

HYDROMELC

Electrostatic charge phenomena : hydrodynamic model and interface conditions.

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Motivation

- In magnetosphere, satellites are surrounded by a plasma : interaction between charged particles of the plasma (ion and electrons) and the spacecraft.
- Ions and electrons of the plasma modify the electrostatic charge of the external surfaces of the spacecraft.
- This leads to charging phenomena. An excessive charging induces the formation of electric arcing which can destroy the experimental devices.

⇒ For the numerical simulation of the spacecraft charging, we need a complete description of the boundary conditions at the interface between the plasma and the spacecraft.

Motivation

Kinetic description of a plasma

- $F_e = F_e(t, x, v)$ and F_i the densities functions of electrons and ions respectively.
- Boltzmann-Poisson system (for $\beta \in \{i, e\}$) :

$$\begin{aligned} \frac{\partial F_\beta}{\partial t} + v \cdot \nabla_x F_\beta - \frac{q_\beta}{m_\beta} \nabla_x V \cdot \nabla_v F_\beta &= Q_\beta(F_i, F_e), \\ -\operatorname{div}_x(\epsilon_0 \nabla_x V) &= q_i n_i + q_e n_e, \\ n_\beta &= \int F_\beta dv. \end{aligned}$$

- Boundary conditions on F_e and F_i at the boundary Γ of the satellite :

$$\gamma^{inc} F_\beta = \mathcal{R}(\gamma^{out} F_\beta) + S \text{ for } v \cdot \eta(x) < 0, \quad x \in \Gamma$$

+ Boundary conditions on V

- Initial conditions.

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- Hydrodynamic limit of Boltzmann-Poisson equations : Euler equation on $(n_\beta, u_\beta, \theta_\beta)$ defined by

$$\begin{cases} n_\beta = \int F_\beta dv \\ n_\beta u_\beta = \int v F_\beta(v) dv \\ n_\beta u_\beta^2 + N n_\beta \theta_\beta = \int |v|^2 F_\beta(v) dv. \end{cases}$$

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- What are the boundary conditions to impose for n_β , u_β , and θ_β ?

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- What are the boundary conditions to impose for n_β , u_β , and θ_β ?
- Difficulties :
 - ▶ imposing only the incoming flux,
 - ▶ taking into account the Knudsen layer.

Objective of the study

Hypothesis

- $N=1$, $\Omega = [-\omega, \omega]$
- Only one specie (ions or electrons)
- BGK operator :

$$Q(F) = M[F] - F, \quad \text{with} \quad M[F](v) = M_{\rho, u, \theta} = \frac{\rho}{\sqrt{2\pi\theta}} \exp\left(-\frac{|v-u|^2}{2\theta}\right)$$

BGK-Poisson model

$$\begin{aligned} \frac{\partial F}{\partial t} + v \cdot \nabla_x F - q \partial_x V \partial_v F &= \frac{1}{\tau} (M[F] - F) \\ -\partial_x^2 V &= q(\rho - 1), \quad q \in \{-1, 1\} \\ V(-\omega) &= V^L, \quad V(\omega) = V^R. \end{aligned}$$

Boundary conditions

$$\gamma^{inc} F(t, -\omega, v) = \mathcal{R}(\gamma^{out} F(t, -\omega, \cdot))(v) + \phi^{data,L}(t, v) \text{ for } v > 0$$

$$\gamma^{inc} F(t, \omega, v) = \mathcal{R}(\gamma^{out} F(t, \omega, \cdot))(v) + \phi^{data,R}(t, v) \text{ for } v < 0$$

- Diffuse reflexion (for the boundary $x = -\omega$)

$$\mathcal{R}(\gamma^{out} F(t, -\omega, \cdot))(v) = \alpha \frac{M_w(v)}{Z_w} \int_{v' < 0} |v'| \gamma^{out} F(t, -\omega, v') dv' \text{ for } v > 0,$$

where $\alpha \in [0, 1]$ and




$$M_w(v) = \frac{\rho_w}{\sqrt{2\pi\theta_w}} e^{-|v|^2/(2\theta_w)}, \quad Z_w = \int_{v > 0} v M_w(v) dv.$$

- Specular reflexion (for the boundary $x = -\omega$)

$$\mathcal{R}(\gamma^{out} F(t, -\omega, \cdot))(v) = \alpha \gamma^{out} F(t, -\omega, -v) \text{ for } v > 0. \quad (1)$$

References

We follow the approach described in :

-  F. Coron, F. Golse, C. Sulem, *A classification of well-posed kinetic layer problems*, Comm. Pure Appl. Math., **41** (4), 409–435 (1988).
-  C. Bardos, F. Golse; Y. Sone, *Half-space problems for the Boltzmann equation : a survey*, Journal of Statistical Physics, **124**, (2)-(4), (2006).
-  C. Besse, S. Borghol, T. Goudon, I. Lacroix-Violet, J.-P. Dudon, *Hydrodynamic regimes, Knudsen layer, numerical schemes : definition of boundary fluxes*, submitted.

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Linearized BGK equation

We set $F = M_*(1 + \delta f)$, with $M_* := M_{(\rho_*, u_*, \theta_*)}$, and we obtain

$$\partial_t f + v \partial_x f = \frac{1}{\tau} L_*(f),$$

with $L_*(f) = \Pi f - f$:

$$\Pi f = \frac{\tilde{\rho}}{\rho_*} + \frac{v - u_*}{\theta_*} \tilde{u} + \left(\frac{|v - u_*|^2}{\theta_*} - 1 \right) \frac{\tilde{\theta}}{2\theta_*} =: m_{(\tilde{\rho}, \tilde{u}, \tilde{\theta})},$$

where $(\tilde{\rho}, \tilde{u}, \tilde{\theta})$ are such that

$$\begin{pmatrix} \tilde{\rho} \\ \tilde{\rho} u_* + \rho_* \tilde{u} \\ \tilde{\rho}(u_*^2 + \theta_*) + 2\rho_* u_* \tilde{u} + \rho_* \tilde{\theta} \end{pmatrix} = \int_{\mathbb{R}} \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} f(\cdot, \cdot, v) M_*(v) dv.$$

We have $\text{Ker}(L_*) = \text{Span} \{1, v, |v|^2\}$.

Linearized Euler system

- Substituting f by the infinitesimal Maxwellian $m_{(\tilde{\rho}, \tilde{u}, \tilde{\theta})}$ leads to the linearized Euler system :

$$\partial_t \begin{pmatrix} \tilde{\rho} \\ \tilde{u} \\ \tilde{\theta} \end{pmatrix} + \begin{pmatrix} u_* & \rho_* & 0 \\ \theta_* & u_* & 1 \\ 0 & 2\theta_* & u_* \end{pmatrix} \partial_x \begin{pmatrix} \tilde{\rho} \\ \tilde{u} \\ \tilde{\theta} \end{pmatrix} = 0. \quad (2)$$

- Eigenvalues : $\{u_* - \sqrt{3\theta_*}, u_*, u_* + \sqrt{3\theta_*}\}$.
- Entropy flux :

$$\mathbf{Q} : \tilde{U} = (\tilde{\rho}, \tilde{u}, \tilde{\theta}) \mapsto \int_{\mathbb{R}} v |m_{(\tilde{\rho}, \tilde{u}, \tilde{\theta})}|^2 M_* dv.$$

- We can split $Ker(L_*)$ according to the sign of the quadratic form \mathbf{Q}

$$Ker(L_*) = \Lambda^+ \oplus \Lambda^- \oplus \Lambda^0.$$

Half space problem

- Ansatz for f :

$$f(t, x, v) = m_{(\tilde{\rho}, \tilde{u}, \tilde{\theta})} + G^L \left(t, \frac{x + \omega}{\tau} \right) + G^R \left(t, \frac{x - \omega}{\tau} \right) + o(\tau).$$

- G^R and G^L stand for boundary layers and are solution of the half space problem

$$\begin{cases} v \partial_z G = L_* G, & z > 0, v \in \mathbb{R} \\ G(0, v) = \Upsilon^{data} & \text{for } v > 0. \end{cases} \quad (3)$$

- *Theorem* : There exists a unique $G \in L^\infty(0, \infty; L^2(\mathbb{R}, M_*(v) dv))$ solution of (3) and a linear mapping (generalized Chandrasekhar functional)

$$\begin{aligned} \mathcal{C}_* : L^2(\mathbb{R}, (1 + |v|)M_*(v) dv) &\rightarrow \Lambda^+ \oplus \Lambda^0 \\ \Upsilon^{data} &\mapsto m_\infty, \end{aligned}$$

where $m_\infty = \lim_{z \rightarrow \infty} G(z)$.

- For $x = -\omega$: G^L is defined by (3) with $\Upsilon^{data} = \gamma^{inc} f(\cdot, -\omega, \cdot) - m_{(\tilde{\rho}, \tilde{u}, \tilde{\theta})}$.

Flux at the boundary (example of $x = -\omega$)

- Decomposition of $m_{(\tilde{\rho}, \tilde{u}, \tilde{\theta})}(t, -\omega, v) \in \text{Ker}(L_*) = \Lambda^+ \oplus \Lambda^0 \oplus \Lambda^- :$

$$m_{(\tilde{\rho}, \tilde{u}, \tilde{\theta})} = m_+ + m_-.$$

- $m_- \in \Lambda^-$: outgoing flux
→ given by the fluid.

$$m_-(v) = \sum_{k \in I^-} \alpha_k \xi_k, \quad \text{where } (\xi_k)_{k \in I^-} \text{ is a basis of } \Lambda^-.$$

- $m_+ \in \Lambda^+ \oplus \Lambda^0$: incoming flux
→ to be imposed as a boundary condition on the Euler system.

- Determination of m_+ : by integration of (3), we get

$$\frac{d}{dz} \int_{\mathbb{R}} v \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} G(z, v) M_*(v) dv = 0$$

- Maxwell approximation : $\gamma^{out} G(0, v) = G(\infty, v)$.
- We obtain :

$$\int_{v>0} v \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} [\gamma^{inc}(f) - m_-] M_* dv = \int_{v>0} v \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} m_+ M_* dv$$

with

$$\gamma^{inc}(f)(v) = \frac{1}{M_*(v)} \mathcal{R}(\gamma^{out}(f M_*)(t, -\omega, \cdot))(v) + \Psi^{data, L}(t, v) \text{ for } v > 0.$$

Numerical examples

We compare

- $(\tilde{\rho}, \tilde{u}, \tilde{\theta})$ given by

$$\begin{pmatrix} \tilde{\rho} \\ \tilde{\rho}u_* + \rho_*\tilde{u} \\ \tilde{\rho}(u_*^2 + \theta_*) + 2\rho_*u_*\tilde{u} + \rho_*\tilde{\theta} \end{pmatrix} = \int_{\mathbb{R}} \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} f M_* dv.$$

where f is the solution of the BGK equation :

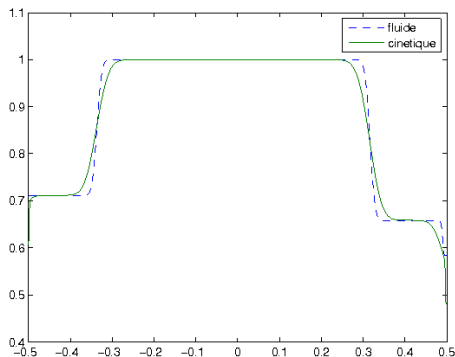
$$\partial_t f + v \partial_x f = \frac{1}{\tau} L_*(f) \quad (+\text{B.C.}),$$

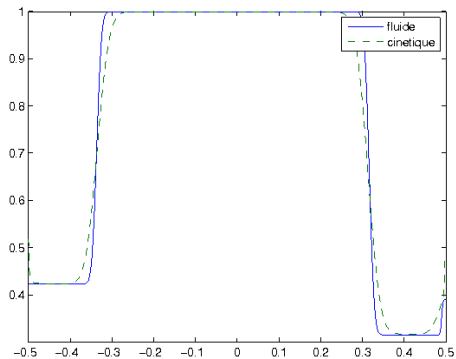
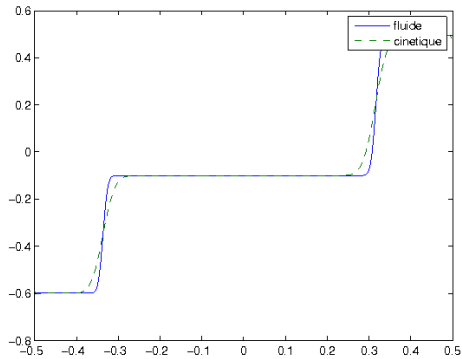
- and $(\tilde{\rho}, \tilde{u}, \tilde{\theta})$ solution of the linearized Euler system :

$$\partial_t \begin{pmatrix} \tilde{\rho} \\ \tilde{u} \\ \tilde{\theta} \end{pmatrix} + \begin{pmatrix} u_* & \rho_* & 0 \\ \frac{\theta_*}{\rho_*} & u_* & 1 \\ 0 & 2\theta_* & u_* \end{pmatrix} \partial_x \begin{pmatrix} \tilde{\rho} \\ \tilde{u} \\ \tilde{\theta} \end{pmatrix} = 0 \quad (+\text{B.C}).$$

Example 1

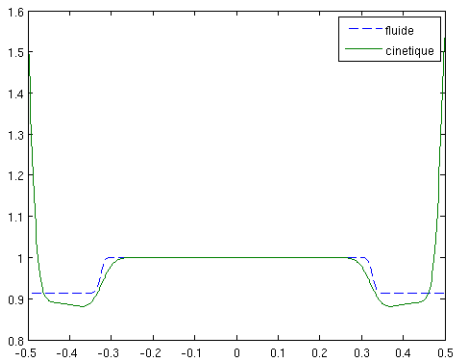
- $(\rho_*, u_*, \theta_*) = (1, -0.1, 1)$
→ signature of $Q : (1, 2)$
- specular reflexion
- $\alpha = 0.1$

FIG.: Density at $t=0.1$ s

FIG.: Temperature at $t=0.1$ sFIG.: Macroscopic velocity at $t=0.1$ s

Example 2

- $(\rho_*, u_*, \theta_*) = (1, 0, 1)$
→ signature of $Q : (1, 1)$
- diffuse reflexion
- $(\rho_w, \theta_w) = (1, 1)$
- $\alpha = 0.8$

FIG.: Density at $t=0.1$ s

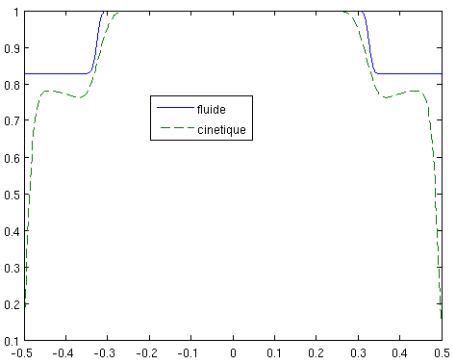


FIG.: Temperature at $t=0.1$ s

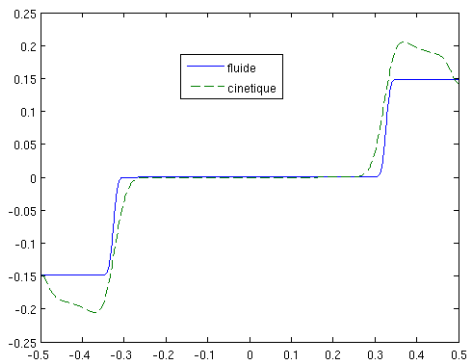


FIG.: Macroscopic velocity at $t=0.1$ s

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Linearization

- We set for equilibrium state : $\mathcal{M}_*(x, v) = M_{(\rho_*, 0, \theta_*)}(v) \frac{e^{-qV_*(x)/\theta_*}}{\int_{-\omega}^{\omega} e^{-qV_*(y)/\theta_*} dy}$.
where $V_* \in C^2(-\omega, \omega)$ is the (unique) solution of the problem

$$\begin{cases} -\partial_x^2 V_* = q \left(\rho_* \frac{e^{-qV_*(x)/\theta_*}}{\int_{-\omega}^{\omega} e^{-qV_*(y)/\theta_*} dy} - 1 \right), \\ V_*(-\omega) = V^L, \quad V_*(\omega) = V^R. \end{cases} \quad (4)$$

- The linearization $F = \mathcal{M}_*(1 + \delta f)$ and $V = V_* + \delta \tilde{V}$ lead to the linearized Boltzmann-Poisson problem :

$$\begin{cases} \partial_t f + v \partial_x f - q \partial_x V_* \partial_v f + q \frac{v}{\theta_*} \partial_x \tilde{V} = \frac{1}{\tau} L_* f, \\ -\partial_x^2 \tilde{V} = q \int_{\mathbb{R}} f \mathcal{M}_* dv, \quad \tilde{V}(-\omega) = \tilde{V}(\omega) = 0. \end{cases}$$

Linearized Euler-Poisson system

Substituting f by the infinitesimal Maxwellian $m_{(\tilde{\rho}, \tilde{u}, \tilde{\theta})}$ leads to :

$$\partial_t \begin{pmatrix} \tilde{\rho} \\ \tilde{u} \\ \tilde{\theta} \end{pmatrix} + \begin{pmatrix} 0 & \rho_* & 0 \\ \frac{\theta_*}{\rho_*} & 0 & 1 \\ 0 & 2\theta_* & 0 \end{pmatrix} \partial_x \begin{pmatrix} \tilde{\rho} \\ \tilde{u} \\ \tilde{\theta} \end{pmatrix} = q \partial_x V_* \begin{pmatrix} \frac{\rho_* \tilde{u}}{\theta_*} \\ \frac{\tilde{\theta}}{\theta_*} \\ 0 \end{pmatrix} - q \partial_x \tilde{V} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

with

$$-\partial_x^2 \tilde{V} = q \tilde{\rho} \frac{e^{-qV_*(x)/\theta_*}}{\int_{-\omega}^{\omega} e^{-qV_*(y)/\theta_*} dy}, \quad \tilde{V}(-\omega) = \tilde{V}(\omega) = 0.$$

Numerical examples

Example 3

- $(\rho_*, u_*, \theta_*) = (1, 0, 0.5)$
- Source term at the left boundary :
 $\phi^{data,L} = m_{(\rho_w^L, 0, \theta_w^L)}$, with
 $(\rho_w^L, \theta_w^L) = (2/1.2, 1.2/2)$
- Boundary condition at the right side :
 $\phi^{data,R} = m_{(\rho_*, 0, \theta_*)}$
- $q = -1$
- $V^R = 1, V^L = 1.$

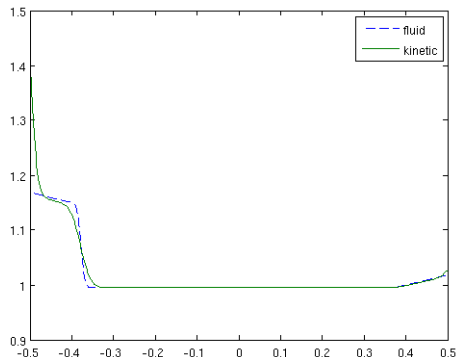
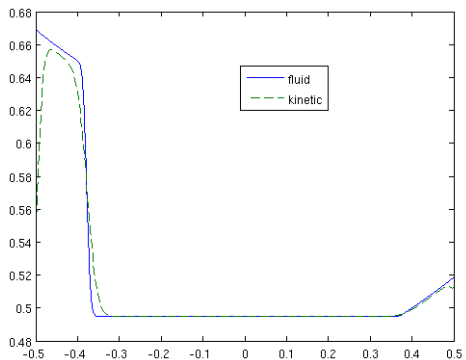
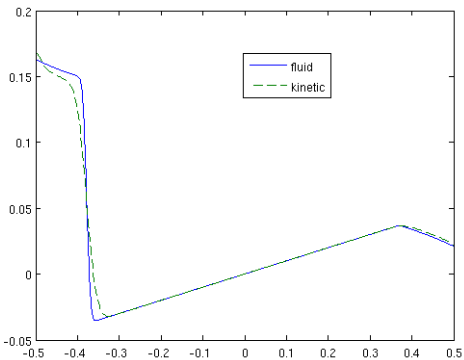
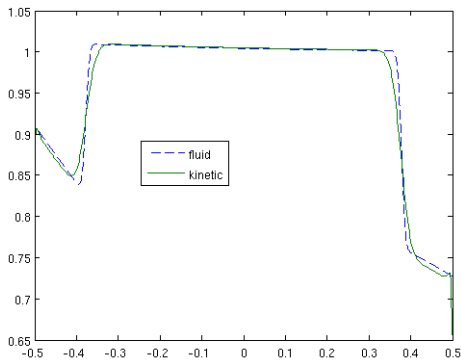


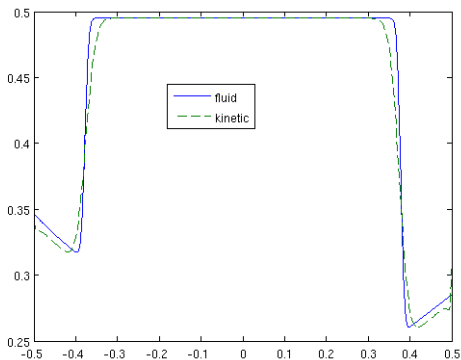
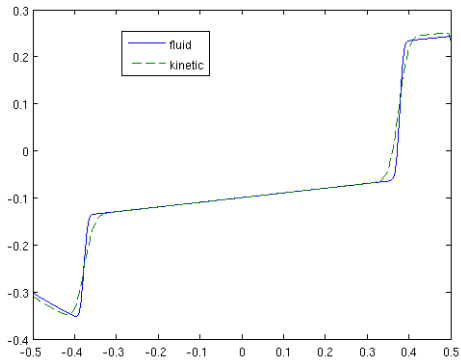
FIG.: Density at t=0.1 s

FIG.: Temperature at $t=0.1$ sFIG.: Macroscopic velocity at $t=0.1$ s

Example 4

- $(\rho_*, u_*, \theta_*) = (1, 0, 0.5)$
- specular reflexion
- $\alpha = 0.4$
- $q = 1$
- $V^R = 0, V^L = 1.$

FIG.: Density at $t=0.1$ s

FIG.: Temperature at $t=0.1$ sFIG.: Macroscopic velocity at $t=0.1$ s

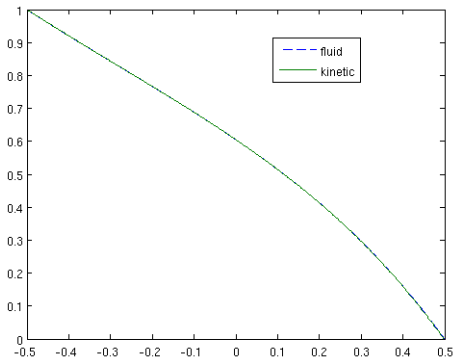


FIG.: Potential V at $t=0.1$ s

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Conclusion

- Comparison between linearized BGK-Poisson equation and linearized Euler system.
- In most of situation, comparisons are quite good !
- Successful addition of the Poisson potential.

Work in progress

- Improvement of the case with diffuse reflexion and α close to 1.
- Implementation of the full Euler-Poisson system and comparison with the BGK-Poisson equation.

Perspectives

- The full ions-electrons system.