mathcal{H}_\theta} &= \mathcal{A}_1(v^\ell,v^3)+\mathcal{A}_2(v^\ell,v^3)\quad\text{with}\\
\mathcal{A}_1(v^\ell,v^3) &\stackrel{\rm def}{=} -\bigl((\mathrm{Id}+2\partial_3^2\Delta^{-1})(v^\ell\partial_\ell v^3)\,|\,\partial_3^2 v^3\bigr)_{\mathcal{H}_\theta}\quad\text{and}\\
\mathcal{A}_2(v^\ell,v^3) &\stackrel{\rm def}{=} -\bigl((\mathrm{Id}+2\partial_3^2\Delta^{-1})(v^\ell\partial_\ell\partial_3 v^3)\,|\,\partial_3 v^3\bigr)_{\mathcal{H}_\theta}.
\end{aligned}
\tag{6.6}
$$

Law of product of Lemma 4.5 implies that

$$
\begin{aligned}
\bigl|\mathcal{A}_1(v^\ell,v^3)\bigr| &\lesssim \|v^\ell\partial_\ell v^3\|_{\mathcal{H}_\theta}\|\partial_3^2 v^3\|_{\mathcal{H}_\theta}\\
&\lesssim \|v^\ell\|_{(B^1_{2,1})_{\rm h}(B^{\frac12-\frac2p}_{2,1})_{\rm v}}\|\partial_\ell v^3\|_{H^{-\frac12+\theta,\frac2p-\theta}}\|\partial_3^2 v^3\|_{\mathcal{H}_\theta}.
\end{aligned}
$$

As we have $\|\partial_\ell v^3\|_{H^{-\frac12+\theta,\frac2p-\theta}}\lesssim\|v^3\|_{H^{\frac12+\theta,\frac2p-\theta}}\leq\|v^3\|_{H^{\frac12+\frac2p}}$, we infer that

$$
\bigl|\mathcal{A}_1(v^\ell,v^3)\bigr|\lesssim\|v^\ell\|_{(B^1_{2,1})_{\rm h}(B^{\frac12-\frac2p}_{2,1})_{\rm v}}\|v^3\|_{H^{\frac12+\frac2p}}\|\partial_3^2 v^3\|_{\mathcal{H}_\theta}.
$$

Because of (6.6), Proposition 4.1 and Lemma 6.2 ensures that

$$\begin{aligned}\big|\big(Q_2(v,v)\,|\,\partial_3 v^3\big)_{\mathcal{H}_\theta}\big| &\lesssim \|v^3\|_{H^{\frac12+\frac2p}}\\ &\times\Big(\big\|\omega_{\frac34}\big\|_{L^2}^{\frac13+\frac2p}\big\|\nabla\omega_{\frac34}\big\|_{L^2}^{1-\frac2p}+\|\partial_3 v^3\|_{\mathcal{H}_\theta}^{\frac2p}\|\nabla\partial_3 v^3\|_{\mathcal{H}_\theta}^{1-\frac2p}\Big)\|\nabla\partial_3 v^3\|_{\mathcal{H}_\theta}.\end{aligned}$$

Applying convexity inequality yields

$$\begin{aligned}\big|\big(Q_2(v,v)\,|\,\partial_3 v^3\big)_{\mathcal{H}_\theta}\big| \leq{}& \frac16\|\nabla\partial_3 v^3\|_{\mathcal{H}_\theta}^2+C\|v^3\|_{H^{\frac12+\frac2p}}^p\|\partial_3 v^3\|_{\mathcal{H}_\theta}^2\\ &+C\|v^3\|_{H^{\frac12+\frac2p}}^2\big\|\omega_{\frac34}\big\|_{L^2}^{\frac{2(p+6)}{3p}}\big\|\nabla\omega_{\frac34}\big\|_{L^2}^{2\left(1-\frac2p\right)}.\end{aligned}\tag{6.7}$$

Finally let us estimate $\big(Q_3(v,v)\,|\,\partial_3 v^3\big)_{\mathcal{H}_\theta}$. Lemma 6.2 implies that

$$\begin{aligned}\big|\big(v^{\rm h}\cdot\nabla_{\rm h}\partial_3 v^3\,|\,\partial_3 v^3\big)_{\mathcal{H}_\theta}\big| &\lesssim \|v^3\|_{H^{\frac12+\frac2p}}\\ &\times\Big(\big\|\omega_{\frac34}\big\|_{L^2}^{\frac13+\frac2p}\big\|\nabla\omega_{\frac34}\big\|_{L^2}^{1-\frac2p}+\|\partial_3 v^3\|_{\mathcal{H}_\theta}^{\frac2p}\|\nabla\partial_3 v^3\|_{\mathcal{H}_\theta}^{1-\frac2p}\Big)\|\nabla\partial_3 v^3\|_{\mathcal{H}_\theta}.\end{aligned}\tag{6.8}$$

To estimate $\big(v^3\partial_3^2 v^3\,|\,\partial_3 v^3\big)_{\mathcal{H}_\theta}$, we write, according to (6.3), that

$$\big|(f\,|\,g)_{\mathcal{H}_\theta}\big|\leq\|f\|_{H^{\theta+\frac2p-\frac32,-\theta}}\|g\|_{H^{\frac12+\theta-\frac2p,-\theta}}.$$

As $\theta>\frac12-\frac2p$, we get, by applying law of product of Lemma 4.5 and then Lemma 4.3, that

$$\begin{aligned}\big|\big(v^3\partial_3^2 v^3\,|\,\partial_3 v^3\big)_{\mathcal{H}_\theta}\big| &\leq \|v^3\partial_3^2 v^3\|_{H^{\theta+\frac2p-\frac32,-\theta}}\|\partial_3 v^3\|_{H^{\frac12+\theta-\frac2p,-\theta}}\\ &\lesssim \|v^3\|_{\left(H^{\frac2p}\right)_{\rm h}\left(B^{\frac12}_{2,1}\right)_{\rm v}}\|\partial_3^2 v^3\|_{\mathcal{H}_\theta}\|\partial_3 v^3\|_{H^{\frac12+\theta-\frac2p,-\theta}}\\ &\lesssim \|v^3\|_{H^{\frac12+\frac2p}}\|\partial_3^2 v^3\|_{\mathcal{H}_\theta}\|\partial_3 v^3\|_{H^{\frac12+\theta-\frac2p,-\theta}},\end{aligned}$$

This along with the interpolation inequality which claims that

$$\|\partial_3 v^3\|_{H^{\frac12+\theta-\frac2p,-\theta}}\leq\|\partial_3 v^3\|_{\mathcal{H}_\theta}^{\frac2p}\|\nabla_h\partial_3 v^3\|_{\mathcal{H}_\theta}^{1-\frac2p},$$

ensures

$$\big|\big(v^3\partial_3^2 v^3\,|\,\partial_3 v^3\big)_{\mathcal{H}_\theta}\big|\lesssim\|v^3\|_{H^{\frac12+\frac2p}}\|\partial_3 v^3\|_{\mathcal{H}_\theta}^{\frac2p}\|\nabla\partial_3 v^3\|_{\mathcal{H}_\theta}^{2-\frac2p}.$$

Due to (6.8) and convexity inequality, we thus obtain

$$\begin{aligned}\big|\big(Q_3(v,v)\,|\,\partial_3 v^3\big)_{\mathcal{H}_\theta}\big| \leq{}& \frac16\|\nabla\partial_3 v^3\|_{\mathcal{H}_\theta}^2+C\|v^3\|_{H^{\frac12+\frac2p}}^p\|\partial_3 v^3\|_{\mathcal{H}_\theta}^2\\ &+C\|v^3\|_{H^{\frac12+\frac2p}}^2\big\|\omega_{\frac34}\big\|_{L^2}^{\frac{2(p+6)}{3p}}\big\|\nabla\omega_{\frac34}\big\|_{L^2}^{2\left(1-\frac2p\right)}.\end{aligned}\tag{6.9}$$

Now we are in a position to complete the proof of Proposition 2.2.

Conclusion of the proof to Proposition 2.2. By resuming the estimates (6.5), (6.7) and (6.9) into (6.1), we obtain

$$
\begin{aligned}
&\frac{d}{dt}\|\partial_3 v^3(t)\|_{\mathcal{H}_\theta}^2+\|\nabla\partial_3 v^3(t)\|_{\mathcal{H}_\theta}^2\\
&\qquad\leq C\Big(\|v^3\|^2_{H^{\frac12+\frac2p}}\big\|\omega_{\frac34}\big\|_{L^2}^{2\left(\frac13+\frac2p\right)}\big\|\nabla\omega_{\frac34}\big\|_{L^2}^{2\left(1-\frac2p\right)}\\
&\qquad\quad+\|v^3\|^p_{H^{\frac12+\frac2p}}\|\partial_3 v^3\|^2_{\mathcal{H}_\theta}+\|v^3\|_{H^{\frac12+\frac2p}}\big\|\omega_{\frac34}\big\|_{L^2}^{2\left(\frac13+\frac1p\right)}\big\|\nabla\omega_{\frac34}\big\|_{L^2}^{2\left(1-\frac1p\right)}\Big).
\end{aligned}\tag{6.10}
$$

On the other hand, Inequality (2.5) claims that $\|\partial_3 v_0^3\|_{\mathcal{H}^\theta}\lesssim\|v_0\|_{H^{\frac12}}$. Thus Gronwall's inequality allows to conclude the proof of Proposition 2.2. □

7. The closure of the estimates to horizontal vorticity and divergence

The main step of the proof of Proposition 2.3 is the proof of the following estimate, for any t in $[0,T^\star[$.

$$
\big\|\omega_{\frac34}(t)\big\|_{L^2}^{2\frac{p+3}{3}}+\big\|\nabla\omega_{\frac34}\big\|_{L^2_t(L^2)}^{2\frac{p+3}{3}}\leq C\|\Omega_0\|_{L^{\frac32}}^{\frac{p+3}{2}}\exp\Big(C\exp\Big(C\int_0^t\|v^3(t)\|^p_{H^{\frac12+\frac2p}}\,dt'\Big)\Big).\tag{7.1}
$$

In order to do it, let us introduce the notation

$$
e(T)\stackrel{\text{def}}{=}C\exp\Big(C\int_0^T\|v^3(t)\|^p_{H^{\frac12+\frac2p}}\,dt\Big).\tag{7.2}
$$

where the constant C may change from line to line. As $(a+b)^{\frac34}\sim a^{\frac34}+b^{\frac34}$, Proposition 2.2 implies that

$$
\begin{aligned}
&\Big(\int_0^t\|\partial_3^2 v^3(t')\|^2_{\mathcal{H}_\theta}\,dt'\Big)^{\frac34}e(T)\lesssim e(T)\big(\|\Omega_0\|_{L^{\frac32}}^{\frac32}+V_1(t)+V_2(t)\big)\quad\text{with}\\
&V_1(t)\stackrel{\text{def}}{=}\Big(\int_0^t\|v^3(t')\|_{H^{\frac12+\frac2p}}\big\|\,\omega_{\frac34}(t')\big\|_{L^2}^{2\left(\frac13+\frac1p\right)}\big\|\nabla\omega_{\frac34}(t')\big\|_{L^2}^{2\left(1-\frac1p\right)}dt'\Big)^{\frac34}\quad\text{and}\\
&V_2(t)\stackrel{\text{def}}{=}\Big(\int_0^t\|v^3(t')\|^2_{H^{\frac12+\frac2p}}\big\|\,\omega_{\frac34}(t')\big\|_{L^2}^{2\left(\frac13+\frac2p\right)}\big\|\nabla\omega_{\frac34}(t')\big\|_{L^2}^{2\left(1-\frac2p\right)}dt'\Big)^{\frac34}.
\end{aligned}\tag{7.3}
$$

Let us estimate the two terms $V_j(t), j=1,2$. Applying Hölder inequality gives

$$
\begin{aligned}
V_1(t)\ &\leq\ \Big(\int_0^t\|v^3(t')\|^p_{H^{\frac12+\frac2p}}\big\|\omega_{\frac34}(t')\big\|_{L^2}^{2\left(\frac13+\frac1p\right)p}dt'\Big)^{\frac34\times\frac1p}\Big(\int_0^t\big\|\nabla\omega_{\frac34}(t')\big\|^2_{L^2}dt'\Big)^{\frac34\left(1-\frac1p\right)}\\
&\leq\ \Big(\int_0^t\|v^3(t')\|^p_{H^{\frac12+\frac2p}}\big\|\,\omega_{\frac34}(t')\big\|_{L^2}^{2\frac{p+3}{3}}dt'\Big)^{\frac{3}{4p}}\Big(\int_0^t\big\|\nabla\omega_{\frac34}(t')\big\|^2_{L^2}dt'\Big)^{\frac34\left(1-\frac1p\right)}.
\end{aligned}
$$

As we have

$$
1-\frac34\Big(1-\frac1p\Big)=\frac{p+3}{4p}\,,
$$

convexity inequality implies that, for any t in $[0,T]$,

$$
e(T)V_1(t)\leq\frac19\int_0^t\big\|\nabla\omega_{\frac34}(t')\big\|^2_{L^2}dt'+e(T)\Big(\int_0^t\|v^3(t')\|^p_{H^{\frac12+\frac2p}}\big\|\omega_{\frac34}(t')\big\|_{L^2}^{2\frac{p+3}{3}}dt'\Big)^{\frac{3}{p+3}}.\tag{7.4}
$$

Now let us estimate the term $V_2(t)$. Applying Hölder inequality yields

$$\begin{aligned} V_2(t) &\leq \biggl(\int_0^t \|v^3(t')\|^p_{H^{\frac12+\frac2p}} \bigl\|\omega_{\frac34}(t')\bigr\|^{2\left(\frac13+\frac2p\right)\frac p2}_{L^2} dt'\biggr)^{\frac34\times\frac2p} \biggl(\int_0^t \bigl\|\nabla\omega_{\frac34}(t')\bigr\|^2_{L^2} dt'\biggr)^{\frac34\left(1-\frac2p\right)} \\ &\leq \biggl(\int_0^t \|v^3(t')\|^p_{H^{\frac12+\frac2p}} \bigl\|\omega_{\frac34}(t')\bigr\|^{2\frac{p+6}{6}}_{L^2} dt'\biggr)^{\frac{3}{2p}} \biggl(\int_0^t \bigl\|\nabla\omega_{\frac34}(t')\bigr\|^2_{L^2} dt'\biggr)^{\frac34\left(1-\frac2p\right)}. \end{aligned}$$

As we have

$$1-\frac34\Bigl(1-\frac2p\Bigr)=\frac{p+6}{4p}\,,$$

convexity inequality implies that

$$e(T)V_2(t) \leq \frac19 \int_0^t \bigl\|\nabla\omega_{\frac34}(t')\bigr\|^2_{L^2} dt' + e(T)\biggl(\int_0^t \|v^3(t')\|^p_{H^{\frac12+\frac2p}} \|\omega_{\frac34}(t')\|^{2\frac{p+6}{6}}_{L^2} dt'\biggr)^{\frac{6}{p+6}}. \tag{7.5}$$

Let us notice that the power of $\bigl\|\omega_{\frac34}\bigr\|_{L^2}$ here is not the same as that in Inequality (7.4). Applying Hölder inequality with

$$q=\frac{p+3}{3}\times\frac{6}{p+6}=2\,\frac{p+3}{p+6}$$

and with the measure $\|v^3(t')\|^p_{H^{\frac12+\frac2p}}\,dt'$ gives

$$\begin{aligned} \biggl(\int_0^t \|v^3(t')\|^p_{H^{\frac12+\frac2p}} \bigl\|\omega_{\frac34}(t')\bigr\|^{2\frac{p+6}{6}}_{L^2} dt'\biggr)^{\frac{6}{p+6}} &\leq \biggl(\int_0^t \|v^3(t')\|^p_{H^{\frac12+\frac2p}} dt'\biggr)^{\left(1-\frac1q\right)\times\frac{6}{p+6}} \\ &\quad\times \biggl(\int_0^t \|v^3(t')\|^p_{H^{\frac12+\frac2p}} \bigl\|\omega_{\frac34}(t')\bigr\|^{2\frac{p+3}{3}}_{L^2} dt'\biggr)^{\frac{3}{p+3}}. \end{aligned}$$

By definition of $e(T)$, we have

$$\biggl(\int_0^t \|v^3(t')\|^p_{H^{\frac12+\frac2p}} dt'\biggr)^{\left(1-\frac1q\right)\times\frac{6}{p+6}} e(T)\leq e(T).$$

Thus we deduce from (7.5) that

$$e(T)V_2(t) \leq \frac19 \int_0^t \bigl\|\nabla\omega_{\frac34}(t')\bigr\|^2_{L^2} dt' + e(T)\biggl(\int_0^t \|v^3(t')\|^p_{H^{\frac12+\frac2p}} \bigl\|\omega|_{\frac34}(t')\bigr\|^{2\left(\frac{p+3}{3}\right)}_{L^2} dt'\biggr)^{\frac{3}{p+3}}.$$

Plugging this inequality and (7.4) into (7.3) gives, for any t in $[0,T]$,

$$\begin{aligned} \biggl(\int_0^t \|\partial_3^2 v^3(t')\|^2_{\mathcal{H}_\theta}\,dt'\biggr)^{\frac34} e(T) &\leq \frac29 \int_0^t \bigl\|\nabla\omega_{\frac34}(t')\bigr\|^2_{L^2} dt' + e(T)\|\Omega_0\|^{\frac32}_{L^{\frac32}} \\ &\quad + e(T)\biggl(\int_0^t \|v^3(t')\|^p_{H^{\frac12+\frac2p}} \bigl\|\omega_{\frac34}(t')\bigr\|^{2\frac{p+3}{3}}_{L^2} dt'\biggr)^{\frac{3}{p+3}}. \end{aligned}$$

Hence thanks to Proposition 2.1, we deduce that

$$\begin{aligned} \frac23\|\omega_{\frac34}(t)\|^2_{L^2}+\frac13\int_0^t \|\nabla\omega_{\frac34}(t')\|^2_{L^2}\,dt' &\leq \|\Omega_0\|^{\frac32}_{L^{\frac32}} e(T) \\ &\quad + e(T)\biggl(\int_0^t \|v^3(t')\|^p_{H^{\frac12+\frac2p}} \|\omega_{\frac34}(t')\|^{2\frac{p+3}{3}}_{L^2} dt'\biggr)^{\frac{3}{p+3}}. \end{aligned}$$

Taking the power $\dfrac{p+3}{3}$ of this inequality and using that $(a+b)^{\frac{p+3}{3}} \sim a^{\frac{p+3}{3}} + b^{\frac{p+3}{3}}$, we obtain for any t in $[0,T]$,

$$\bigl\|\omega_{\frac34}(t)\bigr\|_{L^2}^{2\frac{p+3}{3}} + \biggl(\int_0^t \bigl\|\nabla\omega_{\frac34}(t')\bigr\|_{L^2}^2\,dt'\biggr)^{\frac{p+3}{3}} \leq \|\Omega_0\|_{L^{\frac32}}^{\frac{p+3}{2}} e(T)$$
$$+ e(T)\int_0^t \|v^3(t')\|^p_{H^{\frac12+\frac2p}} \bigl\|\omega_{\frac34}(t')\bigr\|_{L^2}^{2\frac{p+3}{3}}\,dt'.$$

Then Gronwall lemma leads to Inequality (7.1).

On the other hand, it follows from Proposition 2.2 that, for any $t < T^*$,

$$\begin{aligned}\|\partial_3 v^3(t)\|^2_{\mathcal{H}_\theta} &+ \int_0^t \|\nabla\partial_3 v^3(t')\|^2_{\mathcal{H}_\theta}\,dt' \\ &\leq e(t)\biggl(\|\Omega_0\|^2_{L^{\frac32}} + \|v^3\|_{L^p_t(H^{\frac12+\frac2p})}\bigl\|\omega_{\frac34}\bigr\|^{2\frac{p+3}{3p}}_{L^\infty_t(L^2)}\bigl\|\nabla\omega_{\frac34}\bigr\|^{2\left(1-\frac1p\right)}_{L^2_t(L^2)} \\ &\qquad + \|v^3\|^2_{L^p_t(H^{\frac12+\frac2p})}\bigl\|\omega_{\frac34}\bigr\|^{2\frac{p+6}{3p}}_{L^\infty_t(L^2)}\bigl\|\nabla\omega_{\frac34}\bigr\|^{2\left(1-\frac2p\right)}_{L^2_t(L^2)}\biggr).\end{aligned}$$

Resuming the estimate (7.1) into the above inequality concludes the proof of Proposition 2.3.

8. Proof of the end point blow up theorem

The proof of Theorem 1.5 relies on the following lemma.

Lemma 8.1. *Let $(p_{k,\ell})_{1\leq k,\ell\leq 3}$ be a sequence of $]1,\infty[^9$ and $v = (v^1, v^2, v^3)$ be a smooth divergence free vector field. Then for the norm $\|\cdot\|_{\mathcal{B}_p}$ given by Definition 1.1, we have*

$$\bigl|(v\cdot\nabla v|v)_{H^{\frac12}}\bigr| \lesssim \sum_{k,\ell} \|\partial_\ell v^k\|_{\mathcal{B}_{p_{k,\ell}}} \|v\|^{\frac{2}{p_{k,\ell}}}_{H^{\frac12}} \|\nabla v\|^{2-\frac{2}{p_{k,\ell}}}_{H^{\frac12}}.$$

Proof. Let us choose on $H^{\frac12}$ the following inner product

$$(a|b)_{H^{\frac12}} = \sum_{j\in\mathbb{Z}} 2^j (\Delta_j a|\Delta_j b)_{L^2}.$$

We use Bony's decomposition (4.10) to deal with the product function $v\cdot\nabla v$. Namely, we write

$$v^\ell\partial_\ell v^k = T(v^\ell, \partial_\ell v^k) + T(\partial_\ell v^k, v^\ell) + R(v^\ell, \partial_\ell v^k). \tag{8.1}$$

Let us start with the terms $T(\partial_\ell v^k, v^\ell)$. The support of the Fourier transform of the function $S_{j'-1}\partial_\ell v^k \Delta_{j'} v^\ell$ is included in a ring of the type $2^{j'}\widetilde{\mathcal{C}}$. Thus according to Definition 1.1, we have

$$\begin{aligned}\|\Delta_j T(\partial_\ell v^k, v^\ell)\|_{L^2} &\leq \sum_{|j'-j|\leq 4} \|S_{j'-1}\partial_\ell v^k \Delta_{j'} v^\ell\|_{L^2} \\ &\leq \sum_{|j'-j|\leq 4} \|S_{j'-1}\partial_\ell v^k\|_{L^\infty} \|\Delta_{j'} v^\ell\|_{L^2} \\ &\lesssim \|\partial_\ell v^k\|_{\mathcal{B}_{p_{k,\ell}}} 2^{j\left(1-\frac{1}{p_{k,\ell}}\right)} \sum_{|j'-j|\leq 4} 2^{j'\left(1-\frac{1}{p_{k,\ell}}\right)} \|\Delta_{j'} v^\ell\|_{L^2}.\end{aligned}$$

Now let us write that

$$2^j\big|\big(\Delta_j T(\partial_\ell v^k,v^k)\,\big|\,\Delta_j v^k\big)_{L^2}\big| \lesssim \|\partial_\ell v^k\|_{\mathcal{B}_{p_{k,\ell}}}\big(2^{\frac{j}{2}}\|\Delta_j v\|_{L^2}\big)^{\frac{1}{p_{k,\ell}}}\big(2^{\frac{3j}{2}}\|\Delta_j v\|_{L^2}\big)^{1-\frac{1}{p_{k,\ell}}}$$
$$\times\sum_{|j'-j|\leq 4}2^{\frac{j-j'}{2}}\big(2^{\frac{j'}{2}}\|\Delta_{j'}v\|_{L^2}\big)^{\frac{1}{p_{k,\ell}}}\big(2^{\frac{3j'}{2}}\|\Delta_{j'}v\|_{L^2}\big)^{1-\frac{1}{p_{k,\ell}}}.$$

Using the characterization of Sobolev norms in term of Littlewood-Paley theory, we get

$$\sum_{k,\ell=1}^{3}\sum_{j\in\mathbb{Z}}2^j\big|(\Delta_j T(\partial_\ell v^k,v^\ell)\,|\,\Delta_j v^k)_{L^2}\big| \lesssim \sum_{1\leq k,\ell\leq 3}\|\partial_\ell v^k\|_{\mathcal{B}_{p_{k,\ell}}}\|v\|_{H^{\frac12}}^{\frac{2}{p_{k,\ell}}}\|\nabla v\|_{H^{\frac12}}^{2-\frac{2}{p_{k,\ell}}}. \tag{8.2}$$

The terms $R(\partial_\ell v^k,v^\ell)$ are a little bit more delicate. The support of the Fourier transform of $\Delta_{j'}\partial_\ell v^k\widetilde{\Delta}_{j'}v^\ell$ is included in a ball of the type $2^{j'}\widetilde{B}$. Thus we have

$$\Delta_j R(\partial_\ell v^k,v^\ell)=\sum_{j'\geq j-N_0}\Delta_j\big(\Delta_{j'}\partial_\ell v^k\widetilde{\Delta}_{j'}v^\ell\big).$$

Because of the divergence free condition of v, we can write

$$\sum_{\ell=1}^{3}\Delta_j R(\partial_\ell v^k,v^\ell)=\sum_{\ell=1}^{3}\partial_\ell\sum_{j'\geq j-N_0}\Delta_j\big(\Delta_{j'}v^k\widetilde{\Delta}_{j'}v^\ell\big).$$

Using the fact that the Fourier transform of $\Delta_{j'}$ is supported in a ring of the type $2^{j'}\mathcal{C}$, we can write that

$$\Delta_{j'}v^k=\sum_{\ell'=1}^{3}2^{-j'}\widetilde{\Delta}_{j'}^{\ell'}\Delta_{j'}\partial_{\ell'}v^k\quad\text{with}\quad\widetilde{\Delta}_{j'}^{\ell'}a\overset{\text{def}}{=}\mathcal{F}^{-1}\big(\phi^{\ell'}(2^{-j'}\xi)\widehat{a}\big) \tag{8.3}$$

where $\phi^{\ell'}$, for $\ell'=1,2,3$, are functions of $\mathcal{D}(\mathbb{R}^3\setminus\{0\})$ (see for instance page 56 of [1] for the details). We thus obtain

$$\Big\|\sum_{\ell=1}^{3}\Delta_j R(\partial_\ell v^k,v^\ell)\Big\|_{L^2}\lesssim\sum_{\ell=1}^{3}\sum_{j'\geq j-N_0}2^{-(j'-j)}2^{2j'\left(1-\frac{1}{p_{k,\ell}}\right)}\|\partial_\ell v^k\|_{\mathcal{B}_{p_{k,\ell}}}\|\widetilde{\Delta}_{j'}v\|_{L^2},$$

from which, we infer

$$\begin{aligned}
I_R(v) &\overset{\text{def}}{=} \sum_{k,\ell=1}^{3}\sum_{j\in\mathbb{Z}}2^j\big|\big(\Delta_j R(\partial_\ell v^k,v^\ell)\,\big|\,\Delta_j v^k\big)_{L^2}\big|\\
&\lesssim \sum_{k=1}^{3}\sum_{j\in\mathbb{Z}}2^j\Big\|\sum_{\ell=1}^{3}\Delta_j R(\partial_\ell v^k,v^\ell)\Big\|_{L^2}\|\Delta_j v\|_{L^2}\\
&\lesssim \sum_{k,\ell=1}^{3}\|\partial_\ell v^k\|_{\mathcal{B}_{p_{k,\ell}}}\\
&\qquad\times\sum_{\substack{j,j'\in\mathbb{Z}\\ j'\geq j-N_0}}2^{-(j'-j)\left(\frac12+\frac{1}{p_{k,\ell}}\right)}\big(2^{\frac{j'}{2}}\|\widetilde{\Delta}_{j'}v\|_{L^2}\big)^{\frac{1}{p_{k,\ell}}}\big(2^{\frac{3j'}{2}}\|\widetilde{\Delta}_{j'}v\|_{L^2}\big)^{1-\frac{1}{p_{k,\ell}}}\\
&\qquad\qquad\times\big(2^{\frac{j}{2}}\|\Delta_j v\|_{L^2}\big)^{\frac{1}{p_{k,\ell}}}\big(2^{\frac{3j}{2}}\|\Delta_j v\|_{L^2}\big)^{1-\frac{1}{p_{k,\ell}}}.
\end{aligned}$$

Using the convolution law of $\mathbb{Z}$, we deduce that

$$I_R(v) \lesssim \sum_{k,\ell=1}^{3} \|\partial_\ell v^k\|_{\mathcal{B}_{p_{k,\ell}}} \|v\|_{H^{\frac12}}^{\frac{2}{p_{k,\ell}}} \|\nabla v\|_{H^{\frac12}}^{2-\frac{2}{p_{k,\ell}}}. \tag{8.4}$$

To deal with the terms of the form $T(v^\ell, \partial_\ell v^k)$ in (8.1), we use the skew symmetry property of the operator $v\cdot\nabla$. Then we follow [7]. As the support of the Fourier transform of $S_{j'-1}a\Delta_{j'}b$ is included in a ring of the type $2^{j'}\widetilde{\mathcal{C}}$, we write

$$\begin{aligned}
\Delta_j \sum_{j'} S_{j'-1}v^\ell \Delta_{j'}\partial_\ell w &= S_{j-1}v^\ell\Delta_j\partial_\ell w + \sum_{\ell=1}^{2} R^k_{j,\ell}(v,w) \quad \text{with}\\
R^1_{j,\ell}(v^\ell,w) &\stackrel{\text{def}}{=} \sum_{|j'-j|\leq 4} \big[\Delta_j, S_{j'-1}v^\ell\big]\Delta_{j'}\partial_\ell w \quad \text{and}\\
R^2_{j,\ell}(v^\ell,w) &\stackrel{\text{def}}{=} \sum_{|j'-j|\leq 4} (S_{j'-1}v^\ell - S_{j-1}v^\ell)\Delta_j\Delta_{j'}\partial_\ell w.
\end{aligned}$$

By definition of the space $\mathcal{B}_p$ in Definition 1.1, Lemma 2.97 of [1] implies that

$$\begin{aligned}
\|R^1_{j,\ell}(v,w)\|_{L^2} &\lesssim 2^{-j} \sum_{|j'-j|\leq 4} \|\nabla S_{j'-1}v^\ell\|_{L^\infty} \big\|\Delta_{j'}\partial_\ell w\big\|_{L^2}\\
&\lesssim 2^{-j} \sum_{|j'-j|\leq 4} \sum_{\ell'=1}^{3} \|S_{j'-1}\partial_{\ell'}v^\ell\|_{L^\infty} \big\|\Delta_{j'}\partial_\ell w\big\|_{L^2}\\
&\lesssim \sum_{\ell'=1}^{3} 2^{j\left(1-\frac{2}{p_{\ell,\ell'}}\right)} \|\partial_{\ell'}v^\ell\|_{\mathcal{B}_{p_{\ell,\ell'}}} \sum_{|j'-j|\leq 4} \big\|\Delta_{j'}\partial_\ell w\big\|_{L^2}.
\end{aligned} \tag{8.5}$$

In order to estimate $\|R^2_{j,\ell}(v,w)\|_{L^2}$, we use Lemma 4.1 to get

$$\|R^2_{j,\ell}(v^\ell,w)\|_{L^2} \lesssim \sum_{\substack{|j'-j|\leq 4\\ j''\in[j-1,j'-1]}} \|\Delta_{j''}v^\ell\|_{L^\infty}\|\Delta_j\Delta_{j'}\partial_\ell w\|_{L^2}.$$

Notice that (8.3) ensures that

$$\|\Delta_j v^\ell\|_{L^\infty} \lesssim 2^{-j}\sum_{\ell'=1}^{3}\|\Delta_j\partial_{\ell'}v^\ell\|_{L^\infty}.$$

By virtue of Definition 1.1, this implies that

$$\|\Delta_j v^\ell\|_{L^\infty} \lesssim \sum_{\ell'=1}^{3} 2^{j\left(1-\frac{2}{p_{\ell,\ell'}}\right)} \|\partial_{\ell'}v^\ell\|_{\mathcal{B}_{p_{\ell,\ell'}}}.$$

We thus infer that

$$\|R^2_{j,\ell}(v,w)\|_{L^2} \lesssim \sum_{\ell'=1}^{3} 2^{j\left(1-\frac{2}{p_{\ell,\ell'}}\right)} \|\partial_{\ell'}v^\ell\|_{\mathcal{B}_{p_{\ell,\ell'}}} \sum_{|j'-j|\leq 4} \big\|\Delta_{j'}\partial_\ell w\big\|_{L^2}. \tag{8.6}$$

Because of the divergence free on v, we have

$$(S_{j-1}v\cdot\Delta_j w|\Delta_j w)_{L^2} = 0,$$

this together with (8.5) and (8.6) gives rise to

$$\begin{aligned}&\Big|\sum_{k,\ell=1}^{3}\sum_{j\in\mathbb{Z}}2^{j}\big(\Delta_jT(v^\ell,\partial_\ell v^k)\ \big|\ \Delta_jv^k\big)_{L^2}\Big|\\&\quad\lesssim\sum_{1\leq k,\ell\leq 3}\|\partial_\ell v^k\|_{\mathcal{B}_{p_{k,\ell}}}\\&\qquad\times\sum_{\substack{j,j'\in\mathbb{Z}\\|j'-j|\leq 4}}2^{(j-j')\left(\frac12-\frac{1}{p_{k,\ell}}\right)}\big(2^{\frac{j'}{2}}\|\Delta_{j'}v\|_{L^2}\big)^{\frac{1}{p_{k,\ell}}}\big(2^{\frac{3j'}{2}}\|\Delta_{j'}v\|_{L^2}\big)^{1-\frac{1}{p_{k,\ell}}}\\&\qquad\qquad\times\big(2^{\frac{j}{2}}\|\Delta_jv\|_{L^2}\big)^{\frac{1}{p_{k,\ell}}}\big(2^{\frac{3j}{2}}\|\Delta_jv\|_{L^2}\big)^{1-\frac{1}{p_{k,\ell}}}\\&\quad\lesssim\sum_{k,\ell=1}^{3}\|\partial_\ell v^k\|_{\mathcal{B}_{p_{k,\ell}}}\|v\|_{H^{\frac12}}^{\frac{2}{p_{k,\ell}}}\|\nabla v\|_{H^{\frac12}}^{2-\frac{2}{p_{k,\ell}}},\end{aligned}$$

which along with Inequalities (8.1), (8.2) and (8.4) yields the lemma. □

We now turn to the proof of Theorem 1.5 and Theorem 1.4.

Conclusion of the proof of Theorem 1.5. We shall prove that, for any T less than $T^\star$,

$$\|v(T)\|_{H^{\frac12}}^2+\int_0^T\|\nabla v(t)\|_{H^{\frac12}}^2\,dt\leq\|v_0\|_{H^{\frac12}}^2\exp\Big(C\sum_{1\leq k,\ell\leq 3}\int_0^T\|\partial_\ell v^k(t)\|_{\mathcal{B}_{p_{k,\ell}}}^{p_{k,\ell}}\,dt\Big).\tag{8.7}$$

As a matter of fact, we get, by taking $H^{\frac12}$ energy estimate to (NS) and Lemma 8.1, that

$$\begin{aligned}\frac12\frac{d}{dt}\|v(t)\|_{H^{\frac12}}^2+\|\nabla v(t)\|_{H^{\frac12}}^2&=-\big(v\cdot\nabla v|v\big)_{H^{\frac12}}\\&\lesssim\sum_{k,\ell=1}^{3}\|\partial_\ell v^k(t)\|_{\mathcal{B}_{p_{k,\ell}}}\|v(t)\|_{H^{\frac12}}^{\frac{2}{p_{k,\ell}}}\|\nabla v(t)\|_{H^{\frac12}}^{2-\frac{2}{p_{k,\ell}}}.\end{aligned}$$

Using the convexity inequality, we infer that

$$\frac{d}{dt}\|v(t)\|_{H^{\frac12}}^2+\|\nabla v(t)\|_{H^{\frac12}}^2\lesssim\|v(t)\|_{H^{\frac12}}^2\Big(\sum_{k,\ell=1}^{3}\|\partial_\ell v^k(t)\|_{\mathcal{B}_{p_{k,\ell}}}^{p_{k,\ell}}\Big).$$

Gronwall lemma implies (8.7). This completes the proof of Theorem 1.5. □

Conclusion of the proof of Theorem 1.4. We are going to deduce Theorem 1.4 from Theorem 1.5 and Proposition 2.3. Let us star with a Littlewood-Paley vision of Inequality (1.6). t It follows from Lemma 4.1 that

$$\max_{1\leq\ell\leq 3}\|\partial_\ell v^3\|_{\mathcal{B}_p}\lesssim\sup_{j\in\mathbb{Z}}2^{j\left(-1+\frac{2}{p}\right)}\|\Delta_jv^3\|_{L^\infty}\lesssim\sup_{j\in\mathbb{Z}}2^{j\left(\frac12+\frac{2}{p}\right)}\|\Delta_jv^3\|_{L^2}\lesssim\|v^3\|_{H^{\frac12+\frac{2}{p}}},$$

which together with (1.5) ensures that

$$\max_{1\leq\ell\leq 3}\int_0^{T^\star}\|\partial_\ell v^3(t)\|_{\mathcal{B}_p}^p\,dt\lesssim\int_0^{T^\star}\|v^3(t)\|_{H^{\frac12+\frac{2}{p}}}^p\,dt<\infty.\tag{8.8}$$

The same argument yields

$$\forall T<T^\star\,,\ \int_0^T\|\partial_{\rm h}^2\Delta_{\rm h}^{-1}\partial_3v^3(t)\|_{\mathcal{B}_p}^p\,dt\lesssim\int_0^T\|v^3(t)\|_{H^{\frac12+\frac{2}{p}}}^p\,dt.\tag{8.9}$$

While for any integer N, we get by using Lemma 4.1 that, for any function a and $p>\frac{3}{2}$,

$$\begin{aligned}\|a\|_{\mathcal{B}_p} &\leq \sum_{j\leq N} 2^{j\left(-2+\frac{2}{p}\right)}\|\Delta_j a\|_{L^\infty}+\sum_{j>N} 2^{j\left(-2+\frac{2}{p}\right)}\|\Delta_j a\|_{L^\infty}\\ &\lesssim \sum_{j\leq N} 2^{\frac{2j}{p}}\|a\|_{L^{\frac{3}{2}}}+\sum_{j>N} 2^{j\left(-\frac{4}{3}+\frac{2}{p}\right)}\|\nabla a\|_{L^{\frac{9}{5}}}\\ &\lesssim 2^{\frac{2N}{p}}\|a\|_{L^{\frac{3}{2}}}+2^{N\left(-\frac{4}{3}+\frac{2}{p}\right)}\|\nabla a\|_{L^{\frac{9}{5}}}.\end{aligned}$$

Choosing $N=\left[\log_2\left(e+\left(\frac{\|\nabla a\|_{L^{\frac{9}{5}}}}{\|a\|_{L^{\frac{3}{2}}}}\right)^{\frac{3}{4}}\right)\right]$, we obtain

$$\|a\|_{\mathcal{B}_p}\lesssim \|a\|_{L^{\frac{3}{2}}}^{1-\frac{3}{2p}}\|\nabla a\|_{L^{\frac{9}{5}}}^{\frac{3}{2p}}.$$

Applying this inequality with $a=\partial_{\rm h}^2\Delta_{\rm h}^{-1}\omega$, we get

$$\|\partial_{\rm h}^2\Delta_{\rm h}^{-1}\omega\|_{\mathcal{B}_p}\lesssim \|\partial_{\rm h}^2\Delta_{\rm h}^{-1}\omega\|_{L^{\frac{3}{2}}}^{1-\frac{3}{2p}}\|\partial_{\rm h}^2\Delta_{\rm h}^{-1}\nabla\omega\|_{L^{\frac{9}{5}}}^{\frac{3}{2p}}.$$

Once noticed that $L^p=L^p_{\rm v}(L^p_{\rm h})$, we apply Riesz theorem in the horizontal variables to infer that

$$\|\partial_{\rm h}^2\Delta_{\rm h}^{-1}\omega\|_{L^{\frac{3}{2}}}\lesssim\|\omega\|_{L^{\frac{3}{2}}}\quad\text{and}\quad \|\partial_{\rm h}^2\Delta_{\rm h}^{-1}\nabla\omega\|_{L^{\frac{9}{5}}}\lesssim\|\nabla\omega\|_{L^{\frac{9}{5}}}.$$

Then due to (3.7), we deduce that

$$\|\partial_{\rm h}^2\Delta_{\rm h}^{-1}\omega\|_{\mathcal{B}_p}\lesssim \|\omega\|_{L^{\frac{3}{2}}}^{1-\frac{3}{2p}}\|\nabla\omega\|_{L^{\frac{9}{5}}}^{\frac{3}{2p}}\lesssim \|\omega\|_{L^{\frac{3}{2}}}^{1-\frac{3}{2p}}\big\|\nabla\omega_{\frac{3}{4}}\big\|_{L^2}^{\frac{2}{p}},$$

Together with (8.9), this gives, for any T less than $T^\star$,

$$\int_0^T\|\nabla_{\rm h}v^{\rm h}(t)\|_{\mathcal{B}_p}^p\,dt\lesssim\int_0^T\|v^3(t)\|^p_{H^{\frac{1}{2}+\frac{2}{p}}}\,dt+\sup_{t\in[0,T[}\|\omega(t)\|_{L^{\frac{3}{2}}}^{p-\frac{3}{2}}\int_0^T\big\|\nabla\omega_{\frac{3}{4}}(t)\big\|_{L^2}^2\,dt.$$

Proposition 2.3 then implies that

$$\int_0^{T^\star}\|\nabla_{\rm h}v^{\rm h}(t)\|_{\mathcal{B}_p}^p\,dt<\infty. \tag{8.10}$$

Let us observe from (2.1) and (2.2) that the components of $\partial_3 v^{\rm h}$ are sum of terms of the form $\partial_{\rm h}\Delta_{\rm h}^{-1}\partial_3 f$ with $f=\omega$ or $\partial_3 v^3$. On the one hand, we get, by applying Lemma 4.1, that

$$\begin{aligned}\|\Delta_j\partial_3 v^{\rm h}_{\rm div}(t)\|_{L^\infty} &\lesssim \sum_{\substack{k\leq j+N_0\\ \ell\leq j+N_0}}\|\Delta_k^{\rm h}\Delta_\ell^{\rm v}\nabla_{\rm h}\Delta_{\rm h}^{-1}\partial_3^2 v^3(t)\|_{L^\infty}\\ &\lesssim \sum_{\substack{k\leq j+N_0\\ \ell\leq j+N_0}}2^{\frac{\ell}{2}}\|\Delta_k^{\rm h}\Delta_\ell^{\rm v}\partial_3^2 v^3(t)\|_{L^2}\\ &\lesssim \|\partial_3^2 v^3(t)\|_{\mathcal{H}_\theta}\sum_{\substack{k\leq j+N_0\\ \ell\leq j+N_0}}2^{k(\frac{1}{2}-\theta)}2^{\ell(\frac{1}{2}+\theta)}\\ &\lesssim 2^j\|\partial_3^2 v^3(t)\|_{\mathcal{H}_\theta}.\end{aligned}$$

Together with Definition 1.1 this implies

$$\|\partial_3 v^{\rm h}_{\rm div}(t)\|_{\mathcal{B}_2}\lesssim\|\partial_3^2 v^3(t)\|_{\mathcal{H}_\theta}.$$

Proposition 2.3 implies that

$$\int_0^{T^\star} \|\partial_3 v^{\rm h}_{\rm div}(t)\|^2_{\mathcal{B}_2}\,dt \lesssim \int_0^{T^\star} \|\partial_3^2 v^3(t)\|^2_{\mathcal{H}_\theta}\,dt < \infty. \tag{8.11}$$

On the other hand, we deduce from Lemma 4.1 that

$$\begin{aligned}\|\Delta_j\partial_3 v^{\rm h}_{\rm curl}(t)\|_{L^\infty} &\lesssim 2^{\frac{2j}{3}} \sum_{k\leq j+N_0} 2^{\frac{k}{3}} \|\Delta_j\Delta^{\rm h}_k\partial_3\omega(t)\|_{L^{\frac32}}\\ &\lesssim 2^j \|\partial_3\omega(t)\|_{L^{\frac32}},\end{aligned}$$

from which and (3.4), we infer that for any T less than $T^\star$,

$$\begin{aligned}\int_0^T \|\partial_3 v^{\rm h}_{\rm curl}(t)\|^2_{\mathcal{B}_2}\,dt &\lesssim \int_0^T \|\partial_3\omega(t)\|^2_{L^{\frac32}}\,dt\\ &\lesssim \sup_{t\in[0,T]} \big\|\omega_{\frac34}(t)\|\|^{\frac23}_{L^2} \int_0^T \big\|\nabla\omega_{\frac34}(t)\big\|^2_{L^2}\,dt.\end{aligned}$$

Proposition 2.3 then implies that

$$\int_0^{T^\star} \|\partial_3 v^{\rm h}_{\rm curl}(t)\|^2_{\mathcal{B}_2}\,dt < \infty.$$

With Inequalities (8.8), (8.10), (8.11), and by virtue of Theorem 1.5, we conclude the proof of Theorem 1.4. □

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