NOTES DU COURS
”INTRODUCTION AUX ÉQUATIONS AUX DÉRIVÉES PARTIELLES D’ÉVOLUTION”

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Introduction

This text consists in notes of a series of lectures given in the second year of the Master "Mathématiques et applications fondamentales" of the University Pierre et Marie Curie. The purpose of this series of lectures is to expose some basic results and methods in the field of evolution partial differential equations. By basic, we mean that we stay far away from optimal or refined results for the type of problems we look at.

We investigate four different class of problems:

- the first one is the linear transport equations associated with a non smooth vector field which is divergence free.
- the second is the problem of local and global existence of solutions of the incompressible Navier-Stokes equations in a bounded domain
- the third one is the linear Schrödinger equation in the whole space. This is the opportunity to put in light dispersive phenomena and to apply it to a simple example of semi linear Schrödinger equation.
- the fourth one concern linear and quasilinear symmetric systems of order 1. A basic example of such a system is the system related to barotropic gases.

The structure of these notes is the following. 
In the first chapter, we recall the basic theorem on ordinary differential equations (linear and non linear Cauchy-Lipchitz theorem, Peano’s theorem). We explain the link between ordinary differential equations and transport equations in the smooth case. Then we study transport equations associated with non smooth divergence free vector fields.
In the second chapter, we investigate the heat equation and the stationary Stokes equation. The concept of solution of Stokes problem is important. Then we study the heat equation and the evolution Stokes equation.
The third chapter is devoted to the proof of the global existence of weak solution for the incompressible Navier-Stokes equation in a bounded domain in dimension 2 or 3. We prove the uniqueness of such solutions in dimension 2.
The fourth chapter studies the problem of uniqueness and more generally the problem of stability for the three dimensionnal incompressible Navier-Stokes system. We prove also the local existence for small data and the global existence for small data, the initial data being in this case more regular than just of finite energy.
The fifth chapter is devoted to the study of the Schrödinger equation in the whole space. We compute the fundamental solution and prove the Strichartz estimate which is a consequence of some dispersive effect. This is the opportunity to study complex interpolation theory and also refined convolution inequality.
The sixth chapter is devoted to the study of linear symmetric system with bounded smooth variable coefficients.

The seventh chapter is an introduction to the Littlewood-Paley theory which is an important tool of non linear partial differential equation.

The aim of the eighth chapter is the proof of local wellposedness of quasilinear of quasilinear systems.
Chapter 1

Ordinary Differential Equations and transport equations

Introduction

The purpose of this chapter is twofold: first we want to recall very basic theorem, namely linear and non linear Cauchy-Lipschitz theorem) and Peano’s theorem. Secondly, we present the basis of the Di-Perna-Lions theory about the resolution of transport equation associated with non smooth divergence free vector fields. The chapter is organized as follows.

In the first section, we prove the linear Cauchy-Lipschitz theorem using the method of conjugation by an exponential term involving a large parameter. This method provides a very short proof of global existence for a linear ordinary differential equation.

The second section is devoted to the proof of the non linear Cauchy-Lipschitz theorem. Again, we use a conjugation with respect to an exponential term involving a large parameter and the Lipschitz constant of the function defining the ordinary differential equation. The main point is that this method gives immediately the existence on a time interval the length of which is estimated from below in a way which depends only on the supremum norm of the vector field which defines the ordinary differential equation.

In the third section, we prove Peano’s theorem which claims the existence (and not the uniqueness which is false in general) for ordinary differential equations associated with a continuous vector field. The method consists in the regularization of the vector field. Then Ascoli’s theorem provides relative compactness of the family of solutions of the approximated problems and thus a solution.

In the forth section, we establish the basis of Di-Perna-Lions theory which claims that for divergence free vector fields for which the first order derivatives belong to some $L^p$ space for finite $p$, the associated transport equation as a unique solution for intial data in $L^{p'}$. The existence part relies on the same method used to prove Peano’s theorem. The proof of the uniqueness follows a duality method which relies on the fact that the transport problem (which the same transport equation) satisfies the maximum principle.

As a conclusion of this introduction, let us point out that this chapter contains some fundamental techniques for studying evolution partial differential equations:

- the conjugation by an exponential term involving a large parameter;
- fixed point theorem which provides existene and uniqueness together with an approximation scheme;
• relative compactness of solutions of a family of approximated problems; the relative compactness coming from some time regularity;
• duality method which provides uniqueness for linear problem without any idea of contraction.

These methods will be used all along these notes.

1.1 Linear ordinary differential equations

Theorem 1.1.1 Let $E$ be a Banach space, $I$ an open interval of $\mathbb{R}$ and $A$ a continuous map from $I$ to $\mathcal{L}(E)$, the set of continuous linear maps from $E$ into $E$. Let $t_0$ be in $I$, a unique $C^1$ function $x$ from $I$ to $E$ exists such that

$$\begin{cases}
\frac{du}{dt} = A(t)u(t) \\
u(0) = u_0.
\end{cases}$$

(ODE)

Proof. Uniqueness is obvious because as it is a linear equation, it is enough to prove that if $x$ is a solution of (ODE) with initial data 0, then it is identically zero. The set of $t$ in $I$ such that $x(t) = 0$ is a closed subset of $I$. It is open because if $t_1$ is such that $x(t_1) = 0$, then we have by integration

$$\sup_{t \in [t_1-\alpha, t_1+\alpha]} \|x(t)\|_E \leq \alpha \sup_{t \in [t_1-\alpha, t_1+\alpha]} \|A(t)\|_{\mathcal{L}(E)} \sup_{t \in [t_1-\alpha, t_1+\alpha]} \|x(t)\|_E$$

Taking $\alpha$ small enough implies that $x$ is identically 0 near $t_1$. The proof of the existence of solutions of this equation is very simple. Let $\lambda$ be a positive real number, let us introduce the space $E_\lambda$ defined by

$$E_\lambda = \left\{ x \in C(I, E) / \|x\|_{\lambda} \overset{\text{def}}{=} \sup_{t \in I} \|x(t)\| \exp\left(-\lambda \int_0^t A(t')dt'\right) < \infty \right\}$$

with $A(t') \overset{\text{def}}{=} \|A(t')\|_{\mathcal{L}(E)}$. It is an exercise left to the reader to prove that $E_\lambda$ is a Banach space. The solution of (ODE) are the same as the solutions of

$$(\text{Id} - \mathcal{P}_A)u = u_0 \quad \text{with} \quad \mathcal{P}_A u(t) = \int_0^t A(t')u(t')dt'.$$

Let us prove that

$$\|\mathcal{P}_A\|_{\mathcal{L}(E_\lambda)} \leq \frac{1}{\lambda} \quad (1.1)$$

which will imply the theorem because the constant function with value $x_0$ obviously belongs to $E_\lambda$ for any positive real number $\lambda$. By definition of $A$ we have

$$\|\mathcal{P}_A x(t)\| \exp\left(-\lambda \int_0^t A(t')dt'\right) \leq \int_0^t \exp\left(-\lambda \int_{t'}^t A(t'')dt''\right) A(t') \exp\left(-\lambda \int_0^{t'} A(t'')dt''\right) \|x(t')\|dt'.$$
By definition of $|\cdot|_\lambda$, we infer that
\[
\|P_A x(t)\| \exp\left(-\lambda \int_0^t A(t') dt'\right) \leq \|x\|_\lambda \int_0^t \exp\left(-\lambda \int_0^t A(t'') dt''\right) A(t') dt'.
\]
An obvious computation of integral implies that, for any $t$ in $I$,
\[
\|P_A x(t)\| \exp\left(-\lambda \int_0^t A(t') dt'\right) \leq \frac{1}{\lambda} \|x\|_\lambda
\]
which is exactly (1.1); the theorem is proved.  \hfill \Box

## 1.2 Around Cauchy-Lipschitz theorem

Let us state and prove the classical theorem about existence and uniqueness of solution of an ordinary differential equation associated with a Lipschitz function. This uses Picard’s fixed point theorem.

**Theorem 1.2.1 (de Cauchy-Lipschitz)** Let $E$ be a Banach space and $f$ a continuous function on an open subset $U$ of $\mathbb{R} \times E$ with value in $E$. Let us consider a point $(t_0, x_0)$ of $U$ such that an open interval $J_0$ containing $t_0$ and a ball $B_0$ centered in $x_0$ and of radius $R_0$ such that $J_0 \times B_0$ is included in $U$ and such that the two functions
\[
M_0(t) \equiv \sup_{x \in B_0} \|f(t, x)\|_E \quad \text{and} \quad K_0(t) \equiv \sup_{(x_1, x_2) \in B_0^2} \frac{\|f(t, x_1) - f(t, x_2)\|_E}{\|x_1 - x_2\|_E}
\]
are integrable on $J_0$. Then, for any subinterval $J$ of $J_0$ containing $t_0$ which satisfies
\[
\int_J M_0(t) dt \leq R_0,
\]
a unique continuous function $x$ on $J$ with values in $B_0$ exists such that, for any $t$ in $J$
\[
x(t) = x_0 + F(x)(t) \quad \text{with} \quad F(x)(t) \equiv \int_{t_0}^t f(t', x(t')) dt'.
\]

**Proof.** Let us denote by $X$ the set of continuous functions on $J$ with values in $B_0$. Let us consider $x$ in $X$. By definition of $M_0$ and using Condition (1.2), we get, for any $t$ in $J$
\[
\left\| \int_{t_0}^t f(t', x(t')) dt' \right\|_E \leq \left| \int_{t_0}^t \|f(t', x(t'))\|_E dt' \right| \leq \int_J M_0(t) dt \leq R_0.
\]
Thus the function $F$ maps $X$ into itself. Then for any positive $\lambda$, let us consider the metric space $X$ equipped with the distance
\[
d_\lambda(x_1, x_2) \equiv \sup_{t \in J} \left(\exp -\lambda \int_{t_0}^t K_0(t') dt'\right) \|x_1(t) - x_2(t)\|_E.
\]
Let us estimate $d_\lambda(F(x_1), F(x_2))$. In order to do it, let us write that, for $t \geq t_0$,
\[
\|F(x_1)(t) - F(x_2)(t)\|_E \leq \int_{t_0}^t \|f(t', x_1(t')) - f(t', x_2(t'))\|_E dt' \\
\leq \int_{t_0}^t K_0(t') \|x_1(t') - x_2(t')\|_E dt'.
\]
From this we deduce that, by definition of $d_\lambda$,

$$K_\lambda(t) \overset{\text{def}}{=} \exp\left(-\lambda \int_{t_0}^t K_0(t') dt'\right) \|F(x_1)(t) - F(x_2)(t)\|_E$$

$$\leq \int_{t_0}^t K_0(t') \exp\left(-\lambda \int_{t'}^t K_0(t'') dt''\right) \|x_1(t') - x_2(t')\|_E dt'$$

$$\leq d_\lambda(x_1, x_2) \int_{t_0}^t K_0(t') \exp\left(-\lambda \int_{t'}^t K_0(t'') dt''\right) dt'$$

$$\leq \frac{1}{\lambda} d_\lambda(x_1, x_2).$$

This proves that the function $F$ is Lipschitz with constant $1/\lambda$ on $X$. Thus it has a unique fixed point in $X$ and the theorem is proved.

\[ \square \]

**Remark** Let us point out that the concepts of iterative scheme and of Cauchy sequence plays a key role. Moreover, let us notice Condition (1.2) is the only restriction on the interval $J$.

The existence and uniqueness theorem for ordinary differential equations is a local theorem. Let us first define the concept of maximal solution and then investigate what can be necessary conditions for a blow up phenomena. For the sake of simplicity, we are going to define it in the case when the function $F$ is globally defined on the Banach space $\mathbb{R} \times E$. Because of the uniqueness part of Theorem 1.2.1, if $x_1$ and $x_2$ are to solution of $(ODE)$ defined respectively on two interval $J_1$ and $J_2$ for which $t_0$ is an interior point, then they coincides on $J_1 \cap J_2$. Then it is an exercice of calculus to check that, if we are a collection $(x_\lambda)_{\lambda \in \Lambda}$ are solution of $(ODE)$ defined on $J_\lambda$, then the function the function $x$ on $J \overset{\text{def}}{=} \bigcup_{\lambda \in \Lambda} J_\lambda$ by

$$x|_{J_\lambda} = x_\lambda$$

is a solution of $(ODE)$ on $J$.

**Definition 1.2.1** The solution $x$ defined above the called the maximal solution of $(ODE)$.

**Proposition 1.2.1** Let $F$ be a function of $\mathbb{R} \times E$ in $E$ satisfying the hypothesis of Theorem 1.2.1 in any point $x_0$ of $E$. Let us assume in addition that a locally bounded function $M$ from $\mathbb{R}^+$ into $\mathbb{R}^+$ and a locally integrable function $\beta$ from $\mathbb{R}^+$ into $\mathbb{R}^+$ such that

$$\|F(t, u)\| \leq \beta(t)M(\|u\|).$$

then, if the maximal interval of definition is $[T_*, T^*]$, then, if $T^*$ is finite,

$$\limsup_{t \to T^*} \|u(t)\| = \infty,$$

the same being true for $T_*$.

**Proof.** Let us first prove that, if we consider a time $T > T_0$ such that $\|u(t)\|$ is bounded on the interval $[T_0, T]$, then we can define the solution on a larger interval $[T_0, T_1]$ with $T_1 > T$. As the function $u$ is bounded on the interval $[T_0, T]$, the hypothesis on $F$ that, for any $t$ of the interval $[T_0, T]$, we have

$$\|F(t, u(t))\| \leq C\beta(t).$$
The function \( \beta \) being integrable on the interval \([T_0, T]\), we have deduce que, for any \( \varepsilon \) strictement positive, it exists a positive real number \( \eta \) such that, pour tout \( t \) et \( t' \) such that \( T - t < \eta \) and \( T - t' < \eta \),

\[
\| u(t) - u(t') \| < \varepsilon.
\]

The space \( E \) being complete, an element \( u_* \) of \( E \) exists such that

\[
\lim_{t \to T^*} u(t) = u_*.
\]

Applying Theorem 1.2.1, we construct solution of \((ODE)\) on some \([T_0^*, T_1]\) and the continuous function defined by induction on the interval \([T_0, T_1]\) is a solution of the equation \((ODE)\) on the interval \([T_0, T_1]\).

**Corollary 1.2.1** Under the hypothesis of Proposition 1.2.1, if we have in addition that

\[
\| F(t, u) \| \leq M\| u \|^2,
\]

then, if the interval \([T_*, T^*]\) is the maximal interval of definition of \( u \) and if \( T^* \) is finite, then

\[
\int_{t_0}^{T^*} \| x(t) \| \, dt = \infty.
\]

**Proof.** The solution satisfies, for any \( t \geq t_0 \),

\[
\| x(t) \| \leq \| x(t_0) \| + M \int_{t_0}^{t} \| x(t') \|^2 \, dt'.
\]

(1.3)

Gronwall’s Lemma implies that

\[
\| x(t) \| \leq \| x_0 \| \exp \left( M \int_{0}^{t} \| x(t') \|^2 \, dt' \right).
\]

A more precise way of proving this result is the following. Let us define

\[
T \overset{\text{def}}{=} \sup \{ t \in [t_0, T^*] \mid \| x(t) \| \leq 2\| x(t_0) \| \}.
\]

For any \( t \) in \([t_0, T^*]\), we have, using (1.3),

\[
\| x(t) \| \leq \| x(t_0) \| + 4M(t - t_0)\| x(t_0) \|^2.
\]

Thus we infer

\[
\forall t \in \left[ t_0, \min \left\{ T, t_0 + \frac{1}{4M\| x(t_0) \|} \right\} \right], \quad \| x(t) \| \leq 2\| x_0 \|.
\]

Thanks to Proposition 1.2.1, we have

\[
T^* - t_0 \geq \frac{c}{\| x_0 \|}.
\]

Applying again this result at time \( t \in [t_0, T^*] \), we find that

\[
\forall t \in [t_0, T^*], \quad \| x(t) \| \geq \frac{c}{T^* - t}.
\]

This proves the theorem. \( \square \)
1.3 Non linear ordinary differential equations solved by a compactness method

In this section, we are going to work only in $\mathbb{R}^d$. This point is important because we shall use that bounded closed subsets are compact. The main point is the proof of Peano’s theorem.

**Theorem 1.3.1 (Peano)** Let $I$ be an open interval of $\mathbb{R}$. Let us consider a continuous function $f$ from $I \times \mathbb{R}^d$ into $\mathbb{R}^d$. Then, for any point $(t_0, x_0)$ of $I \times \mathbb{R}^d$, an open interval $J \subset I$ containing $t_0$ and a continuous function $x$ on $J$ exists such that

$$(\text{ODE}) \quad x(t) = x_0 + \int_{t_0}^{t} f(t', x(t')) \, dt'.$$

**Proof.** The structure of the proof is at least as interesting as the result itself. This proof will be a model for the proof of existence of weak solutions for the incompressible Navier-Stokes equation we shall study in Chapter 3.

There are three steps in the proof:

- we regularize the function $f$ and we apply Cauchy-Lipschitz’s Theorem to the sequence of regularized functions; have a Commun interval of definition,
- then, we prove that the sequence of those solutions of the regularized problem are relatively compact in the space $C(J, \mathbb{R}^d),$
- as a conclusion, we pass to the limit.

Let us proceed to a classical regularization; let $\chi$ a non negative function of $\mathcal{D}(B(0, 1))$ the integral of which is 1. Let us define $\chi_n(x) = n^d \chi(nx)$ and $f_n(t) = \chi_n * f(t)$. We have

$$\|f_n(t)\|_{L^\infty(K)} \leq \|f(t)\|_{L^\infty(K+B(0,n^{-1}))}.$$ 

Thus two positive real numbers $\alpha$ and $R$ independant of $n$, such that

$$\forall n \in \mathbb{N}, \int_{J_\alpha} \sup_{x \in B(x_0,R+1)} \|f(t,x)\|_{\mathbb{R}^d} \, dt \leq R. \quad (1.4)$$

Thus, using Theorem 1.2.1 and the remark page 10, for all integer $n$, a unique solution $x_n$ of

$$(\text{ODE}_n) \quad x_n(t) = x_0 + \int_{t_0}^{t} f_n(t', x_n(t')) \, dt'.$$

exists on $J_\alpha$.

Now let us prove that the set $X = \{x_n, n \in \mathbb{N}\}$ is relatively compact in $C(J_\alpha; \mathbb{R}^d)$. First of all, we have

$$\forall t \in J, \, X(t) = \{x_n(t), \, n \in \mathbb{N}\} \subset B(x_0, R).$$

As we work on a finite dimensionnal space, $X(t)$ is relatively compact. Moreover, we have

$$\|x_n(t) - x_n(t')\| \leq \left| \int_{t}^{t'} \|f_n(t'')\|_{L^\infty(B(x_0,R))} \, dt'' \right| \leq \left| \int_{t}^{t'} \|f(t'')\|_{L^\infty(B(x_0,R+1))} \, dt'' \right|.$$
Thus, for any positive \(\epsilon\), it exists a positive real number \(\alpha\) such that

\[
\forall (t, t') \in J^2, \quad |t - t'| < \alpha \implies \|x_n(t) - x_n(t')\| < \epsilon.
\]

As the function \(\|f(\cdot, \cdot)\|_{L^\infty(B(x_0, R+1))}\) is integrable on \(J_\alpha\), the family \((x_n)_{n \in \mathbb{N}}\) is equicontinuous on \(J_\alpha\). Ascoli’s Theorem ensures that the set \(X\) is relatively compact in \(C(J_\alpha; \mathbb{R}^d)\). Thus we can extract a subsequence which converge uniformly on \(J_\alpha\) to a function \(x\) of \(C(J; \mathbb{R}^d)\). Let omit to note the extraction in the following.

Now let us pass to the limit. For any \(t\) of \(J_\alpha\), we have

\[
\|f_n(t, x_n(t)) - f(t, x(t))\| \leq \|f_n(t) - f(t)\|_{L^\infty(B(x_0, R))} + \|f(t, x_n(t)) - f(t, x(t))\|.
\]

Thus for any \(t\) of \(J\), we have

\[
\lim_{n \to \infty} f_n(t, x_n(t)) = f(t, x(t)).
\]

Moreover, \(\|f_n(t, x_n(t))\| \leq \|f(t)\|_{L^\infty(B(x_0, R+1))}\). Lebesgue’s Theorem ensures that, for any \(t\), we have

\[
\lim_{n \to \infty} \int_{t_0}^{t} f_n(t', x_n(t')) dt' = \int_{t_0}^{t} f(t', x(t')) dt'.
\]

The theorem is proved. \(\square\)

### 1.4 Transport equations

Let us consider a regular time dependant vector field, say continuous in time and continuously differentiable with respect to the variable \(x\). Let us consider the flow of \(v\), namely the map uniquely defined by

\[
(EDO) \quad \begin{cases} 
\partial_t \psi(t, x) = v(t, \psi(t, x)) \\
\psi(0, x) = x.
\end{cases}
\]

For any time \(t\), the map \(x \mapsto \psi(t, x)\) is a \(C^1\) diffeomorphism. Let us describe the link between the \((ODE)\) and the transport equation \((T)\) given by

\[
(T) \quad \begin{cases} 
\partial_t f + v(t, x) \cdot \nabla f(t, x) = 0 \\
f(0, x) = f_0(x)
\end{cases}
\]

Indeed, the chain rule implies that

\[
\partial_t f(t, \psi(t, x)) = (\partial_t f)(t, \psi(t, x)) + \sum_{j=1}^{d} (\partial_j f)(t, \psi(t, x)) \partial_t \psi^j(t, x)
\]

\[
= (\partial_t f + v \cdot \nabla f)(t, \psi(t, x)).
\]

This implies that \(f\) is solution of \((T)\) if and only if

\[
f(t, \psi(t, x)) = f_0(x) \quad \text{which writes} \quad f(t, x) = f_0(\psi^{-1}(t, x)).
\]

In the case when the vector field \(v\) is rough, the correspondance between flow and solutions of the transport equation is less simple. It turns out that in the case of incompressible flows, the transport equation \((T)\) is easier to solve than the associated ordinary differential equation.
About incompressible flow, let us remark that it is a classical exercise of calculus to prove that
\[ \partial_t \det(D \psi)(t, x) = (\div v)(t, \psi(t, x)) \det(D \psi)(t, x). \]
This implies that the incompressibility of the flow \( \psi \) is equivalent to the fact that \( v \) is divergence free. In this case of incompressibility, the equation \( (T) \) requires only low regularity to be well posed (i.e. to have a unique solution).
Let us define an appropriate space for the vector field \( v \).

**Definition 1.4.1** Let \( p \) be a real number in \([1, \infty]\). We define \( W^{1,p}(\mathbb{R}^d) \) are the space of functions \( a \) in \( L^p(\mathbb{R}^d) \) such that the derivatives of \( a \) belong to \( L^p(\mathbb{R}^d) \).

Let us also define the concept of weak continuity for functions \( f \) which are bounded in time with value in \( L^p(\mathbb{R}^d) \) which means that the function \( t \mapsto \| f(t, \cdot) \|_{L^p(\mathbb{R}^d)} \) is bounded.

**Definition 1.4.2** We denote by \( C_{w,b}([0,T];L^q(\mathbb{R}^q)) \) the set of functions which are bounded with value in \( L^p(\mathbb{R}^d) \) such that for any function \( \varphi \) in \( \mathcal{D}(\mathbb{R}^{1+d}) \), the function
\[
\begin{cases}
  [0,T] & \mapsto \mathbb{R} \\
  f & \mapsto \int_{\mathbb{R}^d} f(t,x)\varphi(t,x)dx
\end{cases}
\]
is continuous.

The purpose of this section is the proof of the following theorem.

**Theorem 1.4.1** Let \( q \) be in \([1, \infty]\). Let us consider a time dependant vector divergence free vector field \( v \) the components of which belong to \( L^1([0,T];W^{1,q}(\mathbb{R}^d)) \). Then for any function \( f_0 \) in \( L^q(\mathbb{R}^d) \), a unique solution \( f \) of \( (T) \) exists in \( C_{w,b}([0,T];L^q(\mathbb{R}^q)) \) which means that the function \( f(t, \cdot) \) is well defined as a function of \( L^q(\mathbb{R}^d) \), that \( \| f(t, \cdot) \|_{L^q(\mathbb{R}^d)} \) is a bounded function and that, for any \( \varphi \) in \( \mathcal{D}(\mathbb{R}^{1+d}) \), the function
\[
  t \mapsto \int_{\mathbb{R}^d} f(t,x)\varphi(t,x)dx
\]
is continuous and we have
\[
  (T_w) \int_{\mathbb{R}^d} f(t,x)\varphi(t,x)dx = \int_{\mathbb{R}^d} f_0(x)\varphi(0,x)dx + \int_0^t \int_{\mathbb{R}^d} f(t',x)(\partial_t \varphi + v \cdot \nabla \varphi)(t',x)dt'dx.
\]

The proof of this theorem is can be splitted into two steps: the first one is an existence theorem the proof of which has some analogy with the proof of Peano’s theorem and use Ascoli’s theorem. The second step is the uniqueness theorem proved using a duality method. Let us state the existence theorem.

**Theorem 1.4.2** Let \( q \) be in \([1, \infty]\). Let us consider a time dependant vector divergence free vector field \( v \) the components of which belong to \( L^1([0,T];L^q(\mathbb{R}^d)) \). Then for any function \( f_0 \) in \( L^q(\mathbb{R}^d) \), a solution \( f \) of \( (T) \) exists in \( C_{w}([0,T];L^q(\mathbb{R}^q)) \).
Proof. As in the proof of Peano’s theorem, we proceed by regularization and compactness. Let us consider a sequence \((v_n)_{n \in \mathbb{N}}\) a smooth vector fields in \(\mathbb{R} \times \mathbb{R}^d\) such that
\[
\lim_{n \to \infty} \|v - v_n\|_{L^1([0,T];L^q(\mathbb{R}^d))} = 0 \tag{1.5}
\]
and a sequence \((f_{0,n})_{n \in \mathbb{N}}\) a smooth compactly supported function which tends to \(f_0\) in \(L^q(\mathbb{R}^d)\). Then let us denote by \(f_n\) the solution of \((T)\) associated with the vector field \(v_n\) and the initial data \(f_{0,n}\). Obviously, we have, for any \(\varphi\) in \(\mathcal{D}(\mathbb{R}^{1+d})\)
\[
\int_{\mathbb{R}^d} f_n(t,x)\varphi(t,x)\,dx = \int_{\mathbb{R}^d} f_{0,n}(x)\varphi(0,x)\,dx + I_n(\varphi)(t) \quad \text{with} \quad I_n(\varphi)(t) \overset{\text{def}}{=} \int_0^t \int_{\mathbb{R}^d} f_n(t',x) (\partial_t \varphi + v_n \cdot \nabla \varphi)(t',x)\,dt'dx. \tag{1.6}
\]
The goal is to pass to the limit in the above expression. Let us first observe that
\[
\lim_{n \to \infty} \int_{\mathbb{R}^d} f_{0,n}(x)\varphi(0,x)\,dx = \int_{\mathbb{R}^d} f_0(x)\varphi(0,x)\,dx. \tag{1.7}
\]
As the vector field \(v\) is divergence free, we have
\[
\forall t \in [0,T], \quad \|f_n(t,\cdot)\|_{L^q(\mathbb{R}^d)} = \|f_{0,n}\|_{L^q(\mathbb{R}^d)}.
\]
This implies in particular that the sequence \((f_n)_{n \in \mathbb{N}}\) is bounded in \(L^q([0,T] \times \mathbb{R}^d)\). As \(q\) is different from 1, we can assume, up to an extraction we omit to note, that the sequence \((f_n)_{n \in \mathbb{N}}\) convergence in the weak* sense to a function \(f\) in \(L^q([0,T] \times \mathbb{R}^d)\). Moreover, as \((f_n)_{n \in \mathbb{N}}\) is bounded in \(L^{\infty}([0,T];L^q(\mathbb{R}^d))\) this implies that the function \(f\) also belongs to \(L^{\infty}([0,T];L^q(\mathbb{R}^d))\). Let us prove that such a function \(f\) is a solution.

The first step is the proof of the following proposition.

**Proposition 1.4.1** The family \((I_n(\varphi))_{n \in \mathbb{N}}\) is relatively compact in \(C([0,T];\mathbb{R})\).

**Proof.** Hölder inequalities imply that
\[
\left| \int_0^t \int_{\mathbb{R}^d} f_n(t',x) \partial_t \varphi(t',x)\,dt'\,dx \right| \leq \|f_n\|_{L^\infty([0,T];L^q(\mathbb{R}^d))} \|\partial_t \varphi\|_{L^1([0,T];L^{q'}(\mathbb{R}^d))} \quad \text{and}
\]
\[
\left| \int_0^t \int_{\mathbb{R}^d} f_n(t',x) (v_n \cdot \nabla \varphi)(t',x)\,dt'\,dx \right| \leq \|f_n\|_{L^\infty([0,T];L^q(\mathbb{R}^d))} \|v_n\|_{L^1([0,T];L^{q'}(\mathbb{R}^d))} \times \|\nabla \varphi\|_{L^\infty([0,T] \times \mathbb{R}^d)}.
\]
Thus \((I_n(\varphi))_{n \in \mathbb{N}}\) is a bounded family of \(L^{\infty}([0,T])\). Now let us prove it is uniformly equicontinuous. Again Hölder inequalities implies that
\[
\left| \int_{t_1}^{t_2} \int_{\mathbb{R}^d} f_n(t',x) \partial_t \varphi(t',x)\,dt'\,dx \right| \leq \|f_n\|_{L^\infty([0,T];L^{q'}(\mathbb{R}^d))} \|\partial_t \varphi\|_{L^\infty([0,T];L^{q'}(\mathbb{R}^d))} \|v_n\|_{L^1([t_1,t_2];L^{q'}(\mathbb{R}^d))} \times |t_1 - t_2| \quad \text{and}
\]
\[
\left| \int_0^t \int_{\mathbb{R}^d} f_n(t',x) (v_n \cdot \nabla \varphi)(t',x)\,dt'\,dx \right| \leq \|f_n\|_{L^\infty([0,T];L^q(\mathbb{R}^d))} \|v_n\|_{L^1([t_1,t_2];L^{q'}(\mathbb{R}^d))} \times \|\nabla \varphi\|_{L^\infty([0,T] \times \mathbb{R}^d)}.
\]
As the sequence \((v_n)_{n \in \mathbb{N}}\) tends to \(v\) in \(L^1([0,T];L^q(\mathbb{R}^d))\), then for any positive \(\varepsilon\), a positive real number \(\alpha_\varepsilon\) exists such that
\[
|t_1 - t_2| < \alpha_\varepsilon \implies \forall n \in \mathbb{N}, \quad \|v_n\|_{L^1([t_1,t_2];L^{q'}(\mathbb{R}^d))} < \varepsilon.
\]
Using Ascoli’s theorem, this proves the proposition. \(\square\)
Continuation of the proof of Theorem 1.4.2. From Ascoli’s theorem, we deduce that, up to an extraction we omit to note, the sequence \((I_n(\varphi))_{n \in \mathbb{N}}\) converges to some continuous function we denote by \(I(\varphi)\). This implies that the sequence of functions

\[
\left( \int_{\mathbb{R}^d} f_n(t, x) \varphi(t, x) dx \right)_{n \in \mathbb{N}}
\]

converge uniformly on \([0, T]\) to \(L(\varphi)(t)\) which satisfies, thanks to (1.6) and

\[
L(\varphi)(t) = \int_{\mathbb{R}^d} f_0(x) \varphi(0, x) dx + I(\varphi)(t). \tag{1.8}
\]

Because on the weak convergence of \((f_n)_{n \in \mathbb{N}}\) to \(f\), this implies that, for any function \(\alpha\) in \(\mathcal{D}([0, T])\), we have

\[
\int_0^T I(\varphi)(t) \alpha(t) dt = \lim_{n \to \infty} \int_0^T \left( \int_{\mathbb{R}^d} f_n(t, x) \varphi(t, x) dx \right) \alpha(t) dt
\]

\[
= \lim_{n \to \infty} \int_{[0,T] \times \mathbb{R}^d} f_n(t, x) \varphi(t, x) \alpha(t) dt dx
\]

\[
= \int_{[0,T] \times \mathbb{R}^d} f(t, x) \varphi(t, x) \alpha(t) dt dx
\]

\[
= \int_{[0,T]} \left( \int_{\mathbb{R}^d} f(t, x) \varphi(t, x) dx \right) \alpha(t) dt.
\]

In order to compute \(I(\varphi)\) it is enough to worry about pointwise convergence. This implies that \(f\) belongs to \(C_{w,0}([0, T]; L^2(\mathbb{R}^d))\) and that

\[
\lim_{n \to \infty} \int_{\mathbb{R}^d} f_n(t, x) \varphi(t, x) dx = \int_{\mathbb{R}^d} f(t, x) \varphi(t, x) dx \text{ uniformly on } [0, T]. \tag{1.9}
\]

Thanks to (1.8), this becomes

\[
\int_{\mathbb{R}^d} f(t, x) \varphi(t, x) dx = \int_{\mathbb{R}^d} f_0(x) \varphi(0, x) dx + I(\varphi)(t). \tag{1.10}
\]

Now let us compute \(I(\varphi)\). In order to get to result, it is enough now to think interm to pointwise convergence. Hölder estimates imply that, for any \(t\) in \([0, T]\),

\[
\left| \int_0^t \int_{\mathbb{R}^d} f_n(t', x) ((v_n - v) \cdot \nabla \varphi)(t', x) dt' dx \right|
\]

\[
\leq \| f_n \|_{L^\infty([0,T];L^q(\mathbb{R}^d))} \| v_n - v \|_{L^1([0,T];L^{q'}(\mathbb{R}^d))} \times \| \nabla \varphi \|_{L^\infty([0,T] \times \mathbb{R}^d)}. \]

Thus

\[
\lim_{n \to \infty} \sup_{t \in [0,T]} \left| I_n(\varphi)(t) - \int_0^t \int_{\mathbb{R}^d} f_n(t', x) (\partial_t \varphi + v \cdot \nabla \varphi)(t', x) dt' dx \right| = 0.
\]

As the sequence \((f_n)_{n \in \mathbb{N}}\) tends weakly \(\star\) to \(f\) as a sequence of functions of \(L^q([0, T] \times \mathbb{R}^d)\), the sequence \((I_n(\varphi))_{n \in \mathbb{N}}\) tends to

\[
\int_0^t f(t', x) (\partial_t \varphi + v \cdot \nabla \varphi)(t', x) dt' dx.
\]

This proves the theorem. \(\square\)
Then we are going to prove a uniqueness theorem with an additional hypothesis of regularity about the vector field $v$.

**Theorem 1.4.3** Let $q$ be in $[1, \infty]$ and $v$ a time depending divergence free vector field the component of which belongs to $L^1([0, T]; W^{1,q}(\mathbb{R}^d))$. Let is $f$ is a solution of $(T_v)$ with initial data $0$ i.e. for all $\varphi$ in $\mathcal{D}(\mathbb{R}^{1+d})$,

$$(T_v) \int_{\mathbb{R}^d} f(t, x) \varphi(t, x) dx = \int_0^t \int_{\mathbb{R}^d} f(t', x) \left( \partial_t \varphi + v \cdot \nabla \varphi \right)(t', x) dt' dx,$$

then $f = 0$.

**Proof.** The strategy is the following: we regularize $f$ in space by convolution by a smooth compactly supported function, we establish the equation $(T_{v, \varepsilon})$ satisfied by the family of approximated solutions and then in this equation, we use as a test function $\varphi^\varepsilon$ which is the solution of

$$(T^\varepsilon) \begin{cases} 
\partial_t \varphi^\varepsilon + v_0(t, x) \cdot \nabla \varphi^\varepsilon(t, x) &= 0 \\
\varphi^\varepsilon(t_0, x) &= \phi(x)
\end{cases}
$$

where $\phi$ is a given smooth compactly supported function on $\mathbb{R}^d$ and $v_0$ a suitable family of approximation of $v$. Let us define

$$a_\varepsilon(x) \stackrel{\text{def}}{=} \frac{1}{\varepsilon^d} \rho\left(\frac{x}{\varepsilon}\right) * a.$$

where $\rho$ is a smooth compactly supported function on $\mathbb{R}^d$, even, non negative and with integral 1. For any compactly supported function $\varphi$ on $\mathcal{D}(\mathbb{R}^{1+d})$, we get, using $\varphi_\varepsilon$ as a test function in $(T_{v, \varepsilon})$ that

$$(T_v) \int_{\mathbb{R}^d} f(t, x) \varphi_\varepsilon(t, x) dx = \int_0^t \int_{\mathbb{R}^d} f(t', x) \left( \partial_t \varphi_\varepsilon + v \cdot \nabla \varphi_\varepsilon \right)(t', x) dt' dx,$$

As the function $\rho$ is even we have

$$\int_{\mathbb{R}^d} f(t, x) \varphi(x) dx = \int_{\mathbb{R}^d} f_\varepsilon(t, x) \varphi(x) dx \quad \text{and}$$

$$\int_0^t \int_{\mathbb{R}^d} f(t', x) \partial_t \varphi_\varepsilon(t', x) dx dt' = \int_0^t \int_{\mathbb{R}^d} f_\varepsilon(t', x) \partial_t \varphi(t', x) dx dt'.$$

By definition of the convolution and because the function $\rho$ is even, we have

$$E \stackrel{\text{def}}{=} \int_0^t \int_{\mathbb{R}^d} f(t', x) \left( v \cdot \nabla \varphi_\varepsilon \right)(t', x) dt' dx - \int_0^t \int_{\mathbb{R}^d} f_\varepsilon(t', x) \left( v \cdot \nabla \varphi \right)(t', x) dt' dx$$

$$= \frac{1}{\varepsilon^d} \sum_{j=1}^d \int_0^t \int_{\mathbb{R}^d} f(t', x) \rho\left(\frac{x - y}{\varepsilon}\right) \left( v^j(t', x) \partial_j \varphi(t', y) \right) dt' dx dy.$$

Using that the vector field $v$ is divergence free, an integration by parts gives

$$E = \int_0^t \int_{\mathbb{R}^d} C^\varepsilon(v, f)(t', y) \varphi(t', y) dt' dy \quad \text{with}$$

$$C^\varepsilon(v, f)(t, y) \stackrel{\text{def}}{=} \frac{1}{\varepsilon^{d+1}} \sum_{j=1}^d \int_{\mathbb{R}^d} (\partial_j \rho)\left(\frac{x - y}{\varepsilon}\right) \left( v^j(t', x) - v^j(t', y) \right) dt' dx.$$

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Now \((T_w)\) writes

\[
\int_{\mathbb{R}^d} f_\varepsilon(t, x)\varphi(t, x)dx = \int_0^t \int_{\mathbb{R}^d} f_\varepsilon(t', x)(\partial_t \varphi + v \cdot \nabla \varphi)(t', x)dt' dx + \int_0^t \int_{\mathbb{R}^d} C_\varepsilon(v, f)(t', x)\varphi(t', x)dx.
\]

(1.13)

The main step of the proof of the following proposition

**Proposition 1.4.2** For any positive real number \(\varepsilon\), we have

\[
\|C_\varepsilon(v, f)\|_{L^1([0, T] \times \mathbb{R}^d)} \leq \|\cdot \|_{L^\infty(\mathbb{R}^d)}\|f\|_{L^\infty([0, T]; L^p(\mathbb{R}^d))}\|\nabla v\|_{L^1([0, T]; L^p(\mathbb{R}^d))}.
\]

(1.14)

Moreover, for any function \(f\) in the space \(L^\infty([0, T]; L^3(\mathbb{R}^d))\) and any divergence free vector field \(v\) in \(L^1([0, T]; L^p(\mathbb{R}^d))\), we have

\[
\lim_{\varepsilon \to 0} \|C_\varepsilon(v, f)\|_{L^1([0, T] \times \mathbb{R}^d)} = 0.
\]

**Proof.** The proof of the first inequality relies on the following lemma, which can be understood as a \(L^p\) version of finite increments inequality.

**Lemma 1.4.1** For any \(p\) in \([1, \infty]\), then for any \(a\) in \(W^{1,p}\) the map

\[
\begin{cases}
\mathbb{R}^d &\rightarrow \ L^p(\mathbb{R}^d) \\
z &\mapsto a(\cdot + z)
\end{cases}
\]

is lipschitzian.

**Proof.** By density, we can assume that \(a\) is a smooth compactly supported function. Then we can write

\[
a(x + z') - a(x + z) = \sum_{j=1}^d (z'_j - z_j) \int_0^1 \partial_j a(x + z + s(z' - z))ds.
\]

Taking the \(L^p\) norm in \(x\) in the above identity gives

\[
\|a(\cdot + z') - a(\cdot + z)\|_{L^p} \leq \sum_{j=1}^d |z'_j - z_j| \|\partial_j a\|_{L^p}.
\]

This proves the lemma.

**Continuation of the proof of Proposition 1.4.2** Changing variable \(x = y + \varepsilon z\) gives

\[
C_\varepsilon(v, f)(t, y) = \frac{1}{\varepsilon} \sum_{j=1}^d \int_{\mathbb{R}^d} \partial_j \rho(z)(v^j(t', y + \varepsilon z) - v^j(t', y)) f(t', y + \varepsilon z)dz.
\]

(1.15)

Hölder inequality implies that

\[
\|C_\varepsilon(v, f)\|_{L^1(\mathbb{R}^d)} \leq \frac{1}{\varepsilon} \sum_{j=1}^d \int_{\mathbb{R}^d} |\partial_j \rho(z)| \|v^j(t', \cdot + \varepsilon z) - v^j(t', \cdot)\|_{L^p(\mathbb{R}^d)}\|f(t', \cdot + \varepsilon z)\|_{L^1(\mathbb{R}^d)}dz.
\]
Indeed, we have, using Proposition 1.4.2 and (1.11).

Continuation of the proof of Theorem 1.4.2

conclusion of Proposition 1.4.2.

As, for any positive $\varepsilon$ we get, for almost every $t \in [0, T]$,

$$\|C^\varepsilon(v, f)(t, \cdot)\|_{L^1(\mathbb{R}^d)} \leq \sum_{j=1}^{d} \int_{\mathbb{R}^d} |z_j| |\partial_j \rho(z)| \|\nabla v^j(t', \cdot)\|_{L^\infty(\mathbb{R}^d)} \|f(t', \cdot)\|_{L^\infty(\mathbb{R}^d)} dz.$$  

This gives (1.14). Now let us prove the second part of the proposition. Because of (1.14), it is enough to prove it for smooth divergence free vector field. Writing a Taylor formula at order 2 in (1.15) gives

$$C^\varepsilon(v, f) = \sum_{j=1}^{d} \sum_{k=1}^{d} C^\varepsilon_{j,k}(v, f) + C^\varepsilon_2(v, f)$$  

with

$$C^\varepsilon_{j,k}(v, f)(t, y) = \partial_k v^j(t, y) \int_{\mathbb{R}^d} z_k \partial_j \rho(z) f(t', y + \varepsilon z) dz$$  

and

$$C^\varepsilon_2(v, f)(t, y) = \varepsilon \int_{t-1}^{t} \int_{\mathbb{R}^d} z_k z_d \partial_j \rho(z) \int_{0}^{1} \frac{1}{(1-t)(\partial_k \partial_j v^j)(t, y + \sigma \varepsilon z)} f(t', y + \varepsilon z) dz.$$

Let us observe that we have

$$\|C^\varepsilon_{j,k}(v, f)\|_{L^1([0, T] \times \mathbb{R}^d)} \leq \varepsilon \int_{0}^{T} \int_{\mathbb{R}^d} \int_{0}^{1} f(t, \cdot + \varepsilon z) \|\partial_j \rho(z)| |z|^2 dz f(t', \cdot) dz dt$$

$$\times \|\nabla^2 v^j(t, \cdot + s + \varepsilon z)\|_{L^\infty(\mathbb{R}^d)} ds dz dt$$

$$\leq C_{v, \rho} \varepsilon \|f\|_{L^\infty([0, T]; L^\infty(\mathbb{R}^d))}. \quad (1.16)$$

We know that, for almost every $t$ in $[0, T]$, we have

$$\lim_{\varepsilon \to 0} \left\| \int_{\mathbb{R}^d} z_k \partial_j \rho(z) f(t, \cdot + \varepsilon z) dz - \left( \int_{\mathbb{R}^d} z_k \partial_j \rho(z) dz \right) f(t, \cdot) \right\|_{L^1(\mathbb{R}^d)} = 0.$$

Because $\rho$ is of integral 1, we have

$$\int_{\mathbb{R}^d} z_k \partial_j \rho(z) dz = 0 \quad \text{if} \quad j \neq k \quad \text{and} \quad \int_{\mathbb{R}^d} z_j \partial_j \rho(z) dz = 1.$$

We get, for almost every $t$,

$$\lim_{\varepsilon \to 0} \left\| \sum_{j=1}^{d} \sum_{k=1}^{d} C^\varepsilon_{j,k}(f)(t, \cdot) - \text{div} \; v(t, \cdot) f(t, \cdot) \right\|_{L^1(\mathbb{R}^d)} = 0$$

As, for any positive $\varepsilon$,

$$\|C^\varepsilon_{j,k}(v, f)(t, \cdot)\|_{L^1(\mathbb{R}^d)} \leq \|\partial_k v^j(t, \cdot)\|_{L^\infty(\mathbb{R}^d)} \|f(t, \cdot)\|_{L^\infty(\mathbb{R}^d)}.$$

Lebesgue theorem and the fact that that $v$ is divergence free together with (1.16) implies the conclusion of Proposition 1.4.2.

Continuation of the proof of Theorem 1.4.2 Passing to the limit in $(T_{w})$ is not sufficient. Indeed, we have, using Proposition 1.4.2 and (1.11)

$$\int_{\mathbb{R}^d} f(t, x) \varphi(t, x) dx = \int_{0}^{t} \int_{\mathbb{R}^d} f(t', x) (\partial_t \varphi + v \cdot \nabla \varphi)(t', x) dt' dx \quad (1.17)$$
for any function \( \varphi \) in \( \mathcal{D}(\mathbb{R}^{1+d}) \). But we cannot solve

\[
\partial_t \varphi + v \cdot \nabla \varphi = 0
\]

with smooth solution \( \varphi \). Thus we need to regularize the vector field \( v \). Let us consider the family \( \varphi^\delta \) given by (1.18) in page 17. Applying (1.18) with \( \varphi^\delta \) gives thanks to (1.11) and (1.13), for some \( t_0 \) in \([0, T]\) and for any positive \( \delta \),

\[
\int_{\mathbb{R}^d} f(t_0, x) \phi_\varepsilon(x) \, dx = \int_0^{t_0} \int_{\mathbb{R}^d} f(t', x) \left( \partial_t \varphi^\delta_\varepsilon + v_\delta \cdot \nabla \varphi^\delta_\varepsilon \right)(t', x) \, dt' \, dx \\
= \int_0^{t_0} \int_{\mathbb{R}^d} f(t', x) \left( (v - v_\delta) \cdot \nabla \varphi^\delta_\varepsilon \right)(t', x) \, dt' \, dx \\
\quad + \int_0^{t_0} \int_{\mathbb{R}^d} f(t', x) \left( \partial_t \varphi^\delta_\varepsilon + v_\delta \cdot \nabla \varphi^\delta_\varepsilon \right)(t', x) \, dt' \, dx \\
= \int_0^{t_0} \int_{\mathbb{R}^d} f_\varepsilon(t', x) \left( (v - v_\delta) \cdot \nabla \varphi^\delta_\varepsilon \right)(t', x) \, dt' \, dx \\
\quad + \int_0^{t_0} \int_{\mathbb{R}^d} C^\varepsilon(v, f)(t', y) \varphi^\delta(t', y) \, dt' \, dy.
\]

Now for fixed positive \( \varepsilon \), let us take the limit when \( \delta \) tends to 0. By integration by parts, we get because \( v \) and \( v_\delta \) are divergence free,

\[
\int_0^{t_0} \int_{\mathbb{R}^d} f_\varepsilon(t', x) \left( (v - v_\delta) \cdot \nabla \varphi^\delta_\varepsilon \right)(t', x) \, dt' \, dx = -\int_0^{t_0} \int_{\mathbb{R}^d} \varphi^\delta(t', x) \left( (v - v_\delta) \cdot \nabla f_\varepsilon \right)(t', x) \, dt' \, dx.
\]

Maximum principle for transport equation ensures that

\[
\forall \delta > 0, \quad \|\varphi^\delta\|_{L^\infty([0, T] \times \mathbb{R}^d)} \leq \|\phi\|_{L^\infty(\mathbb{R}^d)}.
\]

Thus, using the fact that the vector field \( v \) (and thus all the vector fields \( v_\delta \)) is divergence free, we get

\[
\left| \int_0^{t_0} \int_{\mathbb{R}^d} f_\varepsilon(t', x) \left( (v - v_\delta) \cdot \nabla \varphi^\delta_\varepsilon \right)(t', x) \, dt' \, dx \right| \\
\quad \leq \|\phi\|_{L^\infty(\mathbb{R}^d)} \|v - v_\delta\|_{L^1([0, T]; L^1(\mathbb{R}^d))} \|\nabla f_\varepsilon\|_{L^\infty([0, T]; L^q(\mathbb{R}^d))}.
\]

By definition of the regularization process of \( f \), we have

\[
\|\nabla f_\varepsilon\|_{L^\infty([0, T]; L^q(\mathbb{R}^d))} \leq C_{\varepsilon^{-1}} \|f\|_{L^\infty([0, T]; L^q(\mathbb{R}^d))}.
\]

This implies that, for any positive \( \varepsilon \),

\[
\lim_{\delta \to 0} \sup_{t \in [0, T]} \left| \int_0^t \int_{\mathbb{R}^d} f_\varepsilon(t', x) \left( (v - v_\delta) \cdot \nabla \varphi^\delta_\varepsilon \right)(t', x) \, dt' \, dx \right| = 0.
\]

Let \((\delta_n)_{n \in \mathbb{N}}\) be a sequence of positive real numbers which tends to 0. Thanks to (1.18), we can assume (up to an extraction we omit to note) that the sequence \((\varphi^{\delta_n})_{n \in \mathbb{N}}\) converges weakly-* to some function \( \varphi \) in \( L^\infty([0, T] \times \mathbb{R}^d) \). Thus we get, for any \( t \) in \([0, T] \), and any fixed \( \varepsilon \),

\[
\lim_{n \to \infty} \int_0^t \int_{\mathbb{R}^d} C^\varepsilon(v, f)(t', y) \varphi^{\delta_n}(t', y) \, dt' \, dy = \int_0^t \int_{\mathbb{R}^d} C^\varepsilon(v, f)(t', y) \varphi(t', y) \, dt' \, dy.
\]

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Together with (1.19), this gives

\[ \int_{\mathbb{R}^d} f(t_0, x) \phi_x(x) dx = \int_0^t \int_{\mathbb{R}^d} C^{\varepsilon}(v, f)(t', y) \varphi(t', y) dt' dy. \]

Proposition 1.4.2 implies that

\[ \lim_{\varepsilon \to 0} \int_{\mathbb{R}^d} f(t_0, x) \phi_x(x) dx = \int_{\mathbb{R}^d} f(t_0, x) \phi(x) dx = 0. \]

Theorem 1.4.3 is proved.
Chapter 2

The heat flow and the Stokes flow in a bounded domain

Introduction

In this chapter, we first describe the structure of the Laplace operator in a bounded domain with Dirichlet boundary condition. Then we solve the heat flow with Dirichlet boundary condition.

In the second section, we investigate the equivalent of the Dirichlet problem for divergence free vector field still in a bounded domain: the stationary Stokes problem. The main point is that the divergence free constrain gives rise a new concept of solution.

In the third section, we solve the Stokes flow which the equivalent of the heat flow in the context of divergence free vector field. Again the concept of solutions will be different of the one of the heat flow; it takes into account the divergence free condition.

2.1 The Dirichlet problem in a bounded domain

We are going to present the solution of classical linear partial differential equations in a bounded domain with Dirichlet boundary condition. In order to do so, let us recall the basic theorem about the Laplacian on a bounded domain with Dirichlet boundary condition.

Theorem 2.1.1 For any $f$ in $H^{-1}(\Omega)$, a unique solution of the equation $-\Delta u = f$ exists in $H^1_0(\Omega)$ in the following sense:

$$\forall v \in H^1_0(\Omega), \quad (\nabla v, \nabla \varphi)_{L^2(\Omega)} = (f, \varphi)_{H^{-1}(\Omega) \times H^1_0(\Omega)}. \quad (2.1)$$

It also exists a non decreasing sequence $(\lambda_k)_{k \in \mathbb{N}}$ of positive real numbers which tends to infinity and a hilbertian basis of $L^2(\Omega)$ denoted by $(e_k)_{k \in \mathbb{N}}$ such that

$$-\Delta e_k = \lambda_k e_k.$$ 

Moreover, the sequence $(\lambda_k^{-1} e_k)_{k \in \mathbb{N}}$ is an orthonormal basis of $H^1_0(\Omega)$. Finally, if $f$ belongs to $H^{-1}(\Omega)$, then

$$\|f\|_{H^{-1}(\Omega)}^2 = \sum_k \lambda_k^{-2} \langle f, e_k \rangle^2.$$ 

Remark Thus, the space $H^{-1}(\Omega)$ is a Hilbert space and $(\lambda_k e_k)_{k \in \mathbb{N}}$ is a hilbertian basis of $H^{-1}(\Omega)$. 

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Proof of Theorem 2.1.1. The first assertion comes simply from the fact that the dual space of $H^1_0(\Omega)$ is described in two different ways. As $H^1_0(\Omega)$ is a Hilbert space, a unique $u$ exists in $H^1_0(\Omega)$ such that

$$\forall v \in H^1_0(\Omega), \quad (u|v)_{H^1_0} = \langle f, v \rangle$$

As we have $(u|v)_{H^1_0} = \langle -\Delta u, v \rangle_{H^{-1}\times H^1_0}$, we get that $-\Delta u = f$ in $H^{-1}$.

Now let us observe that as the space $L^2(\Omega)$ is continuously included in $H^{-1}(\Omega)$, we can define an operator $B$ as follows:

$$B \left\{ \begin{array}{ccc} L^2 & \longrightarrow & H^1_0(\Omega) \subset L^2(\Omega) \\ f & \longmapsto & u \end{array} \right.$$ 

such that $u$ is the solution in $H^1_0(\Omega)$ of $-\Delta u = f$. The operator $B$ is of course continuous from $L^2(\Omega)$ into $H^1_0(\Omega)$. Thanks to Rellich's theorem (see A.4.3 page 91 for the statement and the proof), the operator $B$ is compact from $L^2(\Omega)$ into $L^2(\Omega)$. Let us prove it is self-adjoint and positive. Let us write that for any couple of functions $(f, g)$ in $L^2(\Omega)$, we have

$$(Bf|g)_{L^2} = \langle g, Bf \rangle_{H^{-1}\times H^1_0}.$$ 

By definition of $B$, we have for any $g$ in $L^2(\Omega)$, $g = \Delta Bg$. Thus we infer that

$$(Bf|g)_{L^2} = \langle \Delta Bg, Bf \rangle_{H^{-1}\times H^1_0}.$$ 

As for any $u$ and $v$ in $H^1_0(\Omega)$, we have

$$\langle \Delta u, v \rangle_{H^{-1}\times H^1_0} = (u|v)_{H^1_0},$$

we deduce that

$$(Bf|g)_{L^2} = (Bf|Bg)_{H^1_0}.$$ 

If $f = g$, this gives in particular $(Bf|f)_{L^2} = \|Bf\|_{H^1_0}^2$. Thus the operator $B$ is compact, self-adjoint and positive. The spectral theorem about compact selfadjoint operator applied to $B$ implies the existence of a non increasing sequence $(\mu_k)_{k \in \mathbb{N}}$ of positive real numbers which tends to 0 and a hilbertian basis of $L^2(\Omega)$ denoted $(e_k)_{k \in \mathbb{N}}$ such that, for any $k$, the function $e_k$ belongs to $L^2(\Omega)$ and such that $Be_k = \mu_k e_k$. This implies that $-\Delta e_k = \mu_k^{-1} e_k$.

We have,

$$\|f\|_{H^{-1}(\Omega)} = \sup_{(c_k) \in Bf} \left| \sum_k \lambda_k^{-1} c_k e_k \right|,$$

where $Bf$ denotes the set of sequences having only a finite number of non zero terms and of $\ell^2$ norm less or equal to 1. Thus

$$\|f\|_{H^{-1}(\Omega)} = \sum_{(c_k) \in Bf} \left| \sum_k \lambda_k^{-1} \langle f, e_k \rangle c_k \right| = \|v\|_{\ell^2(\mathbb{N})}.$$

Theorem 2.1.1 is proved. □

Let us first study the case of the heat equation in a bounded domain with Dirichlet boundary condition (which means that the solution ”vanishes on the boundary”). As we are far away from smooth function, we need a definition of ”weak solution”.

Definition 2.1.1 Let $\Omega$ be a bounded domain of $\mathbb{R}^d$. Let us consider a function $u$ continuous on $\mathbb{R}$ with value in $H^{-1}(\Omega)$ which moreover belongs to $L^1_{\text{loc}}(\mathbb{R}^+; H^1_0(\Omega))$, a distribution $u_0$ in $H^{-1}(\Omega)$ and a function $f$ which belongs to $L^1_{\text{loc}}(\mathbb{R}^+; H^{-1}(\Omega))$.  

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We say that $u$ is the solution of the heat equation with initial data $u_0$ and external force $f$ with Dirichlet boundary condition if, for any function $\phi$ continuously differentiable with value in $H^1_0(\Omega)$,

$$\langle u(t), \phi(t) \rangle_{H^{-1} \times H^1_0} = \langle u_0, \phi(0) \rangle_{H^{-1} \times H^1_0} + \int_0^t \langle \Delta \phi(t'), \partial_t \phi(t'), u(t') \rangle_{H^{-1} \times H^1_0} dt' + \int_0^t \langle f(t'), \phi(t') \rangle_{H^{-1} \times H^1_0} dt'.$$

The symbolic writing of this equation is

$$\begin{align*}
\text{(HE)} & \quad \begin{cases}
\partial_t u - \Delta u = f \\
u_{t=0} = u_0 \\
u_{\partial\Omega} = 0
\end{cases}
\end{align*}$$

This type of solution, if it exists, is unique because of the following proposition.

**Proposition 2.1.1** Let us consider a solution $u$ of the heat equation with initial data $0$ and external force $0$ with Dirichlet boundary condition. Then $u$ is identically $0$.

**Proof.** This hypothesis means exactly that, for any function $\phi$ continuously differentiable with value in $H^1_0(\Omega)$, we have

$$\langle u(t), e_k \rangle_{H^{-1} \times H^1_0} = \int_0^t \langle \Delta e_k, u(t') \rangle_{H^{-1} \times H^1_0} dt'.$$

Let us consider the hilbertian basis $(e_k)_{k \in \mathbb{N}}$ given by Theorem 2.1.1. Let us apply the above identity with a the test function $\phi(t) = e_k$. This gives

$$\langle u(t), e_k \rangle_{H^{-1} \times H^1_0} = \int_0^t \langle \Delta e_k, u(t') \rangle_{H^{-1} \times H^1_0} dt'.$$

As $\Delta e_k = -\lambda_k^2 e_k$, we gives

$$\langle u(t), e_k \rangle_{H^{-1} \times H^1_0} = -\lambda_k^2 \int_0^t \langle e_k, u(t') \rangle_{H^{-1} \times H^1_0} dt'.$$

As for almost every $t$, $u(t)$ belongs to $H^1_0(\Omega)$, we get

$$\langle e_k, u(t) \rangle_{H^{-1} \times H^1_0} = \langle u(t), e_k \rangle_{H^{-1} \times H^1_0} = \langle u(t)|e_k\rangle_{L^2}.$$

This implies that $\langle u(t), e_k \rangle = 0$ for any $t$ and any $k$. The proposition is proved. $\Box$

In order to prove the existence, we need some additional regularity properties on the initial data $u_0$ and the external force $f$. We have the following theorem.

**Theorem 2.1.2** Let $u_0$ be in $L^2(\Omega)$ and $f$ in $L^2_{\text{loc}}(\mathbb{R}^+; H^{-1}(\Omega))$. A (unique) solution exists (in the sense of Definition 2.1.1) which belongs to the space

$$C(\mathbb{R}^+; L^2(\Omega)) \cap L^2_{\text{loc}}(\mathbb{R}^+; H^1_0(\Omega))$$

and which satisfies

$$\frac{1}{2} \|u(t)\|_{L^2}^2 + \int_0^t \|\nabla u(t')\|_{L^2}^2 dt' = \frac{1}{2} \|u_0\|_{L^2}^2 + \int_0^t \langle f(t'), u(t') \rangle dt'.$$
Proof. Because a uniqueness, we can prove that the solution exists on \([0, T]\) for any positive \(T\). By time cutoff, we are reduced to the case when \(f\) belongs to \(L^2(\mathbb{R}^+; H^{-1}(\Omega))\).

The method used is a regularization method. Let us denote by \(E_k\) the orthogonal projection on \(V_k\) the space generated by the first \(k+1\) eigenvectors \(e_\ell\). Let us first prove the following regularization lemma.

**Lemma 2.1.1** For any force \(f\) in \(L^2(\mathbb{R}^+; H^{-1}(\Omega))\), a sequence \((f_k)_{k \in \mathbb{N}}\) of functions of the space \(L^2(\mathbb{R}^+; H^1_0(\Omega))\) exists which converges to \(f\) in \(L^2(\mathbb{R}^+; H^{-1}(\Omega))\) and moreover satisfies

\[
\forall k \in \mathbb{N}, \quad f_k \in C(\mathbb{R}^+; V_k).
\]

**Proof.** Thanks to Theorem 2.2.2 and to the Lebesgue Theorem,

\[
\lim_{k \to \infty} E_k f = f \quad \text{in} \quad L^2(\mathbb{R}^+; H^{-1}(\Omega)).
\]

Then, let us consider a sequence \((\epsilon_n)_{n \in \mathbb{N}}\) of positive real numbers which tends to 0 and a function \(\chi\) smooth and compactly supported from \(\mathbb{R}\) into \(\mathbb{R}\) with integral 1. As

\[
\lim_{n \to \infty} \frac{1}{\epsilon_n} \chi\left(\frac{\cdot}{\epsilon_n}\right) \ast (1_{\mathbb{R}^+} f_k) = 1_{\mathbb{R}^+} f_k \quad \text{in} \quad L^2(\mathbb{R}^+; V_k),
\]

the lemma is proved. \(\square\)

**Continuation of the proof of Theorem 2.1.2** Let us observe that the spaces \(V_k\) are stable under the action of \(\Delta\). Thus the heat equation with initial data \(E_N u_0\) can be written

\[
\frac{d}{dt} U_N = \Delta U_N + f_N
\]

which is a linear ordinary differential equation on \(V_k\) the solution of which is given by

\[
U_N(t) = \sum_{k=0}^{N} \left( e^{-\lambda_k^2 t} (u_0 | e_k)_{L^2} + \int_0^t e^{-\lambda_k^2 (t-t')} (f_N(t'), e_k) dt' \right) e_k.
\] (2.3)

From (2.2), we infer that

\[
\frac{1}{2} \frac{d}{dt} \|U_N(t)\|_{L^2}^2 = (\Delta U_N | U_N(t))_{L^2} + (f_N(t) | U_N(t))
\]

\[
= - (\nabla U_N | \nabla U_N(t))_{L^2} + (f_N(t), U_N(t)).
\]

A time integration gives

\[
\frac{1}{2} \|U_N(t)\|_{L^2}^2 + \int_0^t \|\nabla U_N(t')\|_{L^2}^2 dt' = \frac{1}{2} \|E_N u_0\|_{L^2}^2 + \int_0^t (f_N(t'), U_N(t')) dt'.
\] (2.4)

Now let us prove that \((U_N)_{N \in \mathbb{N}}\) is a Cauchy sequence in \(L^\infty(\mathbb{R}^+; L^2(\Omega)) \cap L^2(\mathbb{R}^+; H^1_0(\Omega))\). By linearity, we have

\[
\frac{d}{dt} (U_{N+P} - U_N) = \Delta (U_{N+P} - U_N) = f_{N+P} - f_N.
\]

Using again the above energy method, we infer that

\[
\frac{1}{2} \|U_{N+P}(t) - U_N(t)\|_{L^2}^2 + \int_0^t \|\nabla U_{N+P} - \nabla U_N(t')\|_{L^2}^2 dt'
\]

\[
= \frac{1}{2} \|E_{N+P} u_0 - E_N u_0\|_{L^2}^2 + \int_0^t \langle f_{N+P} - f_N(t'), U_{N+P}(t') - U_N(t') \rangle dt'.
\]
Using that
\[ \langle g, v \rangle \leq \|g\|_{H^{-1}(\Omega)} \|v\|_{H^1_0(\Omega)} \]
and that \( ab \leq \frac{1}{2} a^2 + \frac{1}{2} b^2 \), we infer that
\[
\|U_{N+P}(t) - U_N(t)\|_{L^2}^2 + \int_0^t \|\nabla U_{N+P} - \nabla U_N(t')\|_{L^2}^2 dt' \\
\leq \|E_{N+P} u_0 - E_N u(0)\|_{L^2}^2 + \int_0^t \|f_{N+P}(t') - f_N(t')\|_{H^{-1}(\Omega)}^2 dt'.
\]
As \((E_N u_0)_{N \in \mathbb{N}}\) is a Cauchy sequence in \(L^2(\Omega)\) and \((f_N)_{N \in \mathbb{N}}\) is a Cauchy sequence in the space \(L^2(\mathbb{R}^+; H^{-1}(\Omega))\), the theorem is proved.

One of the important points to notice is that the solution \(u\) is in some sense explicit, because given by the following formula which comes immediately from (2.3) once noticed that the sequence \((U_N)_{N \in \mathbb{N}}\) is a Cauchy one.

\[
U(t) = \sum_{k \in \mathbb{N}} \left( e^{-\lambda_k^2 t} (u_0 | e_k)_{L^2} + \int_0^t e^{-\lambda_k^2 (t-t')} \langle f_N(t'), e_k \rangle dt' \right) e_k. \tag{2.5}
\]

### 2.2 The stationary Stokes’s problem

This problem is analogous to the Dirichlet problem, but we work on the set of divergence free vector field. The Laplace equation will become the Stokes equation. Let us define the space we are going to work with.

**Definition 2.2.1** Let us denote by \(\mathcal{V}(\Omega)\) (resp. \(\mathcal{V}_r(\Omega)\)) the set of vector fields (resp. of divergence free vector fields) the components of which are in \(H^1_0(\Omega)\). Theses spaces are equipped with the norm

\[
\|u\|_\mathcal{V}^2 \overset{\text{def}}{=} \sum_{j,k=1}^d \|\partial_j u^k\|_{L^2}^2.
\]

Let us denote by \(\mathcal{H}(\Omega)\) the closure of \(\mathcal{V}_r(\Omega)\) in the space \((L^2(\Omega))^d\). Let us denote by \(\mathcal{V}'(\Omega)\) the space \((H^{-1}(\Omega))^d\). For \(f\) in \(\mathcal{V}'\) and \(v\) in \(\mathcal{V}\), we note

\[
\langle f, v \rangle_{\mathcal{V}' \times \mathcal{V}} = \sum_{j=1}^d \langle f^j, v^j \rangle_{H^{-1} \times H^1_0}.
\]

We denote by \((\mathcal{V}_r(\Omega))^0\) the polar set of \(\mathcal{V}_r(\Omega)\), i.e. the set of elements of \(\mathcal{V}'(\Omega)\) such that, for any \(v\) in \(\mathcal{V}_r(\Omega)\), we have \(\langle f, v \rangle_{\mathcal{V}' \times \mathcal{V}} = 0\). When no confusion is possible, we omitted to mention the domain \(\Omega\).

When no confusion is possible, we drop \(\Omega\) in the notations.

**Remark** The space \(\mathcal{H}\) is strictly included in the space of divergence free vector fields the components of which are in \(L^2\). In fact, when the boundary is regular, it is possible to prove that, if \(v\) belongs to the space of vector fields the components of which are in \(L^2(\Omega)\), then the value of \(v \cdot n\) where \(n\) denotes a (local) unit normal vector of the boundary of \(\Omega\) (denoted by \(\partial \Omega\)) at the boundary \(\partial \Omega\) makes sense because of the divergence free condition. After a suitable localization, it can be recued to the following exercise.
Exercice 2.2.1 Let us consider $L^2_\sigma(\mathbb{R}^3)$ the set of vector fields of $\mathbb{R}^3$ the components of which are in $L^2(\mathbb{R}^3)$. Let us prove that, for any $x_3$ and $y_3$ in $\mathbb{R}$, we have

$$\|v^3(\cdot, x_3) - v^3(\cdot, y_3)\|_{H^{-1}(\mathbb{R}^2)} \leq C |x_3 - y_3|^{\frac{1}{2}} \|v\|_{L^2(\mathbb{R}^3)}.$$

Let us state the analogous of Dirichlet theorem in this framework. We have the following theorem, which is the analogue of the classical Dirichlet theorem.

Theorem 2.2.1 Let $f$ be in $\mathcal{V}'$. It exists a unique solution $u$ in the space $\mathcal{V}_\sigma$ of the following equation

$$-\Delta u - f \in \mathcal{V}_\sigma^0$$

which means that, for any vector field $v$ of $\mathcal{V}_\sigma$, we have

$$-\langle \Delta u, v \rangle_{\mathcal{V}' \times \mathcal{V}} = \langle f, v \rangle_{\mathcal{V}' \times \mathcal{V}}. \tag{2.6}$$

This also means that $-\Delta u = f$ in $\mathcal{V}_\sigma'$.

Proof. Let us consider $f|_{\mathcal{V}_\sigma}$. It is of course a continuous linear form on $\mathcal{V}_\sigma$. Riesz theorem applied to the Hilbert space $(\mathcal{V}_\sigma, \| \cdot \|_\sigma)$ implies that a unique vector $u$ exists in $\mathcal{V}_\sigma$ such that

$$\forall h \in \mathcal{V}_\sigma, \ (u|h)_{\mathcal{V}_\sigma} = \langle f, h \rangle_{\mathcal{V}' \times \mathcal{V}}.$$

In fact the scalar product on $\mathcal{V}_\sigma$ is simply the scalar product on $\mathcal{V}$ restricted to $\mathcal{V}_\sigma$. Thus, we get

$$\forall h \in \mathcal{V}_\sigma, \ (u|h)_{\mathcal{V}_\sigma} = -\langle \Delta u, h \rangle_{\mathcal{V}' \times \mathcal{V}} = -\langle \Delta h, u \rangle_{\mathcal{V}' \times \mathcal{V}}. \tag{2.7}$$

This gives the theorem. \hfill \Box

The existence and the uniqueness of a minimum $u$ for the functionnal $F$ can be proved following exactly the same lines as in the case of Dirichlet problem. The fact that the differential of $F$ vanishes at point $u$ implies the relation (2.6).

Remarks

- The fact that a vector field $g$ of $H^{-1}(\Omega)$ belongs the polar set (in the sense of the duality) of $\mathcal{V}_\sigma(\Omega)$ implies in particular that, for any function $\varphi$ of $D(\Omega)$, we have

$$\langle g^i, - \partial_j \varphi \rangle_{H^{-1} \times H^1_0} + \langle g^j, \partial_i \varphi \rangle_{H^{-1} \times H^1_0} = 0$$

which implies that $\partial_j g^i - \partial_i g^j = 0$, i.e. the curl of $g$ is identically 0.

- Very simple domains exist such that it exists a smooth vector field which are of divergence and of curl identically 0 and which is not the gradient of a function on $\Omega$.

Let us consider the domain of the plan $\Omega \overset{\text{def}}{=} \{ x \in \mathbb{R}^2 / 0 < R_1 < |x| < R_2 \}$ and the vector field $f$ defined by $(-\partial_2 \log |x|, \partial_1 \log |x|)$. We have the following lemma.

Proposition 2.2.1 The vector field $f$ is curl free, but it is not the gradient of a distribution.
Proof. The fact that \( f \) is of divergence free is obvious. The fact that its curl is 0 follows from the fact that that the function \( x \mapsto \log |x| \) is harmonic on \( \Omega \). Let us assume that \( f \) is a gradient of some distribution \(-p\). As \( f \) is a smooth function, so is \( p \) is also smooth. Let us consider the flow of \( f = -\nabla p \).

By definition of \( f \), its trajectories are periodic. Let us consider a trajectory \( \gamma \) from a point of \( \Omega \) such that \( f \neq 0 \) (here all points are like this). Thanks to the chain rule, we have

\[
\frac{d}{dt}(p \circ \gamma)(t) = \langle Dp(\gamma(t)), \frac{d\gamma}{dt} \rangle.
\]

By definition of the gradient, we get

\[
\langle Dp(\gamma(t)), \frac{d\gamma}{dt} \rangle = \left( \nabla p(\gamma(t)) \right) \frac{d\gamma}{dt}.
\]

By definition of \( \gamma \), we infer that

\[
\frac{d}{dt}(p \circ \gamma)(t) = -|\nabla p(\gamma(t))|^2.
\]

The fact that the derivative is negative is in contradiction with the periodicity of the trajectory \( \gamma \).

Proposition 2.2.1 is proved.

As shown by the following proposition, belonging to \( V_\sigma^0 \) is stronger than being curl free.

Let us admit the following proposition.

Proposition 2.2.2 If \( f \) belongs to \( V_\sigma^0 \), then it exists \( p \) in \( L^2_{\text{loc}}(\Omega) \) such that \( f = -\nabla p \). If the boundary of \( \Omega \) is a \( C^1 \) hypersurface, then \( p \) belongs to \( L^2(\Omega) \).

Remarks Let us note that the above theorem implies that the vector field \( f \) defined in Proposition 2.2.1 does not belongs to \( V_\sigma^0 \) in spite of the fact that it is curl free. The two notions coincide in so called ”simply connected” domain which means essentially that there no hole in them. Often a generic element of \( V_\sigma^0 \) is denoted by \( \nabla p \).

As in the case of Dirichlet problem, the spectral structure of selfadjoint compact operators will give the following result.

Theorem 2.2.2 A non decreasing sequence \((\lambda_k)_{k \in \mathbb{N}}\) of positive numbers which tends to infinity and an orthonormal basis \((e_k)_{k \in \mathbb{N}}\) of \( \mathcal{H} \) exist such that the sequence \((\lambda_k^{-1}e_k)_{k \in \mathbb{N}}\) is an orthonormal basis of \( V_\sigma \) and such that

\[
-\Delta e_k - \lambda_k^2 e_k \in V_\sigma^0 \quad \text{which means} \quad -\Delta e_k = \lambda_k^2 e_k \quad \text{in} \quad V_\sigma'.
\]

Moreover, if \( f \) is in \( V' \), then

\[
\|f\|^2_{V_\sigma'} = \sum_{k \in \mathbb{N}} \lambda_k^{-2} \langle (f, e_k) \rangle^2 \quad \text{and} \quad \lim_{n \to \infty} \left\| f - \sum_{k=0}^n \langle f, e_k \rangle e_k \right\|_{V_\sigma'} = 0.
\]

Proof. It is very close to the proof of analogous theorem for the Laplacian. As the space \( \mathcal{H} \) is continuously included in \( V' \), we can define the operator \( B \)

\[
B \begin{cases} \mathcal{H} & \rightarrow & V_\sigma \subset \mathcal{H} \\ f & \mapsto & u \end{cases}
\]

such that \( u \) is the solution in \( V_\sigma \) of \(-\Delta u = f \) in \( V'_\sigma \).

The operator \( B \) is of course continuous from \( \mathcal{H} \) into \( V_\sigma \). Thanks to Rellich’s theorem, the operator \( B \) is compact from \( \mathcal{H} \) into \( \mathcal{H} \). Let us prove it is self adjoint and positive. Let us write that for any couple of functions \((f, g)\) in \( \mathcal{H} \), we have

\[
(Bf,g)_{L^2} = (g, Bf)_{V' \times V'}.
\]

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By definition of $B$, we have for any $g$ in $\mathcal{H}$, $g = \Delta Bg$ in $\mathcal{V}_\sigma'$. Thus we infer that
\[
(Bf|g)_\mathcal{H} = (g, Bf)_{\mathcal{V}_\sigma' \times \mathcal{V}_\sigma} = (\Delta Bg, Bf)_{\mathcal{V}_\sigma' \times \mathcal{V}_\sigma}.
\]
As for any $u$ and $v$ in $\mathcal{V}_\sigma'$, we have
\[
(\Delta u, v)_{\mathcal{V}_\sigma' \times \mathcal{V}_\sigma} = (u|v)_{\mathcal{V}_\sigma},
\]
we deduce that
\[
(Bf|g)_{\mathcal{H}} = (Bf|Bg)_{\mathcal{V}_\sigma}.
\]
Thus the operator $B$ is compact, selfadjoint and positive. The spectral theorem applied to $B$ implies the existence of a non increasing sequence $(\mu_k)_{k \in \mathbb{N}}$ of positive real numbers which tends to 0 and a Hilbertian basis of $\mathcal{H}$ denoted $(e_k)_{k \in \mathbb{N}}$ such that, for any $k$, the function $e_k$ belongs to $\mathcal{H}$ and such that $Be_k = \mu_k e_k$ in $\mathcal{V}_\sigma'$. We have,
\[
\|f\|_{\mathcal{V}_\sigma'} = \sup_{(c_k) \in \mathcal{B}_0} \langle f, \sum_k \lambda_k^{-1} c_k e_k \rangle_{\mathcal{V}_\sigma' \times \mathcal{V}}
\]
where $\mathcal{B}_0$ denotes the set of sequences having only a finite number of non zero terms and of $\ell^2$ norm less or equal to 1. Thus
\[
\|f\|_{\mathcal{V}_\sigma'} = \sup_{(c_k) \in \mathcal{B}_0} \sum_k \lambda_k^{-1} \langle f, e_k \rangle c_k = \|(\lambda_k^{-1} \langle f, e_k \rangle)_{k \in \mathbb{N}}\|_{\ell^2(\mathbb{N})}.
\]
The theorem is proved. \hfill \square

**Definition 2.2.2** Let us denote by $\mathcal{H}_k$ the $k+1$ dimensionnal vector space generated by the family $(e_j)_{0 \leq j \leq k}$. For $f$ in $\mathcal{V}'$, let us state
\[
\mathbb{P}_k f \overset{\text{def}}{=} \sum_{j \leq k} \langle f, e_j \rangle_{\mathcal{V}_\sigma' \times \mathcal{V}} e_j \quad \text{and} \quad (2.8)
\]
\[
\mathbb{P} f \overset{\text{def}}{=} \sum_{j \in \mathbb{N}} \langle f, e_j \rangle_{\mathcal{V}_\sigma' \times \mathcal{V}} e_j. \quad (2.9)
\]
The map $\mathbb{P}$ is called the Leray projection on divergence free vector field.

Let us notice that from Theorem 2.2.2, we have, for all non negative integer $k$,
\[
\forall f \in \mathcal{V}' \quad \| \mathbb{P}_k f \|_{\mathcal{V}_\sigma'} \leq \| \mathbb{P} f \|_{\mathcal{V}_\sigma'} \leq \| f \|_{\mathcal{V}'}, \quad (2.10)
\]
\[
\forall f \in (L^2)^d \quad \| \mathbb{P}_k f \|_{\mathcal{H}} \leq \| \mathbb{P} f \|_{\mathcal{H}} \leq \| f \|_{(L^2)^d} \quad \text{and} \quad (2.11)
\]
\[
\forall f \in \mathcal{V} \quad \| \mathbb{P}_k f \|_{\mathcal{V}_\sigma} \leq \| \mathbb{P} f \|_{\mathcal{V}_\sigma} \leq \| f \|_{\mathcal{V}}. \quad (2.12)
\]
The operator $\mathbb{P}_k$ (resp. $\mathbb{P}$) is the orthogonal projection of $(L^2)^d$ on $\mathcal{H}_k$ (resp. $\mathcal{H}$) and also the orthogonal projection of $\mathcal{V}$ on $\mathcal{H}_k$ (resp. $\mathcal{V}_\sigma$).
2.3 The Stokes’s flow

Given a positive viscosity $\nu$, the evolution Stokes problem reads as follows:

$$(ES_\nu) \quad \begin{cases} 
\partial_t u - \nu \Delta u &= f - \nabla p \\
\text{div} \ u &= 0 \\
\ u_{|\partial\Omega} &= 0 \\
\ u_{|t=0} &= u_0 \in \mathcal{H}.
\end{cases}$$

Let us define what a solution of this problem is.

**Definition 2.3.1** Let $u_0$ be in $\mathcal{H}$ and $f$ in $L^2_{\text{loc}}(\mathbb{R}^+; \mathcal{V}^\nu)$. We shall say that $u$ is a solution of $(ES_\nu)$ with initial data $u_0$ and external force $f$ if and only if $u$ belongs to the space

$$C(\mathbb{R}^+; \mathcal{V}^\nu) \cap L^\infty_{\text{loc}}(\mathbb{R}^+; \mathcal{H}) \cap L^2_{\text{loc}}(\mathbb{R}^+; \mathcal{V}_\sigma)$$

and satisfies, for any $\Psi$ in $C^1(\mathbb{R}^+; \mathcal{V}_\sigma)$,

$$\langle u(t), \Psi(t) \rangle_{\mathcal{V}^\nu \times \mathcal{V}_\sigma} = \langle u_0, \Psi(0) \rangle_{\mathcal{H}} + \int_0^t \langle \nu \Delta \Psi(t') + \partial_t \Psi(t'), u(t') \rangle_{\mathcal{V}^\nu \times \mathcal{V}_\sigma} dt' + \int_0^t \langle f(t'), \Psi(t') \rangle_{\mathcal{V}^\nu \times \mathcal{V}_\sigma} dt'.$$

The following theorem holds.

**Theorem 2.3.1** The problem $(ES_\nu)$ has a unique solution in the sense of the above definition. Moreover this solution belongs to $C(\mathbb{R}^+; \mathcal{H})$ and satisfies the following energy equality:

$$\frac{1}{2} \|u(t)\|^2_{\mathcal{H}} + \nu \int_0^t \|u(t')\|^2_{\mathcal{V}_\sigma} dt' = \frac{1}{2} \|u_0\|^2_{\mathcal{H}} + \int_0^t \langle f(t'), u(t') \rangle_{\mathcal{V}^\nu \times \mathcal{V}_\sigma} dt'.$$

**Proof.** In order to prove uniqueness, let us consider some function $u$ in $C(\mathbb{R}^+; \mathcal{V}^\nu) \cap L^2_{\text{loc}}(\mathbb{R}^+; \mathcal{V}_\sigma)$ such that, for all $\Psi$ in $C^1(\mathbb{R}^+; \mathcal{V}_\sigma)$,

$$\langle u(t), \Psi(t) \rangle_{\mathcal{V}^\nu \times \mathcal{V}_\sigma} = \int_0^t \langle \nu \Delta \Psi(t') + \partial_t \Psi(t'), u(t') \rangle_{\mathcal{V}^\nu \times \mathcal{V}_\sigma} dt'.$$

This is valid in particular for the time independent function $\Psi(t) \equiv e_k$ where the family vector fields $(e_k)_{k \in \mathbb{N}}$ is given by Theorem 2.2.2. This gives

$$\langle u(t), e_k \rangle = \nu \int_0^t \langle \Delta e_k, u(t') \rangle dt'.$$

Thanks to the spectral Theorem 2.2.2, we get

$$\langle u(t), e_k \rangle = -\nu \lambda_k^2 \int_0^t \langle e_k, u(t') \rangle dt'.$$

This implies that, for any $k$, $\langle u(t), e_k \rangle = 0$. Thus $u \equiv 0$. \qed

In order to prove existence, we use the following approximation lemma, the proof of which is exactly analogous to the proof of Lemma 2.1.1, and is thus omitted.
Lemma 2.3.1 For any force \( f \) in \( L^2_{\text{loc}}(\mathbb{R}^+; \mathcal{V}') \), a sequence \((f_k)_{k \in \mathbb{N}}\) exists in \( C^1(\mathbb{R}^+; \mathcal{V}_\sigma)\) such that for any integer \( k \) and for any positive \( t \), the vector field \( f_k(t) \) belongs to \( \mathcal{H}_k \), and

\[
\lim_{k \to \infty} \|f_k - f\|_{L^2([0,T]; \mathcal{V}_\sigma')} = 0.
\]

Let us consider a sequence \((f_k)_{k \in \mathbb{N}}\) associated with \( f \) given by the above lemma and then let us consider approximated problem

\[
(ES_{\nu,k}) \begin{cases}
\partial_t u_k - \nu \mathbb{P}_k \Delta u_k = f_k \\
u_k|_{t=0} = \mathbb{P}_k u_0.
\end{cases}
\]  

Again thanks to Theorem 2.2.2, it is a linear ordinary differential equation on \( \mathcal{H}_k \) which has a global solution \( u_k \) which is \( C^1(\mathbb{R}^+; \mathcal{H}_k) \). By an energy estimate in \((ES_{\nu,k})\) we get that

\[
\frac{1}{2} \frac{d}{dt} \|u_k(t)\|_{\mathcal{H}}^2 + \nu \|u_k(t)\|_{\mathcal{V}_\sigma'}^2 = \langle f_k(t), u_k(t) \rangle_{\mathcal{V}_\sigma' \times \mathcal{V}_\sigma}.
\]

A time integration gives

\[
\frac{1}{2} \|u_k(t)\|_{\mathcal{H}}^2 + \nu \int_0^t \|u_k(t')\|_{\mathcal{V}_\sigma'}^2 dt' = \frac{1}{2} \|\mathbb{P}_k u(0)\|_{\mathcal{H}}^2 + \int_0^t \langle f_k(t'), u_k(t') \rangle_{\mathcal{V}_\sigma' \times \mathcal{V}_\sigma} dt'.
\]  

In order to pass to the limit, we write an energy estimate for \( u_k - u_{k+\ell} \), which gives

\[
\delta_{k,\ell}(t) \overset{\text{def}}{=} \frac{1}{2} \|(u_k - u_{k+\ell})(t)\|_{\mathcal{H}}^2 + \nu \int_0^t \|u_k(t') - u_{k+\ell}(t')\|_{\mathcal{V}_\sigma'}^2 dt' \\
\leq \frac{1}{2} \|(\mathbb{P}_k - \mathbb{P}_{k+\ell})u(0)\|_{\mathcal{H}}^2 + \nu \int_0^t \|u_k(t') - u_{k+\ell}(t')\|_{\mathcal{V}_\sigma'}^2 dt' + \frac{\nu}{2} \int_0^t \|(u_k - u_{k+\ell})(t')\|_{\mathcal{V}_\sigma'}^2 dt'.
\]

This implies that

\[
\delta_{k,\ell}(t) \leq \|(\mathbb{P}_k - \mathbb{P}_{k+\ell})u(0)\|_{\mathcal{V}_{\sigma}^2} + \frac{\nu}{2} \int_0^t \|(f_k - f_{k+\ell})(t')\|_{\mathcal{V}_\sigma'}^2 dt'.
\]

This implies that the sequence \((u_k)_{k \in \mathbb{N}}\) is a Cauchy sequence in \( C(\mathbb{R}^+; \mathcal{H}) \cap L^2_{\text{loc}}(\mathbb{R}^+; \mathcal{V}_\sigma) \). Let us denote by \( u \) the limit and prove that \( u \) is a solution in the sense of Definition 2.3.1. As \( u_k \) is a \( C^1 \) solution of the ordinary differential equation \((ES_{\nu,k})\), we have, for any \( \Psi \) in \( C^1(\mathbb{R}^+; \mathcal{V}_\sigma)\),

\[
\frac{d}{dt} \langle u_k(t), \Psi(t) \rangle_{\mathcal{V}_\sigma' \times \mathcal{V}_\sigma} = \nu \langle \Delta u_k(t), \Psi(t) \rangle_{\mathcal{V}_\sigma' \times \mathcal{V}_\sigma} + \langle f_k(t), \Psi(t) \rangle_{\mathcal{V}_\sigma' \times \mathcal{V}_\sigma} + \langle u_k(t), \partial_t \Psi(t) \rangle_{\mathcal{V}_\sigma' \times \mathcal{V}_\sigma}.
\]

Using Relation (2.7), we get by time integration,

\[
\langle u_k(t), \Psi(t) \rangle_{\mathcal{V}_\sigma' \times \mathcal{V}_\sigma} = \langle \mathbb{P}_k u_0 | \Psi(0) \rangle_{\mathcal{H}} + \int_0^t \nu \langle \Delta \Psi(t'), \partial_t \Psi(t'), u_k(t') \rangle_{\mathcal{V}_\sigma' \times \mathcal{V}_\sigma} dt' + \int_0^t \langle f_k(t'), \Psi(t') \rangle_{\mathcal{V}_\sigma' \times \mathcal{V}_\sigma} dt'.
\]

Passing to the limit in the above equality gives the existence part of the theorem. Let us prove the energy equality. As \((u_k)_{k \in \mathbb{N}}\) is a Cauchy sequence in \( L^\infty([0,T]; \mathcal{V}_\sigma) \cap L^2([0,T]; \mathcal{V}_\sigma) \), we get the energy equality passing in the limite into (2.14).
**Remark.** The solution is given by the explicit formula

\[
u(t) = \sum_{j \in \mathbb{N}} U_j(t) e_j \quad \text{with} \quad U_j(t) \overset{\text{def}}{=} e^{-\nu \mu^2 t} (u_0|e_j\rangle_{L^2} + \int_0^t e^{-\nu \mu^2 (t-t')} \langle f(t'), e_j \rangle \, dt'.
\]

(2.15)

The formula shows that the solution to \((ES_\nu)\) depends only on the initial data (of course) and of \(f|_{\mathcal{V}_\nu}\).
Chapter 3

Leray’s Theorem on Navier-Stokes equations in a bounded domain

In this chapter, we shall prove the existence of global solutions for the incompressible Navier-Stokes system in a bounded domain with Dirichlet boundary conditions which can be formally written

\[
\begin{aligned}
&\partial_t v + v \cdot \nabla v - \nu \Delta v = -\nabla p \\
&\text{div } v = 0 \\
&v|_{t=0} = v_0 \\
&v|_{\partial \Omega} = 0.
\end{aligned}
\]

In all this chapter, \( \Omega \) will denoted a bounded domain of \( \mathbb{R}^d \) with \( d \in \{2, 3\} \). Let us point out that no hypothesis is made on regularity of the boundary of \( \Omega \). We want to solve the above system for initial data of finite kinetic energy.

The first section consists in defining the concept of "turbulent" solution, which is a non linear generalization of Definition 2.1.1 page 24, called also Leray solution.

The purpose of the the second section is the proof of the existence of a solution in the spirit of the proof of Peano’s Theorem 1.3.1 page 12. Here we project the equation \((NS)\) on the space \( H_k \) defined in Definition 2.2.2 page 30 This approximated problem is an ordinary differential equation in the finite dimensionnal space \( H_k \). Then we pass to the limit using the smoothing effect of the Stokes operator and the time regularity given by the equation (in fact we use Ascoli’s theorem).

In the third section, we prove that the Leray solution is unique and (even stable) in the case when the space dimension is 2. The main point is that Sobolev embeddings related to the energy spaces are better in dimension 2. The problem of uniqueness in the three dimensionnal case is the purpose of the next chapter.

3.1 The concept of turbulent solution

Let us state now the weak formulation of the incompressible Navier–Stokes system \((NS_v)\).

**Definition 3.1.1** We shall say that \( u \) is a weak solution of the Navier–Stokes equations on \([0, T] \times \Omega \) (\( T \) can be \( \infty \)) with Dirichlet boundary condition, with an initial data \( u_0 \) in \( H \) and an external force \( f \) in \( L^2_{\text{loc}}([0, T]; V^c) \) if and only if \( u \) belongs to the space

\[
C([0, T]; V^c) \cap L^\infty_{\text{loc}}([0, T]; H) \cap L^2_{\text{loc}}([0, T]; V^c)
\]
and for any function $\Psi$ in $C^1([0,T];\mathbb{R}^+;\mathcal{V}_\sigma)$, the vector field $u$ satisfies the following condition:

$$\langle u(t), \Psi(t) \rangle_{\mathcal{V}_\sigma^* \times \mathcal{V}_\sigma} = \langle u_0, \Psi(0) \rangle_{\mathcal{V}_\sigma^* \times \mathcal{V}_\sigma} + \int_0^t \langle \nu \Delta \Psi(t'), u(t') \rangle_{\mathcal{V}_\sigma^* \times \mathcal{V}_\sigma} \, dt' + \int_0^t \langle u(t') \otimes u(t') \rangle_{L^2} \, dt' + \int_0^t \langle f(t'), \Psi(t') \rangle_{\mathcal{V}_\sigma^* \times \mathcal{V}_\sigma} \, dt'$$

$$+ \int_0^t \langle u(t') \otimes u(t') \rangle_{\nabla \Psi(t') \rangle_{L^2}} \, dt'$$

with $(u \otimes u)^{t,k} = u^t u^k$

for any $t$ in $[0,T]$.

Let us point out that this definition make sense because, using Gagliardo–Nirenberg’s inequality stated in Corollary A.2.1 page 88 we get

$$\forall u \in \mathcal{V}, \|u\|_{L^4} \leq \|\nabla u\|_{L^2}^{\frac{d}{2}} \|u\|_{L^2}^{1-\frac{d}{2}}.$$ 

Thus the energy space is included in the space $L^2_{\text{loc}}(\mathbb{R}^+; L^4)$ and as $d$ is less than 4, product of coordinates of $u$ belongs to $L^1_{\text{loc}}(\mathbb{R}^+; L^2)$ The non linear term

$$- \int_0^t \langle u(t') \otimes u(t') \rangle_{L^2} \, dt'$$

is well defined for $u$ in the energy space $\mathcal{E} \overset{\text{def}}{=} L^2_{\text{loc}}(\mathbb{R}^+; \mathcal{H}) \cap L^2_{\text{loc}}(\mathbb{R}^+; \mathcal{V}_\sigma)$.

Let us remark that the above relation means that the equality in $(NS_{\nu})$ must be understood as an equality in the sense of $\mathcal{V}_\sigma$. Now let us state the Leray theorem.

**Theorem 3.1.1** Let $\Omega$ be a domain of $\mathbb{R}^d$ and $u_0$ a vector field in $\mathcal{H}$. Then, there exists a global \( (i.e. \text{ defined on } [0, \infty[) \) weak solution $u$ to $(NS_{\nu})$ in the sense of Definition 3.1.1. Moreover, this solution satisfies the energy inequality for all $t \geq 0$,

$$\frac{1}{2} \|u(t)\|^2_{\mathcal{H}} + \nu \int_0^t \|u(t')\|^2_{\mathcal{V}_\sigma} \, dt' \leq \frac{1}{2} \|u_0\|^2_{\mathcal{H}} + \int_0^t \langle f(t'), u(t') \rangle_{\mathcal{V}_\sigma^* \times \mathcal{V}_\sigma} \, dt'.$$

(3.1)

It is convenient to state the following definition.

**Definition 3.1.2** A solution of $(NS_{\nu})$ in the sense of the above Definition 3.1.1 which moreover satisfies the energy inequality (3.1) is called a Leray solution (or a turbulent solution of $(NS_{\nu})$).

Let us remark that the energy inequality implies a control on the energy.

**Proposition 3.1.1** Any Leray solution $u$ of $(NS_{\nu})$ satisfies

$$\|u(t)\|^2_{\mathcal{H}} + \nu \int_0^t \|u(t')\|^2_{\mathcal{V}_\sigma} \, dt' \leq \|u_0\|^2_{\mathcal{H}} + \frac{1}{\nu} \int_0^t \|f(t')\|^2_{\mathcal{V}_\sigma} \, dt'.$$

Proof. By definition of the norm $\|\cdot\|_{\mathcal{V}_\sigma}$, we have, as almost every positive $t'$, $u(t')$ belongs to $\mathcal{V}_\sigma$, we

$$\langle f(t'), u(t') \rangle_{\mathcal{V}_\sigma^* \times \mathcal{V}_\sigma} \leq \|f(t')\|_{\mathcal{V}_\sigma^*} \|u(t')\|_{\mathcal{V}_\sigma}.$$  

(3.2)

Inequality (3.1) becomes

$$\|u(t)\|^2_{\mathcal{H}} + 2\nu \int_0^t \|u(t')\|^2_{\mathcal{V}_\sigma} \, dt' \leq \|u_0\|^2_{\mathcal{H}} + 2 \int_0^t \|f(t')\|_{\mathcal{V}_\sigma} \|u(t', \cdot)\|_{\mathcal{V}_\sigma} \, dt'.$$

Thus, we get, using the fact that $2ab \leq a^2 + b^2$,

$$\|u(t)\|^2_{\mathcal{H}} + \nu \int_0^t \|u(t')\|^2_{\mathcal{V}_\sigma} \, dt' \leq \|u_0\|^2_{\mathcal{H}} + \frac{1}{\nu} \int_0^t \|f(t')\|^2_{\mathcal{V}_\sigma} \, dt'.$$

Thus the proposition is proved.
3.2 The proof of Leray’s theorem

The structure of the proof is the following:

• first a family of approximated solutions \((u_k)_{k \in \mathbb{N}}\) is built in spaces with finite frequencies by using simple ordinary differential equations results in \(L^2\)-type spaces.

• Next, thank energy estimates, we prove that, for any positive \(T\), this family is weakly compact in \(L^2([0,T];\mathcal{V}_\sigma)\) and strongly compact in \(L^\infty([0,T];\mathcal{V}_\sigma')\).

• Finally the conclusion is obtained by passing to the limit in the weak formulation, taking especially care of the nonlinear terms.

The study of the non linearity is based on the following simple (but key) lemma.

Lemma 3.2.1 Let us define the bilinear map

\[ Q : \mathcal{V} \times \mathcal{V} \rightarrow \mathcal{V}' \quad (u,v) \rightarrow -\text{div}(u \otimes v). \] (3.3)

For any \(u\) and \(v\) in \(\mathcal{V}\), the following estimates hold. For \(d\) in \(\{2,3\}\), a constant \(C\) exists such that, for any \(\varphi\) in \(\mathcal{V}\),

\[ \langle Q(u,v), \varphi \rangle \leq C \|\nabla u\|_{L^2}^d \|\nabla v\|_{L^2}^d \|u\|_{L^2}^{1-d} \|v\|_{L^2}^{1-d} \|\nabla \varphi\|_{L^2}. \]

Moreover for any \(u\) in \(\mathcal{V}_\sigma\) and any \(v\) in \(\mathcal{V}\), \(\langle Q(u,v), v \rangle = 0\).

Proof. The first two inequalities follow directly from Gagliardo–Nirenberg’s inequality stated in Corollary A.2.1 page 88, once noticed that

\[ \|u \otimes v\|_{L^2} \leq \|u\|_{L^4} \|v\|_{L^4} \|\nabla \varphi\|_{L^2}. \]

In order to prove the third assertion, let us assume that \(u\) and \(v\) are two vector fields the components of which belong to \(\mathcal{D}(\Omega)\). Then we deduce from integrations by parts that

\[ \langle Q(u,v), v \rangle = -\int_{\Omega} (\text{div}(u \otimes v) \cdot v)(x) \, dx \]
\[ = -\sum_{\ell,m=1}^d \int_{\Omega} \partial_m(u^m(x)v^\ell(x))v^\ell(x) \, dx \]
\[ = \sum_{\ell,m=1}^d \int_{\Omega} u^m(x)v^\ell(x)\partial_m v^\ell(x) \, dx \]
\[ = -\int_{\Omega} |v(x)|^2 \text{div} u(x) \, dx - \langle Q(u,v), v \rangle. \]

Thus, we have

\[ \langle Q(u,v), v \rangle = -\frac{1}{2} \int_{\Omega} |v(x)|^2 \text{div} u(x) \, dx. \]

The two expressions are continuous on \(\mathcal{V}\) and by definition, \(\mathcal{D}\) is dense in \(\mathcal{V}\). Thus the above formula is true for any \((u,v)\) in \(\mathcal{V} \times \mathcal{V}\), which completes the proof. \(\square\)
Now let us proceed to the proof of Leray’s theorem step by step.

**Construction of approximated solutions**

We use the projections $\mathbb{P}_k$ introduced in Definition 2.2.2. Let us introduce the function

$$F_k \begin{cases} \mathcal{H}_k & \rightarrow \mathcal{H}_k \\ v & \mapsto \mathbb{P}_k(\text{div}(v \otimes v)) \end{cases}.$$  

The properties of $F_k$ will be the consequence of the following lemma which will be very useful.

As a corollary, we get that $F_k$ is locally Lipschitz in $\mathcal{H}_k$ and that $F_k$ satisfies

$$\|F_k(v)\|_{\mathcal{H}_k} \lesssim \lambda_k^d \|v\|_{\mathcal{H}_k}^2.$$  

(3.4)

Thanks to Theorem 2.2.2 and to the above lemma, we can solve the following ordinary differential equation

$$(NS_{v,k}) \begin{cases} \frac{du_k}{dt}(t) = \nu \mathbb{P}_k \Delta u_k(t) + F_k(u_k(t)) + f_k(t) \\ u_k(0) = \mathbb{P}_k u_0 \end{cases}$$

where $(f_k)_{k \in \mathbb{N}}$ a the sequence of approximation of $f$ given by Lemma 2.3.1. Theorem 2.2.2 implies that $\mathbb{P}_k \Delta$ is a linear map from $\mathcal{H}_k$ into itself. Thus the continuity properties of $Q$ and $\mathbb{P}_k$ allow to apply the Cauchy–Lipschitz theorem. This gives the existence of $T_k$ in $[0, +\infty]$ and a unique maximal solution $u_k$ of $(NS_{v,k})$ in $C^\infty([0, T_k]; \mathcal{H}_k)$. To prove that $T_k = +\infty$, let us use the energy estimate

$$\frac{1}{2} \frac{d}{dt} \|u_k(t)\|^2_{L^2} = -\nu \|\nabla u_k(t)\|_{L^2}^2 + (F_k(u_k)|u_k)_{L^2} + (f_k(t)|u_k(t))_{L^2}.$$  

Because of Lemma 3.2.1, we get

$$\frac{1}{2} \frac{d}{dt} \|u_k(t)\|^2_{\mathcal{H}} = -\nu \|\nabla u_k(t)\|_{L^2}^2 + (f_k(t)|u_k(t))_{L^2} = -\nu \|u_k(t)\|^2_{\mathcal{H}_k} + (f_k(t), u_k(t))_{\mathcal{H}_k \times \mathcal{H}_k}.$$  

By integration in time, we infer that

$$\frac{1}{2} \|u_k(t)\|^2_{\mathcal{H}} + \nu \int_0^t \|u_k(t')\|^2_{\mathcal{H}_k} dt' = \frac{1}{2} \|u_k(0)\|^2_{\mathcal{H}} + \int_0^t (f_k(t'), u_k(t'))_{\mathcal{H}_k \times \mathcal{H}_k} dt'.$$  

(3.5)

Using (3.2) and the (well known) fact that $2ab \leq a^2 + b^2$, we get

$$\|u_k(t)\|^2_{\mathcal{H}} + \nu \int_0^t \|u_k(t')\|^2_{\mathcal{H}_k} dt' \leq \|u_k(0)\|^2_{\mathcal{H}} + \nu \int_0^t \|f_k(t')\|^2_{\mathcal{H}_k} dt'.$$  

(3.6)

Using Corollary 1.2.1 page 11, we get that, for all $k$, $T_k$ is infinite and the above estimate is valid for any time $t$.

Now let us write the ordinary differential equation on $\mathcal{H}_k$ in terms of Definition 3.1.1. For any function $\Psi$ in $C^1(\mathbb{R}^+; \mathcal{V}_\sigma)$, the function

$$t \mapsto (u_k(t), \Psi(t))_{\mathcal{H}_k} = (u_k(t), \Psi(t))_{\mathcal{H}_k \times \mathcal{H}_k}$$

is a $C^1$ function and we have

$$\frac{d}{dt} (u_k(t), \Psi(t))_{\mathcal{H}_k} = \left(\frac{du_k}{dt}, \Psi(t)\right)_{\mathcal{H}_k \times \mathcal{H}_k} + \left(u_k(t), \frac{d\Psi}{dt}\right)_{\mathcal{H}_k \times \mathcal{H}_k}.$$  

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Thanks to \((NS_{\nu,k})\), we get
\[
\frac{d}{dt}\langle u_k(t), \Psi(t) \rangle_{\mathcal{V}^\prime \times \mathcal{V}} = \langle \nu \Delta u_k(t) + F_k(u(t)) + f_k(t), \Psi(t) \rangle_{\mathcal{V}^\prime \times \mathcal{V}} + \langle u_k(t), \frac{d\Psi}{dt} \rangle_{\mathcal{V}^\prime \times \mathcal{V}}
\]
\[
= \langle \nu \Delta \Psi(t) + \partial_t \Psi(t), u_k(t) \rangle_{\mathcal{V}^\prime \times \mathcal{V}}
\]
\[
+ \langle u_k(t') \otimes u_k(t') | \nabla \mathbb{P}_k \Psi(t') \rangle_{L^2} + \langle f_k(t'), \Psi(t') \rangle_{\mathcal{V}^\prime \times \mathcal{V}}.
\]
By time integration, this gives,
\[
\langle u_k(t), \Psi(t) \rangle_{\mathcal{V}^\prime \times \mathcal{V}} = (\mathbb{P}_k u_0 | \Psi(0))_\mathcal{H} + \int_0^t \langle \nu \Delta \Psi(t') + \partial_t \Psi(t'), u_k(t') \rangle_{\mathcal{V}^\prime \times \mathcal{V}} dt'
\]
\[
= - \int_0^t (u_k(t') \otimes u_k(t') | \nabla \mathbb{P}_k \Psi(t') \rangle_{L^2} dt' + \int_0^t \langle f_k(t'), \Psi(t') \rangle_{\mathcal{V}^\prime \times \mathcal{V}} dt'.
\] (3.7)
The problem consists now to pass to the limit in the above formula.

**The relative compactness of the family \((u_k)_{k \in \mathbb{N}}**

This is described by the following proposition.

**Proposition 3.2.1** It exists an extraction \(\phi\) and a function \(u\) belonging to the space
\[
C(\mathbb{R}^+; \mathcal{V}^\prime) \cap L^\infty_{\text{loc}}(\mathbb{R}^+; \mathcal{H}) \cap L^2_{\text{loc}}(\mathbb{R}^+; \mathcal{V})
\]
such that for any positive real number \(T\),
\[
\forall \psi \in L^2([0,T]; \mathcal{V}), \quad \lim_{k \to \infty} \int_0^T (u_{\phi(k)}(t) | \psi(t))_{\mathcal{V}} dt = \int_0^T (u(t) | \psi(t))_{\mathcal{V}} dt,
\]
\[
\lim_{k \to \infty} \|u_{\phi(k)}(t) - u\|_{L^\infty([0,T]; \mathcal{V}^\prime)} = 0 \quad \text{and} \quad \lim_{k \to \infty} \|u_{\phi(k)}(t) - u\|_{L^2([0,T]; \mathcal{V})} = 0.
\]

**Proof.** The key point is the proof of the following lemma.

**Lemma 3.2.2** For any positive time \(T\), the set \(U(T) \overset{\text{def}}{=} \{u_k|_{[0,T]}, \ k \in \mathbb{N}\}\) is relatively compact in the set of continuous fonction on \([0,T]\) with value in \(\mathcal{V}^\prime\).

**Proof.** We proof that for any positive \(\varepsilon\), the set \(U(T)\) can be covered by a finite number of ball of radius \(\varepsilon\). In order to do it, we use Theorem 2.2.1 which implies that for any couple \((k_0, k)\) of positive integers such that \(k\) is greater than or equal to \(k_0\), for any non negative real number \(t\), we get
\[
\|u_k(t) - \mathbb{P}_{k_0} u_k(t)\|_{\mathcal{V}^\prime}^2 = \sum_{j \geq k_0+1} \lambda_j^{-2} \langle u_k(t), e_j \rangle^2.
\]
Using that the sequence \((\lambda_j)_j\) is non decreasing, we get, by Theorem 2.2.2,
\[
\|u_k(t) - \mathbb{P}_{k_0} u_k(t)\|_{\mathcal{V}^\prime}^2 \leq \lambda_{k_0+1}^{-2} \sum_j \lambda_j^2 \langle u_k(t), e_j \rangle^2 \leq \lambda_{k_0+1}^{-2} \|u_k\|_{L^\infty([0,T]; \mathcal{H})}^2.
\]
As \(\lim_{j \to \infty} \lambda_j = +\infty\) and as \((u_k)_{k \in \mathbb{N}}\) is a bounded sequence of \(L^\infty([0,T]; \mathcal{H})\), we get that
\[
\forall \varepsilon, \ \exists k_0 / \forall k, \ \|u_k - \mathbb{P}_{k_0} u_k\|_{L^\infty([0,T]; \mathcal{V}^\prime)} < \frac{\varepsilon}{2}.
\] (3.8)
Now, let us prove that the sequence \((\mathbb{P}_{k_0} u_k)_{k \in \mathbb{N}}\) is relatively compact in \(C([0, T]; \mathcal{H}_{k_0})\). As all the norm are equivalent on the finite dimensional vector space \(\mathcal{H}_{k_0}\), let us consider \(\mathcal{H}_{k_0}\) equipped with the norm \(\|\cdot\|_{\mathcal{V}_\sigma'}\). It turns out that
\[
\left\| \mathbb{P}_{k_0} \frac{du_k}{dt}(t) \right\|_{\mathcal{V}_\sigma'} \leq \|u_k(t)\|_{\mathcal{V}_\sigma'} + \|u_k(t)\|_{\mathcal{H}}^{2-\frac{4}{d}} \|u_k(t)\|_{\mathcal{H}}^{\frac{4}{d}}.
\]
Using the energy estimate (3.6), we infer that \(\left(\mathbb{P}_{k_0} \frac{du_k}{dt}\right)_{k \in \mathbb{N}}\) is a bounded sequence of the space \(L^{\frac{4}{2}}([0, T]; \mathcal{V}_\sigma')\) which means that
\[
\forall k \in \mathbb{N}, \quad \left\| \mathbb{P}_{k_0} \frac{du_k}{dt} \right\|_{L^{\frac{4}{2}}([0, T]; \mathcal{V}_\sigma')} \leq T^{1-\frac{4}{2}} \|u_0\|_{\mathcal{H}} + C \|u_0\|_{\mathcal{H}}^2.
\]
This implies that, for any fonction \(u\) in \(C([0, T]; \mathcal{H}_{k_0})\) and any non negative integer \(k\), the weak convergence in \(L^2([0, T]; \mathcal{V}_\sigma)\) implies that
\[
\lim_{n \to \infty} \int_0^T \alpha(t) \lambda_k^2(u_n(t), e_k) dt = \int_0^T \alpha(t) \lambda_k^2(u(t), e_k) dt.
\]
The fact that \((u_n)_{n \in \mathbb{N}}\) converges to \(u\) in \(C([0, T]; \mathcal{V}_\sigma')\) implies that
\[
\lim_{n \to \infty} \int_0^T \alpha(t) \lambda_k^{-1}(u_n(t), e_k) dt = \int_0^T \alpha(t) \lambda_k^{-1}(u(t), e_k) dt.
\]
Thus implies that, for any function \(\alpha\) in \(\mathcal{D}([0, T])\) and any non negative integer \(k\),
\[
\int_0^T \alpha(t) (u(t), e_k) dt = \int_0^T \alpha(t) (v(t), e_k) dt
\]
which gives that, for any \(t\), \((u(t), e_k) = (v(t), e_k)\) and thus \(u \equiv v\).

Now let use a diagonal process. Let us consider an increasing sequence \((T_p)_{p \in \mathbb{N}}\) of positive reals numbers tending to infinity. The above two compactness result imply the existence of a sequence of extraction \((\phi_p)_{p \in \mathbb{N}}\), of a sequence \((\widehat{u}_p)_{p \in \mathbb{N}}\) of functions such that \(\widehat{u}_p\) belongs to \(C([0, T_p]; \mathcal{V}_\sigma')\), and a sequence \((\widetilde{v}_p)_{p \in \mathbb{N}}\) such that \(\widetilde{v}\) belongs to \(L^2([0, T_p]; \mathcal{V}_\sigma)\) which satisfies, for all \(p\),
\[
\lim_{k \to \infty} \|u_{\phi_0 \cdots \phi_p(k)} - \widehat{u}_p\|_{L^\infty([0, T_p]; \mathcal{V}_\sigma')} = 0 \quad \text{and} \quad \forall \psi \in L^2([0, T_p]; \mathcal{V}_\sigma), \lim_{k \to \infty} \int_0^{T_p} (u_{\phi_0 \cdots \phi_p(k)}(t) |\psi(t)|) \mathcal{V}_\sigma dt = \int_0^{T_p} (\widehat{u}_p(t) |\psi(t)|) \mathcal{V}_\sigma dt.
\]

Continuation of the proof of Proposition 3.2.1 Now let is observe that as \((u_k)_{k \in \mathbb{N}}\) is bounded in the Hilbert space \(L^2([0, T]; \mathcal{V}_\sigma)\), it is weakly convergent in this Hilbert space to some \(v\) (up to an omitted extraction). Let us prove that \(u = v\) on \([0, T] \times \Omega\). Let us consider any function \(\alpha\) in \(\mathcal{D}([0, T])\) and any non negative integer \(k\). The weak convergence in \(L^2([0, T]; \mathcal{V}_\sigma)\) implies that
\[
\lim_{n \to \infty} \int_0^T \alpha(t) \lambda_k^2(u_n(t), e_k) dt = \int_0^T \alpha(t) \lambda_k^2(v(t), e_k) dt.
\]
Moreover, as for any \( t \), the sequence \( (u_{k}(t))_{k \in \mathbb{N}} \) is bounded in \( \mathcal{H} \), so, for any \( t \) in \( T_{p} \), \( \tilde{u}_{p}(t) \) belongs to \( \mathcal{H} \). Moreover, because of the uniqueness of the limit we have for any \( p' \) less than \( p \),
\[
\tilde{u}_{p'} = \tilde{u}_{p}[0,T_{p}] \quad \text{and} \quad \tilde{v}_{p'} = \tilde{v}_{p}[0,T_{p}].
\]
Thus we can define \( u \) and \( v \) on \( \mathbb{R}^{+} \) by
\[
u[0,T_{p}] = \tilde{u}_{p} \quad \text{and} \quad v[0,T_{p}] = \tilde{v}_{p}.
\]
As both convergence imply convergence in the sense of distributions we get that \( u = v \). Now let us define \( \phi(k) = \phi_{1} \circ \cdots \circ \phi_{k}(k) \). It is obvious that, for any \( p \),
\[
\lim_{k \to \infty} \|u_{\phi(k)} - u\|_{L^{\infty}(0,T_{p})} = 0 \quad \text{and} \quad \forall \psi \in L^{2}([0,T_{p}]; \mathcal{V}_{\sigma}), \quad \lim_{k \to \infty} \int_{0}^{T_{p}} (u_{\phi(k)}(t)|\psi(t))_{\mathcal{V}_{\sigma}} dt = \int_{0}^{T_{p}} (u(t)|\psi(t))_{\mathcal{V}_{\sigma}} dt.
\]
Let us prove the last convergence of the proposition. As \( d \leq 4 \), using Gagliardo-Nirenberg inequality (see Corollary A.2.1 page 88) we get
\[
\|u_{\phi(k)}(t) - u(t)\|_{L^{4}}^{2} \leq C\|u_{\phi(k)}(t) - u(t)\|_{L^{2}}^{2} \frac{d}{2} \| \nabla u_{\phi(k)}(t) - \nabla u(t)\|_{L^{2}}^{2} \quad (3.10)
\]
Moreover, using Cauchy Schwarz inequality, we can write that
\[
\|v\|_{L^{2}}^{2} = \sum_{j=0}^{\infty} \lambda_{j}^{-1}\langle v, e_{j} \rangle \lambda_{j} \langle v, e_{j} \rangle \leq \left( \sum_{j=0}^{\infty} \lambda_{j}^{-2} \langle v, e_{j} \rangle^{2} \right) \left( \sum_{j=0}^{\infty} \lambda_{j}^{2} \langle v, e_{j} \rangle^{2} \right)^{\frac{1}{2}} \|v\|_{\mathcal{V}_{\sigma}} \|v\|_{\mathcal{V}_{\sigma}}.
\]
Plugging this in (3.10) gives
\[
\|u_{\phi(k)}(t) - u(t)\|_{L^{4}}^{2} \leq C\|u_{\phi(k)}(t) - u(t)\|_{\mathcal{V}_{\sigma}}^{1-\frac{d}{4}} \| \nabla u_{\phi(k)}(t) - \nabla u(t)\|_{L^{2}}^{1+\frac{d}{4}}
\]
By time integration this gives
\[
\|u_{\phi(k)} - u\|_{L^{2}((0,T]; L^{4})}^{2} \leq C\|u_{\phi(k)} - u\|_{L^{\infty}((0,T]; \mathcal{V}_{\sigma})}^{1-\frac{d}{4}} \| \nabla u_{\phi(k)} - \nabla u\|_{L^{2}((0,T]; L^{2})}^{1+\frac{d}{4}}.
\]
As \( d \) is less than 4, Hölder estimate implies that
\[
\|u_{\phi(k)} - u\|_{L^{2}((0,T]; L^{4})} \leq CT^{1-\frac{d}{2}} \|u_{\phi(k)} - u\|_{L^{\infty}((0,T]; \mathcal{V}_{\sigma})}^{1-\frac{d}{4}} \| \nabla u_{\phi(k)} - \nabla u\|_{L^{2}((0,T]; L^{2})}^{1+\frac{d}{4}}.
\]
As \( (u_{k})_{k \in \mathbb{N}} \) is bounded in \( L^{2}((0,T]; \mathcal{V}_{\sigma}) \), We infer using Proposition 3.2.1 that
\[
\lim_{k \to \infty} \|u_{\phi(k)} - u\|_{L^{2}((0,T]; L^{4})} = 0.
\]
The proposition is proved. \( \square \)
Conclusion of the proof Theorem 3.1.1

The problem is to pass to the limit in Relation (3.7). We skip the notation of the extraction \( \varphi \) in that follows.

First of all, because of Proposition 3.2.1 and Relation (2.7), we have

\[
\lim_{k \to \infty} \langle u_k(t), \Psi(t) \rangle_{\mathcal{V}_r \times \mathcal{V}_r} = \langle u(t), \Psi(t) \rangle_{\mathcal{V}_r \times \mathcal{V}_r} \quad \text{and} \quad \tag{3.11}
\]

\[
\lim_{k \to \infty} \int_0^t \langle \partial_t \Psi(t') + \Delta \Psi(t'), u_k(t') \rangle_{\mathcal{V}_r \times \mathcal{V}_r} dt' = \int_0^t \langle \partial_t \Psi(t') + \Delta \Psi(t'), u(t') \rangle_{\mathcal{V}_r \times \mathcal{V}_r} dt'. \tag{3.12}
\]

Now the main problem consists in passing to the limit in the non linear term

\[
NL_k(t) \overset{\text{def}}{=} \int_0^t (u_k(t') \otimes u_k(t'))\nabla \mathbb{P}_k \Psi(t') L_2 dt'.
\]

This implies that

\[
\lim_{k \to \infty} \left( NL_k(t) - \int_0^t (u(t') \otimes u(t'))\nabla \mathbb{P}_k \Psi(t') L_2 dt' \right) = 0. \tag{3.13}
\]

Let us observe that, for any \( d \leq 4 \), \( u \otimes u \) belongs to \( L^1_{\text{loc}}(\mathbb{R}^+; L^2) \). Now let us observe that, for have function \( a \) in \( L^2 \) and any function \( b \) in \( H_0^1 \), we have

\[
\int_{\Omega} a(x)\partial_j b(x) dx = -\langle \partial_j a, b \rangle_{H^{-1} \times H^1_0}.
\]

Thus, for almost every \( t' \), we have

\[
\langle u(t') \otimes u(t') \rangle_{\mathcal{V}_r \times \mathcal{V}_r} = \langle \text{div}(u(t') \otimes u(t')), \mathbb{P}_k \Psi(t') \rangle_{\mathcal{V}_r \times \mathcal{V}_r}.
\]

For any \( t' \), we have

\[
\lim_{k \to \infty} \| \mathbb{P}_k \Psi(t') - \Psi(t') \|_{\mathcal{V}_r} = 0 \quad \text{and} \quad \left| \langle \text{div}(u(t') \otimes u(t')), \mathbb{P}_k \Psi(t') \rangle_{\mathcal{V}_r \times \mathcal{V}_r} \right| \leq \| u(t') \|_{L^2_r}^2 \| \Psi(t') \|_{\mathcal{V}_r}.
\]

Lebesgue theorem implies that

\[
\lim_{k \to \infty} \int_0^t (u(t') \otimes u(t'))\nabla \mathbb{P}_k \Psi(t') L_2 dt' = \int_0^t (u(t') \otimes u(t'))\nabla \Psi(t') L_2 dt'.
\]

Then using (3.11)–(3.13) and observing that \( \mathbb{P}_k u_0 \) tends to \( u_0 \) in \( \mathcal{H} \), we infer that

\[
\langle u(t), \Psi(t) \rangle = \langle u_0(t), \Psi(t) \rangle_{\mathcal{H}} - \int_0^t \langle \partial_t \Psi(t') + \Delta \Psi(t'), u(t') \rangle_{\mathcal{V}_r \times \mathcal{V}_r} dt' \tag{3.14}
\]

\[
= -\int_0^t (u(t') \otimes u(t'))\nabla \Psi(t') L_2 dt' + \int_0^t \langle f(t'), \Psi(t') \rangle_{\mathcal{V}_r \times \mathcal{V}_r} dt' \tag{3.14}
\]

which claims that \( u \) is a (weak) solution of the incompressible Navier-Stokes system.

It remains to prove the energy inequality (3.1). Let us start from Relation (3.5) which can be written

\[
\frac{1}{2} \| u_k(t) \|_{L^2_r}^2 + \nu \int_0^t \| \nabla u_k(t') \|_{L^2_r}^2 dt' = \frac{1}{2} \| u_k(0) \|_{L^2_r}^2 + \int_0^t \langle f_k(t'), u_k(t') \rangle dt'. \tag{3.15}
\]

Let us notice that

\[
\left| \langle f_k(t) - f(t), u_k(t) \rangle_{\mathcal{V}_r \times \mathcal{V}_r} \right| \leq \| f_k(t) - f(t) \|_{\mathcal{V}_r} \| u_k(t) \|_{\mathcal{V}_r}.
\]
As the sequence \((u_k)_{k \in \mathbb{N}}\) is bounded in \(L^2([0, T]; V_\sigma)\) and as the sequence \((f_k)_{k \in \mathbb{N}}\) tends to \(f\) in \(L^2_{loc}(\mathbb{R}^+; V'_\sigma)\), we infer that
\[
\lim_{k \to \infty} \left( \int_0^t \langle f_k(t'), u_k(t') \rangle_{V'_\sigma \times V_\sigma} dt' - \int_0^t \langle f(t'), u_k(t') \rangle_{V'_\sigma \times V_\sigma} dt' \right) = 0.
\]
As \(V_\sigma\) is dense in \(V'_\sigma\) and as \(P_k u_0\) tends to \(u_0\) in \(\mathcal{H}\), we infer that
\[
\lim_{k \to \infty} \left( \frac{1}{2} \|u_k(0)\|_{L^2}^2 + \int_0^t \langle f_k(t'), u_k(t') \rangle_{V'_\sigma \times V_\sigma} dt' \right) = \frac{1}{2} \|u(0)\|_{L^2}^2 + \int_0^t \langle f(t'), u(t') \rangle_{V'_\sigma \times V_\sigma} dt'. \tag{3.16}
\]
Proposition 3.2.1 implies in particular that for any time \(t \geq 0\) and any \(v\) in \(V_\sigma\),
\[
\lim_{k \to \infty} \langle u_k(t)|v\rangle_{\mathcal{H}} = \lim_{k \to \infty} \langle u_k(t), v \rangle_{V'_\sigma \times V_\sigma} = \langle u(t), v \rangle_{V'_\sigma \times V_\sigma} = (u(t)|v\rangle_{\mathcal{H}}.
\]
As \(V_\sigma\) is dense in \(\mathcal{H}\), we get that for any \(t \geq 0\), the sequence \((u_k(t))_{k \in \mathbb{N}}\) converges weakly towards \(u(t)\) in the Hilbert space \(\mathcal{H}\). Hence
\[
\|u(t)\|_{\mathcal{H}}^2 \leq \liminf_{k \to \infty} \|u_k(t)\|_{\mathcal{H}}^2 \quad \text{for all} \quad t \geq 0.
\]
On the other hand, \((u_k)_{k \in \mathbb{N}}\) converges weakly to \(u\) in \(L^2_{loc}([0, T]; V)\), so that for all non negative \(t\), we have
\[
\int_0^t \|\nabla u(t')\|_{L^2}^2 \, dt' \leq \liminf_{k \to \infty} \int_0^t \|\nabla u_k(t')\|_{L^2}^2 \, dt'.
\]
The use of (3.15) and of (3.16) gives energy inequality (3.1) and Leray theorem is proved.

### 3.3 Stability of Leray solutions in dimension two

In a two dimensional domain, the Leray weak solutions are unique and even stable. More precisely, we have the following theorem.

**Theorem 3.3.1** For any data \(u_0\) in \(\mathcal{H}\) and \(f\) in \(L^2_{loc}(\mathbb{R}^+; V')\), the Leray weak solution is unique. Moreover, it belongs to \(C(\mathbb{R}^+; \mathcal{H})\) and satisfies, for any \((s, t)\) such that \(0 \leq s \leq t\),
\[
\frac{1}{2} \|u(t)\|_{\mathcal{H}}^2 + \nu \int_s^t \|u(t')\|_{V_\sigma}^2 \, dt' = \frac{1}{2} \|u(s)\|_{\mathcal{H}}^2 + \int_s^t \langle f(t'), u(t') \rangle_{V'_\sigma \times V_\sigma} \, dt'. \tag{3.17}
\]
Furthermore, the Leray solutions are stable in the following sense. Let \(u\) (resp. \(v\)) be the Leray solution associated with \(u_0\) (resp. \(v_0\)) in \(\mathcal{H}\) and \(f\) (resp. \(g\)) in the space \(L^2_{loc}(\mathbb{R}^+; V')\) then,
\[
\|u - v(t)\|_{\mathcal{H}}^2 + \nu \int_0^t \|u(t') - v(t')\|_{V'_\sigma}^2 \, dt' \leq \left( \|u_0 - v_0\|_{\mathcal{H}}^2 + \frac{1}{\nu} \int_0^t \|f(t')\|_{V'_\sigma}^2 \, dt' \right) \exp \left( \frac{C\|E(t)\|_{\mathcal{H}}^2}{\nu^4} \right)
\]
with
\[
E(t) \overset{\text{def}}{=} \min \left\{ \|u_0\|_{\mathcal{H}}^2 + \frac{1}{\nu} \int_0^t \|f(t')\|_{V'_\sigma}^2 \, dt', \|v_0\|_{\mathcal{H}}^2 + \frac{1}{\nu} \int_0^t \|g(t')\|_{V'_\sigma}^2 \, dt' \right\}. \]
Proof. The main point is that because of the Gagliardo-Nirenberg inequality in dimension 2, the non linear term $Q(u, u)$ can be thought as an good external force for the linear Stokes problem. Indeed, as $u$ belongs to $L^\infty_t(L^2_x; \mathcal{H}) \cap L^{2}_t(L^2_x; \mathcal{V}_L)$, thanks to Lemma 3.2.1 page 37, the non linear term $Q(u, u)$ belongs to $L^2_t(L^2_x; \mathcal{V}')$. Thus $u$ is the solution of $(ES_v)$ with initial data $u_0$ and external force $f + Q(u, u)$. Theorem 2.3.1 immediately implies that $u$ belongs to $C(\mathbb{R}^+; \mathcal{H})$ and satisfies, for any $(s, t)$ such that $0 \leq s \leq t$,

$$
\frac{1}{2} \|u(t)\|^2_{\mathcal{H}} + \nu \int_s^t \|u(t')\|^2_{\mathcal{V}_L} \, dt' = \frac{1}{2} \|u(s)\|^2_{\mathcal{H}} + \int_s^t (f(t'), u(t'))_{\mathcal{V}_L \times \mathcal{V}_L} \, dt' + \int_s^t (Q(u(t'), u(t'))_{\mathcal{V}_L \times \mathcal{V}_L} \, dt'.
$$

Using Lemma 3.2.1, we get the energy equality (3.17).

To prove the stability, let us observe that, by difference $w \mathrel{\stackrel{\text{def}}{=}} u - v$ is the solution of $(ES_v)$ with data $u_0 - v_0$ and external force $f + Q(u, u) - Q(v, v)$. Theorem 2.3.1 implies that

$$
\|w(t)\|^2_{\mathcal{H}} + 2\nu \int_0^t \|w(t')\|^2_{\mathcal{V}_L} \, dt' = \|w(0)\|^2_{\mathcal{H}} + 2 \int_0^t \langle (f - g)(t'), w(t') \rangle_{\mathcal{V}_L \times \mathcal{V}_L} \, dt' + 2 \int_0^t \langle (Q(u, u) - Q(v, v))(t'), w(t') \rangle_{\mathcal{V}_L \times \mathcal{V}_L} \, dt'.
$$

The non linear term is estimated thanks to the following lemma.

**Lemma 3.3.1** In two dimensional domains, if $a$ and $b$ belong to $\mathcal{V}_L$, we have

$$
|\langle Q(a, a) - Q(b, b), a - b \rangle| \leq C \|\nabla(a - b)\|_{L^2}^2 \|a - b\|_{L^2} \|\nabla a\|_{L^2} \|a\|_{L^2}^2.
$$

Proof. Lemma 3.2.1 implies that the quantity $\langle Q(a, b), c \rangle$ is well defined. As

$$
\langle Q(b, b - a), b - a \rangle = 0,
$$

we have

$$
\langle Q(a, a) - Q(b, b), a - b \rangle = \langle Q(a, a) - Q(b, a), a - b \rangle = \langle Q(a - b, a), a - b \rangle.
$$

(3.18)

Using again Lemma 3.2.1, we get the result. \hfill \Box

**Continuation of the proof of Theorem 3.3.1.** Using that $2ab \leq a^2 + b^2$, we get

$$
\|w(t)\|^2_{\mathcal{H}} + \frac{3}{2} \nu \int_0^t \|w(t')\|^2_{\mathcal{V}_L} \, dt' \leq \|w(0)\|^2_{L^2} + \frac{2 \nu}{3} \int_0^t \|(f - g)(t')\|^2_{\mathcal{V}_L} \, dt'
$$

$$
+ C \int_0^t \|w(t')\|^2_{\mathcal{V}_L} \|w(t')\|_{\mathcal{V}_L} \|u(t')\|_{\mathcal{V}_L} \|u(t')\|_{\mathcal{V}_L} \, dt'.
$$

Using (with $\theta = 1/4$) the convexity inequality

$$
ab \leq \theta a^\frac{1}{2} + (1 - \theta)b^\frac{1}{2}
$$

(3.19)

we infer that

$$
\|w(t)\|^2_{\mathcal{H}} + \nu \int_0^t \|w(t')\|^2_{\mathcal{V}_L} \, dt' \leq \|w(0)\|^2_{L^2} + \frac{2 \nu}{3} \int_0^t \|(f - g)(t')\|^2_{\mathcal{V}_L} \, dt'
$$

$$
+ \frac{C}{\nu^\frac{1}{2}} \int_0^t \|w(t')\|^2_{\mathcal{H}} \|u(t')\|^2_{\mathcal{V}_L} \|u(t')\|^2_{\mathcal{V}_L} \, dt'.
$$

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Gronwall’s lemma implies that
\[ \|w(t)\|_{H}^2 + \nu \int_{0}^{t} \|w(t')\|_{V}^2 \, dt' \leq \left( \|w(0)\|_{H}^2 + \frac{2}{\nu} \int_{0}^{t} \|(f-g)(t')\|_{V}^2 \, dt' \right) \times \exp \left( \frac{C}{\nu^2} \sup_{t' \in [0,t]} \|u(t')\|_{H}^2 \int_{0}^{t} \|u(t')\|_{V}^2 \, dt' \right). \]

The energy estimate tells us that
\[ \sup_{t' \in [0,t]} \|u(t')\|_{H}^2 \int_{0}^{t} \|u(t')\|_{V}^2 \, dt' \leq \frac{1}{\nu} \left( \|u_0\|_{H}^2 + \frac{2}{\nu} \int_{0}^{t} \|f(t')\|_{V}^2 \, dt' \right)^2. \]

As \( u \) and \( v \) play the same role, the theorem is proved.

\[ \square \]

**Remarks**

- This chapter must be known.

- If you want to know more about the basis of the subject, you can read the seminal paper of J. Leray "Essai on the movement of a liquide visqueux emplissant the space, Acta Mathematica, 63, 1933, pages 193–248.

- To have a more recent review of results on incompressible Navier-Stokes, we can see the books of P. Constantin and C. Foias Navier-Stokes equations, Chicago University Press, 1988 and of P.-G. Lemarié-Rieusset, Recent developments in the Navier-Stokes problem. Chapman & Hall/CRC, Research Notes in Mathematics, 431, 2002.

- If you are interested in developments related to geophysical fluids, you can see the book of J.-Y. Chemin, B. Desjardins, I. Gallagher and E. Grenier, Mathematical Geophysics; an introduction to rotating fluids and Navier-Stokes equations, Oxford Lecture series in Mathematics and its maps, 32, Oxford University Press, 2006.
Chapter 4

Stability of Navier-Stokes equations in dimension 3

Introduction

This chapter investigates the stability of the turbulent solutions constructed in the previous chapter in the case when $\Omega$ is a bounded domain of $\mathbb{R}^3$. This question remains open in general and is one of the most challenging questions in the field of nonlinear partial differential equations.

In the first section, we establish a regularity criteria for a turbulent solution to be stable among all turbulent solutions.

The purpose of the second section is the proof of the existence of a stable solution in the case when the initial data is more regular (roughly speaking belongs to the Sobolev space $H^1$).

We obtain local in time existence of such a solution which becomes global when the initial data is small enough in a suitable sense.

4.1 A sufficient condition of 3D stability

The stability result is the following.

**Theorem 4.1.1** Let $u$ be a Leray solution associated with initial velocity $u_0$ in $\mathcal{H}$ and external force $f$ in $L^2([0,T];V')$. We assume that $u$ belongs to the space $L^4([0,T];V)$ for some positive $T$. Then $u$ is unique, belongs to $C([0,T];\mathcal{H})$ and satisfies, for any $(s,t)$ such that $0 \leq s \leq t \leq T$,

$$
\frac{1}{2}\|u(t)\|_{\mathcal{H}}^2 + \nu \int_s^t \|u(t')\|_{V_0}^2 \, dt' = \frac{1}{2}\|u(s)\|_{\mathcal{H}}^2 + \int_s^t \langle f(t'), u(t') \rangle_{V_0 \times V_0} \, dt'.
$$

(4.1)

Moreover, let $v$ be any Leray solution associated with $v_0$ in $\mathcal{H}$ and $g$ in $L^2_{loc}([0,T];V')$. Then, for all $t$ in $[0,T]$,

$$
\| (u - v)(t) \|_{\mathcal{H}}^2 + \nu \int_0^t \| u - v(t') \|_{V_0}^2 \, dt' 
\leq \left( \| u_0 - v_0 \|_{\mathcal{H}}^2 + \frac{2}{\nu} \int_0^t \| f - g \|_{V_0'} \, dt' \right) \exp \left( \frac{C}{\nu^7} \int_0^t \| u(t') \|_{V_0'} \, dt' \right).
$$

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Proof. Thanks to Lemma 3.2.1, the fact that $u$ belongs to $L^4([0, T]; V')$ implies that
\[
\|Q(u, u)\|_{L^2([0, T]; V')} \leq C\|u\|_{L^{\infty}([0, T]; L^2)}^{\frac{3}{2}}\|u\|_{L^4([0, T]; H_0^1)}^{\frac{1}{2}} \\
\leq CT^{\frac{k}{2}}\|u\|_{L^{\infty}([0, T]; L^2)}^{\frac{3}{2}}\|u\|_{L^4([0, T]; H_0^1)^k}.
\]
(4.2)
Hence the non linear term $Q(u, u)$ belongs to $L^2([0, T]; V')$. Thus, exactly as in the two dimensional case, $u$ is the solution of $(ES_{\nu})$ with initial data $u_0$ and external force $f + Q(u, u)$. Theorem 2.3.1 immediately implies that $u$ belongs to $C([0, T]; H)$ and satisfies, for any $(s, t)$ such that $0 \leq s \leq t$,
\[
\frac{1}{2}\|u(t)\|_H^2 + \nu \int_s^t \|u(t')\|_{V_s}^2 dt' = \frac{1}{2}\|u(s)\|_H^2 \\
+ \int_s^t \langle f(t'), u(t')\rangle_{V' \times V} dt' + \int_s^t \langle Q(u(t'), u(t')\rangle_{V_s \times V_s} dt'.
\]
Using Lemma 3.2.1, we get the energy equality (4.1).

As $u$ and $v$ are two Leray solutions, we can write that
\[
\delta_{\nu}(t) \overset{\text{def}}{=} \|(u - v)(t)\|_H^2 + 2\nu \int_0^t \|u(t') - v(t')\|_{V_s}^2 dt' \\
= \|u(t)\|_H^2 + 2\nu \int_0^t \|u(t')\|_{V_s}^2 dt' + \|v(t)\|_H^2 + 2\nu \int_0^t \|v(t')\|_{V_s}^2 dt' - 2P(t) \quad \text{with}
\]
\[
P(t) \overset{\text{def}}{=} \langle v(t)|u(t)\rangle_H + 2\nu \int_0^t \langle v(t')|u(t')\rangle_{V_s} dt'.
\]
This can be written
\[
\delta_{\nu}(t) \leq \|u_0\|_H^2 + \|v_0\|_H^2 - 2P(t) + 2 \int_0^t \langle g(t'), v(t')\rangle dt' + 2 \int_0^t \langle f(t'), u(t')\rangle dt'.
\]
(4.3)
The idea is to use the solution $u$ as a test function in Definition 3.1.1 page 35 in order to estimate $-2P(t)$. The following lemma makes it possible.

**Lemma 4.1.1** Let $v$ be a weak solution of $(NS)$ in the sense of Definition 3.1.1 page 35. Then for any function $\psi$ such that
\[
\partial_t \psi \in L^2_{\text{loc}}(\mathbb{R}^+; V_s'), \quad \psi \in C(\mathbb{R}^+; H) \quad \text{and} \quad \psi \in L^4_{\text{loc}}(\mathbb{R}^+; V_s)
\]
we have
\[
\langle v(t), \psi(t)\rangle_{V_s \times V_s} = \langle v_0|\psi(0)\rangle_H + \int_0^t \langle \partial_t \psi(t') + \Delta \psi(t'), v(t')\rangle_{V_s \times V_s} dt' \\
- \int_0^t \langle \text{div}(v(t') \otimes v(t')), \psi(t')\rangle_{V_s \times V_s} dt' + \int_0^t \langle g(t'), \psi(t')\rangle_{V_s \times V_s} dt'.
\]

**Proof.** It relies on density argument. If a function $\psi$ satisfies (4.4), then, by Lebesgue convergence theorem, we get, for any time $T$,
\[
\lim_{k \to \infty} \|\partial_t P_k \psi - \partial_t \psi\|_{L^2([0, T]; V_s')} = \lim_{k \to \infty} \|P_k \psi - \psi\|_{L^\infty([0, T]; H)} = \lim_{k \to \infty} \|P_k \psi - \psi\|_{L^4([0, T]; V_s)} = 0.
\]

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Then the following time regularization

\[ \psi_{\varepsilon,k} \overset{\text{def}}{=} \frac{1}{\varepsilon} \chi\left( \frac{\cdot}{\varepsilon} \right) \ast 1_{[0,T]} \mathbb{P}_{k} \psi \]

approximate \( \psi \) by \( C^1 \) functions with value in \( \mathcal{V}_\sigma \). Then, we have to check that we can pass to the limit in all terms of the definition of weak solution. All the terms are obvious except the term

\[ \int_{0}^{t} \langle \text{div}(v(t') \otimes v(t')), \Psi(t') \rangle_{\mathcal{V}_\sigma \times \mathcal{V}_\sigma} dt'. \]

From Gagliardo–Nirenberg’s inequality stated in Corollary A.2.1 page 88, we get that \( v \) belongs \( L_{loc}^{\infty}(\mathbb{R}^+, L^4) \) and thus \( v \otimes v \) belongs to \( L_{loc}^{\infty}(\mathbb{R}^+; L^2) \). As we have convergence of the family \( \psi_{\varepsilon,k} \) in \( L_{loc}^{\infty}(\mathbb{R}^+; \mathcal{V}_\sigma) \), we can pass to the limit. \( \square \)

**Continuation of the proof of Theorem 4.1.1** The idea is to use \( u \) as a test function for the (weak) solution \( v \). This gives

\[ (v(t)|u(t))_{H} = (v_{0}|u_{0})_{H} + \int_{0}^{t} \left( -\nu(v(t)|u(t))_{\mathcal{V}_\sigma} + \langle \partial_{t}u(t'), v(t') \rangle_{\mathcal{V}_\sigma \times \mathcal{V}_\sigma} \right) dt' \]

\[ - \int_{0}^{t} \langle \text{div} v \otimes v, u(t') \rangle dt' + \int_{0}^{t} \langle g(t'), u(t') \rangle dt'. \]

The fact that \( u \) is a solution of Navier-Stokes with initial data \( u_{0} \) and external force \( f \) means exactly that, for any \( t' \),

\[ \partial_{t}u(t') = \nu \Delta u(t') - \text{div} u \otimes u(t') + f(t') \quad \text{in the space} \quad L_{loc}^{2}(\mathbb{R}^+; \mathcal{V}_\sigma'). \]

As for almost every \( t' \), \( v(t') \) belongs to \( \mathcal{V}_\sigma \), we infer that

\[ \int_{0}^{t} \langle \partial_{t}u(t'), v(t') \rangle_{\mathcal{V}_\sigma \times \mathcal{V}_\sigma} dt' = \int_{0}^{t} \langle \nu \Delta u(t') - \text{div} (u(t') \otimes u(t')) + f(t'), v(t') \rangle_{\mathcal{V}_\sigma \times \mathcal{V}_\sigma} dt'. \]

Moreover, for almost every \( t' \), we have \( (v(t')|u(t'))_{\mathcal{V}_\sigma} = -\langle \Delta u(t'), v(t') \rangle \). This gives

\[ (v(t)|u(t))_{L^2} = (v_{0}|u_{0})_{L^2} - 2 \int_{0}^{t} (v(t')|u(t'))_{\mathcal{V}_\sigma} dt' \]

\[ - \int_{0}^{t} \langle \nu \langle \text{div} v \otimes v, u \rangle + \langle \text{div} u \otimes u, v \rangle \rangle dt' + \int_{0}^{t} \langle (f(t'), v(t')) + \langle g(t'), u(t') \rangle \rangle dt'. \]

Using (4.3), this gives

\[ \delta_{\nu}(t) \leq \|u_{0} - v_{0}\|_{L^2}^{2} - \int_{0}^{t} \left( \langle \text{div} v \otimes v, u \rangle + \langle \text{div} u \otimes u, v \rangle \right) dt' \]

\[ + 2 \int_{0}^{t} \langle (f - g)(t'), (u - v)(t') \rangle dt'. \]

(4.5)

As \( \langle \text{div} v \otimes v, v \rangle = \langle \text{div} u \otimes u, u \rangle = 0 \), we have

\[ \langle \text{div} v \otimes v, u \rangle + \langle \text{div} u \otimes u, v \rangle = \langle \text{div} v \otimes v - u \otimes u, u - v \rangle \]

\[ = \langle \text{div} v \otimes (v + u - v) - u \otimes u, u - v \rangle \]

\[ = \langle \text{div} (v - u \otimes u, u - v) \rangle. \]

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Using the divergence free condition, we
\[ \langle \text{div} \, v \otimes v, u \rangle + \langle \text{div} \, u \otimes u, v \rangle \leq \| u - v \|_{L^4}^2 \| \nabla u \|_{L^2}. \]

Using Gagliardo–Nirenberg’s inequality stated in Corollary A.2.1 page 88 and the convexity inequality (3.19) page 44, we infer
\[ \langle \text{div} \, v \otimes v, u \rangle + \langle \text{div} \, u \otimes u, v \rangle \leq \frac{1}{2} \nu \| \nabla u \|_{L^2}^2 + \frac{C}{L^3} \| \nabla u(t) \|_{L^2}^2 \| u - v \|_{L^2}^2. \]

Moreover, we have
\[ \langle (f - g)(t'), (u - v)(t') \rangle \leq \frac{1}{2} \nu \| \nabla (u - v)(t') \|_{L^2}^2 + \frac{1}{2\nu} \| f(t') - g(t') \|_{V^2}^2. \]

Plugging the above two inequalities in (4.5) gives
\[ \| (u - v)(t) \|_{L^2}^2 + \nu \int_0^t \| \nabla (u - v)(t') \|_{L^2}^2 \, dt' \leq \| u_0 - v_0 \|_{L^2}^2 \]
\[ + \frac{C}{L^3} \int_0^t \| \nabla u(t') \|_{L^2}^2 \| (u - v)(t') \|_{L^2}^2 \, dt' + \frac{1}{2\nu} \int_0^t \| f(t') - g(t') \|_{V^2}^2 \, dt'. \]

Gronwall lemma gives the result.

\[ \square \]

4.2 Existence of stable solutions in a bounded domain

The purpose of this section is the proof of the existence of solutions of the system \((\text{NS}_\nu)\) which are \(L^4\) in time with values in \(V_\sigma\). In order to state (and prove) a sharp theorem, we shall introduce intermediate spaces between the spaces \(V'_\sigma\) and \(V_\sigma\). Then, we shall prove a global existence theorem for small data and then a local in time theorem for large data.

4.2.1 Intermediate spaces

We shall define a family of intermediate spaces between the spaces \(V'_\sigma\) and \(V_\sigma\). This can be done by abstract interpolation theory but we prefer to do it here in an explicit way.

Definition 4.2.1 Let \( s \) be in \([-1, 1]\). We shall denote by \( V^s_\sigma \) the space of vector fields \( u \) in \( V'\) such that
\[ \| u \|_{V^s_\sigma}^2 \overset{\text{def}}{=} \sum_{j \in \mathbb{N}} \mu_j^{2s} \langle u, e_j \rangle^2 < +\infty. \]

Theorem 2.2.2 implies that \( V^0_\sigma = H \) and \( V^1_\sigma = V_\sigma \). Moreover, it is obvious that, when \( s \) is non negative, \( V^s_\sigma \) endowed with the norm \( \| \cdot \|_{V^s_\sigma} \) is a Hilbert space.

The following proposition will be important in the following two paragraphs.

Proposition 4.2.1 The space \( V^{\frac{1}{2}}_\sigma \) is embedded in \( L^3 \) and the space \( \mathbb{P} \, L^{\frac{1}{2}} \) is embedded in \( V^{-\frac{1}{2}}_\sigma \). Moreover precisely, we have
\[ \| a \|_{L^3} \lesssim \| a \|_{V^{\frac{1}{2}}_\sigma} \quad \text{and} \quad \| \mathbb{P} \, a \|_{V^{-\frac{1}{2}}_\sigma} \lesssim \| a \|_{L^{\frac{1}{2}}}. \]
Proof. This proposition can be proved using abstract interpolation theory. We prefer to present here a self contained proof in the spirit of the proof of Theorem A.2.1. Let us consider \( a \) in \( \mathcal{V}^\frac{1}{2} \). Without any loss of generality, we can assume that \( \|a\|_{\mathcal{V}^\frac{1}{2}} \leq 1 \). Let us define, for a positive real number \( \nu \),

\[
a_\nu \overset{\text{def}}{=} \sum_{j / \mu_j < \nu} \langle a, e_j \rangle e_j \quad \text{and} \quad b_\nu \overset{\text{def}}{=} a - a_\nu.
\]

Using the fact that \( \{ x \in \Omega / |a(x)| > \Lambda \} \subset \{ x \in \Omega / |a_\Lambda(x)| > \Lambda/2 \} \cup \{ x \in \Omega / |b_\Lambda(x)| > \Lambda/2 \} \), we can write

\[
\|a\|^3_{L^3} \leq 3 \int_0^{+\infty} \Lambda^2 \text{meas}\{ x \in \Omega / |a_\Lambda(x)| > \Lambda/2 \} \, d\Lambda
\]

\[
+ 3 \int_0^{+\infty} \Lambda^2 \text{meas}\{ x \in \Omega / |b_\Lambda(x)| > \Lambda/2 \} \, d\Lambda
\]

\[
\leq 3 \times 2^6 \int_0^{+\infty} \Lambda^{-4} \|a_\Lambda\|^6_{\nu} \, d\Lambda + 3 \times 2^2 \int_0^{+\infty} \|b_\Lambda\|^2_{L^2} \, d\Lambda. \tag{4.1}
\]

Thanks to Theorem A.2.1, we have, by definition of the \( \| \cdot \|_{\mathcal{V}^\nu} \) norm,

\[
\|a_\Lambda\|^2_{\nu} \leq C \|a\|^2_{\mathcal{V}_{\nu}}
\]

\[
\leq C \sum_{j / \mu_j < \Lambda} \mu_j^2 \langle a, e_j \rangle^2
\]

\[
\leq C \Lambda \sum_{j / \mu_j < \Lambda} \mu_j \langle a, e_j \rangle^2 \leq C \Lambda.
\]

Plugging this estimate into (4.1) gives

\[
\|a\|^3_{L^3} \leq C \int_0^{+\infty} \Lambda^{-2} \|a_\Lambda\|^2_{\nu} \, d\Lambda + C \int_0^{+\infty} \|b_\Lambda\|^2_{L^2} \, d\Lambda.
\]

Using Theorem 2.2.2 and the definition of \( a_\Lambda \) gives

\[
\|a\|^3_{L^3} \leq \sum_{j \in \mathbb{N}} \int_{\mu_j}^{+\infty} \Lambda^{-2} \mu_j^2 \langle a, e_j \rangle^2 \, d\Lambda + C \sum_{j \in \mathbb{N}} \int_0^{\mu_j} \langle a, e_j \rangle^2 \, d\Lambda
\]

\[
\leq C \sum_{j \in \mathbb{N}} \mu_j \langle a, e_j \rangle^2
\]

\[
\leq C.
\]

This proves the first part of the proposition.

The second part is obtained by a duality argument. By definition, we have, for any \( a \) in \( \mathcal{V}' \),

\[
\| \mathbb{P} a \|_{\mathcal{V}_{\nu}^{-\frac{1}{2}}} = \| (\mu_j^{-\frac{1}{2}} \langle a, e_j \rangle)_{j \in \mathbb{N}} \|_2
\]

\[
= \sup_{\alpha \in \mathcal{B}} \sum_{j \in \mathbb{N}} \alpha_j \mu_j^{-\frac{1}{2}} \langle a, e_j \rangle \tag{4.2}
\]

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where $\mathcal{B}$ denotes the sequence which are 0 outside a finite set and of $\ell^2(\mathbb{N})$ norm of size less than 1. Let us write that, for any $\alpha$ in $\mathcal{B}$, for $N$ large enough, we have

$$\sum_{j=0}^{N} \mu_j^{-\frac{3}{2}} \alpha_j \langle a, e_j \rangle = \langle a, \sum_{j=0}^{N} \alpha_j \mu_j^{-\frac{3}{2}} e_j \rangle$$  \hfill (4.3)

As $a$ is in $L^q(\Omega)$ which is included in $\mathcal{V}'$, we have, for any $\phi$ in $\mathcal{V}_\sigma$ because $\varphi$ is in $L^3$,

$$\langle a, \varphi \rangle = \int_{\Omega} a(x) \cdot \varphi(x) \, dx.$$

Applying Hölder inequality in (4.3) gives

$$\left| \sum_{j=0}^{N} \mu_j^{-\frac{3}{2}} \alpha_j \langle a, e_j \rangle \right| \leq \|a\|_{L^3}^\frac{1}{2} \left\| \sum_{j=0}^{N} \mu_j^{-\frac{3}{2}} \alpha_j e_j \right\|_{L^3}.$$

The embedding of $\mathcal{V}_{\sigma}^\frac{1}{2}$ into $L^3$ and the definition of the $\mathcal{V}_\sigma$ norm implies

$$\left| \sum_{j=0}^{N} \mu_j^{-\frac{3}{2}} \alpha_j \langle a, e_j \rangle \right| \leq C \|a\|_{L^3} \left\| \sum_{j=0}^{N} \mu_j^{-\frac{3}{2}} \alpha_j e_j \right\|_{\mathcal{V}_{\sigma}^\frac{1}{2}} \leq C \|a\|_{L^3} \|a\|_{\ell^2} \leq C \|a\|_{L^3}.$$  

This completes the proof of Proposition 4.2.1. \hfill $\Box$

### 4.2.2 The wellposedness result in $\mathcal{V}_{\sigma}^\frac{1}{2}$

The aim of this paragraph is the proof of the following existence theorem with data in $\mathcal{V}_{\sigma}^\frac{1}{2}$.

**Theorem 4.2.1** If the initial data $u_0$ belongs to $\mathcal{V}_{\sigma}^\frac{1}{2}$ and the external force $f$ belongs to the space $L^2_{loc}(\mathbb{R}^+; \mathcal{V}_{\sigma}^{-\frac{1}{2}})$, then a positive time $T$ exists such that a solution $u$ of $(NS, \nu)$ exists in $L^4([0,T]; \mathcal{V}_{\sigma})$. This solution is unique and belongs to $C([0,T]; \mathcal{V}_{\sigma}^\frac{1}{2})$.

Moreover, a constant $c$ exists (which can be chosen independent of the domain $\Omega$) such that, if

$$\|u_0\|_{\mathcal{V}_{\sigma}^\frac{1}{2}} + \frac{1}{\nu} \|f\|_{L^2(\mathbb{R}^+; \mathcal{V}_{\sigma}^{-\frac{1}{2}})} \leq c \nu,$$

then the above solution is global.

**Proof.** For the sake of simplicity, we shall ignore the external force in the proof. Let us observe that the map

$$Q \left\{ \begin{array}{ccc} \mathcal{V}_\sigma \times \mathcal{V}_\sigma & \longrightarrow & \mathcal{V}_\sigma^{-\frac{1}{2}} \\ (u,v) & \longmapsto & \mathcal{P} \text{div}(u \otimes v) \end{array} \right. $$

is a bilinear continuous map. Indeed, as $u$ is divergence free, we can write

$$\text{div}(u \otimes v) = \sum_{k=1}^{3} u^k \partial_k v^\ell.$$
Thus, for any $\omega$, we get
\[ \| \text{div}(u \otimes v) \|_{L^\frac{3}{2}} \lesssim \|u\|_{L^6} \|\nabla v\|_{L^2}. \]

Sobolev embedding $V_\sigma \hookrightarrow L^6$ and dual Sobolev embedding of Proposition 4.2.1 implies that
\[ \| \mathcal{P} \text{div}(u \otimes v) \|_{V_\sigma^{-\frac{1}{2}}} \lesssim \|u\|_{V_\sigma} \|v\|_{V_\sigma}. \tag{4.4} \]

Now let us define the following bilinear operator
\[ B \left\{ \begin{array}{ccc} L^4([0,T];V_\sigma) \times L^4([0,T];V_\sigma) & \longrightarrow & L^4([0,T];V_\sigma) \\ (u,v) & \longmapsto & B(u,v) \end{array} \right. \]

where $B(u,v)$ is the solution of the linear Stokes problem with initial data $0$ and external force $\frac{3}{2}(Q(u,v) + Q(v,u))$. Using (4.4), it turns out that
\[ \| \mathcal{P}(Q(u,v) + Q(v,u)) \|_{L^2([0,T];V_\sigma^{-\frac{1}{2}})} \lesssim \|u\|_{L^4([0,T];V_\sigma)} \|v\|_{L^4([0,T];V_\sigma)}. \]

As $V_\sigma^{-\frac{1}{2}}$ is by construction a subspace of $V_\sigma$, the term $Q(u,v) + Q(v,u)$ can be considered as an external force for the Stokes problem of evolution. Thus the bilinear operator $B$ is well defined. The fact that it maps continuously $L^4([0,T];V_\sigma) \times L^4([0,T];V_\sigma)$ into $L^4([0,T];V_\sigma)$ will be a consequence of the following lemma.

**Lemma 4.2.1** Let us consider $f$ in $L^2([0,T];V_\sigma^{-\frac{1}{2}})$. A constant $C$ exists, such that, for any $p$ in $[4,\infty)$, the solution $\mathcal{L}f$ of the Stokes problem with external force $f$ and initial data $0$ satisfies
\[ \sum_j \mu_j^{1+\frac{1}{p}} \| \langle \mathcal{L}f(t), e_k \rangle \|_{L^p([0,T])}^2 \lesssim \|f\|_{L^2([0,T];V_\sigma^{-\frac{1}{2}})}^2. \]

**Proof.** Using Formula (2.15) page 33, we get
\[ \langle \mathcal{L}f(t), e_j \rangle = \int_0^t e^{-\nu \mu_j^2(t-t')} \langle f(t'), e_j \rangle \, dt'. \]

Using the Young’s inequality, we get
\[ \| \langle \mathcal{L}f(t), e_k \rangle \|_{L^p([0,T])} \leq \frac{1}{\nu^{\frac{1}{2}} \mu_j^{-\frac{3}{4}}} \| \langle f(t), e_j \rangle \|_{L^2([0,T])}. \]

By definition of the $V_\sigma^{-\frac{1}{2}}$ norm, we get the result. \qed

**Continuation of the proof of Theorem 4.2.1** Let us denote by $S(t)u_0$ the solution of the linear Stokes problem with initial data $u_0$ and external force $0$. Using again Formula (2.15) page 33, we get
\[ \langle S(t)u_0, e_j \rangle = \langle u_0, e_j \rangle e^{-\nu \mu_j^2 t}. \]

Thus, for any $p$ in $[4,\infty)$, we get
\[ \sum_j \mu_j^{1+\frac{1}{p}} \| \langle S(t)u_0(t), e_j \rangle \|_{L^p([0,T])}^2 \lesssim \frac{1}{\nu^{\frac{1}{p}}} \sum_j \mu_j \langle u_0, e_j \rangle^2. \tag{4.5} \]
The idea of the Kato theory is the following: \( u \) is a solution of incompressible Navier-Stokes equation in the space \( L^4([0,T];V_v) \) if and only if \( u \) satisfies

\[
 u = S(t)u_0 + B(u, u).
\]

In other terms, \( u \) is a fixed point of the map

\[
 u \mapsto S(t)u_0 + B(u, u).
\]

Now let us observe that, thanks to the Cauchy–Schwarz inequality, for any \( a \) in \( \ell^2(L^4[0,T]) \),

\[
 \int_0^T \|a_j(t)\|_{L^2(N)}^2 dt = \int_0^T \left( \sum_{j \in N} a_j^2(t) \right)^2 dt
 = \sum_{j \in N, k \in N} \int_0^T a_j^2(t)a_k^2(t) dt
 \leq \sum_{j \in N, k \in N} \|a_j\|_{L^4([0,T])}^2 \|a_k\|_{L^4([0,T])}^2
 \leq \left\| \left( \|a_j\|_{L^4([0,T])} \right)_{j \in N} \right\|_{\ell^2}^4.
\]

Let us notice that this is a particular case of the Minkowski inequality. Then we deduce from Lemma 4.2.1 that

\[
 \|B(u, u)\|_{L^4([0,T];V_v)} \leq \frac{C}{\nu^4} \|u\|_{L^4([0,T];V_v)}^2.
\]

Using (4.5) with \( p = 4 \) and Minkowski inequality gives

\[
 \|S(t)u_0\|_{L^4([0,T];V_v)} \leq \frac{1}{\nu^4} \|u_0\|_{V_v^{1/2}}. \tag{4.6}
\]

Using Picard fixed point theorem, if we prove that \( \lim_{T \to 0} \|u_L\|_{L^4([0,T];V_v)} = 0 \) we conclude the proof of the theorem up to the continuity of \( u \). Let us observe that, for any positive \( \varepsilon \), an interger \( j_\varepsilon \) exists such that

\[
 \|\text{Id} - P_{j_\varepsilon} u_0\|_{V_v} \leq \frac{\nu^2 \varepsilon}{2} \quad \text{with} \quad P_k a \overset{\text{def}}{=} \sum_{k' \leq k} \langle a, e_k' \rangle e_k'.
\]

Then Inequality (4.6) implies that

\[
 \|S(t)P_{j_\varepsilon} u_0\|_{L^4([0,T];V_v)} \leq \frac{\varepsilon}{2}. \tag{4.7}
\]

Then we have

\[
 \|S(t)P_{j_\varepsilon} u_0\|_{L^4([0,T];V_v)} \leq \frac{T^{1/2}}{\mu_{j_\varepsilon}^{1/2}} \|S(t)P_{j_\varepsilon} u_0\|_{L^\infty([0,T];V_v)}
 \leq \frac{T^{1/2}}{\mu_{j_\varepsilon}^{1/2}} \|u_0\|_{V_v^{1/2}}.
\]

Together with (4.7), this implies that

\[
 \lim_{T \to 0} \|u_L\|_{L^4([0,T];V_v)} = 0.
\]
In order to prove the continuity in time with value in $V^\frac{1}{2}$ of $u$, let us observe that, as 
\[ \partial_t u = \Delta u + P \text{div} (u \otimes u), \]
we have that $\partial_t u$ belongs to 
\[ L^4([0, T]; V_\sigma') + L^2([0, T]; V_\sigma^{-\frac{1}{2}}) \hookrightarrow L^2([0, T]; V'). \]
Thus $u$ is continuous with value in $V'_\sigma$. Using Lemma 4.2.1 and Inequality (4.2.1) we infer that 
\[ \sum_j \mu_j \| \langle u(t), e_j \rangle \|_{L^\infty([0, T])}^2 < \infty \]
Thus for any positive real number $\varepsilon$, an integer $j_\varepsilon$ exists such that 
\[ \sum_{j > j_\varepsilon} \mu_j \| \langle u(t), e_j \rangle \|_{L^\infty([0, T])}^2 < \frac{\varepsilon^2}{4}. \]
Now, it turns out that for all $(t_1, t_2)$ in $[0, T]^2$, one has 
\[
\| u(t_1) - u(t_2) \|_{V^\frac{1}{2}} \leq \left( \sum_{j > j_\varepsilon} \mu_j \| \langle u(t), e_j \rangle \|_{L^\infty([0, T])} \right)^{\frac{1}{2}} + \left( \sum_{j \leq j_\varepsilon} \mu_j \| u(t_1) - u(t_2), e_j \|_{L^2([0, T])} \right)^{\frac{1}{2}} \\
\leq \frac{\varepsilon}{2} + \mu_{j_\varepsilon} \| u(t_1) - u(t_2) \|_{V'_\sigma}.
\]
As $u$ is continuous in time with value in $V'_\sigma$, the whole Theorem 4.2.1 is proved. \hfill \square

### 4.2.3 Some remarks about stable solutions

In this paragraph, we shall assume that the external force $f$ is identically 0. We shall establish some results about the maximal existence time of the solution constructed in the preceding paragraph.

**Proposition 4.2.2** Let us assume that the initial data $u_0$ belongs to $V_\sigma$. Then the maximal time of existence $T^*$ of the solution $u$ in the space $C((0, T^*]; V^\frac{1}{2}) \cap L^4_{\text{loc}}((0, T^*]; V_\sigma)$ satisfies 
\[ T^* \geq \frac{c\nu^3}{\| \nabla u_0 \|_{L^2}^4}. \]

**Proof.** Let us observe that, if $u_0$ belongs to $V_\sigma$, we have 
\[ \| S(t)u_0 \|_{L^4([0, T]; V_\sigma)} \leq T^\frac{1}{4} \| u_0 \|_{V_\sigma}, \]
This ensures the proposition. \hfill \square

From this proposition, we infer the following corollary.

**Corollary 4.2.1** Let $T^*$ be the maximal time of existence for a solution $u$ of the system $(NS_\nu)$ in the space $C((0, T^*]; V^\frac{1}{2}) \cap L^4_{\text{loc}}((0, T^*]; V_\sigma)$. If $T^*$ is finite, then 
\[ \int_0^{T^*} \| \nabla u(t) \|_{L^2}^4 dt = +\infty \quad \text{and} \quad T^* \leq \frac{c}{\nu^\beta} \| u_0 \|_{L^2}^4. \]
Proof. For almost every $t$, $u(t)$ belongs to $V_2$. Then, thanks to the above proposition, the maximal time of existence of the solution starting at time $t$, which is of course $T^* - t$, satisfies

$$T^* - t \geq \frac{c\nu^3}{\|\nabla u(t)\|_{L^2}^4}.$$ 

This can be written as

$$\|\nabla u(t)\|_{L^2}^4 \geq \frac{c\nu^3}{T^* - t}.$$ 

This gives the first part of the corollary. Taking the square root of the above inequality gives, thanks to the energy estimate,

$$c\nu^{\frac{5}{2}} \int_0^{T^*} \frac{dt}{(T^* - t)^{\frac{1}{2}}} \leq \frac{1}{2} ||u_0||_{L^2}^2.$$ 

The corollary is proved.

Remarks

- Sections 3.3 and 4.2 must be known.

Chapter 5

An example of a dispersive equation: the Schrödinger equation

Introduction

The chapter is devoted to study of the Schrödinger equation in the whole space $\mathbb{R}^d$. This equation writes

$$(LS) \quad i \partial_t u - \Delta u = 0 \quad \text{and} \quad u|_{t=0} = u_0.$$ 

and represents the free evolution of a system of quantum particles in the whole space $\mathbb{R}^d$. Here, the preserved $L^2$ norm does not represent a kinetic energy like in the preceding chapters but the total mass. Indeed if the function $u$ is a solution of $(LS)$, the quantity $|u(t, x)|^2$ represents the density of particles at time $t$ and point $x$.

The purpose of the first section is to define the concept of weak solution of $(LS)$ possibly with an external force $f$. Then we compute explicitly the solution of $(LS)$ in term of Fourier transform and also in term in convolution. In other term, we compute the fundamental solution. From this formula, we deduce a so called dispersive estimates which is a mathematical translation of the fact that free particles spread in the whole space.

The purpose of the second section is to explain the procedure of complex interpolation which allow to extend the two inequalities about an operator between $L^p$ spaces in a family of inequalities.

In the third section, we expose a general method called $TT^*$ method which allows to translate a dispersive estimate in space time estimates known as Strichartz estimates. Based on duality arguments, this proof requires also refined Young inequalities.

In the last section, we use Strichartz estimates to prove the existence ans the uniqueness of solution to a semi-linear Schrödinger equation.

5.1 The solution of some classical linear evolution PDE in $\mathbb{R}^d$

The case of Schrödinger equation follows the same lines.

Definition 5.1.1 Let $u$ be a continuous function from $\mathbb{R}$ with value in $S'(\mathbb{R}^d)$, which means exactly that for any $\phi$ in $S(\mathbb{R}^d)$, that maps defined by

$$t \mapsto \langle u(t), \phi \rangle$$
is continuous. Let \( f \) be a locally integrable map from \( \mathbb{R} \) into \( S'(\mathbb{R}^d) \), which means exactly that for any \( \phi \) in \( S(\mathbb{R}^d) \), that maps defined by

\[
t \mapsto \langle f(t), \phi \rangle
\]

is locally integrable on \( \mathbb{R} \). Let \( u_0 \) be in \( S'(\mathbb{R}^d) \). We say that \( u \) is a solution of (LS) if and only if, for any function \( \phi \) of \( S(\mathbb{R} \times \mathbb{R}^d) \), we have, for any \( t \) in \( \mathbb{R} \),

\[
\langle u(T), \phi(T) \rangle - \langle u_0, \phi(0) \rangle = \int_0^T \langle u(t), i\Delta \phi(t) + \partial_t \phi(t) \rangle dt - i \int_0^T \langle f(t), \phi(t) \rangle dt.
\]

**Proposition 5.1.1** If \( u_0 \) belongs to \( S'(\mathbb{R}^d) \) and \( f \) is locally integrable form \( \mathbb{R} \) into \( S'(\mathbb{R}^d) \), there is a unique solution of (LS) with a given by

\[
u(t) = \mathcal{F}^{-1}\left(e^{-it|\xi|^2}\hat{u}_0(\xi) - i \int_0^t e^{-i(t-t')|\xi|^2}\hat{f}(t', \xi) dt'\right).
\]  

**Proof.** Let us first prove the result using a duality method. As the equation is linear, uniqueness consists only in proving that if, for all \( \phi \) of \( S(\mathbb{R} \times \mathbb{R}^d) \), if

\[
\langle u(T), \phi(T) \rangle = \int_0^T \langle u(t), i\Delta \phi(t) + \partial_t \phi(t) \rangle dt
\]

then \( u(T) = 0 \) in \( S'(\mathbb{R}^d) \) for all time \( T \). Let us consider any function \( \phi_T \) in \( S'(\mathbb{R}^d) \). Let us observe that

\[
\phi(t) = \mathcal{F}^{-1}\left(e^{-it|\xi|^2}\hat{\phi}_T(\xi)\right)
\]

satisfies \( i\Delta \phi(t) + \partial_t \phi(t) = 0 \). Then, Identity (5.2) implies that \( \langle u(T), \phi(T) \rangle = 0 \) and the uniqueness is proved.

Let us proved that the function

\[
u(t) = \mathcal{F}^{-1}(e^{-it|\xi|^2}\hat{u}_0(\xi)).
\]

is a solution of (LS) with initial data \( u_0 \) and external force \( 0 \). In order to do it, let us compute, for any \( \Phi \) in \( C^\infty(\mathbb{R}; S(\mathbb{R}^d)) \),

\[
U(T) = -i \int_0^T \langle u(t), i\Delta \Phi(t) + \partial_t \Phi(t) \rangle dt.
\]

We have, because of the properties of the Fourier transform,

\[
U(T) = \int_0^T \langle u(t), -i\mathcal{F}(i.|\xi|^2 \Phi(t)) \rangle dt + \int_0^T \langle u(t), \partial_t \Phi(t) \rangle dt = \int_0^T \langle \hat{u}(t), -i|\xi|^2 \Phi(t) \rangle dt + \int_0^T \langle \hat{u}(t), \partial_t \Phi(t) \rangle dt.
\]

By definition of \( u \), we get

\[
U(t) = \int_0^T \langle e^{-it|\xi|^2}\hat{u}_0, -i|\xi|^2 \Phi(t) \rangle dt + \int_0^T \langle e^{-it|\xi|^2}\hat{u}_0, \partial_t \Phi(t) \rangle dt = -\langle \hat{u}_0, \int_0^T e^{-it|\xi|^2} (-i|\xi|^2 \Phi(t) + \partial_t \Phi(t)) dt \rangle dt.
\]
As we have
\[ e^{-it|\xi|^2} (-i|\xi|^2 \Phi(t, \xi) + \partial_t \Phi(t, \xi)) = \partial_t (e^{-it|\xi|^2} \Phi(t, \xi)), \] (5.3)
we infer that
\[ U(t) = \langle \hat{u}_0, \int_0^T \partial_t (e^{-it|\xi|^2} \Phi(t)) \rangle dt \]
\[ = \langle \hat{u}_0, e^{-iT|\xi|^2} \Phi(t) \rangle - \langle \hat{u}_0, \Phi(0) \rangle. \]

By definition of \( u \), this means exactly that
\[ U(T) = \langle \hat{u}(T), \Phi(t) \rangle - \langle \hat{u}_0, \Phi(0) \rangle = \langle u(T), \hat{\Phi}(t) \rangle - \langle \hat{u}_0, \hat{\Phi}(0) \rangle. \]

In the case when the external force \( f \) is identically 0, Formula (5.1) is established. In the case when \( u_0 = 0 \), let us compute
\[ V(T) = \int_0^T \langle F(t), i\Delta \Phi(t) + \partial_t \Phi(t) \rangle dt \quad \text{with} \quad F(t) = \int_0^t S(t-t')f(t')dt', \]
where \( S(t)f = \mathcal{F}^{-1}(e^{-it|\xi|^2} f) \). By definition of \( S \), we get
\[ V(T) = \int_0^T \left( \int_0^t \langle f(t')dt', S(t-t') (i\Delta \Phi(t) + \partial_t \Phi(t)) dt' \right) dt. \]

Again by definition of \( S \), we have, using (5.3),
\[ \left( \mathcal{F}S(t-t') (i\Delta \Phi(t) + \partial_t \Phi(t)) \right)(\xi) = e^{-i(t-t')|\xi|^2} (-i|\xi|^2 \hat{\Phi}(t, \xi) + \partial_t \hat{\Phi}(t, \xi)) \]
\[ = \partial_t (e^{-i(t-t')|\xi|^2} \hat{\Phi}(t, \xi)) \]
\[ = \partial_t \mathcal{F}S(t-t') \Phi(t). \]

By definition of \( V \), we infer that
\[ V(T) = -i \int_0^T \int_0^t \langle \partial_t \langle f(t'), S(t-t') \Phi(t) \rangle dt' \rangle dt. \]

Let us observe that
\[ \frac{d}{dt} \langle F(t), \Phi(t) \rangle = \frac{d}{dt} \int_0^t \langle f(t'), S(t-t') \Phi(t) \rangle dt' \]
\[ = \langle f(t), \Phi(t) \rangle + \int_0^t \partial_t \langle f(t'), S(t-t') \Phi(t) \rangle dt'. \]

Thus, we infer that
\[ V(T) = -i \int_0^T \frac{d}{dt} \langle F(t), \Phi(t) \rangle dt' + i \int_0^T \langle f(t), \Phi(t) \rangle dt \]
\[ = -i \langle F(T), \Phi(T) \rangle + i \int_0^T \langle f(t), \Phi(t) \rangle dt. \]

The result is proved. \[\square\]
Let us now give a formula for the solution of the Schrödinger equation.

**Proposition 5.1.2** Let $u_0$ a function in $L^1(\mathbb{R}^d)$. The solution of the linear equation Schrödinger

$$i\partial_t u - \Delta u = 0 \quad \text{and} \quad u_{|t=0} = u_0.$$ 

is given by

$$u(t,x) = \frac{1}{(4\pi it)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{\frac{-|x-y|^2}{4it}} u_0(y) dy.$$ 

with, for $z^{-\frac{d}{2}} \overset{\text{def}}{=} |z|^{-\frac{d}{2}} e^{-i\frac{d}{2}\theta}$ if $z = |z| e^{i\theta}$ with $\theta$ in $[-\pi/2, \pi/2]$.

**Proof.** Proposition 5.1.1 claims that

$$u(t) = F^{-1}(e^{-|\cdot|^2} u_0).$$

Thus the point is to compute the Fourier transform of the purely imaginary gaussian. In order to do it, let us remark that, for any $\xi$ in $\mathbb{R}^d$, the two functions

$$z \longmapsto \int_{\mathbb{R}^d} e^{-i(\xi,x)} e^{-|z|^2} \frac{d\xi}{2\pi} e^{-\frac{|z|^2}{4\xi}}$$

are holomorphic on the domain $D$ of complex numbers with positive real part. As we have for any positive real number $a$,

$$\int_{\mathbb{R}^d} e^{-i(\xi,x)} e^{-a|x|^2} \frac{d\xi}{2\pi} = \left(\frac{\pi}{a}\right)^{\frac{d}{2}} e^{-\frac{|z|^2}{4a}}.$$ 

This claims that the two functions of (5.4) coincide on the intersection of the real line with $D$. Thus they also coincide on the whole domain $D$. Now let $(z_n)_{n \in \mathbb{N}}$ be a sequence of elements of $D$ which converges to $it$ for $t \neq 0$. For any function $\phi$ in $\mathcal{S}$, we have by virtue of Lebesgue’s dominated convergence theorem,

$$\lim_{n \to \infty} \int_{\mathbb{R}^d} e^{-iz_n|x|^2} \phi(x) \, dx = \int_{\mathbb{R}^d} e^{-it|x|^2} \phi(x) \, dx \quad \text{and} \quad \lim_{n \to \infty} \int_{\mathbb{R}^d} e^{-\frac{|\xi|^2}{4zn}} \phi(\xi) \, d\xi = \int_{\mathbb{R}^d} e^{-\frac{|\xi|^2}{4\xi}} \phi(\xi) \, d\xi.$$ 

As we have

$$\mathcal{F}\left(e^{-z_n|\xi|^2}\right) = \left(\frac{\pi}{z_n}\right)_{\frac{d}{2}} e^{-\frac{|\xi|^2}{4z_n}},$$

passing to the limit in $\mathcal{S}'(\mathbb{R}^d)$ when $n$ tends to 0 gives

$$\mathcal{F}(e^{-it|\cdot|^2})(\xi) = \left(\frac{\pi}{z}\right)^{\frac{d}{2}} e^{-\frac{|\xi|^2}{4\xi}}. \quad (5.5)$$

This gives the result. \qed

**Corollary 5.1.1** If the denote by $e^{it\Delta}u_0$ the solution of $(LS)$ with $f \equiv 0$, we have

$$\|e^{it\Delta}u_0\|_{L^\infty} \leq \left(\frac{1}{4\pi|t|}\right)^{\frac{d}{2}} \|u_0\|_{L^1}.$$
5.2 The complex interpolation method in $L^p$ space

We present here the theory of complex interpolation in the particular case of $L^p$ space. It allows to extend inequalities between $L^p$ space. The basic theorem is the following.

**Theorem 5.2.1** Let us consider $(X_k, \mu_k)_{1 \leq k \leq 2}$ two measured spaces and $(p_j, q_j)_{j \in \{0, 1\}}$ two elements of $[1, \infty]^2$. Let us consider an operator $A$ which maps continuously the space $L^{p_j}(X_1)$ into the space $L^{q_j}(X_2)$ for $j$ in $\{0, 1\}$. For any $\theta$ in $[0, 1]$, if

$$
\left( \frac{1}{p_0}, \frac{1}{q_0} \right) \overset{\text{def}}{=} (1 - \theta) \left( \frac{1}{p_0}, \frac{1}{q_0} \right) + \theta \left( \frac{1}{p_1}, \frac{1}{q_1} \right),
$$

then $A$ maps continuously $L^{p_0}(X_1)$ into $L^{q_0}(X_2)$ and

$$
\|A\|_{L^{p_0}(X_1); L^{q_0}(X_2)} \leq A_0 \quad \text{with} \quad A_0 \overset{\text{def}}{=} \|A\|^{1 - \theta}_{L^{p_0}(X_1); L^{q_0}(X_2)} \|A\|^{\theta}_{L^{p_1}(X_1); L^{q_1}(X_2)}.
$$

**Proof.** Let us first point out that there is nothing to prove if $\theta$ is equal to 0 or 1. From now on, we assume that $\theta$ is in $[0, 1[$. This implies that $(p_0, q_0)$ belongs to $]1, \infty^2]$. Thus we consider only $(f, \varphi)$ in $L^1 \cap L^\infty(X_1) \times L^1 \cap L^\infty(X_2)$ such that $\|f\|_{L^{p_0}(X_1)} = \|\varphi\|_{L^{q_0}(X_2)} = 1$. It is enough to prove that

$$
\int_{X_2} (Af)(x_2)\varphi(x_2) d\mu_2(x_2) \leq A_0. \quad (5.6)
$$

Let us consider a complex number $z$ in the strip $S$ of complex numbers the real part of which is between 0 and 1. Let us define

$$
f_z(x_1) \overset{\text{def}}{=} \frac{f(x_1)}{|f(x_1)|^{p_0} \left( \frac{1 + z}{p_0} + \frac{1}{p_1} \right)} \quad \text{and} \quad \varphi_z(x_2) \overset{\text{def}}{=} \frac{\varphi(x_2)}{|\varphi(x_2)|^{q_0} \left( \frac{1 + z}{q_0} + \frac{1}{q_1} \right)}, \quad (5.7)
$$

Obviously, we have $f_0 = f$ and $\varphi_0 = \varphi$. Moreover, for any $t$ in $\mathbb{R}$, we have, for $j$ in $\{0, 1\},$

$$
|f_{j+it}(x_1)| = |f(x_1)|^{p_j} \quad \text{and} \quad |\varphi_{j+it}(x_2)| = |\varphi(x_2)|^{q_j}. \quad (5.8)
$$

It can be checked that the function defined by

$$
F(z) \overset{\text{def}}{=} \int_{X_2} (Af_z)(x_2)\varphi_z(x_2) d\mu_2(x_2)
$$

is holomorphic and bounded on $S$ and continuous on the closure of $S$. Using Lindel"of’s principle (see Lemme A.5.1 page 92), we infer that

$$
F(\theta) \leq M_0^{1 - \theta} M_1^\theta \quad \text{with} \quad M_j \overset{\text{def}}{=} \sup_{t \in \mathbb{R}} |F(j + it)|. \quad (5.9)
$$

Using (5.8), we observe that that $f_{j+it}$ belongs to $L^{p_j}$ and $\|f_{j+it}\|_{L^{p_j}} = 1$. We infer that

$$
M_j \leq \|A\|_{L^{p_j}(X_1); L^{q_j}(X_2)}.
$$

and the lemma is proved. \qed

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Let us give some immediate applications of this result.

**Corollary 5.2.1** For any $p$ in $[1,2]$, the Fourier transform maps $L^p(\mathbb{R}^d)$ into $L^{p'}(\mathbb{R}^d)$ with norm less or equal to $(2\pi)^{d/p}$. 

**Corollary 5.2.2** In $\mathbb{R}^d$, we have, for any $p$ in $[1,2]$, 

$$
\|e^{it\Delta}\|_{L^p(\mathbb{R}^d)} \lesssim \left(\frac{1}{4\pi |t|}\right)^{d\left(\frac{1}{p'} - \frac{1}{2}\right)}.
$$

### 5.3 The duality method and the $TT^*$ argument

This section describes the so-called $TT^*$ argument which is the standard method for converting the dispersive estimates (presented in the previous section) into inequalities involving suitable space-time Lebesgue norms of the solution.

In all this section, we denote by $\| \cdot \|_{L^p(L^q)}$ the norm in $L^p(\mathbb{R}; L^q(\mathbb{R}^d))$. Let us now state the “abstract” Strichartz estimates.

**Theorem 5.3.1** Let $(U(t))_{t \in \mathbb{R}}$ be a bounded family of continuous operators on $L^2(\mathbb{R}^d)$ such that, for some positive real numbers $C_0$ and $C_0$, we have 

$$
\|U(t)U^*(t')f\|_{L^\infty} \leq \frac{C_0}{|t - t'|^\sigma} \|f\|_{L^1}. \tag{5.10}
$$

Then, for any $(p,q)$ in $[2,\infty]^2$ such that 

$$
\frac{2}{p} + \frac{2\sigma}{r} = \sigma \quad \text{and} \quad (p,r,\sigma) \neq (2,\infty,1), \tag{5.11}
$$

we have for some positive constant $C$

$$
\|U(t)u_0\|_{L^p(L^r)} \leq C\|u_0\|_{L^2}.
$$

**Proof.** It is based on a duality argument together with the Hardy-Littlewood-Sobolev inequality stated in Theorem 5.5.2. Let us first notice that 

$$
\|U(t)u_0\|_{L^p(L^r)} = \sup_{\varphi \in B_{p,r}} \left| \int_{\mathbb{R} \times \mathbb{R}^d} U(t)u_0(x)\varphi(t,x) \, dt \, dx \right|
$$

where 

$$
B_{p,r} \overset{\text{def}}{=} \{ \phi \in \mathcal{D}(\mathbb{R}^{1+d}; \mathbb{C}) / \|\phi\|_{L^{p'}(L^{r'})} \leq 1 \}.
$$

By definition of the adjoint operator, we have 

$$
\|U(t)u_0\|_{L^p(L^r)} = \sup_{\varphi \in B_{p,r}} \left| \int_{\mathbb{R}} (U(t)u_0)\varphi(t) \, dt \right|.
$$

By virtue of the Cauchy-Schwarz inequality, we deduce that 

$$
\|U(t)u_0\|_{L^p(L^r)} \leq \|u_0\|_{L^2} \sup_{\varphi \in B_{p,r}} \left| \int_{\mathbb{R}} U^*(t)\varphi(t) \, dt \right|_{L^2}.
$$

(5.12)
Let us write that
\[
\| \int_{\mathbb{R}} U^*(t) \varphi(t) \, dt \|^2_{L^2} = \int_{\mathbb{R}^2} (U^*(t') \varphi(t') | U^*(t) \varphi(t))_{L^2} \, dt' \, dt \\
= \int_{\mathbb{R}^2} (U(t) U^*(t') \varphi(t') \varphi(t))_{L^2} \, dt' \, dt \\
= \int_{\mathbb{R}^2} \langle U(t) U^*(t') \varphi(t'), \overline{\varphi(t)} \rangle \, dt' \, dt. \tag{5.13}
\]

Now, let us observe that, using Theorem 5.2.1, we infer that
\[
\forall r \in [2, \infty], \| U(t) U^*(t') f \|_{L^r} \leq C \frac{1}{|t - t'|^{\frac{d}{2}} \sigma(1 - \frac{d}{2})} \| f \|_{L^{r'}}. \tag{5.14}
\]

Thanks to (5.11) we infer that
\[
\| \int_{\mathbb{R}} U^*(t) \varphi(t) \, dt \|^2_{L^2} \leq C \int_{\mathbb{R}^2} \frac{1}{|t - t'|^\frac{d}{2}} \| \varphi(t') \|_{L^{r'}} \| \varphi(t) \|_{L^r} \, dt' \, dt.
\]

Because \( p > 2 \), the Hardy-Littlewood-Sobolev inequality page 65 gives
\[
\| \int_{\mathbb{R}} U^*(t) \varphi(t) \, dt \|^2_{L^2} \leq C \| \varphi \|^2_{L^{r'}(L^{r'})}.
\]

Thanks to (5.12), the theorem is proved. \( \square \)

Form this theorem, we can deduce Strichartz estimates for the linear Schrödinger equation.

**Theorem 5.3.2** Let us consider \( (p, q) \in [2, \infty] \) such that
\[
\frac{2}{p} + \frac{d}{r} = \frac{d}{2} \quad \text{with} \quad (p, r) \neq (2, \infty). \tag{5.15}
\]

A constant \( C \) exists such that if \( u \) be a solution of \( (LS) \) (in the sense of Definition 5.1.1) with \( u_0 \in L^2(\mathbb{R}^d) \) and \( f \) in \( L^1(\mathbb{R}; L^2(\mathbb{R}^d)) \) given by Proposition 5.1.1 page 58, then
\[
\| u \|_{L^p(\mathbb{R}; L^q(\mathbb{R}^d))} \leq C (\| u_0 \|_{L^2} + \| f \|_{L^1(\mathbb{R}; L^2(\mathbb{R}^d))}).
\]

**Proof.** It is a clear consequence of Theorem 5.3.1 that
\[
\| e^{it \Delta} u_0 \|_{L^p(\mathbb{R})} \leq C \| u_0 \|_{L^2}. \tag{5.16}
\]

Now, we can prove the theorem in the case when \( u_0 = 0 \). In this case, let us write the solution as
\[
u(t) = \int_0^t e^{i(t-t') \Delta} f(t') \, dt'.
\]

We have, for any \( t \),
\[
\| u(t) \|_{L^r(\mathbb{R}^d)} \leq C \int_0^t \| e^{i(t-t') \Delta} f(t') \|_{L^r(\mathbb{R}^d)} \, dt' \\
\leq C \int_{\mathbb{R}} \| e^{i(t-t') \Delta} f_+(t') \|_{L^r(\mathbb{R}^d)} \, dt' 
\]

with \( f_+(t) \overset{\text{def}}{=} 1_{\mathbb{R}^+} f(t) \). Taking the \( L^p \) norm in time in the above inequality, using (5.16) and the translation invariance of the Lebesgue measure on \( \mathbb{R} \) gives
\[
\| u \|_{L^p(\mathbb{R}; L^q(\mathbb{R}^d))} \leq C \int_{\mathbb{R}} \| f_+(t') \|_{L^2(\mathbb{R}^d)} \, dt' \\
\leq C \| f \|_{L^1(\mathbb{R}; L^2(\mathbb{R}^d))}.
\]

This concluded the proof of Theorem 5.3.2. \( \square \)
5.4 An example of application

As an application of the results of the previous section, we here solve the initial boundary value problem for the cubic semilinear Schrödinger equation in \( \mathbb{R}^2 \):

\[
(NLS_3) \quad \begin{cases}
i \partial_t u - \frac{1}{2} \Delta u = P_3(u, \overline{u}) \\
u|_{t=0} = u_0
\end{cases}
\]

where \( P_3 \) is some given homogeneous polynomial of degree 3.

**Theorem 5.4.1** There exists a constant \( c \) such that, for any initial data \( u_0 \) in \( L^2(\mathbb{R}^2) \) satisfying \( \|u_0\|_{L^2} \leq c \), Equation \((NLS_3)\) has a unique solution \( u \) in the space \( L^3(\mathbb{R}; L^6(\mathbb{R}^2)) \) which in addition belongs to \( L^\infty(\mathbb{R}; L^2(\mathbb{R}^2)) \).

**Remark** Let us first have a look on the scaling properties of Equation \((NLS_3)\). If \( u \) is a solution of \((NLS_3)\), then \( u_\lambda(t, x) \overset{\text{def}}{=} \lambda u(\lambda^2 t, \lambda x) \) is also a solution of the same equation. In the scale of Sobolev spaces, \( L^2(\mathbb{R}^2) \) is the only invariant space.

**Proof of Theorem 5.4.1.** Let \( Q \) be the nonlinear functional defined by

\[
\begin{align*}
i \partial_t Q(u) - \frac{1}{2} \Delta Q(u) &= P_3(u, \overline{u}) \\
Q(u)|_{t=0} &= 0.
\end{align*}
\]

The functional \( Q \) maps continuously the space \( L^3(\mathbb{R}; L^6(\mathbb{R}^2)) \) into the space \( L^\infty(\mathbb{R}; L^2(\mathbb{R}^2)) \cap L^3(\mathbb{R}; L^6(\mathbb{R}^2)) \). Indeed Theorem 5.3.2 leads to

\[
\|Q(u)\|_{L^3(\mathbb{R}; L^6(\mathbb{R}^2))} \leq C\|P_3(u, \overline{u})\|_{L^1(\mathbb{R}; L^2(\mathbb{R}^2))} \leq C\|u\|_{L^3(\mathbb{R}; L^6(\mathbb{R}^2))}^3.
\]

As \( Q(u) - Q(v) \) satisfies

\[
\left(i \partial_t + \frac{1}{2} \Delta\right)(Q(u) - Q(v)) = P_3(u, \overline{u}) - P_3(v, \overline{v}),
\]

we get, using Theorem 5.3.2 again,

\[
\|Q(u) - Q(v)\|_{L^\infty(\mathbb{R}; L^2(\mathbb{R}^2)) \cap L^3(\mathbb{R}; L^6(\mathbb{R}^2))} \leq C\|u - v\|_{L^3(\mathbb{R}; L^6(\mathbb{R}^2))} \times \left(\|u\|_{L^3(\mathbb{R}; L^6(\mathbb{R}^2))}^2 + \|v\|_{L^3(\mathbb{R}; L^6(\mathbb{R}^2))}^2\right). \tag{5.17}
\]

Now, it is obvious that \( u \) is a solution of \((NLS_3)\) if and only if \( u \) is a fixed point of the map

\[
F(u) \overset{\text{def}}{=} U(t)u_0 + Q(u).
\]

Applying Theorem 5.3.2 and Estimate (5.17) with \( v = 0 \), we get that

\[
\|F(u)\|_{L^3(\mathbb{R}; L^6)} \lesssim \|u_0\|_{L^2} + \|u\|_{L^3(\mathbb{R}; L^6)}^3.
\]

Thus, if \( 8C^2\|u_0\|_{L^2}^2 \leq 1 \), then the ball \( B(0, 2C\|u_0\|_{L^2}) \) of center 0 and radius \( 2C\|u_0\|_{L^2} \) of the Banach space \( L^3(\mathbb{R}; L^6(\mathbb{R}^2)) \) is invariant by the map \( F \). Using again Inequality (5.17), we get, for any \( u \) and \( v \) in \( B(0, 2C\|u_0\|_{L^2}) \),

\[
\|F(u) - F(v)\|_{L^3(\mathbb{R}; L^6)} \leq 8C^3\|u_0\|_{L^2}^2\|u - v\|_{L^3(\mathbb{R}; L^6)}.
\]
Thus, if in addition
\[ 8C^3\|u_0\|_{L^2}^2 \leq \frac{1}{2}, \]
then Picard’s fixed point theorem implies that a unique solution \( u \) exists in some neighborhood of 0 in \( L^3(\mathbb{R}; L^6) \). Clearly, Inequality (5.17) implies that uniqueness holds true in \( L^3(\mathbb{R}; L^6) \) without any smallness condition.

Finally, the energy estimate entails that this solution belongs to \( L^1(\mathbb{R}; L^2) \). Indeed, multiplying Equation \((NLS_3)\) by \( u \), integrating over \( \mathbb{R}^2 \) then taking the real part, we discover that
\[
\frac{1}{2} \frac{d}{dt} \|u\|_{L^2}^2 = \Im \int \pi P_3(u, u) \, dx,
\]
whence, for all \( t \) in \( \mathbb{R} \),
\[
\|u(t)\|_{L^2} \leq \|u_0\|_{L^2} + C \int_0^t \|u(t')\|_{L^6}^3 \, dt'.
\]
This completes the proof of the theorem.

### 5.5 Refined convolution inequalities

The purpose of this section is to improve the classical Young inequalities which claims that, if
\[
\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}
\]
then \( \|f \ast g\|_{L^r} \leq \|f\|_{L^p} \|g\|_{L^q} \). In order to state these refined inequalities, let us introduce the so-called weak \( L^p \) spaces.

**Definition 5.5.1** For \( p \) in \([1, \infty[\), we denote by \( L^p_w(X, \mu) \) space the set of measure functions such that
\[
\|f\|_{L^p_w(X, \mu)} \overset{\text{def}}{=} \sup_{\lambda > 0} \lambda^p \mu(|f| > \lambda) < \infty,
\]
Let us notice that we have
\[
\mu(|f| > \lambda) \leq \int_{\{|f| > \lambda\}} \left( \frac{|f(x)|}{\lambda} \right)^p \mu(x) \leq \frac{1}{\lambda^p} \|f\|_{L^p}^p.
\]
Thus \( L^p \) is a subset of \( L^p_w \). Typical examples of weak \( L^p \) functions. The function \(|\cdot|^{-\alpha}\) belongs for \( L^p_w(\mathbb{R}^d, dx) \).

**Theorem 5.5.1** Let \((p, q, r)\) be in \([1, \infty[^3\) and satisfy (5.18). A constant \( C \) exists such that
\[
\|f \ast g\|_{L^r} \leq C \|f\|_{L^p} \|g\|_{L^q_w}.
\]
Let us notice that the above theorem implies the well-known **Hardy-Littlewood-Sobolev inequalities** on \( \mathbb{R}^d \).

**Theorem 5.5.2 (Hardy-Littlewood-Sobolev inequality)** Let \( \alpha \) be in \([0, d[\) and \((p, r)\) in \([1, \infty[^2\) satisfy
\[
\frac{1}{p} + \frac{\alpha}{d} = 1 + \frac{1}{r}.
\]
Then a constant \( C \) exists such that
\[
\||\cdot|^{-\alpha} \ast f\|_{L^r(\mathbb{R}^d)} \leq C \|f\|_{L^p(\mathbb{R}^d)}.
\]
The proof of Theorem 5.5.1 relies on the atomic decomposition which is describe in the following proposition.

**Proposition 5.5.1** Let $(X, \mu)$ be a measure space and $p$ be in $[1, \infty[$. Let $f$ be a nonnegative function in $L^p$. Then a sequence of positive real numbers $(c_k)_{k \in \mathbb{Z}}$ and a sequence of nonnegative functions $(f_k)_{k \in \mathbb{Z}}$ (the atoms) exist such that

$$f = \sum_{k \in \mathbb{Z}} c_k f_k$$

where the support of the functions $f_k$ are pairwise disjoint and

$$\mu(\text{Supp } f_k) \leq 2^{k+1},$$

$$\|f_k\|_{L^\infty} \leq 2^{-\frac{k}{p}},$$

$$\frac{1}{2} \|f\|_p^p \leq \sum_{k \in \mathbb{Z}} |c_k|^p \leq 2 \|f\|_p^p.$$ (5.22)

**Remarks**

- Let us notice that, because of Inequalities (5.20) and (5.21), the functions $f_k$ belongs to $L^a$ or any $a$ in $[1, \infty]$ and satisfies

$$\|f_k\|_{L^a} \leq 2^\frac{1}{p} 2^k \left( \frac{1}{2} - \frac{1}{p} \right).$$

(5.23)

- The fact that $f$ belongs to $L^p$ is given by the behavior of the sequence $(c_k)_{k \in \mathbb{Z}}$.

- As inferred by the definition given below, the sequence $(c_k f_k)_{k \in \mathbb{Z}}$ is independent of $p$ and depends only on $f$.

**Proof of Proposition 5.5.1.** If $\mu(f > 0)$ is not finite, let us define, for any $k$ in $\mathbb{Z}$,

$$\lambda_k \overset{\text{def}}{=} \inf \left\{ \lambda \mid \mu(f > \lambda) < 2^k \right\}.$$  

If $\mu(f > 0)$ is finite, let us define by $k_0$ the smallest integer such that

$$\mu(f > 0) \leq 2^{k_0}$$

Then, for $k \leq k_0$, let us define

$$\lambda_k \overset{\text{def}}{=} \inf \left\{ \lambda \mid \mu(f > \lambda) < 2^k \right\}.$$ 

In all that follows in this proof, we shall implicitely consider that, if $\mu(f > 0)$ is finite, then, all the sequence define are 0 for $k \geq k_0$. Let us notice that $(\lambda_k)_{k \in \mathbb{Z}}$ is a decreasing sequence. By definition of the sequence $(\lambda_k)_{k \in \mathbb{Z}}$, we get

$$\forall \lambda < \lambda_k, \mu(f > \lambda) \geq 2^k.$$ (5.24)

The monotonic convergence theorem and the definition of $(\lambda_k)_{k \in \mathbb{Z}}$, we have that

$$\mu(f > \lambda_k) \leq 2^k.$$ (5.25)
Then, let us define 
\[ c_k \overset{\text{def}}{=} 2^k \lambda_k \quad \text{and} \quad f_k \overset{\text{def}}{=} c_k^{-1} 1_{(\lambda_{k+1} < f \leq \lambda_k)} f. \]
It is obvious that \( \| f_k \|_{L^\infty} \leq 2^{-k} \). The point is now the proof of (5.22). As the support of the functions \((f_k)_{k \in \mathbb{Z}}\) are pairwise disjoint, one may write
\[ \| f \|_{L^p}^p = \sum_{k \in \mathbb{Z}} c_k^p \| f_k \|_{L^p}^p. \]
Taking advantage of Inequalities (5.20) and (5.21), we claim that 
\[ \| f_k \|_{L^p} \leq 2 \| f \|_{L^p}^p \]
which claims exactly that \( \sum_{k \in \mathbb{Z}} c_k^p \leq 2 \| f \|_{L^p}^p \).

Moreover, which, owing to the fact that \( f \) is a nonnegative function of \( L^p \), converges to 0 when \( k \) tends to \(+\infty\).

Thanks to (5.25), we have \( \mu(\text{Supp } f_k) \leq 2^{k+1} \). This gives
\[ \sum_{k \in \mathbb{Z}} c_k^p = \sum_{k \in \mathbb{Z}} 2^k \lambda_k^p = p \sum_{k \in \mathbb{Z}} \int_0^\infty 2^k \mathbf{1}_{[0, \lambda_k]}(\lambda) \lambda^{p-1} d\lambda. \]
Using Fubini’s theorem, we get
\[ \sum_{k \in \mathbb{Z}} c_k^p = \int_{0}^\infty \lambda^{p-1} \left( \sum_{k / \lambda_k > \lambda} 2^k \right) d\lambda. \]
By definition of sequence \((\lambda_k)_{k \in \mathbb{Z}}\), having \( \lambda < \lambda_k \) implies that \( \mu(f > \lambda) \geq 2^k \). Thus we infer that
\[ \sum_{k \in \mathbb{Z}} c_k^p \leq p \int_{0}^\infty \lambda^{p-1} \left( \sum_{k / 2^k \leq \mu(f > \lambda)} 2^k \right) d\lambda \leq 2p \int_{0}^\infty \lambda^{p-1} \mu(f > \lambda) d\lambda. \]
Now, the right inequality in (5.22) follows from the fact that, owing to Fubini’s theorem, we have
\[ \| f \|_{L^p}^p = p \int_{0}^\infty \lambda^{p-1} \mu(|f| > \lambda) d\lambda \quad \text{(5.26)} \]
which implies the desired inequality. \( \square \)

**Proof of Theorem 5.5.1** Let \( f \) and \( g \) be two nonnegative measurable functions on \( \mathbb{R}^d \). Let us consider a nonnegative function \( h \) in \( L^r \) and let us define
\[ I(f, g, h) \overset{\text{def}}{=} \int_{\mathbb{R}^{2d}} f(y)g(x-y)h(x) d\mu(x) d\mu(y). \]
Arguing by homogeneity, one can assume that \( \| f \|_{L^p} = \| g \|_{L^q} = \| h \|_{L^r} = 1 \). Defining
\[ C_j \overset{\text{def}}{=} \{ y \in \mathbb{R}^d : 2^{-j+1} \leq g(y) \leq 2^{-j} \} \quad \text{and} \quad g_j \overset{\text{def}}{=} \mathbf{1}_{C_j} g, \]

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we can write

\[ I(f, g, h) \leq \sum_{j \in \mathbb{Z}} I(f, g_j, h). \]

Because \( \|g\|_{L^p_v} = 1 \), we have \( \|g_j\|_{L^s} \leq 2^{\frac{1}{s}-\frac{1}{p}}(\frac{1}{t} - \frac{1}{p}) \) for all \( s \) in \([1, \infty] \). Thus if we directly apply Young’s inequality with \( p, q \) and \( r \), we find that \( I(f, g, h) \leq 2^\frac{1}{r} \) so that series \( \left( (I(f, g_j, h))_{j \in \mathbb{Z}} \right) \) has no reason to converge. In order to bypass this difficulty, one may introduce the atomic decomposition for \( f \) and \( h \) given by Proposition 5.5.1 which leads to

\[ I(f, g, h) = \sum_{j,k,\ell} c_k d_\ell I(f_k, g_j, h_\ell). \]

Using Young’s inequalities, we get for any \((a, b)\) in \([1, \infty]^2\) such that \( b \leq a' \) and for any \((\tilde{f}, \tilde{h})\) in \( L^a \times L^b \),

\[ I(f_k, g_j, h_\ell) \leq \|f_k\|_{L^a} \|g_j\|_{L^b} \|h_\ell\|_{L^{b'}} \quad \text{with} \quad \frac{1}{a} + \frac{1}{b} = 1 + \frac{1}{c}. \]

Using Proposition 5.5.1, this gives

\[ I(f_k, g_j, h_\ell) \leq 4 \times 2^j \left( \frac{1}{a} - \frac{1}{p} \right) 2^k \left( \frac{1}{b} - \frac{1}{q} \right) 2^\ell \left( \frac{1}{c} - \frac{1}{r} \right) = 4 \times 2^j \left( \frac{1}{a} - \frac{1}{p} \right) 2^k \left( \frac{1}{b} - \frac{1}{q} \right) 2^\ell \left( \frac{1}{c} - \frac{1}{r} \right). \]

Using the condition (5.18) on \((p, q, r)\) and \((a, b, c)\) implies

\[ I(f_k, g_j, h_\ell) \leq 4 \times 2^j \left( \frac{1}{a} - \frac{1}{p} \right) 2^k \left( \frac{1}{b} - \frac{1}{q} \right) 2^\ell \left( \frac{1}{c} - \frac{1}{r} \right) \leq 4 \times 2^j \left( \frac{1}{a} - \frac{1}{p} \right) 2^k \left( \frac{1}{b} - \frac{1}{q} \right) 2^\ell \left( \frac{1}{c} - \frac{1}{r} \right). \quad (5.27) \]

As \((p, q, r)\) is in \([1, \infty]^2\), a positive real number \( \varepsilon \) exists, if

\[ \frac{1}{a} = \frac{1}{p} - \varepsilon \sgn(k - \ell) \quad \text{and} \quad \frac{1}{b} = \frac{1}{q} - \varepsilon \sgn(j - \ell) \]

then \((a, b)\) is in \([1, \infty]^2\) and \( \frac{1}{a'} + \frac{1}{b'} \geq 1 \). With this choice of \( a \) and \( b \), (5.27) becomes

\[ I(f_k, g_j, h_\ell) \leq 4 \times 2^{-|j| + |k| + |\ell| - \varepsilon |k - \ell|}. \]

Now, using Young’s inequality for \( \mathbb{Z} \) equipped with the counting measure, we deduce that

\[ I(f, g, h) \leq 4 \sum_{k,\ell} c_k d_\ell 2^{-|k|} \sum_{j} 2^{-|j|} \leq \frac{C}{\varepsilon^2} \sum_{k,\ell} c_k d_\ell 2^{-|k|} \leq \frac{C}{\varepsilon^2} \|(c_k)\|_{L^p} \|\|(d_\ell)\|_{L^{p'}}. \]

Condition (5.18) implies that \( r' \leq p' \) and thus

\[ I(f, g, h) \leq \frac{C}{\varepsilon^2} \|(c_k)\|_{L^p} \|\|(d_\ell)\|_{L^{p'}}. \]

The theorem is proved.
Chapter 6

Linear symmetric systems

6.1 Definition and examples

Let us first define the concept in the framework of linear system with variable coefficients. Let $I$ be a closed interval of $\mathbb{R}$ which has $0$ as an interior point. Let us consider a family $\mathcal{A}$ of smooth bounded functions $(\mathcal{A}_k)_{0 \leq k \leq d}$ from $I \times \mathbb{R}^d$ into the space of $N \times N$ matrices with real coefficients. Let us assume that all their derivatives in the $x$ variable are bounded. We consider the system

\[
(\mathcal{L}S) \begin{cases}
\partial_t U + \sum_{k=1}^{d} \mathcal{A}_k \partial_k U + \mathcal{A}_0 U = F \\
U|_{t=0} = U_0.
\end{cases}
\]

Let us introduce the following notations. If $U$ in $\mathcal{D}'(\mathbb{R}^d)$ and $\varphi$ in $\mathcal{D}(\mathbb{R}^d)$,

\[
\langle U, \varphi \rangle = \sum_{i=1}^{N} \langle U^i, \varphi^i \rangle
\]

and if $U$ and $V$ belongs to $(L^2(\mathbb{R}^d))^N$,

\[
(U|V)_{L^2} = \sum_{i=1}^{N} \int_{\mathbb{R}^d} U^i(x)V^i(x)dx.
\]

**Definition 6.1.1** A function $U$ in $(C(I; L^2))^N$ is a solution of $(LS)$ if and only if for any $\varphi$ in $\mathcal{D}(\mathbb{R} \times \mathbb{R}^d)$, for any $t$ in $I$,

\[
\langle U(t), \varphi(t) \rangle = \int_{0}^{t} \langle U(t'), \partial_t \varphi(t') \rangle dt' + \langle U_0, \varphi(0) \rangle + \int_{0}^{t} \sum_{k=1}^{d} \langle \mathcal{A}_k U(t'), \partial_k \varphi(t') \rangle dt' + \int_{0}^{t} \langle \mathcal{A}_0 U(t'), \varphi(t') \rangle dt' + \int_{0}^{t} \langle F(t'), \varphi(t') \rangle dt'.
\]

Let us define the concept of symmetric system.

**Definition 6.1.2** The above system $(LS)$ is symmetric if and only if for any $k \in \{1, \ldots, d\}$ and any $(t, x)$ in $I \times \mathbb{R}^d$ the matrices $\mathcal{A}_k(t, x)$ are symmetric, which means that for any $k$, we have $\mathcal{A}_{k,j,i}(t, x) = \mathcal{A}_{k,j,i}(t, x)$.
Let us of a look to an example coming from the gas dynamics. The unknown is the couple \((\rho, v)\) which satisfies
\[
\begin{align*}
\partial_t \rho + v \cdot \nabla \rho + \rho \text{div } v &= 0 \\
\partial_t v + v \cdot \nabla v + \frac{1}{\rho} \nabla p &= 0
\end{align*}
\]
with \(p = A \rho^\gamma\). Here, \(\rho\) is a scalar function with values in \(\mathbb{R}_+^\star\) and represents the density of the particles of the gas at time \(t\) in the point \(x\) and \(v\) a time dependant vector field which describes the speed of a particule located in \(x\) at time \(t\).

It will be clear later on that we have to change the unknowns defining
\[
c \overset{\text{def}}{=} \frac{2}{\gamma - 1} \left( \frac{\partial p}{\partial \rho} \right)^{\frac{1}{2}} = \frac{(4\gamma A)^{\frac{1}{2}}}{\gamma - 1} \rho^{\frac{\gamma - 1}{2}}.
\]
The first equation becomes
\[
\partial_t c + v \cdot \nabla c + \frac{\gamma - 1}{2} c \text{div } v = 0.
\]
About the second one, let us observe that
\[
\frac{\gamma - 1}{2} c \nabla c = \frac{1}{\rho} \nabla p.
\]
The Euler system related to gas dynamics becomes
\[
\begin{align*}
\partial_t c + v \cdot \nabla c + \frac{\gamma - 1}{2} c \text{div } v &= 0 \\
\partial_t v + v \cdot \nabla v + \frac{\gamma - 1}{2} c \nabla c &= 0.
\end{align*}
\]
Let us assume that the solution is "small", i.e. is a perturbation of magnitude \(\varepsilon\) of a stationary flat state \(v = 0\) and \(c = \bar{c}\), by an easy computation of the coefficients of the powers of \(\varepsilon\), we infer
\[
\begin{align*}
\partial_t c + \frac{\gamma - 1}{2} \bar{c} \text{div } v &= 0 \\
\partial_t v + \frac{\gamma - 1}{2} \bar{c} \nabla c &= 0.
\end{align*}
\]
An obvious computation ensures that
\[
\partial_t^2 c - \left( \frac{\gamma - 1}{2} \right)^2 \varepsilon^2 \Delta c = 0.
\]
This equation is called "acoustic waves equation".

The reason why this definition is fundamental is the following. Let us consider a solution of \((LS)\) and let us look to the evolution of its energy. This question leads to the following formal computation which will be made rigorous in the following section.
\[
\frac{1}{2} \frac{d}{dt} \|U(t)\|_{L^2}^2 = -\sum_{k=1}^{d} \left( A_k \partial_k U \middle| U \right)_{L^2} - (A_0 U | U)_{L^2} + (F | U)_{L^2}
\]
By integration by part, we get that
\[-(A_k \partial_k U | U)_{L^2} = - \sum_{i,j} \int_{\mathbb{R}^d} A_{k,i,j} \partial_k U^i U^j \, dx\]
\[= \sum_{i,j} \int_{\mathbb{R}^d} A_{k,i,j} U^i \partial_k U^j \, dx + \sum_{i,j} \int_{\mathbb{R}^d} \partial_k A_{k,i,j} U^i U^j \, dx.\]

If the system \((LS)\) is symmetric, then we have
\[- \sum_{k=1}^d \left( A_k \partial_k U | U \right)_{L^2} = \frac{1}{2} ((\text{div} A) U | U)_{L^2} \quad \text{with} \quad (\text{div} A)_{i,j} \overset{\text{def}}{=} \sum_{k=1}^d \partial_k A_{k,i,j}\]

This implies that
\[\left| \sum_{k=1}^d \left( A_k(t) \partial_k U(t) | U(t) \right)_{L^2} \right| \leq \frac{1}{2} \| \text{div} A(t) \|_{L^\infty} \| U(t) \|_{L^2}^2.\]

Thus we get that
\[\frac{d}{dt} \| U(t) \|_{L^2}^2 \leq a_0(t) \| U \|_{L^2}^2 + (F|U)_{L^2} \quad \text{with} \quad a_0(t) \overset{\text{def}}{=} \| \text{div} A(t, \cdot) \|_{L^\infty} + 2 \| A_0(t, \cdot) \|_{L^\infty}. \quad (6.3)\]

The purpose of this section is to study linear symmetric systems. First, we want to solve them and then to study basic properties of their solutions. In this section, for \(s \in \mathbb{N}\) we shall state
\[|U(t)|_s^2 \overset{\text{def}}{=} \sum_{1 \leq |\alpha| \leq d} \| \partial_x^\alpha U^f(t) \|_{L^2}^2.\]

### 6.2 The wellposedness of linear symmetric systems

The goal of this paragraph is the proof of the following wellposedness theorem.

**Theorem 6.2.1** Let \((LS)\) be a linear symmetric system. Then, if \(U_0\) belongs to \(H^s\) and if \(F\) is a continuous function with value in \(H^s\), then a unique solution of \((S)\) exists in the space \(C^0(I; H^s) \cap C^1(I; H^{s-1})\).

**Proof.** It requires four steps:

- We first prove a-priori estimates for smooth enough solutions of the system \((S)\).
- Then we apply Friedrichs method.
- Then we pass to the limit in the case of smooth enough initial data and we get existence in any case by smoothing of the initial data.
- Finally, we get uniqueness using existence for the adjoint system.

A priori estimates use in a crucial way the symmetry hypothesis and are true only for smooth enough solutions.
Lemma 6.2.1 For any non negative integer \( s \), a locally bounded function \( a_s \) exists such that for any function \( U \) in \( C^0(I, H^{s+1}) \cap C^1(I, H^s) \), we have for any \( t \) in \( I \),

\[
|U(t)|_s 
\leq |U(0)|_s \exp \int_0^t a_s(t') dt' + \int_0^t |F(t')|_s \exp \left( \int_{t'}^t a_s(t'') dt'' \right) dt',
\]

with

\[
F = \partial_t U + \sum_{k=1}^d A_k \partial_k U + A_0 u.
\]

**Proof.** To start with, let us prove this lemma for \( s = 0 \). Let us consider a function \( U \) in the space \( C^0(I; H^1) \cap C^1(I; L^2) \). By definition of \( F \), we have

\[
\frac{1}{2} \frac{d}{dt} |U(t)|_0^2 = (\partial_t U |_0)_{L^2}^2
\]

\[
= (F|U)_{L^2} - (A_0 U|U)_{L^2} - \sum_{k=1}^d (A_k \partial_k U|U)_{L^2}.
\]

As the system \((LS)\) is symmetric and \( U \) belongs to \( C^0(I; H^1) \cap C^1(I; L^2) \), computations done page 71 which lead to (6.3) are rigorous. Thus we have

\[
\frac{d}{dt} |U(t)|_0^2 \leq a_0(t) |U(t)|_0^2 + 2 |F(t)|_0 |U(t)|_0
\]

with \( a_0(t) \) defined as \( \| \text{div} A(t, \cdot) \|_{L^\infty} + 2 \| A_0(t, \cdot) \|_{L^\infty} \). By Gronwall lemma, we get

\[
|U(t)|_0 \leq |U(0)|_0 \exp \int_0^t a_0(t') dt' + \int_0^t |F(t')|_0 \exp \left( \int_{t'}^t a_0(t'') dt'' \right) dt'
\]

and the lemma is proved is the case when \( s = 0 \).

**Remark** Let us point out that the above computations are also valid when the matrices \((A_k)\) have \( C^1 \) coefficients and \( A_0 \) has \( C^0 \) coefficients.

Let us study the case when \( s \) is any non negative integer. To do so, we shall proceed by induction on the integer \( s \). Let us assume that Lemma 6.2.1 is proved for some \( s \). Let \( U \) be a function in \( C^0(I, H^{s+1}) \cap C^1(I, H^s) \). Let us introduce the function (with \( N(d+1) \) components) \( \tilde{U} \) defined by

\[
\tilde{U} = (U, \partial_1 U, \cdots, \partial_d U).
\]

As, for any \( j \) in \( \{1, \ldots, d\} \),

\[
F = \partial_t U + \sum_{k=1}^d A_k \partial_k U + A_0 U,
\]

we obtain by differentiation of the equation,

\[
\partial_t (\partial_j U) = - \sum_{k=1}^d A_k \partial_k \partial_j U - \sum_{k=1}^d (\partial_j A_k) \cdot \partial_k U - \partial_j (A_0 U) - \partial_j F.
\]
We may write

$$\partial_t \tilde{U} = - \sum_{k=1}^{d} B_k \partial_k \tilde{U} - B_0 \tilde{U} + \tilde{F} \quad \text{with}$$

$$\tilde{F} = (F, \partial_1 F, \ldots, \partial_d F) \quad \text{and}$$

$$B_k = \begin{pmatrix} A_k & 0 \\ 0 & A_k \end{pmatrix}. \quad (6.8)$$

The coefficients of $B_0$ are computed from those of $A_k \ (k = 0, \ldots, d)$ and their first order derivatives. The induction hypothesis allows to conclude the proof of Lemma 6.2.1. □

**Remark** Let us point out that the proof of the inequalities of Lemma 6.2.1 done above demands exactly one more derivative than in the statement of Theorem 6.2.1.

**Second step of the proof of Theorem 6.2.1.** This leads us to use a smoothing method, the Friedrich method. It consists in smoothing both the initial data and the system itself. More precisely, let us consider the system $(LS_n)$ defined by

$$(LS_n) \left\{ \begin{array}{l} \partial_t U_n + \sum_{k=1}^{d} \mathbb{E}_n (A_k \partial_k U_n) + \mathbb{E}_n (A_0 U_n) = \mathbb{E}_n F \\
\mathbb{E}_n U_{t=0} = \mathbb{E}_n U_0 \end{array} \right.$$  

where $\mathbb{E}_n$ is the cutoff operator defined on $L^2$ by

$$\mathbb{E}_n u \overset{\text{def}}{=} F^{-1} (1_{B(0,n)} \hat{u}). \quad (6.9)$$

This is nothing else that the orthogonal projection of $L^2$ on the closed space $L^2_n$ of the $L^2$ functions the Fourier transform of which are supported in the ball of center 0 and radius $n$. Fourier Plancherel tells us in particular that the operator $\partial_k$ is a continuous on $L^2_n$. As the functions $A_k$ are bounded it turns out that the linear operator

$$V \mapsto \sum_{k=1}^{d} \mathbb{E}_n (A_k \partial_k V) + \mathbb{E}_n (A_0 V)$$

is continuous on $L^2_n$. Thus the system $(LS_n)$ is a linear system of ordinary differential equations on $L^2_n$. This implies the existence of a unique function $U_n$ continuous on $I$ with value in $L^2_n$ which is a solution of $(LS_n)$. Moreover as the functions $A_k$ are smooth functions in $(t, x)$, using the equation $(LS_n)$, we get that $U_n$ is a smooth function on $I$ with value in $H^s$ for any integer $s$.

Let us prove that the functions $U_n$ satisfy the energy estimates of Lemma 6.2.1. More precisely, we have the following lemma.

**Lemma 6.2.2** For any non negative integer $s$, a locally bounded function $a_s$ exists such that for any $n \in N$ and any $t$ in $I$ we have,

$$|U_n(t)|_s \leq |\mathbb{E}_n U(0)|_s \exp \int_0^t a_s(t') dt' + \int_0^t |\mathbb{E}_n F(t')|_s \exp \left( \int_{t'}^t a_s(t'') dt'' \right) dt',$$

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Lemma 6.2.2 tells us that the sequence \((\mathcal{L}S_n)\) is a Cauchy sequence in \(L^2\) and that \(\mathbb{E}_n U_n = U_n\). 

\[
\frac{d}{dt} U_n(t)_0^2 = -2 \sum_{k=1}^{d} (A_k \partial_k U_n | U_n)_{L^2} - 2(A_0 U_n | U_n)_{L^2} - 2(\mathbb{E}_n F | U_n)_{L^2}.
\]

We proceed exactly as in the proof of Lemma 6.2.1. As the system \((\mathcal{L}S)\) is symmetric and \(U_n\) belongs to \(C^0(I; H^1) \cap C^1(I; L^2)\), computations done page 71 which lead to (6.3) are rigourous. Thus we have

\[
\frac{d}{dt} |U_n(t)|^2_0 \leq a_0(t) |U_n(t)|^2_0 + 2|\mathbb{E}_n F(t)|_0 |U_n(t)|_0 \tag{6.10}
\]

with \(a_0(t) \overset{\text{def}}{=} \| \text{div} \mathcal{A}(t, \cdot) \|_{L^\infty} + 2\| \mathcal{A}_0 (t, \cdot) \|_{L^\infty}\). Gronwall Lemma implies that

\[
|U_n(t)|_0 \leq |\mathbb{E}_n U_0|_0 \exp \int_0^t a_0(t') dt' + \int_0^t |\mathbb{E}_n F(t')|_0 \exp \left( \int_{t'}^t a_0(t'') dt'' \right) dt'.
\]

The proof of the lemma for any integer \(s\) works exactly as the one of Lemma 6.2.1 and is omitted. \(\Box\)

**Third step of the proof of Theorem 6.2.1.** The third step consists in the proof of the following wellposedness result.

**Proposition 6.2.1** Let \(s \geq 3\). We consider the linear symmetric system 

\[
(LS) \begin{cases} 
\partial_t U + \sum_{k=1}^{d} A_k \partial_k U + A_0 U = F \\
U(0) = U_0
\end{cases}
\]

with \(F \in C(I; H^s)\) and \(U_0 \in H^s\). A unique solution \(U\) exists in \(C(I, H^{s-2}) \cap C^1(I; H^{s-3})\) which moreover satisfies the energy estimate

\[
\forall \sigma \leq s, \forall t \in I, \ |U(t)|_\sigma \leq |U_0|_\sigma \exp \int_0^t a_s(t') dt' + \int_0^t |F(t')|_\sigma \exp \left( \int_{t'}^t a_s(t'') dt'' \right) dt'.
\]

**Proof.** Let us consider the sequence \((U_n)_{n \in \mathbb{N}}\) of solution of \((\mathcal{L}S_n)\). We shall prove that \((U_n)_{n \in \mathbb{N}}\) is a Cauchy sequence in \(L^\infty(I; H^{s-2})\). In order to do so, let us state \(V_{n,p} \overset{\text{def}}{=} U_{n+p} - U_n\). We have

\[
\begin{cases} 
\partial_t V_{n,p} + \sum_{k=1}^{d} \mathbb{E}_{n+p}(A_k \partial_k V_{n,p}) + \mathbb{E}_{n+p}(A_0 V_{n,p}) = F_{n,p} \\
V_{n,p}(0) = (\mathbb{E}_{n+p} - \mathbb{E}_n) U_0
\end{cases}
\]

with \(F_{n,p} \overset{\text{def}}{=} \sum_{k=1}^{d} (\mathbb{E}_{n+p} - \mathbb{E}_n)(A_k \partial_k U_n) - (\mathbb{E}_{n+p} - \mathbb{E}_n)(A_0 U_n) + (\mathbb{E}_{n+p} - \mathbb{E}_n) F\).

Lemma 6.2.2 tells us that the sequence \((U_n)_{n \in \mathbb{N}}\) is bounded in \(L^\infty(I; H^s)\). Moreover we have

\[
|(\mathbb{E}_{n+p} - \mathbb{E}_n) a|_{\sigma-1} \leq \frac{C}{n} |a|_\sigma.
\]

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Thus we have
\[
| (E_{n+p} - E_n) \left( A_k \partial_t U_n(t) \right) |_{s-2} \leq \frac{C'}{n} \sup_k | (E_{n+p} - E_n) \left( A_k \partial_t U_n(t) \right) |_{s-1} \leq \frac{C'}{n} |U_n(t)|_s.
\]

The same arguments give
\[
\left| (E_{n+p} - E_n)(A_0 U_n(t)) + (E_{n+p} - E_n)F(t) \right|_{s-2} \leq \frac{C'}{n} \left( |U_n(t)|_s + |F(t)|_s \right). \quad (6.12)
\]

Energy estimate implies that
\[
| V_{n,p}(t) |_{s-2} \leq \frac{C}{n} \exp \left( t \int_0^t a_s(t') dt' \right).
\]

Thus the sequence \((U_n)_{n \in \mathbb{N}}\) is a Cauchy one in \(L^\infty(I; H^{s-2})\). Moreover, using (6.11) and (6.12), we infer that \((\partial_t U_n)_{n \in \mathbb{N}}\) is a Cauchy sequence in \(L^\infty(I; H^{s-3})\). Let us denote by \(U\) the limit of \((U_n)_{n \in \mathbb{N}}\). Of course, \(U\) belongs to \(C(I; H^{s-2}) \cap C^1(I; H^{s-3})\). Let us check that this function \(U\) is solution of \((LS)\). As \(F\) belongs to \(C(I; H^s)\), we have that
\[
\lim_{n \to \infty} E_n F = F \quad \text{in} \quad L^\infty(I; H^s). \quad (6.13)
\]

As the sequence \((U_n)_{n \in \mathbb{N}}\) is bounded in \(L^\infty(I; H^s)\), we have that
\[
\| (E_n - I)A_k(U) \partial_t U_n \|_{L^\infty(I; H^{s-2})} \leq \frac{C}{n}.
\]
Thus \(U\) is a solution of \((LS)\). To conclude, let us point out that the sequence \((U_n)_{n \in \mathbb{N}}\) is bounded in \(L^\infty(I; H^s)\). Using interpolation inequality, we get that for any \(s' < s\), the sequence \((U_n)_{n \in \mathbb{N}}\) is a Cauchy one in \(C(I, H^{s'})\). Thus \(U\) belongs to \(C(I, H^{s'})\). Using the fact that \(U\) is a solution of \((LS)\), we get that \(U\) belongs to \(C(I, H^{s'}) \cap C^1(I; H^{s'-1})\). But as \((U_n)_{n \in \mathbb{N}}\) is bounded in \(L^\infty(I; H^s)\), it weakly converges to \(U\) in \(L^\infty(I; H^s)\). Using (6.13), the fact that \((E_n U_0)_{n \in \mathbb{N}}\) converges to \(U_0\) in \(H^s\) and that
\[
\| U \|_{L^\infty([0, t]; H^s)} \leq \limsup_{n \to \infty} \| U_n \|_{L^\infty([0, t]; H^s)}
\]
we get, passing to the limit in Lemma 6.2.2 that
\[
|U_n(t)|_s \leq |E_n U(0)|_s \exp \left( s \int_0^t a_s(t') dt' \right) \quad \text{for any} \quad s > 0.
\]

This concludes the proof of Proposition 6.2.1.

**Last step of the proof of Theorem 6.2.1.** Let us consider the sequence \((\tilde{U}_n)_{n \in \mathbb{N}}\) of solutions of
\[
\begin{cases}
\frac{\partial \tilde{U}_n}{\partial t} + \sum_{k=1}^d A_k \partial_t \tilde{U}_n + A_0 \tilde{U}_n = E_n F \\
\tilde{U}_n|_{t=0} = E_n U_0.
\end{cases}
\]

Thanks to Proposition 6.2.1 this solution does exist in \(C^1(I, H^s)\) for any positive real number \(s\). Let us state \(V_{n,p} \overset{\text{def}}{=} U_{n+p} - U_n\). It satisfies
\[
\begin{cases}
\frac{\partial \tilde{V}_{n,p}}{\partial t} + \sum_{k=1}^d A_k \partial_t \tilde{V}_{n,p} + A_0 \tilde{V}_{n,p} = (E_{n+p} - E_n)F \\
\tilde{V}_{n,p}|_{t=0} = (E_{n+p} - E_n)U_0.
\end{cases}
\]
Lemma 6.2.1 implies that

$$|\widetilde{V}_{n,p}(t)|_s \leq |(E_{n+p} - E_n)U(0)|_s \exp \int_0^t a_s(t')dt'$$

$$+ \int_0^t |(E_{n+p} - E_n)F(t')|_s \exp \left(\int_0^t a_s(t')dt'\right)dt.$$ 

As the function $F$ is continuous from $I$ into $H^s$, the sequence $(E_n F)_{n \in \mathbb{N}}$ converges to $F$ in the space $L^\infty([0,T];H^s)$. As $U_0$ belongs to $H^s$, the sequence $(E_n U_0)_{n \in \mathbb{N}}$ converges to $U_0$ in $H^s$. Thus the sequence $(\tilde{U}_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $L^\infty(I;H^s)$. It converges to a function $U$ of $C(I;H^s)$ which is of course solution of the system $(LS)$. The fact that $\partial_t U$ belongs to $C(I;H^{s-1})$ comes immediately from the fact that $U$ is solution of $(S)$.

The existence part of Theorem 6.2.1 and also the uniqueness when $s \geq 1$ is now proved. The following proposition will conclude the proof of Theorem 6.2.1.

**Proposition 6.2.2** Let $U$ be a solution $C^0(I;L^2)$ of the symmetric system $(LS)$.

$$(LS) \quad \begin{cases}
\partial_t U + \sum_{k=1}^d A_k \partial_k U + A_0 U = 0 \\
U|_{t=0} = 0.
\end{cases}$$

Then $U \equiv 0$.

**Proof.** In order to prove this proposition, we shall use a duality method. Let $\psi$ be a function of $\mathcal{D}([0,T] \times \mathbb{R}^d)$. Let us consider the solution of

$$(tLS) \quad \begin{cases}
-\partial_t \varphi - \sum_{k=1}^d \partial_k (A_k \varphi) + tA_0 \varphi = \psi \\
\varphi|_{t=T} = 0.
\end{cases}$$

The system $(tLS)$ can be understood as the adjoint system of the system $(LS)$. As we have

$$\partial_k (A_k \varphi) = A_k \partial_k \varphi + \partial_k A_k \varphi,$$

the system $(tLS)$ becomes

$$(tS) \quad \begin{cases}
-\partial_t \varphi - \sum_{k=1}^d A_k \partial_k \varphi + \tilde{A_0} \varphi = \psi \\
\varphi|_{t=T} = 0 \\
\text{with } \tilde{A_0} = tA_0 - \sum_{k=1}^d \partial_k A_k.
\end{cases}$$

This is obviously a linear symmetric system. The existence part of Theorem 6.2.1 tells us that a solution $\varphi$ of $(tLS)$ exists in $C^1(I,H^s)$ for any $s$ in $\mathbb{N}$. Thus we have

$$\langle U, \psi \rangle = \left\langle U, -\partial_t \varphi - \sum_{k=1}^d A_k \partial_k \varphi + \tilde{A_0} \varphi \right\rangle$$

$$= -\int_I \langle U(t, \cdot), \partial_t \varphi(t, \cdot) \rangle dt - \sum_{k=1}^d \int_I \langle U(t), \partial_k (A_k \varphi)(t) \rangle dt$$

$$+ \int_{I \times \mathbb{R}^d} U(t, x)tA_0 \varphi(t, x)dt\,dx.$$ 

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Considering the weak regularity of $U$, each integration by part must be justified. Using Theorem 6.3.2 page 79 below (the proof of which is totally independant of Proposition 6.2.2), we have that for any $t$ in $I$, the function $\varphi(t, \cdot)$ belongs to $\mathcal{D}(\mathbb{R}^d)$. By definition of the derivative of distributions, we have

$$
\langle U(t), \partial_k(A_k \varphi)(t) \rangle = \sum_{i,j} \langle U^i(t), \partial_k(A_{k,i,j} \varphi^j)(t) \rangle
$$

$$
= \sum_{i,j} \langle \partial_k U^i(t), A_{k,i,j} \varphi^j(t) \rangle.
$$

Because the matrices $A_k$ are symmetric, we have for any $t$ in $I$,

$$
\langle U(t), \partial_k(A_k \varphi)(t) \rangle = \langle A_k \frac{\partial U(t)}{\partial x_k}, \varphi(t) \rangle.
$$

It turns out that

$$
\langle U, \psi \rangle = - \int_I \langle U(t, \cdot), \partial_t \varphi(t, \cdot) \rangle dt - \sum_{k=1}^d A_k \partial_k U - A_0 U, \varphi \rangle.
$$

In order to justify the time integration by part, let us observe that $U$ belongs to $C^1(I, H^{-1})$. Indeed, as for smooth function we have

$$
\langle A_k \partial_k V, \varphi \rangle = - \langle V, \partial_k^t A_k \varphi \rangle - \langle V, ^t A_k \partial_k \varphi \rangle
$$

$$
\leq (\|A_k(t, \cdot)\|_{L^\infty} + a_0(t)) \|V\|_{L^2} \|\varphi\|_{H^1}.
$$

This implies that $\partial_t U$ belongs to $C^0(I; H^{-1})$. Now, let us use the smoothing operator $\mathbb{E}_n$ defined by (6.9). The function $\mathbb{E}_n U$ belongs to $C^1(I; H^s)$ for any $s$. Using this with $s$ greater than $d/2$ implies that for any $x$, the function

$$
t \mapsto \mathbb{E}_n U(t, x)
$$

exists and is a $C^1$ function on $I$. This implies that

$$
- \int_I \mathbb{E}_n U(t, x) \frac{\partial \varphi}{\partial t}(t, x) dt = - \mathbb{E}_n U(T, x) \varphi(T, x) + \mathbb{E}_n U(0, x) \varphi(0, x)
$$

$$
+ \int_I \frac{\partial \mathbb{E}_n U}{\partial t}(t, x) \varphi(t, x) dt.
$$

Using the fact that $U_0 = 0$ and that $\varphi(T, \cdot) = 0$, we get that

$$
- \int_I \mathbb{E}_n U(t, x) \partial_t \varphi(t, x) dt = \int_I \partial_t(\mathbb{E}_n U)(t, x) \varphi(t, x) dt.
$$

By integration in the variable $x$ and interchanging time and space integration, we get that

$$
- \int_I \langle \mathbb{E}_n U(t, \cdot), \partial_t \varphi(t, \cdot) \rangle dt = \int_I \langle \partial_t(\mathbb{E}_n U)(t, \cdot), \varphi(t, \cdot) \rangle dt.
$$

As $U$ is a function of $C(I; L^2) \cap C^1(I; H^{-1})$, we have

$$
\lim_{n \to \infty} \mathbb{E}_n U = U \quad \text{in} \quad L^\infty(I, L^2) \quad \text{and} \quad \lim_{n \to \infty} \mathbb{E}_n \partial_t U = \partial_t U \quad \text{in} \quad L^\infty(I, H^{-1}).
$$
Passing to the limit in the above equality gives

\[-\langle U, \partial_t \varphi \rangle = \int_I \langle \partial_t U(t, \cdot), \varphi(t, \cdot) \rangle dt\]

and thus

\[\langle U, \psi \rangle = \int_I \left( \partial_t U(t, \cdot) + \sum_{k=1}^d A_k \partial_k U(t, \cdot) + A_0 U(t, \cdot), \varphi(t, \cdot) \right) dt\].

Assume \(U\) is solution of (LS) with \(F = 0\), then \(U \equiv 0\) which ends the proof of the proposition \(\Box\)

The whole Theorem 6.2.1 is now proved. \(\Box\)

### 6.3 Finite propagation speed

The phenomena of finite propagation speed is described by the following theorem.

**Theorem 6.3.1** Let \((LS)\) be a symmetric system. A constant \(C_0\) exists such that, for any positive real number \(R\) and any data \(F \in C^0(I, L^2)\) and \(U_0 \in L^2\) such that

\[F(t, x) \equiv 0 \quad \text{when} \quad |x| < R - C_0 t \quad \text{and} \quad U_0(x) \equiv 0 \quad \text{when} \quad |x| < R. \quad (6.14)\]

then the unique solution \(U\) of the system \((LS)\) in \(C^0(I, L^2)\) with data \(F\) and \(U_0\) satisfies

\[U(t, x) \equiv 0 \quad \text{when} \quad |x| < R - C_0 t.\]

An other form of this statement is given by the following corollary.

**Corollary 6.3.1** If the data \(F\) and \(U_0\) satisfy

\[F(t, x) \equiv 0 \quad \text{for} \quad |x| > R + C_0 t \quad \text{and} \quad U_0(x) \equiv 0 \quad \text{for} \quad |x| > R,\]

then the solution \(U\) satisfies

\[U(t, x) \equiv 0 \quad \text{when} \quad |x| > R + C_0 t.\]

**Proof of Theorem 6.3.1.** To start with, let us regularize the data \(U_0\) and \(F\) perturbing their support as less as possible. Let \(\chi\) be a function of \(D(B(0,1))\) the integral of which is 1. For any positive \(\epsilon\), we state

\[\chi_\epsilon(x) \overset{\text{def}}{=} \frac{1}{\epsilon^d} \chi \left( \frac{x}{\epsilon} \right).\]

Now let us consider the data

\[U_{0,\epsilon} \overset{\text{def}}{=} \chi_\epsilon \ast U_0 \quad \text{and} \quad F_\epsilon(t, \cdot) \overset{\text{def}}{=} \chi_\epsilon \ast F(t, \cdot).\]

Of course, we have

\[\text{Supp } U_{0,\epsilon} \subset \text{Supp } U_0 + B(0, \epsilon) \quad \text{and} \quad F_\epsilon(t, \cdot) \subset \text{Supp } F(t, \cdot) + B(0, \epsilon).\]

The support hypothesis are satisfied for \(U_{0,\epsilon}\) and \(F_\epsilon\) with \(R + \epsilon\) instead of \(R\) and the associated solution \(U_\epsilon\) is \(C^1(I; H^s)\) for any \(s \in \mathbb{N}\). Thus it is enough to prove Theorem 6.3.1 with those regular solutions, namely the following statement.
Theorem 6.3.2 Let (LS) be a symmetric system. A constant $C_0$ exists such that, for any positive real number $R$ and any data $F \in C^0(I,H^1) \cap C^1(I,L^2)$ and $U_0 \in H^1$ such that

$$F(t,x) \equiv 0 \quad \text{when} \quad |x| < R - C_0 t \quad \text{and} \quad U_0(x) \equiv 0 \quad \text{when} \quad |x| < R. \quad (6.15)$$

then the unique solution $U$ of the system (LS) in $C^0(I,H^1) \cap C^1(I,L^2)$ with data $F$ and $U_0$ satisfies

$$U(t,x) \equiv 0 \quad \text{when} \quad |x| < R - C_0 t.$$ 

Proof. The method used is weighted energy estimates. More precisely, for $\tau$ greater than 1,

let us introduce $U_{\tau}(t,x) \overset{\text{def}}{=} e^{\tau \phi(t,x)} U(t,x)$.

with $\phi(t,x) = -t + \psi(x)$. The function $\psi$ is a smooth real valued function on $\mathbb{R}^d$ which will be choosen later on.

$$\partial_t U_{\tau} + \sum_{k=1}^d A_k \partial_k U_{\tau} + B_{\tau} U_{\tau} = F_{\tau}$$

with

$$F_{\tau}(t,x) \overset{\text{def}}{=} e^{\tau \phi(t,x)} F(t,x) \quad \text{and} \quad B_{\tau} \overset{\text{def}}{=} A_0 + \tau \left( \partial_t \psi \text{Id} + \sum_{k=1}^d \partial_k \psi A_k \right)$$

Considering the form of the function $\phi$, we have

$$B_{\tau} = A_0 - \tau \left( \text{Id} - \sum_{k=1}^d \partial_k \psi \right).$$

Thus a constant $K > 0$ exists such that for any $(t,x) \in I \times \mathbb{R}^d$, any vector $W \in \mathbb{R}^N$ and any positive real number $\tau$, we have

$$||\nabla \psi||_{L^\infty} \leq K \Rightarrow (B_{\tau}(t,x)W|\vec{W}) \leq (A_0(t,x)W|W).$$

Then let us write the energy estimate and use the above inequality and relation (6.3); we get

$$\frac{d}{dt} |U_{\tau}(t)|_0^2 = -2 \sum_{k=1}^d (A_k \partial_k U_{\tau})_2 L^2 - 2(B_{\tau} U_{\tau})_2 L^2 + 2(F_{\tau} U_{\tau})_2 L^2 \leq a_0(t)|U_{\tau}|_0^2 + (F_{\tau} U_{\tau})_2 L^2$$

Using Gronwall Lemma, we get

$$|U_{\tau}(t)|_0 \leq |U_{\tau}(0)|_0 \exp \int_0^t a_0(t') dt' + \int_0^t |F_{\tau}(t')|_0 \exp \left( \int_0^t a_0(t'') dt'' \right) dt'. \quad (6.16)$$

Let us point out that the above inequality is independant of $\tau$. Now let us state $C_0 = 1/K$ and let us pick up a smooth function $\psi = \psi(|x|)$ such that

$$-2\varepsilon + K(R - |x|) \leq \psi(x) \leq -\varepsilon + K(R - |x|) \quad \text{and} \quad \|
abla \psi\|_{L^\infty} \leq K. \quad (6.17)$$

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Then we have
\[ \forall (t, x) \in I \times \mathbb{R}^d, \ |x| \geq R - C_0 t \implies -t + \psi(x) \leq -\varepsilon. \]

When \( \tau \) tends to \(+\infty\) in the inequality \((6.16)\), we get that
\[ \forall t \in I, \ \lim_{\tau \to +\infty} \int_{\mathbb{R}^d} e^{2\tau \psi(t,x)} |u(t,x)|^2 dx = 0. \]

Thus \( U(t, x) \equiv 0 \) on the open set \( t < \psi(x) \). But, if \((t_0, x_0)\) satisfies \( |x_0| < R - C_0 t_0 \), it is possible to pick up a function \( \psi \) satisfying \((6.17)\) and such that \( t_0 < \psi(x_0) \). This proves the theorem. \( \square \)
Appendix A

Sobolev spaces

Introduction

In this course, we shall restrict ourselves to Sobolev spaces modeled on $L^2$. These spaces definitely play a crucial role in the study of partial differential equations, linear or not. The key tool will be the Fourier transform. This chapter is not treated during the lectures. It is here as a convenient reference for the members of the audience of the lectures.

A.1 Definition of Sobolev spaces on $\mathbb{R}^d$

Definition A.1.1 Let $s$ be a real number, a tempered distribution $u$ belongs to the Sobolev space of index $s$, denoted $H^s(\mathbb{R}^d)$, or simply $H^s$ if no confusion is possible, if and only if

$$\hat{u} \in L^2_{\text{loc}}(\mathbb{R}^d) \quad \text{and} \quad \hat{u}(\xi) \in L^2(\mathbb{R}^d; (1 + |\xi|^2)^s d\xi).$$

and we note

$$\|u\|_{H^s}^2 = \int_{\mathbb{R}^d} (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 d\xi.$$ 

Proposition A.1.1 For any $s$ real number, the space $H^s$, equipped with the norm $\| \cdot \|_{H^s}$, is a Hilbert space.

Proof. The fact that the norm $\| \cdot \|_{H^s}$ comes from the scalar product

$$(u|v)_{H^s} \overset{\text{def}}{=} \int_{\mathbb{R}^d} (1 + |\xi|^2)^s \overline{\hat{u}(\xi)} \hat{v}(\xi) d\xi$$

is obvious. Let us prove that this space is complete. Let $(u_n)_{n \in \mathbb{N}}$ a Cauchy sequence of $H^s$. By definition of the norm, the sequence $(\hat{u}_n)_{n \in \mathbb{N}}$ is a Cauchy sequence of $L^2(\mathbb{R}^d; (1 + |\xi|^2)^s d\xi)$. Thus, a function $\tilde{u}$ exists in the space $L^2(\mathbb{R}^d; (1 + |\xi|^2)^s d\xi)$ such that

$$\lim_{n \to \infty} \|\hat{u}_n - \hat{u}\|_{L^2(\mathbb{R}^d; (1 + |\xi|^2)^s d\xi)} = 0. \quad (A.1)$$

In particular, the sequence $(\hat{u}_n)_{n \in \mathbb{N}}$ tends to $\hat{u}$ in the space $S'$ of tempered distributions. Let us define $u = F^{-1}\tilde{u}$. As the Fourier transform is an isomorphism of $S'$, the sequence $(u_n)_{n \in \mathbb{N}}$ tends to $u$ in the space $S'$, and also in $H^s$ thanks to (A.1).

Shortly said, this is nothing more than observing that the Fourier transform is an isometric isomorphism from $H^s$ onto $L^2(\mathbb{R}^d; (1 + |\xi|^2)^s d\xi)$. \qed
Proposition A.1.2 Let $s$ be a non negative integer, the space $H^s(\mathbb{R}^d)$ is the space of functions $u$ of $L^2$ all the derivatives of which of order less or equal to $m$ are distributions which belongs to $L^2$. Moreover, the space $H^m$ equipped with the norm

$$
\|u\|_{H^m}^2 \overset{\text{def}}{=} \sum_{|\alpha| \leq m} \|\partial^\alpha u\|^2_{L^2}
$$

is a Hilbert space and this norm is equivalent to the norm $\|\cdot\|_{H^s}$.

Proof. The fact that

$$
\|u\|_{H^m}^2 = (u|u)_{H^m} \quad \text{with} \quad (u|v)_{H^m} \overset{\text{def}}{=} \sum_{|\alpha| \leq m} \int_{\mathbb{R}^d} \partial^\alpha u(x) \partial^\alpha v(x) dx.
$$

ensures that the norm $\|\cdot\|_{H^m}$ comes from a scalar product. Moreover, a constant $C$ exists such that

$$
\forall \xi \in \mathbb{R}^d, \quad C^{-1} \left(1 + \sum_{0<|\alpha| \leq m} |\xi|^{2|\alpha|}\right) \leq (1 + |\xi|^2)^s \leq C \left(1 + \sum_{0<|\alpha| \leq m} |\xi|^{2|\alpha|}\right). \quad (A.2)
$$

As the Fourier transform is, up to a constant, an isometric isomorphism from $L^2$ onto $L^2$, we have

$$
\partial^\alpha u \in L^2 \iff \xi^\alpha \hat{u} \in L^2.
$$

Thus, we have deduce that

$$
u \in H^m \iff \forall \alpha / |\alpha| \leq m, \partial^\alpha u \in L^2.
$$

Inequality (A.2) ensures the equivalence of the two norms using again the fact that the Fourier transform is a isometric isomorphism up to a constant. The proposition is proved. \qed

Exercice A.1.1 Prove that the space $\mathcal{S}$ is continuously included in the space $H^s$ for any real $s$.

Exercice A.1.2 Prove that the mass of Dirac $\delta_0$ belongs to the space $H^{-\frac{d}{2}-\varepsilon}$ for any positive real number $\varepsilon$. Prove that $\delta_0$ does not belong to the space $H^{-\frac{d}{2}}$.

Exercice A.1.3 Prove that, for any distribution to support compact $u$, it exists a real number $s$ such that $u$ belongs to the Sobolev space $H^s$.

Exercice A.1.4 Prove that the constant 1 does not belong to $H^s$ for any real number $s$.

Proposition A.1.3 Let $s$ a real number of the interval $]0,1[$. Prove that the space $H^s$ is the space des functions $u$ of $L^2$ such that

$$
\int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|u(x+y) - u(x)|^2}{|y|^{2s+d}} dxdy.
$$

Moreover, a constant $C$ exists such that, for any function $u$ of $H^s$, we have

$$
C^{-1}\|u\|_{H^s}^2 \leq \|u\|_{L^2}^2 + \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|u(x+y) - u(x)|^2}{|y|^{2s+d}} dxdy \leq C\|u\|_{H^s}^2.
$$
Proof. Thanks to Fourier-Plancherel identity, we can write that
\[
\int_{\mathbb{R}^d} \frac{|u(x+y) - u(x)|^2}{|y|^{d+2s}} \, dx = \int_{\mathbb{R}^d} \frac{|e^{i(y|\xi|)} - 1|^2}{|y|^{d+2s}} |\hat{u}(\xi)|^2 \, d\xi < \infty.
\]
It turns out that
\[
\int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|u(x+y) - u(x)|^2}{|y|^{d+2s}} \, dx \, dy = \int_{\mathbb{R}^d} F(\xi) |\hat{u}(\xi)|^2 \, d\xi \quad \text{with}
\]
\[
F(\xi) \overset{\text{def}}{=} \int_{\mathbb{R}^d} \frac{|e^{i(y|\xi|)} - 1|^2}{|y|^{2s}} \, dy \cdot |y|^d.
\]
By an obvious change of variable, we see that the function $F$ is radial and homogeneous of degree $2s$. Thus
\[
F(\xi) = |\xi|^{2s} \int_{\mathbb{R}^d} \frac{|e^{iy_1} - 1|^2}{|y|^{2s}} \, dy \cdot |y|^d.
\]
This concludes the proof of the proposition. \(\square\)

Let us prove now an interpolation inequality which will be very useful.

**Proposition A.1.4** If $s = \theta s_1 + (1-\theta)s_2$ with $\theta$ in $[0,1]$, then, we have
\[
\|u\|_{H^s} \leq \|u\|_{H^{s_1}}^{\theta} \|u\|_{H^{s_2}}^{1-\theta}.
\]

The proof consists in applying H"older inequality with the measure $|\hat{u}(\xi)|^2 \, d\xi$ and the two functions $(1 + |\xi|^2)^{\theta s_1}$ and $(1 + |\xi|^2)^{(1-\theta)s_2}$.

**Theorem A.1.1** Let $s$ a real quelconque;

- the space $\mathcal{D}(\mathbb{R}^d)$ is dense in $H^s(\mathbb{R}^d)$,
- the multiplication by a function of $S$ is a continuous function of $H^s$ into itself.

**Proof.** In order to prove the first point of this theorem, let us consider a distribution $u$ of $H^s$ such that, for any test function $\varphi$, we have $(\varphi|u)_{H^s} = 0$. This means that, for any test function $\varphi$, we have
\[
\int_{\mathbb{R}^d} \hat{\varphi}(\xi)(1 + |\xi|^2)^s \hat{u}(\xi) \, d\xi = 0.
\]
which means that $(1 + |\cdot|^2)^s \hat{u} = 0$ as a tempered distribution. As the multiplication by the function $(1 + |\cdot|^2)^{-s}$ is continuous in $S'$, we have that $\hat{u} = 0$ as a tempered distribution. Thus $u \equiv 0$.

Let us prove now the second second point of the theorem. This proof is presented here just for culture. We know that
\[
\hat{\varphi} \hat{u} = (2\pi)^{-d} \hat{\varphi} \ast \hat{u}.
\]
The point is to estimate the $L^2$ norm of the function defined by
\[
U(\xi) = (1 + |\xi|^2)^{\frac{s}{2}} \int_{\mathbb{R}^d} |\hat{\varphi}(\xi - \eta)| \times |\hat{u}(\eta)| \, d\eta.
\]
We shall use the following lemma.
Lemma A.1.1 For any \((a, b)\) in \(\mathbb{R}^d\), for any \(s \in \mathbb{R}\), we have
\[
(1 + |a + b|^2)^{-\frac{s}{2}} \leq 2^{\frac{|s|}{2}} (1 + |a|^2)^{-\frac{s}{2}} (1 + |b|^2)^{-\frac{s}{2}}.
\]

Proof. Let us first observe that
\[
1 + |a + b|^2 \leq 1 + 2(|a|^2 + |b|^2) \leq 2(1 + |a|^2)(1 + |b|^2).
\]
Taking the power \(s/2\) of this inequality, we find the result for non negative \(s \geq 0\). In the case when \(s\) is negative, we have
\[
(1 + |b|^2)^{-\frac{s}{2}} \leq 2^{-\frac{s}{2}} (1 + |a + b|^2)^{-\frac{s}{2}} (1 + |b|^2)^{-\frac{s}{2}}.
\]
Thus the result is proved.

Continuation of the proof of Theorem A.1.1 Lemma A.1.1 implies that
\[
U(\xi) \leq \int_{\mathbb{R}^d} (1 + |\xi - \eta|^2)^{\frac{|d|}{2}} \hat{\varphi}(\xi - \eta) \times (1 + |\eta|^2)^{\frac{d}{2}} |\hat{u}(\eta)| d\eta.
\]
Young’s law implies that \(\|U_2\|_{L^2} \leq C\|u\|_{H^s}\); this concludes the proof of the theorem.

Exercice A.1.5 Let \(\mathcal{F}L^1 = \{u \in \mathcal{S}' / \hat{u} \in L^1\}\). Prove that, for any non negative real number \(s\), the product is a bilinear continuous map from \(\mathcal{F}L^1 \cap H^s \times \mathcal{F}L^1 \cap H^s\) into \(\mathcal{F}L^1 \cap H^s\). What happens when \(s\) is greater than \(d/2\)?

Exercice A.1.6 Let \(s\) a real number greater than \(1/2\). Prove that the map \(\gamma\) defined by
\[
\gamma \left\{ \begin{array}{l}
\mathcal{D}(\mathbb{R}^d) \rightarrow \mathcal{D}(\mathbb{R}^{d-1}) \\
\varphi \mapsto \gamma(\varphi) : (x_2, \ldots, x_d) \mapsto \varphi(0, x_2, \ldots, x_d)
\end{array} \right.
\]
can be extended in a continuous onto map from \(H^s(\mathbb{R}^d)\) onto \(H^{s-\frac{1}{2}}(\mathbb{R}^{d-1})\).

Hint: Write
\[
\mathcal{F}_{\mathbb{R}^{d-1}} \hat{\varphi}(0, \xi_2, \ldots, \xi_d) = (2\pi)^{-1} \int_{\mathbb{R}} \hat{\varphi}(\xi_1, \xi_2, \ldots, \xi_d)d\xi_1.
\]
and for the fact that the map in onto, observe that, if
\[
u = (2\pi)^{-(n-1)} C_n \mathcal{F}^{-1} \left( \frac{1 + |\xi'|^2}{{2(1 + |\xi|^2)^s}^{\frac{1}{2}}} \hat{\nu}(\xi') \right),
\]
then \(\nu \in H^s\) and \(\gamma(\nu) = \nu\).

Let us prove a theorem which describes the dual of the space \(H^s\).

Theorem A.1.2 The bilinear form \(B\) defined by
\[
\begin{array}{ll}
B \left\{ \begin{array}{l}
\mathcal{S} \times \mathcal{S} \rightarrow \mathbb{C} \\
(u, \varphi) \mapsto \int_{\mathbb{R}^d} u(x) \varphi(x) dx
\end{array} \right.
\end{array}
\]
can be extended as a bilinear form continuous from \(H^{-s} \times H^s\) to \(\mathbb{C}\). Moreover, the map \(\delta_B\) defined by
\[
\delta_B \left\{ \begin{array}{l}
H^{-s} \rightarrow (H^s)' \\
u \mapsto \delta_B(\nu) : \varphi \mapsto B(u, \varphi)
\end{array} \right.
\]
is a linear and isometric isomorphism (up to a constant), which means that the bilinear form \(B\) identifies the space \(H^{-s}\) to the dual space of \(H^s\).
Proof. The important point of the proof of this theorem is inverse Fourier formula which ensures that, for any couple \((u, \varphi)\) of functions of \(S\), we have

\[
B(u, \varphi) = \int_{\mathbb{R}^d} u(x)\varphi(x) dx
= \int_{\mathbb{R}^d} u(x)\mathcal{F}(\mathcal{F}^{-1}\varphi)(x) dx
= \int_{\mathbb{R}^d} \hat{u}(\xi)(\mathcal{F}^{-1}\varphi)(\xi) d\xi
= (2\pi)^{-d} \int_{\mathbb{R}^d} \hat{u}(\xi)\hat{\varphi}(-\xi) d\xi.
\] (A.3)

Multiplying and dividing by \((1 + |\xi|^2)^{\frac{1}{2}}\), we immediately get thanks to Cauchy-Schwarz inequality,

\[
|B(u, \varphi)| \leq (2\pi)^{-d} \|u\|_{H^s} \|\varphi\|_{H^{-s}}.
\]

Thus the first point of the theorem. The fact that the map \(\delta_B\) is one to one comes from the fact that if, for any function \(\varphi\) in \(S\), we have \(B(u, \varphi) = 0\), then \(u = 0\). We shall prove that \(\delta_B\) is one to one and onto.

Let \(\phi\) and \(\varphi\) be in \(S_0\). One can write that

\[
\left|\int_{\mathbb{R}^d} \phi(x)\varphi(x) dx\right| = \left|\int_{\mathbb{R}^d} (\mathcal{F}^{-1}\phi)(\xi)(\mathcal{F}\varphi)(\xi) d\xi\right|
= (2\pi)^{-d} \left|\int_{\mathbb{R}^d} |\xi|^{-s} \hat{\phi}(-\xi)|\xi|^s \hat{\varphi}(\xi) d\xi\right|
\leq (2\pi)^{-d} \|\phi\|_{H^{-s}} \|\varphi\|_{H^s}.
\]

As \(S_0\) is dense in \(\dot{H}^s\) when \(|\sigma| < d/2\), then one can extend \(B\) to \(\dot{H}^{-s} \times \dot{H}^s\). Of course, if \((u, \phi)\) is in \(\dot{H}^{-s} \times S\) then \(B(u, \phi) = \langle u, \phi \rangle\).

Let \(L\) be a linear functional on \(\dot{H}^s\). Consider the linear functional \(L_s\) defined by

\[
L_s : \begin{cases} 
L^2(\mathbb{R}^d) & \longrightarrow \mathbb{C} \\
f & \mapsto \langle L, \mathcal{F}^{-1}((1 + |\cdot|^{2})^{-\frac{1}{2}} f)\rangle.
\end{cases}
\]

It is obvious that

\[
\sup_{\|f\|_{L^2} = 1} |\langle L_s, f \rangle| = \sup_{\|f\|_{L^2} = 1} |\langle L, \mathcal{F}^{-1}((1 + |\cdot|^{2})^{-\frac{1}{2}} f)\rangle|
= \sup_{\|\phi\|_{H^s} = 1} |\langle L, \phi \rangle|
= \|L\|_{(\dot{H}^s)'}.
\]

Riesz representation theorem implies that a function \(g\) exists in \(L^2\) such that

\[
\forall h \in L^2, \quad \langle L_s, h \rangle = \int_{\mathbb{R}^d} g(\xi)h(\xi) d\xi.
\]

Stating \(\phi(\xi) \overset{\text{def}}{=} \mathcal{F}^{-1}((1 + |\xi|^2)^{-\frac{1}{2}} h)(-\xi)\), we infer that, for any \(\phi\) in \(H^s\)

\[
\langle L, \phi \rangle = \langle L_s, h \rangle
= \int_{\mathbb{R}^d} g(\xi)(1 + |\xi|^2)^{\frac{1}{2}} \hat{\phi}(-\xi) d\xi.
\]

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Then stating $u \overset{\text{def}}{=} F^{-1}((1 + |\cdot|^2)^{\frac{3}{2}} g)$, we infer that

$$\langle L, \phi \rangle = \int_{\mathbb{R}^d} \hat{u}(\xi) \hat{\phi}(-\xi) d\xi$$

and the theorem is proved.

\[ \square \]

### A.2 Sobolev embeddings

The purpose of this section is the study of embedding properties of Sobolev spaces $H^s(\mathbb{R}^d)$ into $L^p$ spaces. Let us prove the following theorem.

**Theorem A.2.1** If $s$ is greater than $d/2$, then the space $H^s$ is continuously included in the space of continuous functions which tend to 0 at infinity. If $s$ is a positive real number less than $d/2$, then the space $H^s$ is continuously included in $L^2$ and we have

$$k f k_{L^p} \leq C k f k_{H^s} \quad \text{with} \quad k f k_{H^s} \overset{\text{def}}{=} \left( \int_{\mathbb{R}^d} |\xi|^{2s}|\hat{f}(\xi)|^2 d\xi \right)^{\frac{1}{2}}.$$

**Proof.** The first point of this theorem is very easy to prove. Let us use the fact that $k u k_{L^1} \leq (2\pi)^{-d} k \hat{u} k_{L^1}$ (A.4)

Indeed, if $s$ is greater than $d/2$, we have,

$$|\hat{u}(\xi)| \leq (1 + |\xi|^2)^{-s/2}(1 + |\xi|^2)^{-s/2} |\hat{u}(\xi)|.$$ (A.5)

The fact that $s$ is greater than $d/2$ implies that the function

$$\xi \mapsto (1 + |\xi|^2)^{-s/2}$$

belongs to $L^2$. Thus, we have

$$k \hat{u} k_{L^1} \leq \left( \int (1 + |\xi|^2)^{-s} d\xi \right)^{\frac{1}{2}} k f k_{H^s}.$$ 

The first point of the theorem is proved.

The proof of the second point is more delicate. A way to understand the index $p = 2d/(d - 2s)$ is the use of a scaling argument. Let us consider a function $v$ on $\mathbb{R}^d$ and let us denote by $v_\lambda$ the function $v_\lambda(x) = v(\lambda x)$. We have

$$k v_\lambda k_{L^p} = \lambda^{-\frac{d}{p}} k v k_{L^p}$$

and also

$$\int |\xi|^{2s} |\hat{v}_\lambda(\xi)|^2 d\xi = \lambda^{-2d} \int |\xi|^{2s} |\hat{v}(\lambda^{-1} \xi)|^2 d\xi$$

$$= \lambda^{-d + 2s} k v k_{H^s}^2,$$

with

$$k v k_{H^s}^2 \overset{\text{def}}{=} \int_{\mathbb{R}^d} |\xi|^{2s} |\hat{v}(\xi)|^2 d\xi.$$
The two quantities $\| \cdot \|_{L^p}$ and $\| \cdot \|_{H^s}$ have the same scaling, which means that they have the same behaviour with respect to changes of unit. Thus, it make sense to compare them. Multiplying $f$ by a positive real number, it is enough to prove the inequality in the case when $\| f \|_{H^s} = 1$. On utilise then the fact that for any $p$ de the interval $[1, +\infty]$, we have, for any function measurable $f$,

$$
\| f \|_{L^p}^p = p \int_0^\infty \lambda^{p-1} m(|f| > \lambda) d\lambda.
$$

Let us decompose $f$ in a low and in a high frequencies by writing $f = f_{1,A} + f_{2,A}$ with $f_{1,A} = \mathcal{F}^{-1}(1_{B(0,A)} \hat{f})$ and $f_{2,A} = \mathcal{F}^{-1}(1_{B^c(0,A)} \hat{f})$. (A.6) As the support of the Fourier transform of $f_{1,A}$ is compact, the function $f_{1,A}$ is bounded and more precisely,

$$
\| f_{1,A} \|_{L^\infty} \leq (2\pi)^{-d} \| \widehat{f_{1,A}} \|_{L^1} \leq (2\pi)^{-d} \int_{B(0,A)} |\xi|^{-s} |\xi|^s \hat{f}(\xi) d\xi \leq (2\pi)^{-d} \left( \int_{B(0,A)} |\xi|^{-2s} d\xi \right)^{1/2} \leq \frac{C}{(d - 2s)^{1/2}} A^{\frac{d}{2} - s}.
$$

(A.7) The triangle inequality implies that, for any positive real number $A$,

$$
(|f| > \lambda) \subset (2|f_{1,A}| > \lambda) \cup (2|f_{2,A}| > \lambda).
$$

Using Inequality (A.7), we have

$$
A = A_\lambda \overset{\text{def}}{=} \left( \frac{\lambda(d - 2s)^{1/2}}{4C} \right)^{\frac{2}{d}} \implies m \left( |f_{1,A}| > \frac{\lambda}{2} \right) = 0.
$$

Thus we deduce that

$$
\| f \|_{L^p}^p = p \int_0^\infty \lambda^{p-1} m(2|f_{2,A}| > \lambda) d\lambda.
$$

It is well known (this is Bienaimé-Tchebychev inequality) that

$$
m \left( |f_{2,A}\lambda| > \frac{\lambda}{2} \right) = \int_{|f_{2,A}\lambda| > \frac{\lambda}{2}} dx \leq \int_{|f_{2,A}\lambda| > \frac{\lambda}{2}} \frac{4|f_{2,A}\lambda(x)|^2}{\lambda^2} dx \leq \frac{4 \| f_{2,A}\lambda \|_{L^2}^2}{\lambda^2}.
$$

For such a choice of $A$, we have

$$
\| f \|_{L^p}^p \leq 4p \int_0^\infty \lambda^{p-3} \| f_{2,A}\lambda \|_{L^2}^2 d\lambda.
$$

(A.8)
As the Fourier transform is (up to a constant) an isometric isomorphism of $L^2$, we have

$$\|f_{2, A_\lambda}\|_{L^2}^2 = (2\pi)^{-d} \int_{|\xi| \geq A_\lambda} |\hat{f}(\xi)|^2 d\xi.$$  

Thanks to Inequality (A.8), we get

$$\|f\|_{L^p}^p \leq 4p(2\pi)^{-d} \int_{\mathbb{R}^d} \lambda^{p-3} \mathbf{1}_{\{(\lambda, \xi) / |\xi| \geq A_\lambda\}} (\lambda, \xi) |\hat{f}(\xi)|^2 d\lambda d\xi.$$  

By definition of $A_\lambda$, we have

$$|\xi| \geq A_\lambda \iff \lambda \leq \frac{4C}{(d-2s)^\frac{p}{2}} |\xi|^\frac{s}{p}.$$  

Fubini’s theorem implies that

$$\|f\|_{L^p}^p \leq 4p(2\pi)^{-d} \int_{\mathbb{R}^d} \left( \int_0^C \lambda^{p-3} d\lambda \right) |\hat{f}(\xi)|^2 d\xi \leq 4p(2\pi)^d \left( \frac{4C}{(d-2s)^\frac{p}{2}} \right)^{p-2} \int_{\mathbb{R}^d} |\xi|^\frac{d(p-2)}{p} |\hat{f}(\xi)|^2 d\xi.$$  

As $2s = \frac{d(p-2)}{p}$, the theorem is proved.

**Corollary A.2.1** Let $p$ be in $]2, \infty[$, and $s$ greater than $s_p \overset{\text{def}}{=} \left( \frac{1}{2} - \frac{1}{p} \right)$. We have

$$\|u\|_{L^p} \leq C \|u\|_{L^2}^{1-\theta} \|u\|_{\dot{H}^s}^\theta \quad \text{with} \quad \theta = \frac{s_p}{s}.$$  

**Proof.** It is an application of the above theorem together with the fact that

$$\|u\|_{\dot{H}^{s_1} \cap (1-s_2)_{\mathcal{H}}} \leq \|u\|_{\dot{H}^{s_1}}^{1-\theta} \|u\|_{\dot{H}^{s_2}}^\theta.$$  

### A.3 Homogeneous Sobolev spaces

**Definition A.3.1** Let $s$ be a real number, the homogeneous Sobolev space $\dot{H}^s$ is the space of tempered distributions such that $\hat{u}$ belongs to $L_{\text{loc}}^1$ and satisfies

$$\|u\|_{\dot{H}^s}^2 \overset{\text{def}}{=} \int_{\mathbb{R}^d} |\xi|^{2s} |\hat{u}(\xi)|^2 d\xi < \infty.$$  

These spaces (or at least their norms) naturally appeared in the proof of Theorem A.2.1. The $\| \cdot \|_{\dot{H}^s}$ norm has the following scaling property

$$\|f(\lambda \cdot)\|_{\dot{H}^s} = \lambda^{-\frac{d}{2} + s} \|f\|_{\dot{H}^s}.$$  

These spaces are different from the inhomogeneous $H^s$ spaces. Let us notice that if $s$ is positive, then $H^s$ is included in $\dot{H}^s$ but that if $s$ is negative, then $\dot{H}^s$ is included in $H^s$. The inhomogeneous spaces is a decreasing family of spaces (with respect to the index $s$). The homogeneous ones are not comparable together.

We shall only consider these homogeneous spaces in the case when $s$ is less than the half dimension.
**Proposition A.3.1** If \( s < d/2 \), then the space \( \hat{H}^s \) is a Banach space.

**Proof.** Let \((u_n)_{n \in \mathbb{N}}\) a Cauchy sequence of \( \hat{H}^s \). The sequence \((\hat{a}_n)_{n \in \mathbb{N}}\) is a Cauchy one in the Banach space \( L^2(\mathbb{R}^d \setminus \{0\}; |\xi|^{2s}d\xi) \). Let \( f \) be its limit. It is clear that \( f \) belongs to \( L^1_{\text{loc}}(\mathbb{R}^d \setminus \{0\}) \). Moreover,

\[
\int_{B(0,1)} |f(\xi)|d\xi \leq \left( \int_{\mathbb{R}^d} |\xi|^{2s}f(\xi)^2d\xi \right)^{\frac{1}{2}} \left( \int_{B(0,1)} |\xi|^{-2s}d\xi \right)^{\frac{1}{2}} < \infty
\]

because \( s \) is less than the half-dimension. Thus \( \hat{f} \) belongs to \( S' \) and to \( L^1_{\text{loc}} \). Thus \( u \stackrel{\text{def}}{=} F^{-1} f \) is well defined, belongs to \( \hat{H}^s \), and is the limit of the sequence \((u_n)_{n \in \mathbb{N}}\) in the sense of the norm \( \hat{H}^s \).

**Exercice A.3.1**

1) Prove that the space

\[ B \stackrel{\text{def}}{=} \{ u \in S'(\mathbb{R}^d), \hat{u} \in L^1(B(0,1);d\xi) \cap L^2(\mathbb{R}^d;|\xi|^{2s}d\xi) \} \]

equipped with the norme \( N(u) \stackrel{\text{def}}{=} \| \hat{u} \|_{L^1(B(0,1))} + \| u \|_{\hat{H}^s} \) is a Banach space.

2) Let \( s \geq d/2 \). Give an example of a sequence \((f_n)_{n \in \mathbb{N}}\) of \( B \), bounded in \( \hat{H}^s(\mathbb{R}^d) \), such that

\[ \lim_{n \to \infty} N(f_n) = +\infty. \]

3) Then deduce that \( (\hat{H}^s, \| \cdot \|_{\hat{H}^s}) \) is not a Banach space.

**Exercice A.3.2** Prove that, if \( k \in \mathbb{N} \), then we have

\[ \dot{H}^{-k}(\mathbb{R}^d) = \left\{ u \in S'(\mathbb{R}^d), \ u = \sum_{|\alpha| = k} \partial^\alpha f_\alpha \ \text{with} \ f_\alpha \in L^2 \right\}. \]

Prove that a constant \( C \) exists such that

\[ C^{-1} \| u \|_{\dot{H}^{-k}} \leq \inf \left\{ \left( \sum_{|\alpha| = k} \| f_\alpha \|_{L^2} \right)^{\frac{1}{2}} / u = \sum_{|\alpha| = k} \partial^\alpha f_\alpha \right\} \leq C \| u \|_{\dot{H}^{-k}}. \]

**A.4 The spaces \( H^1_0(\Omega) \) and \( H^{-1}(\Omega) \)**

**Definition A.4.1** Let \( \Omega \) a domain of \( \mathbb{R}^d \), the space \( H^1_0(\Omega) \) is defined as the closure of \( \mathcal{D}(\Omega) \) in the sense of the norm \( H^1(\mathbb{R}^d) \).

The space \( H^{-1}(\Omega) \) is the set of distributions \( u \) on \( \Omega \) such that

\[ \| u \|_{H^{-1}(\Omega)} \stackrel{\text{def}}{=} \sup_{f \in \mathcal{D}(\Omega)} \frac{|\langle u, f \rangle|}{\| f \|_{H^1_0(\Omega)}^1} < \infty. \]

**Proposition A.4.1** The space \( H^1_0(\Omega) \) is a Hilbert space equipped with the norm

\[ \left( \| u \|_{L^2(\Omega)}^2 + \| \nabla u \|_{L^2(\Omega)}^2 \right)^{\frac{1}{2}}. \]

The proof is an easy exercize left to the reader. The space \( H^{-1}(\Omega) \) can be indentified to the dual space of \( H^1_0(\Omega) \) thanks to the following theorem.
Theorem A.4.1  The bilinear map defined by
\[
B \left\{ \begin{array}{c}
H^{-1}(\Omega) \times \mathcal{D}(\Omega) \\ (u, \varphi)
\end{array} \right\} \longrightarrow \mathbb{C}
\]
\[
(u, \varphi) \longmapsto \langle u, \varphi \rangle
\]
can be extended to a bilinear continuous map from \( H^{-1}(\Omega) \times H^1_0(\Omega) \) into \( \mathbb{C} \), still denoted by \( B \). Moreover, the map \( \delta_B \) defined by
\[
\delta_B \left\{ \begin{array}{c}
H^{-1}(\Omega) \\ u
\end{array} \right\} \longrightarrow (H^1_0(\Omega))' 
\]
\[
u \longmapsto \delta_B(u)(\varphi) \overset{\text{def}}{=} B(u, \varphi)
\]
is a linear isometric isomorphism between the space \( H^{-1}(\Omega) \) and the dual space of \( H^1_0(\Omega) \).

Proof. The fact that the bilinear map \( B \) can be extended because \( B \) is uniformly continuous. Let \( \ell \) a linear form continuous on \( H^1_0(\Omega) \). Its restriction on \( \mathcal{D}(\Omega) \) is a distribution \( u \) on \( \Omega \) such that
\[
\forall \varphi \in \mathcal{D}(\Omega), \quad \langle u, \varphi \rangle = \langle \ell, \varphi \rangle.
\]
By definition of the norm on \( (H^1_0(\Omega))' \), the theorem is proved. \( \square \)

Remark  The space \( H^{-1}(\Omega) \) is exactly the set of the restrictions to \( \Omega \) of the distributions which belongs to \( H^{-1}(\mathbb{R}^d) \). Indeed, let us observe that any linear form on \( H^1_0(\Omega) \) can be extended to \( H^1(\mathbb{R}^d) \) simply by the extension which is 0 on the orthogonal of the closed subspace \( H^1_0(\Omega) \) is the space \( H^1(\mathbb{R}^d) \).

Theorem A.4.2 (Poincaré Inequality) Let \( \Omega \) be bounded open subset of \( \mathbb{R}^d \). A constant \( C \) exists such that
\[
\forall \varphi \in H^1_0(\Omega), \quad \| \varphi \|_{L^2} \leq C \left( \sum_{j=1}^{d} \| \partial_j \varphi \|_{L^2}^2 \right)^{\frac{1}{2}}.
\]

Proof. Let \( R \) a positive real number such that \( \Omega \) is included in \( ]-R, R[ \times \mathbb{R}^{d-1} \). Then, for any test function \( \varphi \), we have
\[
\varphi(x_1, \cdots, x_d) = \int_{-R}^{x_1} \frac{\partial \varphi}{\partial y_1}(y_1, x_2, \cdots, x_d) dy_1.
\]
Cauchy-Schwarz Inequality implies that
\[
|\varphi(x_1, \cdots, x_d)|^2 \leq 2R \int_{-R}^{x_1} \left| \frac{\partial \varphi}{\partial y_1}(y_1, x_2, \cdots, x_d) \right|^2 dy_1.
\]
By integration in \( x_1 \), we get
\[
\int_{\Omega} |\varphi(x_1, \cdots, x_d)|^2 dx_1 \leq 2R \int_{\Omega \times ]-R, R[} \left| \frac{\partial \varphi}{\partial y_1}(y_1, x_2, \cdots, x_d) \right|^2 dy_1.
\]
Then, integrating with respect to the other \( d-1 \) variables, we find
\[
\int_{\Omega} |\varphi(x_1, \cdots, x_d)|^2 dx \leq 2R \int_{\Omega \times ]-R, R[} \left| \frac{\partial \varphi}{\partial y_1}(y_1, x_2, \cdots, x_d) \right|^2 dy_1 dx_2 \cdots dx_d
\]
\[
\leq 4R^2 \sum_{j=1}^{d} \| \partial_j \varphi \|_{L^2}^2.
\]
As \( \mathcal{D}(\Omega) \) is dense in \( H^1_0(\Omega) \), the theorem is proved. \( \square \)

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It obviously implies the following corollary.

**Corollary A.4.1** The space $H_0^1(\Omega)$ equipped with the norm

$$
u \mapsto \left( \sum_{j=1}^{d} \| \partial_j \nu \|_{L^2}^2 \right)^{\frac{1}{2}} \overset{\text{def}}{=} \| \nabla \nu \|_{L^2}$$

is a Hilbert space and this norm is equivalent to the previous one.

In order to conclude this chapter, let us prove the following very important compactness theorem.

**Theorem A.4.3** For any positive real number $s$, $M$ and $R$, the set

$$B_{M,R}^s \overset{\text{def}}{=} \left\{ u \in H^s(\mathbb{R}^d) / \| u \|_{H^s} \leq M \text{ and } u|_{B(0,R)} = 0 \right\}$$

is relatively compact in $L^2$.

**Proof.** It is enough to prove that for any positive real number $\varepsilon$, the set $B_{M,R}^s$ can be covered by a finite number of $L^2$ balls of radius $\varepsilon$. The first step consists in a cut off in Fourier space, the second step in a cutoff in physical space and then the theorem follows from Ascoli’s compactness theorem.

Let us consider a function $\chi$ in $\mathcal{S}(\mathbb{R}^d)$ such that the Fourier transform has value 1 on the ball centered at 0 and of radius 1 and has values between 0 and 1. Let us consider the family $(\chi_\alpha)_{\alpha > 0}$ defined by

$$\chi_\alpha(x) \overset{\text{def}}{=} \frac{1}{\alpha^d} \chi\left( \frac{x}{\alpha} \right).$$

Let us observe that, for any $u$ in $H^s$ the $H^s$ norm of which is less than or equal to $M$ satisfies

$$\| u - \chi_\alpha * u \|_{L^2} = (2\pi)^d \int_{\mathbb{R}^d} (1 - \widehat{\chi}(\alpha \xi))^2 (1 + |\xi|^2)^{-s} (1 + |\xi|^2)^{2s} |\widehat{u}(\xi)|^2 d\xi$$

$$\leq \left( 1 + \frac{1}{\alpha^s} \right)^{-2s} \| u \|_{H^s}^2$$

$$\leq \alpha^{2s} M^2.$$

The we get that

$$\alpha \leq \left( \frac{\varepsilon}{3M} \right)^{\frac{1}{2}} \implies \forall u \in B_{M,R}^s, \| u - \chi_\alpha * u \|_{L^2} \leq \frac{\varepsilon}{3}. \quad (A.9)$$

Let us notice that this part does not use the hypothesis of compact support.

For the cutoff in the physical space, let us observe that

$$|x| \geq 2R \implies \forall y \in B(0,R), \ |x - y| \geq \frac{1}{2} |x|. $$

Thus we infer that, for any $\alpha$ and any $u$ in $B_{M,R}^s$

$$|x| \left| \chi_\alpha * u(x) \right| \leq 2 \alpha^{1-d} \int_{B(0,R)} \frac{|x - y|}{\alpha} \left| \chi \left( \frac{x - y}{\alpha} \right) u(y) \right| dy.$$

We deduce from law of convolution that

$$\forall u \in B_{M,R}^s, \| \left( |x| \chi_\alpha \right) * u \|_{L^2} \leq C \alpha M$$

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\[ \alpha \leq \frac{\varepsilon R}{3CM} \implies \forall u \in B_{M,R}^s, \|u - \chi_{\alpha \varepsilon} * u\|_{L^2(B(0,R))} \leq \frac{\varepsilon}{3}. \] (A.10)

Now let us notice that, for any positive \( \alpha \)

\[ \|\chi_{\alpha} * u\|_{L^\infty} \leq C\alpha^{-\frac{2}{3}} M \quad \text{and} \quad \|\partial_j(\chi_{\alpha} * u)\|_{L^\infty} \leq C\alpha^{-\frac{2}{3}} M. \]

Thus, because of Ascoli’s theorem, for any positive \( \alpha \) the set \( \{ \chi_{\alpha} * u \}_{B(0,2R)} \) can be recovered by a family number of balls of radius \( \varepsilon/(3\text{Vol}B(0,2R)^{\frac{1}{3}}) \) in \( L^\infty(B(0,2R)) \). But balls of radius \( \varepsilon/(3\text{Vol}B(0,2R)^{\frac{1}{3}}) \) are included in balls of radius \( \varepsilon/3 \) in \( L^2(B(0,2R)) \). Then choosing

\[ \alpha_\varepsilon \overset{\text{def}}{=} \min \left\{ \left( \frac{\varepsilon}{3M} \right)^{\frac{1}{3}}, \frac{\varepsilon R}{3CM} \right\} \]

we get the results thanks to Assertions (A.9) and (A.10).

\[ \square \]

### A.5 The Lindelhöf principle

**Lemma A.5.1**
Liste des questions de cours pour l’examen

• Théorème de Cauchy-Lipschiz (le théorème 1.2.1 page 9 et sa démonstration)
• Théorème de Peano (le théorème 1.3.1 page 12 et sa démonstration)
• Théorème d’unicité pour les équations de transport (le théorème 1.4.3 page 17 et sa démonstration)
• Le chapitre 3 dans son entier
• Le théorème de Fujita-Kato (le théorème 4.2.1 page 52 et sa démonstration)
• Le théorème 5.3.1 page 62 et sa démonstration.