AN OVERVIEW OF THE NONLINEAR SCHRÖDINGER EQUATION

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1. INTRODUCTION

We consider the model nonlinear Schrödinger equation

$$i\partial_t u + \Delta u + \eta |u|^\alpha u = 0, \qquad t \geq 0, x \in \mathbb{R}^N \tag{NLS}$$

where $\alpha > 0$ and $\eta \in \mathbb{R}$ (or, possibly, $\eta \in \mathbb{C}$). One could consider a more general nonlinearity, but this is the canonical example. Here $u = u(t,x)$ is a complex-valued function. We consider solutions that vanish at (space) infinity, $u(t,x) \to 0$ as $|x| \to \infty$ in some appropriate sense, for instance $u(t,\cdot) \in L^2(\mathbb{R}^N)$. If $\eta > 0$, then the equation is called **focusing**, while if $\eta < 0$ it is called **defocusing**.

For motivation from physics, see for instance the book [126] by Sulem and Sulem.

NLS is part of a family of nonlinear dispersive equations including the nonlinear wave ($m = 0$) or Klein-Gordon ($m \neq 0$) equation

$$\partial_{tt} u = \Delta u - m^2 u + \eta |u|^\alpha u,$$

and the (generalized) Korteweg-de Vries (KdV) equation

$$\partial_t u + \partial_x^3 u + u^p \partial_x u = 0,$$

where $p \in \mathbb{N}$, $p \geq 1$.

These equations share several structural properties:

- They are semilinear (perturbations of a linear equation by a lower order nonlinear term).
- They are conservative (one or more conservation laws, like energy).
- They are dispersive (despite of the conservation laws, solutions of the linear equation disperse, i.e. they converge to zero locally in space at time infinity).

The solutions of these three equations share many of their properties. Some properties are easier to prove for one equation and more difficult (sometimes open) for another equation of the family.

We will focus on a few fundamental topics only. Essentially: local well-posedness, finite-time blowup, standing waves, asymptotic behavior of global solutions.

We will **not** discuss several important issues, such as:

- Weak solutions constructed by compactness
- The "critical" cases (see the books by Bourgain [14] and Tao [128])
- NLS set on domains (exterior, bounded, etc) or manifolds
- NLS in fractional Sobolev spaces (H^s theory)
- The many extensions and variants of NLS: the equation with potentials and/or with inhomogeneous nonlinearities, Hartree-type nonlinearities, fractional Laplacian, etc
- Solutions that do not vanish as $|x| \to \infty$: kinks, Gross-Pitaevskii ($|u| \to 1$ as $|x| \to \infty$), etc

Note that papers are becoming more and more technically involved and long. Following closely the recent developments certainly implies a considerable time investment. For instance, the last-to-date breakthrough (finite-time blowup for defocusing supercritical NLS) consists of a 168p paper [96] (construction of smooth self-similar solutions of the compressible Euler equation) followed by a 107p paper [97] (application to (NLS) via its hydrodynamical formulation).

2. Invariances and conservation laws

Several transformations (invariances) leave the set of solutions of equation (NLS) invariant, i.e. when such a transformation is applied to a solution, the result is also a solution. The standard invariances are

$$u(t,x) \mapsto \overline{u(-t,x)}, \quad \gamma \in \mathbb{R} \quad \text{(time inversion)}$$
$$u(t,x) \mapsto e^{i\gamma} u(t,x), \quad \gamma \in \mathbb{R} \quad \text{(phase)}$$
$$u(t,x) \mapsto u(t+t_0, x+x_0), \quad t_0 \in \mathbb{R}, x_0 \in \mathbb{R}^N \quad \text{(space-time transl.)}$$
$$u(t,x) \mapsto u(t, x-\beta t) e^{i\frac{\beta}{2}\cdot(x-\beta t)}, \quad \beta \in \mathbb{R}^N \quad \text{(Galilean)}$$
$$u(t,x) \mapsto \lambda^{\frac{2}{\alpha}} u(\lambda^2 t, \lambda x), \quad \lambda > 0 \quad \text{(scaling)}$$

In view of the time inversion invariance, solving (NLS) backwards is equivalent to solving (NLS) forward, up to comlex conjugation. Therefore, in the sequel **we will in general consider only positive time**.

At least formally, equation (NLS) has the following conservation laws:

$$\int_{\mathbb{R}^N} |u(t,x)|^2 = \int_{\mathbb{R}^N} |u(0,x)|^2 \quad \textbf{(mass)} \tag{2.1}$$
$$E(u(t)) = E(u(0)) \quad \textbf{(energy)} \tag{2.2}$$
$$\text{where } E(w) =: \frac{1}{2}\int_{\mathbb{R}^N} |\nabla w(,x)|^2 - \frac{\eta}{\alpha+1}\int_{\mathbb{R}^N} |w(x)|^{\alpha+2}$$
$$\Im \int_{\mathbb{R}^N} \overline{u(t,x)} \nabla u(t,x) = \Im \int_{\mathbb{R}^N} \overline{u(0,x)} \nabla u(0,x) \quad \textbf{(momentum)} \tag{2.3}$$

Invariances and conservation laws are related by the Noether theorem (see [126, Section 2.3]):

- Phase invariance corresponds to conservation of mass
- Time-translation invariance corresponds to conservation of energy
- Space-translation invariance corresponds to conservation of momentum.

Note that the equation (NLS) has the above invariances (except time inversion) for every $\eta \in \mathbb{C}$, but **the conservation laws are valid only if** $\eta \in \mathbb{R}$.

The **pseudo-conformal transformation**

$$u(t,x) \mapsto (1-bt)^{-\frac{N}{2}} u\Big(\frac{t}{1-bt}, \frac{x}{1-bt}\Big) e^{-i\frac{b|x|^2}{4(1-bt)}}, \quad b \in \mathbb{R} \tag{2.4}$$

also transforms a solution of (NLS) to a solution of the same equation, but **only when** $\alpha = \frac{4}{N}$. In the general case, it transforms a solution of (NLS) to a solution of the nonautonomous equation

$$i\partial_t u + \Delta u + \eta(1-bt)^{\frac{N\alpha-4}{2}}|u|^\alpha u = 0.$$

The pseudo-conformal transformation corresponds to the **variance identity**

$$\frac{d^2}{dt^2}\int_{\mathbb{R}^N}|x|^2|u(t,x)|^2 = 16E(u(t)) - \eta\frac{4(N\alpha-4)}{\alpha+2}\int_{\mathbb{R}^N}|u(t,x)|^{\alpha+2}. \tag{2.5}$$

Alternatively, we can write (2.5) in the form

$$\frac{d^2}{dt^2}\int_{\mathbb{R}^N}|x|^2|u(t,x)|^2 = 4N\alpha E(u(t)) - 2(N\alpha-4)\int_{\mathbb{R}^N}|\nabla u(t,x)|^2. \tag{2.6}$$

(Note that the last term in the above identities vanishes if $\alpha = \frac{4}{N}$.) Identity (2.5) is equivalent to the **pseudo-conformal conservation law**

$$\begin{aligned}\int_{\mathbb{R}^N}|(x+2it\nabla)u(t,x)|^2 - \frac{8\eta t^2}{\alpha+2}\int_{\mathbb{R}^N}|u(t,x)|^{\alpha+2}\\ = \int_{\mathbb{R}^N}|xu(0,x)|^2\\ + \frac{4(N\alpha-4)}{\alpha+2}\eta\int_0^t s\int_{\mathbb{R}^N}|u(s,x)|^{\alpha+2}dxds,\end{aligned} \tag{2.7}$$

which is an exact conservation law when $\alpha = \frac{4}{N}$. (Here also, it is essential that $\eta \in \mathbb{R}$.)

Remark 2.1. The above conservation laws have some important consequences. (Recall that the conservation laws are formal as of now, so their implications are only valid for the solutions for which the conservation laws are actually valid.)

(i) The conservation of mass (2.1) implies that the solutions of (NLS) are bounded in $L^2(\mathbb{R}^N)$ uniformly in time.
(ii) In the defocusing case $\eta < 0$, the conservation of energy (2.2) implies that if u is a solution of (NLS), then $\|\nabla u(t)\|_{L^2}$ is bounded uniformly in time.
(iii) The above two observations imply that in the defocusing case $\eta < 0$, the solutions of (NLS) are bounded in $H^1(\mathbb{R}^N)$ uniformly in time.
(iv) In the focusing case $\eta > 0$, the solutions of (NLS) are bounded in $H^1(\mathbb{R}^N)$ uniformly in time, provided $\alpha < \frac{4}{N}$. This follows easily from the conservation of charge and energy, together with the

Gagliardo-Nirenberg inequality (see [42])

$$\int_{\mathbb{R}^N} |u|^{\alpha+2} \le C\Big(\int_{\mathbb{R}^N} |\nabla u|^2\Big)^{\frac{N\alpha}{4}} \Big(\int_{\mathbb{R}^N} |u|^2\Big)^{\frac{4-(N-2)\alpha}{4}}, \tag{2.8}$$

which is valid for all $\alpha \ge 0$ such that $(N-2)\alpha \le 4$.

(v) In the focusing case $\eta > 0$ and if $\alpha = \frac{4}{N}$, then by (2.8) the solutions of (NLS) with sufficiently small L^2 norm are bounded in $H^1(\mathbb{R}^N)$ uniformly in time. This follows again from (2.8).

(vi) Suppose $(N-2)\alpha \le 4$. Conservation of mass and energy together with (2.8) imply that

$$\|u(t)\|_{H^1}^2 \le [2E(u(0)) + \|u(0)\|_{L^2}^2] + C\|u(t)\|_{H^1}^{\alpha+2}.$$

It follows easily from a trapping argument that if $\|u(0)\|_{H^1}$ is sufficiently small, then $\|u(t)\|_{H^1}$ remains bounded uniformly in time. This is relevant in the focusing case when $\alpha \ge \frac{4}{N}$.

3. The linear Schrödinger equation

The Cauchy problem for the linear Schrödinger equation

$$i\partial_t u + \Delta u = 0 \tag{3.1}$$

is easily solved by using the (space) Fourier transform $\mathcal{F}$. One obtains the ordinary differential equation for $\widehat{u} = \mathcal{F}u$

$$i\partial_t \widehat{u} - 4\pi^2|\xi|^2\widehat{u} = 0,$$

hence the solution (with the initial condition $u(0) = \varphi$)

$$\widehat{u}(t,\xi) = e^{-i4\pi^2 t|\xi|^2}\widehat{\varphi}(\xi), \tag{3.2}$$

or equivalently

$$u(t,\cdot) = [\mathcal{F}^{-1}(e^{-i4\pi^2 t|\xi|^2})] \star \varphi, \tag{3.3}$$

and

$$[\mathcal{F}^{-1}(e^{-i4\pi^2 t|\xi|^2})](x) = (4\pi i t)^{-\frac{N}{2}} e^{i\frac{|x|^2}{4t}}. \tag{3.4}$$

It is not difficult to prove that for every $\varphi \in \mathcal{S}'(\mathbb{R}^N)$ there exists a unique solution $u \in C(\mathbb{R}, \mathcal{S}'(\mathbb{R}^N))$ of (3.1) with the initial condition $u(0) = \varphi$, given by the above formulas. We denote by

$$u(t) = e^{it\Delta}\varphi$$

this solution. $(e^{it\Delta})_{t\in\mathbb{R}}$ is the Schrödinger group. In particular, by (3.2)

$$|\widehat{u}(t,\xi)| = |\widehat{\varphi}(\xi)|$$

for all $t \in \mathbb{R}$, $\xi \in \mathbb{R}^N$, so that

$$\|u(t)\|_{L^2} = \|\varphi\|_{L^2}, \quad t \in \mathbb{R}, \tag{3.5}$$

and more generally

$$\|D^\beta u(t)\|_{L^2} = \|D^\beta \varphi\|_{L^2}, \quad t \in \mathbb{R},$$

for every multi-index β. **Therefore all the Sobolev spaces $H^s(\mathbb{R}^N)$, $s \in \mathbb{R}$ are invariant by the linear flow, with conserved norm.** (Note that $L^p(\mathbb{R}^N)$ is not invariant by the flow if $p \neq 2$.)

Since $\|\mathcal{F}^{-1}(e^{-i4\pi^2 t|\xi|^2})\|_{L^\infty} = (4\pi|t|)^{-\frac{N}{2}}$ by (3.4), it follows from (3.3) that

$$\|u(t)\|_{L^\infty} \le (4\pi|t|)^{-\frac{N}{2}} \|\varphi\|_{L^1}, \quad t \neq 0. \tag{3.6}$$

This is the **dispersion estimate**, which expresses the fact that the solution of (3.1) converges to 0 locally in space as $|t| \to \infty$. Interpolation of (3.5) and (3.6) yields the general dispersion estimate

$$\|u(t)\|_{L^r} \le (4\pi|t|)^{-N(\frac{1}{2}-\frac{1}{r})} \|\varphi\|_{L^{r'}}, \quad t \neq 0. \tag{3.7}$$

valid for all $2 \le r \le \infty$ (where $\frac{1}{r'} = 1 - \frac{1}{r}$).

The dispersion estimate is pointwise in time. In 1977, Strichartz established a space-time estimate (the so-called Strichartz estimate) that is now fundamental in the study of (NLS) (and of other dispersive equations). See [125], then [49, 136, 24, 67]. We now state the Strichartz estimates as in [67]. We say that $(q,r) \in \mathbb{R}^2$ is an **admissible pair** if

$$2 \le q, r \le \infty, \quad (N-2)r \le 2N, \quad r < \infty \text{ if } N = 2, \quad \frac{2}{q} = N\Big(\frac{1}{2} - \frac{1}{r}\Big).$$

(The notation "admissible pair" is introduced in [25, Definition 1]). The homogeneous Strichartz estimate states that if $\varphi \in L^2(\mathbb{R}^N)$, then $u(t) = e^{it\Delta}\varphi$ satisfies $u \in L^q(\mathbb{R}, L^r(\mathbb{R}^N))$ for every admissible pair (q,r). Moreover, there exists a constant C such that

$$\|u\|_{L^q(\mathbb{R}, L^r(\mathbb{R}^N))} \le C\|\varphi\|_{L^2}, \tag{3.8}$$

for all $u \in L^2(\mathbb{R}^N)$. In space dimension $N \neq 2$, the constant C can be chosen independently of the admissible pair (q,r). The inhomogeneous Strichartz estimate concerns solutions of the linear, inhomogeneous Schrödinger equation

$$i\partial_t u + \Delta u + f = 0, \quad u(0) = 0,$$

where $f = f(t,x)$. By Duhamel's formula, it concerns, equivalently, the function

$$u(t) = \int_0^t e^{i(t-s)\Delta} f(s)\, ds.$$

The inhomogeneous Strichartz estimate states that if (γ, ρ) is an admissible pair, if (γ', ρ') is the pair of conjugate exponents, and if $f \in$

$L^{\gamma'}(\mathbb{R}, L^\rho(\mathbb{R}^N))$, then $u \in L^q(\mathbb{R}, L^r(\mathbb{R}^N))$ for every admissible pair (q, r). Moreover, there exists a constant C such that

$$\|u\|_{L^q(\mathbb{R}, L^r(\mathbb{R}^N))} \le C\|f\|_{L^{\gamma'}(\mathbb{R}, L^{\rho'}(\mathbb{R}^N))}, \tag{3.9}$$

for all $f \in L^{\gamma'}(\mathbb{R}, L^\rho(\mathbb{R}^N))$. In space dimension $N \neq 2$, the constant C can be chosen independently of the admissible pairs (q, r) and (γ, ρ).

Note that (3.9) is not the more general form of the nonhomogeneous Strichartz estimate. In particular, one can prove estimates for non-admissible pairs. See [27, 66, 131, 43, 127, 73, 74].

4. Local well-posedness

One can use the estimates of the Schrödinger group for solving the Cauchy problem for (NLS) by a perturbation argument (fixed point). Given and initial value φ, the appropriate formulation of the corresponding Cauchy problem

$$\begin{cases} i\partial_t u + \Delta u + \eta |u|^\alpha u = 0, \\ u(0, \cdot) = \varphi(\cdot), \end{cases} \tag{4.1}$$

is Duhamel's formula

$$u(t) = e^{it\Delta}\varphi + i\eta \int_0^t e^{i(t-s)\Delta}(|u|^\alpha u)(s)\, ds. \tag{4.2}$$

A crucial point in such problems is the choice of the space in which one applies the perturbation argument. As observed in the previous section, the appropriate spaces for the linear Schrödinger equation are the L^2-based Sobolev spaces $H^s(\mathbb{R}^N)$. Other spaces might also be used, but only as auxiliary spaces, namely the L^p spaces associated to the dispersion estimate, and the $L^q L^r$ spaces associated to the Strichartz estimates.

We begin with a very simple case. If $m > \frac{N}{2}$, then $H^m(\mathbb{R}^N) \hookrightarrow L^\infty(\mathbb{R}^N)$, and $H^m(\mathbb{R}^N)$ is a Banach algebra, see [1]. Assuming $\alpha \in 2\mathbb{N}$, i.e. α is an even integer, we may write $|u|^\alpha u = u^{\frac{\alpha}{2}+1}\overline{u}^{\frac{\alpha}{2}}$, and we deduce that the $u \mapsto |u|^\alpha u$ maps $H^m(\mathbb{R}^N) \to H^m(\mathbb{R}^N)$ and $\| |u|^\alpha u\|_{H^m} \le C\|u\|_{H^m}^{\alpha+1}$. Moreover,

$$\begin{aligned} \| |u|^\alpha u - |v|^\alpha v\|_{L^2} &\le C(\|u\|_{L^\infty}^\alpha + \|v\|_{L^\infty}^\alpha)\|u - v\|_{L^2} \\ &\le C(\|u\|_{H^m}^\alpha + \|v\|_{H^m}^\alpha)\|u - v\|_{L^2}. \end{aligned}$$

Since $e^{it\Delta}$ is an isometry of $H^m(\mathbb{R}^N)$, one can apply a standard argument based on the Banach fixed point theorem to prove the following result. (See [48] and [17, Theorem 4.10.1]).

Theorem 4.1. *Suppose* $\alpha \in 2\mathbb{N}$ *and let* $m \in \mathbb{N}$, $m > \frac{N}{2}$. *For every* $\varphi \in H^m(\mathbb{R}^N)$, *there exist* $T > 0$ *and a unique solution* $u \in C([0,T], H^m(\mathbb{R}^N))$ *of* (4.1). *Moreover, the following properties hold.*

(i) *The solution can be extended to a maximal interval of existence* $[0, T_{\max})$ *with* $0 < T_{\max} \le \infty$.

(ii) *(Blow-up alternative) If* $T_{\max} < \infty$, *then* $\|u(t)\|_{H^m} \to \infty$ *as* $t \uparrow T_{\max}$. *More precisely,*

$$\liminf_{t \uparrow T_{\max}} (T_{\max} - t)^{\frac{1}{\alpha}} \|u(t)\|_{H^m} > 0. \tag{4.3}$$

Moreover, $\limsup \|u(t)\|_{L^\infty} = \infty$ *as* $t \uparrow T_{\max}$.

(iii) u *depends continuously on* φ *in the following sense. The function* $T_{\max}$ *is lower semicontinuous* $H^m(\mathbb{R}^N) \to (0, \infty]$. *Moreover, if* $\varphi_n \to \varphi$ *in* $H^m(\mathbb{R}^N)$ *and if* u_n *is the maximal solution of* (4.1) *with the initial value* φ_n, *then* $u_n \to u$ *in* $C([0,T], H^m(\mathbb{R}^N))$ *for every* $0 < T < T_{\max}$.

(iv) *There is conservation of mass and energy, i.e.* (2.1) *and* (2.2) *hold for all* $0 \le t < T_{\max}$.

A similar result holds if $\alpha \notin 2\mathbb{N}$, under the assumption $\frac{N}{2} < m \le [\alpha] + 1$ ($\frac{N}{2} < m \le [\alpha]$ if α is an odd integer), where $[\alpha]$ is the integer part of α. See [17], Theorem 4.10.1 and Remark 4.10.3.

Theorem 4.1 is fairly elementary, but it is not obvious to obtain sufficient conditions on the initial value so that the corresponding solution is global (i.e. $T_{\max} = \infty$). Indeed, Remark 2.1 provides sufficient conditions so that solutions of (4.1) are bounded in $H^1(\mathbb{R}^N)$ on the interval $[0, T_{\max})$. For instance, in the defocusing case $\eta < 0$, all solutions are bounded in $H^1(\mathbb{R}^N)$. In dimension $N = 1$, one can choose $m = 1$, so that by the blow-up alternative, $T_{\max} = \infty$ for every initial value $\varphi \in H^1(\mathbb{R})$. In higher dimension, however, one must choose $m \ge 2$ in Theorem 4.1, and it is not clear that the H^1 bound on the solutions will prevent blowup of the H^m norm. This is true in the H^1-subcritical case $(N-2)\alpha < 4$ in space dimensions $N \le 7$ (see [48]). It seems that the case of larger dimensions $N \ge 8$ is open. In the energy supercritical case (but still for the defocusing equation), it can happen in higher dimension that some solutions blow up in finite time in H^m (even though they are bounded in H^1). See [97].

In view of the above observations, it is tempting to solve the Cauchy problem (4.1) in a space corresponding to the mass and energy, i.e. the space $H^1(\mathbb{R}^N)$. In order to solve locally the Cauchy problem in H^1, it seems natural to impose that the therm $\int |u|^{\alpha+2}$ in the energy be controlled by the H^1 norm. By Sobolev's embedding, this yields the condition $\alpha \le \frac{4}{N-2}$. The same condition appears if one requires that $\|u\|_{H^1}$ controls $\| |u|^\alpha u\|_{H^{-1}}$. (Recall that if $u \in H^1$, then $\Delta u \in H^1$, so this is also a natural condition by the equation.) The exponent $\alpha = \frac{4}{N-2}$ also arises from a scaling argument. Indeed, by scaling invariance, if u is a solution of (NLS) with the initial value φ, then for all $\lambda > 0$, $u_\lambda(t,x) = \lambda^{\frac{2}{\alpha}} u(\lambda^2 t, \lambda x)$ is also a solution of (NLS), with the initial value $\varphi_\lambda(x) = \lambda^{\frac{2}{\alpha}} \varphi(\lambda x)$. Moreover, $\|\nabla \varphi_\lambda\|_{L^2} = \lambda^{\frac{4-(N-2)\alpha}{2\alpha}} \|\nabla \varphi\|_{L^2}$, so that $\|\nabla \varphi_\lambda\|_{L^2} = \|\nabla \varphi\|_{L^2}$ for all $\lambda > 0$ if and only if $\alpha = \frac{4}{N-2}$. Thus on expects the exponent $\alpha = \frac{4}{N-2}$ to be critical for the H^1 theory.

Using the dispersion estimates and/or the Strichartz estimates, one can prove the following local well-posedness result, in the H^1-subcritical case. See Ginibre-Velo [46], Kato [64, 65]. For the lower estimate (4.5), see [26, Theorem 1.1 (vii)]. See also Proposition 6.5.1 (variance identity) and Theorem 7.2.1 (pseudo-conformal conservation law) in [17].

Theorem 4.2. *Suppose*

$$\alpha > 0, \quad (N-2)\alpha < 4. \tag{4.4}$$

Given any $\varphi \in H^1(\mathbb{R}^N)$, there exist $T > 0$ and a unique solution $u \in C([0,T], H^1(\mathbb{R}^N)) \cap C^1([0,T], H^{-1}(\mathbb{R}^N))$ of (4.1). Moreover, the following properties hold.

(i) *The solution can be extended to a maximal interval of existence $[0, T_{\max})$ with $0 < T_{\max} \le \infty$.*
(ii) *$u \in L^q((0,T), W^{1,r}(\mathbb{R}^N))$ for every $0 < T < T_{\max}$ and every admissible pair (q,r).*
(iii) *(Blow-up alternative) If $T_{\max} < \infty$, then $\|u(t)\|_{H^1} \to \infty$ as $t \uparrow T_{\max}$. More precisely,*
$$\liminf_{t \uparrow T_{\max}} (T-t)^{\frac{1}{\alpha} - \frac{N-2}{4}} \|u(t)\|_{H^1} > 0. \tag{4.5}$$
(iv) *u depends continuously on φ in the following sense. The function $T_{\max}$ is lower semicontinuous $H^1(\mathbb{R}^N) \to (0,\infty]$. Moreover, if $\varphi_n \to \varphi$ in $H^1(\mathbb{R}^N)$ and if u_n is the maximal solution of (4.1) with the initial value φ_n, then $u_n \to u$ in $C([0,T], H^1(\mathbb{R}^N))$ and in $L^q((0,T), W^{1,r}(\mathbb{R}^N))$ for every $0 < T < T_{\max}$ and every admissible pair (q,r).*
(v) *There is conservation of mass and energy, i.e. (2.1) and (2.2) hold for all $0 \le t < T_{\max}$.*
(vi) *If $|\cdot|\varphi \in L^2(\mathbb{R}^N)$, then $|\cdot|u \in C([0,T_{\max}), L^2(\mathbb{R}^N))$, the map $t \mapsto \int_{\mathbb{R}^N} |x|^2 |u(t,x)|^2$ belongs to $C^2([0,T_{\max}))$, and the variance identity (2.5) and the pseudo-conformal conservation law (2.7) hold for all $0 \le t < T_{\max}$.*

5. Standing waves in the focusing case

A **standing wave** (or **solitary wave**, or **stationary state**) is a solution of (NLS) of the form

$$u(t,x) = e^{i\omega t}\phi(x). \tag{5.1}$$

Such a solution, if it exists, has the constant profile ϕ, modulated by the periodic, space independent phase $e^{i\omega t}$. It is in particular a global solution of (NLS). Moreover, it is a genuinely nonlinear effect, since $|u(t,x)| = |\phi(x)|$ is time-independent, so there is no dispersion.

Clearly, u given by (5.1) is a solution of (NLS) if and only if ϕ is a solution of the nonlinear elliptic equation

$$-\Delta\phi + \omega\phi = \eta|\phi|^\alpha\phi. \tag{5.2}$$

We assume that $(N-2)\alpha < 4$**, and we look for solutions** $\phi \in H^1(\mathbb{R}^N)$, giving H^1 solutions of (NLS) in the energy-subcritical case. Multiplying the equation by $\overline{\phi}$ and taking the real part yields

$$\int_{\mathbb{R}^N} |\nabla\phi|^2 + \omega\int_{\mathbb{R}^N} |\phi|^2 = \eta\int_{\mathbb{R}^N} |\phi|^{\alpha+2}. \tag{5.3}$$

Moreover, Pohozaev's identity yields (see [12, Proposition 1])

$$\frac{N-2}{2}\int_{\mathbb{R}^N} |\nabla\phi|^2 + \frac{N\omega}{2}\int_{\mathbb{R}^N} |\phi|^2 = \frac{N\eta}{\alpha+2}\int_{\mathbb{R}^N} |\phi|^{\alpha+2}. \tag{5.4}$$

It follows from (5.3) and (5.4) that

$$E(\phi) = \frac{N\alpha-4}{2N\alpha}\int_{\mathbb{R}^N} |\nabla\phi|^2, \tag{5.5}$$

and

$$E(\phi) + \frac{\omega}{2}\int_{\mathbb{R}^N} |\phi|^2 = \frac{1}{N}\int_{\mathbb{R}^N} |\nabla\phi|^2. \tag{5.6}$$

If $\omega \le 0$**, then there is no nontrivial solution** $\phi \in H^1(\mathbb{R}^N)$ **of** (5.2). This is a delicate result, see [63, 2] (see also [18, p. 52]). If $\omega > 0$ and $\eta \le 0$, then there is no nontrivial solution $\phi \in H^1(\mathbb{R}^N)$ by (5.3).

We now assume $\omega > 0$ **and** $\eta > 0$.

In dimension $N = 1$, a simple calculation shows that there is the positive, even, radially decreasing, exponentially decaying solution of (5.2)

$$\phi(x) = \Bigl(\frac{\omega(\alpha+2)}{2\eta}\Bigr)^{\frac{1}{\alpha}}\Bigl(\cosh\Bigl(\frac{\alpha}{2}\sqrt{\omega}x\Bigr)\Bigr)^{-\frac{2}{\alpha}}.$$

All other (complex valued) solutions of (5.2) have the form $e^{i\theta_0}\phi(x-x_0)$ for some $\theta_0, x_0 \in \mathbb{R}$ (see e.g. [17, Theorem 8.1.6]).

In dimension $N \geq 2$, there always exists a positive, radially symmetric and decreasing solution of (5.2). It can be constructed by variational methods (constrained minimization) [122, 12, 11], or by ODE methods [61, 85]. Radially symmetric, positive solutions of (5.2) are unique, see [76, 84]. In fact, any positive solution of (5.2) is radially symmetric about some point of $\mathbb{R}^N$, see [44]. Therefore, if we denote by Φ the (unique) radially symmetric, positive solution of (5.2), then any positive solution of (5.2)has the form $\Phi(x - x_0)$ for some $x_0 \in \mathbb{R}$. As opposed to the one-dimensional case, there are infinitely many genuinely different solutions of (5.2) if $N \geq 2$. Again, these solutions can be constructed by variational arguments [122, 13, 11, 7] (and they are genuinely different because their H^1 norm is unbounded), or by ODE methods [61, 85] (and they are genuinely different by nodal considerations). In particular, for every integer $m \geq 0$, there exists a radially symmetric solution of (5.2) with exactly m zeros. This is not yet the end of the story, since there are many non-symmetric (in particular, non-radial) solutions of (5.2), see [4, 5, 86, 60].

The asymptotic behavior as $|x| \to \infty$ of the solutions $\phi \in H^1(\mathbb{R}^N)$ of (5.2) is precisely known. First, it follows from standard arguments that $\phi \in C^2(\mathbb{R}^N)$ and that ϕ has exponential decay at infinity, see e.g. [17, Theorem 8.1.1]. Then it follows from [101, Theorem 4.3] that

$$\phi(x) = |x|^{-\frac{N-1}{2}} e^{-|x|\sqrt{\omega}} \Big[f\Big(\frac{x}{|x|}\Big) + g\Big(|x|, \frac{x}{|x|}\Big)\Big], \tag{5.7}$$

where $f \in L^2(S^{N-1})$, $f \neq 0$; and $\int_{S^{N-1}} |g(r,\theta|^2 d\theta = O(r^{-\gamma})$ for some $\gamma > 0$.

Equation (5.2) is not scaling invariant, however if ϕ is a solution of (5.2), then for all $\lambda > 0$,

$$\phi_\lambda(x) = \lambda^{\frac{2}{\alpha}} \phi(\lambda x) \tag{5.8}$$

is a solution of

$$-\Delta\phi_\lambda + \lambda^2 \omega \phi_\lambda = \eta |\phi_\lambda|^\alpha \phi_\lambda. \tag{5.9}$$

In other words, if ϕ is the profile of a standing wave with frequency ω, then ϕ_λ is the profile of a standing wave with frequency $\lambda^2\omega$.

The (unique) positive, radially symmetric solution of (5.2) is called the **ground state**. This solution is also characterized by several variational criteria. For further reference, we denote by Q the ground state for $\omega = \eta = 1$ and $\alpha = \frac{4}{N}$, i.e. Q is the unique positive, radially symmetric solution of

$$-\Delta Q + Q = |Q|^{\frac{4}{N}} Q. \tag{5.10}$$

Given $\omega > 0$, we let Q_ω be the ground state with frequency ω, i.e.

$$Q_\omega(x) = \omega^{\frac{1}{\alpha}} Q(x\sqrt{\omega}). \tag{5.11}$$

An important variational characterization of Q is due to Weinstein [133] and is related to the best constant in the Gagliardo-Nirenberg estimate (2.8) in the case $\alpha = \frac{4}{N}$. More precisely,

$$\frac{1}{2+\frac{4}{N}} \int_{\mathbb{R}^N} |u|^{2+\frac{4}{N}} \le \frac{1}{2}\Big(\int_{\mathbb{R}^N} Q^2\Big)^{-\frac{2}{N}} \Big(\int_{\mathbb{R}^N} |\nabla u|^2\Big)\Big(\int_{\mathbb{R}^N} |u|^2\Big)^{\frac{2}{N}}, \tag{5.12}$$

for all $u \in H^1(\mathbb{R}^N)$. It follows that if $\alpha = \frac{4}{N}$, then

$$E(u) \ge \frac{1}{2}\|\nabla u\|_{L^2}^2 \left(1 - \frac{\|u\|_{L^2}^{\frac{2}{N}}}{\|Q\|_{L^2}^{\frac{2}{N}}}\right), \tag{5.13}$$

for all $u \in H^1(\mathbb{R}^N)$.

By Galilean invariance, if u is a standing wave (5.1), then

$$u_\beta(t,x) = e^{i\omega t} e^{i\frac{\beta}{2}\cdot(x-\beta t)}\phi(x-\beta t), \tag{5.14}$$

is also a solution of (NLS) for all $\beta \in \mathbb{R}^N$. We can write

$$u_\beta(t,x) = e^{i\omega t}\psi_\beta(x-\beta t) \quad \text{where} \quad \psi_\beta(x) = e^{i\frac{\beta}{2}\cdot x}\phi(x),$$

and we see that u_β is the profile ψ_β traveling with the velocity β and modulated by the periodic, space independent phase $e^{i\omega t}$. Such a solution is called **traveling wave**, or **soliton**.

In the case $\alpha = \frac{4}{N}$, we can apply the pseudo-conformal transformation (2.4) to the particular solution $u(t,x) = e^{it}Q(x)$of (NLS). Choosing $b = \frac{1}{T}$ with $T > 0$, we obtain the solution v of (NLS) given by

$$v_T(t,x) = \Big(\frac{T}{T-t}\Big)^{\frac{N}{2}} e^{i\frac{tT}{T-t}} e^{-i\frac{|x|^2}{4(T-t)}} Q\Big(\frac{xT}{T-t}\Big), \tag{5.15}$$

for $t < T$ and $x \in \mathbb{R}^N$. It follows that $v_T \in C((-\infty, T), H^1(\mathbb{R}^N))$ is a solution of (NLS) for $t < T$. Moreover,

$$\|v_T(t)\|_{L^r} = \Big(\frac{T}{T-t}\Big)^{\frac{N}{2}(1-\frac{2}{r})} \|Q\|_{L^r}, \quad t < T, \tag{5.16}$$

for every $1 \le r \le \infty$ and

$$\|\nabla v_T(t)\|_{L^2}^2 = \Big(\frac{T}{T-t}\Big)^2 \|\nabla Q\|_{L^2}^2 + \frac{1}{4T^2}\int_{\mathbb{R}^N} |x|^2 Q(x)^2 dx, \quad t < T. \tag{5.17}$$

Thus we see that

$$(T-t)\|\nabla v_T(t)\|_{L^2} \underset{t\to T}{\longrightarrow} T\|\nabla Q\|_{L^2}. \tag{5.18}$$

In particular, v_T blows up at $t = T$ twice faster than the lower bound given by (4.5). This explicit example of blowup is taken from [135].

Remark 5.1. Note that the above observation concerning (5.15) apply not only for the ground state Q, but also when Q is replaced by any nontrivial H^1 solution of (5.2).

6. Global existence vs. finite-time blowup

We consider the equation (NLS) in the energy subcritical case $(N-2)\alpha < 4$. Given any $\varphi \in H^1(\mathbb{R}^N)$, there exists a unique H^1 solution u of (4.1) defined on the maximal interval $[0, T_{\max})$, and we look for sufficient conditions for global existence ($T_{\max} = \infty$) or finite-time blowup ($T_{\max} < \infty$). Since the mass and energy are conserved, we can apply Remark 2.1.

We see that in the defocusing case $\eta < 0$, all solutions are global and uniformly bounded (with respect to time) in $H^1(\mathbb{R}^N)$. (The original result is [46].)

We now study the focusing case $\eta > 0$.

In the mass-subcritical case $\alpha < \frac{4}{N}$. all solutions are global and uniformly bounded (with respect to time) in $H^1(\mathbb{R}^N)$. (The original result is [46].)

In the mass-critical case, if $\|\varphi\|_{L^2}$ is sufficiently small, then the corresponding solution u is global and uniformly bounded (with respect to time) in $H^1(\mathbb{R}^N)$. In fact, we can use (5.13) to obtain an explicit condition, see [133]. Applying (5.13) and conservation of mass and energy, we see that

$$E(\varphi) = E(u) \geq \frac{1}{2}\|\nabla u\|_{L^2}^2 \left(1 - \frac{\|u\|_{L^2}^{\frac{2}{N}}}{\|Q\|_{L^2}^{\frac{2}{N}}}\right) = \frac{1}{2}\|\nabla u\|_{L^2}^2 \left(1 - \frac{\|\varphi\|_{L^2}^{\frac{2}{N}}}{\|Q\|_{L^2}^{\frac{2}{N}}}\right);$$

and so, if

$$\|\varphi\|_{L^2} < \|Q\|_{L^2}, \tag{6.1}$$

then $\|\nabla u\|_{L^2}$ is bounded, so $T_{\max} = \infty$ by the blowup alternative and the solution is uniformly bounded (with respect to time) in $H^1(\mathbb{R}^N)$.

In the mass-supercritical case $\alpha > \frac{4}{N}$, we deduce from Remark 2.1 that if $\|\varphi\|_{H^1}$ is sufficiently small, then the corresponding solution of (4.1) is global and uniformly bounded (with respect to time) in $H^1(\mathbb{R}^N)$.

Thus we see that finite-time blowup may only occur in the mass-critical and mass-supercritical cases, for sufficiently large initial data.

We note that the condition (6.1) for global existence in the mass-critical case is optimal. Indeed, the explicit solution (5.15) blows up in finite time and has exactly the mass $\|Q\|_{L^2}$, see (5.16).

We now examine sufficient conditions for finite-time blowup. All such conditions are based on the variance identity (2.5) or some modification of it. The first applications of (2.5) are due to [139] (see in particular inequality (3.7) and the comments that follow) in the case $N = 3$, $\alpha = 2$ and [51] in the general case. The main result is the following.

Theorem 6.1. *Suppose $\eta > 0$, $\alpha \geq \frac{4}{N}$, $(N-2)\alpha < 4$. Let $\varphi \in H^1(\mathbb{R}^N)$ and let u be the corresponding solution of (4.1) defined on the maximal interval $[0, T_{\max})$. If $E(\varphi) < 0$ and $|\cdot|\varphi \in L^2(\mathbb{R}^N)$, then $T_{\max} < \infty$.*

The proof is quite elementary. By (2.5) and conservation of energy, the C^2 function $f(t) = \int_{\mathbb{R}^N} |x|^2 |u(t,x)|^2$ satisfies

$$f''(t) \leq 16E(u(t)) = 16E(\varphi) < 0;$$

And so,

$$0 \leq f(t) \leq f(0) + tf'(0) + 8t^2 E(\varphi),$$

for all $0 \leq t < T_{\max}$, so that $T_{\max} \leq \frac{f'(0)+\sqrt{f'(0)^2+8f(0)(-E(\varphi))}}{16(-E(\varphi))} < \infty$.

Whether or not the assumption $|\cdot|\varphi \in L^2(\mathbb{R}^N)$ can be removed in Theorem 6.1 is a long standing open problem. It can be removed in the one-dimensional mass-critical case $\alpha = 4$ (see [107]) and for radially symmetric solutions in dimensions $N \geq 2$ under the extra assumption $\alpha < 4$ if $N = 2$ (see [108]). It also can be removed in the mass critical case $\alpha = \frac{4}{N}$, $N \geq 2$, assuming a spectral condition, and for initial values φ satisfying $\|\varphi\|_{L^2} \leq \|Q\|_{L^2} + \varepsilon$ for some sufficiently small $\varepsilon > 0$, see [90]. **In all other cases, the problem is open.**

7. Stability of standing waves in the focusing case

According to the "soliton resolution conjecture", a global solution of (NLS) should split into a sum of solitons traveling at different velocities and a dispersive part. Of course, this formulation is vague in what concerns the topologies involved and the regularity of the solutions under consideration. For (NLS), the soliton resolution conjecture is essentially open, except in the one-dimensional, cubic case. Indeed, the equation is then completely integrable and can be studied by inverse scattering methods [35, 38, 62, 106, 138, 140, 141].

In the focusing case $\eta > 0$, it is natural to study the **stability** of the standing waves of Section 5.

The standing wave $u(t,x) = e^{i\omega t}\phi(x)$ is stable if an initial value close (in $H^1(\mathbb{R}^N)$) to ϕ (the initial value of u) produces a solution of (NLS) which remains close to u in some appropriate sense. The "appropriate sense" must sake into account the invariances of the equation. Indeed, by Galilean invariance and uniqueness, for any $\beta \in \mathbb{R}^N$ the initial value $\phi_\beta(x) = e^{i\frac{\beta\cdot x}{2}}\phi(x)$ produces the solution $u_\beta(t,x) = e^{i\frac{\beta\cdot(x-\beta t}{2}}e^{i\omega t}\phi(x-\beta t)$. We have $\|\phi_\beta - \phi\|_{H^1} \to 0$ as $|\beta| \to 0$, but u_β does not remain close to u for all time. However, u_β remains close to u modulo space translations and multiplication by a constant phase. In other words, u_β remains close to the orbit $\mathcal{O}(\phi) = \{e^{i\theta}\phi(\cdot - y); \theta \in \mathbb{R}, y \in \mathbb{R}^N\}$. So we say that the standing wave $u(t,x) = e^{i\omega t}\phi(x)$ is **(orbitally) stable** if an initial value close to ϕ in $H^1(\mathbb{R}^N)$ produces a solution that remains close (in $H^1(\mathbb{R}^N)$) to $\mathcal{O}(\phi)$. Otherwise, the standing wave is called **(orbitally) unstable**. This means that there exists an ε-neighborhood of $\mathcal{O}(\phi)$ such that some initial values arbitrarily close to ϕ produce solutions of (NLS) that eventually leave this neighborhood. One also uses a stronger notion of instability. The standing wave $u(t,x) = e^{i\omega t}\phi(x)$ is **unstable by blowup** if initial values arbitrarily close to ϕ (in $H^1(\mathbb{R}^N)$) can produce solutions of (NLS) that blow up in finite time.

In the mass-critical case $\alpha = \frac{4}{N}$, **all standing waves** $v(t) = e^{i\omega t}\phi$ with $\phi \in H^1(\mathbb{R}^N)$, $\phi \neq 0$ **are unstable by blowup**. More precisely, for every $\varepsilon > 0$, there exists $\varphi \in H^1(\mathbb{R}^N)$ such that $\|\varphi - \phi\|_{H^1} \le \varepsilon$ and the solution u of (4.1) blows up in finite time. In particular, even though $u(0)$ can by arbitrarily close to $v(0)$, $u(t)$ does not remain close to $v(t)$ for all time. This is proved in [133]. The argument is very simple. It follows from (5.5) that $E(\phi) = 0$. Therefore, if $\varphi = (1+\delta)\phi$ with $\delta > 0$, then

$$E(\varphi) = (1+\delta)^2 E(\phi) - \eta\frac{[(1+\delta)^{2+\frac{4}{N}} - (1+\delta)^2]}{2+\frac{4}{N}}\int_{\mathbb{R}^N}|\phi|^{2+\frac{4}{N}} < 0.$$

Since ϕ has exponential decay by (5.7), it has finite variance, so that u blows up in finite time.

Still in the mass-critical case, one can prove another form of instability for the ground state Q. Indeed, if $\|\varphi\|_{L^2} < \|Q\|_{L^2}$, then we know that the corresponding solution u of (4.1) is global. In fact, one can prove that u scatters (see [68, 69] for the radial case and [36] for the general case), and in particular that $\|u(t)\|_{L^p} \to 0$ as $t \to \infty$ for all $2 < p < \frac{2N}{N-2}$. In particular, u does not remain close to the standing wave $e^{it}Q$. This is true for $\varphi = (1-\varepsilon)Q$, $0 < \varepsilon < 1$, which can be arbitrarily close to Q.

In the mass-supercritical case $\alpha > \frac{4}{N}$, standing waves have positive energy by (5.5), so that the above argument does not apply. One can prove that the **ground state is unstable by blowup**. This follows from the variance identity and an appropriate variational characterization of the ground state. See [10], and also [17, Theorem 8.2.2] for details. One can expect that the other (non ground state) standing waves are unstable, maybe not by blowup. The general case is open, but radially symmetric standing waves unstable, see [52, Theorem 3.2]. The proof relies on the analysis of an appropriate linearized operator.

As in the mass-critical case, one can prove instability of the ground state by scattering, see [59, 37, 40, 3].

Even when the standing waves are unstable, there may be directions of stability, i.e. stable manifolds. In fact, there should be in general a finite number of instability directions. The study of the stability directions of the ground state is related to the study of the asymptotic stability of the ground states. This is a very delicate issue, for which one can consult the review articles by Kowalczyk, Martel and Muñoz [75], and by Cuccagna and Maeda [33] There are very few asymptotic stability results available, and they concern the cubic 3-dimensional equation. The main result is the existence of a local center-stable manifold around the ground state, such that for all initial data in this manifold, the solution decomposes into a moving soliton and a dispersive part [120, 8, 104]. This is related to the soliton resolution conjecture.

In the mass-subcritical case $\alpha < \frac{4}{N}$, the **ground state is orbitally stable** [21]. This follows easily from conservation of mass and energy, and an appropriate variational characterization of the ground state. The result also follows from the general conditions obtained in the works [53, 54] and based on the study of an appropriate linearized operator. Orbital stability of the ground state ϕ means that if φ is sufficiently close to ϕ, then there exist parameters $\omega(t) \in \mathbb{R}$ and $x(t) \in \mathbb{R}^N$ such that the solution $u(t)$ of (4.1) remains close for all time to $e^{i\omega(t)}\phi(\cdot - x(t))$. The "modulation equations" satisfied by the parameters $\omega(t)$ and $x(t)$ are studied in [134]. A fundamental ingredient is the study of the spectral properties of the linearized operators

$$L^+ = -\Delta + \omega - (\alpha + 1)\phi^\alpha,$$
$$L^- = -\Delta + \omega - \phi^\alpha.$$

Note that in particular $L^-\phi = 0$ and $L^+(\partial_{x_j}\phi) = 0$ for $1 \le j \le N$. In fact, L^- is a self-adjoint, nonnegative operator on $L^2(\mathbb{R}^N)$ with null space $\mathbb{R}\phi$, and L^+ is a self-adjoint operator on $L^2(\mathbb{R}^N)$ whose null space is span $\{\partial_{x_j}\phi; 1 \le j \le N\}$.

One can expect that the standing waves that are not ground states are unstable, but it seems that **no result of this type is available**.

It seems that there is only one asymptotic stability result available in the mass subcritical case. It concerns the cubic, one-dimensional equation, and is based on the inverse scattering method. See [34, Theorem 1.3].

Some solutions of (NLS) decompose as $t \to \infty$ in the form of a sum of solitons traveling at different velocities. Such solutions are called **multi-solitons**. See [87] (mass-critical case), [81] (mass-subcritical case), [32] (mass-supercritical case). More precisely, let $0 < \alpha < \frac{4}{N-2}$. For any $k \in \{1, \dots, K\}$, $K \in \mathbb{N}$, let $\omega_k^0 > 0$, $v_k \in \mathbb{R}^N$, $x_k^0 \in \mathbb{R}^N$, and $\gamma_k^0 \in \mathbb{R}$. Assume that, for any $k \neq k'$, $v_k \neq v_{k'}$. Let $R_k(x,t) = Q_{\omega_k^0}(x - x_k^0 - tv_k)e^{i(\frac{1}{2}v_k \cdot x - \frac{1}{4}|v_k|^2 t + \omega_K^0 t + \gamma_k^0)}$ be a solitary wave of (NLS) moving on the line $x = x_k^0 + tv_k$. There exists an H^1 solution u of (NLS) such that, for all $t > 0$,

$$\Big\| u(t) - \sum_{k=1}^{K} R_k(t) \Big\|_{H^1} \leq Ce^{-\theta_0 t}$$

for some $\theta_0 > 0$ and $C > 0$. Construction is made by solving a Cauchy problem at infinity. The standing waves are at distance of order t, and they are exponentially decaying, so their interaction is weak.

Several generalizations and extensions: Similar construction with excited states instead of the ground state [31]. Multiple existence (non-uniqueness) in the one-dimensional, mass-supercritical case [30]. Multi-solitons with infinitely many terms [78, 79].

Some solutions involve solitons moving at approximately the same velocity. Their interaction is strong and the construction is much more delicate. More precisely, in the case $\alpha \neq \frac{4}{N}$, there exists a solution u of (NLS) such that

$$\| u(t) - e^{\gamma(t)}(Q(\cdot - x(t)) + Q(\cdot + x(t)) \|_{H^1} \leq \frac{C}{t}$$

where $|x(t)| = (1 + O(1)) \log t$. See [105].

The existence of such a solution is **ruled out** by the variance identity in the **mass-critical case**. (We would have $\int |x|^2 |u|^2 \sim (\log t)^2$ but $\frac{d^2}{dt^2} \int |x|^2 |u|^2$ is constant.) A different phenomenon takes place in this case. More precisely [83], given an integer $K \geq 1$, there exists a solution u of (NLS) such that

$$\Big\| u(t) - e^{i\gamma(t)} \sum_{k=1}^{K} \mu(t) Q(\mu(t)(\cdot - x_k(t))) \Big\|_{H^1} \underset{t \to \infty}{\longrightarrow} 0,$$

where $\gamma(t) \in \mathbb{R}$ is some phase parameter, the translation parameters $|x_k(t)$ converge as $t \to \infty$ to the vertices of a K-sided regular polygon, and

$$\frac{\mu(t)}{\log t} \underset{t\to\infty}{\longrightarrow} 1.$$

So $u(t)$ is asymptotically the sum of K solitons at logarithmic distance, but the solitons are concentrated logarithmically by $\mu(t)$. This is **most interesting!** Indeed, the solution u is **global, but unbounded in** H^1, since

$$\frac{\|\nabla u(t)\|_{L^2}}{\log t} \underset{t\to\infty}{\longrightarrow} \sqrt{K}\|\nabla Q\|_{L^2}. \tag{7.1}$$

Applying the pseudo-conformal transformation, one obtains a solution $\widetilde{u}$ of (NLS) that blows up at $t = 1$ with

$$\frac{1-t}{|\log(1-t)|}\|\nabla \widetilde{u}(t)\|_{L^2} \underset{t\to 1}{\longrightarrow} \sqrt{K}\|\nabla Q\|_{L^2}. \tag{7.2}$$

In particular, the solution $\widetilde{u}$ **blows up faster than the pseudo-conformal rate** (5.18).

Global, unbounded solutions satisfying a lower bound like (7.1) **do not exist in the mass-supercritical case**. Indeed, in this case a global solution satisfies

$$\liminf_{t\to\infty} \|\nabla u(t)\|_{L^2} < \infty.$$

This can be seen by integrating twice (2.6) to obtain $\int_0^t \int_0^s \|\nabla u(\sigma)\|_{L^2}^2 \leq Ct^2$. **Does there exist any global, unbounded solution in the mass-supercritical case?**

Note that the multi-soliton results fit in the soliton resolution conjecture. However, there is no dispersive term. It seems that there is **no available results of a solution in the form of a multi-soliton plus a dispersive term. (Except in the integrable case.)**

Also, the **stability of the multi-soliton** solutions seems to be essentially open.

8. Scattering in the defocusing case

In the defocusing case $\eta < 0$, there are no solitons, so the solution resolution conjecture would mean that every solution is dispersive. One can expect that the solutions of (NLS) behave like a solution of the linear Schrödinger equation. This is the scattering theory, according to which a solution of (NLS) behaves as $t \to \infty$ like a solution of the linear equation in the sense that there exists a scattering state $u^+ \in H^1(\mathbb{R}^N)$, such that

$$\|u(t) - e^{it\Delta}u^+\|_{H^1} \underset{t\to\infty}{\longrightarrow} 0. \tag{8.1}$$

When a solution satisfies (8.1) we say that this **solution scatters**.

A heuristic argument indicates that scattering can only be the rule if $\alpha > \frac{2}{N}$. Indeed, $\|e^{it\Delta}u^+\|_{L^\infty}$ decays at most like $t^{-\frac{N}{2}}$ (see e.g.: Lemma, p. 69 in [121]; Decay Lemma, p. 228 in [66]; and Proposition 8.1 in [28] for three different proofs), and this decay will make the potential $|u|^\alpha$ integrable in time only if $\alpha > \frac{2}{N}$. In fact, if $\alpha \le \frac{2}{N}$, then scattering cannot be expected, see [121, Theorem 3.2 and Example 3.3, p. 68], and [6] (one-dimensional case). More precisely, if $\alpha \le \frac{2}{N}$ ($\alpha \le 1$ in dimension $N = 1$) and if $\varphi \in \Sigma$ where

$$\Sigma = H^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N, |x|^2 dx), \tag{8.2}$$

and u is the corresponding solution of (4.1), then there does not exist any $u^+ \in L^2$ such that $\|u(t) - e^{it\Delta}u^+\|_{L^2} \to 0$ as $t \to \infty$. See e.g. [17, Theorem 7.6.2]. (In the one-dimensional case and $1 < \alpha \le 2$, there is no limit in Σ.)

Thus we see that it is only when $\alpha > \frac{2}{N}$ that one can expect scattering. Showing scattering for all initial values (in a certain space) is the problem of **asymptotic completeness**. To be more precise, asymptotic completeness is for a certain space (H^1, or Σ, for example), and it means that for all initial value in this space, $u(t) - e^{it\Delta}u^+ \to 0$ **in the same space**. (Essential for constructing the scattering operator.)

Since asymptotic completeness involves arbitrarily large initial values, it cannot be the result of a fixed point argument, so it requires an *a priori* estimate.

Two different estimates have been used. The pseudo-conformal conservation law (2.7) in the case $\eta < 0$ and $\alpha \ge \frac{4}{N}$ immediately yields the decay estimate

$$t^2 \|u(t)\|_{L^{\alpha+2}}^{\alpha+2} \le C,$$

for initial values in Σ. Using this decay estimate, one can prove asymptotic completeness (in Σ) if $\alpha \ge \alpha_0$ where $\alpha_0 = \frac{N+2+\sqrt{N^2+12N+4}}{2N}$. See [47] (case $\alpha \ge \frac{4}{N}$) and [129] (case $\alpha \ge \alpha_0$).

The other estimate is Morawetz's estimate, see [80]. It implies that in space dimension $N \ge 3$,

$$\int_0^\infty \int_{\mathbb{R}^N} \frac{|u(t,x)|^{\alpha+2}}{|x|} < \infty,$$

for all initial values in $H^1(\mathbb{R}^N)$. This implies sufficient decay to prove asymptotic completeness in $H^1(\mathbb{R}^N)$ for $\alpha > \frac{4}{N}$, see [50]. A modified Morawetz estimate was introduced in [102], from which one can extend the previous result to the cases $N = 1, 2$.

Morawetz's estimate has given rise to various extensions (interaction Morawetz estimates), which have become an important tool in the study of (NLS), see e.g. [29, 113].

In the case $\alpha > \frac{2}{N}$, there is a general result by Tsutsumi and Yajima [130] which proves that for every initial value in Σ there exists a scattering state $u^+ \in L^2(\mathbb{R}^N)$ such that $u(t) - e^{it\Delta}u^+ \to 0$ in $L^2(\mathbb{R}^N)$.

However, if $\alpha \le \alpha_0$ (still in the defocusing case), then **no asymptotic completeness result** in any space seem to be available.

The case $\alpha < \frac{2}{N}$ seems to be completely open. For every initial value $\varphi \in H^1(\mathbb{R}^N)$, the solution u of (4.1) is global and uniformly bounded in $H^1(\mathbb{R}^N)$. Moreover,

$$\|u(t)\|_{L^p} \underset{t\to\infty}{\longrightarrow} 0,$$

for every $2 < p < \frac{2N}{N-2}$ by [132]. If the initial value is in Σ, then some explicit decay rate can be derived from the pseudo-conformal conservation law [17, Theorem 7.3.1 (ii)]. Moreover, u does not decay faster than the solutions of the linear Schrödinger equation, since $\liminf t^{N(\frac{1}{2}-\frac{1}{q})}\|u(t)\|_{L^q} > 0$ as $t \to \infty$ for all $2 < q \le \infty$, see [9, Theorem 2.3]. In particular, by [17, Theorem 7.3.1 (ii)] and [9, Theorem 2.3],

$$c \le t^{\frac{N\alpha}{2(\alpha+2)}}\|u(t)\|_{L^{\alpha+2}} \le C,$$

for t large, with $0 < c < C < \infty$. On the other hand, u does not scatter, but what is the precise behavior of u for large time? Is some kind of modified scattering possible?

Another interesting **question** : assuming $\alpha_0 < \alpha < \frac{4}{N}$, what is the asymptotic behavior of a solution with initial value $\varphi \in H^1(\mathbb{R}^N)$, $\varphi \notin \Sigma$?

9. Low-energy scattering in the focusing case

One can establish **low energy scattering**, i.e. scattering for small solutions in an appropriate sense. This is usually obtained by a fixed point argument producing solutions defined for all $t \ge 0$ and having a certain decay, which implies scattering. This argument works of course equally well in **the focusing and defocusing cases**.

There are intrinsic **limitations**, in particular because there may exist arbitrarily **small standing waves**, which do not scatter. So the possibility of low energy scattering depends on the power α and the norm which one chooses. (And there is always the limitation $\alpha > \frac{2}{N}$.)

Using the scaling (5.8), we see that there are arbitrarily small standing waves in $H^1(\mathbb{R}^N)$ if $\alpha < \frac{4}{N}$, so that low energy scattering in $H^1(\mathbb{R}^N)$ can only be true if $\alpha \ge \frac{4}{N}$. And il fact, there is actually low energy scattering in $H^1(\mathbb{R}^N)$, see [123, 124].

In the smaller space Σ given by (8.2), there are arbitrarily small standing waves if $\alpha < \frac{4}{N+2}$, so that $\alpha \ge \frac{4}{N+2}$ is a necessary condition of scattering for small initial data in Σ. There is indeed low energy scattering in Σ under the slightly stronger assumption $\alpha > \frac{4}{N+2}$, see [27, Theorem 4.2].

Low energy scattering in some appropriate space (with sufficient regularity and decay) under the condition $\alpha > \frac{2}{N}$, is established in [123, 124, 27, 45, 103] in dimension $N = 1, 2, 3$.

Results are only partial in the general case $N \geq 4$, see [45, 103]. The difficulty is the following. Using the scaling (5.8) we see that there are arbitrarily small standing waves in the space $L^2(\mathbb{R}^N, |x|^{2m}dx)$ if $\alpha < \frac{4}{N+2m}$. Thus we see that in order to rule out the possibility of small standing waves for α close to $\frac{2}{N}$, we would have to choose $m \geq \frac{N}{2}$. However, for the Schrödinger equation, space decay and regularity are related, so we would have to work in a space like $H^m(\mathbb{R}^N) \cap L^2(\mathbb{R}^N, |x|^{2m}dx)$. The problem is now that the nonlinearity is not sufficiently smooth to be differentiated m times!

A (very partial) solution to the above differentiability problem is given in [22]. Indeed, the defect of regularity of the nonlinearity $|u|^\alpha u$ is **only at** $u = 0$, so it is not seen by **solutions that do not vanish**.

The strategy in [22] is the following. First, applying the pseudo-conformal transformation (2.4) to the Cauchy problem (4.1), one obtains the non-autonomous equation

$$\begin{cases} i\partial_t v + \Delta v + \eta(1-bt)^{\frac{N\alpha-4}{2}}|v|^\alpha v = 0, \\ v(0) = \psi, \end{cases} \tag{9.1}$$

where $\psi(x) = e^{-i\frac{b|x|^2}{4}}\varphi(x)$. In addition the behavior of $u(t,\cdot)$ as $t \to \infty$ is related to the behavior of $v(t,\cdot)$ as $t \to \frac{1}{b}$. In particular, u **scatters** in Σ if and only if $v(t)$ **has a limit** in Σ as $t \to \frac{1}{b}$.

Note that if $\alpha > \frac{2}{N}$, then the non-autonomous term $(1-bt)^{\frac{N\alpha-4}{2}}$ is integrable at $t = \frac{1}{b}$, and that

$$\|(1-b\,\cdot)^{\frac{N\alpha-4}{2}}\|_{L^1(0,\frac{1}{b})} \leq \frac{C}{b} \underset{b\to\infty}{\longrightarrow} 0.$$

Next, fix $k > \frac{N}{2}$, $n > \frac{N}{2}+1$, $n > \frac{N}{2\alpha}$, $2m > k+n+1$, set $J = 2m+2+k+n$ and define $\mathcal{X}$ by

$$\begin{aligned} \mathcal{X} = \{u \in H^J(\mathbb{R}^N); \langle x\rangle^n D^\beta u \in L^\infty(\mathbb{R}^N), 0 \leq |\beta| \leq 2m \\ \langle x\rangle^n D^\beta u \in L^2(\mathbb{R}^N), 2m+1 \leq |\beta| \leq 2m+2+k, \\ \langle x\rangle^{J-|\beta|} D^\beta u \in L^2(\mathbb{R}^N), 2m+2+k < |\beta| \leq J\} \end{aligned} \tag{9.2}$$

with the obvious norm. In particular, $\langle x\rangle^{-\mu} \in \mathcal{X}$ for $\mu \geq n$. It follows from Taylor's formula, Sobolev's embedding and energy estimates that

$(e^{it\Delta})_{t\in\mathbb{R}}$ is a continuous group on the space $\mathcal{X}$. By direct calculations, if $u\in\mathcal{X}$ satisfies $K\inf_{x\in\mathbb{R}^N}\langle x\rangle^n|u(x)|\ge 1$, for some $K>0$, then

$$\|\,|u|^\alpha u\|_{\mathcal{X}}\le C(1+K\|u\|_{\mathcal{X}})^{2J}\|u\|_{\mathcal{X}}^{\alpha+1}$$

and a similar estimate holds for $\|\,|u|^\alpha u-|v|^\alpha v\|_{\mathcal{X}}$.

We can now use a simple contraction mapping argument $C([0,\frac{1}{b}],\mathcal{X})$. It follows that if $\alpha>\frac{2}{N}$, if $\psi\in\mathcal{X}$ satisfies

$$\inf_{x\in\mathbb{R}^N}\langle x\rangle^n|\psi(x)|>0, \tag{9.3}$$

and if $b>0$ is sufficiently large, then there exist a unique solution $u\in C([0,\frac{1}{b}],\mathcal{X})$ of (9.1). In terms of the original equation, this means that if $\psi\in\mathcal{X}$ satisfies (9.3) and if $b>0$ is sufficiently large, then **the solution of** (4.1) with $\varphi(x)=e^{i\frac{b|x|^2}{4}}\psi(x)$ **scatters**. Here there is no smallness condition on φ, but instead it must be sufficiently oscillatory.

In the case $\alpha=\frac{2}{N}$, **modified scattering** is expected, i.e. the existence of a phase $\theta(t,x)$ such that $u(t,x)$ behaves like $e^{i\theta(t,x)}e^{it\Delta}u^+$. There are only partial results, see [110, 16, 58, 119, 57, 72] One can, however, use the strategy of [22], as is done in [23]. In the case $\alpha=\frac{2}{N}$, the power of $(1-bt)$ in (9.1) is $(1-bt)^{-1}$, which is **not integrable**. However,

$$\int_0^t(1-bs)^{-1-\mu}ds\le\frac{1}{b\mu}(1-bt)^{-\mu}$$

for every $\mu>0$ and $t<\frac{1}{b}$. It follows that if a certain norm of $e^{i(t-s)\Delta}|v(s)|^\alpha v(s)$ is estimated by $(1-bs)^{-\mu}$, then the integral in (9.1) is estimated in that norm by the same power $(1-bt)^{-\mu}$. Concretely, this means that we can control a certain growth of $v(t)$ as $t\to\frac{1}{b}$. Technically, this is achieved by introducing an appropriate cascade of exponents. It follows that if $\psi\in\mathcal{X}$ satisfies (9.3) and if $b>0$ is sufficiently large, then **the solution of** (4.1) with $\varphi(x)=e^{i\frac{b|x|^2}{4}}\psi(x)$ behaves as $t\to\infty$ like

$$z(t,x)=(1+bt)^{-\frac{N}{2}}e^{i\Phi(t,\cdot)}w_0\Big(\frac{x}{1+bt}\Big)$$

where $w_0\ne0$ and Φ is real valued. Equivalently, $u(t)$ behaves like $e^{i\frac{\lambda}{b}|w_0(\frac{x}{1+bt})|^{\frac{2}{N}}\log(1+bt)}e^{t\Delta}u^+$, i.e. a free solution modulated by a phase, where $u^+=e^{i\frac{b|x|^2}{4}}e^{-i\frac{1}{b}\Delta}w_0$. See Theorem 1.1 and Remark 1.3 (vi) in [23] for details. Here again, there is no smallness condition on φ, but instead it must be sufficiently oscillatory.

As for the defocusing equation, **the case $\alpha<\frac{2}{N}$ is completely open**.

10. Asymptotic behavior for the dissipative NLS

In this section, we briefly comment the case of the dissipative (NLS), i.e. when $\Im\eta > 0$. Since $\Im\eta \neq 0$, mass and energy are not conserved. However, the mass is decreasing. It is not difficult to deduce that in the mass-subcritical case all solutions are global. Concerning the long time asymptotic behavior of the solutions, if $\alpha > \frac{2}{N}$, then solutions (at least for a certain class of initial values) are expected to scatter. In fact, all the low energy scattering results mentioned in the preceding section apply equally well to the case where η is an arbitrary complex number.

We now consider the case $\alpha \le \frac{2}{N}$. It turns out that, due to the dissipative nature of the nonlinear term, the solutions tend to decay faster than the solutions of the free Schrödinger equation.

In particular, in the case $\alpha = \frac{2}{N}$, a large class of solutions have the decay rate $(t \log t)^{-\frac{N}{2}}$ as $t \to \infty$, see [118, 70, 71]. One can also argue as in [23]. It follows that if $\psi \in \mathcal{X}$ satisfies (9.3) and if $b > 0$ is sufficiently large, then **the solution of** (4.1) with $\varphi(x) = e^{i\frac{b|x|^2}{4}}\psi(x)$ behaves as $t \to \infty$ like

$$z(t,x) = (1+bt)^{-\frac{N}{2}} e^{i\Theta(t,\cdot)} \Psi\Big(t, \frac{\cdot}{1+bt}\Big) w_0\Big(\frac{\cdot}{1+bt}\Big) \tag{10.1}$$

where $w_0 \neq 0$, Φ is real valued and $\Psi(t,x)$ goes to 0 as $t \to \infty$ like $[\log(1+bt)]^{-\frac{N}{2}}$. This implies that the limit

$$\lim_{t\to\infty} (t \log t)^{\frac{N}{2}} \|u(t)\|_{L^\infty} = (\alpha \Im\eta)^{-\frac{N}{2}}$$

exists (and is independent of the initial value). See [23, Theorem 1.2] for details.

In the case $\alpha < \frac{2}{N}$ several results are available under some "dissipative" condition between $\Im\eta$ and $\Re\eta$ (see [71, 56]), or if α is "close" to $\frac{2}{N}$ (see [70, 55]). Here also, one can argue as in [23]. It follows that if

$$\frac{2}{N+2} < \alpha < \frac{2}{N},$$

(and assuming stronger conditions on the integers n, k, m) if $\psi \in \mathcal{X}$ satisfies (9.3) and if $b > 0$ is sufficiently large, then **the solution of** (4.1) with $\varphi(x) = e^{i\frac{b|x|^2}{4}}\psi(x)$ behaves as $t \to \infty$ like $z(t,x)$ given by (10.1), but where now $\Psi(t,x)$ goes to 0 as $t \to \infty$ like $(1+bt)^{-\frac{2-N\alpha}{2\alpha}}$. This implies that the limit

$$\lim_{t\to\infty} t^{\frac{1}{\alpha}} \|u\|_{L^\infty} = \Big(\frac{2-N\alpha}{2\alpha|\Im\lambda|}\Big)^{\frac{1}{\alpha}}$$

exists (and is independent of the initial value). See [19, 20] for details.

11. Nature of finite-time blowup in the focusing case

In this section, we consider the focusing ($\eta > 0$), mass-critical or supercritical case ($\alpha \geq \frac{4}{N}$) of (NLS). In particular, some solutions blow up in finite time.

What mechanism makes a solution of (NLS) blow up in finite time, and what singularity develops at the blow-up time are questions that have been studied by a number of authors during the past 40 years, and that are still being actively studied. See for instance the survey articles [82] and [116].

11.1. **The mass-critical case.** We first consider the mass-critical case $\alpha = \frac{4}{N}$. The (only) general result is the lower estimate (4.5) which implies that if a finite-energy solution of (NLS) blows up at the time T, then

$$\|\nabla u(t)\|_{L^2} \geq \frac{\delta}{\sqrt{T-t}}, \tag{11.1}$$

with $\delta > 0$ for t close to T.

The first precise description of blowup is Weinstein's solution (5.15), which satisfies

$$\|\nabla v_T(t)\|_{L^2} \sim \frac{C}{T-t} \tag{11.2}$$

as $t \to T$, with $C = T\|\nabla Q\|_{L^2}$. In other words, this solution blows up **twice faster** than the universal lower estimate (11.1). It is a blowing up solution of **minimal mass**, since $\|v_T(t)\|_{L^2} = \|Q\|_{L^2}$, while $\|\varphi\|_{L^2} < \|Q\|_{L^2}$ implies global existence. In fact, v_T is the only solution (up to the invariances of the equation) that blows up in finite time on the critical sphere, see Merle [88, 89].

Blowup at the rate (11.2) is achieved by a whole class of solutions. More precisely, there is the following result of Bourgain and Wang [15].

Theorem 11.1. *Suppose $N = 1$ or $N = 2$. There exists an integer $A \geq [\frac{N}{2}] + 1$ such that if*

$$\psi \in X_A := \{u \in H^A(\mathbb{R}^N);\ (1 + |x|^A)u \in L^2(\mathbb{R}^N)\},$$

and $D^\beta \psi(0) = 0$ for $|\beta| \leq A - [\frac{N}{2}] - 1$, then there exist $\delta > 0$ and a solution u of (NLS) on $[1-\delta, 1)$ which satisfies $u(t) - v_1(t) \to \psi$ in $H^1(\mathbb{R}^N)$ as $t \uparrow 1$. In particular, u blows up at $t = 1$ with the rate (11.2).

The strategy of proof of Theorem 11.1 is the following. Since $\psi \in H^1(\mathbb{R}^N)$, there exist $\nu > 0$ and a solution $z \in C([1-\nu, 1], H^1(\mathbb{R}^N))$ of (NLS) such that $z(1) = \psi$. Then one looks for a solution of the form

$u = v_1 + z + w$. This last equation for w is solved by a perturbation argument, after applying the pseudo-conformal transformation to send $t = 1$ to $t = \infty$. An important ingredient is the study of the linearized operators in [134].

It turns out that the Bourgain-Wang solutions are unstable if $\|\psi\|_{X_A}$ is sufficiently small. Indeed, there exist initial values arbitrarily close to $u(1-\delta)$ such that the corresponding solution of (NLS) scatters, and other initial values arbitrarily close to $u(1-\delta)$ such that the corresponding solution of (NLS) blows up in finite time, but with a blow-up rate different from (11.2). See [99] for a precise statement.

On the other hand, it was **conjectured** in [77] on the basis of formal arguments and numerical computations that the solutions of (NLS) in the mass-critical case should blow up with a different rate, more precisely $\|\nabla u(t)\|_{L^2}$ should behave like $[(T-t)^{-1} \log|\log(T-t)|]^{\frac{1}{2}}$ generically. The **first rigorous proof** of such a blow-up regime is due to Perelman [111]: If $N = 1$, then for initial values in some open set (in the space Σ given by (8.2)) of perturbations of the ground state Q, the corresponding solution u of (4.1) blows up in finite time T and

$$u(t, x) = e^{i\mu(t)} \lambda(t)^{\frac{1}{2}} (Q(\lambda(t)x) + \chi(t, \lambda(t)x)),$$

where $\lambda(t)\sqrt{\frac{\log|\log(T-t)|}{T-t}} \to C \in (0, \infty)$ as $t \to T$ and $\chi(t, \cdot)$ is small in $L^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$.

A complete description of the blowup near the critical sphere was obtained by Merle and Raphaël in a series of papers [90, 91, 92, 114, 93, 94, 41].

Theorem 11.2. *Suppose $N \le 5$. There exists $\delta > 0$ such that if $\varphi \in H^1(\mathbb{R}^N)$ satisfies*

$$\|Q\|_{L^2} < \|\varphi\|_{L^2} < \|Q\|_{L^2} + \delta, \tag{11.3}$$

and $E(\varphi) \le 0$, then the corresponding solution u of (4.1) blows up in a finite time T, and

$$u(t, \cdot) - \lambda(t)^{-\frac{N}{2}} Q\Big(\frac{\cdot - x(t)}{\lambda(t)}\Big) e^{i\gamma(t)} \underset{t \uparrow T}{\longrightarrow} u^\star \tag{11.4}$$

in $L^2(\mathbb{R}^N)$, where $u^\star \in L^2(\mathbb{R}^N)$, but $u^\star \notin L^p(\mathbb{R}^N)$ for $p > 2$, and the parameters $\lambda(t) > 0$, $\gamma(t) \in \mathbb{R}$ and $x(t) \in \mathbb{R}^N$ satisfy $x(t) \to x(T) \in \mathbb{R}^N$ and

$$\lambda(t)\sqrt{\frac{\log|\log(T-t)|}{T-t}} \underset{t \uparrow T}{\longrightarrow} \sqrt{2\pi}. \tag{11.5}$$

In addition,

$$\frac{\|\nabla u(t)\|_{L^2}}{\|\nabla Q\|_{L^2}}\sqrt{\frac{T-t}{\log|\log(T-t)|}}\underset{t\uparrow T}{\longrightarrow}\frac{1}{\sqrt{2\pi}}. \tag{11.6}$$

The conclusions of Theorem 11.2 are the result of **several long and delicate articles**. Concerning the construction of the blowing-up solutions, the strategy consists first in constructing a good approximated solution in the form of a bubble concentrating the ground state. Then one needs a control of the remainder, which is obtained via energy and virial-type estimates. Some **orthogonality conditions** on linearized operators (**spectral property**) are required for two purposes. First, to derive the modulation equations for the parameters like $\lambda(t)$ and $x(t)$. Second, to control the (infinite dimensional) remainder. There is some flexibility on the choice of these orthogonality conditions, but in the end one must check that they are satisfied. This is where the assumption $N \le 5$ comes from. Appropriate conditions are satisfied in dimension $N = 1$. That appropriate conditions hold in space dimensions $2 \le N \le 5$ have been established by a **computer-assisted proof** [41]. The recent article [137] establishes a certain spectral property in space dimensions $2 \le N \le 12$.

As opposed to the Bourgain-Wang solutions, **the** $\log\log$ **blowup of Theorem 11.2 is stable**. More precisely, suppose that φ satisfies (11.3) and the corresponding solution u of (4.1) blows up at the finite time T with the estimate $\|\nabla u(t)\|_{L^2} \le C[(T-t)^{-1}\log|\log(T-t)|]^{\frac{1}{2}}$. It follows that there exists $\delta > 0$ such that if $\|\widetilde{\varphi}-\varphi\|_{H^1} \le \delta$, then the corresponding solution $\widetilde{u}$ of (4.1) satisfies all the conclusions of Theorem 11.2. (Of course, this applies to u itself.)

In addition to the above, it follows from [114] that if φ satisfies (11.3) and the corresponding solution u of (4.1) blows up at the finite time T, then either u satisfies the conclusions of Theorem 11.2, or else

$$\liminf_{t\uparrow T}\,(T-t)\|\nabla u(t)\|_{L^2} > 0.$$

In other words, in the neighborhood of the critical sphere, blowup occurs either at the $\log\log$ regime, or else at a blow-up rate which is at least $C(T-t)^{-1}$.

Theorem 11.2 and the other results in [90, 91, 92, 114, 93, 94, 41] give a precise description of what happens in the neighborhood of the critical sphere, however, many important questions remain open. Here are some of them.

- In the neighborhood of the critical sphere, solutions blow up either at the $\log\log$ regime, or else at least at the pseudo-conformal rate $(T-$

$t)^{-1}$. The solutions constructed in [83] blows up at the rate $|\log(T-t)|(T-t)^{-1}$ which is strictly faster than the pseudo-conformal rate, but these solutions are not in the neighborhood of the critical sphere (they are the sum of at least two interacting bubbles). Therefore, on may ask if the pseudo-conformal rate is the maximal blow-up rate in the neighborhood of the critical sphere or if some solutions blow up faster.

- Away from the critical sphere, it seems that very little is known. The solutions of Bourgain and Wang [15], and the solutions of Martel and Raphaël [83] can have arbitrarily large L^2 norm. Fan [39] constructed arbitrarily large solutions that blow up at the $\log\log$ rate (by "glueing" several Merle-Raphaël solutions that blow up at different points in space). (See also [112, Corollary 1].) On the other hand it seems that there is no general description or classification.
- Is there any lower estimate for blowup? (Better than (11.1).) This is true in the neighborhood of the critical sphere, but can any (larger) solution blow up more slowly than the $\log\log$ rate?
- Is there any upper estimate for blowup? Can any solution blow up faster than the $|\log(T-t)|(T-t)^{-1}$ rate of [83]?
- Can any solution blow up at a rate between the $\log\log$ and the pseudo-conformal $(T-t)^{-1}$ rates? (This is ruled out in the neighborhood of the critical sphere, but might be possible elsewhere.)

11.2. **The mass-supercritical case.** We now consider the case $\alpha > \frac{4}{N}$. The first general result is the **lower estimate** (4.5), i.e.

$$\|\nabla u(t)\|_{L^2} \geq c(T-t)^{-\frac{4-(N-2)\alpha}{4\alpha}}, \tag{11.7}$$

for t close to T, if u is a solution of (NLS) that blows up at T.

Another **general lower bound** is proved in [95] under some restrictive assumptions. Suppose $N=2$ and $2<\alpha<5$ or $N\geq 3$ and $\frac{4}{N}<\alpha<\frac{4}{N-2}$. There exists $\gamma=\gamma(N,\alpha)$ such that if $\varphi\in H^1(\mathbb{R}^N)$ is radially symmetric, and the corresponding solution u of (4.1) blows up at $T<\infty$, then $\|u(t)\|_{L^{\frac{N\alpha}{2}}}\geq |\log(T-t)|^\gamma$ for t close to T.

Also, integrating twice the variance identity (2.6), one obtains easily that if $\varphi\in\Sigma$ given by (8.2) and the corresponding solution u of (4.1) blows up at the finite time T, then there holds the **upper estimate**

$$\int_0^T (T-t)\|\nabla u(t)\|_{L^2}^2 dt < \infty. \tag{11.8}$$

In particular, for some sequence $t_n\to T$,

$$(T-t_n)\|\nabla u(t_n)\|_{L^2} \underset{n\to\infty}{\longrightarrow} 0.$$

This upper estimate does not match the lower estimate (11.7).

A **stronger upper bound** than (11.8) is obtained in [100], by using a localized variance identity. Suppose $N = 2$ and $2 < \alpha < 5$ or $N \geq 3$ and $\frac{4}{N} < \alpha < \frac{4}{N-2}$. If $\varphi \in H^1(\mathbb{R}^N)$ is radially symmetric and the corresponding solution u of (4.1) blows up at the finite time T, then

$$\int_t^T (T-\tau)\|\nabla u(\tau)\|_{L^2}^2 d\tau \leq C(\varphi)(T-t)^{\frac{2\sigma}{1+\sigma}}, \tag{11.9}$$

where $\sigma = \frac{4-\alpha}{\alpha(N-1)}$. In particular, for some sequence $t_n \to T$,

$$\|\nabla u(t_n)\|_{L^2} \leq C(T-t_n)^{-\frac{1}{1+\sigma}}.$$

Note that $\frac{1}{1+\sigma} > \frac{4-(N-2)\alpha}{4\alpha}$, so that there is a gap between this upper bound and the lower bound (11.7). However, it turns that both bounds are achieved by some solutions (at least for α sufficiently close to $\frac{4}{N}$).

If $1 \leq N \leq 5$ and α is sufficiently close to $\frac{4}{N}$, then there exists an open set of initial values in $H^1(\mathbb{R}^N)$ such that the corresponding solution of (4.1) blows up in finite time with the self-similar rate

$$\|\nabla u(t)\|_{L^2} \sim \frac{1}{(T-t)^{\frac{4-(N-2)\alpha}{4\alpha}}}.$$

see [98] for a precise statement.

On the other hand, if $N = 2$ and $2 < \alpha < 5$ or $N \geq 3$ and $\frac{4}{N} < \alpha < \frac{4}{N-2}$, then there exist "collapsing ring solutions" that satisfy

$$\|\nabla u(t)\|_{L^2} \sim \frac{1}{(T-t)^{\frac{1}{1+\sigma}}}.$$

see [100] for a precise statement. See also [115, 117].

Ortoleva and Perelman [109] have described some infinite-time and finite-time blow-up solutions for the $3D$ energy critical focusing (NLS): A global solution that blows up as $t \to \infty$, by concentrating the ground state $W(x) = (1 + \frac{1}{3}|x|^2)^{-\frac{1}{2}}$. More precisely, for every sufficiently small $\nu, \mu \in \mathbb{R}$ and any $\delta > 0$, there exist $T > 0$ and a global, finite-energy solution $u \in C([T,\infty), \dot{H}^1(\mathbb{R}^3))$ of (NLS) such that

$$u(t,x) = e^{i\mu \log t} t^{\frac{\nu}{2}} W(t^\nu x) + \zeta(t,x),$$

where $\|\zeta(t)\|_{\dot{H}^1} \leq \delta$ and $\|\zeta(t)\|_{L^\infty} \leq Ct^{-\frac{1+\nu}{2}}$. See [109, Theorem 1.1]. If we choose $\nu > 0$, then although the $\dot{H}^1$ norm of u is bounded, this corresponds to blowup by concentration of W. The proof is based on the construction of a "good" approximate solution, and the estimate of the remainder by energy estimates. In [109, Remark 1.5] the authors announce that by the same techniques one can prove finite-time blowup.

More precisely, they claim that for every $\mu \in \mathbb{R}$ and $\nu > 1$, there exist $T > 0$ and a finite-energy solution u of (NLS) which blows up at T, of the form

$$u(t,x) = e^{i\mu \log(T-t)}(T-t)^{-\frac{1+2\nu}{4}} W((T-t)^{-\frac{1+2\nu}{2}} x) + \zeta(t,x),$$

where $\|\zeta(t)\|_{\dot H^1}$ is arbitrarily small. (Note that the first term has a fixed $\dot H^1$ norm, which is precisely $\|W\|_{\dot H^1}$).

There are many open problems concerning finite-time blowup in the mass-supercritical case. In particular, one can ask the following questions.

- The lower possible blow-up rate (11.7) is achieved by an open set of solutions, but for slightly supercritical nonlinearities, see [98]. What about general $\alpha > \frac{4}{N}$?
- The faster blowup rate given by (11.9) is also achieved, see [98], but is there any solution blowing up at an intermediate rate?
- What are the stable blow-up rates besides the slow rate (11.7)?
- What are the possible blow-up mechanisms?

We conclude this section with a finite-time blowup result of a completely different nature. Indeed, it concerns the **defocusing case**, but in the **energy-supercritical** case. As observed earlier, in this case, the control of the H^1 norm by the conservation laws is not necessarily sufficient to imply global existence. It turns out that some solutions can indeed blow up in finite time, as shows the following result.

Theorem 11.3 ([97])**.** *Assume $N = 5, \alpha = 8$; or $N = 6, \alpha = 4$; or $N = 8, \alpha = 2$; or $N = 9, \alpha = 2$. Assume further that $\eta < 0$. (So that equation (NLS) is energy-supercritical and defocusing.) It follows that there exist smooth, radially symmetric initial values $\varphi \in H^\infty(\mathbb{R}^N)$ such that the corresponding solution of (NLS) (given by Theorem 4.1) blows up in finite time.*

More precisely, there exists a sequence $(r_k)_{k\ge 1} \subset (2, \frac{4+\alpha N}{4+\alpha\sqrt{N}})$ with $r_k \to \frac{4+\alpha N}{4+\alpha\sqrt{N}}$ such that for all $k \ge 1$ there exists a finite-codimensional manifold of radially symmetric initial values $\varphi \in H^\infty(\mathbb{R}^N)$ such that the corresponding solution of (NLS) blows up in a finite time time $0 < T < \infty$ at $x = 0$ and

$$(T-t)^{\frac{1}{\alpha}\left(1+\frac{r_k-2}{r_k}\right)} \|u(t)\|_{L^\infty} \underset{t\uparrow T}{\longrightarrow} c > 0.$$

The proof relies on the construction of smooth self-similar solutions of the compressible Euler equation [96]. This is applied to (NLS) by using its hydrodynamical formulation $u(t,x) = \rho(t,x)e^{i\phi(t,x)}$.

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