Iterative positive polynomial interpolation

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Introduction

Problem

Given $f \geq 0$ on $[0,1]$, let find $n+1$ quadrature points

$$0 \leq x_0 < x_1 < \cdots < x_{n-1} < x_n \leq 1$$

such that the polynomial interpolant $p_n \in P_n := P_n(\mathbb{R})$

$$p_n(x_i) = f(x_i), \quad 0 \leq i \leq n,$$

is such that $p_n \in P_n^+ := \{p_n \in P_n : \quad p_n(x) \geq 0, \quad \forall x \in [0,1]\}$.

Motivations

- Oscillations in numerical schemes (VF for transport equations...)
- Physical motives (positive reconstruction of a density ...)
- ...

Introduction

Our assumptions

We introduce a parameter $0 < h \leq 1$ and consider the problem on $[0, h]$.

$$\Rightarrow \quad \text{Interpolation of } f_h(x) = f(xh) \text{ at } n+1 \text{ points of } [0,1].$$

We assume

$$(H) \qquad f \in W^{1,\infty}(0,1) \qquad \text{and} \qquad \inf_{x \in [0,1]} f(x) > 0. \tag{1}$$

Lukacs Theorem

Theorem [Lukacs]

- If $\mathbf{n} = \mathbf{2p}$, then $p_n \in P_n^+$ if and only if the polynomial can be expressed as

$$p_n(x) = a_p(x)^2 + x(1-x)\, b_{p-1}(x)^2$$

with $a_p \in \mathcal{P}_p$ and $b_{p-1} \in \mathcal{P}_{p-1}$.

- If $\mathbf{n} = \mathbf{2p+1}$, then $p_n \in P_n^+$ if and only if the polynomial can be expressed as

$$p_n(x) = x\, a_p(x)^2 + (1-x)\, b_p(x)^2$$

with $a_p, b_p \in \mathcal{P}_p$.

Remark

Non-uniqueness : $1 = 1^2 + 0^2 x(1-x) = (1-2x)^2 + 2^2 x(1-x)$.

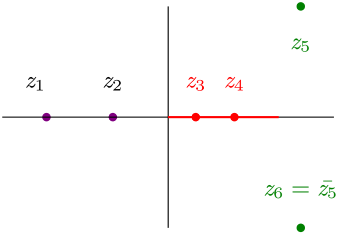
Proof [Després]

- $n = 2 : \; p_2(x) = \Bigg(\underbrace{\sqrt{p_2(0)}(1-x) - \sqrt{p_2(1)}x}_{\text{oscillating polynomial}} \Bigg)^2 + x(1-x)b_0^2.$

Proof [Després]

- $n = 2:\ p_2(x) = \left(\underbrace{\sqrt{p_2(0)}(1-x) - \sqrt{p_2(1)}x}_{\text{oscillating polynomial}}\right)^2 + x(1-x)b_0^2.$
- $n = 2p:$

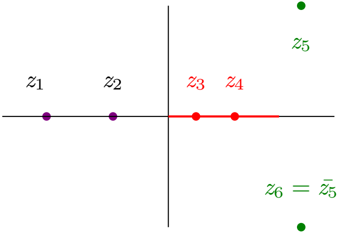
$$\begin{aligned} p_n(x) &= \prod_{k=1}^{p} p_k(x), \text{ with } p_k \in P_2^+ \\ &= \prod_{k=1}^{p} \left| a_{1,k}(x) + i\sqrt{x(1-x)}b_{0,k} \right|^2 \\ &= a_p(x)^2 + x(1-x)b_{p-1}^2 \end{aligned}$$



Proof [Després]

- $n=2: \; p_2(x) = \Big(\underbrace{\sqrt{p_2(0)}(1-x) - \sqrt{p_2(1)}x}_{\text{oscillating polynomial}} \Big)^2 + x(1-x)b_0^2.$
- $n = 2p:$

$$\begin{aligned} p_n(x) &= \prod_{i=1}^{p} p_i(x), \text{ with } p_i \geq 0 \text{ on } [0,1] \\ &= \prod_{i=1}^{p} \left| a_{1,i}(x) + i\sqrt{x(1-x)}b_{0,i} \right|^2 \\ &= a_p(x)^2 + x(1-x)b_{p-1}^2 \end{aligned}$$



- $n = 2p+1:$

$$p(x) := xp_n(x) = \hat{a}_{p+1}(x)^2 + x(1-x)b_p^2$$

$$\text{then } \hat{a}_{p+1}(x) = xa_p(x)$$

$$\Rightarrow p_n(x) = xa_p(x)^2 + (1-x)b_p^2$$

Cubic polynomials

According to Lukacs representation, $p_3 \in P_3^+$ writes

$$p_3(x) = x a_1(x)^2 + (1-x) b_1(x)^2$$

Sufficient criterion

If $a_1, b_1 \in P_1$ and α, β are such that

$$\begin{cases} a_1(\alpha) = -\sqrt{\dfrac{f(\alpha)}{\alpha}} \\ a_1(\beta) = 0 \\ a_1(1) = \sqrt{f(1)} \end{cases} \quad \text{and} \quad \begin{cases} b_1(0) = -\sqrt{f(0)} \\ b_1(\alpha) = 0 \\ b_1(\beta) = \sqrt{\dfrac{f(\beta)}{1-\beta}} \end{cases} \tag{2}$$

then $0 < \alpha < \beta < 1$ and $p_3(x) = x a_1(x)^2 + (1-x) b_1(x)^2$ is a positive cubic polynomial that interpolates f at 0, α, β and 1.

Fixed point problem

$$\begin{cases} b_1(\alpha) = 0 \\ a_1(\beta) = 0 \end{cases} \Leftrightarrow \begin{cases} (\alpha-\beta)\sqrt{f(0)} + \alpha\sqrt{\dfrac{f(\beta)}{1-\beta}} = 0. \\ (\beta-\alpha)\sqrt{f(1)} + (\beta-1)\sqrt{\dfrac{f(\alpha)}{\alpha}} = 0 \end{cases} \begin{matrix} \Leftrightarrow G(\alpha,\beta) = (\alpha,\beta) \\ \Leftrightarrow \tau(\alpha) = 0 \end{matrix}$$

with

$$G(x,y) := (\varphi(y), \psi(x)),$$

$$\varphi(y) = \frac{y\sqrt{(1-y)f(0)}}{\sqrt{(1-y)f(0)} + \sqrt{f(y)}} \quad \text{and} \quad \psi(x) = \frac{x\sqrt{xf(1)} + \sqrt{f(x)}}{\sqrt{xf(1)} + \sqrt{f(x)}}.$$

$$\tau(x) = x - \varphi(\psi(x)).$$

The existence of α, β results from the following

Lemma

One has $\tau \in C^0([0,1] : [-1,1])$ and $\tau(0) = 0$ with $\tau(1) = 1$. Assuming that f satisfy Hypothesis (H), one that $\frac{d}{d\alpha}\tau(0^+) = -\infty$.

Study of G_h

Rescaling

We replace f by $f_h(\cdot) = f(h\cdot)$, G become G_h.

Properties of G_0

Due to Lipschitz regularity of f we have $G_h \underset{h\to 0}{\longrightarrow} G_0$ uniformely on $[0,1]$, with

$$G_0(x,y) = \Big(K(y), 1 - K(1-x)\Big) \qquad K(z) = \frac{z\sqrt{1-z}}{1+\sqrt{1-z}},$$

Lemma

The function G_0 has the following properties

(i) *it has a unique fixed point in $(0,1)^2$, which is $\underline{X} = (\underline{\alpha}, \underline{\beta}) = \left(\frac{1}{4}, \frac{3}{4}\right) \in F$*

(ii) *it leaves invariant the domain $F := \left[\frac{1}{5}, \frac{1}{3}\right] \times \left[\frac{2}{3}, \frac{4}{5}\right]$,*

(iii) *it is contractant on F for any ℓ^p subordinate norm,*

Algorithm to compute the cubic nodes

Algorithm

Given $X^0 = \underline{X} = (\frac{1}{4}, \frac{3}{4})$, we consider the fixed point method

$$X^{m+1} := G_h(X^m). \tag{3}$$

Theorem (Convergence of the fixed point)

Let f satisfy Hypothesis (H). There exist $h_0 > 0$ such that for all $0 \leq h \leq h_0$, the sequence $(X^m)_{m\geq 0}$ given by (3) remains in the domain $F \subset]0,1[^2$ and converges to a fixed point of G_h denoted as $X_h^\infty \in F$. Moreover one has the inequality for all $m \geq 0$

$$\|X_h^\infty - X^m\| \leq C\left(\frac{h}{2h_0}\right)^{m+1}. \tag{4}$$

Proof based (among other arguments) on $G_h(F) \subset F$ and the estimate

$$\|\nabla G_h(X)\| \leq C^*(h + \|X - \underline{X}\|), \qquad X \in F.$$

Accuracy of the approximate interpolants in P_3^+

Theorem

Let $f \in W^{q,\infty}(0,1)$, $1 \leq q \leq 4$, satisfy (H), and let $h_0 > 0$ be given by previous Theorem. Then for all $0 \leq h \leq h_0$ and all $m \geq 0$, the cubic polynomial

$$p_3^m(x) := x a_1[\alpha^m](x)^2 + (1-x) b_1[\beta^m](x)^2$$

where $(\alpha^m, \beta^m) = X^m$ satisfies

$$\|p_3^m - f_h\| \leq C h^{\min(q, 2(m+1))}$$

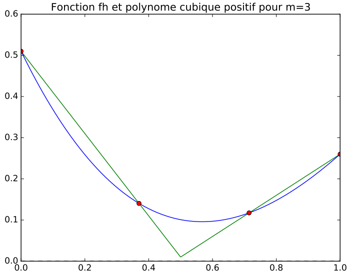
for a constant C depending on $f^{(q)}$.

Numerical illustrations

Example 1

$$f(x) = 0.01 + |x - 0.5| \text{ for } x < 0.5, \quad f(x) = 0.01 + \frac{1}{2}|x - 0.5| \text{ for } 0.5 \leq x.$$

m	α^m	β^m
0	0.250000,	0.750000
1	0.369641	0.750000
2	0.369641	0.714574
3	0.376562	0.714574
4	0.376562	0.712361
5	0376923	0.7123618



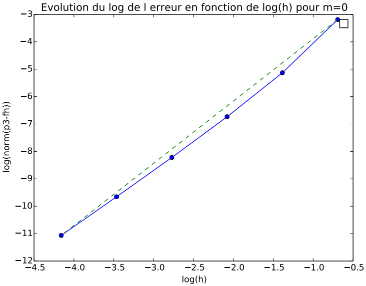
Numerical illustrations

Example 2

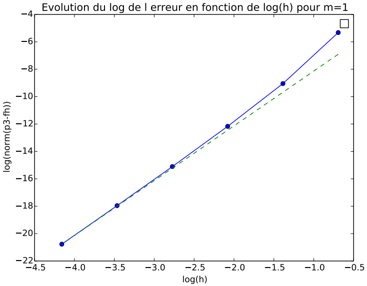
$$f_h(x) = \frac{1}{1 - hx}, \quad 0 \leq x \leq 1, \quad h = 1/2, 1/4, 1/8, \ldots$$

We want to verify the estimate

$$\|p_3^m - f_h\| \leq C h^{\min(q, 2(m+1))}$$



$a \simeq 2.2$



$a \simeq 4.4$

Odd case : $n = 2p + 1$

Proposition : sufficient criterion for positive interpolation

Let $f \in W^{1,\infty}(0,1)$ satisfy (H), and let $h \geq 0$. If $a_p, b_p \in \mathcal{P}_p$ and the $2p$ nodes $\alpha_0 < \ldots < \alpha_{p-1}$, $\beta_1 < \cdots < \beta_p$ in $]0,1[$ are such that

$$b_p(\alpha_i) = a_p(\beta_{i+1}) = 0 \qquad \text{for} \quad 0 \leq i \leq p-1, \tag{5}$$

and such that, setting $\beta_0 = 0$ and $\alpha_p = 1$,

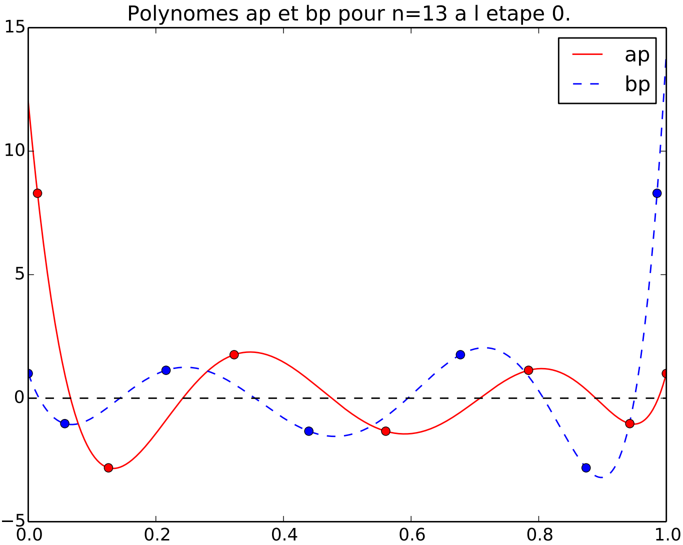
$$a_p(\alpha_i) = (-1)^{i+p}\sqrt{\frac{f(h\alpha_i)}{\alpha_i}}, \quad b_p(\beta_i) = (-1)^{i+p}\sqrt{\frac{f(h\beta_i)}{1-\beta_i}} \quad \text{for} \quad 0 \leq i \leq p, \tag{6}$$

then we have $0 = \beta_0 < \alpha_0 < \beta_1 < \cdots < \beta_p < \alpha_p = 1$ and the polynomial

$$p_n(x) = x a_p(x)^2 + (1-x) b_p(x)^2 \in P_n^+$$

interpolates $f_h = f(h\,\cdot)$ on the $n+1$ nodes $\beta_0, \alpha_0, \cdots, \beta_p, \alpha_p$

Oscillating polynomials a_p and b_p



Fixed point problem

Let $I_p = \{(x_1, \dots x_p) \subset (0,1)^p,\ 0 < x_1 < \cdots < x_p < 1\}$ and for

$$(\alpha, \beta) = (\alpha_0, \dots, \alpha_{p-1}; \beta_1, \dots, \beta_p) \in I_p^2,$$

let $a_p[\alpha]$ and $b_p[\beta]$ be the polynomials solving the interpolation problems (15),

$$a_p[\alpha](x) = \sum_{0 \le i \le p} (-1)^{i+p} \sqrt{\frac{f(h\alpha_i)}{\alpha_i}} \prod_{0 \le j \ne i \le p} \frac{x - \alpha_j}{\alpha_i - \alpha_j} \tag{7}$$

and

$$b_p[\beta](x) = \sum_{0 \le i \le p} (-1)^{i+p} \sqrt{\frac{f(h\beta_i)}{1 - \beta_i}} \prod_{0 \le j \ne i \le p} \frac{x - \beta_j}{\beta_i - \beta_j}. \tag{8}$$

Finally define $\Theta_{p,h} : I_p^2 \longrightarrow \mathbb{R}^{2p}$ by

$$\Theta_{p,h}(\alpha, \beta) = \left(b_p[\beta](\alpha_0), \dots, b_p[\beta](\alpha_{p-1}), a_p[\alpha](\beta_1), \dots, a_p[\alpha](\beta_p)\right). \tag{9}$$

Then the sufficient criterion of previous Proposition applies as soon as $(\alpha, \beta) \in I_p^2$ satisfies

$$\Theta_{p,h}(\alpha, \beta) = 0. \tag{10}$$

Algorithm

Modified Newton-Raphson algorithm A

Given a starting point $X^0 \in I_p^2$, we compute

$$X^{m+1} = X^m - J_p(X^0)^{-1}\Theta_{p,h}(X^m) \tag{11}$$

where

$$J_p(X^0) = \nabla\Theta_{p,0}(X^0) = \begin{pmatrix} \nabla_\alpha b_p[\beta](\alpha) & \nabla_\beta b_p[\beta](\alpha) \\ \nabla_\alpha a_p[\alpha](\beta) & \nabla_\beta a_p[\alpha](\beta) \end{pmatrix}\Bigg|_{(\alpha,\beta)=X^0} \in \mathbb{R}^{2p\times 2p}.$$

Definition of the starting point X^0

Same idea as before : for $h = 0$ we seek two polynomials $a_p, b_p \in P_p$ such that

$$xa_p(x)^2 + (1-x)b_p(x)^2 = 1 \text{ for all } x \in [0,1].$$

Definition of the starting point X^0

Chebyshev polynomials $(T_p, U_p) \in P_p \times P_{p-1}$

$$T_p(\cos\theta) = \cos(p\theta) \quad \text{and} \quad U_p(\cos\theta) = \frac{\sin(p\theta)}{\sin\theta}, \quad p \geq 0$$

Lemma Let

$$\begin{cases} \underline{a}_p(x) = T_p\,(2x-1) - 2(1-x)\,U_p\,(2x-1) \\ \underline{b}_p(x) = T_p\,(2x-1) + 2x U_p\,(2x-1) \end{cases}.$$

Then

i) **Root property**: $\underline{a}_p$ and $\underline{b}_p$ have p simple roots in $]0,1[$

$$\underline{\beta}_i := \frac{1}{2}\left[1 - \cos\left(\frac{2i\pi}{2p+1}\right)\right], \quad i = 1,\dots,p \text{ and } \underline{\alpha}_i := \frac{1}{2}\left[1 - \cos\left(\frac{(2i+1)\pi}{2p+1}\right)\right]$$

ii) **Interlacing and symmetry of the nodes**: with $\underline{\beta}_0 = 0$ and $\underline{\alpha}_p = 1$,

$$0 = \underline{\beta}_0 < \underline{\alpha}_0 < \underline{\beta}_1 < \cdots < \underline{\beta}_p < \underline{\alpha}_p = 1 \qquad \underline{\alpha}_i + \underline{\beta}_{p-i} = 1, \text{ for } 0 \leq i \leq p.$$

iii) Weighted sum of squares: for all x, we have $x\underline{a}_p(x)^2 + (1-x)\underline{b}_p(x)^2 = 1$.

iv) the polynomials $\underline{a}_p$ and $\underline{b}_p$ correspond to the ones defined according to (7)-(8) with a constant function $f = 1$.

Study of $J_p(X^0)$

Lemma

The reference Jacobian matrix has the form $J_p(X^0) = \begin{pmatrix} D_\alpha & 0 \\ 0 & D_\beta \end{pmatrix}$ where

$$D_\alpha = \mathrm{diag}\big(\underline{b}'_p(\underline{\alpha}_i) : i = 0, \ldots, p-1\big), \quad D_\beta = \mathrm{diag}\big(\underline{a}'_p(\underline{\beta}_i) : i = 1, \ldots, p\big)$$

are two diagonal matrices with non zero entries given by

$$\begin{cases} \underline{a}'_p(\underline{\beta}_i) = \dfrac{2p\cos(p\underline{\eta}_i)}{\cos\underline{\eta}_i + 1} + \dfrac{\sin(p\underline{\eta}_i)}{\sin\underline{\eta}_i}\left(2p + \dfrac{2}{\cos\underline{\eta}_i + 1}\right) & \text{for } i = 1, \ldots, p \\ \underline{b}'_p(\underline{\alpha}_i) = \dfrac{2p\cos(p\underline{\theta}_i)}{\cos\underline{\theta}_i - 1} + \dfrac{\sin(p\underline{\theta}_i)}{\sin\underline{\theta}_i}\left(2p - \dfrac{2}{\cos\underline{\theta}_i - 1}\right) & \text{for } i = 0, \ldots, p-1 \end{cases}$$

with $\underline{\eta}_i = \frac{(2(p-i)+1)\pi}{2p+1}$ and $\underline{\theta}_i = \frac{2(p-i)\pi}{2p+1}$.

Algorithm with guaranteed node separation

Issue

To avoid $\Theta_{p,h}$ to become unbounded one must guarantee that the approximated nodes stay away from each other !

Separation operator $S_{p,\varepsilon}$

For some given $0 \leq \varepsilon \leq 1/(2(p+1))$ let

$$I_{p,\varepsilon} = \{(x_1, \dots x_p) \subset [\varepsilon, 1-\varepsilon]^p : \varepsilon \leq x_i - x_{i-1} \quad \text{for} \quad 1 \leq i \leq p\}$$

We introduce

$$S_{p,\varepsilon} : \mathbb{R}^{2p} \to I_{p,\varepsilon}$$

such that for $X \in I_{p,2\varepsilon}$ $S_{p,\varepsilon} X = X$.

Algorithm A_ε with guaranteed node separation

$$X^{m+1} = S_{p,\varepsilon} G_h(X^m) \qquad \text{where} \qquad G_h(X) := X - J_p(X^0)^{-1} \Theta_{p,h}(X)$$

Convergence of the modified Newton algorithm

Theorem Let f satisfy (H) and $\varepsilon = \varepsilon(n) > 0$ be such that

$$\varepsilon \leq \min\left(\frac{1}{2(p+1)}, \min_{1\leq i\leq 2p+1}\left(\frac{\underline{\gamma}_i - \underline{\gamma}_{i-1}}{4}\right)\right)$$

Then there exist $h_0 > 0$ such that for all $0 \leq h \leq h_0$ the following properties hold:

i) the algorithms A and A_ε with starting point $\underline{\mathrm{X}}$ compute the same iterates $(X^m)_{m\geq 0}$ which belong to the set $(I_{p,2\varepsilon})^2 \subset]0,1[^{2p}$

ii) the sequence $(X^m)_{m\geq 0}$ converges towards a fixed point of G_h in the ball $B(X^0,\varepsilon)$, $X_h^\infty = (\alpha_0^\infty, \dots, \alpha_{p-1}^\infty; \beta_1^\infty, \dots \beta_p^\infty)$, consisting of interlaced nodes bounded away from each other and from the end nodes

$$(\alpha_0^\infty, \beta_1^\infty, \dots, \alpha_{p-1}^\infty, \beta_p^\infty) \in I_{2p,2\varepsilon}$$

iii) the error estimate

$$\|X_h^\infty - X^m\| \leq 2\left(\frac{h}{2h_0}\right)^{m+1} \tag{12}$$

holds for all $m \geq 0$.

Accuracy

Theorem

Let $f \in W^{q,\infty}(]0,1[)$, $1 \leq q \leq n+1$, satisfy (H), and let $h_0 > 0$ be given by the previous Theorem. Then for all $0 \leq h \leq h_0$ and all $m \geq 0$, the polynomial

$$p_n^m(x) := x a_p^m(x)^2 + (1-x) b_p^m(x)^2$$

satisfies

$$\|p_n^m - f_h\| \leq C h^{\min(q,2(m+1))} \tag{13}$$

for a constant C depending on f and n.

Extension to the case $n = 2p$

Sufficient criterion

Let $f \in W^{1,\infty}(\,]0,1[\,)$ satisfy (H), and let $h \geq 0$. If $a_p \in \mathcal{P}_p$, $b_{p-1} \in \mathcal{P}_{p-1}$ and the $2p-1$ nodes $\alpha_1 < \ldots < \alpha_{p-1}$, $\beta_1 < \cdots < \beta_p$ in $]0,1[$ are such that

$$\begin{aligned} a_p(\beta_i) &= 0 \qquad \text{for} \quad 1 \leq i \leq p, \\ b_p(\alpha_i) &= 0 \qquad \text{for} \quad 1 \leq i \leq p-1, \end{aligned} \tag{14}$$

and such that, setting $\alpha_0 = 0$ and $\alpha_p = 1$,

$$\begin{aligned} a_p(\alpha_i) &= (-1)^{i+p}\sqrt{f(h\alpha_i)}, \qquad \text{for} \quad 0 \leq i \leq p, \\ b_p(\beta_i) &= (-1)^{i+p}\sqrt{\frac{f(h\beta_i)}{\beta_i(1-\beta_i)}} \qquad \text{for} \quad 1 \leq i \leq p, \end{aligned} \tag{15}$$

then we have $0 = \alpha_0 < \beta_1 < \alpha_1 < \cdots < \beta_p < \alpha_p = 1$ and the polynomial $p_n(x) = a_p(x)^2 + x(1-x)b_{p-1}(x)^2 \in \mathcal{P}_n^+$ interpolates $f_h = f(h\,\cdot)$ on the $n+1$ nodes $\alpha_0, \beta_1, \cdots, \beta_p, \alpha_p$.

Extension to the case $n = 2p$

Theorem Given $p \in \mathbb{N}$ let

$$\underline{a}_p(x) = T_p\,(2x-1) \qquad \underline{b}_{p-1}(x) = 2\,U_p\,(2x-1)\,.$$

and

$$\underline{\alpha}_i := \frac{1}{2}\left[1 - \cos\left(\frac{i\pi}{p}\right)\right]\ i = 0, \dots, p \qquad \underline{\beta}_i := \frac{1}{2}\left[1 - \cos\left(\frac{(2i-1)\pi}{2p}\right)\right],\ i = 1, \dots, p$$

We have the following properties.

i) Interlacing of the nodes: we have

$$0 = \underline{\alpha}_0 < \underline{\beta}_1 < \underline{\alpha}_1 < \cdots < \underline{\beta}_p < \underline{\alpha}_p = 1 \tag{16}$$

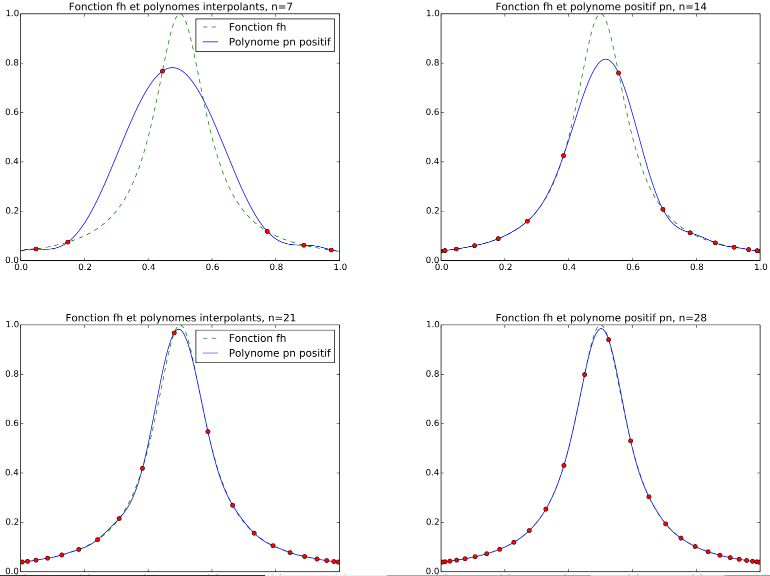
ii) Root property: $\underline{a}_p$ (respectively $\underline{b}_{p-1}$) has p (respectively $p-1$) simple roots in $]0,1[$, which coincide with $\underline{\beta} = (\underline{\beta}_1, \dots, \underline{\beta}_p)$ and $\underline{\alpha} = (\underline{\alpha}_1, \dots, \underline{\alpha}_{p-1})$ respectively.

iv) Weighted sum of squares: for all x, we have

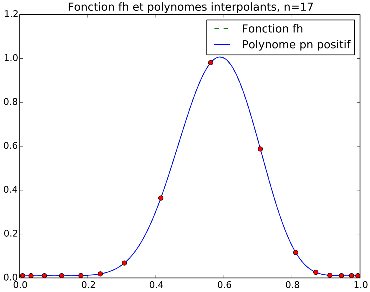
$$\underline{a}_p(x)^2 + x(1-x)\underline{b}_p(x)^2 = 1. \tag{17}$$

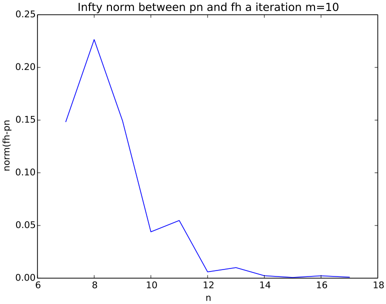
v) the polynomials $\underline{a}_p$ and $\underline{b}_p$ correspond to the ones defined according to (7)-(8) with a constant function $f = 1$.

Example : $f(x) = \frac{1}{1+25(2x-1)^2} - \frac{1}{27}$, $h = 1$, $m = 10$



Example : $f(x) = 10^5\, x^{10}\, (1-x)^7 + 0.01$



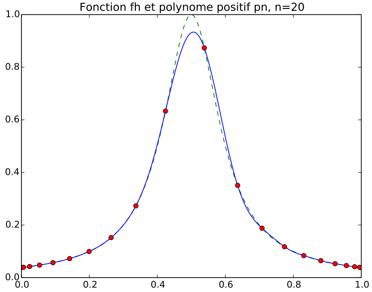


Comparaison with Lagrange interpolation

Basic idea : take p the interpolation polynomiale of $\sqrt{f}$ and set $\tilde{p}(x) = p(x)^2$.

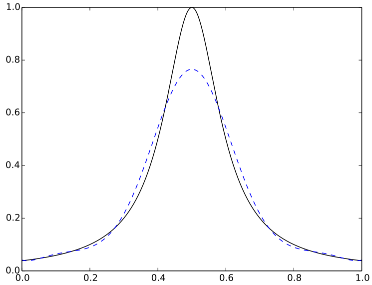
Test with $f(x) = \frac{1}{1+25(2x-1)^2} - \frac{1}{27}$

Positiv polynomial, with $n = 20$



$$\|f_h - p_n\|_\infty = 0.0715292$$

Lagrange interpolation with Tchebychev point of $\sqrt{f}$, $n = 10$



$$\|f_h - p_n\|_\infty = 0.2347271$$

Conclusion

Results

- Modified Newton algorithm for interpolation of a function $f \in W^{1,\infty}$, such that $\inf_{[0,1]} f(x) > 0$ on $[0,h]$, by a positive polynomial of arbitrary order.
- Weak number of iteration m are necessary in pratice.
- Theorical convergence and accurary for $h \to 0$... but in practice good also for $h = 1$! (up to a rescaling of the Jacobien matrix...)
- Good behaviour on some examples when $\lambda = \inf_{[0,1]} f(x) \to 0$
- Application to non-conservative and conservative semi-Lagrangian approximation of a 1D advection equation (see preprint... soon avalaible!)

Perspectives

- Application to 1d numerical schemes (remapping in particle methods...)
- Algorithm for polynomial $\in [0,1]$?
- Positive interpolation in 2d?

Thank you for your attention!