Introduction to partial differential equations

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Foreword

Partial differential equations (PDEs) are a fundamental tool for modeling natural phenomena, and anyone working in the field of applied mathematics should know its bases. The purpose of these lectures is to present three classes of PDEs (elliptic, parabolic, hyperbolic), to introduce their fundamental properties, and to introduce the mathematical tools that are necessary for their study. Prerequisites for these lectures are the bases of functional analysis, of distribution theory, and of Sobolev spaces.

PDEs have been used since the mid-18$^{\text{th}}$ century for modeling various phenomena arising in particular in mechanics, physics, biology, chemistry, etc. It is an extremely vast domain of study, even if one consider only linear equations.

We consider here typical examples in the three main classes of PDEs: elliptic, parabolic, hyperbolic. More specifically, we consider Laplace's equation

$$-\Delta u + \lambda u = f$$

the heat equation

$$\frac{\partial u}{\partial t} - \Delta u = f$$

and the one-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = f$$

We note when these equations are set on a bounded domain, one needs boundary conditions, which are in general imposed by the physical phenomena that these equations model. For instance, a vibrating string fixed at its ends. Classical boundary conditions are Dirichlet (one imposes u on the boundary of the domain), Neumann (one imposes the normal derivative of u on the boundary of the domain), and Robin (a combination of the preceding two conditions). The boundary conditions can be homogeneous (for instance $u = 0$ in the case of Dirichlet) or nonhomogeneous (for instance $u = g$, for a given function g, in the case of Dirichlet). In these lectures, we only use homogeneous Dirichlet boundary conditions, which are in general technically simpler and do not require regularity conditions on the domain.

For each of these three examples, we first consider the case of the equation set on the whole space, in which one can use explicit formulas given by the Fourier transform; then the case of a bounded domain with boundary conditions, in which one can use Fourier series.

For Laplace's equation, we study the following questions: existence, regularity, maximum principle, spectral decomposition of the Laplacian.

For the heat equation, we study the following questions: existence, regularity, smoothing effect, maximum principle, large time behavior.

For the one-dimensional wave equation, we study the following questions: existence, regularity, finite speed of propagation, large time behavior.

Nonlinear PDEs is an extremely vast domain, and many different phenomena can take place. If we have time, we will study a few fundamental phenomena, only for semilinear equations. More precisely, we will study the existence (by variational methods) for cettain semilinear elliptic equations. For the semilinear heat equation, we will study local existence and the blowup alternative, global existence, finite-time blowup.

Notation

a.a.	almost all
a.e.	almost everywhere
$u \star v$	the convolution in $\mathbb{R}^N$, i.e. $$u \star v(x) = \int_{\mathbb{R}^N} u(y)v(x-y)\,dy = \int_{\mathbb{R}^N} u(x-y)v(y)\,dy$$
$\mathcal{F}$	the Fourier transform in $\mathbb{R}^N$, defined by $$\mathcal{F}u(\xi) = \int_{\mathbb{R}^N} e^{-2\pi i x\cdot\xi} u(x)\,dx$$
$\overline{\mathcal{F}}$	$= \mathcal{F}^{-1}$, given by $\overline{\mathcal{F}}v(x) = \int_{\mathbb{R}^N} e^{2\pi i \xi\cdot x} v(\xi)\,d\xi$
$\widehat{u}$	$= \mathcal{F}u$
$\overline{E}$	the closure of the subset E of the topological space X
$C(E,F)$	the space of continuous functions from the topological space E to the topological space F
$C^k(E,F)$	the space of k times continuously differentiable functions from the topological space E to the topological space F
$C_{\mathrm{b}}(E,F)$	the Banach space of continuous, bounded functions from the topological space E to the Banach space F, with the topology of uniform convergence
$C_{\mathrm{c}}(E,F)$	the space of continuous, compactly supported functions from the topological space E to the topological space F
$\mathcal{L}(E,F)$	the Banach space of linear, continuous operators from the Banach space E to the Banach space F, equipped with the norm topology
$\mathcal{L}(E)$	the space $\mathcal{L}(E,E)$
$X^\star$	the topological dual of the space X
$X \hookrightarrow Y$	if $X \subset Y$ with continuous injection
Ω	an open subset of $\mathbb{R}^N$
$\overline{\Omega}$	the closure of Ω in $\mathbb{R}^N$
$\partial\Omega$	the boundary of Ω, i.e. $\partial\Omega = \overline{\Omega} \setminus \Omega$
$\omega \subset\subset \Omega$	if $\overline{\omega} \subset \Omega$ and $\overline{\omega}$ is compact
x^+	$= \max\{x, 0\}$ for $x \in \mathbb{R}$. The positive part of x, i.e. $x^+ = x$ if $x \ge 0$ and $x^+ = 0$ if $x \le 0$
x^-	$= \max\{-x, 0\}$ for $x \in \mathbb{R}$. The negative part of x, i.e. $x^- = 0$ if $x \ge 0$ and $x^- = -x$ if $x \le 0$
$\lvert x\rvert$	$= x^+ + x^-$ for $x \in \mathbb{R}$. The absolute value of x, i.e. $\lvert x\rvert = x$ if $x \ge 0$ and $\lvert x\rvert = -x$ if $x \le 0$
x^α	$= \prod_{j=1}^{N} x_j^{\alpha_j}$ for $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{N}^N$ and $x \in \mathbb{R}^N$
$\partial_i u$	$= u_{x_i} = \dfrac{\partial u}{\partial x_i}$

$\partial_r u$	$= u_r = \dfrac{\partial u}{\partial r} = \dfrac{1}{r} x \cdot \nabla u$, where $r = \lvert x\rvert$
D^α	$= \dfrac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \dfrac{\partial^{\alpha_N}}{\partial x_N^{\alpha_N}}$ for $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{N}^N$
∇u	$(\partial_1 u, \cdots, \partial_N u)$
Δ	$= \sum_{i=1}^N \dfrac{\partial^2}{\partial x_i^2}$
$C_{\mathrm{c}}(\Omega)$	the space of continuous functions $\Omega \to \mathbb{R}$ with compact support
$C_{\mathrm{c}}^k(\Omega)$	the space of functions of $C^k(\Omega)$ with compact support
$C_{\mathrm{b}}(\Omega)$	the Banach space of continuous, bounded functions $\Omega \to \mathbb{R}$, equipped with the topology of uniform convergence
$C_{\mathrm{b}}^m(\Omega)$	the Banach space of $u \in C_{\mathrm{b}}(\Omega)$ such that $D^\alpha u \in C_{\mathrm{b}}(\Omega)$ for all $\alpha \in \mathbb{N}^N$ with $\lvert\alpha\rvert \le m\}$, equipped with the norm $\lVert u\rVert_{C_{\mathrm{b}}^m(\Omega)} = \sum_{\lvert\alpha\rvert\le m} \lVert D^\alpha u\rVert_{L^\infty}$
$C(\overline{\Omega})$	the space of continuous functions $\overline{\Omega} \to \mathbb{R}$. When Ω is bounded, $C(\overline{\Omega})$ is a Banach space when equipped with the topology of uniform convergence
$C_{\mathrm{b,u}}(\overline{\Omega})$	the Banach space of uniformly continuous and bounded functions $\overline{\Omega} \to \mathbb{R}$ equipped with the topology of uniform convergence
$C_{\mathrm{b,u}}^m(\overline{\Omega})$	the Banach space of functions $u \in C_{\mathrm{b,u}}(\overline{\Omega})$ such that $D^\alpha u \in C_{\mathrm{b,u}}(\overline{\Omega})$, for every multi-index α such that $\lvert\alpha\rvert \le m$. $C_{\mathrm{b,u}}^m(\overline{\Omega})$ is equipped with the norm of $W^{m,\infty}(\Omega)$
$C^{m,\alpha}(\overline{\Omega})$	for $0 \le \alpha < 1$, the Banach space of functions $u \in C_{\mathrm{b,u}}^m(\overline{\Omega})$ such that $\lVert u\rVert_{C^{m,\alpha}} = \lVert u\rVert_{W^{m,\infty}} + \sup_{x,y\in\Omega,\ \lvert\beta\rvert=m} \dfrac{\lvert D^\beta u(x) - D^\beta u(y)\rvert}{\lvert x-y\rvert^\alpha} < \infty.$
$\mathcal{D}(\Omega)$	$= C_{\mathrm{c}}^\infty(\Omega)$, the Fréchet space of C^∞ functions $\Omega \to \mathbb{R}$ (or $\Omega \to \mathbb{C}$) compactly supported in Ω, equipped with the topology of uniform convergence of all derivatives on compact subsets of Ω
$C_0(\Omega)$	the closure of $C_{\mathrm{c}}^\infty(\Omega)$ in $L^\infty(\Omega)$. $C_0(\Omega)$ is the set of $u \in C(\overline{\Omega})$ such that $u(x) = 0$ for all $x \in \partial\Omega$ and such that $u(x) \to 0$ as $\lvert x\rvert \to \infty$ (if Ω is unbounded)
$C_0(\mathbb{R}^N)$	the closure of $C_{\mathrm{c}}^\infty(\mathbb{R}^N)$ in $L^\infty(\Omega)$. $C_0(\mathbb{R}^N)$ is the set of $u \in C(\mathbb{R}^N)$ such that $u(x) \to 0$ as $\lvert x\rvert \to \infty$
$C_0^m(\Omega)$	the closure of $C_{\mathrm{c}}^\infty(\Omega)$ in $W^{m,\infty}(\Omega)$
$\mathcal{D}'(\Omega)$	the space of distributions on Ω, that is the topological dual of $\mathcal{D}(\Omega)$
$\mathcal{S}(\mathbb{R}^N)$	the Schwartz space, i.e. the space of $u \in C^\infty(\mathbb{R}^N, \mathbb{R})$ (or $C^\infty(\mathbb{R}^N, \mathbb{C})$) such that for every nonnegative integer m, $p_m(u) = \sup_{\lvert\alpha\rvert\le m} \sup_{x\in\mathbb{R}^N} (1 + \lvert x\rvert^2)^{m/2} \lvert D^\alpha u(x)\rvert < \infty.$ $\mathcal{S}(\mathbb{R}^N)$ is a Fréchet space when equipped with the seminorms p_m
$\mathcal{S}'(\mathbb{R}^N)$	the space of tempered distributions on $\mathbb{R}^N$, that is the topological dual of $\mathcal{S}(\mathbb{R}^N)$. $\mathcal{S}'(\mathbb{R}^N)$ is a subspace of $\mathcal{D}'(\mathbb{R}^N)$
p'	the conjugate of p given by $\dfrac{1}{p} + \dfrac{1}{p'} = 1$
$L^p(\Omega)$	the Banach space of (classes of) measurable functions $u : \Omega \to \mathbb{R}$ (or $\Omega \to \mathbb{C}$) such that $\int_\Omega \lvert u(x)\rvert^p \, dx < \infty$ if $1 \le p < \infty$, or $\operatorname{ess\,sup}_{x\in\Omega} \lvert u(x)\rvert <$

∞ if $p = \infty$. $L^p(\Omega)$ is equipped with the norm

$$\|u\|_{L^p} = \begin{cases} \left(\int_\Omega |u(x)|^p \, dx \right)^{\frac{1}{p}} & \text{if } p < \infty \\ \operatorname{ess\,sup}_{x \in \Omega} |u(x)| & \text{if } p = \infty. \end{cases}$$

$L^p_{\mathrm{loc}}(\Omega)$ — the set of measurable functions $u : \Omega \to \mathbb{R}$ (or $\Omega \to \mathbb{C}$) such that $u_{|\omega} \in L^p(\omega)$ for all $\omega \subset\subset \Omega$

$W^{m,p}(\Omega)$ — the space of (classes of) measurable functions $u : \Omega \to \mathbb{R}$ (or $\Omega \to \mathbb{C}$) such that $D^\alpha u \in L^p(\Omega)$ in the sense of distributions, for every multi-index $\alpha \in \mathbb{N}^N$ with $|\alpha| \le m$. $W^{m,p}(\Omega)$ is a Banach space when equipped with the norm $\|u\|_{W^{m,p}} = \sum_{|\alpha| \le m} \|D^\alpha u\|_{L^p}$

$W^{m,p}_{\mathrm{loc}}(\Omega)$ — the set of measurable functions $u : \Omega \to \mathbb{R}$ (or $\Omega \to \mathbb{C}$) such that $u_{|\omega} \in W^{m,p}(\omega)$ for all $\omega \subset\subset \Omega$

$W^{m,p}_0(\Omega)$ — the closure of $C^\infty_{\mathrm{c}}(\Omega)$ in $W^{m,p}(\Omega)$

$W^{-m,p'}(\Omega)$ — the topological dual of $W^{m,p}_0(\Omega)$

$H^m(\Omega)$ — $= W^{m,2}(\Omega)$. $H^m(\Omega)$ is equipped with the equivalent norm

$$\|u\|_{H^m} = \Big(\sum_{|\alpha| \le m} \int_\Omega |D^\alpha u(x)|^2 \, dx \Big)^{\frac{1}{2}},$$

and $H^m(\Omega)$ is a Hilbert space for the scalar product $(u, v)_{H^m} = \displaystyle\int_\Omega \operatorname{Re}\left(u(x)\overline{v(x)}\right) dx$

$H^m_{\mathrm{loc}}(\Omega)$ — $= W^{m,2}_{\mathrm{loc}}(\Omega)$

$H^m_0(\Omega)$ — $= W^{m,2}_0(\Omega)$

$H^{-m}(\Omega)$ — $= W^{-m,2}(\Omega)$

$|u|_{m,p,\Omega}$ — $= \sum_{|\alpha| = m} \|D^\alpha u\|_{L^p(\Omega)}$

$H^{s,p}(\mathbb{R}^N)$ — for $s \in \mathbb{R}$ and $1 \le p \le \infty$, the set of $u \in \mathcal{S}'(\mathbb{R}^N)$ such that $\mathcal{F}^{-1}[(1 + |\cdot|^2)^{\frac{s}{2}} \widehat{u}] \in L^p(\mathbb{R}^N)$. $H^{s,p}(\mathbb{R}^N)$ is a Banach space for the norm $\|u\|_{H^{s,p}} = \|\mathcal{F}^{-1}[(1 + |\cdot|^2)^{\frac{s}{2}} \widehat{u}]\|_{L^p}$

$\ell^p(\mathbb{N})$ — the Banach space of sequences $(u_n)_{n \in \mathbb{N}}$ such that $\displaystyle\sum_{n=1}^\infty |u_n|^p < \infty$ if $1 \le p < \infty$, or $\sup_{n \ge 1} |u_n| < \infty$ if $p = \infty$. $\ell^p(\mathbb{N})$ is equipped with the norm

$$\|u\|_{\ell^p} = \begin{cases} \left(\sum_{n=1}^\infty |u_n|^p \right)^{\frac{1}{p}} & \text{if } p < \infty \\ \sup_{n \ge 1} |u_n| & \text{if } p = \infty. \end{cases}$$

$\ell^p(\mathbb{Z})$ — similar to $\ell^p(\mathbb{N})$, but for sequences $(u_n)_{n \in \mathbb{Z}}$

CHAPTER 1

Laplace's equation

1.1. Existence and interior regularity

1.1.1. On $\mathbb{R}^N$. Given $\lambda \in \mathbb{R}$, we consider the model Laplace equation

$$-\Delta u + \lambda u = f \tag{1.1.1}$$

set on the whole space $\mathbb{R}^N$. Suppose that for some $f \in \mathcal{S}'(\mathbb{R}^N)$, there is a solution $u \in \mathcal{S}'(\mathbb{R}^N)$ of (1.1.1) . Applying the Fourier transform, we deduce that

$$(\lambda + 4\pi^2|\xi|^2)\widehat{u} = \widehat{f} \tag{1.1.2}$$

in $\mathcal{S}'(\mathbb{R}^N)$. (See Section B.6.) Classically, we look for "localized" solutions, for instance $u \in L^2(\mathbb{R}^N)$. Consider a very nice f, for instance the Gaussian $f(x) = e^{-\pi a|x|^2}$. It follows that $\widehat{f}(\xi) = a^{-\frac{N}{2}} e^{-\frac{\pi}{a}|\xi|^2}$. (See formula (B.6.9).) Therefore, if there exists a solution $u \in \mathcal{S}'(\mathbb{R}^N)$, then

$$\widehat{u}(\xi) = a^{-\frac{N}{2}}(\lambda + 4\pi^2|\xi|^2)^{-1} e^{-\frac{\pi}{a}|\xi|^2}.$$

If $\lambda < 0$, then $\widehat{u}$ is too singular at $|\xi| = \frac{\sqrt{-\lambda}}{2\pi}$ to be in $L^2(\mathbb{R}^N)$. If $\lambda = 0$, one can also find $f \in L^2(\mathbb{R}^N)$ for which there is no solution in $L^2(\mathbb{R}^N)$. Therefore, we only consider the case $\lambda > 0$. We have the following existence result.

Theorem 1.1.1. *Suppose $\lambda > 0$.*

(i) *Given $f \in \mathcal{S}'(\mathbb{R}^N)$, there exists a unique solution $u \in \mathcal{S}'(\mathbb{R}^N)$ of* (1.1.1)*, given by* (1.1.2).

(ii) *If $f \in L^p(\mathbb{R}^N)$ for some $1 < p < \infty$, then $u \in W^{2,p}(\mathbb{R}^N)$. More generally, if $f \in H^{s,p}(\mathbb{R}^N)$ for some $s \in \mathbb{R}$ and $1 < p < \infty$, then $u \in H^{s+2,p}(\mathbb{R}^N)$. Moreover, there exists a constant C indépendant of f such that $\|u\|_{H^{s+2,p}} \le C\|f\|_{H^{s,p}}$.*

Proof. As observed above, if $u \in \mathcal{S}'(\mathbb{R}^N)$ is a solution of (1.1.1), the u is given by (1.1.2). This proves uniqueness. Moreover, given $f \in \mathcal{S}'(\mathbb{R}^N)$, formula (1.1.2) determines an element $u \in \mathcal{S}'(\mathbb{R}^N)$. (See Remark B.6.10 (v).) This proves property (i).

Suppose now $f \in H^{s,p}(\mathbb{R}^N)$ for some $s \in \mathbb{R}$ and $1 < p < \infty$. It follows in particular that $\mathcal{F}^{-1}[(\lambda + 4\pi^2|\cdot|^2)^{\frac{s}{2}}\widehat{f}] \in L^p(\mathbb{R}^N)$. (See Proposition B.6.28.) Applying (1.1.2), we deduce that

$$\mathcal{F}^{-1}[(\lambda + 4\pi^2|\cdot|^2)^{\frac{s+2}{2}}\widehat{u}] = \mathcal{F}^{-1}[(\lambda + 4\pi^2|\cdot|^2)^{\frac{s}{2}}\widehat{f}] \in L^p(\mathbb{R}^N)$$

so that $u \in H^{s+2,p}(\mathbb{R}^N)$ and $\|u\|_{H^{s+2,p}} \le C\|f\|_{H^{s,p}}$. This completes the proof. □

Remark 1.1.2. Formula (1.1.2) yields interesting informations on the solution u of (1.1.1). Indeed, let $\lambda > 0$ and $f \in \mathcal{S}'(\mathbb{R}^N)$. If u is the solution u of (1.1.1), then u is given by formula (1.1.2), i.e.

$$\widehat{u} = (\lambda + 4\pi^2|\cdot|^2)^{-1}\widehat{f}$$

It follows that (see Theorem B.6.14 (iv))

$$u = \mathcal{F}^{-1}[(\lambda + 4\pi^2|\cdot|^2)^{-1}] \star f$$

With the notation in Remark B.6.16, we have $\mathcal{F}^{-1}[(\lambda+4\pi^2|\cdot|^2)^{-1}] = F^1_\lambda$, which is a radially symmetric and decreasing function of $L^1(\mathbb{R}^N)$. Therefore, if $f \in L^p(\mathbb{R}^N)$ for some $1 \le p < \infty$ and f is nonnegative, radially symmetric and decreasing, then the solution u of (1.1.1) is also nonnegative, radially symmetric and decreasing. See Proposition A.2.2.

The following result describes the local regularity of u in terms of the local regularity of f.

THEOREM 1.1.3. *Let $\lambda \in \mathbb{R}$ and $f \in \mathcal{S}'(\mathbb{R}^N)$, and suppose $u \in \mathcal{S}'(\mathbb{R}^N)$ satisfies (1.1.1) in $\mathcal{S}'(\mathbb{R}^N)$. If $u \in L^p_{\rm loc}(\mathbb{R}^N)$ and $f \in W^{m,p}_{\rm loc}(\mathbb{R}^N)$ for some $m \ge 0$ and $1 < p < \infty$, then $u \in W^{m+2,p}_{\rm loc}(\mathbb{R}^N)$. In particular, if $f \in C^\infty(\mathbb{R}^N)$, then $u \in C^\infty(\mathbb{R}^N)$.*

PROOF. Let $1 < p < \infty$ and suppose $u \in W^{\ell,p}_{\rm loc}(\mathbb{R}^N)$ for some $0 \le \ell \le m+1$. Given $\theta \in C^\infty_{\rm c}(\mathbb{R}^N)$, we have in $\mathcal{S}'(\mathbb{R}^N)$

$$-\Delta(\theta u) + \theta u = \theta f - \nabla\theta\cdot\nabla u - u\Delta\theta + (1-\lambda)\theta u$$

Since $u \in W^{\ell,p}_{\rm loc}(\mathbb{R}^N)$, we see that $-\nabla\theta\cdot\nabla u - u\Delta\theta + (1-\lambda)\theta u \in W^{\ell-1,p}(\mathbb{R}^N)$; and since $f \in W^{m,p}_{\rm loc}(\mathbb{R}^N)$, we also have $\theta f \in W^{m,p}(\mathbb{R}^N)$. Applying Theorem 1.1.1 (ii) (with $\lambda = 1$), we deduce that $\theta u \in W^{\ell+1,p}(\mathbb{R}^N)$. θ being arbitrary, it follows that $u \in W^{\ell+1,p}_{\rm loc}(\mathbb{R}^N)$.

We now argue by induction. Starting with $\ell = 0$, we obtain $u \in W^{1,p}_{\rm loc}(\mathbb{R}^N)$. Iterating, it follows that $u \in W^{\ell+1,p}_{\rm loc}(\mathbb{R}^N)$ for all $\ell \in \mathbb{N}$ such that $\ell \le m+1$. Hence the result follows. ☐

REMARK 1.1.4. Note that Theorem 1.1.3 does not impose any estimate of the derivatives of f. For instance, let $f(x) = (1+|x|^2)^{-N}\sin(e^{|x|^2})$ et $\lambda = 1$. Since $f \in L^2(\mathbb{R}^N)$, there exists a solution $u \in H^2(\mathbb{R}^N)$ of (1.1.1). The derivatives of f become very large as $|x| \to \infty$, however $f \in L^2 \cap C^\infty$, so that $u \in H^2(\mathbb{R}^N) \cap C^\infty(\mathbb{R}^N)$.

REMARK 1.1.5. Note that we can consider real-valued functions as well as complex-valued functions.

1.1.2. On a domain (connected open set), bounded or not. Let Ω be an open, connected subset of $\mathbb{R}^N$. We look for solutions of

$$\begin{cases} -\Delta u + \lambda u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1.3}$$

If Ω is sufficiently smooth (of class C^2) and $\mathbf{u} \in C^1(\overline{\Omega}, \mathbb{R}^N)$, then we have the following Green's formula

$$\int_\Omega \nabla\cdot\mathbf{u}(x)\,dx = \int_{\partial\Omega} \mathbf{u}(\sigma)\cdot\mathbf{n}(\sigma)\,d\sigma$$

where $\mathbf{n}(\sigma)$ is the outwards normal at $\sigma \in \partial\Omega$ and $d\sigma$ is the surface measure on $\partial\Omega$. Suppose u is a classical solution of (1.1.3), i.e. $u \in C^2(\overline{\Omega})$, and let $\varphi \in C^1(\overline{\Omega})$. We have

$$-\varphi\Delta u = -\nabla\cdot(\varphi\nabla u) + \nabla u\cdot\nabla\varphi$$

so that Green's formula yields

$$-\int_\Omega \varphi\Delta u = \int_\Omega \nabla u\cdot\nabla\varphi - \int_{\partial\Omega} \varphi(\sigma)\nabla u(\sigma)\cdot\mathbf{n}(\sigma)\,d\sigma$$

Therefore, if $\varphi \in C^1_{\rm c}(\Omega)$, then $\varphi(\sigma) = 0$ for all $\sigma \in \partial\Omega$ so that

$$-\int_\Omega \Delta u\varphi = \int_\Omega \nabla u\cdot\nabla\varphi$$

Thus we see that if u is a classical solution of (1.1.3), then

$$\int_\Omega (\nabla u \cdot \nabla \varphi + \lambda u \varphi - f\varphi) = 0 \tag{1.1.4}$$

for all $\varphi \in C^1_c(\Omega)$, hence by density for all $\varphi \in H^1_0(\Omega)$.

Note that the term $\int_\Omega uf$ is well defined for all $f \in L^2(\Omega)$, and that

$$\int_\Omega uf = \langle u, f\rangle_{H^1_0, H^{-1}}$$

where $\langle\cdot,\cdot\rangle_{H^1_0,H^{-1}}$ is the duality bracket $H^1_0 - H^{-1}$. (Here, we identify $L^2(\Omega)$ with its dual, and we use the fact that $(H^1_0(\Omega))^\star = H^{-1}(\Omega)$.)

Thus we see that if u is a classical solution of (1.1.3), then

$$\int_\Omega (\nabla u \cdot \nabla \varphi + \lambda u \varphi) = \langle \varphi, f\rangle_{H^1_0, H^{-1}} \tag{1.1.5}$$

for all $\varphi \in H^1_0(\Omega)$. Note that identity (1.1.5) makes sense for all $u, \varphi \in H^1_0(\Omega)$ and $f \in H^{-1}(\Omega)$, where Ω is any domain.

We use identity (1.1.5) as the definition of "weak solution": Given $f \in H^{-1}(\Omega)$, we say that u is a weak solution of (1.1.3) iff

$$\begin{cases} u \in H^1_0(\Omega) \\ \text{(1.1.5) holds for all } \varphi \in H^1_0(\Omega) \end{cases} \tag{1.1.6}$$

The important fact at this point is that if Ω is smooth and u is a classical solution of (1.1.3), then u is a weak solution. The boundary condition $u_{|\partial\Omega} = 0$ is understood in the weak sense $u \in H^1_0(\Omega)$.

The motivation of this approach is that we will see that, under very general conditions, there always exists a unique weak solution, whereas there does not always exist a classical solution.

Several simple methods are available for proving the existence of a weak solution, such as Lax-Milgram's theorem ([**20**, Theorem 2.1]), or minimization (variational methods). In the present case, the operator Δ is symmetric, and one can use an even simpler technique, based on the Riesz representation theorem.

We set

$$\overline{\lambda}(\Omega) = \inf\Big\{ \int_\Omega |\nabla u|^2 ;\ u \in H^1_0(\Omega) \text{ et } \|u\|_{L^2} = 1 \Big\} \tag{1.1.7}$$

so that $\overline{\lambda}(\Omega) \ge 0$ and

$$\int_\Omega |\nabla u|^2 \ge \overline{\lambda}(\Omega) \int_\Omega |u|^2$$

for all $u \in H^1_0(\Omega)$. On the other hand, we know that $\overline{\lambda}(\mathbb{R}^N) = 0$ (Remark B.3.21), and that $\overline{\lambda}(\Omega) > 0$ if Ω is bounded (by Poincaré's inequality (B.3.73)). In addition, we will see later (Section 1.4) that if Ω is bounded, then $\overline{\lambda}(\Omega)$ is the first eigenvalue of $-\Delta$ in $H^1_0(\Omega)$.

Given $0 \le \theta \le 1$, we have

$$\begin{aligned} \int (|\nabla u|^2 + \lambda |u|^2) &= \theta \int |\nabla u|^2 + (1-\theta) \int |\nabla u|^2 + \lambda \int |u|^2 \\ &\ge \theta \int |\nabla u|^2 + [(1-\theta)\overline{\lambda}(\Omega) + \lambda] \int |u|^2 \\ &= \theta \int |\nabla u|^2 + [\overline{\lambda(\Omega)} + \lambda - \theta \overline{\lambda}(\Omega)] \int |u|^2 \end{aligned}$$

for all $u \in H^1_0(\Omega)$. Suppose

$$\lambda > -\overline{\lambda}(\Omega) \tag{1.1.8}$$

and let

$$\theta = \begin{cases} 1 & \lambda \geq 1 \\ \frac{\lambda + \overline{\lambda}(\Omega)}{1+\overline{\lambda}(\Omega)} & -\overline{\lambda}(\Omega) < \lambda < 1 \end{cases}$$

It follows that $\theta > 0$ and

$$\overline{\lambda(\Omega)} + \lambda - \theta\overline{\lambda}(\Omega) = \begin{cases} \lambda & \lambda \geq 1 \\ \theta & -\overline{\lambda}(\Omega) < \lambda < 1 \end{cases}$$

In particular, $\overline{\lambda(\Omega)} + \lambda - \theta\overline{\lambda}(\Omega) \geq \theta$, so that

$$\int (|\nabla u|^2 + \lambda |u|^2) \geq \theta \int (|\nabla u|^2 + |u|^2) = \theta \|u\|_{H^1}^2 \tag{1.1.9}$$

Therefore, the scalar product

$$((u, v)) = \int (\nabla u \cdot \nabla v + \lambda uv)$$

defines on $H_0^1(\Omega)$ a norm which is equivalent to the canonical norm. Thus we may equip $H_0^1(\Omega)$ with the scalar product $((\cdot, \cdot))$ provided (1.1.8) holds.

Theorem 1.1.6. *Assume* (1.1.8). *Given* $f \in H^{-1}(\Omega)$, *there exists a unique weak solution* u *of* (1.1.3). *In addition, there exist constants* $0 < c < C < \infty$ *independent of* f *such that*

$$c\|f\|_{H^{-1}} \leq \|u\|_{H^1} \leq C\|f\|_{H^{-1}} \tag{1.1.10}$$

Proof. Let $f \in H^{-1}(\Omega)$, and note that u is a weak solution of (1.1.3) iff

$$((u, \varphi)) = \langle \varphi, f \rangle_{H_0^1, H^{-1}} \tag{1.1.11}$$

for all $\varphi \in H_0^1(\Omega)$. By the Riesz representation theorem, there exists a unique $u \in H_0^1(\Omega)$ such that (1.1.11) holds for all $\varphi \in H_0^1(\Omega)$. This proves the existence and uniqueness part. Moreover, it follows from Propositions B.6.29 and B.6.24 that

$$\|f\|_{H^{-1}} = \| -\Delta u + \lambda u\|_{H^{-1}} \leq \|\Delta u\|_{H^{-1}} + |\lambda| \, \|u\|_{H^{-1}} \leq C\|u\|_{H^1}$$

which proves the first inequality in (1.1.10). Letting $\varphi = u$ in (1.1.11), we obtain

$$\|u\|_{H^1}^2 \leq C((u, u)) \leq C\|u\|_{H^1}\|f\|_{H^{-1}}$$

which yields the second inequality in (1.1.10). □

Remark 1.1.7. Here are some comments on Theorem 1.1.6.

(i) We do not impose any condition on the domain Ω, neither of smoothness, nor of boundedness.
(ii) If Ω then $\overline{\lambda}(\Omega) > 0$ so we may choose $\lambda = 0$.
(iii) We may apply the above very elementary technique because the Laplacian is a symmetric operator, so that $((u, v))$ defined a scalar product on $H_0^1(\Omega)$. For non-symmetric operators, on can apply for instance Lax-Milgram's theorem.
(iv) A classical solution is a weak solution, and weak solutions are unique. It is therefore natural to consider weak solutions. It may happen that there is no classical solution, while there is a weak solution. For example, let U be the unit ball in $\mathbb{R}^3$, and $\Omega = U \setminus \{0\}$. Let $u \in C_c^\infty(U)$ such that $u \equiv 1$ in a neighborhood of 0, and let $f = -\Delta u \in C_c^\infty(\Omega)$. We have $u \in H_0^1(\Omega)$, since $H_0^1(\Omega) = H_0^1(U)$. Therefore, u is a weak solution of

$$\begin{cases} -\Delta u = f \\ u_{|\partial\Omega} = 0 \end{cases} \tag{1.1.12}$$

Since u does not vanish on $\partial\Omega$ (because u does not vanish at 0), this means that there is no classical solution of (1.1.12). (Otherwise, it would equal u, by uniqueness.)

REMARK 1.1.8. If we consider Laplace's equation with Neumann boundary conditions, i.e.

$$\begin{cases} -\Delta u + \lambda u = f & \text{in } \Omega \\ \dfrac{\partial u}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1.13}$$

then Green's formula leads to the following definition of a weak solution

$$\begin{cases} u \in H^1(\Omega) \\ \displaystyle\int_\Omega (\nabla u \cdot \nabla\varphi + \lambda u\varphi - f\varphi) = 0 \quad \forall \varphi \in H^1(\Omega) \end{cases}$$

Applying again the Riesz representation theorem, it is immediate that if $\lambda > 0$ and $f \in L^2(\Omega)$, then there exists a unique weak solution u.

In analogy with Theorem 1.1.1, one might expect extensions of Theorem 1.1.6 in two directions. On the one hand, if f is more regular, for instance $f \in H^m(\Omega)$ for some $m \geq 0$, then one might expect accordingly u to be more regular, i.e. $u \in H^{m+2}(\Omega)$. On the other hand, one might consider for instance $f \in L^p(\Omega)$ with $p \neq 2$ and expect the existence of a weak solution, in some appropriate sense, in an L^p-based space such as $W_0^{1,p}(\Omega)$. It turns out that both these extensions are possible only under certain smoothness assumptions on Ω. Moreover, the proofs are considerably more difficult than in the case of the equation set on the whole space $\mathbb{R}^N$. On these issues, see for instance [**14**] and the references therein.

We present below a result on the interior regularity, similar to Theorem 1.1.3.

THEOREM 1.1.9. *Let $\lambda \in \mathbb{R}$, $f \in H^{-1}(\Omega)$, and suppose $u \in H_0^1(\Omega)$ satisfies (1.1.5). If $f \in W_{\mathrm{loc}}^{m,p}(\Omega)$ for some $m \geq 0$ and $1 < p < \infty$, then $f \in W_{\mathrm{loc}}^{m+2,p}(\Omega)$. In particular, if $f \in C^\infty(\Omega)$, then $u \in C^\infty(\Omega)$.*

PROOF. Consider $\theta \in C_{\mathrm{c}}^\infty(\Omega)$. Since $u \in H_0^1(\Omega)$, we see that $\theta u \in H_0^1(\Omega)$ and

$$\nabla(\theta u) = \theta\nabla u + u\nabla\theta.$$

Define $w : \mathbb{R}^N \to \mathbb{R}$ by

$$w(x) = \begin{cases} \theta(x)u(x) & x \in \Omega \\ 0 & x \notin \Omega. \end{cases}$$

It follows that $w \in H^1(\mathbb{R}^N)$ and

$$\nabla w = \begin{cases} \nabla(\theta u) & \Omega \\ 0 & \mathbb{R}^N \setminus \Omega \end{cases}$$

Given $\varphi \in \mathcal{S}(\mathbb{R}^N)$, we have

$$\begin{aligned} \int_{\mathbb{R}^N} \nabla w \cdot \nabla\varphi = \int_\Omega \nabla(\theta u) \cdot \nabla\varphi &= \int_\Omega (\theta\nabla u + u\nabla\theta) \cdot \nabla\varphi \\ &= \int_\Omega \nabla u \cdot (\theta\nabla\varphi) + \int_\Omega u\nabla\theta \cdot \nabla\varphi \\ &= \int_\Omega \nabla u \cdot \nabla(\theta\varphi) - \int_\Omega \nabla u \cdot (\varphi\nabla\theta) + \int_\Omega u\nabla\theta \cdot \nabla\varphi \end{aligned}$$

Applying (1.1.5) with φ replaced by $(\theta\varphi)_{|\Omega} \in H_0^1(\Omega)$, we calculate the first term in the right-hand side of the last inequality as follows

$$\int_\Omega \nabla u \cdot (\theta\nabla\varphi) = -\lambda \int_{\mathbb{R}^N} w\varphi + \int_\Omega f\theta\varphi$$

so that

$$\int_{\mathbb{R}^N} \nabla w \cdot \nabla \varphi = -\lambda \int_{\mathbb{R}^N} w\varphi + \int_\Omega f\theta\varphi - \int_\Omega \nabla u \cdot (\varphi \nabla \theta) + \int_\Omega u \nabla \theta \cdot \nabla \varphi$$

hence

$$\int_{\mathbb{R}^N} (\nabla w \cdot \nabla \varphi + w\varphi) = (1-\lambda) \int_{\mathbb{R}^N} w\varphi + \int_\Omega f\theta\varphi - \int_\Omega \nabla u \cdot (\varphi \nabla \theta) + \int_\Omega u \nabla \theta \cdot \nabla \varphi$$

We integrate the last term by parts, which is possible because $u\nabla\theta \in (H^1_0(\Omega))^N$. Since $\nabla \cdot (u \nabla \theta) = \nabla u \cdot \nabla \theta + u \Delta \theta$, we obtain

$$\begin{aligned}\int_{\mathbb{R}^N} (\nabla w \cdot \nabla \varphi + w\varphi) = (1-\lambda) \int_{\mathbb{R}^N} u\theta\varphi + \int_\Omega f\theta\varphi - \int_\Omega \nabla u \cdot (\varphi \nabla \theta) \\ - \int_\Omega (\nabla u \cdot \nabla \theta + u \Delta \theta)\varphi = \int_{\mathbb{R}^N} F\varphi\end{aligned}$$

where

$$F(x) = \begin{cases} f\theta - 2\nabla u \cdot \nabla \theta + [(1-\lambda)\theta - \Delta\theta]u & \Omega \\ 0 & \mathbb{R}^N \setminus \Omega \end{cases}$$

We first show that $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$. Since $u \in H^1_0(\Omega)$, there is nothing to prove if $p \le 2$, so we assume $2 < p < \infty$. We consider the unique integer $\ell \ge 1$ such that

$$\frac{1}{2} - \frac{\ell}{N} < \frac{1}{p} \le \frac{1}{2} - \frac{\ell - 1}{N}$$

and we define $q_j \ge 2$ for $0 \le j \le \ell$ by

$$\frac{1}{q_j} = \frac{1}{2} - \frac{j}{N}$$

In particular, $q_0 = 2$, so that $u \in W^{1,q_0}_{\mathrm{loc}}(\Omega)$. We show by induction that $u \in W^{1,q_\ell}_{\mathrm{loc}}(\Omega)$. Suppose $u \in W^{1,q_j}_{\mathrm{loc}}(\Omega)$ for some $0 \le j \le \ell - 1$. Note that all terms in F constructed above have a compact support in Ω. Therefore, since $f\theta \in L^p(\mathbb{R}^N)$, we have $f\theta \in L^{q_j}(\mathbb{R}^N)$. Moreover, $u \in u \in W^{1,q_j}_{\mathrm{loc}}(\Omega)$, so that $-2\nabla u \cdot \nabla \theta + [(1-\lambda)\theta - \Delta\theta]u \in L^{q_j}(\mathbb{R}^N)$. Therefore, $F \in L^{q_j}(\mathbb{R}^N)$, so that by Theorem 1.1.1 (ii), $w \in W^{2,q_{j+1}}(\mathbb{R}^N)$. By Sobolev's embedding, we deduce that $w \in W^{1,q_j}(\mathbb{R}^N)$. Since θ and φ are arbitrary, this implies that $u \in W^{1,q_{j+1}}_{\mathrm{loc}}(\Omega)$. Thus $u \in W^{1,q_\ell}_{\mathrm{loc}}(\Omega)$, and applying once more this regularity argument, we obtain $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$.

We finally argue as in the proof of Theorem 1.1.3. Suppose $u \in W^{\ell,p}_{\mathrm{loc}}(\Omega)$ for some $1 \le \ell \le m+1$. It follows that F constructed above belongs to $W^{\ell-1,p}(\mathbb{R}^N)$. Therefore, $w \in W^{\ell+1,p}(\mathbb{R}^N)$ by Theorem 1.1.1 (ii), so that $u \in W^{\ell+1,p}_{\mathrm{loc}}(\Omega)$. Iterating, we deduce that $u \in W^{m+2,p}_{\mathrm{loc}}(\Omega)$, which is the desired result. □

Remark 1.1.10. All the results in this section are also true, with the same proofs, for complex-valued functions. Note that the new scalar product on $H^1_0(\Omega)$ is defined in this case by

$$((u,v)) = \Re \int_\Omega (\nabla u \cdot \nabla \overline{v} + \lambda u \overline{v})$$

for $uv \in H^1_0(\Omega)$.

1.2. The maximum principle

The maximum principle is a fundamental property of the Laplace equation, and we establish below several forms of it. Throughout this section, we assume that Ω is an open, connected subset of $\mathbb{R}^N$, bounded or not.

1.2.1. The weak maximum principle. We begin with the following weak maximum principle, which is very simple but on which are based the more elaborated forms of the maximum principle.

We recall that if $f \in H^{-1}(\Omega)$, then $f \geq 0$ (in the sense of H^{-1}) if

$$\langle f, \varphi \rangle_{H^{-1}, H_0^1} \geq 0$$

for all $\varphi \in H_0^1(\Omega)$ such that $\varphi \geq 0$ a.e.

THEOREM 1.2.1. *Suppose* (1.1.8). *Let* $u \in H^1(\Omega)$ *and assume*

$$-\Delta u + \lambda u \geq 0 \quad \textit{(respectively, } \leq 0\textit{)}$$

in the sense of $H^{-1}(\Omega)$. *If* $u^- \in H_0^1(\Omega)$ *(respectively,* $u^+ \in H_0^1(\Omega)$*), then* $u \geq 0$ *(respectively,* $u \leq 0$*) a.e. in* Ω.

PROOF. We prove the first part of the result, the second follows by changing u to $-u$. Since $u^- \geq 0$, we have

$$\langle -\Delta u + \lambda u, -u^- \rangle_{H^{-1}, H_0^1} \leq 0$$

which we rewrite, using formuls (B.1.8) and (B.1.7), in the form

$$\int_\Omega [\nabla u \cdot \nabla(-u^-) + \lambda u(-u^-)] \leq 0.$$

Note that $u = u^+ - u^-$, so that $u(-u^-) = (u^-)^2$; and that $\nabla u = \nabla u^+ - \nabla u^-$, so that $\nabla u \cdot \nabla(-u^-) = |\nabla u^-|^2$ (by (B.2.4). It follows that

$$\int_\Omega [|\nabla(u^-)|^2 + \lambda |u^-|^2] \leq 0.$$

Since $\lambda > -\overline{\lambda}(\Omega)$, we deduce by applying (1.1.9) that $u^- = 0$, so that $u \geq 0$ a.e. □

REMARK 1.2.2. Here are some comments on the preceding result.

(i) Since $u \in H^1(\Omega)$, we know that $u^- \in H^1(\Omega)$, by Remark B.2.4. Therefore, the condition $u^- \in H_0^1(\Omega)$ is a weak way of saying $u^- = 0$ on $\partial\Omega$, which means $u \geq 0$ on $\partial\Omega$.

(ii) Let $f \in H^{-1}(\Omega)$, and $u \in H_0^1(\Omega)$ the weak solution of (1.1.3) given by Theorem 1.1.6. It follows that if $f \geq 0$, then $u \geq 0$ a.e. on Ω. Indeed, since $u \in H_0^1(\Omega)$, we have $u^- \in H_0^1(\Omega)$, by Remark B.2.4.

1.2.2. The strong maximum principle. There are several forms of the strong maximum principle. We begin with the simplest, which is valid in any domain.

THEOREM 1.2.3. *Assume* (1.1.8). *Let* $u \in H^1(\Omega) \cap C(\Omega)$ *satisfy* $-\Delta u + \lambda u \geq 0$ *(respectively,* ≤ 0*) in the sense of* $H^{-1}(\Omega)$. *If* $u^- \in H_0^1(\Omega)$ *(respectively,* $u^+ \in H_0^1(\Omega)$*) and if* $u \not\equiv 0$, *then* $u > 0$ *(respectively,* $u < 0$*) in* Ω.

The proof of Theorem 1.2.3 is based on the following simple lemma.

LEMMA 1.2.4. *Let* $0 < \rho < R < \infty$ *and set* $\omega = \{\rho < |x| < R\}$. *Let* $\lambda \in \mathbb{R}$ *and suppose* $\beta > \max\{0, N-2\}$ *satisfies* $\beta(\beta - N + 2) \geq |\lambda| R^2$. *If* v *is defined by* $v(x) = |x|^{-\beta} - R^{-\beta}$ *for* $\rho \leq |x| \leq R$, *then the following properties hold.*

(i) $v \in C^\infty(\overline{\omega})$.

(ii) $v(x) = 0$ *if* $|x| = R$.

(iii) $\rho^{-\beta} > v(x) \geq \beta R^{-(\beta+1)}(R - |x|)$ *if* $\rho \leq |x| \leq R$.

(iv) $-\Delta v + \lambda v \leq 0$ *in* ω.

PROOF. Properties (i), (ii) and (iii) are immediate. Next,

$$\begin{aligned}-\Delta v+\lambda v &= -\beta(\beta-N+2)|x|^{-(\beta+2)}+\lambda|x|^{-\beta}-\lambda R^{-\beta}\\ &\le -\beta(\beta-N+2)R^{-2}|x|^{-\beta}+|\lambda|\,|x|^{-\beta},\end{aligned}$$

and (iv) easily follows. □

PROOF OF THEOREM 1.2.3. We prove the first part of the result, the second follows by changing u to $-u$. We first note that by Theorem 1.2.1 (and Remark 1.2.2 (ii)), $u \ge 0$ a.e. in Ω. Since $u \in C(\Omega)$ and $u \not\equiv 0$, the set

$$O = \{x \in \Omega;\ u(x) > 0\},$$

is a nonempty open subset of Ω. Ω being connected, we need only show that O is a closed subset of Ω. Suppose $(y_n)_{n\ge 0} \subset O$ and $y_n \to y \in \Omega$ as $n \to \infty$. Let $R > 0$ be such that $B(y, 2R) \subset \Omega$, and fix n_0 large enough so that $|y - y_{n_0}| < R$. Since $u(y_{n_0}) > 0$, there exist $0 < \rho < R$ and $\varepsilon > 0$ such that $u(x) \ge \varepsilon$ for $|x - y_{n_0}| = \rho$. Set $U = \{\rho < |x - y_{n_0}| < R\}$ and let $w(x) = u(x) - \varepsilon\rho^\beta v(x - y_{n_0})$ for $x \in U$, where β and v are as in Lemma 1.2.4. It follows that $w \in H^1(U) \cap C(\overline{U})$. Moreover, $-\Delta w + \lambda w \ge 0$ by property (iv) of Lemma 1.2.4. Also, since $u \ge 0$ in Ω, we deduce from property (ii) of Lemma 1.2.4 that $w(x) \ge 0$ if $|x - y_{n_0}| = R$. Furthermore, $w(x) \ge 0$ if $|x - y_{n_0}| = \rho$ by property (iii) of Lemma 1.2.4 and because $u(x) \ge \varepsilon$. Thus we may apply Theorem 1.2.1 and we deduce that $w(x) \ge 0$ for $x \in U$. In particular, $u(y) \ge \varepsilon\rho^\beta v(y - y_{n_0}) > 0$ by property (iii) of Lemma 1.2.4, so that $y \in O$. Therefore, O is closed, which completes the proof. □

We now state a stronger version of the maximum principle, which requires a certain amount of regularity of the domain. More precisely, we assume the following geometric condition. (See Figure 1.)

$$\begin{cases}\Omega \text{ is a connected, open, bounded subset of } \mathbb{R}^N, \text{and}\\ \exists \eta,\nu>0 \text{ s.t. } \forall x\in\Omega \text{ with } \mathrm{d}(x,\partial\Omega)\le\eta, \exists y\in\Omega\\ \text{s.t. } x\in B(y,\eta),\ B(y,\eta)\subset\Omega \text{ and } \eta-|x-y|\ge\nu\mathrm{d}(x,\partial\Omega)\end{cases} \tag{1.2.1}$$

Assumption (1.2.1) means that one cannot have outwards corners. It is easy to see that it is satisfied if Ω is of class C^2. Indeed, let $\gamma(z)$ denote the unit outwards normal to $\partial\Omega$ at $z \in \partial\Omega$. Since Ω is bounded, $\partial\Omega$ is uniformly C^2, so that there exists $\eta > 0$ such that $B(z - \eta\gamma(z), \eta) \subset \Omega$ for every $z \in \partial\Omega$. If $x \in \Omega$ and $\mathrm{d}(x, \partial\Omega) \le \eta$, let $z \in \partial\Omega$ be such that $|x - z| = \mathrm{d}(x, \partial\Omega)$. It follows that $x - z$ is parallel to $\gamma(y)$. Thus, if we set $y = z - \eta\gamma(z)$, we see that $x \in B(y, \eta)$, $B(y, \eta) \subset \Omega$ and $\eta - |x - y| = |z - x| = \mathrm{d}(x, \partial\Omega)$.

THEOREM 1.2.5. *Assume (1.2.1) and (1.1.8). Suppose $u \in H^1(\Omega) \cap C(\Omega)$ satisfies $-\triangle u + \lambda u \ge 0$ (respectively, ≤ 0) in $H^{-1}(\Omega)$. If $u^- \in H^1_0(\Omega)$ (respectively, $u^+ \in H^1_0(\Omega)$) and if $u \not\equiv 0$, then there exists $\mu > 0$ such that $u(x) \ge \mu\mathrm{d}(x, \partial\Omega)$ (respectively, $u(x) \le -\mu\mathrm{d}(x, \partial\Omega)$) in Ω, where $\mathrm{d}(x, \partial\Omega)$ is the distance of x to $\partial\Omega$.*

PROOF. We prove the first part of the result, the second follows by changing u to $-u$. Let $0 < \varepsilon \le \eta/2$ and consider $\Omega_\varepsilon = \{x \in \Omega;\ \mathrm{d}(x, \partial\Omega) \ge \varepsilon\}$. We fix $\varepsilon > 0$ sufficiently small so that Ω_ε is a nonempty, compact subset of Ω. It follows from Theorem 1.2.3 that there exists $\delta > 0$ such that

$$u(x) \ge \delta \quad \text{for all} \quad x \in \Omega_\varepsilon. \tag{1.2.2}$$

We now consider $x_0 \in \Omega$ such that $\mathrm{d}(x_0, \partial\Omega) < \varepsilon$, and we let $y_0 \in \Omega$ satisfy (1.2.1). Since $B(y_0, \eta) \subset \Omega$ and $\eta \ge 2\varepsilon$, we see that $\mathrm{d}(z, \partial\Omega) \ge \varepsilon$ for all $z \in B(y_0, \eta/2)$. It then follows from (1.2.2) that

$$u(z) \ge \delta \quad \text{for all} \quad z \in B(y_0, \eta/2). \tag{1.2.3}$$

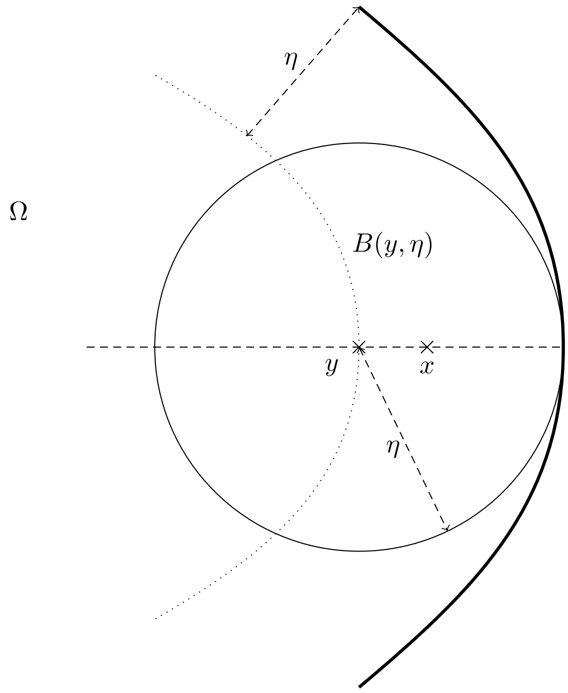

FIGURE 1. The geometric condition (1.2.1)

We let $\rho = \eta/2$, $R = \eta$ and $U = \{\rho < |x - y_0| < R\}$, so that $x_0 \in U$. Let $w(x) = u(x) - \varepsilon\rho^\beta v(x - y_0)$ for $x \in U$, where β and v are as in Lemma 1.2.4. It follows that $w \in H^1(U) \cap C(\overline{U})$. Moreover, $-\Delta w + \lambda w \geq 0$ by property (iv) of Lemma 1.2.4. Also, $w(x) \geq 0$ if $|x - y_0| = R$ by property (ii) of Lemma 1.2.4 and because $u \geq 0$ in Ω. Furthermore, $w(x) \geq 0$ if $|x - y_0| = \rho$ by property (iii) of Lemma 1.2.4 and (1.2.3). Thus we may apply Theorem 1.2.1 and we deduce that $w(x) \geq 0$ for $x \in U$. In particular,

$$u(x_0) \geq \beta\varepsilon\rho^\beta R^{-(\beta+1)}(R - |x_0 - y_0|) \geq \nu\beta\varepsilon\rho^\beta R^{-(\beta+1)}\mathrm{d}(x_0, \partial\Omega),$$

where the first inequality above follows from of Lemma 1.2.4 (iii) and the second from (1.2.1). Since $x_0 \in \Omega \setminus \Omega_\varepsilon$ is arbitrary, we see that there exists $\mu > 0$ such that $u(x) \geq \mu\mathrm{d}(x, \partial\Omega)$ for all $x \in \Omega \setminus \Omega_\varepsilon$. On the other hand, (1.2.2) implies that there exists $\mu' > 0$ such that $u(x) \geq \mu'\mathrm{d}(x, \partial\Omega)$ for all $x \in \Omega_\varepsilon$. This completes the proof. □

1.3. $L^p(\Omega)$ et $C_0(\Omega)$ regularity

Throughout this section, Ω is an open, connected subset of $\mathbb{R}^N$. Unless otherwise specified, Ω may be bounded or not.

1.3.1. L^p regularity.

We begin with a simple estimate.

THEOREM 1.3.1. *Soient $\lambda > 0$, $f \in H^{-1}(\Omega)$ et soit $u \in H^1_0(\Omega)$ la solution de (1.1.5). Si $f \in L^p(\Omega)$ pour un certain $p \in [1, \infty]$, alors $u \in L^p(\Omega)$ et $\lambda\|u\|_{L^p} \leq \|f\|_{L^p}$.*

PROOF. Let $\varphi \in C^1(\mathbb{R}, \mathbb{R})$. Assume that $\varphi(0) = 0$, $\varphi' \geq 0$ and $\varphi' \in L^\infty(\mathbb{R})$. It follows from Proposition B.2.1 that $\varphi(u) \in H^1_0(\Omega)$ and that $\nabla\varphi(u) = \varphi'(u)\nabla u$ a.e.

in Ω. We let $\varphi = \varphi(u)$ in (1.1.5), and we obtain

$$\int_\Omega \varphi'(u)|\nabla u|^2\,dx + \lambda\int_\Omega u\varphi(u)\,dx = \int_\Omega f\varphi(u)\,dx$$

so that

$$\lambda\int_\Omega u\varphi(u)\,dx \le \int_\Omega f\varphi(u)\,dx$$

Assuming further $|\varphi(u)| \le |u|^{p-1}$, we see that $|\varphi(u)| \le (u\varphi(u))^{\frac{p-1}{p}}$, so that

$$\lambda\int_\Omega u\varphi(u)\,dx \le \|f\|_{L^p}\|\varphi(u)\|_{L^{\frac{p}{p-1}}} \le \|f\|_{L^p}\Big(\int_\Omega u\varphi(u)\,dx\Big)^{\frac{p-1}{p}}$$

Therefore,

$$\lambda\Big(\int_\Omega u\varphi(u)\,dx\Big)^{\frac{1}{p}} \le \|f\|_{L^p} \tag{1.3.1}$$

We now consider separately two cases.

Case 1: $1 \le p \le 2$ Given $\varepsilon > 0$, let $\varphi(u) = u(\varepsilon + u^2)^{\frac{p-2}{2}}$. It is easy to see that φ satisfies the previous conditions, and we deduce from (1.3.1) that

$$\lambda\Big(\int_\Omega u^2(\varepsilon + u^2)^{\frac{p-2}{2}}\,dx\Big)^{\frac{1}{p}} \le \|f\|_{L^p}$$

Letting $\varepsilon \downarrow 0$ and applying Fatou, we obtain the desired estimate

Case 2: $2 < p \le \infty$ We apply a duality argument. Let $h \in C_c^\infty(\Omega)$ and $v \in H_0^1(\Omega)$ the solution of (1.1.5) with f replaced by h. It follows that

$$\begin{aligned}\int_\Omega uh &= \langle u, -\Delta v + \lambda v\rangle_{H_0^1,H^{-1}} = \langle -\Delta u + \lambda u, v\rangle_{H^{-1},H_0^1}\\ &= \langle f, v\rangle_{H^{-1},H_0^1} = \int_\Omega fv\end{aligned}$$

Therefore,

$$\Big|\int_\Omega uh\Big| \le \|f\|_{L^p}\|v\|_{L^{p'}} \le \frac{1}{\lambda}\|f\|_{L^p}\|h\|_{L^{p'}}$$

by Case 1, since $p' < 2$. Since $h \in C_c^\infty(\Omega)$ is arbitrary, we deduce that $\|u\|_{L^p} \le \lambda^{-1}\|f\|_{L^p}$. □

One can improve the above estimates by using Sobolev's inequalities.

Theorem 1.3.2. *Let $\lambda > 0$, $f \in H^{-1}(\Omega)$ and $u \in H_0^1(\Omega)$ the solution of (1.1.5). Assume $f \in L^p(\Omega)$ for some $1 < p < \infty$.*

(i) *If $N \ge 3$ and $p = N/2$, then $u \in L^r(\Omega)$ for all $r \in [p, \infty)$, and there exists a constant $C(r)$ independent of f such that $\|u\|_{L^r} \le C(r)\|f\|_{L^p}$.*

(ii) *If $N \ge 3$ and $1 < p < N/2$, then $u \in L^p(\Omega) \cap L^{\frac{Np}{N-2p}}(\Omega)$, and there exists constant C independent of f such that $\|u\|_{L^r} \le C\|f\|_{L^p}$ for all $r \in [p, \frac{Np}{N-2p}]$.*

Proof. Let $1 < q < \infty$ such that $(q-1)p' \ge 1$. We obtain an estimate by using $\varphi = |u|^{q-2}u$ in (1.1.5). In fact, as in the preceding theorem, a regularization is necessary, and one should really use the test function

$$\varphi_\varepsilon = \begin{cases} u(\varepsilon + |u|^2)^{\frac{q-2}{2}} & q \le 2\\ u|u|^{q-2}(1 + \varepsilon|u|^2)^{\frac{2-q}{2}} & q > 2\end{cases}$$

then $\varepsilon \downarrow 0$ in the estimate that one obtains. For simplicity, we only give the formal calculations, letting $\varphi = |u|^{q-2}u$ in (1.1.5). Since $\nabla\varphi = (q-1)|u|^{q-2}\nabla u$, we obtain

$$\nabla u \cdot \nabla\varphi = (q-1)|u|^{q-2}|\nabla u|^2$$

On the other hand,

$$\nabla(|u|^{\frac{q-2}{2}}u) = \frac{q}{2}|u|^{\frac{q-2}{2}}\nabla u$$

so that

$$\nabla u \cdot \nabla\varphi = \frac{4(q-1)}{q^2}|\nabla(|u|^{\frac{q-2}{2}}u)|^2$$

Identity (1.1.5) therefore gives

$$\frac{4(q-1)}{q^2}\int_\Omega |\nabla(|u|^{\frac{q-2}{2}}u)|^2 + \lambda\int_\Omega |u|^q = \int_\Omega f|u|^{q-2}u \le \|f\|_{L^p}\|u\|^{q-1}_{L^{(q-1)p'}}$$

Applying Sobolev's inequality to the function $|u|^{\frac{q-2}{2}}u$, we obtain

$$\int_\Omega |\nabla(|u|^{\frac{q-2}{2}}u)|^2 \ge c\|\,|u|^{\frac{q-2}{2}}u\|^2_{L^{\frac{2N}{N-2}}} = c\|u\|^q_{\frac{2Nq}{N-2}}$$

so that

$$\|u\|^q_{L^{\frac{Nq}{N-2}}} \le C\frac{q^2}{q-1}\|f\|_{L^p}\|u\|^{q-1}_{L^{(q-1)p'}}, \tag{1.3.2}$$

for all $1 < q < \infty$ satisfying $(q-1)p' \ge 1$. To prove (i), we choose $p = \frac{N}{2}$ and we apply (1.3.2) with $q > \frac{N}{2}$. We obtain

$$\|u\|^q_{L^{\frac{Nq}{N-2}}} \le C\frac{q^2}{q-1}\|f\|_{L^{N/2}}\|u\|^{q-1}_{L^{\frac{N(q-1)}{N-2}}}. \tag{1.3.3}$$

Applying Hölder's inequality and Theorem 1.3.1, we deduce

$$\|u\|^{q-1}_{L^{\frac{N(q-1)}{N-2}}} \le \|u\|^{\frac{(2q-N)q}{2q-N+2}}_{L^{\frac{Nq}{N-2}}}\|u\|^{\frac{N-2}{2q-N+2}}_{L^{\frac{N}{2}}} \le \|u\|^{\frac{(2q-N)q}{2q-N+2}}_{L^{\frac{Nq}{N-2}}}\|f\|^{\frac{N-2}{2q-N+2}}_{L^{\frac{N}{2}}}.$$

Substitution into (1.3.3) yields

$$\|u\|_{L^{\frac{Nq}{N-2}}} \le C(q)\|f\|_{L^{\frac{N}{2}}}.$$

Estimate (i) is a consequence of the above inequality, combined with Theorem 1.3.1, since q can be arbitrarily large. Finally, to prove (ii), we lat $q = \frac{(N-2)p}{N-2p}$, so that $1 < \frac{Nq}{(N-2)} = (q-1)p' = \frac{Np}{(N-2p)}$. Inequality (1.3.2) yields

$$\|u\|_{L^{\frac{Np}{N-2p}}} \le C'\|f\|_{L^p}$$

which, together with Theorem 1.3.1, proves (ii). □

The preceding results do not apply when $\lambda \le 0$. One can, however, obtain certain estimates by more elaborated techniques. We begin with an L^∞ estimate.

THEOREM 1.3.3. *Assume (1.1.8). Let $f \in H^{-1}(\Omega)$ and $u \in H^1_0(\Omega)$ the solution of (1.1.5). If $f \in L^p(\Omega)$ for some $\max\{1, \frac{N}{2}\} < p \le \infty$, then $u \in L^\infty(\Omega)$. Moreover, given any $1 \le r < \infty$, there exists a constant C independant of f such that*

$$\|u\|_{L^\infty} \le C(\|f\|_{L^p} + \|u\|_{L^r}) \tag{1.3.4}$$

In particular,

$$\|u\|_{L^\infty} \le C(\|f\|_{L^p} + \|f\|_{H^{-1}}) \tag{1.3.5}$$

PROOF. By homogeneity, we may assume that $\|u\|_{L^r} + \|f\|_{L^p} \le 1$. Since $-u$ solves the same equation as u, with f replaced by $-f$ (which satisfies the same assumptions), it is sufficient to estimate $\|u^+\|_{L^\infty}$. Set $T = \|u^+\|_{L^\infty} \in [0,\infty]$ and assume that $T > 0$. For $t \in (0,T)$, set

$$v(t) = (u-t)^+$$

We It follows from Corollary B.2.6 that $v(t) \in H_0^1(\Omega)$ and

$$\nabla v(t) = \begin{cases} \nabla u & u > t \\ 0 & u \le t \end{cases} \tag{1.3.6}$$

Set now

$$\alpha(t) = |\{x \in \Omega, u(x) > t\}|$$

for $t > 0$. Note that $\alpha(t)$ is always finite. In particular, since $v(t) \in L^2(\Omega)$ is supported in $\{x \in \Omega, u(x) > t\}$, we have $v(t) \in L^1(\Omega)$. We set

$$\beta(t) = \int_\Omega v(t)\, dx$$

Integrating the measurable function $1_{\{u>s\}}(x)$ on $(t, \infty) \times \Omega$ and applying Fubini, we obtain

$$\beta(t) = \int_t^\infty \alpha(s)\, ds$$

so that $\beta \in W^{1,1}_{\mathrm{loc}}(0, \infty)$ and

$$\beta'(t) = -\alpha(t) \tag{1.3.7}$$

for a.a. $t > 0$.

The idea of the proof is to obtain a differential inequality on $\beta(t)$ which implies that $\beta(t)$ must vanish for t large enough. Lettin $\varphi = v(t)$ in (1.1.5), we obtain

$$\int_\Omega \nabla u \cdot \nabla v(t) + \lambda \int_\Omega u v(t) = \langle f, v(t) \rangle_{H^{-1}, H_0^1}$$

Applying (1.3.6) and the property $v(t) \in L^1(\Omega)$, we see that

$$\int_\Omega \{ |\nabla v(t)|^2 + \lambda |v(t)|^2 \}\, dx = \int_\Omega (f - t\lambda) v(t)\, dx$$

By (1.1.9), this yields

$$\|v(t)\|_{H^1}^2 \le C \int_\Omega (f - t\lambda) v(t)\, dx \le C \int_\Omega (|f| + t|\lambda|) v(t)\, dx \tag{1.3.8}$$

Note that

$$\int_\Omega |f| v(t) \le \|f\|_{L^p} \|v(t)\|_{L^{p'}} \le \|v(t)\|_{L^{p'}}$$

so that, applying (1.3.8),

$$\|v(t)\|_{H^1}^2 \le C(1+t)(\|v(t)\|_{L^{p'}} + \|v(t)\|_{L^1}) \tag{1.3.9}$$

Since $p > \min\{1, \frac{N}{2}\}$, we may fix

$$2p' < \rho < \frac{2N}{(N-2)^+}$$

so that $H_0^1(\Omega) \hookrightarrow L^\rho(\Omega)$. On the other hand, it follows from Hölder's inequality that $\|v(t)\|_{L^1} \le \alpha(t)^{1-\frac{1}{\rho}} \|v(t)\|_{L^\rho}$ and $\|v(t)\|_{L^{p'}} \le \alpha(t)^{\frac{1}{p'} - \frac{1}{\rho}} \|v(t)\|_{L^\rho}$. Therefore, we deduce from (1.3.9) that

$$\|v(t)\|_{L^\rho}^2 \le C(1+t)(\alpha(t)^{\frac{1}{p'} - \frac{1}{\rho}} + \alpha(t)^{1-\frac{1}{\rho}}) \|v(t)\|_{L^\rho}$$

Since $\beta(t) = \|v(t)\|_{L^1} \le \alpha(t)^{1-\frac{1}{\rho}} \|v(t)\|_{L^\rho}$, we obtain

$$\beta(t) \le C(1+t)(\alpha(t)^{1 + \frac{1}{p'} - \frac{2}{\rho}} + \alpha(t)^{2 - \frac{2}{\rho}})$$

which we rewrite in the form

$$\beta(t) \le C(1+t) F(\alpha(t))$$

where $F(s) = s^{1+\frac{1}{p'}-\frac{2}{\rho}} + s^{2-\frac{2}{\rho}}$. It follows that

$$-\alpha(t) + F^{-1}\Big(\frac{\beta(t)}{C(1+t)}\Big) \le 0 \tag{1.3.10}$$

Since $-\alpha(t) = \beta'(t)$ by (1.3.7), we deduce from (1.3.10) that

$$z' + \frac{\psi(z(t))}{C(1+t)} \le 0$$

where $z(t) = \frac{\beta(t)}{C(1+t)}$ and $\psi(s) = F^{-1}(s) + Cs$. This yields

$$\int_s^t \frac{d\sigma}{C(1+\sigma)} \le \int_{z(t)}^{z(s)} \frac{d\sigma}{\psi(\sigma)},$$

If $0 < s < t < T$. Si $T \le 1$, then $\|u^+\|_{L^\infty} \le 1$. Otherwise, we obtain

$$\int_1^t \frac{d\sigma}{C(1+\sigma)} \le \int_{z(t)}^{z(1)} \frac{d\sigma}{\psi(\sigma)}$$

for $1 < t < T$, which implies

$$\int_1^T \frac{d\sigma}{C(1+\sigma)} \le \int_0^{z(1)} \frac{d\sigma}{\psi(\sigma)}$$

Note that $F(s) \approx s^{1+\frac{1}{p'}-\frac{2}{\rho}}$ as $s \downarrow 0$, and $1 + \frac{1}{p'} - \frac{2}{\rho} > 1$, so that $\frac{1}{\psi}$ is integrable at 0. Since $\frac{1}{1+\sigma}$ is not integrable at infinity, this proves that $T = \|u^+\|_{L^\infty} < \infty$. Moreover, $\|u^+\|_{L^\infty}$ is estimated in terms of $z(1)$. Since

$$z(1) = \frac{1}{C}\int_\Omega (u-1)^+ \le \frac{1}{C}\int_{\{u>1\}} u \le \frac{1}{C}\int_{\{u>1\}} u^r \le \frac{1}{C}$$

this proves the desired estimate. □

Corollary 1.3.4. *Let $\lambda > 0$, $f \in H^{-1}(\Omega)$ and $u \in H^1_0(\Omega)$ the solution of (1.1.5). If $f \in L^p(\Omega)$ for some $1 < p < \infty$, $p > \frac{N}{2}$, then $u \in L^\infty(\mathbb{R}^N) \cap C(\Omega)$ and there exists a constant C independent of f such that $\|u\|_{L^\infty} \le C\|f\|_{L^p}$.*

Proof. It follows from Theorem 1.3.1 that $\lambda\|u\|_{L^p} \le \|f\|_{L^p}$, so that by Theorem 1.3.3 $u \in L^\infty(\Omega)$ and

$$\|u\|_{L^\infty} \le C(\|f\|_{L^p} + \|u\|_{L^p}) \le C(1+\lambda^{-1})\|f\|_{L^p} \tag{1.3.11}$$

To prove that $u \in C(\Omega)$, consider $(f_n)_{n\ge1} \subset C_c^\infty(\Omega)$ such that $f_n \to f$ in $L^p(\Omega)$, and let $(u_n)_{n\ge1}$ be the corresponding solutions of (1.1.5). It follows from Theorem 1.1.9 that $u_n \in C(\Omega)$. Since $u_n \to u$ in $L^\infty(\Omega)$ by (1.3.11), we obtain $u \in C(\Omega)$. □

In the case where Ω has finite measure, we deduce from Theorem 1.3.3 that Theorem 1.3.2 holds under assumption (1.1.8), and not just for $\lambda > 0$.

Corollary 1.3.5. *Assume $N \ge 3$, (1.1.8) and $|\Omega| < \infty$. Let $f \in H^{-1}(\Omega)$ and $u \in H^1_0(\Omega)$ the solution of (1.1.5). If $f \in L^p(\Omega)$ for some $1 < p \le \infty$, then the following properties hold.*

(i) *If $p = \frac{N}{2}$, then $u \in L^r(\Omega)$ for all $r \in [p, \infty)$, and there exists a constant $C(r)$ independent of f such that $\|u\|_{L^r} \le C(r)(\|f\|_{H^{-1}} + \|f\|_{L^p})$.*

(ii) *If $1 < p < N/2$, then $u \in L^{\frac{Np}{N-2p}}(\Omega)$, and there exists a constant C independent of f such that $\|u\|_{L^{\frac{Np}{N-2p}}} \le C(\|f\|_{H^{-1}} + \|f\|_{L^p})$.*

PROOF. It follows from Theorem 1.1.6 that $\|u\|_{H^1} \le C\|f\|_{H^{-1}}$. Therefore, $\|u\|_{L^{\frac{2N}{N-2}}} \le C\|f\|_{H^{-1}}$ by Sobolev's embedding. We fix

$$1 < q_0 < \frac{N}{2}, \quad q_0 \le \frac{2N}{N-2}, \quad q_0 \le p \tag{1.3.12}$$

and, since $|\Omega| < \infty$, we deduce that

$$\|u\|_{L^{q_0}} \le C\|f\|_{H^{-1}} \tag{1.3.13}$$

We now write equation (1.1.3) in the form $-\Delta u + u = g$ with $g = f + (1 - \lambda)u$. Given any $1 < q < \frac{N}{2}$, $q \le p$, we have $\|f\|_{L^q} \le C\|f\|_{L^p}$ (since $|\Omega| < \infty$). Therefore, if $u \in L^q(\Omega)$, then $g \in L^q(\Omega)$ and $\|g\|_{L^q} \le C(\|f\|_{L^p} + \|u\|_{L^q})$, so that by Theorem 1.3.2 (ii), $u \in L^{\frac{Nq}{N-2q}}(\Omega)$ and

$$\|u\|_{L^{\frac{Nq}{N-2q}}} \le C(\|f\|_{L^p} + \|u\|_{L^q}) \tag{1.3.14}$$

Since $\frac{Nq}{N-2q} \ge \frac{N}{N-2}q$ and $|\Omega| < \infty$, we deduce that if $1 < q < \frac{N}{2}$, then

$$\|u\|_{L^{\frac{N}{N-2}q}} \le C(\|f\|_{L^p} + \|u\|_{L^q}) \tag{1.3.15}$$

Let $k \in \mathbb{N}$ be defined by

$$\Big(\frac{N}{N-2}\Big)^k q_0 \le p < \Big(\frac{N}{N-2}\Big)^{k+1} q_0$$

where q_0 is given by (1.3.12), and set

$$q_j = \Big(\frac{N}{N-2}\Big)^j q_0$$

for $0 \le j \le k+1$. It follows from (1.3.15) that if $u \in L^{q_j}(\Omega)$ for some $0 \le j \le k$, then $u \in L^{q_{j+1}}(\Omega)$ and $\|u\|_{L^{q_{j+1}}} \le C(\|f\|_{L^p} + \|u\|_{L^{q_j}})$. Since $u \in L^{q_0}(\Omega)$ and $\|u\|_{L^{q_0}} \le C\|f\|_{H^{-1}}$ by (1.3.13), we deduce from an obvious iteration argument that $u \in L^{q_{k+1}}(\Omega)$ and $\|u\|_{L^{q_{k+1}}} \le C(\|f\|_{L^p} + \|f\|_{H^{-1}})$. Since $q_{k+1} \ge p$ and $|\Omega| < \infty$, we deduce in particular that $u \in L^p(\Omega)$ and $\|u\|_{L^p} \le C(\|f\|_{L^p} + \|f\|_{H^{-1}})$. Properties (i) abd-(ii) now follows easily from (1.3.14). ☐

REMARK 1.3.6. The previous results do not apply to the case $p = 1$. If $N = 1$, then $\|u\|_{L^\infty} \le C\|f\|_{H^{-1}}$, simply by Sobolev's embedding $H^1_0(\Omega) \hookrightarrow L^\infty(\Omega)$. If $N \ge 2$ and $\lambda > 0$, one can deduce certin estimates by a duality argument. Fix $q > \frac{N}{2}$. Let $h \in C^\infty_{\mathrm{c}}(\Omega)$ and $\varphi \in H^1_0(\Omega)$ the weak solution of $-\Delta\varphi + \lambda\varphi = h$. It follows from Corollary 1.3.4 that $\|\varphi\|_{L^\infty} \le C\|h\|_{L^q}$. On the other hand

$$\langle f, \varphi\rangle_{H^{-1},H^1_0} = \langle -\Delta u + \lambda u, \varphi\rangle_{H^{-1},H^1_0} = \langle -\Delta\varphi + \lambda\varphi, u\rangle_{H^{-1},H^1_0} = \langle h, u\rangle_{H^{-1},H^1_0}$$

so that

$$\Big|\int_\Omega hu\Big| \le \|f\|_{L^1}\|\varphi\|_{L^\infty} \le C\|f\|_{L^1}\|h\|_{L^q}$$

Since $h \in C^\infty_{\mathrm{c}}(\Omega)$ is arbitrary, we deduce that $\|u\|_{L^{q'}} \le C\|f\|_{L^1}$. Moreover, $q > \frac{N}{2}$ is also arbitrary, so that $\|u\|_{L^r} \le C(r)\|f\|_{L^1}$ for $1 \le r < \frac{N}{N-2}$.

REMARK 1.3.7. Theorems 1.3.2 and 1.3.3 can be established by proving $W^{2,p}$ regularity and the using Sobolev embedding $W^{2,p}(\Omega) \hookrightarrow L^q(\Omega)$. See [14], for instance. This last approach is, however, considerably more difficult, and requires regularity assumptions on Ω.

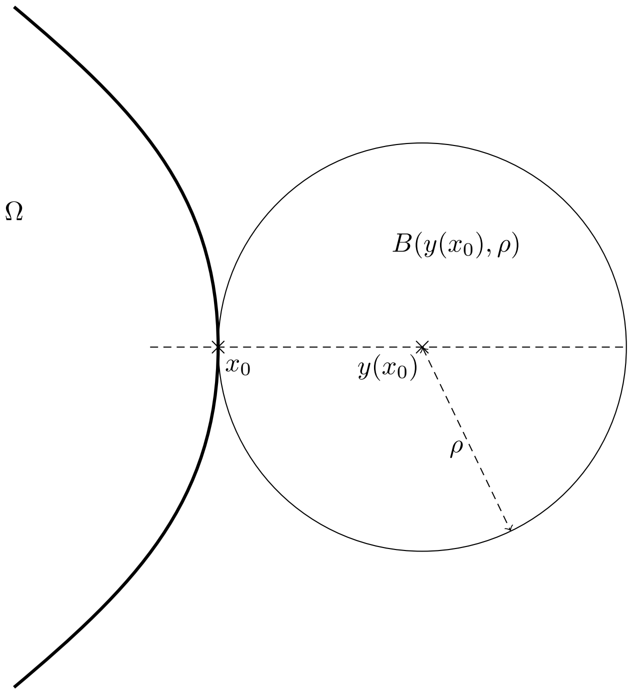

FIGURE 2. The geometric condition (1.3.16)

1.3.2. C_0 regularity. We show, assuming a mild geometric assumption on Ω (which means that there is no invards corner), that the solutions of (1.1.5) with $\lambda > 0$ and $f \in L^\infty(\Omega)$, are estimated by the distance to $\partial\Omega$.

THEOREM 1.3.8. *Assume either $N = 1$, or else $N \geq 2$ and*

$$\begin{cases} \exists \rho > 0 \ s.t. \ \forall x_0 \in \partial\Omega, \exists y(x_0) \in \mathbb{R}^N \\ \text{with } |x_0 - y(x_0)| = \rho \text{ and } B(y_0, \rho) \cap \Omega = \emptyset \end{cases} \tag{1.3.16}$$

(See Figure 2.) Let $\lambda > 0$, $f \in H^{-1}(\Omega)$ and $u \in H_0^1(\Omega)$ the solution of (1.1.5). If $f \in L^\infty(\Omega)$, then

$$|u(x)| \leq C\|f\|_{L^\infty} \mathrm{d}(x, \partial\Omega), \tag{1.3.17}$$

for all $x \in \Omega$, where the constant C is independent of f.

PROOF. By homogeneity, we may assume $|f| \leq 1$, so that $|u| \leq \lambda^{-1}$ by Theorem 1.3.1. Suppose further $N \geq 2$, the case $N = 1$ being immediate. We construct a *local barrier* at each point of $\partial\Omega$. Given $c > 0$, set

$$w(x) = h(|x|)$$

where

$$h(r) = \begin{cases} \frac{1}{4}(\rho^2 - r^2) + c\log(r/\rho) & N = 2 \\ \frac{1}{2N}(\rho^2 - r^2) + c(\rho^{2-N} - r^{2-N}) & N \geq 3 \end{cases}$$

(Here $\rho > 0$ is given by (1.3.16).) It is easy to see that

$$-\Delta w = 1 \text{ in } \mathbb{R}^N \setminus \{0\}$$

and that for if c is sufficiently large, then there exist $\rho < \rho_0 < \rho_1$ and a constant K such that

$$w(x) > 0 \text{ for } \rho < |x| \leq \rho_1, \quad w(x) \geq \lambda^{-1} \text{ for } \rho_0 \leq |x| \leq \rho_1 \tag{1.3.18}$$

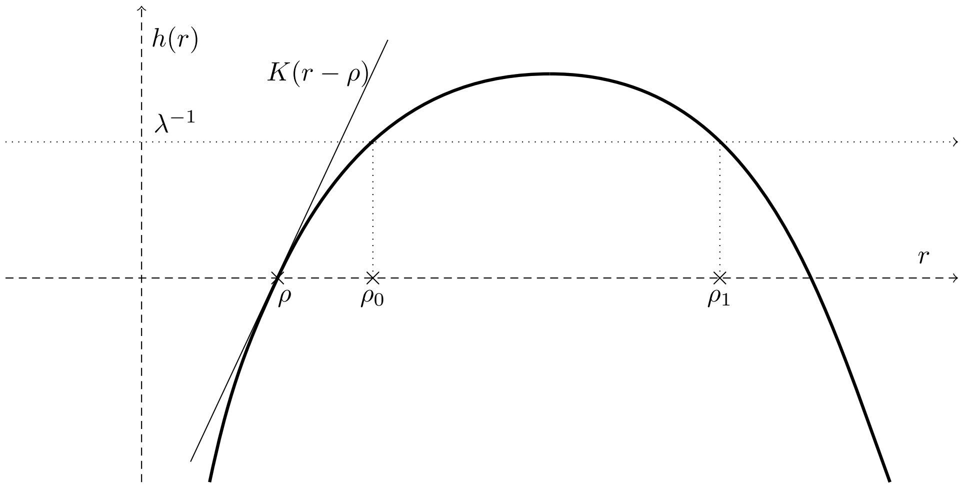

FIGURE 3. The function $h(r)$

and

$$w(x) \le K(|x| - \rho) \text{ for } \rho \le |x| \le \rho_1. \tag{1.3.19}$$

(See Figure 3.) Let now $\widetilde{x} \in \Omega$ such that $2\mathrm{d}(\widetilde{x}, \partial\Omega) < \rho_1 - \rho$, and let $x_0 \in \partial\Omega$ be such that $|\widetilde{x} - x_0| \le 2\mathrm{d}(\widetilde{x}, \partial\Omega)$. Set $\widetilde{\Omega} = \{x \in \Omega;\, \rho < |x - y(x_0)| < \rho_1\}$ and $v(x) = w(x - y(x_0))$ for $x \in \widetilde{\Omega}$. We note that $|\widetilde{x} - y(x_0)| > \rho$ by the geometric condition (1.3.16). Moreover,

$$|\widetilde{x} - y(x_0)| \le |\widetilde{x} - x_0| + |x_0 - y(x_0)| < \rho_1 - \rho + \rho = \rho_1,$$

so that $\widetilde{x} \in \widetilde{\Omega}$. Next, it follows from (1.3.18)-(1.3.19) that $v > 0$ on $\widetilde{\Omega}$ and that

$$\begin{aligned} 0 \le v(\widetilde{x}) &\le K(|\widetilde{x} - y(x_0)| - \rho) \le K(|\widetilde{x} - x_0| + |x_0 - y(x_0)| - \rho) \\ &= K|\widetilde{x} - x_0| \le 2K\mathrm{d}(\widetilde{x}, \partial\Omega) \end{aligned}$$

On the other hand

$$-\Delta(u - v) + \lambda(u - v) = f - (1 + \lambda v) \le f - 1 \le 0,$$

in $\widetilde{\Omega}$. We claim that

$$(u - v)^+ \in H^1_0(\widetilde{\Omega}). \tag{1.3.20}$$

It then follows from the maximum principle that $u \le v$ in $\widetilde{\Omega}$. In particular, $u(\widetilde{x}) \le v(\widetilde{x}) \le 2K\mathrm{d}(\widetilde{x}, \partial\Omega)$. Changing u to $-u$, one obtains as well that $-u \le v$, so that $|u(\widetilde{x})| \le 2K\mathrm{d}(\widetilde{x}, \partial\Omega)$ for a.a. $x \in \widetilde{\Omega}$. Therefore, $|u(\widetilde{x})| \le 2K\mathrm{d}(\widetilde{x}, \partial\Omega)$. Since $\widetilde{x}$ is arbitrary, we deduce that if $x \in \Omega$ such that $2\mathrm{d}(x, \partial\Omega) < \rho_1 - \rho$, then $|u(x)| \le 2K\mathrm{d}(x, \partial\Omega)$. For $x \in \Omega$ such that $2\mathrm{d}(x, \partial\Omega) \ge \rho_1 - \rho$, we have $u(x) \le \lambda^{-1} \le 2\lambda^{-1}(\rho_1 - \rho)^{-1}\mathrm{d}(x, \partial\Omega)$, and the result follows.

It now remains to establish the claim (1.3.20). Let $\varphi \in C^\infty_c(\mathbb{R}^N)$ satisfy $0 \le \varphi \le 1$, $\varphi \equiv 1$ on the set $\{|x - y(x_0)| \le \rho_0\}$ and $\varphi \equiv 0$ on the set $\{|x - y(x_0)| \ge \rho_1\}$. We note that by (1.3.18), $u \le \lambda^{-1} \le v$, thus $\varphi u - v \le u - v \le 0$ on $\widetilde{\Omega} \cap \{|x - y(x_0)| \ge \rho_0\}$. Therefore, $(\varphi u - v)^+ = (u - v)^+ = 0$ on $\widetilde{\Omega} \cap \{|x - y(x_0)| \ge \rho_0\}$. On $\widetilde{\Omega} \cap \{|x - y(x_0)| \le \rho_0\}$, $\varphi u - v = u - v$, so that $(u - v)^+ = (\varphi u - v)^+$ in $\widetilde{\Omega}$. Let now $(u_n)_{n \ge 0} \subset C^\infty_c(\Omega)$ satisfy $u_n \to u$ in $H^1(\Omega)$ as $n \to \infty$. It follows (see Proposition B.2.3) that $(\varphi u_n - v)^+ \to (\varphi u - v)^+ = (u - v)^+$ in $H^1(\widetilde{\Omega})$. Thus, we need only verify that $(\varphi u_n - v)^+ \in H^1_0(\widetilde{\Omega})$. This follows from Remark B.1.10 (i), because $\varphi u_n = 0$ and $v \ge 0$ on $\partial\widetilde{\Omega}$. □

1.4. Spectral decomposition of the Laplacian

Throughout this section Ω is an open, connected, bounded subset of $\mathbb{R}^N$, and we equip $H^1_0(\Omega)$ with the norm $\|u\|_{H^1_0} = \|\nabla u\|_{L^2}$ (see Corollary B.3.20), and $H^{-1}(\Omega)$ with the dual norm.

We recall that if H is an infinite-dimensional Hilbert space, and if $L \in \mathcal{L}(H)$ is a self-adjoint, compact, nonnegative (in the sense that $(Au, u) \geq 0$ for all $u \in H$) operator, then the eigenvalues of L are a nonincreasing sequence $(\mu_n)_{n\geq 1} \subset (0, \infty)$ such that $\mu_n \to 0$ as $n \to \infty$, and there exists a Hilbert basis $(\varphi_n)_{n\geq 1}$ of H composed of eigenvectors of L, i.e. $L\varphi_n = \mu_n \varphi_n$. (See for instance [5, Theorems 6.8 and 6.11].) Note that eigenvalues may be repeated in the sequence $(\mu_n)_{n\geq 1}$, since some eigenvalues may be multiple.

THEOREM 1.4.1. *Let Ω be a bounded domain of $\mathbb{R}^N$. There exists a nondecreasing sequence $(\lambda_n)_{n\geq 1} \subset (0, \infty)$ such that $\lambda_n \to \infty$ as $n \to \infty$, and a Hilbert basis $(\varphi_n)_{n\geq 1}$ of $L^2(\Omega)$ such that $(\varphi_n)_{n\geq 1} \subset H^1_0(\Omega)$ and*

$$-\Delta \varphi_n = \lambda_n \varphi_n \tag{1.4.1}$$

in $H^{-1}(\Omega)$. In particular,

$$\|\nabla \varphi_n\|^2_{L^2} = \lambda_n \tag{1.4.2}$$

In addition, $\varphi_n \in L^\infty(\Omega) \cap C^\infty(\Omega)$, for all $n \geq 1$, and there exist a constant C and an integer ℓ such that

$$\|\varphi_n\|_{L^\infty} \leq C(1 + \lambda_j)^\ell \tag{1.4.3}$$

for all $n \geq 1$.

PROOF. Let $f \in H^{-1}(\Omega)$, and $u \in H^1_0(\Omega)$ the solution of $-\Delta u = f$ in $H^{-1}(\Omega)$. We set $u = Kf$. Theorem 1.1.6 shows that K is bounded $H^{-1}(\Omega) \to H^1_0(\Omega)$, hence $L^2(\Omega) \to H^1_0(\Omega)$. Since Ω is bounded, the embedding $H^1_0(\Omega) \hookrightarrow L^2(\Omega)$ is compact (see Theorem B.4.4), therefore K is a compact operator $L^2(\Omega) \to L^2(\Omega)$. We claim that K is self-adjoint. Indeed, let $f, g \in L^2(\Omega)$ and set $u = Kf$, $v = Kg$. It follows that

$$\begin{aligned}(Kf, g)_{L^2} - (f, Kg)_{L^2} &= (u, g)_{L^2} - (f, v)_{L^2} \\ &= (-\Delta v, u)_{H^{-1}, H^1_0} - (-\Delta u, v)_{H^{-1}, H^1_0} = 0\end{aligned}$$

by formula (B.1.8). Next, if $f \in L^2(\Omega)$ and $u = Kf$, then

$$(Kf, f)_{L^2} = (u, -\Delta u)_{H^1_0, H^{-1}} = \int_\Omega |\nabla u|^2 \geq c\|u\|^2_{H^1} \geq c\|f\|^2_{H^{-1}}$$

so that K is a positive operator. Therefore, the eigenvalues of K are a nonincreasing sequence $(\mu_n)_{n\geq 1} \subset (0, \infty)$ such that $\mu_n \to 0$ as $n \to \infty$, and there exists a Hilbert basis $(\varphi_n)_{n\geq 1}$ of H composed of eigenvectors of K, i.e. $K\varphi_n = \mu_n \varphi_n$. In particular, $\mu_n \varphi_n = K\varphi_n \in H^1_0(\Omega)$, so that $(\varphi_n)_{n\geq 1} \subset H^1_0(\Omega)$. In addition, $K\varphi_n = \mu_n \varphi_n$ means that (1.4.1) holds, where

$$\lambda_n = \frac{1}{\mu_n} \underset{n\to\infty}{\longrightarrow} \infty$$

Formula (1.4.2) follows immediately from (1.4.1) and (B.1.8).

To prove that $\varphi_n \in L^\infty(\Omega)$ with the estimate (1.4.3), we write (1.4.1) in the form

$$-\Delta \varphi_n + \varphi_n = (\lambda_n + 1)\varphi_n \tag{1.4.4}$$

We claim that there exist $p > \max\{1, \frac{N}{2}\}$, a constant C and an integer ℓ such that

$$\|\varphi_n\|_{L^p} \leq C(1 + \lambda_j)^\ell \tag{1.4.5}$$

for all $n \geq 0$. The conclusion then follows from Corollary 1.3.4. We now prove the claim (1.4.5). Since $\|\varphi_n\|_{L^2} = 1$, this is verified with $p = 2$ if $N \leq 3$, so we now assume $N \geq 4$. Let the integer $k \geq 1$ be defined by

$$2\Big(\frac{N}{N-2}\Big)^{k-1} \leq \frac{N}{2} < 2\Big(\frac{N}{N-2}\Big)^{k}$$

fix $1 < q \leq 2$ such that

$$q\Big(\frac{N}{N-2}\Big)^{k-1} < \frac{N}{2} < q\Big(\frac{N}{N-2}\Big)^{k} \tag{1.4.6}$$

and set

$$q_j = q\Big(\frac{N}{N-2}\Big)^{j-1}, \quad j = 1, \cdots k+2 \tag{1.4.7}$$

Suppose $\varphi_n \in L^{q_j}(\Omega)$ for some $1 \leq j \leq k+1$. Since $1 < q_j < \frac{N}{2}$, It follows from Theorem 1.3.2 that $\varphi_n \in L^{\frac{Nq_j}{N-2q-J}}(\Omega)$ and $\|\varphi_n\|_{L^{\frac{Nq_j}{N-2q_j}}} \leq C(1+\lambda_n)\|\varphi_n\|_{L^{q_j}}$. Moreover, $q_j \geq 1$, so that

$$\frac{Nq_j}{N-2q_j} \geq \frac{N}{N-2} q_j = q_{j+1}$$

Thus we see that $\varphi_n \in L^{q_{j+1}}(\Omega)$ and $\|\varphi_n\|_{L^{q_{j+1}}} \leq C(1+\lambda_n)\|\varphi_n\|_{L^{q_j}}$. Since $\|\varphi_n\|_{L^{q_1}} \leq C\|\varphi_n\|_{L^2} \leq C$, an obvious induction shows that $\varphi_n \in L^{q_{k+1}}(\Omega)$ and $\|\varphi_n\|_{L^{q_{k+1}}} \leq C(1+\lambda_n)^k$. Since $q_{k+1} > \frac{N}{2}$, the claim (1.4.5) follows.

Finally, since $\varphi_n \in H^1_{\mathrm{loc}}(\Omega)$, it follows from (1.4.4) and Theorem 1.1.9 that $\varphi_n \in H^3_{\mathrm{loc}}(\Omega)$. Applying again Theorem 1.1.9, we obtain $\varphi_n \in H^5_{\mathrm{loc}}(\Omega)$. An obvious iteration shows that $\varphi_n \in H^m_{\mathrm{loc}}(\Omega)$ for all $m \geq 0$, and so $\varphi_n \in C^\infty(\Omega)$ by Corollary B.3.17. □

Corollary 1.4.2. *Let Ω be a bounded domain of $\mathbb{R}^N$, and let $(\lambda_n)_{n\geq 1} \subset (0,\infty)$ and $(\varphi_n)_{n\geq 1}$ be given by Theorem 1.4.1. If Ω satisfies the geometric condition (1.3.16), then there exist a constant C and an integer ℓ such that*

$$|\varphi_n| \leq C(1+\lambda_n)^\ell \mathrm{d}(\cdot, \partial\Omega) \tag{1.4.8}$$

for all $n \geq 1$, and in particular $\varphi_n \in C_0(\Omega)$.

Proof. Estimate (1.4.8) follows from Theorem 1.3.8 applied to the equation (1.4.4), together with estimate (1.4.3). Moreover, $\varphi_n \in C(\Omega)$ by Theorem 1.4.1, and continuity at the boundary follows from (1.4.8). □

Proposition 1.4.3. *Let Ω be a bounded domain of $\mathbb{R}^N$, and let $(\lambda_n)_{n\geq 1} \subset (0,\infty)$ and $(\varphi_n)_{n\geq 1}$ be given by Theorem 1.4.1. Let $u \in H^{-1}(\Omega)$, and set*

$$\alpha_j = \langle u, \varphi_j \rangle_{H^{-1}, H^1_0} \tag{1.4.9}$$

for $j \geq 1$.

(i) *$u \in L^2(\Omega) \Leftrightarrow$ the series $u = \sum \alpha_j \varphi_j$ is convergent in $L^2(\Omega) \Leftrightarrow \sum_{j=1}^\infty \alpha_j^2 < \infty$. In this case, $\|u\|^2_{L^2} = \sum_{j=1}^\infty \alpha_j^2$, i.e. $\|u\|_{L^2} = \|(\alpha_j)_{j\geq 1}\|_{\ell^2}$.*

(ii) *$u \in H^1_0(\Omega)$ if and only if $\sum_{j=1}^\infty \lambda_j \alpha_j^2 < \infty$. In this case, the series $u = \sum \alpha_j \varphi_j$ is convergent in $H^1_0(\Omega)$, and $\sum_{j=1}^\infty \lambda_j \alpha_j^2 = \|\nabla u\|^2_{L^2}$, i.e. $\|u\|_{H^1_0} = \|(\lambda_j^{\frac{1}{2}} \alpha_j)_{j\geq 1}\|_{\ell^2}$.*

(iii) *The series $u = \sum \alpha_j \varphi_j$ is convergent in $H^{-1}(\Omega)$, $\sum_{j=1}^\infty \lambda_j^{-1} \alpha_j^2 < \infty$, and $\|u\|^2_{H^{-1}} = \sum_{j=1}^\infty \lambda_1^{-1} a_j^2$, i.e. $\|u\|_{H^{-1}} = \|(\lambda_j^{-\frac{1}{2}} \alpha_j)_{j\geq 1}\|_{\ell^2}$.*

PROOF. Since $(\varphi_j)_{j\ge1}$ is a Hilbert basis of $L^2(\Omega)$, Property (i) is immediate. Consider now the isometric isomorphism $T:\ell^2(\mathbb{N})\to L^2(\Omega)$ defined by

$$TA=\sum_{j=1}^{\infty}\alpha_j\varphi_j,$$

where $A=(\alpha_j)_{j\ge1}$. Consider the space

$$\mathcal{V}=\{A\in\ell^2(\mathbb{N});\ \sum\lambda_j\alpha_j^2<\infty\},$$

equipped with the norm $\|A\|_{\mathcal{V}}=(\sum_{j=1}^{\infty}\lambda_j\alpha_j^2)^{\frac12}$. It is clear that $\mathcal{V}$ is a Banach (in fact, a Hilbert) space. We first observe that $T(\mathcal{V})\subset H_0^1(\Omega)$ and that

$$\|A\|_{\mathcal{V}}=\|\nabla TA\|_{L^2},\tag{1.4.10}$$

for all $A\in\mathcal{V}$. Indeed, given $A\in\mathcal{V}$, let $(A_n)_{n\ge0}\subset\mathcal{V}$ be defined by $\alpha_{n,j}=\alpha_j$ if $j\le n$ and $\alpha_{n,j}=0$ if $j>n$. We see that $A_n\to A$ in $\mathcal{V}$ (hence in $\ell^2(\mathbb{N})$). In addition,

$$\begin{aligned}\|\nabla TA_n\|_{L^2}^2&=(-\Delta TA_n,TA_n)_{H^{-1},H_0^1}=\Bigl(\sum_{j=1}^{n}\lambda_j\alpha_j\varphi_j,\sum_{j=1}^{n}\alpha_j\varphi_j\Bigr)_{H^{-1},H_0^1}\\&=\Bigl(\sum_{j=1}^{n}\lambda_j\alpha_j\varphi_j,\sum_{j=1}^{n}\alpha_j\varphi_j\Bigr)_{L^2}=\sum_{j=1}^{n}\lambda_j\alpha_j^2=\|A_n\|_{\mathcal{V}}^2.\end{aligned}\tag{1.4.11}$$

Similarly, if $m>n\ge1$, then

$$\|\nabla(TA_m-TA_n)\|_{L^2}^2=\sum_{j=n+1}^{m}\lambda_j\alpha_j^2\underset{n\to\infty}{\longrightarrow}0.\tag{1.4.12}$$

Since $TA_n\in H_0^1(\Omega)$ (finite sum of elements of $H_0^1(\Omega)$), $(TA_n)_{n\ge1}$ is a Cauchy sequence in $H_0^1(\Omega)$; and since $TA_n\to TA$ in $L^2(\Omega)$, we deduce that $TA\in H_0^1(\Omega)$ and $A_n\to A$ in $\mathcal{V}$. Identity (1.4.10) is obtained by letting $n\to\infty$ in (1.4.11). We now prove that $T\mathcal{V}=H_0^1(\Omega)$. Since $\mathcal{V}$ is a Banach space and $T:\mathcal{V}\to H_0^1(\Omega)$ is isometric, $T\mathcal{V}$ is a closed subspace of $H_0^1(\Omega)$. If $\mathcal{V}\neq H_0^1(\Omega)$, then there exists $w\in H_0^1(\Omega)$, $w\neq0$ such that $(TA,w)_{H_0^1}=0$ for all $A\in\mathcal{V}$. Given $j\ge1$, we may choose $TA=\varphi_j$, so that $(\varphi_j,w)_{H_0^1}=0$. Since

$$\begin{aligned}(\varphi_j,w)_{H_0^1}&=(\nabla\varphi_j,\nabla w)_{L^2}=(-\Delta\varphi_j,w)_{H^{-1},H_0^1}\\&=\lambda_j(\varphi_j,w)_{H^{-1},H_0^1}=\lambda_j(\varphi_j,w)_{L^2},\end{aligned}$$

and $j\ge1$ is arbitrary, we deduce that $w=0$. This contradiction shows that $T\mathcal{V}=H_0^1(\Omega)$, from which Property (ii) easily follows.

Finally, we observe that if we identify $\ell^2(\mathbb{N})$ with its dual, then the dual of $\mathcal{V}$ is $\mathcal{W}=\{A\in\ell^2(\mathbb{N});\ \sum\lambda_j^{-1}\alpha_j^2<\infty\}$, equipped with the natural scalar product. One deduces easily that T can be extended to an isometric isomorphism $\mathcal{W}\to(H_0^1(\Omega))^\star=H^{-1}(\Omega)$. Hence Property (iii). □

REMARK 1.4.4. Here are some comments on Proposition 1.4.3.

(i) Given $j\ge1$, the map $u\mapsto\alpha_j$, where α_j is defined by (1.4.9), is continuous $H^{-1}(\Omega)\to\mathbb{R}$. It follows easily that if $I\subset\mathbb{R}$ is an interval and $u\in C(I,H^{-1}(\Omega))$, then the map $t\mapsto\alpha_j(t)$ defined by (1.4.9) is continuous. Moreover, if u is differentiable at some $t\in I$, then every $j\ge1$, α_j is differentiable at t, and

$$\frac{d\alpha_j}{dt}=\Bigl\langle\frac{du}{dt},\varphi_j\Bigr\rangle_{H^{-1},H_0^1}\tag{1.4.13}$$

Therefore, if $u \in C^1(I, H^{-1}(\Omega))$, then the map $t \mapsto \alpha_j(t)$ is C^1 and (1.4.13) holds for all $t \in I$.

(ii) Let $u^1, u^2 \in H^{-1}(\Omega)$, and let $a_j^\ell = \langle u^\ell, \varphi_j \rangle_{H^{-1},H_0^1}$. It follows that

$$(u^1, u^2)_{H^{-1}} = \sum_{j=1}^{\infty} \lambda_j^{-1} a_j^1 a_j^2$$

$$(u^1, u^2)_{L^2} = \sum_{j=1}^{\infty} a_j^1 a_j^2 \quad \text{if } u^1, u^2 \in L^2(\Omega)$$

$$(u^1, u^2)_{H_0^1} = \sum_{j=1}^{\infty} \lambda_j a_j^1 a_j^2 \quad \text{if } u^1, u^2 \in H_0^1(\Omega)$$

$$\langle u^1, u^2 \rangle_{H^{-1},H_0^1} = \sum_{j=1}^{\infty} a_j^1 a_j^2 \quad \text{if } u^2 \in H_0^1(\Omega)$$

PROPOSITION 1.4.5. *Let Ω be a bounded domain of $\mathbb{R}^N$ and $\overline{\lambda}(\Omega)$ be defined by (1.1.7). It follows that there exists $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega) \cap C(\Omega)$, $\varphi > 0$ in Ω, $\|\varphi\|_{L^2} = 1$, such that*

$$-\Delta\varphi = \overline{\lambda}(\Omega)\varphi, \tag{1.4.14}$$

in $H^{-1}(\Omega)$. In addition, the following properties hold.

(i) *φ is the unique nonnegative solution of*

$$u \in S, \quad J(u) = \inf_{v \in S} J(v), \tag{1.4.15}$$

where

$$J(v) = \frac{1}{2} \int_\Omega |\nabla v|^2, \tag{1.4.16}$$

and $S = \{v \in H_0^1(\Omega);\ \|v\|_{L^2} = 1\}$.

(ii) *If $\psi \in H_0^1(\Omega)$ is a solution of (1.4.14), then there exists a constant $c \in \mathbb{R}$ such that $\psi = c\varphi$.*

PROOF. We proceed in five steps.

STEP 1. Problem (1.4.15) has a solution $u \in H_0^1(\Omega)$, $u \geq 0$. We recall that $J \in C^1(H_0^1(\Omega), \mathbb{R})$ and $J'(u) = -\Delta u$ for all $u \in H_0^1(\Omega)$. (See Corollary B.1.21.) Moreover, one verifies easily that if $F(u) = \|u\|_{L^2}^2$, then $F \in C^1(H_0^1(\Omega), \mathbb{R})$ and $F'(u) = 2u$ for all $u \in H_0^1(\Omega)$. On the other hand, it follows from (1.1.7) that

$$\inf_{v \in S} J(v) = \frac{\overline{\lambda}(\Omega)}{2}. \tag{1.4.17}$$

Let $(v_n)_{n \geq 0}$ be a minimizing sequence of (1.4.15). Setting $u_n = |v_n|$, it follows that $u_n \geq 0$ and that $(u_n)_{n \geq 0}$ is also a minimizing sequence. (Apply (B.2.4).) In addition, $\|u_n\|_{L^2} = 1$, so that $(u_n)_{n \geq 0}$ is bounded in $H_0^1(\Omega)$. Therefore, there exist a subsequence $(u_{n_k})_{k \geq 1}$ and $u \in H_0^1(\Omega)$ such that $u_n \to u$ in $L^2(\Omega)$ and $\|\nabla u\|_{L^2} \leq \liminf \|\nabla u_n\|_{L^2}$. In particular, $u \geq 0$ and u is a solution of (1.4.15).

STEP 2. If $u \in H_0^1(\Omega)$ and $\|u\|_{L^2} = 1$, then u is a solution of (1.4.15) iff u is a solution of equation (1.4.14). Suppose u is a solution of (1.4.15). Then there exists a Lagrange multiplier λ such that $-\Delta u = \lambda u$. One can see this easily as follows. Let $w \in H_0^1(\Omega)$ such that $(w, u)_{L^2} = 0$, and set $v = (1 + t^2\|w\|_{L^2}^2)^{-\frac{1}{2}}(u + tw)$, with $t \in \mathbb{R}$. It follows that $\|v\|_{L^2} = 1$, so that $J(u) \leq J(v)$. This means that

$$t(J(u) - J(v)) \leq \int_\Omega \nabla u \cdot \nabla w$$

Letting $t \downarrow 0$, we obtain

$$\int_\Omega \nabla u \cdot \nabla w \geq 0$$

then replacing w by $-w$

$$\int_\Omega \nabla u \cdot \nabla w = 0$$

Consider now $\varphi \in H_0^1(\Omega)$. We may write $\varphi = \eta u + w$ with $(w, u)_{L^2} = 0$ (choose $\eta = (\varphi, u)_{L^2}$). Setting $\lambda = \|\nabla u\|_{L^2}^2$, we see that

$$\langle -\Delta u - \lambda u, \varphi \rangle_{H^{-1}, H_0^1} = \int_\Omega (\nabla u \cdot \nabla \varphi - \lambda u \varphi) = \eta \int_\Omega |\nabla u|^2 - \eta\lambda = 0$$

Since $\varphi \in H_0^1(\Omega)$ is arbitrary, this shows that $-\Delta u = \lambda u$ in $H^{-1}(\Omega)$. Taking the $H^{-1} - H_0^1$ duality product with u, we obtain $2J(u) = \lambda$, so that $\lambda = \overline{\lambda}(\Omega)$ by (1.4.17). Therefore, u is a solution of (1.4.14). Conversely, suppose u is a solution of (1.4.14). Taking the $H^{-1} - H_0^1$ duality product with u, we obtain $2J(u) = \Lambda$, so that by (1.4.17), u is a solution of (1.4.15).

STEP 3. If $u \in H_0^1(\Omega)$, $u \geq 0$, $u \not\equiv 0$, is a solution of (1.4.14), then $u \in L^\infty(\Omega) \cap C(\Omega)$ and $u > 0$ on Ω. Regularity is proved like for Theorem 1.4.1. The fact that $u > 0$ is a consequence of the strong maximum principle.

STEP 4. If $u, v \geq 0$ are two solutions of (1.4.15), then $u = v$. Let $w = u - v$ and assume by contradiction the $w \not\equiv 0$. Since u and v are solutions of (1.4.14), so is w. It follows from Step 2 that $w/\|w\|_{L^2}$ is a solution of (1.4.15), so that $z = |w|/\|w\|_{L^2}$ is also a solution. Steps 2 et 3 show that $z > 0$ in Ω. In particular, w does not vanish in Ω. Therefore, either $w > 0$, or else $w < 0$. Thus either $0 \leq v < u$, or else $0 \leq u < v$, which is absurd since $\|u\|_{L^2} = \|v\|_{L^2}$.

STEP 5. Conclusion. Let $\varphi = u$ with u given by Step 1. Steps 2 et 3 show that φ is a solution of (1.4.14), that $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega) \cap C(\Omega)$, $\varphi > 0$ in Ω and $\|\varphi\|_{L^2} = 1$. Property (i) is then a consequence of Step 4. Finally, suppose $\psi \in H_0^1(\Omega)$, $\psi \not\equiv 0$ is a solution of (1.4.14). Setting $z = |\psi|/\|\psi\|_{L^2}$, we deduce from Step 2 that z is a positive solution of (1.4.15). Hence $z = \varphi$, so that $|\psi| = \|\psi\|_{L^2}\varphi$. In particular, ψ has constant sign. Therefore, $\psi = \pm\|\psi\|_{L^2}\varphi$, which proves (ii). □

COROLLARY 1.4.6. *Let Ω be a bounded domain of $\mathbb{R}^N$, and let $(\lambda_n)_{n\geq 1} \subset (0, \infty)$ and $(\varphi_n)_{n\geq 1}$ be given by Theorem 1.4.1.*

(i) *$\lambda_1 = \overline{\lambda}(\Omega)$, where $\overline{\lambda}(\Omega)$ is defined by (1.1.7).*
(ii) *λ_1 is a simple eigenvalue, and either $\varphi_1 > 0$ or else $\varphi_1 < 0$ in Ω.*
(iii) *Suppose Ω satisfies the geometric conditions (1.3.16) and (1.2.1). If φ_1 is chosen such that $\varphi_1 > 0$, then*

$$c\mathrm{d}(\cdot, \partial\Omega) \leq \varphi_1(\cdot) \leq C\mathrm{d}(\cdot, \partial\Omega) \tag{1.4.18}$$

for some constants $0 < c \leq C < \infty$.
(iv) *If $\psi \in H_0^1(\Omega)$ satisfies $-\Delta\psi = \lambda\psi$ for some $\lambda \in \mathbb{R}$, and if $\psi \geq 0$, then there exists a constant $c \in \mathbb{R}$ such that $\psi = c\varphi_1$.*

PROOF. It follows from Proposition 1.4.5 that $\overline{\lambda}(\Omega)$ is an eigenvalue of $-\Delta$ on $H_0^1(\Omega)$, so $\overline{\lambda}(\Omega) = \lambda_\ell$ for some $\ell \geq 1$. In particular, $\overline{\lambda}(\Omega) \geq \lambda_1$. On the other hand, $\|\varphi_1\|_{L^2} = 1$ and $\|\nabla\varphi_1\|_{L^2}^2 = \lambda_1$ by (1.4.2), so that $\overline{\lambda}(\Omega) \leq \lambda_1$ by (1.1.7). Thus $\overline{\lambda}(\Omega) = \lambda_1$, which proves (i).

By Proposition 1.4.5 (ii), all nontrivial solutions of $-\Delta u = \lambda_1 u$ are multiples of a given one, so that λ_1 is a simple eigenvalue. Moreover, again by Proposition 1.4.5, these solutions are either positive or else negative on Ω, which proves (ii).

The upper estimate in (1.4.18) follows from (1.4.8), and the lower estimate follows from Theorem 1.2.5 (applied to the equation $-\Delta\varphi_1 + \varphi_1 = (1+\lambda_1)\varphi_1$), since $\varphi_1 > 0$.

Finally, suppose $\psi \in H_0^1(\Omega)$ satisfies $-\Delta\psi = \lambda\psi$ for some $\lambda \in \mathbb{R}$, and $\psi \geq 0$, $\psi \not\equiv 0$. It follows that ψ is an eigenvector of $-\Delta$. If $\lambda \neq \lambda_1$, then $\int_\Omega \psi\varphi_1 = 0$, which is absurd because $\varphi_1 > 0$ and $\psi \geq 0$, $\psi \not\equiv 0$. Therefore, $\lambda = \lambda_1$ and Property (iv) follows from Property (1.4.18). □

REMARK 1.4.7. In dimension 1, let $\Omega = (0, \ell)$ with $\ell > 0$. Let

$$\varphi_j(x) = \sqrt{\frac{2}{\ell}} \sin\Big(\frac{j\pi}{\ell}x\Big)$$

for $j \geq 1$. It follows that $\varphi_j'' = \lambda_j\varphi_j$ with $\lambda_j = (\frac{j\pi}{\ell})^2$. Moreover, every solution $u \in H_0^1(\Omega)$ of the equation $-u'' = \lambda u$ is given by the above formula for some $j \geq 1$. This shows that the spectral decomposition of the Laplacian on $H_0^1(0,\ell)$ is (φ_j, λ_j) where φ_j and λ_j are as above. In particular, if $\Omega = (0,\pi)$, then $\varphi_j(x) = \sin(jx)$ and $\lambda_j = j^2$, and the above analysis corresponds to the Fourier series.

REMARK 1.4.8. Suppose $N = 3$, and let $\Omega = B_R$, the ball of center 0 and radius R in $\mathbb{R}^3$. Let

$$\varphi(r) = \frac{1}{r}(2\pi R)^{\frac{-1}{2}} \sin\Big(\frac{\pi}{R}r\Big)$$

so that $\varphi > 0$ and $\|\varphi\|_{L^2(\Omega)} = 1$. It is not difficult to check that $\varphi \in H_0^1(\Omega)$ and that $-\Delta\varphi = (\frac{\pi}{R})^2\varphi$. Since $\varphi > 0$, it follows that $\varphi = \varphi_1$ and $(\frac{\pi}{R})^2 = \lambda_1$. Note that for every integer $k \geq 1$, $\psi(r) = \frac{1}{r}(2\pi R)^{\frac{-1}{2}}\sin(\frac{k\pi}{R}r)$ is an eigenvector of $-\Delta$ corresponding to the eigenvalue $\lambda = (\frac{k\pi}{R})^2$. However, this does not give all the eigenvectors and eigenvalues of $-\Delta$, since there are eigenvalues corresponding to non-radial eigenvectors.

It follows from Remarks 1.4.7 and 1.4.8 that the first eigenvector of $-\Delta$ in $H_0^1(B_R)$ is radially symmetric and decreasing in dimensions $N = 1$ and $N = 3$. In fact, this property is true in any dimension, as shows the following result.

PROPOSITION 1.4.9. *Let $\Omega = B_R$, the ball of center 0 and radius R in $\mathbb{R}^N$, and let λ_1 be the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$ and $\varphi_1 > 0$ a corresponding first eigenvector.*

(i) *$\varphi_1 \in C^\infty(\overline{\Omega})$ is radially symmetric and decreasing.*

(ii) *There exist two constants $c = c(N)$ and $C = C(N)$ such that*

$$cR^{-1}(R - |x|) \leq \varphi_1(x) \leq CR^{-1}(R-|x|) \tag{1.4.19}$$

for all $x \in \Omega$ if φ_1 is normalized by the condition $\varphi_1(0) = 1$, and

$$cR^{-\frac{N+2}{2}}(R - |x|) \leq \varphi_1(x) \leq CR^{-\frac{N+2}{2}}(R-|x|) \tag{1.4.20}$$

for all $x \in \Omega$ if φ_1 is normalized by the condition $\|\varphi_1\|_{L^2} = 1$

(iii) *There exists a constant $\lambda = \lambda(N)$ such that $\lambda_1 = R^{-2}\lambda$.*

PROOF. Consider the solution of the ODE

$$\begin{cases} z'' + \frac{N-1}{r}z' + u = 0 \\ z(0) = 1, z'(0) = 0 \end{cases} \tag{1.4.21}$$

We claim that

$$\exists \rho > 0 \text{ s.t. } z \text{ is positive, decreasing on } (0,\rho),\ z(\rho) = 0 \text{ and } z'(\rho) < 0 \tag{1.4.22}$$

We complete the proof, assuming the claim (1.4.22). It follows that there exist two constants $0 < c < C < \infty$ (depending only on N) such that

$$c(\rho - r) \leq z(r) \leq C(\rho - r) \quad 0 \leq r \leq \rho$$

Setting

$$\psi(x) = z\Big(\frac{\rho}{R}|x|\Big)$$

we deduce that

$$\frac{c\rho}{R}(R-|x|) \le \psi(x) \le \frac{C\rho}{R}(R-|x|) \quad 0 \le r \le R \tag{1.4.23}$$

Moreover, ψ is radially symmetric and decreasing, and one verifies easily that $\psi \in C^\infty(\overline{\Omega})$ (so that $\psi \in H^1_0(\Omega)$, since $\psi_{|\partial\Omega} = 0$), and that $-\Delta\psi = (\frac{\rho}{R})^2\psi$. Therefore, ψ is a multiple of the φ_1 and $\lambda_1 = (\frac{\rho}{R})^2$, by Corollary 1.4.2 (iv). Since $\psi(0) = 1$, estimate (1.4.19) follows from (1.4.23). Estimate (1.4.20) also follows from (1.4.23) (replacing ψ by $\frac{\psi}{\|\psi\|_{L^2}}$), since $\|\psi\|_{L^2} \approx R^{\frac{N}{2}}$. This completes the proof.

We now prove the claim (1.4.22). It follows easily from the equation (1.4.21) that $z'(r) < 0$ for $r > 0$ sufficiently small, and that z' cannot vanish while z remains > 0. If z vanishes, then $z' \ne 0$ (otherwise $z \equiv 0$), and (1.4.22) is proved. Otherwise, $z' < 0$ and $z > 0$ for all $r > 0$. It follows that z decreases to a nonnegative limit as $r \to \infty$, and it one deduces easily from the equation (1.4.21) that this limit must be 0. Next, we write (1.4.21) in the form $(r^{N-1}z')' = -r^{N-1}z$. Integrating on $(0, r)$ and since z is decreasing, we obtain

$$-r^{N-1}z'(r) = \int_0^r s^{N-1}z(s) \ge z(r)\int_0^r s^{N-1} = \frac{1}{N}r^N z(r)$$

This means that $z' + rz \le 0$, so that $(e^{\frac{r^2}{2}}z)' \le 0$. Thus $0 \le z(r) \le e^{-\frac{r^2}{2}}$, and in particular

$$e^r z(r) \underset{r\to\infty}{\longrightarrow} 0$$

On the other hand,

$$[e^r z(r)]' = e^r[z'' + 2z' + z] = e^r\Big(2 - \frac{N-1}{r}\Big)z' \le 0$$

for $r \ge \frac{N-1}{2}$, since $z' < 0$. We deduce that $e^r z(r)$ is a positive, concave function for r large, that converges to 0 at infinity. This is absurd, so the claim (1.4.22) is proved. □

REMARK 1.4.10. Suppose Ω is a bounded domain of $\mathbb{R}^N$ and let λ satisfy (1.1.8) (i.e. $\lambda > -\lambda_1$). Given $f \in H^{-1}(\Omega)$, it follows from Theorem 1.1.6 that there exists a unique weak solution $u \in H^1_0(\Omega)$ of (1.1.8). The solution u can be expressed in terms of the spectral decomposition of the Laplacian. More precisely, let $\beta_j = \langle f, \varphi_j\rangle_{H^{-1},H^1_0}$ and $\alpha_j = \langle u, \varphi_j\rangle_{H^{-1},H^1_0}$ for $j \ge 1$. Letting $\varphi = \varphi_j$ in (1.1.5) and applying (B.1.8), we deduce that $(\lambda + \lambda_j)\alpha_j = \beta_j$. In other words,

$$u = \sum_{j=1}^\infty \frac{\langle f, \varphi_j\rangle_{H^{-1}}}{\lambda + \lambda_j}\varphi_j$$

where the series is convergent in $H^1_0(\Omega)$ (see Proposition 1.4.3).

CHAPTER 2

The heat equation

The heat equation is the prototype parabolic equation. Among its fundamental properties are the maximum principle, and the smoothing effect. We first study the equation set on the whole space $\mathbb{R}^N$, by using the Fourier transform. Then we study the equation set on a bounded domain, with Dirichlet boundary conditions, by using Fourier series associated with the spectral decomposition of the Laplacian.

2.1. The heat equation on $\mathbb{R}^N$

We consider the following (Cauchy) problem

$$\begin{cases} u_t = \Delta u & t > 0, x \in \mathbb{R}^N \\ u(0,x) = u_0(x) & x \in \mathbb{R}^N \end{cases} \tag{2.1.1}$$

2.1.1. Existence and regularity. We begin with an existence and uniqueness result in the space $\mathcal{S}'(\mathbb{R}^N)$.

THEOREM 2.1.1. *Given any $u_0 \in \mathcal{S}'(\mathbb{R}^N)$, there exists a unique solution $u \in C^1([0,\infty), \mathcal{S}'(\mathbb{R}^N))$ of equation (2.1.1). This solution is given, for all $t > 0$, by*

$$\widehat{u}(t) = e^{-4\pi^2 t|\xi|^2} \widehat{u_0} \tag{2.1.2}$$

where $\widehat{u} = \mathcal{F}u$ is the Fourier transform of u or, equivalently, by

$$u(t,x) = G_t \star u_0 \tag{2.1.3}$$

where $G \in C^\infty((0,\infty), \mathcal{S}(\mathbb{R}^N))$ is defined by

$$G_t(x) = (4\pi t)^{-\frac{N}{2}} e^{-\frac{|x|^2}{4t}} \tag{2.1.4}$$

(G_t is called the Gauss kernel.) Moreover, the following properties hold.

(i) *$u \in C^\infty([0,\infty), \mathcal{S}'(\mathbb{R}^N))$*
(ii) *For all $t > 0$, $u(t)$ is a function of $C^\infty(\mathbb{R}^N)$.*
(iii) *if $u_0 \in \mathcal{S}(\mathbb{R}^N)$, then $u \in C^\infty([0,\infty), \mathcal{S}(\mathbb{R}^N))$*
(iv) *$\partial_t^m u = \Delta^m u$ for all $t \geq 0$ and all $m \in \mathbb{N}$.*

REMARK 2.1.2. Note that the convolution (2.1.3) is well defined for all $t > 0$ (convolution of a tempered distribution with a fonction of $\mathcal{S}(\mathbb{R}^N)$, see Theorem B.6.14).

PROOF OF THEOREM 2.1.1. We first prove uniqueness. This is the more delicate part of the proof, as we claim uniqueness in the large class $C^1([0,\infty), \mathcal{S}'(\mathbb{R}^N))$. We consider $u, v \in C^1([0,\infty), \mathcal{S}'(\mathbb{R}^N))$ two solutions of (2.1.1). Setting $w = u - v$ and $\widehat{w} = \mathcal{F}w$, it follows that $\widehat{w} \in C^1([0,\infty), \mathcal{S}'(\mathbb{R}^N))$ (see Remark B.6.12) and that $\widehat{w}$ solves the equation

$$\widehat{w}_t + 4\pi^2|\xi|^2\widehat{w} = 0$$

(See (B.6.42).) Moreover $\widehat{w}(0) = 0$. If the above equation were an ODE, it would be immediate that $\widehat{w} = 0$. However, the equation makes sense in $C^1([0,\infty, \mathcal{S}'(\mathbb{R}^N))$, so some care must be taken. We fix $\theta \in \mathcal{S}(\mathbb{R}^N)$ and $\tau > 0$, and we define $\varphi \in C^1([0,\tau],\mathcal{S}(\mathbb{R}^N))$ by

$$\varphi(t,\xi) = e^{-4\pi^2|\xi|^2(\tau-t)}\theta(\xi)$$

for $0 \le t \le \tau$ and $\xi \in \mathbb{R}^N$, so that $\varphi'(t) = 4\pi^2|\cdot|^2\varphi(t)$. It follows from (B.6.50) that

$$\begin{aligned}\frac{d}{dt}\langle \widehat{w}(t),\varphi(t)\rangle &= \langle \widehat{w}'(t),\varphi(t)\rangle + \langle \widehat{w}(t),\varphi'(t)\rangle \\ &= -\langle 4\pi^2|\cdot|^2\widehat{w}(t),\varphi(t)\rangle + \langle \widehat{w}(t),4\pi^2|\cdot|^2\varphi(t)\rangle = 0\end{aligned}$$

by (B.6.33). Therefore, $\langle \widehat{w}(\tau),\varphi(\tau)\rangle = \langle \widehat{w}(0),\varphi(0)\rangle = 0$, since $\widehat{w}(0) = 0$. This means that $\langle \widehat{w}(\tau),\theta\rangle = 0$. Since $\theta \in \mathcal{S}(\mathbb{R}^N)$ and $\tau > 0$ are arbitrary, we conclude that $\widehat{w} = 0$. Therefore, $w = 0$, hence $u = v$.

We next prove existence. Let $u_0 \in \mathcal{S}'(\mathbb{R}^N)$, so that $\widehat{u_0} \in \mathcal{S}'(\mathbb{R}^N)$. It follows from Remark B.6.10 (v) that $e^{-4\pi^2t|\xi|^2}\widehat{u_0} \in \mathcal{S}'(\mathbb{R}^N)$ for all $t \ge 0$. Therefore, formula (2.1.2) defines $u(t) \in \mathcal{S}'(\mathbb{R}^N)$ for all $t \ge 0$. In addition, it is not difficult to check that, given any $\varphi \in \mathcal{S}(\mathbb{R}^N)$

$$\text{the map } t \mapsto e^{-4\pi^2t|\xi|^2}\varphi \text{ is } C^\infty : [0,\infty) \to \mathcal{S}(\mathbb{R}^N) \tag{2.1.5}$$

It follows (see e.g. Proposition B.6.15) that formula (2.1.2) defines a function $\widehat{u} \in C^\infty([0,\infty),\mathcal{S}'(\mathbb{R}^N))$, hence $u \in C^\infty([0,\infty),\mathcal{S}'(\mathbb{R}^N))$. Moreover, $\widehat{u}$ satisfies the equation $\widehat{u}_t + 4\pi^2|\xi|^2\widehat{u} = 0$, which means that $u_t = \Delta u$. Since $\widehat{u}(0) = \widehat{u_0}$, we have $u(0) = u_0$, so that u is a solution of (2.1.1).

So far, we have established existence, uniqueness, formula (2.1.2), and the regularity property (i). Property (iii) is an immediate consequence of (2.1.5) (with $\varphi = \widehat{u_0}$).

Next, we prove formula (2.1.3) and the regularity property (ii). We apply the inverse Fourier transform to formula (2.1.2), and we note that for all $t > 0$

$$\mathcal{F}^{-1}[e^{-4\pi^2t|\cdot|^2}] = G_t$$

where G_t is defined by (2.1.4). (Apply (B.6.9) with $a = 4\pi t$.) Since

$$u = \mathcal{F}^{-1}\widehat{u} = \mathcal{F}^{-1}[e^{-4\pi^2t|\cdot|^2}] \star \mathcal{F}^{-1}\widehat{u_0} = \mathcal{F}^{-1}[e^{-4\pi^2t|\cdot|^2}] \star u_0$$

(see Theorem B.6.14 (iv)) we see that for every $t > 0$, (2.1.2) is equivalent to (2.1.3). It follows in particular (see Theorem B.6.14 (i)) that $u(t) \in C^\infty(\mathbb{R}^N)$ for all $t > 0$.

Finally, we prove Property (iv). Let $t \ge 0$ and $h \ne 0$ such that $t+h \ge 0$. Given $\varphi \in \mathcal{S}(\mathbb{R}^N)$,

$$\langle u_t(t+h) - u_t(t),\varphi\rangle = \langle \Delta u(t+h) - \Delta u(t),\varphi\rangle = \langle u(t+h) - u(t),\Delta\varphi\rangle$$

Letting $h \to 0$, we obtain

$$\langle \partial_t^2 u,\varphi\rangle = \langle u_t,\Delta\varphi\rangle = \langle \Delta u,\Delta\varphi\rangle = \langle \Delta^2 u,\varphi\rangle$$

Thus we see that $\partial_t^2 u = \Delta^2 u$, and Property (iv) follows by an obvious iteration argument. ☐

REMARK 2.1.3. Here are some comments on Theorem 2.1.1

(i) Let $u_0 \in \mathcal{S}'(\mathbb{R}^N)$ and u the corresponding solution of (2.1.1). Given $\tau > 0$, it follows that $v(t) = u(\tau + t)$ satisfies $v_t = \Delta v$ in $C([0,\infty),\mathcal{S}'(\mathbb{R}^N))$ and $v(0) = u(\tau)$. By uniqueness, it follows that the solution v of (2.1.1) with the initial condition $v(0) = u(\tau)$ is given by $v(t) = u(t + \tau)$.

(ii) The proof of uniqueness shows in fact the following stronger uniqueness property. If $T > 0$ and $u \in C^1([0,T],\mathcal{S}'(\mathbb{R}^N))$ satisfies $u_t = \Delta u$ for all $0 \le t \le T$ and if $u(0) = 0$, then $u(t) = 0$ for all $0 \le t \le T$.

We now describe some regularity properties that follow from formulas (2.1.2) and (2.1.3). The results below show that the regularity in Sobolev spaces is preserved by the heat equation. Moreover, they show that the heat equation has a remarkable smoothing effect, in the sense that for $t > 0$, $u(t)$ is considerably smoother than the initial value $u(0)$.

Corollary 2.1.4. *Let $u_0 \in \mathcal{S}'(\mathbb{R}^N)$ and $u \in C^1([0,\infty),\mathcal{S}'(\mathbb{R}^N))$ the corresponding solution of (2.1.1). If $u_0 \in L^p(\mathbb{R}^N)$ for some $1 \le p \le \infty$, then $u(t) \in L^q(\mathbb{R}^N)$ all $t > 0$ and all $p \le q \le \infty$. Moreover,*

$$\|u(t)\|_{L^q} \le (4\pi t)^{-\frac{N}{2}(\frac{1}{p}-\frac{1}{q})}\|u_0\|_{L^p} \tag{2.1.6}$$

for all $t > 0$ and all $p \le q \le \infty$.

Proof. Note that

$$\|G_t\|_{L^r} = r^{-\frac{N}{2r}}(4\pi t)^{-\frac{N}{2}(1-\frac{1}{r})} \le (4\pi t)^{-\frac{N}{2}(1-\frac{1}{r})} \tag{2.1.7}$$

for all $1 \le r \le \infty$. Applying Young's inequality to formula (2.1.3), we obtain

$$\|u(t)\|_{L^q} \le \|G_t\|_{L^r}\|u_0\|_{L^q}$$

where r is given by $1 - \frac{1}{r} = \frac{1}{p} - \frac{1}{q}$. Hence the result. □

Remark 2.1.5. Here are some comments on Corollary 2.1.4.

(i) If $u_0 \in L^p(\mathbb{R}^N)$ with $1 \le p \le \infty$, $u_0 \ge 0$, $u_0 \not\equiv 0$, then formula (2.1.3) shows that $u(t,x) > 0$ for all $t > 0$ and $x \in \mathbb{R}^N$.

(ii) Note that $u(t)$ is the convolution of u_0 with the radially symmetric and decreasing kernel G_t. It follows that if u_0 is radially symmetric, then so is $u(t)$ for all $t \ge 0$; and if u_0 is radially symmetric and decreasing, then so is $u(t)$ for all $t \ge 0$. See Proposition A.2.2.

(iii) It follows from (2.1.6) that if, for instance, $u_0 \in L^1(\mathbb{R}^N)$, then $\|u(t)\|_{L^\infty} \le Ct^{-\frac{N}{2}}$. This decay cannot, in general, be improved. Indeed, if $u_0(x) = e^{-|x|^2}$, then $u(t,x) = (1+4t)^{-\frac{N}{2}}e^{-\frac{|x|^2}{1+at}}$ (see Remark 2.1.10 (iv) below). In particular, $\|u(t)\|_{L^\infty} = (1+4t)^{-\frac{N}{2}}$, which is precisely the decay given by (2.1.6).

(iv) Let $u_0 \in L^1(\mathbb{R}^N)$ and u the corresponding solution of (2.1.1). It follows from (2.1.6) that $\|u(t)\|_{L^p} \to 0$ as $t \to \infty$ for every $p > 1$. In general, $\|u(t)\|_{L^1} \not\to 0$ as $t \to \infty$. Indeed,

$$\int_{\mathbb{R}^N} u(t,x)\,dx = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} G_t(y)u_0(x-y) = \Big(\int_{\mathbb{R}^N} G_t\Big)\Big(\int_{\mathbb{R}^N} u_0\Big) = \int_{\mathbb{R}^N} u_0$$

If $u_0 \ge 0$ and $u_0 \not\equiv 0$ then $u(t) > 0$ by Property (i), so that $\|u(t)\|_{L^1} = \int u(t) = \int u_0 > 0$.

Corollary 2.1.6. *Let $u_0 \in \mathcal{S}'(\mathbb{R}^N)$ and $u \in C^1([0,\infty),\mathcal{S}'(\mathbb{R}^N))$ the corresponding solution of (2.1.1). If $u_0 \in H^{s,p}(\mathbb{R}^N)$ for some $s \in \mathbb{R}$ and $1 < p < \infty$, then the following properties hold.*

(i) *$u \in C([0,\infty),H^{s,p}(\mathbb{R}^N))) \cap C((0,\infty),H^{s,q}(\mathbb{R}^N)))$ for all $p \le q \le \infty$, and*

$$\|u(t)\|_{H^{s,q}} \le (4\pi t)^{-N(\frac{1}{p}-\frac{1}{q})}\|u_0\|_{H^{s,p}} \tag{2.1.8}$$

for all $t > 0$.

(ii) *$u \in C^k([0,\infty),H^{s-2k,p}(\mathbb{R}^N)) \cap C^k((0,\infty),H^{s-2k,q}(\mathbb{R}^N))$ for all $k \in \mathbb{N}$ and $p \le q \le \infty$, and*

$$\|\partial_t^k u(t)\|_{H^{s-2k,q}} \le Ct^{-N(\frac{1}{p}-\frac{1}{q})}\|u_0\|_{H^{s,p}} \tag{2.1.9}$$

for all $t > 0$, where the constant C is independent of $t > 0$ and u_0.

(iii) *$u \in C^\infty((0,\infty), H^{\sigma,q}(\mathbb{R}^N))$ for all $\sigma \in \mathbb{R}$ and $q \ge p$. In addition*

$$\|D^\beta \partial_t^k u(t)\|_{H^{s,q}} \le C t^{-\frac{N}{2}(\frac{1}{p}-\frac{1}{q})-\frac{|\beta|}{2}-k} \|u_0\|_{H^{s,p}} \tag{2.1.10}$$

for all multi-indices β, all $k \in \mathbb{N}$ and all $t > 0$. In particular, $u \in C^\infty((0,\infty)\times \mathbb{R}^N)$. Moreover,

$$\|\partial_t^k u(t)\|_{H^{s+m,q}} \le C t^{-\frac{N}{2}(\frac{1}{p}-\frac{1}{q})-k} (1 + t^{-\frac{m}{2}}) \|u_0\|_{H^{s,p}} \tag{2.1.11}$$

for all $m, k \in \mathbb{N}$ and all $t > 0$.

PROOF. Fix $s \in \mathbb{R}$ and $1 \le p < \infty$. Suppose first $u_0 \in \mathcal{S}(\mathbb{R}^N)$, so that $u \in C^\infty([0,\infty), \mathcal{S}(\mathbb{R}^N))$. Let β be a multi-index. It follows from (2.1.3) that

$$D^\beta u(t) = (D^\beta G_t) \star u_0 \tag{2.1.12}$$

Let $\sigma \in \mathbb{R}$ and $p \le q \le \infty$, and let $1 \le r \le \infty$ be given by

$$\frac{1}{r} = \frac{1}{q} - \frac{1}{p} + 1 \tag{2.1.13}$$

Since $\partial_t^\mu u = (\Delta)^\mu u$ by Theorem 2.1.1 (iv), it follows from (B.6.98) that

$$\|D^\beta \partial_t^\mu u(t)\|_{H^{\sigma-2\mu,q}} = \|(\Delta)^\mu D^\beta u(t)\|_{H^{\sigma-2\mu,q}} \le \|D^\beta u(t)\|_{H^{\sigma,q}}$$

Applying (2.1.12) and (B.6.91) we deduce that

$$\|D^\beta \partial_t^\mu u(t)\|_{H^{\sigma-2\mu,q}} \le \|D^\beta G_t\|_{H^{\sigma-s,r}} \|u_0\|_{H^{s,p}} \tag{2.1.14}$$

for all $t > 0$. (Note that $D^\beta G_t \in \mathcal{S}(\mathbb{R}^N) \subset H^{\sigma-s,r}(\mathbb{R}^N)$, so that $\|D^\beta G_t\|_{H^{\sigma-s,r}} < \infty$.) Since $H^{0,r}(\mathbb{R}^N) = L^r(\mathbb{R}^N)$, we see by letting $\sigma = s$ in (2.1.14) that

$$\|D^\beta \partial_t^\mu u(t)\|_{H^{s-2\mu,q}} \le \|D^\beta G_t\|_{L^r} \|u_0\|_{H^{s,p}} \tag{2.1.15}$$

In particular, if $\beta = 0$ and $\mu = 0$, we obtain by applying (2.1.7) and (2.1.13) that

$$\|u(t)\|_{H^{s,q}} \le (4\pi t)^{-\frac{N}{2}(\frac{1}{p}-\frac{1}{q})} \|u_0\|_{H^{s,p}} \tag{2.1.16}$$

for all $t > 0$. Moreover, writing $G_t(x) = t^{-\frac{N}{2}} \Phi(\frac{x}{\sqrt{t}})$, where $\Phi(y) = (4\pi)^{-\frac{N}{2}} e^{-\frac{|y|^2}{4}}$, we see that $D^\beta G_t(x) = t^{-\frac{N}{2}} t^{-\frac{|\beta|}{2}} D^\beta \Phi(\frac{x}{\sqrt{t}})$. Hence

$$\|D^\beta G_t\|_{L^r} = t^{-\frac{N}{2}(1-\frac{1}{r})} t^{-\frac{|\beta|}{2}} \|D^\beta \Phi\|_{L^r} = t^{-\frac{N}{2}(\frac{1}{p}-\frac{1}{q})} t^{-\frac{|\beta|}{2}} \|D^\beta \Phi\|_{L^r}$$

and it follows from (2.1.15) that

$$\|D^\beta \partial_t^\mu u(t)\|_{H^{s-2\mu,q}} \le C t^{-\frac{N}{2}(\frac{1}{p}-\frac{1}{q})} t^{-\frac{|\beta|}{2}} \|u_0\|_{H^{s,p}} \tag{2.1.17}$$

for all $t > 0$.

We now consider $u_0 \in H^{s,p}(\mathbb{R}^N)$ with $s \in \mathbb{R}$ and $1 \le p < \infty$. By density of $\mathcal{S}(\mathbb{R}^N)$ in $H^{s,p}(\mathbb{R}^N)$ (see Remark B.6.23 (vii)), there exists a sequence $(u_0^n)_{n\ge1} \subset \mathcal{S}(\mathbb{R}^N)$ such that

$$\|u_0^n - u_0\|_{H^{s,p}} \underset{n\to\infty}{\longrightarrow} 0 \tag{2.1.18}$$

We denote by u^n the corresponding solutions of (2.1.1). It follows from (2.1.2) that

$$u^n(t) \underset{n\to\infty}{\longrightarrow} u(t) \quad \text{in } \mathcal{S}'(\mathbb{R}^N) \tag{2.1.19}$$

for all $t \ge 0$. We first apply (2.1.16) with $q = p$, and u_0 replaced by u_0^n, then by $u_0^m - u_0^n$, and we obtain

$$\|u^n(t)\|_{H^{s,p}} \le \|u_0^n\|_{H^{s,p}} \tag{2.1.20}$$

$$\|u^m(t) - u^n(t)\|_{H^{s,p}} \le \|u_0^m - u_0^n\|_{H^{s,p}} \tag{2.1.21}$$

It follows from (2.1.21) that, given any $T > 0$, $(u^n)_{n\ge1}$ is a Cauchy sequence in the Banach space $C([0,T], H^{s,p}(\mathbb{R}^N))$. Applying (2.1.19), we deduce that $u \in C([0,T], H^{s,p}(\mathbb{R}^N))$ and $u^n \to u$ in $C([0,T], H^{s,p}(\mathbb{R}^N))$. Letting $n \to \infty$ in (2.1.20) yields (2.1.8) for $q = p$ (since $T > 0$ is arbitrary). This proves the case $q = p$ of Property (i). For $q > p$, we apply (2.1.16) with u_0 replaced by u_0^n and $u_0^m - u_0^n$, respectively, and we obtain

$$\|u^n(t)\|_{H^{s,q}} \le (4\pi t)^{-\frac{N}{2}(\frac{1}{p}-\frac{1}{q})}\|u_0^n\|_{H^{s,p}} \tag{2.1.22}$$

$$\|u^m(t) - u^n(t)\|_{H^{s,q}} \le (4\pi t)^{-\frac{N}{2}(\frac{1}{p}-\frac{1}{q})}\|u_0^m - u_0^n\|_{H^{s,p}} \tag{2.1.23}$$

It follows that, given any $0 < \tau < T < \infty$, $(u^n)_{n\ge1}$ is a Cauchy sequence in the Banach space $C([\tau, T], H^{s,q}(\mathbb{R}^N))$, and we conclude as above that Property (i) holds as well in the case $q > p$.

Property (ii) is proved like Property (i), but instead of applying (2.1.16), one applies (2.1.17) with $\beta = 0$.

To prove Property (iii), we first fix $\varepsilon > 0$, $\sigma \in \mathbb{R}$ and $p \le q \le \infty$. Applying (2.1.14) with $\beta = 0$, $\mu = 0$ and $t = \varepsilon$, we obtain

$$\|u(\varepsilon)\|_{H^{\sigma,q}} \le \|G_\varepsilon\|_{H^{\sigma-s,r}}\|u_0\|_{H^{s,p}} \tag{2.1.24}$$

Arguing by density as above, we deduce that $u(\varepsilon) \in H^{\sigma,q}(\mathbb{R}^N)$. Note that $\varepsilon > 0$, $\sigma \in \mathbb{R}$ and $q \ge p$ are arbitrary, so that the first statement of Property (iii) follows from Property (ii), since the solution $v(t)$ of (2.1.1) with the initial value $v(0) = u(\varepsilon)$ is $v(t) = u(t+\varepsilon)$. (See Remark 2.1.3.) To prove the estimate (2.1.10), we note that $D^\beta\partial_t^k u(t) = D^\beta\Delta^k u(t)$, so that by (2.1.17) (applied with $\mu = 0$ and β replaced by $\widetilde{\beta}$ with $|\widetilde{\beta}| = |\beta| + 2k$) yields (2.1.10). Finally, (2.1.11) follows from (2.1.10) and the fact that $\|u\|_{H^{s+m,p}} \approx \sum_{|\beta|\le m}\|D^\beta u\|_{H^{s,p}}$ by Proposition B.6.30. □

COROLLARY 2.1.7. *Let $u_0 \in C_0(\mathbb{R}^N)$ and $u \in C^1([0,\infty), \mathcal{S}'(\mathbb{R}^N))$ the corresponding solution of (2.1.1). It follows that $u \in C([0,\infty), C_0(\mathbb{R}^N))$.*

PROOF. We first consider the case $u_0 \in C_c^\infty(\mathbb{R}^N)$. In particular, $u_0 \in H^N(\mathbb{R}^N)$, so that $u \in C([0,\infty), H^N(\mathbb{R}^N))$ by Corollary 2.1.6. Since $H^N(\mathbb{R}^N) \hookrightarrow C_0(\mathbb{R}^N)$, we see that $u \in C([0,\infty), C_0(\mathbb{R}^N))$.

The general case follows by density and estimate (2.1.6). Indeed, let $u_0 \in C_0(\mathbb{R}^N)$. Let $(u_0^n)_{n\ge1} \subset C_c^\infty(\mathbb{R}^N)$ satisfy $u_0^n \to u_0$ in $C_0(\mathbb{R}^N)$ and let u^n be the corresponding solutions of (2.1.1). We have seen that $u^n \in C([0,\infty), C_0(\mathbb{R}^N))$. On the other hand, it follows from (2.1.6) (with $p = q = \infty$) that $u^n \to u$ in $L^\infty((0,\infty)\times\mathbb{R}^N)$. In particular, $u^n \to u$ in $L^\infty((0,T)\times\mathbb{R}^N)$ for every $0 < T < \infty$. Since $C([0,T], C_0(\mathbb{R}^N))$ is a Banach space with the norm of $L^\infty((0,T)\times\mathbb{R}^N)$, we conclude that $u \in C([0,T], C_0(\mathbb{R}^N))$ for all $0 < T < \infty$. □

2.1.2. The heat semigroup. The heat semigroup, often denoted $(e^{t\Delta})_{t\ge0}$, is the family of operators $\mathcal{S}'(\mathbb{R}^N) \to \mathcal{S}'(\mathbb{R}^N)$ defined as follows for every $t \ge 0$: given $u_0 \in \mathcal{S}'(\mathbb{R}^N)$, $\mathcal{T}(t)u_0$ is the solution at time t of equation (2.1.1).

PROPOSITION 2.1.8. *If $(\mathcal{T}(t))_{t\ge0}$ is as defined above, then the following properties hold.*

(i) $\mathcal{T}(t) \in \mathcal{L}(\mathcal{S}'(\mathbb{R}^N))$ *and* $\mathcal{T}(t) \in \mathcal{L}(\mathcal{S}(\mathbb{R}^N))$ *for all* $t \ge 0$.
(ii) $\mathcal{T}(0) = I$.
(iii) $\mathcal{T}(t+s) = \mathcal{T}(t)\mathcal{T}(s)$, *for all* $s, t \ge 0$.
(iv) $\forall u_0 \in \mathcal{S}'(\mathbb{R}^N)$, *the map* $t \mapsto \mathcal{T}(t)u_0$ *is* $C^\infty([0,\infty), \mathcal{S}'(\mathbb{R}^N))$ *and* $\frac{d}{dt}\mathcal{T}(t)u_0 = \Delta\mathcal{T}(t)u_0$ *for all* $t \ge 0$.
(v) $\forall u_0 \in \mathcal{S}(\mathbb{R}^N)$, *the map* $t \mapsto \mathcal{T}(t)u_0$ *is* $C^\infty([0,\infty), \mathcal{S}(\mathbb{R}^N))$.
(vi) $\langle\mathcal{T}(t)\varphi, \psi\rangle_{\mathcal{S}',\mathcal{S}} = \langle\varphi, \mathcal{T}(t)\psi\rangle_{\mathcal{S}',\mathcal{S}}$ *for all* $\varphi \in \mathcal{S}'(\mathbb{R}^N)$ *and* $\psi \in \mathcal{S}(\mathbb{R}^N)$.

PROOF. Properties (ii), (iv) and (v) follow from Theorem 2.1.1, and Property (iii) from Remark 2.1.3 (i). Next, we note that given $t > 0$, $\mathcal{T}(t)u = G_t \star u$ by formula (2.1.3). Since convolution with a function of $\mathcal{S}(\mathbb{R}^N)$ is a continuous map $\mathcal{S}(\mathbb{R}^N) \to \mathcal{S}(\mathbb{R}^N)$ and $\mathcal{S}'(\mathbb{R}^N) \to \mathcal{S}'(\mathbb{R}^N)$ (see Remark B.6.4 (v) and Theorem B.6.14), Property (ii) follows. Finally, given $\varphi \in \mathcal{S}'(\mathbb{R}^N)$, $\psi \in \mathcal{S}(\mathbb{R}^N)$ and $t > 0$, it follows from Theorem B.6.14 that

$$\langle \mathcal{T}(t)\varphi, \psi \rangle_{\mathcal{S}',\mathcal{S}} = \langle G_t \star \varphi, \psi \rangle_{\mathcal{S}',\mathcal{S}} = \langle \varphi, G_t \star \psi \rangle_{\mathcal{S}',\mathcal{S}} = \langle \varphi, \mathcal{T}(t)\psi \rangle_{\mathcal{S}',\mathcal{S}}$$

which proves Property (vi). □

We now describe the regularity and smoothing properties of the heat equation in terms of the heat semigroup.

PROPOSITION 2.1.9. *If $(\mathcal{T}(t))_{t\geq 0}$ is as defined above, and $s \in \mathbb{R}$, $1 < p < \infty$, then the following properties hold.*

(i) *$(\mathcal{T}(t))_{t\geq 0} \subset \mathcal{L}(H^{s,p}(\mathbb{R}^N))$ and $\|\mathcal{T}(t)\|_{\mathcal{L}(H^{s,p})} \leq 1$.*

(ii) *$\forall u_0 \in H^{s,p}(\mathbb{R}^N)$, the map $t \mapsto \mathcal{T}(t)u_0$ is $C^k : [0,\infty) \to H^{s-2k,p}(\mathbb{R}^N)$ for all $k \in \mathbb{N}$, and $\frac{d^k}{dt^k}\mathcal{T}(t)u_0 = \Delta^k \mathcal{T}(t)u_0$.*

(iii) *For all $t > 0$, $\sigma \geq s$ and $q \geq p$, $\mathcal{T}(t) \in \mathcal{L}(H^{s,p}(\mathbb{R}^N), H^{\sigma,q}(\mathbb{R}^N))$. In addition, if $p \leq q < \infty$, then there exists a constant C such that*

$$\|\mathcal{T}(t)\|_{\mathcal{L}(H^{s,p},H^{\sigma,q})} \leq Ct^{-\frac{N}{2}(\frac{1}{p}-\frac{1}{q})}(1 + t^{-\frac{\sigma-s}{2}}) \tag{2.1.25}$$

for all $t > 0$.

(iv) *Given any $u_0 \in H^{s,p}(\mathbb{R}^N)$,*

$$\frac{\mathcal{T}(h) - I}{h} u_0 \underset{h\downarrow 0}{\longrightarrow} \Delta u_0 \tag{2.1.26}$$

in $H^{s-2,p}(\mathbb{R}^N)$.

PROOF. This follows immediately from Corollary 2.1.6, except for the estimate (2.1.25) which we prove now. The case $\sigma = s$ follows from (2.1.9), so we suppose now $\sigma > s$. Let $n \geq \sigma - s$ be an integer. Applying (2.1.11) with $k = s = 0$, first with $m = 0$, then with $m = n$, we obtain

$$\|\mathcal{T}(t)\|_{\mathcal{L}(L^p,L^q)} \leq Ct^{-\frac{N}{2}(\frac{1}{p}-\frac{1}{q})}, \quad \|\mathcal{T}(t)\|_{\mathcal{L}(L^p,H^{n,q})} \leq Ct^{-\frac{N}{2}(\frac{1}{p}-\frac{1}{q})}(1 + t^{-\frac{n}{2}})$$

Moreover, we deduce from (B.6.101) (with s replaced by $\sigma - s$, $\ell = 0$, $m = n$, and p replaced by q) that

$$\|u\|_{H^{\sigma-s,q}} \leq C\|u\|_{L^q}^{\frac{n-\sigma+s}{n}} \|u\|_{H^{n,q}}^{\frac{\sigma-s}{n}}$$

Therefore, we obtain

$$\|\mathcal{T}(t)\|_{\mathcal{L}(L^p,H^{\sigma-s,q})} \leq Ct^{-\frac{N}{2}(\frac{1}{p}-\frac{1}{q})}[(1 + t^{-\frac{n}{2}})]^{\frac{\sigma-s}{n}} \leq Ct^{-\frac{N}{2}(\frac{1}{p}-\frac{1}{q})}(1 + t^{-\frac{\sigma-s}{2}})$$

This means that

$$\|\mathcal{T}(t)v_0\|_{H^{\sigma-s,q}} \leq Ct^{-\frac{N}{2}(\frac{1}{p}-\frac{1}{q})}(1 + t^{-\frac{\sigma-s}{2}})\|v_0\|_{L^q}$$

for all $v_0 \in L^q$. Let now $u_0 \in H^{s,p}(\mathbb{R}^N)$ and $v_0 = J_s u_0 \in L^p(\mathbb{R}^N)$, with the notation (B.6.80). Applying (B.6.83), we obtain

$$\begin{aligned}\|\mathcal{T}(t)J_s u_0\|_{H^{\sigma-s,q}} &\leq Ct^{-\frac{N}{2}(\frac{1}{p}-\frac{1}{q})}(1 + t^{-\frac{\sigma-s}{2}})\|J_s u_0\|_{L^q} \\ &= Ct^{-\frac{N}{2}(\frac{1}{p}-\frac{1}{q})}(1 + t^{-\frac{\sigma-s}{2}})\|u_0\|_{H^{s,q}}\end{aligned}$$

Next, it follows easily from (B.6.80) and (2.1.2) that $\mathcal{T}(t)J_s u_0 = J_s \mathcal{T}(t)u_0$. Applying now (B.6.85), we finally obtain

$$\begin{aligned}\|\mathcal{T}(t)u_0\|_{H^{\sigma,q}} &= \|J_s\mathcal{T}(t)u_0\|_{H^{\sigma-s,q}} = \|\mathcal{T}(t)J_s u_0\|_{H^{\sigma-s,q}} \\ &\leq Ct^{-\frac{N}{2}(\frac{1}{p}-\frac{1}{q})}(1 + t^{-\frac{\sigma-s}{2}})\|u_0\|_{H^{s,q}}\end{aligned}$$

which proves (2.1.25). □

REMARK 2.1.10. Here are some particular solutions of (2.1.1) .

(i) Given any $c \in \mathbb{R}$, $u(t,x) \equiv c$ is the solution of (2.1.1) with $u_0(x) \equiv c$. In other words, $\mathcal{T}(t)c = c$.
(ii) Since $\Delta(|x|^2) = 2N$, it follows that $u(t,x) \equiv 2Nt + |x|^2$ is the solution of (2.1.1) with $u_0(x) \equiv |x|^2$. In other words, $\mathcal{T}(t)|x|^2 = 2Nt + |x|^2$.
(iii) Let δ_0 be the Dirac mass at 0. In particular, $\delta_0 \in \mathcal{S}'(\mathbb{R}^N)$ and $\langle \delta_0, \varphi \rangle_{\mathcal{S}',\mathcal{S}} = \varphi(0)$. As is well known, $f \star \delta_0 = f$ for all $f \in \mathcal{S}(\mathbb{R}^N)$ and in particular, $G_t \star \delta_0 = G_t$. It follows that the solution of (2.1.1) with $u_0 = \delta_0$ is $u(t) = G_t$. In other words, $\mathcal{T}(t)\delta_0 = G_t$.
(iv) Let u_0 be a Gaussian, i.e. $u_0(x) = e^{-a|x|^2}$ for some $a > 0$. Setting $\tau = \frac{1}{4a}$, we have $u_0 = (\frac{a}{\pi})^{-\frac{N}{2}} G_\tau$. On the other hand, $\mathcal{T}(t)G_\tau = G_{\tau+t}$ by Proposition 2.1.8 (iii) and Property (iii) above, so that $\mathcal{T}(t)u_0 = (\frac{a}{\pi})^{-\frac{N}{2}} G_{\tau+t}$. Therefore, $\mathcal{T}(t)e^{-a|x|^2} = (1+4at)^{-\frac{N}{2}} e^{-\frac{a|x|^2}{1+4at}}$.

2.1.3. The nonhomogeneous equation and Duhamel's formula. We consider the nonhomogeneous heat equation

$$\begin{cases} u_t = \Delta u + f & 0 \le t \le T, x \in \mathbb{R}^N \\ u(0,x) = u_0(x) & x \in \mathbb{R}^N \end{cases} \tag{2.1.27}$$

where $T > 0$ and f is a given function $[0,T] \times \mathbb{R}^N \to \mathbb{R}$.

THEOREM 2.1.11. *Let $T > 0$, $1 < p < \infty$ and $\sigma \in \mathbb{R}$. Given $u_0 \in H^{s,p}(\mathbb{R}^N)$ and $f \in C([0,T], H^{s,p}(\mathbb{R}^N))$, there exists a unique solution $u \in C([0,T], H^{s,p}(\mathbb{R}^N)) \cap C^1([0,T], H^{s-2,p}(\mathbb{R}^N))$ of equation (2.1.27), and u given by Duhamel's formula*

$$u(t) = \mathcal{T}(t)u_0 + \int_0^t \mathcal{T}(t-s)f(s)\,ds \tag{2.1.28}$$

for all $0 \le t \le T$. Moreover,

$$\|u(t)\|_{H^{s,p}} \le \|u_0\|_{H^{s,p}} + \int_0^t \|f(s)\|_{H^{s,p}} ds \tag{2.1.29}$$

for all $0 \le t \le T$.

For the proof of Theorem 2.1.11, we will use the following lemma.

LEMMA 2.1.12. *Let $T > 0$, $1 < p < \infty$, $\sigma \in \mathbb{R}$ and $f \in C([0,T], H^{s,p}(\mathbb{R}^N))$. If*

$$w(t) = \int_0^t \mathcal{T}(t-s)f(s)\,ds \tag{2.1.30}$$

for all $0 \le t \le T$, then $w \in C([0,T], H^{s,p}(\mathbb{R}^N)) \cap C^1([0,T], H^{s-2,p}(\mathbb{R}^N))$, $w(0) = 0$ and $w_t = \Delta w + f$ for all $0 \le t \le T$. Moreover,

$$\|u(t)\|_{H^{s,p}} \le \int_0^t \|f(s)\|_{H^{s,p}} ds \tag{2.1.31}$$

for all $0 \le t \le T$. In addition, if $p \le q < \infty$ and $\sigma \ge s$ satisfy

$$\frac{N}{2}\Big(\frac{1}{p} - \frac{1}{q}\Big) + \frac{\sigma - s}{2} < 1 \tag{2.1.32}$$

then $w \in C([0,T], H^{\sigma,q}(\mathbb{R}^N))$ and

$$\|u(t)\|_{H^{\sigma,q}} \le C \int_0^t (t-s)^{-\frac{N}{2}(\frac{1}{p}-\frac{1}{q})}(1 + (t-s)^{-\frac{\sigma-s}{2}})\|f(s)\|_{H^{s,p}} ds \tag{2.1.33}$$

for all $0 \le t \le T$.

Proof. Given $0 < t \le T$, it follows from Proposition 2.1.9 that the map $s \mapsto \mathcal{T}(t-s)f(s)$ is continuous on $[0,t] \to H^{s,p}(\mathbb{R}^N)$, so that the integral in (2.1.30) defines an element $w(t) \in H^{s,p}(\mathbb{R}^N)$. Moreover, we deduce from Proposition 2.1.9 (i) that estimate (2.1.31) holds. We now show that $w \in C([0,T], H^{s,p}(\mathbb{R}^N))$. Given $0 \le t < t+h \le T$, it follows from (2.1.30) and the semigroup property $\mathcal{T}(h)\mathcal{T}(t-s) = \mathcal{T}(t+h-s)$ (see Proposition 2.1.8 (iii)) that

$$w(t+h) - w(t) = [\mathcal{T}(h) - I]w(t) + \int_t^{t+h} \mathcal{T}(t+h-s)f(s)\,ds \tag{2.1.34}$$

It follows easily (applying once again Proposition 2.1.9) that, given any $0 \le t < T$, $w(t+h) \to w(t)$ in $H^{s,p}(\mathbb{R}^N)$ ad $h \to 0$. Hence $w \in C([0,T), H^{s,p}(\mathbb{R}^N))$, and $w(0) = 0$. Continuity at T requires a slightly different argument. Given $0 \le h \le T$, we write

$$w(T) - w(T-h) = \int_{T-h}^T \mathcal{T}(T-s)f(s)\,ds + \int_0^T \Psi_h(s)\,ds$$

where

$$\Psi_h(s) = \begin{cases} \mathcal{T}(T-s)f(s) - \mathcal{T}(T-h-s)f(s) & 0 \le s \le T-h \\ 0 & T-h \le s \le T \end{cases}$$

Therefore,

$$\|w(T) - w(T-h)\|_{H^{s,p}} \le h \sup_{0 \le s \le T} \|f(s)\|_{H^{s,p}} + \int_0^T \|\Psi_h(s)\|_{H^{s,p}} ds$$

Since $\|\Psi_h(s)\|_{H^{s,p}} \le 2\sup_{0\le s\le T} \|f(s)\|_{H^{s,p}}$ and $\|\Psi_h(s)\|_{H^{s,p}} \to 0$ as $h \to 0$, continuity at T follows by dominated convergence. Next, given $0 \le t < t+h \le T$, we deduce from (2.1.34) that

$$\frac{u(t+h) - u(t)}{h} = \frac{\mathcal{T}(h) - I}{h} u(t) + \frac{1}{h}\int_t^{t+h} \mathcal{T}(t-s)f(s)\,ds$$

Letting $h \downarrow 0$ and applying Proposition 2.1.9, we obtain

$$\lim_{h\downarrow 0} \frac{u(t+h) - u(t)}{h} = \Delta u(t) + f(t)$$

where the limit is in $H^{s-2,p}(\mathbb{R}^N)$. This shows that $u : [0,T) \to H^{s-2,p}(\mathbb{R}^N)$ is right-differentiable at every $0 \le t < T$ and $\frac{d^+u}{dt} = \Delta u + f$. Since $\Delta u + f \in C([0,T], H^{s-2,p}(\mathbb{R}^N))$, we deduce (see Theorem A.1.3 and Remark A.1.4) that $u \in C^1([0,T], H^{s-2,p}(\mathbb{R}^N))$ and $\frac{du}{dt} = \Delta u + f$. This proves the first part of the lemma.

Let now $p \le q < \infty$ and $\sigma \ge s$ satisfy (2.1.32). Given $0 < t \le T$, it follows from Proposition 2.1.9 that the map $s \mapsto \mathcal{T}(t-s)f(s)$ is continuous on $[0,t) \to H^{\sigma,q}(\mathbb{R}^N)$ and (see (2.1.25))

$$\|\mathcal{T}(t-s)f(s)\|_{H^{\sigma,q}} \le C(-s)t^{-\frac{N}{2}(\frac{1}{p}-\frac{1}{q})}(1 + (t-s)^{-\frac{\sigma-s}{2}})$$

Thus we see that condition (2.1.32) ensures that the function $s \mapsto \mathcal{T}(t-s)f(s)$ is integrable $(0,t) \to H^{\sigma,q}(\mathbb{R}^N)$. Thus $w(t) \in H^{\sigma,q}(\mathbb{R}^N)$ for all $t \in [0,T]$ and w satisfies estimate (2.1.33). Continuity is proved as above. □

Proof of Theorem 2.1.11. We first prove uniqueness. Given two solutions $u, v \in C^1([0,T], \mathcal{S}'(\mathbb{R}^N))$, we set $w = u - v$. It follows that $w \in C^1([0,T], \mathcal{S}'(\mathbb{R}^N))$ satisfies $w_t = \Delta w$ on $[0,T]$ and $w(0) = 0$. Therefore, $w = 0$ by Remark 2.1.3 (ii), hence $u = v$.

To prove existence and Duhamel's formula (2.1.28), it suffices to consider the cases $f(t) \equiv 0$ and $u_0 = 0$ independently. (The general solution is then the sum of these two solutions.) The result in the case $f(t) \equiv 0$ follows from Proposition 2.1.9, while in the case $u_0 = 0$ it follows from Lemma 2.1.12. □

We now state a smoothing effect for the solutions of (2.1.27).

THEOREM 2.1.13. *Let $T > 0$, $1 < p < \infty$ and $s \in \mathbb{R}$. Let $u_0 \in H^{s,p}(\mathbb{R}^N)$ and $f \in C([0,T], H^{s,p}(\mathbb{R}^N))$, and let $u \in C([0,T], H^{s,p}(\mathbb{R}^N))$ be the corresponding solution of (2.1.27). If $q \le p < \infty$ and $s \le \sigma < s+2$ satisfy (2.1.32), then $u \in C((0,T], H^{\sigma,q}(\mathbb{R}^N))$. In particular, $u \in C((0,T], H^{\sigma,p}(\mathbb{R}^N))$ for all $s \le \sigma < s+2$ and $u \in C((0,T], H^{s,q}(\mathbb{R}^N))$ for all $p \le q < \frac{Np}{(N-2p)^+}$.*

PROOF. This follows from Corollary 2.1.6 and Lemma 2.1.12. □

2.2. The heat equation on a bounded domain

We consider the heat equation with Dirichlet boundary conditions

$$\begin{cases} u_t = \Delta u & t > 0, x \in \Omega \\ u(t,x) = 0 & t > 0, x \in \partial\Omega \\ u(0,x) = u_0(x) & x \in \Omega \end{cases} \tag{2.2.1}$$

on a bounded domain $\Omega \subset \mathbb{R}^N$. As for the Laplace equation, the boundary condition is interpreted in a weak sense. We will use the spectral decomposition of the Laplacian (see Section 1.4) to construct solutions of (2.2.1). In particular, we consider the eigenvalues $(\lambda_j)_{j\ge1}$ and a corresponding sequence of eigenvectors $(\varphi_j)_{j\ge1}$ as given by Theorem 1.4.1. Moreover, we equip $H_0^1(\Omega)$ with the (equivalent) norm $\|u\|_{H_0^1} = \|\nabla u\|_{L^2}$, and $H^{-1}(\Omega)$ with the corresponding dual norm.

2.2.1. Existence and regularity.

Our first result of this section is as follows.

THEOREM 2.2.1. *Given any $u_0 \in H^{-1}(\Omega)$, there exists a unique solution*

$$u \in C([0,\infty), H^{-1}(\Omega)) \cap C((0,\infty), H_0^1(\Omega)) \cap C^1((0,\infty), H^{-1}(\Omega)) \tag{2.2.2}$$

of (2.2.1). In addition, u is given by the formula

$$u(t) = \sum_{j=1}^{\infty} e^{-t\lambda_j} a_j^0 \varphi_j \tag{2.2.3}$$

where the sequence $(a_j^0)_{j\ge1}$ is defined by

$$a_j^0 = \langle u_0, \varphi_j \rangle_{H^{-1},H_0^1} \quad j \ge 1 \tag{2.2.4}$$

Moreover, $u \in C^\infty((0,\infty), H_0^1(\Omega))$, with the estimate

$$\Big\| \frac{d^k u}{dt^k} \Big\|_{H_0^1} \le \frac{(k+1)!}{t^{k+1}} \|u_0\|_{H^{-1}} \tag{2.2.5}$$

Furthermore, the following properties hold.

(i) *$\|u(t)\|_{H^{-1}} \le e^{-t\lambda_1} \|u_0\|_{H^{-1}}$ for all $t > 0$.*
(ii) *If $u_0 \in L^2(\Omega)$, then $u \in C([0,\infty), L^2(\Omega))$ and $\|u(t)\|_{L^2} \le e^{-t\lambda_1} \|u_0\|_{L^2}$ for all $t > 0$.*
(iii) *If $u_0 \in H_0^1(\Omega)$, then $u \in C([0,\infty), H_0^1(\Omega))$ and $\|u(t)\|_{H_0^1} \le e^{-t\lambda_1} \|u_0\|_{H_0^1}$ for all $t > 0$.*

PROOF. We first prove uniqueness, so we consider $u_0 \in H^{-1}(\Omega)$ and a solution u of (2.2.1) in the class (2.2.2). Let

$$a_j(t) = \langle u(t), \varphi_j \rangle_{H^{-1},H_0^1} \quad j \ge 1$$

so that $a_j \in C([0,\infty)) \cap C^1((0,\infty))$ by Remark 1.4.4. Moreover, $u_t = \Delta u$ in $H^{-1}(\Omega)$ for all $t > 0$ so that, applying (1.4.13),

$$a_j' = \langle \Delta u, \varphi_j \rangle_{H^{-1},H_0^1} = \langle u, \Delta\varphi_j \rangle_{H^{-1},H_0^1} = -\lambda_j \langle u, \varphi_j \rangle_{H^{-1},H_0^1} = -\lambda_j a_j$$

for all $t > 0$. Integrating the above equation yields

$$a_j(t) = e^{-t\lambda_j} a_j^0 \tag{2.2.6}$$

Therefore, u is given by (2.2.3), which proves uniqueness.

We now prove the existence and regularity properties. Let $u_0 \in H^{-1}(\Omega)$, $(a_j^0)_{j\ge1}$ defined by (2.2.4), and $(a_j(t))_{j\ge1}$ defined by (2.2.6). We see that $a_j \in C^\infty([0,\infty))$ and

$$\frac{d^k a_j}{dt^k} = (-1)^k \lambda_j^k e^{-t\lambda_j} a_j^0 \tag{2.2.7}$$

In particular,

$$|a_j(t)| \le e^{-t\lambda_j} |a_j^0| \le e^{-t\lambda_1} |a_j^0| \tag{2.2.8}$$

since $\lambda_j \ge \lambda_1$ for all $j \ge 1$. In the rest of the proof, we apply repeatedly Proposition 1.4.3 without further mention. Since

$$\sum \lambda_j^{-1} |a_j^0|^2 = \|u_0\|_{H^{-1}}^2 < \infty \tag{2.2.9}$$

we deduce from (2.2.8) that $e^{2t\lambda_1} \sum \lambda_j^{-1} |a_j(t)|^2 \le \|u_0\|_{H^{-1}}^2$. Thus we see that formula (2.2.3) defines a function $u(t) \in H^{-1}(\Omega)$ for all $t \ge 0$, and $\|u(t)\|_{H^{-1}} \le e^{-t\lambda_1} \|u_0\|_{H^{-1}}$. Moreover,

$$\|u(t) - u(s)\|_{H^{-1}}^2 = \sum_{j=1}^\infty \lambda_j^{-1} |a_j(t) - a_j(s)|^2$$

Applying (2.2.8) and dominated convergence (for sequences), we deduce that $u \in C([0,\infty), H^{-1}(\Omega))$. Next, $m! e^x \ge x^m$ for all $x \ge 0$ and $m \in \mathbb{N}$, so that $e^{-x} \le m! x^{-m}$. Therefore, it follows from (2.2.7) that

$$\Big|\frac{d^k a_j}{dt^k}\Big| \le \lambda_j^k e^{-t\lambda_j} |a_j^0| \le \lambda_j^k (k+1)! (t\lambda_j)^{-k-1} |a_j^0| = (k+1)! t^{-k-1} \lambda_j^{-1} |a_j^0|$$

Therefore,

$$\begin{aligned}\sum_{j=1}^\infty \lambda_j \Big|\frac{d^k a_j(t)}{dt^k}\Big|^2 &\le [(k+1)! t^{-k-1}]^2 \sum_{j=1}^\infty \lambda_j^{-1} |a_j^0|^2 \\ &= [(k+1)! t^{-k-1}]^2 \|u_0\|_{H^{-1}}^2\end{aligned} \tag{2.2.10}$$

For $k = 0$, this yields

$$\sum_{j=1}^\infty \lambda_j |a_j|^2 \le t^{-2} \|u_0\|_{H^{-1}}^2$$

This proves that $u(t) \in H_0^1(\Omega)$ for all $t > 0$, as well as estimate (2.2.5) for $k = 0$, then by dominated convergence for sequences, that $u \in C((0,\infty), H_0^1(\Omega))$. Next, given $s, t > 0$,

$$\|u(t) - u(s)\|_{H^1}^2 = \sum_{j=1}^\infty \lambda_j |a_j(t) - a_j(s)|^2 = \sum_{j=1}^\infty \lambda_j \Big| \int_s^t \frac{da_j}{dt} \Big|^2$$

Applying (2.2.10), we conclude easily that $u : (0,\infty) \to H_0^1(\Omega)$ is differentiable at every $t > 0$. Moreover (see Remark 1.4.4 (i)),

$$\frac{du}{dt} = \sum_{j=1}^\infty \frac{da_j}{dt} \varphi_j$$

One concludes as above that $u \in C^1((0,\infty), H^1_0(\Omega))$ and that estimate (2.2.5) holds for $k = 1$. An obvious iteration argument shows that $u \in C^k((0,\infty), H^1_0(\Omega))$ for all $k \geq 2$ and that estimate (2.2.5) holds.

So far, we have proved the fists part of the theorem, as well as Property (i). We finally prove Properties (ii) and (iii). We deduce from (2.2.3) and(2.2.8) that

$$\|u(t)\|_{L^2}^2 = \sum_{j=1}^{\infty} |a_j(t)|^2 \leq e^{-2\lambda_1} \sum_{j=1}^{\infty} |a_j^0|^2 = \|u_0\|_{L^2}^2$$

Moreover, it follows easily from formula (2.2.3) and dominated convergence for sequences that $u \in C([0,\infty), L^2(\Omega))$. This proves Property (ii), and Property (iii) is proved likewise. □

REMARK 2.2.2. Here are some comments on Theorem 2.2.1.

(i) Property (i) (as well as (ii) and (iii)) show that all solutions of (2.2.1) decay like $e^{-t\lambda_1}$ as $t \to \infty$. This is in contrast with the case of the heat equation set on $\mathbb{R}^N$, where the decay is in general power-like. (See Remark 2.1.5 (iii).)

(ii) Estimate (2.2.5) quantifies the smoothing effect of the heat equation. For instance, for $k = 0$, $\|u(t)\|_{H^1_0} \leq t^{-1}\|u_0\|_{H^{-1}}$. One can deduce from formula (2.2.3) other estimates of the same type. For instance, $\sqrt{y}e^{-y} \leq 1$ for $y \geq 0$, so that $\sqrt{\lambda_j}e^{-t\lambda_j} \leq t^{-\frac{1}{2}}$. With the notation (2.2.6), this implies that $\sqrt{\lambda_j}|a_j(t)| \leq t^{-\frac{1}{2}}|a_j^0|$. It follows easily that $\|u(t)\|_{L^2} \leq t^{-\frac{1}{2}}\|u_0\|_{H^{-1}}$; and if $u_0 \in L^2(\Omega)$, then $\|u(t)\|_{H^1_0} \leq t^{-\frac{1}{2}}\|u_0\|_{L^2}$.

(iii) The proof of uniqueness shows in fact a stronger uniqueness property: If $T > 0$ and $u \in C([0,T), H^{-1}(\Omega)) \cap C((0,T), H^1_0(\Omega)) \cap C^1((0,T), H^{-1}(\Omega))$ satisfies $u_t = \Delta u$ in $H^{-1}(\Omega)$ for all $0 \leq t \leq T$ and if $u(0) = 0$, then $u(t) = 0$ for all $0 \leq t \leq T$.

We next prove an interior regularity property.

PROPOSITION 2.2.3. *Let Ω be any open subset of $\mathbb{R}^N$. Let $T > 0$, and let $u \in C((0,T), H^1(\Omega)) \cap C^1((0,T), H^{-1}(\Omega))$ satisfy $u_t = \Delta u$ in $H^{-1}(\Omega)$ for all $0 < t < T$. It follows that $u \in C^\infty((0,T) \times \Omega)$.*

PROOF. Let $\varphi \in C_c^\infty((0,T) \times \Omega)$ and define the function v on $[0,T) \times \mathbb{R}^N$ by

$$v(t,x) = \begin{cases} \varphi(t,x)u(t,x) & x \in \Omega \\ 0 & x \notin \Omega \end{cases} \tag{2.2.11}$$

It follows easily that $v \in C([0,T), H^1(\mathbb{R}^N)) \cap C^1([0,T), H^{-1}(\mathbb{R}^N))$, $v(0) = 0$, and

$$v_t - \Delta v = f \tag{2.2.12}$$

where

$$f(t,x) = \begin{cases} u(\varphi_t - \Delta\varphi) - 2\nabla u \cdot \nabla\varphi & 0 < t < T, x \in \Omega \\ 0 & 0 < t < T, x \notin \Omega \end{cases} \tag{2.2.13}$$

In particular, $f \in C([0,T], L^2(\mathbb{R}^N))$. Applying Theorem 2.1.13, we deduce that $v \in C([0,T), H^{\frac{3}{2}}(\mathbb{R}^N))$.

We now prove by induction on m that for all $m \in \mathbb{N}$ and $\varphi \in C_c^\infty((0,T) \times \Omega)$, the function v defined by (2.2.11) belongs to $C([0,T), H^{\frac{m}{2}}(\mathbb{R}^N))$. Indeed, we have seen above that the property is true for $m = 3$. Assume it holds up to some $m \geq 3$, and let $\varphi \in C_c^\infty((0,T) \times \Omega)$. Consider $\psi \in C_c^\infty((0,T) \times \Omega)$ such that $\psi = 1$ on the support of φ. By the induction assumption, ψu (extended by 0 outside Ω) belongs to $C([0,T), H^{\frac{m}{2}}(\mathbb{R}^N))$. Moreover, since $\psi = 1$ on the support of φ, we may replace u by ψu in (2.2.13). Since $\nabla(\psi u) \in C([0,T), H^{\frac{m}{2}-1}(\mathbb{R}^N))$, it follows easily that $f \in C([0,T), H^{\frac{m}{2}-1}(\mathbb{R}^N))$. Applying Theorem 2.1.13, we obtain $v \in C([0,T), H^{\frac{m}{2}+\frac{1}{2}}(\mathbb{R}^N))$, which closes the induction argument.

It follows from what precedes that, given any $\varphi \in C_c^\infty((0,T)\times\Omega)$, we have $\Delta v + f \in C([0,T), H^\ell(\mathbb{R}^N))$ for all $\ell \in \mathbb{N}$, and equation (2.2.12) implies that $v \in C^1([0,T), H^\ell(\mathbb{R}^N))$ for all $\ell \in \mathbb{N}$. Then one deduces easily that, given any $\varphi \in C_c^\infty((0,T)\times\Omega)$, we have $\Delta v + f \in C^1([0,T), H^\ell(\mathbb{R}^N))$ for all $\ell \in \mathbb{N}$; and so $v \in C^2([0,T), H^\ell(\mathbb{R}^N))$ for all $\ell \in \mathbb{N}$. An obvious iteration argument shows that, given any $\varphi \in C_c^\infty((0,T)\times\Omega)$, we have $v \in C^\infty([0,T), H^\ell(\mathbb{R}^N))$ for all $\ell \in \mathbb{N}$. By Sobolev's embedding, we conclude that $v \in C^\infty((0,T)\times\mathbb{R}^N)$, which completes the proof since $\varphi \in C_c^\infty((0,T)\times\Omega)$ is arbitrary. ☐

The following property gives some information on the behavior at $t=0$.

Proposition 2.2.4. *Let Ω be any open subset of $\mathbb{R}^N$. Let $u_0 \in C_c^\infty(\Omega)$, $T>0$, and let $u \in C([0,T), H^1(\Omega)) \cap C^1((0,T), H^{-1}(\Omega))$ satisfy $u_t = \Delta u$ in $H^{-1}(\Omega)$ for all $0<t<T$ and $u(0)=u_0$. It follows that $u(t) \to u_0$ as $t \downarrow 0$ in $L^\infty(\omega)$ for any open set $\omega \subset\subset \Omega$.*

Proof. The proof is very similar to the proof of Proposition 2.2.3 above. Let $\varphi \in C_c^\infty(\Omega)$ and define the function v on $[0,T)\times\mathbb{R}^N$ by

$$v(t,x) = \begin{cases} \varphi(x)u(t,x) & x \in \Omega \\ 0 & x \notin \Omega \end{cases} \tag{2.2.14}$$

It follows easily that $v \in C([0,T), H^1(\mathbb{R}^N)) \cap C^1((0,T), H^{-1}(\mathbb{R}^N))$, $v(0)=0$, and

$$v_t - \Delta v = f \tag{2.2.15}$$

where

$$f(t,x) = \begin{cases} -u\Delta\varphi - 2\nabla u\cdot\nabla\varphi & 0<t<T, x\in\Omega \\ 0 & 0<t<T, x\notin\Omega \end{cases} \tag{2.2.16}$$

In particular, $f \in C([0,T), L^2(\mathbb{R}^N))$. Applying Theorem 2.1.13, we deduce that $v \in C([0,T), H^{\frac{3}{2}}(\mathbb{R}^N))$.

We now prove by induction on m that for all $m \in \mathbb{N}$ and $\varphi \in C_c^\infty(\Omega)$, the function v defined by (2.2.14) belongs to $C([0,T), H^{\frac{m}{2}}(\mathbb{R}^N))$. Indeed, we have seen above that the property is true for $m=3$. Assume it holds up to some $m \ge 3$, and let $\varphi \in C_c^\infty(\Omega)$. Consider $\psi \in C_c^\infty(\Omega)$ such that $\psi = 1$ on the support of φ. By the induction assumption, ψu (extended by 0 outside Ω) belongs to $C([0,T), H^{\frac{m}{2}}(\mathbb{R}^N))$. Moreover, since $\psi=1$ on the support of φ, we may replace u by ψu in (2.2.16). Since $\nabla(\psi u) \in C([0,T), H^{\frac{m}{2}-1}(\mathbb{R}^N))$, it follows easily that $f \in C([0,T), H^{\frac{m}{2}-1}(\mathbb{R}^N))$. Applying Theorem 2.1.13, we obtain $v \in C([0,T), H^{\frac{m}{2}+\frac{1}{2}}(\mathbb{R}^N))$, which closes the induction argument.

Finally, for $m > \frac{N}{2}$, we have $H^m(\mathbb{R}^N) \hookrightarrow L^\infty(\mathbb{R}^N)$, and we deduce that $\varphi u(t) \to \varphi u_0$ in $L^\infty(\Omega)$ as $t \downarrow 0$, for every $\varphi \in C_c^\infty(\Omega)$, which proves the desired result. ☐

2.2.2. The heat semigroup. The heat semigroup on Ω with Dirichlet boundary conditions, often denoted $(e^{t\Delta})_{t\ge 0}$, is the family of operators $H^{-1}(\Omega) \to H^{-1}(\Omega)$ defined as follows for every $t \ge 0$: given $u_0 \in H^{-1}(\Omega)$, $\mathcal{T}(t)u_0$ is the solution at time t of equation (2.2.1), given by Theorem 2.2.1. In particular,

$$\mathcal{T}(t)u_0 = \sum_{j=1}^{\infty} e^{-t\lambda_j} \langle u_0, \varphi_j \rangle_{H^{-1}, H_0^1} \varphi_j \tag{2.2.17}$$

by formula (2.2.3).

Proposition 2.2.5. *Let $(\mathcal{T}(t))_{t\ge 0}$ be as above, and let $\mathcal{X}$ be either of the spaces $H^{-1}(\Omega)$, $L^2(\Omega)$, $H_0^1(\Omega)$.*

(i) $(\mathcal{T}(t))_{t\geq 0} \subset \mathcal{L}(\mathcal{X})$ *and* $\|\mathcal{T}(t)\|_{\mathcal{L}(\mathcal{X})} \leq e^{-t\lambda_1}$.
(ii) $\mathcal{T}(0) = I$ *and* $\mathcal{T}(t+s) = \mathcal{T}(t)\mathcal{T}(s)$ *in* $\mathcal{L}(\mathcal{X})$, *for all* $s, t \geq 0$.
(iii) $\forall u_0 \in \mathcal{X}$, *the map* $t \mapsto \mathcal{T}(t)u_0$ *is continuous* $[0,\infty) \to \mathcal{X}$.
(iv) $(\mathcal{T}(t))_{t\geq 0} \subset \mathcal{L}(H^{-1}(\Omega), H_0^1(\Omega))$. *Moreover,* $\|\mathcal{T}(t)\|_{\mathcal{L}(H^{-1},H_0^1)} \leq t^{-1}$ *and* $\|\mathcal{T}(t)\|_{\mathcal{L}(H^{-1},L^2)} \leq t^{-\frac{1}{2}}$.
(v) $(\mathcal{T}(t))_{t\geq 0} \subset \mathcal{L}(L^2(\Omega), H_0^1(\Omega))$ *and* $\|\mathcal{T}(t)\|_{\mathcal{L}(L^2,H_0^1)} \leq t^{-\frac{1}{2}}$.
(vi) $(\mathcal{T}(t)u_0, v_0)_{\mathcal{X}} = (u_0, \mathcal{T}(t)v_0)_{\mathcal{X}}$ *for all* $t \geq 0$ *and* $u_0, v_0 \in \mathcal{X}$. *In other words,* $\mathcal{T}(t)$ *is self-adjoint on* $\mathcal{X}$.

PROOF. Properties (i) and (iii), (v) and (v) follow from Theorem 2.2.1 and Remark 2.2.2 (ii); and Property (ii) is an immediate consequence of uniqueness (cf. Remark 2.1.3 (i)). To prove Property (vi), we note that by formula (2.2.17), $\mathcal{T}(t)u_0 = \sum_{j=1}^{\infty} e^{-t\lambda_j} a_j^0 \varphi_j$, where $a_j^0 = \langle u_0, \varphi_j \rangle_{H^{-1},H_0^1}$. Therefore, if $b_j^0 = \langle v_0, \varphi_j \rangle_{H^{-1},H_0^1}$, so that $v_0 = \sum_{j=1}^{\infty} j_j^0 \varphi_j$, then (see Remark 1.4.4 (ii))

$$(\mathcal{T}(t)u_0, v_0)_{L^2} = \sum_{j=1}^{\infty} e^{-t\lambda_j} a_j^0 b_j^0 = (u_0, \mathcal{T}(t)v_0)_{L^2}$$

Self-adjointness in $H^{-1}(\Omega)$ and in $H_0^1(\Omega)$ are proved similarly. □

2.2.3. The nonhomogeneous equation and Duhamel's formula. We consider the nonhomogeneous heat equation

$$\begin{cases} u_t = \Delta u + f & 0 < t < T, x \in \Omega \\ u(t,x) = 0 & t > 0, x \in \partial\Omega \\ u(0,x) = u_0(x) & x \in \Omega \end{cases} \tag{2.2.18}$$

where $T > 0$ and f is a given function $[0,T] \times \Omega \to \mathbb{R}$.

THEOREM 2.2.6. *Given* $T > 0$, $u_0 \in L^2(\Omega)$ *and* $f \in C([0,T], L^2(\Omega))$, *there exists a unique solution*

$$u \in C([0,T], L^2(\Omega)) \cap C((0,T), H_0^1(\Omega)) \cap C^1((0,T), H^{-1}(\Omega)) \tag{2.2.19}$$

of equation (2.2.18), *and* u *given by Duhamel's formula*

$$u(t) = \mathcal{T}(t)u_0 + \int_0^t \mathcal{T}(t-s)f(s)\,ds \tag{2.2.20}$$

for all $0 \leq t \leq T$. *Moreover,*

$$\|u(t)\|_{L^2} \leq e^{-t\lambda_1}\|u_0\|_{L^2} + \int_0^t e^{-(t-s)\lambda_1}\|f(s)\|_{L^2} ds \tag{2.2.21}$$

for all $0 \leq t \leq T$.

PROOF. We first prove uniqueness. Let u, v in the class (2.2.19) be two solutions of (2.2.18). It follows that $w = u - v$ satisfies $w_t = \Delta w$ in $H^{-1}(\Omega)$ for all $0 < t < T$, and $w(0) = 0$, so that $w = 0$ by Remark 2.2.2 (iii).

We now prove existence and formula (2.2.20). As a matter of fact, we need only prove that formula (2.2.20) defines u in the class (2.2.19) and that u is a solution of (2.2.18). Moreover, $\mathcal{T}(t)u_0$ belongs to the class (2.2.19), satisfies the homogeneous heat equation, and the estimate $\|\mathcal{T}(t)u_0\|_{L^2} \leq e^{-t\lambda_1}\|u_0\|_{L^2}$ by Theorem 2.2.1, so we may assume $u_0 = 0$ and consider

$$u(t) = \int_0^t \mathcal{T}(t-s)f(s)\,ds \tag{2.2.22}$$

Note that, given any $0 < t \le T$, the map $s \mapsto \boldsymbol{\mathcal{T}}(t-s)f(s)$ is continuous $[0,t] \to L^2(\Omega)$, so that the integral (2.2.22) defines an element $u(t) \in L^2(\Omega)$. In addition, since $\langle \boldsymbol{\mathcal{T}}(t-s)f(s), \varphi_j\rangle_{H^{-1},H^1_0} = e^{-(t-s)\lambda_j}\langle f(s), \varphi_j\rangle_{H^{-1},H^1_0}$, it follows that

$$\langle u(t), \varphi_j\rangle_{H^{-1},H^1_0} = \int_0^t e^{-(t-s)\lambda_j}\langle f(s), \varphi_j\rangle_{H^{-1},H^1_0}$$

In other words, setting

$$a_j(t) = \langle u(t), \varphi_j\rangle_{H^{-1},H^1_0}$$
$$b_j(t) = \langle f(t), \varphi_j\rangle_{H^{-1},H^1_0}$$

we have

$$a_j(t) = \int_0^t e^{-(t-s)\lambda_j} b_j(s)\, ds \tag{2.2.23}$$

for all $0 \le t \le T$ and $j \ge 1$. In particular,

$$\begin{aligned}\|u(t)\|_{L^2} = \|(a_j(t))_{j\ge1}\|_{\ell^2} &\le \int_0^t \|(e^{-(t-s)\lambda_j} b_j(s))_{j\ge1}\|_{\ell^2} \\ &\le \int_0^t e^{-(t-s)\lambda_1}\|(b_j(s))_{j\ge1}\|_{\ell^2} = \int_0^t e^{-(t-s)\lambda_1}\|f(s)\|_{L^2}\end{aligned}$$

which proves (2.2.21). Next, it follows from (2.2.23) that

$$|a_j(t)| \le A_j$$

where

$$A_j = \int_0^T |b_j(s)|\, ds$$

We have

$$\|(A_j)_{j\ge1}\|_{\ell^2} \le \int_0^T \|(b_j)_{j\ge1}\|_{\ell^2} < \infty$$

On the other hand, each b_j is a continuous function of t, hence so is a_j, and we deduce by dominated convergence (for sequences) that $(a_j)_{j\ge1} \in C([0,T], \ell^2(\mathbb{N}))$, i.e. $u \in C([0,T], L^2(\Omega))$. Similarly, using the property $\lambda_j^{\frac12} e^{-(t-s)\lambda_j} \le C(t-s)^{-\frac12}$, we see that

$$\lambda_j^{\frac12}|a_j(t)| \le B_j$$

where

$$B_j = \int_0^T (t-s)^{-\frac12}|b_j(s)|\, ds$$

We have

$$\|(B_j)_{j\ge1}\|_{\ell^2} \le \int_0^T (t-s)^{-\frac12}\|(b_j)_{j\ge1}\|_{\ell^2} < \infty$$

and we deduce by dominated convergence that $(\lambda_j^{\frac12} a_j)_{j\ge1} \in C([0,T], \ell^2(\mathbb{N}))$, i.e. $u \in C([0,T], H^1_0(\Omega))$. Moreover,

$$a_j' = -\lambda_j a_j + b_j$$

and it follows that $(\lambda_j^{-\frac12} a_j')_{j\ge1} \in C([0,T], \ell^2(\mathbb{N}))$, i.e. $u \in C^1([0,T], H^{-1}(\Omega))$. Thus we see that u belongs to the class (2.2.19). Finally,

$$\langle u_t - \Delta u - f, \varphi_j\rangle_{H^{-1},H^1_0} = a_j' + \lambda_j a_j + b_j = 0$$

for all $j \ge 1$, so that $u_t - \Delta u - f = 0$ in $H^{-1}(\Omega)$ for all $0 \le t \le T$. Since $u(0) = 0$ (because $a_j(0) = 0$), we see that u is a solution of (2.2.18). $\square$

2.3. The maximum principle

Throughout this section, we assume Ω is an open, connected subset of $\mathbb{R}^N$, bounded or not.

2.3.1. The weak maximum principle. We begin with a weak form of the maximum principle. As for Laplace's equation, this first result, together with the construction of explicit solutions, yields stronger forms of the maximum principle.

THEOREM 2.3.1. *Let $T > 0$, and let $u \in C((0,T), H^1(\Omega)) \cap C^1((0,T), H^{-1}(\Omega))$ and $f \in C((0,T), H^{-1}(\Omega))$ satisfy*

$$u_t - \Delta u = f \tag{2.3.1}$$

in $H^{-1}(\Omega)$ for all $t \in (0,T)$. Assume further that:

(i) $u \in C([0,T), L^2(\Omega))$

(ii) *There exists $v \in C((0,T), H^1_0(\Omega))$ such that $u(t) \le v(t)$ a.e. on Ω, for all $0 < t < T$*

(iii) *$f = g + h$, with $g \in C((0,T), H^{-1}(\Omega))$, $g(t) \le 0$ for a.a. $t \in (0,T)$, and $h \in C((0,T), L^2(\Omega))$, $h(t) \le C|u(t)|$ a.e. on $(0,T) \times \Omega$*

(iv) *$u(0) \le 0$ a.e. on Ω*

It follows that $u(t) \le 0$ a.e. on Ω, for all $t \in (0,T)$.

The proof of Theorem 2.3.1 uses the following result, in the spirit of Corollary A.1.6.

LEMMA 2.3.2. *Let $T > 0$ and $u \in C((0,T), H^1(\Omega)) \cap C^1((0,T), H^{-1}(\Omega))$. Suppose there exists $v \in C((0,T), H^1_0(\Omega))$ such that $u(t) \le v(t)$ a.e. on Ω for all $t \in (0,T)$. It follows that $u^+ \in C((0,T), H^1(\Omega))$, that the map $t \mapsto \|u^+(t)\|^2_{L^2}$ is C^1 on $(0,T)$, and*

$$\frac{d}{dt}\|u^+(t)\|^2_{L^2} = 2\langle u_t(t), u^+(t)\rangle_{H^{-1},H^1_0} \tag{2.3.2}$$

for all $t \in (0,T)$.

PROOF. Note that $u^+(t) \in H^1(\Omega)$ for all $0 < t < T$, by Proposition B.2.3. Moreover, since $|u^+| = u^+ \le v$, it follows from Corollary B.2.5 (replacing u by u^+) that $u^+(t) \in H^1_0(\Omega)$ for all $t \in (0,T)$. Applying again Proposition B.2.3, we conclude that $u^+ \in C((0,T), H^1(\Omega))$.

Next, it follows from an easy calculation (considering separately the possible signs of x, y) that

$$\big|(y^+)^2 - (x^+)^2 - 2(y-x)x^+\big| \le (y-x)^2$$

for all $x, y \in \mathbb{R}$. We deduce that, given $t \in (0,T)$ and $h \ne 0$ such that $t+h \in (0,T)$,

$$\Big|\|u^+(t+h)\|^2_{L^2} - \|u^+(t)\|^2_{L^2} - 2\int_\Omega (u(t+h)-u(t))u^+(t)\,dx\Big| \le \int_\Omega (u(t+h)-u(t))^2$$

which we rewrite in the form

$$\begin{aligned}\big|\|u^+(t+h)\|^2_{L^2} - \|u^+(t)\|^2_{L^2} &- 2\langle u(t+h)-u(t)), u^+(t)\rangle_{H^{-1},H^1_0}\big| \\ &\le \langle u(t+h)-u(t), u(t+h)-u(t)\rangle_{H^{-1},H^1_0}\end{aligned}$$

The conclusion easily follows by dividing through by h and letting $t - s \to 0$. □

PROOF OF THEOREM 2.3.1. Since u satisfies the assumptions of Lemma 2.3.2, it follows in particular that $u^+ \in C((0,T), H^1_0(\Omega))$. Therefore, we may take the $H^{-1} - H^1_0$ duality product of (2.3.1) with u^+ and we obtain

$$\langle u_t(t), u^+(t)\rangle_{H^{-1},H^1_0} - \langle \Delta u(t), u^+(t)\rangle_{H^{-1},H^1_0} = \langle f(t), u^+(t)\rangle_{H^{-1},H^1_0}$$

We deduce from (B.1.8) and (B.2.2) that

$$\langle \Delta u(t), u^+(t)\rangle_{H^{-1},H^1_0} = -\int_\Omega \nabla u^+ \cdot \nabla u = -\int_\Omega |\nabla u^+|^2 \le 0$$

Moreover, by assumption (iii),

$$\begin{aligned}\langle f(t), u^+(t)\rangle_{H^{-1},H^1_0} &\le \langle h(t), u^+(t)\rangle_{H^{-1},H^1_0}\\ &\le C\int_\Omega |u(t)|u^+(t)\,dx = C\int_\Omega u^+(t)^2\,dx.\end{aligned}$$

Applying Lemma 2.3.2, we find

$$\frac{d}{dt}\int_\Omega u^+(t)^2\,dx \le C\int_\Omega u^+(t)^2\,dx,$$

for all $t \in (0,T)$. We fix $t \in (0,T)$ and integrate on (s,t) with $0 < s < t$ to obtain

$$\|u^+(t)\|_{L^2}^2 \le \|u^+(s)\|_{L^2}^2 + \int_s^t \|u^+(\sigma)\|_{L^2}^2 d\sigma$$

It follows from assumption (iv) that $\|u^+(s)\|_{L^2}^2 \to 0$ as $s \downarrow 0$, so that

$$\|u^+(t)\|_{L^2}^2 \le \int_0^t \|u^+(\sigma)\|_{L^2}^2 d\sigma$$

Applying Gronwall' lemma (Lemma A.2.1) we deduce that $u^+(t) = 0$ all $t \in (0,T)$, hence $u \le 0$. □

2.3.2. The strong maximum principle. We apply Theorem 2.3.1 to obtain stronger versions of the maximum principle.

THEOREM 2.3.3. *Let $T > 0$, and let $u \in C((0,T), H^1(\Omega)) \cap C^1((0,T), H^{-1}(\Omega))$ satisfy $u_t - \Delta u \ge 0$ in $H^{-1}(\Omega)$ for all $t \in (0,T)$. Suppose further*

(i) *$u(t) \in C(\Omega)$ for all $0 < t < T$*
(ii) *$u \ge 0$ on $(0,T) \times \Omega$*
(iii) *$u \in C([0,T), L^2(\Omega))$*
(iv) *$u(0) \ge 0$ a.e. on Ω and $u(0) \not\equiv 0$.*

It follows that $u(t,x) > 0$ on $(0,T) \times \Omega$.

We will use the following lemma.

LEMMA 2.3.4. *Given $0 < \rho < R$, there exists a function $v : [0,\infty) \times B_R \to \mathbb{R}$ with the following properties.*

(i) *$v \in C([0,\infty), L^2(B_R)) \cap C^1((0,\infty), H^1_0(B_R)) \cap C^\infty((0,\infty) \times \overline{B_R})$*
(ii) *$v(0) \le 1_{B_\rho}$*
(iii) *$v_t - \Delta v \le 0$ on $(0,\infty) \times B_R$*
(iv) *$v(t)$ is radially symmetric and decreasing, for all $t > 0$*
(v) *There exist a constant $c > 0$ depending only on N such that*

$$v(t,x) \ge c\rho^N R^{-1} e^{-\frac{C}{R^2}t} t^{-\frac{N}{2}} e^{-\frac{R^2}{t}}(R - |x|) \tag{2.3.3}$$

on $(0,\infty) \times B_R$

PROOF. We let $z_0 = 1_{B_\rho}$ and consider the solution z of

$$\begin{cases} z_t = \Delta z & t > 0, x \in \mathbb{R}^N \\ z(0) = z_0 \end{cases}$$

In particular, $z \in C^\infty((0,\infty) \times \mathbb{R}^N$ by Corollary 2.1.6, and $z(t)$ is radially symmetric and decreasing for all $t \ge 0$, by Remark 2.1.5 (ii). Moreover, since $z(t) = G_t \star z_0$

with the notation (2.1.4), it follows from elementary calculations that there exists a constant $c > 0$ depending only on N such that

$$z(t,x) \geq ct^{-\frac{N}{2}} e^{-\frac{R^2}{t}} \rho^N \quad t > 0, x \in B_R \tag{2.3.4}$$

Next, we let λ be the first eigenvalue of $-\Delta$ in $H^1_0(B_R)$, and $\varphi > 0$ a corresponding eigenvector, normalized by $\varphi(0) = 1$. In particular, $\varphi \in C^\infty(\overline{B_R})$ is radially symmetric and decreasing, and there exist two constants $c = c(N)$ and $C = C(N)$ such that

$$\varphi(x) \geq cR^{-\frac{N+2}{2}}(R - |x|) \tag{2.3.5}$$

for all $x \in B_R$ and

$$\lambda \leq \frac{C}{R^2} \tag{2.3.6}$$

by Proposition 1.4.9. We now let

$$v(t,x) = e^{-t\lambda}\varphi(x)z(t,x) > 0 \quad t > 0, x \in B_R$$

It follows from (2.3.4), (2.3.5) and (2.3.6) that v satisfies (2.3.3). Moreover, one verifies easily that v is in the regularity class (i), $v(0) = \varphi 1_{B_\rho} \leq 1_{B_\rho}$, and

$$\begin{aligned} v_t - \Delta v &= e^{-t\lambda}[(-\Delta\varphi - \lambda\varphi)z + \varphi(z_t - \Delta z) - 2\nabla\varphi \cdot \nabla z] \\ &= -2e^{-t\lambda}\nabla\varphi \cdot \nabla z \end{aligned}$$

Both φ and z being radially symmetric and decreasing, we see that $\nabla\varphi \cdot \nabla z = \partial_r\varphi\partial_r z \geq 0$; and so, $v_t - \Delta v \leq 0$. This completes the proof. □

PROOF OF THEOREM 2.3.3. We proceed in three steps.

STEP 1. Let $0 < t_0 < T$, $x_0 \in \Omega$, and $R > 0$ such that $B(x_0, R) \subset \Omega$. If $u(t_0, x_0) > 0$, then $u(t,x) > 0$ for all $t > t_0$ and $x \in B(x_0, R)$. To prove this, we may assume without loss of generality that $x_0 = 0$ (by space-translation invariance of the equation). We first observe that, since $u(t_0) \in C(\Omega)$ by assumption (i), and $u(t_0, 0) > 0$, there exist $0 < \rho \leq R$ and $\varepsilon > 0$ such that

$$u(t_0, x) \geq \varepsilon, \quad x \in B_\rho \tag{2.3.7}$$

We consider the function v given by Lemma 2.3.4 and we set

$$w(t) = -u(t)_{|B_R} + \varepsilon v(t - t_0)$$

It easily follows that $w \in C^1((t_0, T], H^1(B_R)) \cap C([t_0, T], L^2(B_R))$, $w_t - \Delta w \leq 0$ in $H^{-1}(B_R)$, $w(t_0) \leq 0$. Moreover, $w(t) \leq \varepsilon v(t - t_0)$ and $v \geq 0$, so that $w^+(t) \leq \varepsilon v(t - t_0)$. Since $v \in C((0,T), H^1_0(B_R))$, we deduce (see Corollary B.2.5) that $w^+ \in C((t_0, T), H^1_0(B_R))$. Thus we may apply Theorem 2.3.1 and we deduce that $w \leq 0$. Therefore, $u(t) \geq \varepsilon v(t - t_0)$ on $(0,T) \times B_R$, which proves the desired conclusion.

STEP 2. We show that for every $0 < t < T$, there exists $x_0 \in \Omega$ such that $u(t, x_0) > 0$. Indeed, let $0 < t < T$. By assumptions (ii), (iii) and (iv), there exists $0 < \tau \leq t$ such that $u(\tau) \geq 0$ and $u(\tau) \not\equiv 0$. Therefore, there exists x_0 such that $u(\tau, x_0) > 0$. By Step 1, $u(t, x_0) > 0$.

STEP 3. Conclusion. Fix $0 < t_0 < T$. We need to show that $u(t_0) > 0$ on Ω. We have $u(t_0) \in C^\infty(\Omega)$ and, by Step 2, there exists $x \in \Omega$ such that $u(t_0, x)$. Therefore,

$$\mathcal{O} = \{x \in \Omega;\, u(t_0, x) > 0\}$$

is a nonempty, open subset of Ω. Since Ω is connected, the result follows if we show that $\mathcal{O}$ is closed. Let $(x_n)_{n\geq 1} \subset \mathcal{O}$ such that $x_n \to x_0 \in \Omega$, and let $R > 0$ be sufficiently small so that $B(x_0, R) \subset \Omega$. Fix n_0 sufficiently large so that $|x_0 - x_{n_0}| \leq \frac{R}{2}$. Since $u(t_0, x_{n_0}) > 0$ (because $x_{n_0} \in \mathcal{O}$) and $u \in C^\infty((0,T) \times \Omega)$, there exists $0 < \tau < t_0$ such that $u(\tau, x_{n_0}) > 0$. Note that $B(x_{n_0}, \frac{R}{2}) \subset B(x_0, R) \subset \Omega$. Therefore, $u(t_0, x) > 0$ for all $x \in B(x_{n_0}, \frac{R}{2})$ by Step 1. In particular, $u(t_0, x_0) > 0$, hence $x_0 \in \mathcal{O}$. Thus $\mathcal{O}$ is closed, which completes the proof. □

We now give a stronger version of the maximum principle, under a geometric assumption on the domain Ω.

THEOREM 2.3.5. *Suppose Ω satisfies the geometric condition (1.2.1). Under the assumptions of Theorem 2.3.3, it follows that there exists a function $c : (0, T) \to (0, \infty)$ such that*

$$u(t, x) \ge c(t)\mathrm{d}(x, \partial\Omega), \quad x \in \Omega, t \in (0, T) \tag{2.3.8}$$

where $\mathrm{d}(x, \partial\Omega)$ is the distance of x to $\partial\Omega$.

PROOF. Let η, ν be given by assumption (1.2.1). Let $0 < \varepsilon \le \eta/2$ and consider $\Omega_\varepsilon = \{x \in \Omega;\ \mathrm{d}(x, \partial\Omega) \ge \varepsilon\}$. We fix $\varepsilon > 0$ sufficiently small so that Ω_ε is a nonempty, compact subset of Ω. Fix $0 < t_0 < T$. It follows from Theorem 2.3.3 and $u(t_0) \in C(\Omega)$ that there exists $\delta > 0$ such that

$$u(t_0, x) \ge \delta, \quad x \in \Omega_\varepsilon \tag{2.3.9}$$

and

$$u(t_0/2, x) \ge \delta, \quad x \in \Omega_\varepsilon \tag{2.3.10}$$

We now consider $x_0 \in \Omega$ such that $\mathrm{d}(x_0, \partial\Omega) < \varepsilon$, and we let $y_0 \in \Omega$ satisfy (1.2.1). Since $B(y_0, \eta) \subset \Omega$ and $\eta \ge 2\varepsilon$, we see that $\mathrm{d}(z, \partial\Omega) \ge \varepsilon$ for all $z \in B(y_0, \eta/2)$. In particular, $z \in \Omega_\varepsilon$, and it follows from (2.3.10) that

$$u(t_0/2, z) \ge \delta \quad \text{for all} \quad z \in B(y_0, \eta/2). \tag{2.3.11}$$

We let $\rho = \eta/2$, $R = \eta$, we consider the function v given by Lemma 2.3.4 and we set

$$w(t, x) = -u(t + t_0/2, y) + \delta v(t, y - y_0), \quad y \in B(y_0, R), 0 \le t < T - t_0/2$$

It easily follows (see Step 1 of the proof of Theorem 2.3.3) that w satisfies the assumptions of Theorem 2.3.1, and we deduce that $w \le 0$. In particular, for $t = t_0/2$,

$$u(t_0, y) = u(t_0/2 + t_0/2, y) \ge \delta v(t_0/2, y - y_0)$$

for $|y| \le R$ so that, applying (2.3.3),

$$u(t_0, y) \ge \delta c\rho^N R^{-1} e^{-\frac{Ct_0}{2R^2}} (t_0/2)^{-\frac{N}{2}} e^{-\frac{2R^2}{t_0}} (R - |y - y_0|)$$

Since $|x_0 - y_0| < \eta = R$ by (1.2.1), we may let $x = x_0$ in the above inequality. Moreover, it follows from (1.2.1) that

$$R - |x_0 - y_0| = \eta - |x_0 - y_0| \ge \nu \mathrm{d}(x_0, \partial\Omega)$$

and we obtain

$$u(t_0, x_0) \ge \nu\delta c\rho^N R^{-1} e^{-\frac{Ct_0}{2R^2}} (t_0/2)^{-\frac{N}{2}} e^{-\frac{2R^2}{t_0}} \mathrm{d}(x_0, \partial\Omega) \tag{2.3.12}$$

Let now $x_0 \in \Omega$. If $\mathrm{d}(x_0, \partial\Omega) \ge \varepsilon$, it follows from (2.3.9) that $u(t_0, x_0) \ge \delta \ge \mu \mathrm{d}(x_0, \partial\Omega)$, where

$$\mu = \frac{\delta}{\sup_{x\in\Omega} \mathrm{d}(x, \partial\Omega)}$$

If $\mathrm{d}(x_0, \partial\Omega) < \varepsilon$, then we deduce from (2.3.12) that $u(t_0, x_0) \ge \mu' \mathrm{d}(x_0, \partial\Omega)$, where $\mu' = \nu\delta c\rho^N R^{-1} e^{-\frac{Ct_0}{2R^2}} (t_0/2)^{-\frac{N}{2}} e^{-\frac{2R^2}{t_0}}$. This completes the proof. □

2.3.3. Some applications of the maximum principle. We begin with a simple, but very useful, application of the weak maximum principle.

PROPOSITION 2.3.6. *Let Ω be a bounded domain and $(\mathcal{T}(t))_{t\ge 0}$ the corresponding heat semigroup, as defined in Proposition 2.2.5. If $u_0 \in L^2(\Omega) \cap L^p(\Omega)$ for some $1 \le p \le \infty$, then $\mathcal{T}(t)u_0 \in L^q(\Omega)$ for all $p \le q \le \infty$, and*

$$\|\mathcal{T}(t)u_0\|_{L^q} \le (4\pi t)^{-\frac{N}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \|u_0\|_{L^p} \tag{2.3.13}$$

for all $t > 0$.

PROOF. Let $u_0 \not\equiv 0$ and set $u(t) = \mathcal{T}(t)u_0$. Suppose first $p < \infty$. By density, we need only consider the case $u_0 \in H^1_0(\Omega) \cap L^p(\Omega)$. Let $\psi \in H^1(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$ be defined by

$$\psi(x) = \begin{cases} |u_0(x)| & x \in \Omega \\ 0 & x \in \mathbb{R}^N \setminus \Omega \end{cases}$$

and let $v(t) = G_t \star \psi$ where G_t is defined by (2.1.4). It follows from Corollary 2.1.6 that $v \in C([0,\infty), H^1(\mathbb{R}^N)) \cap C^1([0,\infty), H^{-1}(\mathbb{R}^N)) \cap C^\infty((0,\infty) \times \mathbb{R}^N)$ and $v_t - \Delta v = 0$. In addition, $v(t) > 0$ on $(0,\infty) \times \mathbb{R}^N$ (since $G_t > 0$). Therefore, if we set $w(t) = v(t)_{|\Omega}$, then $w \in C([0,\infty), H^1(\Omega)) \cap C^1([0,\infty), H^{-1}(\Omega))$, $w_t - \Delta v = 0$ in $H^{-1}(\Omega)$ and $w(t) > 0$ on $(0,\infty) \times \Omega$. It follows easily that both $z_1(t) = u(t) - w(t)$ and $z_2(t) = -u(t) - w(t)$ satisfy the assumptions of Theorem 2.3.1 with $f = 0$ and $v = |u|$. Consequently, $z_1(t), z_2(t) \le 0$, so that $|u(t)| \le w(t)$ a.e. on Ω for all $t \ge 0$. In particular, $\|u(t)\|_{L^q(\Omega)} \le \|w(t)\|_{L^q(\Omega)} \le \|v(t)\|_{L^q(\mathbb{R}^N)}$, and the result is then a consequence of Corollary 2.1.4. In the case $p = \infty$ (hence $q = \infty$), we apply the inequality with $p < \infty$ and $q = \infty$, then we let $p \uparrow \infty$. ☐

COROLLARY 2.3.7. *Let Ω be a bounded domain and $(\mathcal{T}(t))_{t\ge 0}$ the corresponding heat semigroup, as defined in Proposition 2.2.5. Given any $1 \le p < \infty$, it follows that for any $t \ge 0$, $\mathcal{T}(t)$ can be uniquely extended to an operator of $\mathcal{L}(L^p(\Omega))$ such that $\|\mathcal{T}(t)\|_{\mathcal{L}(L^p)} \le 1$, which we also denote by $\mathcal{T}(t)$. Moreover, the map $t \mapsto \mathcal{T}(t)u_0$ is continuous $[0,\infty) \to L^p(\Omega)$, for every $u_0 \in L^p(\Omega)$.*

PROOF. That $\mathcal{T}(t)$ can be uniquely extended to an operator of $\mathcal{L}(L^p(\Omega))$ such that $\|\mathcal{T}(t)\|_{\mathcal{L}(L^p)} \le 1$ is an immediate consequence of estimate (2.3.13) with $q = p$, and the fact that $L^2(\Omega) \cap L^p(\Omega)$ is dense in $L^p(\Omega)$. To prove the continuity of the map $t \mapsto \mathcal{T}(t)u_0$, consider $u_0 \in L^p(\Omega)$, $t \ge 0$, and $(t_n)_{n\ge 1} \subset [0,\infty)$ such that $t_n \to t$ as $n \to \infty$. We need to show that $\|\mathcal{T}(t_n)u_0 - \mathcal{T}(t)u_0\|_{L^p} \to 0$. Let $\varepsilon > 0$. Since $C_c(\Omega)$ is dense in $L^p(\Omega)$, there exists $v_0 \in C_c(\Omega)$ such that $\|u_0 - v_0\|_{L^p} \le \frac{\varepsilon}{4}$; and so

$$\|\mathcal{T}(t_n)(u_0 - v_0) - \mathcal{T}(t)(u_0 - v_0)\|_{L^p} \le \frac{\varepsilon}{2} \tag{2.3.14}$$

by (2.3.13). Moreover, again by (2.3.13), $\|\mathcal{T}(t_n)v_0 - \mathcal{T}(t)v_0\|_{L^\infty} \le 2\|v_0\|_{L^\infty}$. Since $\|\mathcal{T}(t_n)v_0 - \mathcal{T}(t)v_0\|_{L^2} \to 0$ as $n \to \infty$ by Proposition 2.2.5 (iii), we conclude that $\|\mathcal{T}(t_n)v_0 - \mathcal{T}(t)v_0\|_{L^p} \to 0$. Thus $\|\mathcal{T}(t_n)v_0 - \mathcal{T}(t)v_0\|_{L^p} \le \frac{\varepsilon}{2}$ for all sufficiently large n, and the result follows by combining with (2.3.14). ☐

We now apply the weak maximum principle to prove that the heat semigroup operates on $C_0(\Omega)$.

PROPOSITION 2.3.8. *Let Ω be a bounded domain and $(\mathcal{T}(t))_{t\ge 0}$ the corresponding heat semigroup, as defined in Proposition 2.2.5. Suppose further Ω satisfies the geometric condition (1.3.16). It follows that for any $t \ge 0$, $\mathcal{T}(t)$ can be uniquely extended to an operator of $\mathcal{L}(C_0(\Omega))$ such that $\|\mathcal{T}(t)\|_{\mathcal{L}(C_0(\Omega))} \le 1$, which we also denote by $\mathcal{T}(t)$. Moreover, the map $t \mapsto \mathcal{T}(t)u_0$ is continuous $[0,\infty) \to C_0(\Omega)$, for every $u_0 \in C_0(\Omega)$.*

PROOF. We first consider the case $u_0 \in C_c^\infty(\Omega)$, and we set $u(t) = \mathcal{T}(t)u_0$. Let λ_1 be the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$, and $\varphi_1 > 0$ a corresponding eigenvector. (See Section 1.4.) Recall that (see (1.4.18))

$$\varphi_1(\cdot) \le C\mathrm{d}(\cdot, \partial\Omega) \tag{2.3.15}$$

for some constant $C < \infty$. Moreover, $\varphi_1 > 0$ in Ω and $\varphi_1 \in C^\infty\Omega)$. Since u_0 has compact support, it follows that there exists a constant A such that $|u_0| \le A\varphi_1$. Since $\mathcal{T}(t)\varphi_1 = e^{-t\lambda_1}\varphi_1$, we deduce from the weak maximum principle (Theorem 2.3.1) that

$$|u(t)| \le Ce^{-t\lambda_1}\varphi_1 \tag{2.3.16}$$

Estimates (2.3.16) and (2.3.15) yield

$$|u(t,x)| \le AC\mathrm{d}(\cdot, \partial\Omega) \tag{2.3.17}$$

On the other hand, it follows from Propositions 2.2.3 and 2.2.4 that, given any open set $\omega \subset\subset \Omega$, $u \in C([0,\infty), C(\overline{\omega}))$. This, together with the estimate (2.3.17) (which controls what happens near $\partial\Omega$), imply that $u \in C([0,\infty), C_0(\Omega))$.

Since $C_c^\infty(\Omega)$ is dense in $C_0(\Omega)$, the result easily follows from the above property, together with estimate (2.3.13) (with $p = q = \infty$). (See the proof of Corollary 2.1.7.) □

REMARK 2.3.9. One can apply Proposition 2.3.8 (under the assumption that Ω satisfies the geometric condition (1.3.16)) to extend the heat semigroup to the space $M(\Omega)$ of bounded measures. We recall that $M(\Omega)$ is the topological dual of $C_0(\Omega)$, equipped with the weak* topology. Given $u_0 \in M(\Omega)$ and $t \ge 0$, one defines $\mathcal{T}(t)u_0 \in M(\Omega)$ by

$$\langle \mathcal{T}(t)u_0, \varphi\rangle_{M,C_0} = \langle u_0, \mathcal{T}(t)\varphi\rangle_{M,C_0}$$

for all $\varphi \in C_0(\Omega)$. It is not difficult to verify that $\mathcal{T}(t)u_0$ defined as above is indeed an element of $M(\Omega)$, and that this definition is consistent with the previous definition. (That is, if $u_0 \in M(\Omega) \cap H^{-1}(\Omega)$, then $\mathcal{T}(t)u_0$ defined above is the same as $\mathcal{T}(t)u_0$ defined in Section 2.2.2.) Moreover, the map $t \mapsto \mathcal{T}(t)u_0$ is continuous $[0,\infty) \to M(\Omega)$. In addition, $\mathcal{T}(t) \in C_0(\mathbb{R}^N)$ for all $t > 0$ and

$$\|\mathcal{T}(t)u_0\|_{L^\infty} \le (4\pi t)^{-\frac{N}{2}} \|u_0\|_{M(\Omega)} \tag{2.3.18}$$

To prove this last statement, we use the property that $C_c^\infty(\Omega)$ is dense in $M(\Omega)$ for the strong topology. If $u_0 \in C_c^\infty(\Omega)$, then $\mathcal{T}(t)u_0 \in C_0(\Omega)$ by Proposition 2.3.8, and (2.3.18) is (2.3.13) with $p = 1$ and $q = \infty$, since $\|u_0\|_{M(\Omega)} = \|u_0\|_{L^1}$. By density, we conclude that $\mathcal{T}(t)u_0 \in C_0(\Omega)$ for all $u_0 \in M(\Omega)$ and that (2.3.18) holds.

CHAPTER 3

The one-dimensional wave equation

3.1. The wave equation on the line

We consider the wave equation on the line, for which we can use explicit formulas due to D'Alembert. Since the wave equation is of second order in time, we must prescribe both the initial value and the initial velocity. Thus we consider the following Cauchy problem

$$\begin{cases} u_{tt} = u_{xx} & t \in \mathbb{R}, x \in \mathbb{R} \\ u(0,x) = u_0(x) & x \in \mathbb{R} \\ u_t(0,x) = v_0(x) & x \in \mathbb{R} \end{cases} \tag{3.1.1}$$

In order to make the presentation simpler, we consider "localized solutions", i.e. in $L^2(\mathbb{R})$, although this is not necessary due to the finite speed of propagation. We will use the following elementary result

LEMMA 3.1.1. *Let $s \geq 0$ and $\varphi \in H^s(\mathbb{R})$, and let $u(t) \in H^s(\mathbb{R})$ for $t \in \mathbb{R}$ be defined by $u(t,\cdot) = \varphi(\cdot + t)$ (respectively, $u(t,\cdot) = \varphi(\cdot - t)$). It follows that $u \in C(\mathbb{R}, H^s(\mathbb{R})) \cap C^1(\mathbb{R}, H^{s-1}(\mathbb{R}))$, and $u_t = \varphi'(\cdot + t)$ (respectively, $u_t = \varphi'(\cdot - t)$).*

PROOF. This is easily seen by using Fourier. We consider the case $u(t) = \varphi(x+t)$, the other case being similar. Since

$$\widehat{u}(t,\xi) = \int_{\mathbb{R}} e^{-2\pi i x\xi} \varphi(x+t) dx$$

we see by setting $y = x + t$ that

$$\widehat{u}(t,\xi) = \int_{\mathbb{R}} e^{-2\pi i(y-t)\xi} \varphi(y) dy = e^{2\pi i t\xi} \widehat{\varphi}(\xi)$$

Therefore

$$\begin{aligned} \|u(t+h) - u(t)\|_{H^s}^2 &= \int_{\mathbb{R}} |e^{2\pi i(t+h)\xi} - e^{2\pi i t\xi}|(1+\xi^2)^s |\widehat{\varphi}(\xi)|^2 \\ &= \int_{\mathbb{R}} |e^{2\pi i h\xi} - 1|(1+\xi^2)^s |\widehat{\varphi}(\xi)|^2 \end{aligned}$$

Since $\varphi \in H^s(\mathbb{R})$, we have $(1+\xi^2)^{\frac{s}{2}} \widehat{\varphi}(\xi) \in L^2(\mathbb{R})$, so the right-hand side converges to 0 as $h \to 0$, by dominated convergence. Hence $u \in C(\mathbb{R}, H^s(\mathbb{R}))$. Moreover, $u_t = \varphi'(x+t)$, and $\varphi' \in H^{s-1}(\mathbb{R})$. By the preceding result (replacing s by $s-1$) we deduce that $u_t \in C(\mathbb{R}, H^{s-1}(\mathbb{R}))$. □

REMARK 3.1.2. Applying twice Lemma 3.1.1, we see that if $\varphi \in H^s(\mathbb{R})$, $s \geq 1$, then $u \in C(\mathbb{R}, H^s(\mathbb{R})) \cap C^1(\mathbb{R}, H^{s-1}(\mathbb{R})) \cap C^2(\mathbb{R}, H^{s-2}(\mathbb{R}))$, and $u_{tt} = \varphi''(\cdot \pm t)$. Since $u_{xx} = \varphi''(\cdot \pm t)$, we see that u is a solution of the wave equation.

We have the following existence and uniqueness result.

THEOREM 3.1.3. *Given $u_0 \in H^1(\mathbb{R})$ and $v_0 \in L^2(\mathbb{R})$, there exists a unique solution*

$$u \in C(\mathbb{R}, H^1(\mathbb{R})) \cap C^1(\mathbb{R}, L^2(\mathbb{R})) \cap C^2(\mathbb{R}, H^{-1}(\mathbb{R})) \tag{3.1.2}$$

of (3.1.1)*. Moreover, the following properties hold.*

(i) *u is given by the formula*

$$u(t,x) = \frac{1}{2}(u_0(x+t) + u_0(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} v_0(s)\,ds \tag{3.1.3}$$

for all $t, x \in \mathbb{R}$.

(ii) *u satisfies the following estimates*

$$\|u(t)\|_{L^2} \le \|u_0\|_{L^2} + t\|v_0\|_{L^2} \tag{3.1.4}$$

$$\|u_x(t)\|_{L^2} \le \|\partial_x u_0\|_{L^2} + \|v_0\|_{L^2} \tag{3.1.5}$$

$$\|u_t(t)\|_{L^2} \le \|\partial_x u_0\|_{L^2} + \|v_0\|_{L^2} \tag{3.1.6}$$

for all $t \in \mathbb{R}$

(iii) *If, in addition, $u_0 \in H^2(\mathbb{R})$ and $v_0 \in H^1(\mathbb{R})$, then $u \in C(\mathbb{R}, H^2(\mathbb{R})) \cap C^1(\mathbb{R}, H^1(\mathbb{R})) \cap C^2(\mathbb{R}, L^2(\mathbb{R}))$.*

(iv) *We have conservation of energy, that is*

$$\int_{\mathbb{R}} (|u_x|^2 + |u_t|^2) = \int_{\mathbb{R}} (|(u_0)_x|^2 + |v_0|^2) \tag{3.1.7}$$

for all $t \in \mathbb{R}$

Proof. We first prove uniqueness, and we consider u, v in the class (3.1.2) two solutions of (3.1.1). Setting $w = u - v$, it follows that w belongs to the class (3.1.2) and that

$$\begin{cases} w_{tt} = \Delta w \\ w(0) = w_t(0) = 0 \end{cases}$$

Setting $z = (1 + 4\pi^2|\xi|^2)^{-\frac{1}{2}}\widehat{w}$, we deduce that $z \in C^2([0,\infty), L^2(\mathbb{R}))$ and that z satisfies

$$z_{tt} + 4\pi^2|\xi|^2 z = 0$$

in $L^2(\mathbb{R})$ for all $t \in \mathbb{R}$ (see (B.6.42)), with the initial conditions $z(0) = z_t(0) = 0$. We integrate the equation twice in time to obtain

$$z(t,\xi) = -4\pi^2|\xi|^2 \int_0^t \int_0^s z(\tau,\xi)\,d\tau ds$$

The above equation is, for every $t \in \mathbb{R}$, an identity in $L^2(\mathbb{R})$. In particular, we may integrate it on B_R with $0 < R < \infty$, and obtain for $t \ge 0$

$$\begin{aligned}
\int_{B_R} |z(t,\xi)|\,d\xi &\le 4\pi^2R^2 \Big| \int_{B_R} \int_0^t \int_0^s z(\sigma,\tau)\,d\tau ds d\xi \Big| \\
&\le 4\pi^2R^2 \int_0^t \int_0^s \int_{B_R} |z(\tau,\xi)|\,d\xi d\tau ds \\
&\le 4\pi^2R^2 \int_0^t \int_0^t \int_{B_R} |z(\tau,\xi)|\,d\xi d\tau ds \\
&\le 4\pi^2R^2 t \int_0^t \int_{B_R} |z(\tau,\xi)|\,d\xi d\tau
\end{aligned}$$

We deduce by applying Gronwall's lemma that $z(t,\xi) = 0$ a.e on $(0,\infty) \times B_R$. Since $R > 0$ is arbitrary, this implies that $z(t,\xi) = 0$ a.e on $(0,\infty) \times \mathbb{R}$. One shows by similar calculations that $z(t,\xi) = 0$ a.e on $(-\infty, 0) \times \mathbb{R}$. This implies that $\widehat{w}(t) = 0$ for all $t \in \mathbb{R}$, hence $w(t) \equiv 0$. This completes the proof of uniqueness.

We now prove the existence part. Set

$$U(t,x) = \frac{1}{2}(u_0(x+t) + u_0(x-t))$$

It follows from Lemma 3.1.1 that $U \in C(\mathbb{R}, H^1(\mathbb{R})) \cap C^1(\mathbb{R}, L^2(\mathbb{R})) \cap C^2(\mathbb{R}, H^{-1}(\mathbb{R}))$ and U is a solution of the wave equation. Moreover, $U(0) = u_0$ and $U_t(0) = 0$. Next, set

$$V(t,x) = \frac{1}{2} \int_{x-t}^{x+t} v_0(s)\, ds$$

It follows that $V(0) = 0$ and, for $t > 0$

$$\begin{aligned} \int_{\mathbb{R}} |V(t,x)|^2\, ds &= \frac{1}{4} \int_{\mathbb{R}} \Big(\int_{x-t}^{x+t} v_0(s)\, ds \Big)^2 dx \le \frac{t}{2} \int_{\mathbb{R}} dx \int_{x-t}^{x+t} |v_0(s)|^2 ds \\ &= \frac{t}{2} \int_{\mathbb{R}} ds \int_{s-t}^{s+t} |v_0(s)|^2 dx = t^2 \int_{\mathbb{R}} |v_0(s)|^2 \end{aligned} \tag{3.1.8}$$

where we used Fubini. A similar calculation for $t < 0$ shows that $V(t) \in L^2(\mathbb{R})$ for all $t \in \mathbb{R}$. One shows a well that $V \in C(\mathbb{R}, L^2(\mathbb{R}))$. Moreover,

$$V_x(t,x) = \frac{1}{2}(v_0(x+t) + v_0(x-t))$$

and it follows from Lemma 3.1.1 that $V_x \in C(\mathbb{R}, L^2(\mathbb{R}))$. Thus we conclude that $V \in C(\mathbb{R}, H^1(\mathbb{R}))$. In addition

$$V_t(t,x) = \frac{1}{2}(v_0(x+t) - v_0(x-t))$$

and so $V_t(0) = v_0$; and Lemma 3.1.1 shows that $V_t \in C(\mathbb{R}, L^2(\mathbb{R})) \cap C^1(\mathbb{R}, H^{-1}(\mathbb{R}))$. It is immediate that V is a solution of the wave equation, thus we see that $u = U + V$ has the desired properties. This proves the existence part, as well as formula (3.1.3).

Estimates (3.1.5) and (3.1.6) are immediate consequences of formula (3.1.3). Estimate (3.1.4) follows from formula (3.1.3), together with (3.1.8) for the estimate of $\|V(t)\|_{L^2}$. This proves Property (ii).

Property (iii) is a consequence of formula (3.1.3) and Lemma 3.1.1.

It remains to prove the conservation of energy. We first suppose $u_0 \in H^2(\mathbb{R})$ and $v_0 \in H^1(\mathbb{R})$. The equation is satisfied in $L^2(\mathbb{R})$, so we may multiply it by $u_t \in H^1(\mathbb{R})$. It follows that

$$\int_{\mathbb{R}} u_{tt} u_t + \int_{\mathbb{R}} u_x u_x t = 0$$

i.e.

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} (|u_x|^2 + |u_t|^2) = 0$$

which proves (3.1.7). In the general case, we approximate u_0 in H^1 by a sequence $(u_0^n)_{n \ge 1} \subset H^2(\mathbb{R})$, and v_0 in L^2 by a sequence $(v_0^n)_{n \ge 1} \subset H^1(\mathbb{R})$. We therefore obtain a sequence of solutions $(u^n)_{n \ge 1}$. It follows from (3.1.4)–(3.1.6) (applied with u_0 and v_0 replaced by $u_0 - u_0^n$ and $v_0 - v_0^n$) that $u^n \to u$ in $C([-T,T], H^1(\mathbb{R})) \cap C^1([-T,T], L^2(\mathbb{R}))$ for all $T < \infty$. Therefore, we can apply the conservation of energy to u^n, and then let $n \to \infty$ to obtain the conservation of energy for u. ☐

REMARK 3.1.4. Here are some observations.

(i) Formula (3.1.3) shows that, given $t, x \in \mathbb{R}$, the solution $u(t,x)$ depends on the initial data u_0 and v_0 in the interval $[x - |t|, x + |t|]$.

(ii) It follows from the preceding observation that if u_0 and v_0 have compact support, say in $[-R, R]$, then $u(t)$ also have compact support, more precisely in $[-R - |t|, R + |t|]$.

(iii) Given $0 < R < \infty$, it follows easily from formula (3.1.3) that, as $t \to \pm\infty$, $\|u_x(t)\|_{L^2(-R,R)} + \|u_t(t)\|_{L^2(-R,R)} \to 0$. This means that, on every bounded interval, the solution becomes more and more flat as $t \to \pm\infty$.

REMARK 3.1.5. Some special solutions.

(i) Given $\varphi \in H^1(\mathbb{R})$, $u(t,x) = \varphi(x \pm t)$ is a solution of the wave equation. It is a given profile, that travels at constant velocity, either to the right or to the left.
(ii) Given $\varphi \in H^1(\mathbb{R})$, $u(t,x) = \frac{1}{2}(\varphi(x-t) + \varphi(x+t))$ is a solution of the wave equation. It is the sum of two half profiles, one of which travels to the left, the other to the right, with constant velocity.

3.2. The wave equation on an interval

We now consider the wave equation on the interval $\Omega = (0, \ell)$, with Dirichlet boundary conditions

$$\begin{cases} u_{tt} = u_{xx} & t \in \mathbb{R}, x \in \Omega \\ u(t,0) = u(t,\ell) = 0 & t \in \mathbb{R} \\ u(0,x) = u_0(x) & x \in \Omega \\ u_t(0,x) = v_0(x) & x \in \Omega \end{cases} \tag{3.2.1}$$

We recall that in this case, the eigenvalues of $-\Delta$ in $H^1_0(\Omega)$ are $\lambda_j = (\frac{j\pi}{\ell})^2$ and the corresponding eigenfunctions

$$\varphi_j(x) = \sqrt{\frac{2}{\ell}} \sin\Big(\frac{j\pi}{\ell} x\Big)$$

See Remark 1.4.7.

THEOREM 3.2.1. *Let $u_0 = \sum a^0_j \varphi_j \in H^1_0(\Omega)$ and $v_0 = \sum b^0_j \varphi_j \in L^2(\Omega)$. It follows that (3.2.1) has a unique solution $u \in C(\mathbb{R}, H^1_0(\Omega)) \cap C^1(\mathbb{R}, L^2(\Omega)) \cap C^2(\mathbb{R}, H^{-1}(\Omega))$. Moreover, u is given by the formula*

$$u(t) = \sum_{j=1}^{\infty} a_j(t) \varphi_j \tag{3.2.2}$$

with

$$a_j(t) = a^0_j \cos\Big(\frac{j\pi}{\ell} t\Big) + \frac{\ell b^0_j}{\pi j} \sin\Big(\frac{j\pi}{\ell} t\Big) \tag{3.2.3}$$

In addition, there is conservation of energy, i.e.

$$\int_0^\ell (|u_x|^2 + |u_t|^2) = \int_0^\ell (|(u_0)_x|^2 + |v_0|^2) \tag{3.2.4}$$

for all $t \in \mathbb{R}$

PROOF. We first prove uniqueness. Let u be a solution. In particular, the function $t \mapsto \langle u(t), \varphi_j \rangle_{H^{-1}, H^1_0}$ is C^2 and

$$\frac{d^2}{dt^2} \langle u, \varphi_j \rangle_{H^{-1}, H^1_0} = \langle u_{xx}, \varphi_j \rangle_{H^{-1}, H^1_0} = \langle u, \partial_{xx} \varphi_j \rangle_{H^{-1}, H^1_0} = -\lambda_j \langle u, \varphi_j \rangle_{H^{-1}, H^1_0}$$

Therefore, if we write u in the form (3.2.2), then $a_j = \langle u, \varphi_j \rangle_{H^{-1}, H^1_0}$, so that $a_j \in C^2(\mathbb{R})$ and $a''_j + \lambda_j a_j = 0$. It follows that a_j is given by formula (3.2.3). Therefore, a_j is uniquely determined, hence u is uniquely determined.

Let now u be defined by (3.2.2)-(3.2.3). It follows easily (see the proof of Theorem 2.2.1) that u has the required regularity and that u is a solution of (3.2.1).

We finally prove conservation of energy. Since $a''_j + \lambda_j a_j = 0$, we have $\lambda_j a_j^2 + (a'_j)^2 = \lambda_j (a^0_J)^2 + (b^0_J)^2$; and so

$$\sum_{j=1}^{\infty} \lambda_j a_j^2 + (a'_j)^2 = \sum_{j=1}^{\infty} \lambda_j (a^0_J)^2 + (b^0_J)^2$$

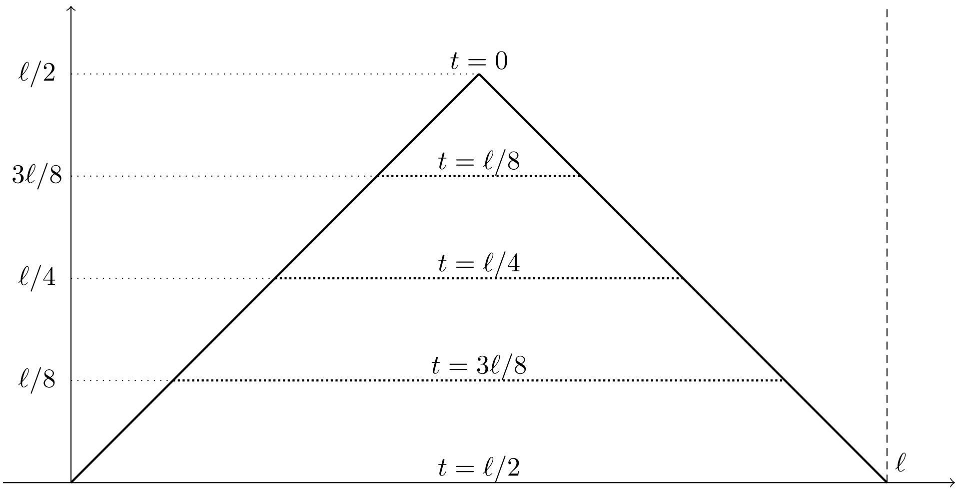

Figure 1. A solution of the wave equation

which yields (3.2.4). □

Corollary 3.2.2. *For every $u_0 \in H_0^1(\Omega)$ and $v_0 \in L^2(\Omega)$, the corresponding solution $u \in C(\mathbb{R}, H_0^1(\Omega)) \cap C^1(\mathbb{R}, L^2(\Omega)) \cap C^2(\mathbb{R}, H^{-1}(\Omega))$ of (3.2.1) is periodic (in time) with period 2ℓ.*

Proof. u is given by formula (3.2.2)-(3.2.3). Note that $\cos(\frac{j\pi}{\ell}t)$ and $\sin(\frac{j\pi}{\ell}t)$ are both periodic with period $\frac{2\ell}{j}$, hence also periodic with period 2ℓ. Therefore, all the coefficients a_j in (3.2.3) are 2ℓ-periodic, hence so is u. □

Remark 3.2.3. Here are some particular solutions of the wave equation on $(0, \ell)$.

(i) $u(t,x) = \cos(\frac{j\pi}{\ell}t)\sin(\frac{j\pi}{\ell}x)$ is a solution, corresponding to $u_0 = \sin(\frac{j\pi}{\ell}x)$ and $v_0 = 0$.

(ii) Let

$$\theta(x) = \begin{cases} x & 0 \le x \le \frac{\ell}{2} \\ \ell - x & 0 \le \frac{\ell}{2} \le x \le \ell \end{cases}$$

The solution of the wave equation with $u_0 = \theta$ and $v_0 = 0$ is given by (see Figure 1)

$$u(t,x) = \begin{cases} x & 0 \le x \le \frac{\ell}{2} - t \\ \frac{\ell}{2} - t & \frac{\ell}{2} - t \le x \le \frac{\ell}{2} + t \\ \ell - x & 0 \le \frac{\ell}{2} + t \le x \le \ell \end{cases}$$

for $0 \le t \le \frac{\ell}{2}$. For $\frac{\ell}{2} \le t \le \ell$, it is given by

$$u(t) = -u(\ell - t)$$

for $\ell \le t \le 2\ell$, it is given by

$$u(t) = u(2\ell - t)$$

and then it is reproduced by 2ℓ-periodicity.

APPENDIX A

Some useful results

We collect in this appendix some useful results.

A.1. Functional analysis

THEOREM A.1.1 (The Banach fixed point theorem). *Let* (E, d) *be a nonempty complete metric space and* $F : E \to E$*. Suppose* F *is a strict contraction, i.e. there exists a constant* $0 \le k < 1$ *such that* $\mathrm{d}(F(x), F(y)) \le k\mathrm{d}(x, y)$ *for all* $x, y \in E$*. It follows that there exists a unique* $x \in E$ *such that* $F(x) = x$*.*

PROOF. Fix $x_0 \in E$ and define x_n recursively by $x_{n+1} = F(x_n)$. Given $n \ge 1$, we have $\mathrm{d}(x_{n+1}, x_n) = \mathrm{d}(F(x_n), F(x_{n-1})) \le k\mathrm{d}(x_n, x_{n-1})$, and an immediate iteration argument shows that

$$\mathrm{d}(x_{n+1}, x_n) \le k^n \mathrm{d}(x_1, x_0), \quad n \ge 0$$

Given $0 \le n < m$, we deduce that

$$\mathrm{d}(x_m, x_n) \le \sum_{j=0}^{m-n-1} \mathrm{d}(x_{n+j+1}, x_{n+j}) \le \mathrm{d}(x_1, x_0) \sum_{j=0}^{m-n-1} k^{n+j} \le \frac{k^n}{1-k} \mathrm{d}(x_1, x_0)$$

It follows that $(x_n)_{n \ge 0}$ is a Cauchy sequence, so it has a limit $x \in E$. Passing to the limit in the equality $x_{n+1} = F(x_n)$, we deduce that $x = F(x)$. Uniqueness follows from the contraction property. □

THEOREM A.1.2 (Lax-Milgram [20, Theorem 2.1]). *Let* H *be a Hilbert space and consider a bilinear functional* $b : H \times H \to \mathbb{R}$*. If there exist* $C < \infty$ *and* $\alpha > 0$ *such that*

$$\begin{cases} |b(u, v)| \le C\|u\| \, \|v\|, & (u, v) \in H \times H \ \textit{(continuity)}, \\ |b(u, u)| \ge \alpha\|u\|^2, & u \in H \ \textit{(coerciveness)}, \end{cases}$$

then, for every $f \in H^\star$ *(the dual space of* H*), the equation*

$$b(u, v) = \langle f, v \rangle_{H^\star, H} \quad \textit{for all} \quad v \in H, \tag{A.1.1}$$

has a unique solution $u \in H$*.*

PROOF. By the Riesz representation theorem, there exists $\varphi \in H$ such that

$$\langle f, v \rangle_{H^\star, H} = (\varphi, v)_H,$$

for all $v \in H$. Furthermore, given $u \in H$, the map $v \mapsto b(u, v)$ defines an element of $H^\star$; and so, by the Riesz representation theorem, there exists an element of H, which we denote by Au, such that

$$b(u, v) = (Au, v)_H,$$

for all $v \in H$. It is clear that $A : H \to H$ is a linear operator such that

$$\begin{cases} \|Au\|_H \le C\|u\|_H, \\ (Au, u)_H \ge \alpha\|u\|_H^2, \end{cases}$$

for all $u \in H$. We see that (A.1.1) is equivalent to $Au = \varphi$. Given $\rho > 0$, this last equation is equivalent to

$$u = Tu, \tag{A.1.2}$$

where $Tu = u + \rho\varphi - \rho Au$. It is clear that $T : H \to H$ is continuous. Moreover, $Tu - Tv = (u - v) - \rho A(u - v)$; and so,

$$\begin{aligned}\|Tu - Tv\|_H^2 &= \|u - v\|_H^2 + \rho^2 \|A(u - v)\|_H^2 - 2\rho (A(u - v), u - v)_H \\ &\leq (1 + \rho^2 C^2 - 2\rho\alpha)\|u - v\|_H^2.\end{aligned}$$

Choosing $\rho > 0$ small enough so that $1 + \rho^2 C^2 - 2\rho\alpha < 1$, T is a strict contraction. By Banach's fixed point theorem, we deduce that T has a unique fixed point $u \in H$, which is the unique solution of (A.1.2). □

THEOREM A.1.3. *Let X be a Banach space, $T > 0$ and $f \in C([0, T), X)$. It f is right-differentiable for all $t \in [0, T)$ and $\frac{d^+ f}{dt} \in C([0, T), X)$, then $f \in C^1([0, T), X)$ and $\frac{df}{dt} = \frac{d^+ f}{dt}$.*

PROOF. Set

$$g(t) = f(t) - f(0) - \int_0^t \frac{d^+ f}{dt}\, ds,$$

for all $t \in [0, T)$. It follows that $g \in C([0, T), X)$, $g(0) = 0$, g is right-differentiable for all $t \in [0, T)$ and $\frac{d^+ g}{dt} = 0$. Let now $\xi \in X^\star$, and set $h(t) = \langle \xi, g(t)\rangle_{X^\star, X}$. We have $h \in C([0, T))$, $h(0) = 0$, h is right-differentiable for all $t \in [0, T)$ and $\frac{d^+ h}{dt} = 0$. We show that $h \equiv 0$. To see this, let $\varepsilon > 0$, set $h_\varepsilon(t) = h(t) - \varepsilon t$, and let us show that $h_\varepsilon \leq 0$. Otherwise, there exists $t \in [0, T)$ such that $h_\varepsilon(t) > 0$. Let $\tau = \inf\{t \in [0, T); \, h_\varepsilon(t) > 0\}$. We have $h_\varepsilon(\tau) = 0$, and there exists $t_n \downarrow \tau$ such that $h_\varepsilon(t_n) > 0$. It follows that

$$\limsup_{t \downarrow \tau} \frac{h_\varepsilon(t) - h_\varepsilon(\tau)}{t - \tau} \geq 0.$$

On the other hand, we have $\frac{d^+ h_\varepsilon}{dt} = -\varepsilon$, which is a contradiction. Therefore, $h_\varepsilon \leq 0$. Since $\varepsilon > 0$ is arbitrary, we have $h \leq 0$. Applying the same argument to $-h$, we obtain as well $h \geq 0$, hence $h \equiv 0$. Therefore, given $t \in [0, T)$, we have $\langle \xi, g(t)\rangle_{X^\star, X} = 0$ for all $\xi \in X^\star$; and so, $g(t) \equiv 0$. The result follows easily. □

REMARK A.1.4. In Theorem A.1.3 above, if $\frac{d^+ f}{dt} \in C([0, T], X)$, then $f \in C^1([0, T], X)$. This follows easily from the formula

$$f(t) = f(0) + \int_0^t f'(s)\, ds$$

for all $0 \leq t < T$.

THEOREM A.1.5. *If X and Y are two Banach spaces such that $X \hookrightarrow Y$ with dense embedding, then, the following properties hold.*

(i) *$Y^\star \hookrightarrow X^\star$, where the embedding e is defined by $\langle ef, x\rangle_{X^\star, X} = \langle f, x\rangle_{Y^\star, Y}$, for all $x \in X$ and $f \in Y^\star$.*
(ii) *If X is reflexive, then the embedding $Y^\star \hookrightarrow X^\star$ is dense.*
(iii) *If the embedding $X \hookrightarrow Y$ is compact and X is separable, then the embedding $Y^\star \hookrightarrow X^\star$ is compact. More precisely, if $(y'_n)_{n\geq 0} \subset Y^\star$ and $\|y'_n\|_{Y^\star} \leq M$, then there exist a subsequence $(n_k)_{k\geq 0}$ and $y' \in Y^\star$ with $\|y'\|_{Y^\star} \leq M$ such that $y'_{n_k} \to y'$ in $X^\star$ as $k \to \infty$.*

Proof. (i) Given $y' \in Y^\star$, set $ey'(x) = \langle y', x\rangle_{Y^\star,Y}$ for all $x \in X \hookrightarrow Y$. Since

$$|ey'(x)| \le \|y'\|_{Y^\star}\|x\|_Y \le C\|y'\|_{Y^\star}\|x\|_X,$$

we see that $e \in \mathcal{L}(Y^\star, X^\star)$. Suppose that $ey' = ez'$, for some $y', z' \in Y^\star$. It follows that $\langle y' - z', x\rangle_{Y^\star,Y} = 0$, for every $x \in X$. By density, we deduce that $\langle y' - z', y\rangle_{Y^\star,Y} = 0$, for every $y \in Y$; and so $y' = z'$. Thus e is injective and (i) follows.

(ii) Assume to the contrary that $\overline{Y^\star} \neq X^\star$. Then there exists $x_0 \in X^{\star\star} = X$ such that $\langle y', x_0\rangle_{X^\star,X} = 0$, for every $y' \in Y^\star$ (see e.g. [5, Corollary 1.8, p. 8]). Let $E = \mathbb{R}x_0 \subset Y$, and let $f \in E^\star$ be defined by $f(\lambda x_0) = \lambda$, for $\lambda \in \mathbb{R}$. We have $\|f\|_{E^\star} = 1$, and by the Hahn-Banach theorem (see e.g. [5, Corollary 1.2, p. 3]) there exists $y' \in Y^\star$ such that $\|y'\|_{Y^\star} = 1$ and $\langle y', x_0\rangle_{Y^\star,Y} = 1$, which is a contradiction, since $\langle y', x_0\rangle_{Y^\star,Y} = \langle y', x_0\rangle_{X^\star,X} = 0$.

(iii) Let $B_{X^\star}$ (respectively, B_X, $B_{Y^\star}$, B_Y) be the unit ball of $X^\star$ (respectively, X, $Y^\star$, Y). Consider a sequence $(y'_n)_{n\ge 0} \subset B_{Y^\star}$. Since $Y^\star$ is the dual of a separable Banach space, it follows (see e.g. [5, Corollary 3.30, p. 76]) that there exist a subsequence, which we still denote by $(y'_n)_{n\ge 0}$, and an element $y' \in B_{Y^\star}$ such that $y'_n \to y'$ in $Y^\star$ weak*. We show that $\|y'_n - y'\|_{X^\star} \to 0$, which proves the desired result. We note that

$$\|y'_n - y'\|_{X^\star} = \sup_{x\in B_X} |\langle y'_n - y', x\rangle_{X^\star,X}| = \sup_{x\in B_X} |\langle y'_n - y', x\rangle_{Y^\star,Y}|, \tag{A.1.3}$$

by (i). Let $\varepsilon > 0$. Since B_X is a relatively compact subset of Y, we see that there exists a (finite) sequence $(x_j)_{1\le j\le \ell} \subset B_X$ such that for every $x \in B_X$, there exists $1 \le j \le \ell$ such that $\|x - x_j\|_Y \le \varepsilon$. Given $x \in B_X$ and $1 \le j \le \ell$ as above, we deduce that

$$\begin{aligned}
|\langle y'_n - y', x\rangle_{Y^\star,Y}| &\le |\langle y'_n - y', x - x_j\rangle_{Y^\star,Y}| + |\langle y'_n - y', x_j\rangle_{Y^\star,Y}| \\
&\le \varepsilon\|y'_n - y'\|_{Y^\star} + |\langle y'_n - y', x_j\rangle_{Y^\star,Y}| \\
&\le 2\varepsilon + |\langle y'_n - y', x_j\rangle_{Y^\star,Y}|.
\end{aligned}$$

Aplying now (A.1.3), we deduce that

$$\|y'_n - y'\|_{X^\star} \le 2\varepsilon + \sup_{1\le j\le \ell} |\langle y'_n - y', x_j\rangle_{Y^\star,Y}|.$$

Since $y'_n \to y'$ in $Y^\star$ weak*, we conclude that

$$\limsup_{n\to\infty} \|y'_n - y'\|_{X^\star} \le 2\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, the result follows. ☐

Corollary A.1.6. *Let H be a Hilbert space, and identify $H^\star = H$ by the Riesz representation theorem. Let X be a Banach space such that $X \hookrightarrow H$ with dense embedding, so that*

$$X \hookrightarrow H = H^\star \hookrightarrow X^\star$$

by Theorem A.1.5. If $T > 0$ and $u \in C((0,T), X) \cap C^1((0,T), X^\star)$, then the map $t \mapsto \|u(t)\|_H^2$ is C^1 on $(0,T)$, and

$$\frac{d}{dt}\|u(t)\|_H^2 = 2\langle u_t(t), u(t)\rangle_{X^\star,X} \tag{A.1.4}$$

for all $t \in (0,T)$.

Proof. Set $F(t) = \|u(t)\|_H^2$. Since $u \in C((0,T), X)$ and $X \hookrightarrow H$, we see that $F \in C((0,T))$. In addition, given $0 < s, t < T$, $s \neq t$, it follows from Theorem A.1.5 (i) that

$$\frac{F(t) - F(s)}{t - s} = \Big(\frac{u(t) - u(s)}{t - s}, u(t) + u(s)\Big)_H = \Big\langle \frac{u(t) - u(s)}{t - s}, u(t) + u(s)\Big\rangle_{X^\star,X}$$

which proves (A.1.4). □

If X is a separable Banach space, then its dual $X^\star$ needs not be separable. (For example $X = L^1(\Omega)$ is separable, but it dual $L^\infty(\Omega)$ is not). However, $X^\star$ is weak* separable, as shows the following result.

Lemma A.1.7. *Let X be a separable Banach space and let $X^\star$ be its dual. There exists a sequence $(x'_n)_{n\in\mathbb{N}} \subset X^\star$ such that for every $x' \in X^\star$, there exists a subsequence $(x'_{n_k})_{k\in\mathbb{N}}$ with the following properties:*

(i) $x'_{n_k} \to x'$ *weak* as* $k \to \infty$.
(ii) $\|x'_{n_k}\|_{X^\star} \le \|x'\|_{X^\star}$.
(iii) $\|x'_{n_k}\|_{X^\star} \to \|x'\|_{X^\star}$ *as* $k \to \infty$.

Proof. $B' = \{x' \in X^\star;\, \|x\|_{X^\star} \le 1\}$ equipped with the weak* topology of $X^\star$, is a compact metric space. In particular, B' is separable and we denote by $(y'_n)_{n\in\mathbb{N}}$ a dense sequence in B'. Given $x' \in X^\star$, there exists a sequence $(n_k)_{k\in\mathbb{N}}$ such that $y'_{n_k} \to x'/\|x'\|_{X^\star}$ weak* as $k \to \infty$. Consider now a sequence $(\lambda_k)_{k\in\mathbb{N}}$ such that $\lambda_k \to \|x'\|_{X^\star}$ as $k \to \infty$ and $0 < \lambda_k \le \|x'\|_{X^\star}$. It follows that $\lambda_k y'_{n_k} \to x'$ weak* as $k \to \infty$. Furthermore, $\|\lambda_k y'_{n_k}\|_{X^\star} \le |\lambda_k| \|y'_{n_k}\|_{X^\star} \le \|x'\|_{X^\star}$. Since also $\|x'\|_{X^\star} \le \liminf \|\lambda_k y'_{n_k}\|_{X^\star}$ as $k \to \infty$, the result follows with $(x'_n)_{n\in\mathbb{N}} = \cup_{\substack{\lambda\in\mathbb{Q}\\ n\in\mathbb{N}}} \{\lambda y'_n\}$. □

Lemma A.1.8. *Let $X \hookrightarrow Y$ be two Banach spaces and let $(x_n)_{n\in\mathbb{N}}$ be a sequence of X. If $x_n \rightharpoonup x$ in X, as $n \to \infty$, then $x_n \rightharpoonup x$ in Y, as $n \to \infty$.*

Proof. The embedding is continuous $X \to Y$; and so, it is also continuous $X \to Y$ for the weak topologies. The result follows. □

Lemma A.1.9. *Let $X \hookrightarrow Y$ be two Banach spaces and let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence of X such that $x_n \rightharpoonup y$ in Y, as $n \to \infty$, for some $y \in Y$. If X is reflexive, then $y \in X$ and $x_n \rightharpoonup y$ in X, as $n \to \infty$.*

Proof. Let us first prove that $y \in X$. There exists $x \in X$ and a subsequence n_k such that $x_{n_k} \rightharpoonup x$ in X, as $k \to \infty$. Therefore, by Lemma A.1.8, $x_{n_k} \rightharpoonup x$ in Y, as $k \to \infty$. It follows that $y = x \in X$.

Let us prove that $x_n \rightharpoonup y$ in X by contradiction. If not, there exists $x' \in X^\star$, $\varepsilon > 0$ and a subsequence n_k such that $|\langle x', x_{n_k} - y\rangle| \ge \varepsilon$, for every $k \in \mathbb{N}$. On the other hand, there exists $x \in X$ and a subsequence n_{k_j} such that $x_{n_{k_j}} \rightharpoonup x$ in X as $j \to \infty$. In particular, $x = y$; and so $x_{n_{k_j}} \rightharpoonup y$ in X as $j \to \infty$, which is a contradiction. □

Corollary A.1.10. *Let $X \hookrightarrow Y$ be two Banach spaces. If Y is separable and X is reflexive, then X is separable.*

Proof. Let B be the closed unit ball of X. Since $B \subset Y$ and Y is separable, it follows that B is separable for the Y norm. Therefore, there exists a sequence $(x_n)_{n\in\mathbb{N}} \subset B$ such that for every $x \in X$, there exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ which converges to x strongly in Y, hence weakly in X by Lemma A.1.9. Therefore, B is contained in, hence equal to the weak X closure of the set $(x_n)_{n\in\mathbb{N}}$. In particular, B is also the weak X closure of the convex hull C of the set $(x_n)_{n\in\mathbb{N}}$. Since the weak and strong closures of convex sets coincide, it follows that C is strongly X dense in B. Since the convex hull of a countable set is clearly separable, it follows that B is separable, which completes the proof. □

Corollary A.1.11. *Let $X \hookrightarrow Y$ be two Banach spaces, let I be a bounded, open interval of $\mathbb{R}$, and let $u : \overline{I} \to Y$ be a weakly continuous function. Assume that there exists a dense subset E of I such that*

(i) $u(t) \in X$, *for all* $t \in E$,

(ii) $\sup\{\|u(t)\|_X, t \in E\} = K < \infty$.

If X is reflexive, then $u(t) \in X$ for all $t \in \overline{I}$ and $u : \overline{I} \to X$ is weakly continuous.

PROOF. Let $t \in \overline{I}$ and let $(t_n)_{n\in\mathbb{N}} \subset E$ converge to t, as $n \to \infty$. Since $u(t_n) \rightharpoonup u(t)$ in Y, it follows from Lemma A.1.9 that $u(t) \in X$ and that

$$\|u(t)\|_X \leq \liminf_{n\to\infty} \|u(t_n)\|_X \leq K.$$

Let now $t \in \overline{I}$ and let $(t_n)_{n\in\mathbb{N}} \subset \overline{I}$ converge to t, as $n \to \infty$. Since $u(t_n) \rightharpoonup u(t)$ in Y and $u(t_n)$ is bounded in X, it follows from Lemma A.1.9 that $u(t_n) \rightharpoonup u(t)$ in X. Hence the result. □

THEOREM A.1.12. *Let Y be a Banach space, and X be a topological vector space If $X \hookrightarrow Y$ with dense embedding and I is a bounded, closed interval of $\mathbb{R}$ then $C(I,X)$ is a dense subspace of $C(I,Y)$.*

PROOF. Let $f \in C(I,Y)$. One can approximate f in $C(I,Y)$ by a sequence of piecewise linear functions $I \to Y$. Thus we may assume $f \in C(I,Y)$ is piecewise linear. It is now easy to approximate f in $C(I,Y)$ by a sequence of piecewise linear functions $I \to X$. □

It is sometimes convenient to consider the intersection and sum of Banach spaces. Consider two Banach spaces X_1 and X_2 that are subsets of a Hausdorff topological vector space $\mathcal{X}$. Let

$$X_1 \cap X_2 = \{x \in \mathcal{X};\, x \in X_1,\, x \in X_2\},$$

and

$$X_1 + X_2 = \{x \in \mathcal{X};\, \exists x_1 \in X_1,\, \exists x_2 \in X_2,\, x = x_1 + x_2\}.$$

Define

$$\|x\|_{X_1\cap X_2} = \|x\|_{X_1} + \|x\|_{X_2}, \text{ for } x \in X_1 \cap X_2,$$

and

$$\|x\|_{X_1+X_2} = \mathrm{Inf}\{\|x_1\|_{X_1} + \|x_2\|_{X_2};\, x = x_1 + x_2\}, \text{ for } x \in X_1 + X_2.$$

We have the following result.

PROPOSITION A.1.13. *$(X_1\cap X_2, \|\,\|_{X_1\cap X_2})$ and $(X_1+X_2, \|\,\|_{X_1+X_2})$ are Banach spaces. If furthermore $X_1 \cap X_2$ is a dense subset of both X_1 and X_2, then the following properties hold:*

(i) $(X_1 \cap X_2)^\star = X_1^\star + X_2^\star$ *and* $(X_1 + X_2)^\star = X_1^\star \cap X_2^\star$;

(ii) $\langle f, x_1 + x_2\rangle_{X_1^\star\cap X_2^\star, X_1+X_2} = \langle f, x_1\rangle_{X_1^\star,X_1} + \langle f, x_2\rangle_{X_2^\star,X_2}$, *for all* $f \in X_1^\star \cap X_2^\star$ *and* $(x_1, x_2) \in X_1 \times X_2$;

(iii) $\langle f_1 + f_2, x\rangle_{X_1^\star+X_2^\star, X_1\cap X_2} = \langle f_1, x\rangle_{X_1^\star,X_1} + \langle f_2, x\rangle_{X_2^\star,X_2}$, *for all* $(f_1, f_2) \in X_1^\star \times X_2^\star$ *and* $x \in X_1 \cap X_2$;

(iv) *if X_1 and X_2 are reflexive, then $X_1 \cap X_2$ and $X_1 + X_2$ are reflexive.*

PROOF. Property (i), (ii) and (iii) follow from [4, Lemma 2.3.1, Theorem 2.7.1, proof of Theorem 2.7.1]. To prove Property (iv) we note that (by (i)) it is sufficient to show that $X_1\cap X_2$ is reflexive. By applying Eberlein-Šmulian's theorem, we need to show that every bounded sequence $(x_n)_{n\in\mathbb{N}} \subset X_1 \cap X_2$ has a weakly convergent subsequence. Since x_n is bounded in both X_1 and X_2, there exists $x \in X_1 \cap X_2$ and a subsequence, which we still denote by $(x_n)_{n\in\mathbb{N}}$, such that $x_n \rightharpoonup x$, in X_1 and in X_2. Given $(f_1, f_2) \in X_1^\star \times X_2^\star$, we have

$$\langle f_1, x_n\rangle_{X_1^\star,X_1} + \langle f_2, x_n\rangle_{X_2^\star,X_2} \underset{n\to\infty}{\longrightarrow} \langle f_1, x\rangle_{X_1^\star,X_1} + \langle f_2, x\rangle_{X_2^\star,X_2}.$$

Applying Property (iii), we deduce that $x_n \rightharpoonup x$ in $X_1 \cap X_2$. □

REMARK A.1.14. It is clear that the definition of the spaces $X_1 \cap X_2$ and $X_1 + X_2$ as well as their properties described in Proposition A.1.13 are independent of the Hausdorff space $\mathcal{X}$. It follows that an element of $X_1 + X_2$ is equal to zero if and only if it is equal to zero in some Hausdorff space containing $X_1 \cup X_2$. In particular, if X_1 and X_2 are spaces of distributions on some open set $\Omega \subset \mathbb{R}^N$, then an element of $X_1 + X_2$ is equal to zero if and only if it is equal to zero in $\mathcal{D}'(\Omega)$.

We recall below a quite useful result concerning L^p spaces.

LEMMA A.1.15. *Let Ω be an open subset of $\mathbb{R}^N$, $N \geq 1$ and let $1 < p \leq \infty$. Consider $u : \Omega \to \mathbb{R}$ and let $(u_n)_{n \in \mathbb{N}}$ be a bounded sequence of $L^p(\Omega)$ such that $u_n \to u$ a.e. as $n \to \infty$. It follows that $u \in L^p(\Omega)$ and $u_n \to u$ as $n \to \infty$ in $L^q(\Omega')$, for every $\Omega' \subset \Omega$ of finite measure and every $q \in [1, p)$. In particular, $u_n \to u$ as $n \to \infty$, in $L^p(\Omega)$ weak if $p < \infty$, and in $L^\infty(\Omega)$ weak* if $p = \infty$.*

PROOF. By extending the functions by 0 outside Ω, we may assume that $\Omega = \mathbb{R}^N$. Observe that by Fatou's lemma, we have $u \in L^p(\mathbb{R}^N)$. Let $\Omega' \subset \mathbb{R}^N$ have a finite measure and let $q \in [1, p)$. Consider $\varepsilon > 0$. By Egorov's theorem, there exists a measurable subset E of Ω' such that $u_n \to u$ uniformly on $\Omega' \setminus E$ and

$$|E|^{\frac{p-q}{p}} \sup_{n \geq 0} \Big(\int_{\mathbb{R}^N} |u_n - u|^p \Big)^{\frac{q}{p}} \leq \varepsilon / 2.$$

Let n_0 be large enough so that $|u_n - u|^q \leq \varepsilon / 2 |\Omega'|$ on $\Omega' \setminus E$, for $n \geq n_0$. It follows that

$$\begin{aligned} \int_{\Omega'} |u_n - u|^q &= \int_E |u_n - u|^q + \int_{\Omega' \setminus E} |u_n - u|^q \\ &\leq |E|^{\frac{p-q}{p}} \Big(\int_E |u_n - u|^p \Big)^{\frac{q}{p}} + |\Omega' \setminus E| \sup_{\Omega' \setminus E} |u_n - u|^q \leq \varepsilon. \end{aligned}$$

Hence the result, since ε is arbitrary. □

A.2. Inequalities

Below is the standard form of Gronwall's lemma (see [**15**, Lemma, p. 293], [**3**, Lemma 1]).

THEOREM A.2.1 (Gronwall's lemma). *Let $T > 0$, $A \geq 0$ and let $f \in C([0, T])$ be a nonnegative function. If the nonnegative function $\varphi \in C([0, T])$ satisfies*

$$\varphi(t) \leq A + \int_0^t f(s) \varphi(s) \, ds$$

for every $t \in [0, T]$, then

$$\varphi(t) \leq A \, \exp\Big(\int_0^t f(s) \, ds \Big),$$

for every $t \in [0, T]$. In particular, if

$$\varphi(t) \leq \int_0^t f(s) \varphi(s) \, ds$$

for every $t \in [0, T]$, then $\varphi(t) \equiv 0$.

PROOF. Set $h(t) = \psi(t) \exp(-\int_0^t f(s) \, ds)$, where $\psi(t) = A + \int_0^t f(s) \varphi(s) \, ds$. It follows that $\psi, h \in C^1([0, T])$, $\psi, h \geq 0$, and

$$\begin{aligned} h'(t) &= (\psi'(t) - f(t)\psi(t)) \exp\Big(- \int_0^t f(s) \, ds \Big) \\ &= (f(t)\varphi(t) - f(t)\psi(t)) \exp\Big(- \int_0^t f(s) \, ds \Big) \leq 0. \end{aligned}$$

It follows that $h(t) \le h(0)$, from which the result follows. $\square$

We complete this section with a useful property of the convolution. We recall that the convolution $f \star g$ of two functions f, g defined on $\mathbb{R}^N$ is given by

$$f \star g(x) = \int_{\mathbb{R}^N} f(x-y)g(y)\,dy$$

The following proposition provides useful information.

Proposition A.2.2. *Let $1 \le p, q < \infty$ such that $\frac{1}{p} + \frac{1}{q} \ge 1$. Let $f \in L^p(\mathbb{R}^N)$ and $g \in L^q(\mathbb{R}^N)$, so that $f \star g \in L^r(\mathbb{R}^N)$ with $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$ by Young's inequality.*

(i) *If $f, g \ge 0$, then $f \star g \ge 0$*
(ii) *If f and g are radially symmetric, then so is $f \star g$*
(iii) *If f and g are radially symmetric and radially decreasing, then so is $f \star g$*

Proof. Parts (i) and (ii) are quite standard. Part (iii) is not, and is taken from [29].

By density and Young's inequality, we need only consider the case $f, g \in C_c(\mathbb{R}^N)$. Property (i) is immediate. To prove (ii), suppose f, g are radially symmetric, and consider a rotation $\mathcal{R}$. Since $|\det \mathcal{R}| = 1$, we see that

$$\begin{aligned} f \star g(\mathcal{R}x) &= \int_{\mathbb{R}^N} f(\mathcal{R}x - y)g(y)\,dy = \int_{\mathbb{R}^N} f(\mathcal{R}(x - \mathcal{R}^{-1}y))g(\mathcal{R}\mathcal{R}^{-1}y)\,dy \\ &= \int_{\mathbb{R}^N} f(x - \mathcal{R}^{-1}y)g(\mathcal{R}^{-1}y)\,dy = \int_{\mathbb{R}^N} f(x-z)g(z)\,dz = f \star g(x) \end{aligned}$$

The rotation $\mathcal{R}$ being arbitrary, this proves Property (ii).

We now prove (iii), so we assume that f and g are radially symmetric and nonincreasing. In particular, $f \star g$ is radially symmetric by Property (ii).

We first consider the one-dimensional case $N = 1$. We need to prove that $h = f \star g$ is nonincreasing on $(0, \infty)$. Given $x, s \ge 0$, we have

$$\begin{aligned} h(x+2s) - h(x) &= \int_{-\infty}^{\infty} [f(x+2s-y) - f(x-y)]g(y)\,dy \\ &= \int_{-\infty}^{\infty} [f(y+s) - f(y-s)]g(x+s-y)\,dy \\ &= I_- + I_+ \end{aligned}$$

where

$$\begin{aligned} I_- &= \int_{-\infty}^{0} [f(y+s) - f(y-s)]g(x+s-y)\,dy \\ I_+ &= \int_{0}^{\infty} [f(y+s) - f(y-s)]g(x+s-y)\,dy \end{aligned}$$

Changing y to $-y$ in I_-, then using the property that f is even, we obtain

$$\begin{aligned} I_- &= \int_{0}^{\infty} [f(-y+s) - f(-y-s)]g(x+s+y)\,dy \\ &= \int_{0}^{\infty} [f(y-s) - f(y+s)]g(x+s+y)\,dy \end{aligned}$$

so that

$$h(x+2s) - h(x) = \int_{0}^{\infty} [f(y+s) - f(y-s)][g(x+s-y) - g(x+s+y)]\,dy \quad \text{(A.2.1)}$$

We claim that

$$f(y+s)-f(y-s)\le 0 \quad y,s\ge 0 \tag{A.2.2}$$

$$g(x+s-y)-g(x+s+y)\le 0 \quad x,y,s\ge 0 \tag{A.2.3}$$

Formulas (A.2.1), (A.2.2) and (A.2.3) imply that $h(x+2s)\le h(x)$ for all $x,s\ge 0$, which is the desired conclusion. To prove (A.2.2), we observe that if $y\ge s$, then $0\le y-s\le y+s$, so that $f(y-s)\le f(y+s)$ since f is radially decreasing. If $0\le y\le s$, then $0\le s-y\le y+s$, so that $f(y+s)\le f(s-y)=f(y-s)$, again because f is radially decreasing. Similarly, to prove (A.2.3), we observe that if $y\ge x+s$, then $0\le x+s-y\le x+s+y$, so that $g(x+s-y)\le g(x+s+y)$ since g is radially decreasing. If $0\le y\le x+s$, then $0\le -x-s+y\le x+s+y$, so that $g(x+s+y)\le g(-x-s+y)=g(x+s-y)$, again because g is radially decreasing.

We finally consider the case $N\ge 2$, and we set $h=f\star g$. Given $x_1\in\mathbb{R}$, we let $x=(x_1,0)\in\mathbb{R}^N$. Since h is radially symmetric, we need only show that $h(x)$ is a nonincreasing function of $x_1>0$. To see this, we write

$$\begin{aligned} h(x) &= \int_{\mathbb{R}^{N-1}}\int_{\mathbb{R}} f(x_1-y_1,-y')g(y_1,y')\ dy_1dy' \\ &= \int_{\mathbb{R}^{N-1}}\Big(\int_{\mathbb{R}} f(x_1-y_1,-y')g(y_1,y')dy_1\Big)dy' \\ &= \int_{\mathbb{R}^{N-1}} f^{y'}\star g^{y'}(x_1)\,dy' \end{aligned}$$

where, for every $y'\in\mathbb{R}^{N-1}$ $f^{y'},g^{y'}\mathbb{R}\to\mathbb{R}$ are defined by

$$f^{y'}(x_1)=f(x_1,-y') \quad g^{y'}(x_1)=g(x_1,y')$$

Since f and g are radially decreasing on $\mathbb{R}^N$, it follows that $f^{y'}(x_1)$ and $g^{y'}(x_1)$ are radially decreasing. Therefore, $f^{y'}\star g^{y'}(x_1)$ is radially decreasing, hence a nonincreasing function of $x_1\ge 0$. This completes the proof. □

APPENDIX B

Sobolev spaces

Throughout this section, Ω is an open subset of $\mathbb{R}^N$. We study the basic properties of the Sobolev spaces $W^{m,p}(\Omega)$ and $W_0^{m,p}(\Omega)$, and particularly the spaces $H^1(\Omega)$ and $H_0^1(\Omega)$ (which correspond to $m = 1$ and $p = 2$). For a more detailed study, see for example Adams and Fournier [1]. We consider the case of real-valued functions, see Section B.5 for the necessary modification in the case of complex-valued function.

B.1. Definitions and basic properties

We begin with the definition of "weak" derivatives. Let $u \in C^m(\Omega)$, $m \geq 1$. If $\alpha \in \mathbb{N}^N$ is a multi-index such that $|\alpha| \leq m$, it follows from Green's formula that

$$\int_\Omega D^\alpha u\varphi = (-1)^{|\alpha|} \int_\Omega u D^\alpha \varphi, \tag{B.1.1}$$

for all $\varphi \in C_c^m(\Omega)$. We note that both integrals in (B.1.1) make sense since $D^\alpha u\varphi \in C_c(\Omega)$ and $uD^\alpha\varphi \in C_c(\Omega)$. As a matter of fact, the right-hand side makes sense as soon as $u \in L^1_{\mathrm{loc}}(\Omega)$ and the left-hand side makes sense as soon as $D^\alpha u \in L^1_{\mathrm{loc}}(\Omega)$. This motivates the following definition.

Definition B.1.1. Let $u \in L^1_{\mathrm{loc}}(\Omega)$ and let $\alpha \in \mathbb{N}^N$. We say that $D^\alpha u \in L^1_{\mathrm{loc}}(\Omega)$ if there exists $u_\alpha \in L^1_{\mathrm{loc}}(\Omega)$ such that

$$\int_\Omega u_\alpha \varphi = (-1)^{|\alpha|} \int_\Omega u D^\alpha \varphi, \tag{B.1.2}$$

for all $\varphi \in C_c^{|\alpha|}(\Omega)$. Such a function u_α is then unique and we set $D^\alpha u = u_\alpha$. If $u_\alpha \in L^p_{\mathrm{loc}}(\Omega)$ (respectively, $u \in L^p(\Omega)$) for some $1 \leq p \leq \infty$, we say that $D^\alpha u \in L^p_{\mathrm{loc}}(\Omega)$ (respectively, $D^\alpha u \in L^p(\Omega)$).

The Sobolev spaces $W^{m,p}(\Omega)$ are defined as follows.

Definition B.1.2. Let $m \in \mathbb{N}$ and $p \in [1, \infty]$. We set

$$W^{m,p}(\Omega) = \{u \in L^p(\Omega);\ D^\alpha u \in L^p(\Omega) \text{ for } |\alpha| \leq m\}.$$

For $u \in W^{m,p}(\Omega)$, we set

$$\|u\|_{W^{m,p}} = \sum_{|\alpha| \leq m} \|D^\alpha u\|_{L^p},$$

which defines a norm on $W^{m,p}(\Omega)$. We set

$$H^m(\Omega) = W^{m,2}(\Omega),$$

and we equip $H^m(\Omega)$ with the scalar product

$$(u, v)_{H^m} = \sum_{|\alpha| \leq m} \int_\Omega D^\alpha u D^\alpha v \, dx,$$

which defines on $H^m(\Omega)$ the norm

$$\|u\|_{H^m} = \Big(\sum_{|\alpha|\le m} \|D^\alpha u\|_{L^2}^2 \Big)^{\frac{1}{2}},$$

which is equivalent to the norm $\|\cdot\|_{W^{m,2}}$.

Proposition B.1.3. *$W^{m,p}(\Omega)$ is a Banach space and $H^m(\Omega)$ is a Hilbert space. If $p<\infty$, then $W^{m,p}(\Omega)$ is separable, and if $1<p<\infty$, then $W^{m,p}(\Omega)$ is reflexive.*

Proof. Let $k = 1+N+\cdots+N^m = (N^{m+1}-1)/(N-1)$ ($k=m+1$ if $N=1$), and consider the operator $T : W^{m,p}(\Omega) \to L^p(\Omega)^k$ defined by

$$Tu = (D^\alpha u)_{|\alpha|\le m}.$$

It is clear that T is isometric and that T is injective. Therefore, $W^{m,p}(\Omega)$ can be identified with the subspace $T(W^{m,p}(\Omega))$ of $L^p(\Omega)^k$.

We claim that $T(W^{m,p}(\Omega))$ is closed. Indeed, suppose $(u_n)_{n\ge 0}$ is such that $u_n \to u$ in $L^p(\Omega)$ and $D^\alpha u_n \to u_\alpha$ in $L^p(\Omega)$ for $1 \le |\alpha| \le m$. Applying (B.1.1) to u_n and letting $n\to\infty$, we deduce that $D^\alpha u \in L^p(\Omega)$ and that $D^\alpha u = u_\alpha$; and so, $u \in W^{m,p}(\Omega)$. Therefore, $T(W^{m,p}(\Omega))$ is a Banach space, and so is $W^{m,p}(\Omega)$. If $p<\infty$, then $L^p(\Omega)^k$ is separable. Thus so is $T(W^{m,p}(\Omega))$, hence $W^{m,p}(\Omega)$. Finally, if $1<p<\infty$, then $L^p(\Omega)^k$ is reflexive. Thus so is $T(W^{m,p}(\Omega))$, hence $W^{m,p}(\Omega)$. □

Remark B.1.4. Here are some simple consequences of Definition B.1.2.

(i) It is not difficult to prove that if $u \in W^{m,p}(\Omega)$ and if $v \in C^m(\Omega)\cap W^{m,\infty}(\Omega)$, then $uv \in W^{m,p}(\Omega)$ and Leibniz's formula holds. More precisely,

$$D^\alpha(uv) = \sum_{\alpha_1+\alpha_2=\alpha} C_{\alpha_1,\alpha_2} D^{\alpha_1}u D^{\alpha_2}v \tag{B.1.3}$$

with constants C_{α_1,α_2} independent of u,v. In particular, there exists a constant C such that

$$\|uv\|_{W^{m,p}} \le C\|u\|_{W^{m,p}}\|v\|_{W^{m,\infty}} \tag{B.1.4}$$

for all $u \in W^{m,p}(\Omega)$ and $v \in C^m(\Omega)\cap W^{m,\infty}(\Omega)$. (The terms $D^{\alpha_1}uD^{\alpha_2}v$ in (B.1.3) are well defined, as the product of the function $D^{\alpha_1}u \in L^p(\Omega)$ with the function $D^{\alpha_2}v \in L^\infty(\Omega)$.)

(ii) If $|\Omega|<\infty$ and $p \ge q$, then $L^p(\Omega) \hookrightarrow L^q(\Omega)$. It follows that $W^{m,p}(\Omega) \hookrightarrow W^{m,q}(\Omega)$.

Remark B.1.5. One can show that if $p<\infty$, then $W^{m,p}(\Omega)\cap C^\infty(\Omega)$ is dense in $W^{m,p}(\Omega)$ (see [1, Theorem 3.17 p. 67]).

We now define the subspaces $W_0^{m,p}(\Omega)$ for $p<\infty$. Formally, $W_0^{m,p}(\Omega)$ is the subspace of functions of $W^{m,p}(\Omega)$ that vanish, as well of their derivatives up to order $m-1$, on $\partial\Omega$.

Definition B.1.6. Let $1\le p<\infty$ and let $m\in\mathbb{N}$. We denote by $W_0^{m,p}(\Omega)$ the closure of $C_c^\infty(\Omega)$ in $W^{m,p}(\Omega)$, and we set $H_0^m(\Omega) = W_0^{m,2}(\Omega)$.

Remark B.1.7. It follows from Proposition B.1.3 that $W_0^{m,p}(\Omega)$ is a separable Banach space and that $W_0^{m,p}(\Omega)$ is reflexive if $p>1$. In addition, $H_0^m(\Omega)$ is a separable Hilbert space.

In general $W_0^{m,p}(\Omega) \ne W^{m,p}(\Omega)$, however both spaces coincide when $\partial\Omega$ is "small" (see [1, Sections 3.24–3.39]). In particular, we have the following result.

Theorem B.1.8. *If $1\le p<\infty$ and $m\in\mathbb{N}$, then $W_0^{m,p}(\mathbb{R}^N) = W^{m,p}(\mathbb{R}^N)$.*

The proof of Theorem B.1.8 makes use of the following lemma.

LEMMA B.1.9. *Let $\rho \in C_c^\infty(\mathbb{R}^N)$, $\rho \ge 0$, with $\operatorname{supp} \rho \subset \{x \in \mathbb{R}^N;\ |x| \le 1\}$ and $\|\rho\|_{L^1} = 1$. For $n \in \mathbb{N}$, $n \ge 1$, set $\rho_n(x) = n^N \rho(nx)$. ($(\rho_n)_{n\ge1}$ is called a smoothing sequence.) Then the following properties hold.*

(i) *For every $u \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$, $\rho_n \star u \in C^\infty(\mathbb{R}^N)$.*

(ii) *If $u \in L^p(\mathbb{R}^N)$ for some $p \in [1,\infty]$, then $\rho_n \star u \in L^p(\mathbb{R}^N)$ and $\|\rho_n \star u\|_{L^p} \le \|u\|_{L^p}$. If $p < \infty$ or if $p = \infty$ and $u \in C_{\mathrm{b,u}}(\mathbb{R}^N)$, then $\rho_n \star u \to u$ in $L^p(\mathbb{R}^N)$ as $n \to \infty$.*

(iii) *If $u \in W^{m,p}(\mathbb{R}^N)$ for some $p \in [1,\infty]$ and $m \in \mathbb{N}$, then $\rho_n \star u \in W^{m,p}(\mathbb{R}^N)$ and $D^\alpha(\rho_n \star u) = \rho_n \star D^\alpha u$ for $|\alpha| \le m$. In particular, if $p < \infty$ or if $p = \infty$ and $D^\alpha u \in C_{\mathrm{b,u}}(\mathbb{R}^N)$ for all $|\alpha| \le m$, then $\rho_n \star u \to u$ in $W^{m,p}(\mathbb{R}^N)$ as $n \to \infty$.*

PROOF. (i) Since

$$\rho_n \star u(x) = \int_{\mathbb{R}^N} \rho_n(x-y)u(y)\,dy,$$

it is clear that $\rho_n \star u \in C(\mathbb{R}^N)$. One deduces easily from the above formula that $D^\alpha(\rho_n \star u) = (D^\alpha \rho_n) \star u$, and the result follows.

(ii) The first part of property (ii) follows from Young's inequality, since

$$\|\rho_n\|_{L^1} = \int_{\mathbb{R}^N} \rho_n(x)\,dx = n^N \int_{\mathbb{R}^N} \rho(nx)\,dx = \int_{\mathbb{R}^N} \rho(y)\,dy = 1.$$

Consider now $u \in C_{\mathrm{b,u}}(\mathbb{R}^N)$ and set $u_n = \rho_n \star u$. We have

$$u_n(x) = \int_{\mathbb{R}^N} \rho_n(y)u(x-y)\,dy, \quad u(x) = \int_{\mathbb{R}^N} \rho_n(y)u(x)\,dy;$$

and so,

$$u_n(x) - u(x) = \int_{\mathbb{R}^N} \rho_n(y)(u(x-y) - u(x))\,dy.$$

Therefore,

$$|u_n(x) - u(x)| \le \int_{\mathbb{R}^N} \rho_n(y)|u(x-y) - u(x)|\,dy \le \sup_{|y|\le 1/n} |u(x-y) - u(x)|,$$

since $\operatorname{supp} \rho_n \subset \{y;\ |y| \le 1/n\}$. Since u is uniformly continuous, we have

$$\sup_{x\in\mathbb{R}^N} \sup_{|y|\le1/n} |u(x-y) - u(x)| \underset{n\to\infty}{\longrightarrow} 0;$$

and so, $u_n \to u$ uniformly. Consider next $u \in L^p(\mathbb{R}^N)$, with $p < \infty$, and let $\varepsilon > 0$. There exists $v \in C_c(\mathbb{R}^N)$ such that $\|u - v\|_{L^p} \le \varepsilon/3$. Furthermore, it follows from what precedes that for n large enough, we have $\|v - \rho_n \star v\|_{L^p} \le \varepsilon/3$. (Since $\rho_n \star v \to v$ uniformly and $\rho_n \star v$ is supported in a fixed compact subset of $\mathbb{R}^N$.) Finally, it follows from the inequality of (ii) that $\|\rho_n \star v - \rho_n \star u\|_{L^p} \le \|u - v\|_{L^p} \le \varepsilon/3$. Writing

$$u - \rho_n \star u = u - v + v - \rho_n \star v + \rho_n \star v - \rho_n \star u,$$

we deduce that $\|u - \rho_n \star u\|_{L^p} \le \varepsilon$. Since $\varepsilon > 0$ is arbitrary, this completes the proof of property (ii).

(iii) For any $v : \mathbb{R}^N \to \mathbb{R}$, we set $\widetilde{v}(x) = v(-x)$. Given $u \in W^{m,p}(\mathbb{R}^N)$ and $\varphi \in C_c^m(\mathbb{R}^N)$, we have

$$\int_{\mathbb{R}^N} (\rho_n \star u) D^\alpha \varphi = \int_{\mathbb{R}^N} u(\widetilde{\rho_n} \star D^\alpha \varphi) = \int_{\mathbb{R}^N} u D^\alpha(\widetilde{\rho_n} \star \varphi).$$

By definition of $D^\alpha u$, we obtain

$$\int_{\mathbb{R}^N} (\rho_n \star u) D^\alpha \varphi = (-1)^{|\alpha|} \int_{\mathbb{R}^N} D^\alpha u (\widetilde{\rho_n} \star \varphi) = (-1)^{|\alpha|} \int_{\mathbb{R}^N} (\rho_n \star D^\alpha u) \varphi.$$

This means that $D^\alpha(\rho_n \star u) \in L^p(\mathbb{R}^N)$ and that $D^\alpha(\rho_n \star u) = \rho_n \star D^\alpha u$; and so, $\rho_n \star u \in W^{m,p}(\mathbb{R}^N)$. The convergence property follows from property (ii). □

PROOF OF THEOREM B.1.8. Let $u \in W^{m,p}(\mathbb{R}^N)$ and $\varepsilon > 0$. It follows from properties (iii) and (i) of Lemma B.1.9 that there exists $v \in W^{m,p}(\mathbb{R}^N) \cap C^\infty(\mathbb{R}^N)$ such that $\|u - v\|_{W^{m,p}} \le \varepsilon/2$. Fix now $\eta \in C^\infty_{\mathrm{c}}(\mathbb{R}^N)$ such that $0 \le \eta \le 1$, $\eta(x) = 1$ for $|x| \le 1$ and $\eta(x) = 0$ for $|x| \ge 2$. Set $\eta_n(x) = \eta(x/n)$ and let $v_n = \eta_n v$. It is clear that $v_n \in C^\infty_{\mathrm{c}}(\mathbb{R}^N)$, and we claim that $\eta_n v \to v$ in $W^{m,p}(\mathbb{R}^N)$ as $n \to \infty$. Indeed, it follows from Leibniz' formula (see Remark B.1.4 (i)) that

$$D^\alpha(\eta_n v) = \sum_{\beta + \gamma = \alpha} D^\beta \eta_n D^\gamma u.$$

Since $\|D^\gamma \eta_n\|_{L^\infty} \le C_\gamma n^{-|\gamma|}$, it follows that all the terms with $|\beta| > 0$ converge to 0 as $n \to \infty$. The remaining term in the sum is $\eta_n D^\alpha v$ which, by dominated convergence, converges to $D^\alpha v$ in $L^p(\mathbb{R}^N)$. We deduce that $D^\alpha(\eta_n v) \to D^\alpha v$ in $L^p(\mathbb{R}^N)$, which proves the claim. Therefore, there exists $w \in C^\infty_{\mathrm{c}}(\mathbb{R}^N)$ such that $\|v - w\|_{W^{m,p}} \le \varepsilon/2$. This implies that $\|u - w\|_{W^{m,p}} \le \varepsilon$, and the result follows. □

REMARK B.1.10. We describe below some useful properties of the Sobolev space $W^{m,p}_0(\Omega)$.

(i) If $u \in W^{m,p}(\Omega)$ and if $\operatorname{supp} u$ is included in a compact subset of Ω, then $u \in W^{m,p}_0(\Omega)$. This is easily shown by using the regularization and truncation argument described above.

(ii) It follows easily from Remark B.1.4 (i) and Property (i) above that that if $u \in W^{m,p}_0(\Omega)$ and if $v \in C^m(\Omega) \cap W^{m,\infty}(\Omega)$, then $uv \in W^{m,p}_0(\Omega)$. Moreover, formulas (B.1.3) and (B.1.4) hold.

(iii) If $u \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$ and if $u_{|\partial\Omega} = 0$, then $u \in W^{1,p}_0(\Omega)$. Indeed, if u has a bounded support, let $F \in C^1(\mathbb{R})$ satisfy $|F(t)| \le |t|$, $F(t) = 0$ for $|t| \le 1$ and $F(t) = t$ for $|t| \ge 2$. Setting $u_n(x) = n^{-1}F(nu(x))$, it follows from Proposition B.2.1 below that $u_n \in W^{1,p}(\Omega)$. In addition, one verifies easily (see (B.2.1) below) that $u_n \to u$ in $W^{1,p}(\Omega)$ as $n \to \infty$. Since $\operatorname{supp} u_n \subset \{x \in \Omega;\ |u(x)| \ge n^{-1}\}$, $\operatorname{supp} u_n$ is a compact subset of Ω, thus $u_n \in W^{1,p}_0(\Omega)$ by (i) above; and so $u \in W^{1,p}_0(\Omega)$. If $\operatorname{supp} u$ is unbounded, we approximate u by $\xi_n u$ where $\xi_n \in C^\infty_{\mathrm{c}}(\mathbb{R}^N)$ is such that $\xi_n(x) = 1$ for $|x| \le n$.

(iv) If $u \in W^{1,p}_0(\Omega) \cap C(\overline{\Omega})$ and if Ω is of class C^1, then $u_{|\partial\Omega} = 0$ (see [5, Theorem 9.17, p. 288],). Note that this property is false if Ω is not smooth enough. For example, one can show that if $\Omega = \mathbb{R}^N \setminus \{0\}$ and $N \ge 2$, then $H^1_0(\Omega) = H^1(\Omega)$. In particular, if $u \in C^\infty_{\mathrm{c}}(\mathbb{R}^N)$ and $u(0) \ne 0$, then $u \in H^1_0(\Omega)$ but $u \ne 0$ on $\partial\Omega$.

(v) Let $u \in L^1_{\mathrm{loc}}(\Omega)$ and define $\widetilde{u} \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ by

$$\widetilde{u}(x) = \begin{cases} u(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \notin \Omega. \end{cases}$$

If $u \in W^{1,p}_0(\Omega)$, then $\widetilde{u} \in W^{1,p}(\mathbb{R}^N)$. This is immediate by the definition of $W^{1,p}_0(\Omega)$. More generally, if $u \in W^{m,p}_0(\Omega)$, then $\widetilde{u} \in W^{m,p}(\mathbb{R}^N)$. Conversely, if $\widetilde{u} \in W^{1,p}(\mathbb{R}^N)$ and if Ω is of class C^1 (as in part (iv) above, the smoothness assumption on Ω is essential), then $u \in W^{1,p}_0(\Omega)$ (see [5, Proposition 9.18, p. 289]).

PROPOSITION B.1.11. *Let $1 \leq p \leq \infty$ and let $u \in W^{1,p}(\Omega)$. Let $\omega \subset \Omega$ be a connected, open set. If $\nabla u = 0$ a.e. on ω, then there exists a constant c such that $u = c$ a.e. on ω.*

PROOF. Let $x \in \omega$ and let $\rho > 0$ be such that $B(x, \rho) \subset \omega$. We claim that there exists c such that $u = c$ a.e. on $B(x, \rho)$. The result follows by Connectedness. To prove the claim, we argue as follows. Let $0 < \varepsilon < \rho$ and let $\eta \in C_c^\infty(\mathbb{R}^N)$ satisfy $\eta \equiv 1$ on $B(x, \rho - \varepsilon)$, $\operatorname{supp} \eta \subset B(x, \rho)$, and $0 \leq \eta \leq 1$. Setting $v = \eta u$, we deduce that $v \in W_0^{1,1}(B(x, \rho))$ and that $\nabla v = 0$ a.e. on $B(x, \rho - \varepsilon)$. We now extend v by 0 outside $B(x, \rho)$ and we call $\overline{v}$ the extension. Let $(\rho_n)_{n\geq 0}$ be a smoothing sequence and fix $n > 1/(\rho - \varepsilon)$. We have $w_n = \rho_n \star \overline{v} \in C_c^\infty(\mathbb{R}^N)$. Furthermore, since $\nabla w_n = \rho_n \star \nabla \overline{v}$, and since $\operatorname{supp} \rho_n \subset B(0, 1/n)$, it follows that $\nabla w_n = 0$ on $B(x, \rho - \varepsilon - 1/n)$. In particular, there exists c_n such that $w_n \equiv c_n$ on $B(x, \rho - \varepsilon - 1/n)$. Since $w_n \to \overline{v}$ in $L^1(\mathbb{R}^N)$, we deduce in particular that for any $\mu < \rho - \varepsilon$, there exists $c(\mu)$ such that $\overline{v} \equiv c(\mu)$ on $B(x, \rho - \varepsilon - \mu)$. Therefore, $c(\mu)$ is independent of μ and we have $\overline{v} \equiv c$ on $B(x, \rho - \varepsilon)$. for some constant c. It follows that c is independent of ε, and the claim follows by letting $\varepsilon \downarrow 0$. □

PROPOSITION B.1.12. *Let $u \in W^{m,\infty}(\mathbb{R}^N)$ for some $m \geq 0$. If $D^\alpha u \in C_{b,u}(\mathbb{R}^N)$ for all $|\alpha| \leq m$, then $u \in C_{b,u}^m(\mathbb{R}^N)$. In other words, the distributional derivatives of u are the classical derivatives.*

PROOF. Let $(\rho_n)_{n\geq 0}$ be a smoothing sequence and set $u_n = \rho_n \star u$. It follows from Lemma B.1.9 that $u_n \in C^\infty(\mathbb{R}^N) \cap W^{m,\infty}(\mathbb{R}^N)$. Moreover, it is clear that $D^\alpha u_n = \rho_n \star (D^\alpha u)$ is uniformly continuous on $\mathbb{R}^N$ for all $|\alpha| \leq m$ and $n \geq 0$. Thus $(u_n)_{n\geq 0} \subset C_{b,u}^m(\mathbb{R}^N)$. Moreover, it follows from Lemma B.1.9 that $D^\alpha u_n \to D^\alpha u$ in $L^\infty(\mathbb{R}^N)$, i.e. $u_n \to u$ in $W^{m,\infty}(\mathbb{R}^N)$. Since $C_{b,u}^m(\mathbb{R}^N)$ is a Banach space, it is a closed subset of $W^{m,\infty}(\mathbb{R}^N)$, and we deduce that $u \in C_{b,u}^m(\mathbb{R}^N)$. □

We next introduce the local Sobolev spaces.

DEFINITION B.1.13. Given $m \in \mathbb{N}$ and $1 \leq p \leq \infty$, we set $W_{\mathrm{loc}}^{m,p}(\Omega) = \{u \in L_{\mathrm{loc}}^1(\Omega);\ D^\alpha u \in L_{\mathrm{loc}}^p(\Omega)$ for all $|\alpha| \leq m\}$.

PROPOSITION B.1.14. *Let $m \in \mathbb{N}$ and $1 \leq p \leq \infty$, let. If $u \in L_{\mathrm{loc}}^1(\Omega)$, then the following properties are equivalent.*

(i) *$u \in W_{\mathrm{loc}}^{m,p}(\Omega)$.*
(ii) *$u_{|\omega} \in W^{m,p}(\omega)$ for all $\omega \subset\subset \Omega$.*
(iii) *$\phi u \in W_0^{m,p}(\Omega)$ ($\varphi \in W^{m,\infty}(\Omega)$ if $p = \infty$) for all $\phi \in C_c^\infty(\Omega)$.*

PROOF. (i)⇒(ii). This is immediate.

(ii)⇒(iii). Suppose $u \in W_{\mathrm{loc}}^{m,p}(\Omega)$ and let $\phi \in C_c^\infty(\Omega)$. If $\omega \subset\subset \Omega$ contains $\operatorname{supp} \phi$, then ϕu has compact support in ω. Since $u \in W^{m,p}(\omega)$, we know (see Remark B.1.4 (i)) that $\phi u \in W^{m,p}(\omega)$. If $p < \infty$, then $\phi u \in W_0^{m,p}(\Omega)$ by Remark B.1.10 (i).

(iii)⇒(i). Suppose $\phi u \in W^{m,p}(\Omega)$ for all $\phi \in C_c^\infty(\Omega)$. Given $|\alpha| \leq m$, we define $u_\alpha \in L_{\mathrm{loc}}^p(\Omega)$ as follows. Let $\omega \subset\subset \Omega$ and let $\phi \in C_c^\infty(\Omega)$ satisfy $\phi(x) = 1$ for all $x \in \omega$. We set $(u_\alpha)_{|\omega} = D^\alpha(\phi u)_{|\omega}$ and we claim that $(u_\alpha)_{|\omega}$ is independent of the choice of ϕ, so that u_α is well-defined. Indeed, if $\psi \in C_c^\infty(\Omega)$ is such that $\psi(x) = 1$ on ω, then for all $\varphi \in C_c^\infty(\omega)$,

$$\int_\omega D^\alpha(\psi u - \phi u)\varphi = (-1)^{|\alpha|} \int_\omega (\psi - \phi) u D^\alpha \varphi = 0,$$

so that $D^\alpha \psi u = D^\alpha \phi u$ a.e. in ω. It remains to show that $u_\alpha = D^\alpha u$. Indeed, let $|\alpha| \le m$ and $\varphi \in C_c^{|\alpha|}(\Omega)$. Let $\phi \in C_c^\infty(\Omega)$ satisfy $\phi(x) = 1$ on $\operatorname{supp} \varphi$. We have

$$(-1)^{|\alpha|} \int_\Omega u_\alpha \varphi = \int_\Omega \phi u D^\alpha \varphi = \int_\Omega u D^\alpha \varphi,$$

and the result follows. □

We now introduce the Sobolev spaces of negative index.

Definition B.1.15. Given $m \in \mathbb{N}$ and $1 \le p < \infty$, we define $W^{-m,p'}(\Omega) = (W_0^{m,p}(\Omega))^\star$, and we denote by $\langle \cdot, \cdot \rangle_{W^{-m,p'}, W_0^{m,p}}$ the corresponding duality pairing. For $p = 2$, we set $H^{-m}(\Omega) = W^{-m,2}(\Omega) = (H_0^m(\Omega))^\star$ and we denote by $\langle \cdot, \cdot \rangle_{H^{-m}, H_0^m}$, the corresponding duality pairing. .

Remark B.1.16. Here are some comments on Definition B.1.15.

(i) It follows from Remark B.1.7 that $W^{-m,p'}(\Omega)$ is a Banach space. If $p > 1$, then $W^{-m,p'}(\Omega)$ is reflexive and separable. $H^{-m}(\Omega)$ is a separable Hilbert space.

(ii) Let $v \in C^m(\Omega) \cap W^{m,\infty}(\Omega)$ and $u \in W^{-m,p'}(\Omega)$. Given $\varphi \in W_0^{m,p}(\Omega)$, it follows from Remark B.1.10 (ii) that $v\varphi \in W_0^{m,p}(\Omega)$ and $\|v\varphi\|_{W^{m,p}} \le C\|v\|_{W^{m,\infty}} \|\varphi\|_{W^{m,p}}$. Therefore, one can define $vu \in W^{-m,p'}(\Omega)$ by

$$\langle vu, \varphi \rangle_{W^{-m,p'}, W_0^{m,p}} = \langle u, v\varphi \rangle_{W^{-m,p'}, W_0^{m,p}}$$

for all $\varphi \in W_0^{m,p}(\Omega)$. Moreover, it follows from (B.1.4) that $\|vu\|_{W^{-m,p'}} \le C\|v\|_{W^{m,\infty}} \|u\|_{W^{-m,p'}}$

(iii) It follows from the dense embedding $C_c^\infty(\Omega) \hookrightarrow W_0^{m,p}(\Omega)$ that $W^{-m,p'}(\Omega)$ is a space of distributions on Ω. In particular, we see that $\langle u, \varphi \rangle_{W^{-m,p'}, W_0^{m,p}} = \langle u, \varphi \rangle_{\mathcal{D}', \mathcal{D}}$ for every $u \in W^{-m,p'}(\Omega)$ and $\varphi \in C_c^\infty(\Omega)$. Like any distribution, an element of $W^{-m,p'}(\Omega)$ can be localized. Indeed, if $u \in \mathcal{D}'(\Omega)$ and Ω' is an open subset of Ω, then one defines $u_{|\Omega'}$ as follows. Given any $\varphi \in C_c^\infty(\Omega')$, let $\tilde{\varphi} \in C_c^\infty(\Omega)$ be equal to φ on Ω' and to 0 on $\Omega \setminus \Omega'$. Then $\Psi(\varphi) = \langle u, \widetilde{\varphi} \rangle_{\mathcal{D}'(\Omega), \mathcal{D}(\Omega)}$ defines a distribution $\Psi \in \mathcal{D}'(\Omega')$, and one sets $u_{|\Omega'} = \Psi$. Note that this is consistent with the usual restriction of functions. Since $\|\tilde{\varphi}\|_{W_0^{m,p}(\Omega')} \le \|\varphi\|_{W_0^{m,p}(\Omega)}$, we see that if $u \in W^{-m,p'}(\Omega)$, then $u_{|\Omega'} \in W^{-m,p'}(\Omega')$ and $\|u_{|\Omega'}\|_{W^{-m,p'}(\Omega')} \le \|u\|_{W^{-m,p'}(\Omega)}$.

(iv) An element $f \in W^{-m,p'}(\Omega)$ being a distribution, it makes sense to say that $f \in W^{-m,p'}(\Omega) \cap L^q(\Omega)$ for some $1 \le q \le \infty$. This means that there exists a function $g \in L^q(\Omega)$ such that

$$\langle f, \varphi \rangle_{W^{-m,p'}, W_0^{m,p}} = \int_\Omega g\varphi$$

for all $\varphi \in W_0^{m,p}(\Omega)$.

Definition B.1.17. Given $m \in \mathbb{N}$ and $1 \le p < \infty$, we define $W^{-m,p'}_{\mathrm{loc}}(\Omega) = \{u \in \mathcal{D}'(\Omega);\ u_{|\omega} \in W^{-m,p'}(\omega) \text{ for all } \omega \subset\subset \Omega\}$. (See Remark B.1.16 (iii) for the definition of $u_{|\omega}$.) For $p = 2$, we set $H^{-m}_{\mathrm{loc}}(\Omega) = W^{-m,2}_{\mathrm{loc}}(\Omega)$.

Proposition B.1.18. *Let $m \in \mathbb{N}$ and $1 < p < \infty$. It follows that $W_0^{m,p}(\Omega) \hookrightarrow L^p(\Omega) \hookrightarrow W^{-m,p}(\Omega)$, with dense embeddings, where the embedding $e : L^p(\Omega) \to W^{-m,p}(\Omega)$ is defined by*

$$eu(\varphi) = \int_\Omega u(x)\varphi(x)\,dx, \tag{B.1.5}$$

for all $\varphi \in W_0^{m,p'}(\Omega)$ and all $u \in L^p(\Omega)$.

Proposition B.1.18 is an immediate application of a general abstract result, see Theorem A.1.5.

REMARK B.1.19. Proposition B.1.18 calls for the following comments.

(i) Formula (B.1.5) means that if $u \in L^p(\Omega)$ and $v \in W_0^{m,p'}(\Omega)$, then

$$\langle u, v\rangle_{W^{-m,p}, W_0^{m,p'}} = \int_\Omega uv \tag{B.1.6}$$

In particular,

$$\langle u, v\rangle_{H^{-m}, H_0^m} = \int_\Omega uv \tag{B.1.7}$$

for all $u \in L^2(\Omega)$ and $v \in H_0^m(\Omega)$.

(ii) Note that any Hilbert space can be identified, via the Riesz representation theorem, with its dual. By defining the embedding $e : L^2(\Omega) \to H^{-1}(\Omega)$ by (B.1.5), we implicitly identify $L^2(\Omega)$ with its dual. If we identify $H_0^1(\Omega)$ with its dual, so that $H^{-1}(\Omega) = H_0^1(\Omega)$, then Proposition B.1.18 becomes absurd. This means that we cannot, at the same time, identify $L^2(\Omega)$ with its dual and $H_0^1(\Omega)$ with its dual, and use the canonical embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$.

PROPOSITION B.1.20. *If $1 < p < \infty$ and $-\Delta$ is defined by*

$$\langle -\Delta u, \varphi\rangle_{W^{-1,p}, W_0^{1,p'}} = \int_\Omega \nabla u \cdot \nabla\varphi, \tag{B.1.8}$$

for all $\varphi \in W_0^{1,p'}(\Omega)$, then $-\Delta \in \mathcal{L}(W^{1,p}(\Omega), W^{-1,p}(\Omega))$. In particular, $-\Delta \in \mathcal{L}(H^1(\Omega), H^{-1}(\Omega))$.

PROOF. We note that

$$\Big|\int_\Omega \nabla u \cdot \nabla\varphi\Big| \le \|\nabla u\|_{L^p}\|\nabla\varphi\|_{L^{p'}} \le \|u\|_{W^{1,p}}\|\varphi\|_{W_0^{1,p'}},$$

for all $u \in W_0^{1,p}(\Omega), \varphi \in W_0^{1,p'}(\Omega)$. It follows that (B.1.8) defines an element of $W^{-1,p}(\Omega)$ (note also that this definition is consistent with the classical definition) and that $\|-\Delta u\|_{W^{-1,p}} \le \|u\|_{W^{1,p}}$, i.e. $-\Delta \in \mathcal{L}(W^{1,p}(\Omega), W^{-1,p}(\Omega))$. □

COROLLARY B.1.21. *Let*

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2, \tag{B.1.9}$$

for $u \in H_0^1(\Omega)$. Then $J \in C^1(H_0^1(\Omega), \mathbb{R})$ and

$$J'(u) = -\Delta u, \tag{B.1.10}$$

for all $u \in H_0^1(\Omega)$.

PROOF. We have

$$J(u+v) - J(u) - \langle -\Delta u, v\rangle_{H^{-1}, H_0^1} = \frac{1}{2}\int_\Omega |\nabla v|^2,$$

from which the result follows. □

We end this section with a useful density result.

PROPOSITION B.1.22. *Let m, j be nonnegative integers and let $1 \le p, q < \infty$. The following properties hold:*

(i) *$C_c^\infty(\Omega)$ is dense in $W_0^{m,p}(\Omega) \cap W_0^{j,q}(\Omega)$;*
(ii) *if $q > 1$, then $C_c^\infty(\Omega)$ is dense in $W_0^{m,p}(\Omega) \cap W^{-j,q'}(\Omega)$;*
(iii) *if $p, q > 1$, then $C_c^\infty(\Omega)$ is dense in $W^{-m,p'}(\Omega) \cap W^{-j,q'}(\Omega)$;*
(iv) *$C_c^\infty(\Omega)$ is dense in $W_0^{m,p}(\Omega) \cap C_0(\Omega)$;*

(v) *if $p > 1$, then $C_c^\infty(\Omega)$ is dense in $W^{-m,p'}(\Omega) \cap C_0(\Omega)$;*

PROOF. Let $X = W_0^{m,p}(\Omega) \cap W_0^{j,q}(\Omega)$. It follows from Proposition A.1.13 that

$$X^\star = W^{-m,p'}(\Omega) + W^{-j,q'}(\Omega).$$

Suppose that $f \in X^\star$ is such that $\langle f, \varphi \rangle_{X^\star, X} = 0$ for all $\varphi \in C_c^\infty(\Omega)$ and write $f = f_1 + f_2$ with $f_1 \in W^{-m,p'}(\Omega)$ and $f_2 \in W^{-j,q'}(\Omega)$. We have (see Proposition A.1.13)

$$\begin{aligned} \langle f, \varphi \rangle_{X^\star, X} &= \langle f_1, \varphi \rangle_{W^{-m,p'}, W_0^{m,p}} + \langle f_2, \varphi \rangle_{W^{-j,q'}, W_0^{j,q}} \\ &= \langle f_1, \varphi \rangle_{\mathcal{D}', \mathcal{D}} + \langle f_2, \varphi \rangle_{\mathcal{D}', \mathcal{D}} \\ &= \langle f_1 + f_2, \varphi \rangle_{\mathcal{D}', \mathcal{D}} = \langle f, \varphi \rangle_{\mathcal{D}', \mathcal{D}}. \end{aligned}$$

It follows that $f = 0$ in $\mathcal{D}'(\Omega)$, hence in $X^\star$ (see Remark A.1.14). Therefore, $C_c^\infty(\Omega)$ is dense in X. This proves property (i), and properties (ii) and (iii) are proved by the same argument. (Note that if $p > 1$, then $W_0^{m,p}(\Omega)$ is reflexive; and so, $(W^{-m,p'}(\Omega))^\star = W_0^{m,p}(\Omega)$.) Properties (iv) and (v) are also proved by the same argument, since the dual of $C_0(\Omega)$ is also a space of distributions (since $C_c^\infty(\Omega)$ is dense in $C_0(\Omega)$). □

B.2. The chain rule and applications

We now study the chain rule, and we begin with a simple result.

PROPOSITION B.2.1. *Let $F \in C^1(\mathbb{R}, \mathbb{R})$ satisfy $F(0) = 0$ and $\|F'\|_{L^\infty} = L < \infty$, and consider $1 \le p \le \infty$. If $u \in W^{1,p}(\Omega)$, then $F(u) \in W^{1,p}(\Omega)$ and*

$$\nabla F(u) = F'(u) \nabla u, \tag{B.2.1}$$

a.e. in Ω. Moreover, if $p < \infty$, then the mapping $u \mapsto F(u)$ is continuous $W^{1,p}(\Omega) \to W^{1,p}(\Omega)$. Furthermore, if $p < \infty$ and $u \in W_0^{1,p}(\Omega)$, then $F(u) \in W_0^{1,p}(\Omega)$.

PROOF. We proceed in four steps.

STEP 1. The case $u \in C_c^1(\Omega)$. It is immediate that $F(u) \in C_c^1(\Omega)$ and that (B.2.1) holds.

STEP 2. The case $u \in W_0^{1,p}(\Omega)$. Suppose $p < \infty$, let $u \in W_0^{1,p}(\Omega)$ and let $(u_n)_{n \ge 0} \subset C_c^\infty(\Omega)$ satisfy $u_n \to u$ in $W_0^{1,p}(\Omega)$ as $n \to \infty$. By possibly extracting a subsequence, we may assume that

$$|u_n| + |\nabla u_n| \le f \in L^p(\Omega),$$

and that

$$u_n \to u, \quad \nabla u_n \to \nabla u,$$

a.e. in Ω. It follows from Step 1 that $F(u_n) \in C_c^1(\Omega) \subset W_0^{1,p}(\Omega)$ and that $\nabla F(u_n) = F'(u_n) \nabla u_n$. In particular,

$$|\nabla F(u_n)| \le L |\nabla u_n| \le L f.$$

Since $F'(u_n) \nabla u_n \to F'(u) \nabla u$ a.e., we obtain $\nabla F(u_n) \to F'(u) \nabla u$ in $L^p(\Omega)$. Moreover, since $|F(u_n) - F(u)| \le L |u_n - u|$, we have $F(u_n) \to F(u)$ in $L^p(\Omega)$. This implies that $F(u_n) \to F(u)$ in $W_0^{1,p}(\Omega)$ and that (B.2.1) holds.

STEP 3. The case $u \in W^{1,p}(\Omega)$. We have $F(u) \in L^p(\Omega)$. Furthermore, given $\varphi \in C_c^1(\Omega)$, let $\xi \in C_c^1(\Omega)$ satisfy $\xi = 1$ on $\operatorname{supp} \varphi$. By Remark B.1.4 (i) and Remark B.1.10 (i), we have $\xi u \in W_0^{1,q}(\Omega)$ for all $1 \le q < \infty$ such that $q \le p$. It follows from Step 2 that

$$\int_\Omega F(u) \nabla \varphi = \int_\Omega F(\xi u) \nabla \varphi = -\int_\Omega \varphi F'(\xi u) \nabla (\xi u) = -\int_\Omega \varphi F'(u) \nabla u.$$

Since clearly $F'(u)\nabla u \in L^p(\Omega)$, we deduce that $F(u) \in W^{1,p}(\Omega)$ and that (B.2.1) holds.

STEP 4. Continuity. Suppose $p < \infty$ and $u_n \to u$ in $W^{1,p}(\Omega)$. We show that $F(u_n) \to F(u)$ in $W^{1,p}(\Omega)$ by contradiction. Thus we assume that $\|F(u_n) - F(u)\|_{W^{1,p}} \geq \varepsilon > 0$. We have $F(u_n) \to F(u)$ in $L^p(\Omega)$. By possibly extracting a subsequence, we may assume that $u_n \to u$ and $\nabla u_n \to \nabla u$ a.e. It follows by dominated convergence that $F'(u_n)\nabla u_n \to F'(u)\nabla u$ in $L^p(\Omega)$. Thus $F(u_n) \to F(u)$ in $W^{1,p}(\Omega)$, which is absurd. □

REMARK B.2.2. One can prove the following stronger result. If $F : \mathbb{R} \to \mathbb{R}$ is (globally) Lipschitz continuous and if $F(0) = 0$, then for every $u \in W^{1,p}(\Omega)$, we have $F(u) \in W^{1,p}(\Omega)$. Moreover, $\nabla F(u) = F'(u)\nabla u$ a.e. This formula makes sense, since F' exists a.e. and $\nabla u = 0$ a.e. on the set $\{x \in \Omega;\ u(x) \in A\}$ where $A \subset \mathbb{R}$ is any set of measure 0. Furthermore, the mapping $u \to F(u)$ is continuous $W^{1,p}(\Omega) \to W^{1,p}(\Omega)$ if $p < \infty$. Finally, if $p < \infty$ and $u \in W^{1,p}_0(\Omega)$, then $F(u) \in W^{1,p}_0(\Omega)$. The proof is rather delicate and makes use in particular of Lebesgue's points theory. (See [23].) We will establish below a particular case of that result.

PROPOSITION B.2.3. *Set* $u^+ = \max\{u, 0\}$ *for all* $u \in \mathbb{R}$ *and let* $1 \leq p \leq \infty$. *If* $u \in W^{1,p}(\Omega)$, *then* $u^+ \in W^{1,p}(\Omega)$. *Moreover,*

$$\nabla u^+ = \begin{cases} \nabla u & \text{if } u > 0, \\ 0 & \text{if } u \leq 0, \end{cases} \tag{B.2.2}$$

a.e. If $p < \infty$, *then the mapping* $u \mapsto u^+$ *is continuous* $W^{1,p}(\Omega) \to W^{1,p}(\Omega)$. *Furthermore, if* $u \in W^{1,p}_0(\Omega)$, *then* $u^+ \in W^{1,p}_0(\Omega)$

PROOF. We proceed in four steps.

STEP 1. If $p < \infty$ and $u \in W^{1,p}_0(\Omega)$, then $u^+ \in W^{1,p}_0(\Omega)$ and (B.2.2) holds. Given $\varepsilon > 0$, let

$$\varphi_\varepsilon(u) = \begin{cases} \sqrt{\varepsilon^2 + u^2} - \varepsilon & \text{if } u \geq 0, \\ 0 & \text{if } u \leq 0. \end{cases}$$

It follows from Proposition B.2.1 that $\varphi_\varepsilon(u) \in W^{1,p}_0(\Omega)$ and $\nabla\varphi_\varepsilon(u) = \varphi'_\varepsilon(u)\nabla u$ a.e. We deduce easily that $\varphi_\varepsilon(u) \to u^+$ and that $\nabla\varphi_\varepsilon(u)$ converges to the right-hand side of (B.2.2) in $L^p(\Omega)$ as $\varepsilon \downarrow 0$. Thus $u^+ \in W^{1,p}_0(\Omega)$ and (B.2.2) holds.

STEP 2. If $u \in W^{1,p}(\Omega)$, then $u^+ \in W^{1,p}(\Omega)$ and (B.2.2) holds. Using Step 1, this is proved by the argument in Step 3 of the proof of Proposition B.2.1.

STEP 3. If $a \in \mathbb{R}$ and $u \in W^{1,p}(\Omega)$, then $\nabla u = 0$ a.e. on the set $\{x \in \Omega;\ u(x) = a\}$. Consider a function $\eta \in C^\infty_c(\mathbb{R})$ such that $\eta(x) = 1$ for $|x| \leq 1$, $\eta(x) = 0$ for $|x| \geq 2$ and $0 \leq \eta \leq 1$. For $n \in \mathbb{N}$, $n \geq 1$, set

$$g_n(x) = \eta(n(x - a)),$$

and

$$h_n(x) = \int_0^x g_n(s)\, ds.$$

It follows from Proposition B.2.1 that $h_n(u) \in W^{1,p}(\Omega)$ and that $\nabla h_n(u) = g_n(u)\nabla u$ a.e. Therefore,

$$-\int_{\mathbb{R}} h_n(u)\nabla\varphi = \int_{\mathbb{R}} g_n(u)\varphi\nabla u,$$

for all $\varphi \in C^1_c(\Omega)$. Since $|h_n| \le n^{-1}\|\eta\|_{L^1}$, the left-hand side of the above inequality tends to 0 as $n \to \infty$. Therefore,

$$\int_{\mathbb{R}} g_n(u)\varphi\nabla u \underset{n\to\infty}{\longrightarrow} 0.$$

Note that $g_n(u) \to 1_{\{x\in\Omega;\, u(x)=a\}}$. Since $0 \le g_n \le 1$, we deduce that

$$\int_{\mathbb{R}} 1_{\{x\in\Omega;\, u(x)=a\}}\varphi\nabla u = 0,$$

for all $\varphi \in C^1_c(\Omega)$; and so, $1_{\{x\in\Omega;\, u(x)=a\}}\nabla u = 0$ a.e. The result follows.

STEP 4. Continuity. Suppose $p < \infty$ and let $u_n \to u$ in $W^{1,p}(\Omega)$ as $n \to \infty$. We have $|u^+ - u_n^+| \le |u - u_n|$, so that $u_n^+ \to u^+$ in $L^p(\Omega)$. Therefore, we need only show that for any subsequence, which we still denote by $(u_n)_{n\ge 0}$, there exists a subsequence $(u_{n_k})_{k\ge 0}$ such that $\nabla u^+_{n_k} \to \nabla u^+$ in $L^p(\Omega)$ as $k \to \infty$. We may extract a subsequence $(u_{n_k})_{k\ge 0}$ such that $u_{n_k} \to u$ and $\nabla u_{n_k} \to \nabla u$ a.e., and such that

$$|u_{n_k}| + |\nabla u_{n_k}| \le f \in L^p(\Omega).$$

Set

$$\begin{aligned}
&A_0 = \{x \in \Omega;\, u(x) = 0\},\\
&A^+ = \{x \in \Omega;\, u(x) > 0\}, \quad A^+_k = \{x \in \Omega;\, u_{n_k}(x) > 0\},\\
&A^- = \{x \in \Omega;\, u(x) < 0\}, \quad A^-_k = \{x \in \Omega;\, u_{n_k}(x) < 0\}.
\end{aligned}$$

For a.a. $x \in A^+$, we have $x \in A^+_k$ for k large, thus $\nabla u^+_{n_k}(x) = \nabla u_{n_k}(x) \to \nabla u(x) = \nabla u^+(x)$. For a.a. $x \in A^-$, we have $x \in A^-_k$ for k large, hence $\nabla u^+_{n_k}(x) = 0 = \nabla u^+(x)$. For $x \in A_0$, we have $u(x) = 0$, so that by Step 3, $\nabla u(x) = 0$ a.e. Since $\nabla u_{n_k} \to \nabla u = 0$ a.e. on A_0, we deduce in particular that $|\nabla u^+_{n_k}| \le |\nabla u_{n_k}| \to 0$ a.e. in A_0. Thus $\nabla u^+_{n_k} \to 0 = \nabla u^+$ a.e. on A_0. It follows that

$$\nabla u^+_{n_k} \to \begin{cases} \nabla u & \text{if } u > 0,\\ 0 & \text{if } u \le 0,\end{cases}$$

a.e., and the result follows by dominated convergence. This completes the proof. □

REMARK B.2.4. Let $u^- = \max\{0, -u\}$. Since $u^- = (-u)^+$, we may draw similar conclusions for u^-. In particular, if $u \in W^{1,p}(\Omega)$, then $u^- \in W^{1,p}(\Omega)$. Moreover,

$$\nabla u^- = \begin{cases} -\nabla u & \text{if } u < 0,\\ 0 & \text{if } u \ge 0,\end{cases} \tag{B.2.3}$$

a.e. If $p < \infty$, then the mapping $u \to u^-$ is continuous $W^{1,p}(\Omega) \to W^{1,p}(\Omega)$. Furthermore, if $u \in W^{1,p}_0(\Omega)$, then $u^- \in W^{1,p}_0(\Omega)$. Note in particular that (B.2.2) and (B.2.3) imply that

$$\nabla u^+ \cdot \nabla u^- = 0 \tag{B.2.4}$$

a.e. Since $|u| = u^+ + u^-$, we deduce the following properties. If $u \in W^{1,p}(\Omega)$, then $|u| \in W^{1,p}(\Omega)$. Moreover,

$$\nabla |u| = \begin{cases} \nabla u & \text{if } u > 0,\\ -\nabla u & \text{if } u < 0,\\ 0 & \text{if } u = 0,\end{cases} \tag{B.2.5}$$

a.e. Note in particular that

$$|\nabla |u|\,| = |\nabla u|, \tag{B.2.6}$$

a.e. If $p < \infty$, then the mapping $u \to |u|$ is continuous $W^{1,p}(\Omega) \to W^{1,p}(\Omega)$. Furthermore, if $u \in W^{1,p}_0(\Omega)$, then $|u| \in W^{1,p}_0(\Omega)$.

COROLLARY B.2.5. *Let* $1 \le p < \infty$*, let* $u \in W^{1,p}(\Omega)$ *and* $v \in W_0^{1,p}(\Omega)$*. If* $|u| \le |v|$ *a.e., then* $u \in W_0^{1,p}(\Omega)$*.*

PROOF. It follows from Remark B.2.4 that $|v| \in W_0^{1,p}(\Omega)$. Let $(w_n)_{n\ge 0} \subset C_c^\infty(\Omega)$ satisfy $w_n \to |v|$ in $W^{1,p}(\Omega)$ as $n \to \infty$. It follows that $w_n - u^+ \to |v| - u^+$ in $W^{1,p}(\Omega)$, so that $(w_n - u^+)^+ \to (|v| - u^+)^+$ in $W^{1,p}(\Omega)$ by Proposition B.2.3. Since $(w_n - u^+)^+ \le w_n^+$, we see that $(w_n - u^+)^+$ has compact support; and so $(w_n - u^+)^+ \in W_0^{1,p}(\Omega)$. We deduce that $(|v| - u^+)^+ \in W_0^{1,p}(\Omega)$. Since $(|v| - u^+)^+ = |v| - u^+$, we see that $u^+ \in W_0^{1,p}(\Omega)$. One shows as well that $u^- \in W_0^{1,p}(\Omega)$, and the result follows. □

COROLLARY B.2.6. *Let* $1 \le p \le \infty$ *and let* $M \ge 0$*. If* $u \in W^{1,p}(\Omega)$*, then* $(u - M)^+ \in u \in W^{1,p}(\Omega)$ *and*

$$\nabla(u - M)^+ = \begin{cases} \nabla u & \text{if } u(x) > M, \\ 0 & \text{if } u(x) \le M, \end{cases} \tag{B.2.7}$$

a.e. in Ω*. If* $p < \infty$*, then the mapping* $u \mapsto (u - M)^+$ *is continuous* $W^{1,p}(\Omega) \to W^{1,p}(\Omega)$*. Moreover, if* $u \in W_0^{1,p}(\Omega)$*, then* $(u - M)^+ \in W_0^{1,p}(\Omega)$*.*

PROOF. The last property is a consequence of Corollary B.2.5, since $(u-M)^+ \le u^+ \in W_0^{1,p}(\Omega)$. Next, observe that if Ω is bounded, then the conclusions are a consequence of Proposition B.2.3, because $u - M \in W^{1,p}(\Omega)$ whenever $u \in W^{1,p}(\Omega)$. In particular, we see that for an arbitrary Ω, if $u \in W^{1,p}(\Omega)$, then $(u - M)^+ \in W^{1,p}_{\mathrm{loc}}(\Omega)$ and (B.2.7) holds. In particular, $|\nabla(u - M)^+| \le |\nabla u| \in L^p(\Omega)$. Since $(u - M)^+ \le u^+ \in L^p(\Omega)$, we see that $(u - M)^+ \in W^{1,p}(\Omega)$ and that (B.2.7) holds.

It now remains to show the continuity of the mapping $u \mapsto (u - M)^+$ when $p < \infty$. By the above observation, we may assume that Ω is unbounded. Given $R > 0$, let $\Omega_R = \{x \in \Omega;\ |x| < R\}$ and $U_R = \Omega \setminus \Omega_R$. We argue by contradiction, and we consider a sequence $(u_n)_{n\ge 0} \subset W^{1,p}(\Omega)$ and $u \in W^{1,p}(\Omega)$ such that $u_n \underset{n\to\infty}{\longrightarrow} u$ in $W^{1,p}(\Omega)$ and $\|(u_n - M)^+ - (u - M)^+\|_{W^{1,p}} \ge \varepsilon > 0$. Note that

$$|(u_n - M)^+ - (u - M)^+| \le |u_n - u| \underset{n\to\infty}{\longrightarrow} 0,$$

in $L^p(\Omega)$, so that we may assume $\|\nabla(u_n - M)^+ - \nabla(u - M)^+\|_{L^p} \ge \varepsilon > 0$. By possibly extracting a subsequence, we may also assume that there exists $f \in L^p(\Omega)$ such that $|\nabla u_n| + |\nabla u| \le f$ a.e. In particular, it follows from (B.2.7) that $|\nabla(u_n - M)^+ - \nabla(u - M)^+| \le |\nabla u_n| + |\nabla u| \le f$ a.e. Therefore, by dominated convergence, we may choose R large enough so that

$$\|\nabla(u_n - M)^+ - \nabla(u - M)^+\|_{L^p(U_R)} \le \frac{\varepsilon}{4}.$$

Finally, since Ω_R is bounded, it follows that $\|\nabla(u_n - M)^+ - \nabla(u - M)^+\|_{L^p(\Omega_R)} \to 0$ as $n \to \infty$. Therefore, for n large enough,

$$\|\nabla(u_n - M)^+ - \nabla(u - M)^+\|_{L^p(\Omega_R)} \le \frac{\varepsilon}{4}.$$

We deduce that $\|\nabla(u_n - M)^+ - \nabla(u - M)^+\|_{L^p(\Omega)} \le \varepsilon/2$, which yields a contradiction. This completes the proof. □

COROLLARY B.2.7. *Let* $1 \le p \le \infty$*,* $(u_n)_{n\ge 0} \subset W^{1,p}(\Omega)$ *and* $u \in W^{1,p}(\Omega)$*. If* $u_n \to u$ *in* $W^{1,p}(\Omega)$ *as* $n \to \infty$*, then there exist a subsequence* $(u_{n_k})_{k\ge 0}$ *and* $v \in W^{1,p}(\Omega)$ *such that* $|u_{n_k}| \le v$ *a.e. in* Ω *for all* $k \ge 0$*. If, in addition,* $p < \infty$ *and* $(u_n)_{n\ge 0} \subset W_0^{1,p}(\Omega)$*, then one can choose* $v \in W_0^{1,p}(\Omega)$*.*

PROOF. Let the subsequence $(u_{n_k})_{k\ge0}$ satisfy $\|u_{n_k}-u\|_{W^{1,p}}\le 2^{-k-1}$, so that $\|u_{n_{k+1}}-u_{n_k}\|_{W^{1,p}}\le 2^{-k}$. It follows from Remark B.2.4 that $|u_{n_{k+1}}-u_{n_k}|\in W^{1,p}(\Omega)$ and that $\|\,|u_{n_{k+1}}-u_{n_k}|\,\|_{W^{1,p}}\le 2^{-k}$. Thus, the series

$$v=|u_{n_0}|+\sum_{j\ge0}|u_{n_{j+1}}-u_{n_j}|,$$

is normally convergent in $W^{1,p}(\Omega)$. Since

$$u_{n_{k+1}}=u_{n_0}+\sum_{j=0}^{k}(u_{n_{j+1}}-u_{n_j}),$$

we see that $|u_{n_{k+1}}|\le v$. The result follows, using again Remark B.2.4 in the case $p<\infty$ and $(u_n)_{n\ge0}\subset W_0^{1,p}(\Omega)$. □

COROLLARY B.2.8. *Let $1\le p<\infty$, $0\le A,B\le\infty$ and set*

$$E=\{u\in W_0^{1,p}(\Omega);\ -A\le u\le B\ a.e.\},$$
$$F=\{u\in C_{\rm c}^{\infty}(\Omega);\ -A\le u\le B\}.$$

It follows that $E=\overline{F}$, where the closure is in $W_0^{1,p}(\Omega)$. In particular, $\{u\in W_0^{1,p}(\Omega);\ u\ge0\ a.e.\}$ is the closure in $W_0^{1,p}(\Omega)$ of $\{u\in C_{\rm c}^{\infty}(\Omega);\ u\ge0\}$.

PROOF. We have $F\subset E$. Since E is clearly closed in $W_0^{1,p}(\Omega)$, we deduce that $\overline{F}\subset E$. We now show the converse inclusion. Let $u\in E$ and let $(u_n)_{n\ge0}\subset C_{\rm c}^{\infty}(\Omega)$ be such that $u_n\to u$ in $W_0^{1,p}(\Omega)$. Set

$$v_n=\max\{-A,\min\{u_n,B\}\}=u_n+(u_n+A)^- -(u_n-B)^+.$$

It follows from Corollary B.2.6 that $v_n\in W_0^{1,p}(\Omega)$ and that

$$v_n\underset{n\to\infty}{\longrightarrow}u+(u+A)^- -(u-B)^+=u,$$

in $W_0^{1,p}(\Omega)$. Thus if $(v_n)_{n\ge0}\subset\overline{F}$, then the conclusion follows. Since clearly $v_n\in C_{\rm c}(\Omega)$, we need only show the following property: if $w\in E\cap C_{\rm c}(\Omega)$, then $w\in\overline{F}$. To see this, let $(\rho_n)_{n\ge0}$ be a smoothing sequence and set $\widetilde{w}_n=\rho_n\star\widetilde{w}$, where $\widetilde{w}$ is the extension of w by 0 outside Ω. Since w has compact support in Ω, we see that if n is sufficiently large, then $\widetilde{w}_n$ also has compact support in Ω. Moreover, $\widetilde{w}_n\in C_{\rm c}^{\infty}(\Omega)$, so that if $w_n=(\widetilde{w}_n)_{|\Omega}$, then $w_n\in C_{\rm c}^{\infty}(\Omega)$. In addition, $\widetilde{w}_n\to\widetilde{w}$ in $W^{1,p}(\mathbb{R}^N)$, so that $w_n\to w$ in $W_0^{1,p}(\Omega)$. It remains to show that $-A\le\widetilde{w}_n\le B$, which is immediate since $-A\le\widetilde{w}\le B$. This completes the proof. □

B.3. Sobolev's inequalities

In this section, we establish some Sobolev-type inequalities and embeddings. It is convenient to make the following definition.

DEFINITION B.3.1. Given an integer $m\ge0$, $1\le p\le\infty$ and Ω and open subset of $\mathbb{R}^N$, we set

$$|u|_{m,p,\Omega}=\sum_{|\alpha|=m}\|D^\alpha u\|_{L^p(\Omega)}.$$

When there is no risk of confusion, we set

$$|u|_{m,p}=|u|_{m,p,\Omega},$$

i.e. we omit the dependence on Ω.

We begin with inequalities for smooth functions on $\mathbb{R}^N$. The following result is the main inequality of this section.

THEOREM B.3.2 (Gagliardo-Nirenberg's inequality). *Consider* $1 \le p, q, r \le \infty$ *and let* j, m *be two integers,* $0 \le j < m$. *If*

$$\frac{1}{p} - \frac{j}{N} = a\Big(\frac{1}{r} - \frac{m}{N}\Big) + \frac{1-a}{q}, \tag{B.3.1}$$

for some $a \in [j/m, 1]$ *(*$a < 1$ *if* $r = N/(m-j) > 1$*), then there exists a constant* $C = C(N, m, j, a, q, r)$ *such that*

$$|u|_{j,p} \le C |u|_{m,r}^{a} \|u\|_{L^q}^{1-a}, \tag{B.3.2}$$

for all $u \in C_c^m(\mathbb{R}^N)$.

The proof of Theorem B.3.2 uses various important inequalities. The fundamental ingredients are Sobolev's inequality (Theorem B.3.5), Morrey's inequality (Theorem B.3.8), and an inequality for intermediate derivatives (Theorem B.3.10). We begin with the following first-order Sobolev inequality.

THEOREM B.3.3. *Let* $N \ge 1$. *For every* $u \in C_c^1(\mathbb{R}^N)$, *we have*

$$\|u\|_{L^{\frac{N}{N-1}}} \le \frac{1}{2} \prod_{j=1}^{N} \Big\| \frac{\partial u}{\partial x_j} \Big\|_{L^1}^{\frac{1}{N}}. \tag{B.3.3}$$

In particular,

$$\|u\|_{L^{\frac{N}{N-1}}} \le \frac{1}{2N} |u|_{1,1}, \tag{B.3.4}$$

for all $u \in C_c^1(\mathbb{R}^N)$.

PROOF. We proceed in three steps.

STEP 1. The case $N = 1$. Given $x \in \mathbb{R}$, we have

$$u(x) = \int_{-\infty}^{x} u'(s)\, ds;$$

and so,

$$|u(x)| \le \int_{-\infty}^{x} |u'(s)|\, ds.$$

As well,

$$|u(x)| \le \int_{x}^{+\infty} |u'(s)|\, ds,$$

so that by summing up the two above inequalities,

$$|u(x)| \le \frac{1}{2} \int_{-\infty}^{+\infty} |u'(s)|\, ds,$$

which yields (B.3.3) (and (B.3.4)) in the case $N = 1$.

STEP 2. Proof of (B.3.3). We assume $N \ge 2$. For any $1 \le j \le N$, it follows from Step 1 that

$$|u(x)| \le \frac{1}{2} \int_{\mathbb{R}} |\partial_j u(x_1, \dots, x_{j-1}, s, x_{j+1}, \dots, x_N)|\, ds;$$

and so,

$$|u(x)|^N \le 2^{-N} \prod_{j=1}^{N} \int_{\mathbb{R}} |\partial_j u(x_1, \dots, x_{j-1}, s, x_{j+1}, \dots, x_N)|\, ds.$$

Taking the $(N-1)^{\text{th}}$ root and integrating on $\mathbb{R}^N$, we obtain

$$\int_{\mathbb{R}^N} |u(x)|^{\frac{N}{N-1}}\, dx \leq 2^{-\frac{N}{N-1}} \int_{\mathbb{R}^N} \prod_{j=1}^{N} \Big(\int_{\mathbb{R}} |\partial_j u(x_1,\dots,x_{j-1},s,x_{j+1},\dots,x_N)|\, ds \Big)^{\frac{1}{N-1}}.$$

We observe that the right-hand side is the product of N functions, each of which depends only on $N-1$ of the variables $x_1,\dots,x_N$ (with a permutation). Therefore, integrating in each of the variables $x_1,\dots,x_N$, we may apply Hölder's inequality

$$\int_{\mathbb{R}} a_1^{\frac{1}{N-1}} \cdots a_{N-1}^{\frac{1}{N-1}} \leq \prod_{\ell=1}^{N-1} \Big(\int_{\mathbb{R}} a_\ell \Big)^{\frac{1}{N-1}}.$$

For example, if we first integrate in x_1, we obtain

$$\begin{aligned}
\int_{\mathbb{R}} dx_1 \prod_{j=1}^{N} \Big(\int_{\mathbb{R}} |\partial_j u(x_1,\dots,x_{j-1},s,x_{j+1},\dots,x_N)|\, ds \Big)^{\frac{1}{N-1}} \\
= \Big(\int_{\mathbb{R}} |\partial_1 u(s,x_2,\dots,x_N)|\, ds \Big)^{\frac{1}{N-1}} \\
\times \int_{\mathbb{R}} \prod_{j=2}^{N} \Big(\int_{\mathbb{R}} |\partial_j u(x_1,\dots,x_{j-1},s,x_{j+1},\dots,x_N)|\, ds \Big)^{\frac{1}{N-1}} \\
\leq \Big(\int_{\mathbb{R}} |\partial_1 u(s,x_2,\dots,x_N)|\, ds \Big)^{\frac{1}{N-1}} \\
\times \prod_{j=2}^{N} \Big(\int_{\mathbb{R}^2} |\partial_j u(x_1,\dots,x_{j-1},s,x_{j+1},\dots,x_N)|\, ds dx_1 \Big)^{\frac{1}{N-1}}.
\end{aligned}$$

Integrating successively in each of the variables $x_1,\dots,x_N$, we obtain finally the estimate (B.3.3).

STEP 3. Proof of (B.3.4). We claim that if $(a_j)_{1\leq j\leq N} \in \mathbb{R}^N$ with $a_j \geq 0$, then

$$\Big(\prod_{j=1}^{N} a_j \Big)^{\frac{1}{N}} \leq \frac{1}{N} \sum_{j=1}^{N} a_j. \tag{B.3.5}$$

The estimate (B.3.4) is a consequence of (B.3.3) and (B.3.5). The claim (B.3.5) follows if show that

$$\max_{|x|^2=1} \prod_{j=1}^{N} x_j^2 = N^{-N}. \tag{B.3.6}$$

To prove (B.3.6), we observe that if the maximum is achieved at x, then there exists a Lagrange multiplier $\lambda \in \mathbb{R}$ such that $F'(x) = \lambda x$, where $F(x) = x_1^2 \dots x_N^2$. This implies that

$$2x_i \prod_{j\neq i} x_j^2 = \lambda x_i,$$

for all $1 \leq i \leq N$. Since none of the x_i vanishes (for the maximum is clearly positive), this implies that $x_1^2 = \cdots = x_N^2$, from which (B.3.6) follows. □

COROLLARY B.3.4. *Let $1 \leq r \leq N$ ($r < N$ if $N \geq 2$). If $r^* > r$ is defined by*

$$\frac{1}{r^*} = \frac{1}{r} - \frac{1}{N},$$

then

$$\|u\|_{L^{r^*}} \le c_{N,r} |u|_{1,r}, \tag{B.3.7}$$

for every $u \in C^1_c(\mathbb{R}^N)$*, with* $c_{N,r} = (N-1)r/2N(N-r)$*. (We use the convention that* $(N-1)/(N-1) = 1$ *if* $N = 1$*.)*

PROOF. The case $N = 1$ follows from Theorem B.3.3, so we assume $N \ge 2$. Let

$$t = \frac{N-1}{N} r^* = \frac{(N-1)r}{N-r}.$$

Since $r \ge 1$, we have $t \ge 1$. We observe that

$$\frac{Nt}{N-1} = (t-1)r' = r^*,$$

and we apply (B.3.4) with u replaced by $|u|^{t-1}u$, and we obtain

$$\|u\|^t_{L^{r^*}} \le (2N)^{-1} |\, |u|^{t-1} u|_{1,1}. \tag{B.3.8}$$

It follows from (B.2.1) that $\partial_j(|u|^{t-1}u) = t|u|^{t-1}\partial_j u$ for all $1 \le j \le N$. Therefore, by Hölder's inequality,

$$\|\partial_j(|u|^{t-1}u)\|_{L^1} \le t\|u\|^{t-1}_{L^{(t-1)r'}} \|\partial_j u\|_{L^r} = t\|u\|^{t-1}_{L^{r^*}} \|\partial_j u\|_{L^r}.$$

Thus $|\, |u|^{t-1}u|_{1,1} \le t\|u\|^{t-1}_{L^{r^*}} |u|_{1,r}$, and we deduce from (B.3.8) that

$$\|u\|^t_{L^{r^*}} \le (2N)^{-1} t \|u\|^{t-1}_{L^{r^*}} |u|_{1,r}, \tag{B.3.9}$$

and (B.3.7) follows. □

The following Sobolev's inequality is now a consequence of Corollary B.3.4.

THEOREM B.3.5 (Sobolev's inequality). *Let* $m \le N$ *be an integer, let* $1 \le r \le N/m$ *(*$r < N/m$ *if* $N \ge 2$*), and let* $r^* > r$ *be defined by*

$$\frac{1}{r^*} = \frac{1}{r} - \frac{m}{N}.$$

If

$$c_{N,m,r} = \frac{[(N-1)r]^m}{(2N)^m \displaystyle\prod_{1 \le \ell \le m} (N - \ell r)}, \tag{B.3.10}$$

then

$$\|u\|_{L^{r^*}} \le c_{N,m,r} |u|_{m,r}, \tag{B.3.11}$$

for all $u \in C^m_c(\mathbb{R}^N)$*. (We use the convention that* $(N-1)/(N-1) = 1$ *if* $N = 1$*.)*

PROOF. We argue by induction on m. By Corollary B.3.4, (B.3.11) holds for $m = 1$. Suppose it holds up to some $m \ge 1$. We suppose that $m + 1 < N$ and we show (B.3.11) at the level $m+1$. Let $1 \le r < N/(m+1)$ and let r^* be defined by

$$\frac{1}{r^*} = \frac{1}{r} - \frac{m+1}{N}.$$

Define p by

$$\frac{1}{p} = \frac{1}{r^*} + \frac{1}{N} = \frac{1}{r} - \frac{m}{N}, \tag{B.3.12}$$

so that $r < p < r^*$. It follows from Corollary B.3.4 and the first identity in (B.3.12) that

$$\|u\|_{L^{r^*}} \le c_{N,p} |u|_{1,p}.$$

Next, it follows from the second identity in (B.3.12) and (B.3.11) applied to $\partial_j u$ that

$$\|\partial_j u\|_{L^p} \le c_{N,m,r} |\partial_j u|_{m,r},$$

for all $1 \le j \le N$. We deduce that

$$|u|_{1,p} \le c_{N,m,r} |u|_{m+1,r},$$

and (B.3.11) at the level $m+1$ follows with $c_{N,m+1,r} = c_{N,p} c_{N,m,r}$, i.e. (B.3.10). □

REMARK B.3.6. Note that when $N \ge 2$, the inequality $\|u\|_{L^\infty} \le C|u|_{1,N}$ does not hold, for any constant C. Indeed, given $0 < \theta < 1 - 1/N$, let $f \in C^\infty(0,\infty)$ satisfy $f(r) = |\log r|^\theta$ for $r \le 1/2$ and $f(r) = 0$ for $r \ge 1$. Let $(f_n)_{n\ge1} \subset C^\infty([0,\infty))$ be such that $f_n(r) = f(r)$ for $r \ge 1/n$ and $0 \le f_n(r) \le f(r)$ and $|f_n'(r)| \le |f'(r)|$ for all $r > 0$. Setting $u_n(x) = f_n(|x|)$, one verifies easily that $\|u_n\|_{L^\infty} \to \infty$ and $\limsup \|\nabla u_n\|_{L^N} < \infty$ as $n \to \infty$. More generally, a similar example with $0 < \theta < 1 - m/N$ shows that the inequality $\|u\|_{L^\infty} \le C|u|_{m,N/m}$ does not hold, for any constant C if $1 \le m < N$.

The following result, in the same spirit as Theorem B.3.3 (case $N = 1$) shows that the inequality $\|u\|_{L^\infty} \le C|u|_{N,1}$ holds in any dimension.

THEOREM B.3.7. *Given any $N \ge 1$,*

$$\|u\|_{L^\infty} \le 2^{-N} |u|_{N,1}, \tag{B.3.13}$$

for all $u \in C^N_c(\mathbb{R}^N)$.

PROOF. Let $y \in \mathbb{R}^N$ Integrating $\partial_1 \cdots \partial_N u$ in x_1 on $(-\infty, y_1)$ yields

$$\partial_2 \cdots \partial_N u(y_1, x_2, \dots, x_N) = \int_{-\infty}^{y_1} \partial_1 \cdots \partial_N u \, dx_1.$$

Integrating successively in the variables $x_2, \dots, x_N$, we obtain

$$u(y) = \int_{-\infty}^{y_1} \cdots \int_{-\infty}^{y_N} \partial_1 \cdots \partial_N u \, dx_1 \cdots dx_N.$$

Therefore,

$$|u(y)| \le \int_{-\infty}^{y_1} \int_{-\infty}^{y_2} \cdots \int_{-\infty}^{y_N} |\partial_1 \cdots \partial_N u| \, dx_1 \cdots dx_N. \tag{B.3.14}$$

We observe that instead on integrating in x_1 on $(-\infty, y_1)$, we might have integrated on (y_1, ∞), thus obtaining

$$|u(y)| \le \int_{y_1}^{\infty} \int_{-\infty}^{y_2} \cdots \int_{-\infty}^{y_N} |\partial_1 \cdots \partial_N u| \, dx_1 \cdots dx_N. \tag{B.3.15}$$

Summing up (B.3.14) and (B.3.15), we obtain

$$|u(y)| \le (1/2) \int_{-\infty}^{\infty} \int_{-\infty}^{y_2} \cdots \int_{-\infty}^{y_N} |\partial_1 \cdots \partial_N u| \, dx_1 \cdots dx_N.$$

Rereating this argument for each of the variables, we deduce that

$$|u(y)| \le 2^{-N} \int_{\mathbb{R}^N} |\partial_1 \cdots \partial_N u| \le |u|_{N,1},$$

and the result follows since y is arbitrary. □

In the case $p > N$, we have the following result.

THEOREM B.3.8 (Morrey's inequality). *If $r > N \ge 1$, then there exists a constant $c(N)$ such that*

$$|u(x) - u(y)| \le c(N) \frac{r}{r - N} |x - y|^{1 - \frac{N}{r}} |u|_{1,r}, \tag{B.3.16}$$

for all $u \in C^1_c(\mathbb{R}^N)$. Moreover, if $1 \le q \le \infty$ and $a \in [0,1)$ is defined by

$$0 = a\Big(\frac{1}{r} - \frac{1}{N}\Big) + \frac{1 - a}{q},$$

then

$$\|u\|_{L^\infty} \le c(N)\frac{r}{r-N}|u|_{1,r}^a\|u\|_{L^q}^{1-a}, \tag{B.3.17}$$

for all $u \in C_c^1(\mathbb{R}^N)$.

PROOF. In the following calculations, we denote by $c(N)$ various constants that may change from line to line but depend only on N. Let $z \in \mathbb{R}^N$ and $\rho > 0$, and set $B = B(z,\rho)$. Consider $x \in B$. We have

$$u(y) - u(y) = \int_0^1 \frac{d}{dt}u(x+t(y-x))\,dt = \int_0^1 (y-x)\cdot\nabla u(x+t(y-x))\,dt,$$

for all $y \in B$. Integrating on B and dividing by $|B|$, we obtain

$$\frac{1}{|B|}\int_B u(y)\,dy - u(x) = \frac{1}{|B|}\int_0^1\int_B (y-x)\cdot\nabla u(x+t(y-x))\,dy\;dt.$$

Since

$$\begin{aligned}\Big|\int_B (y-x)\cdot\nabla u(x+&t(y-x))\,dy\Big|\\ &\le \Big(\int_B |x-y|^{r'}\,dy\Big)^{\frac{1}{r'}}\Big(\int_B |\nabla u(x+t(y-x))|^r\,dy\Big)^{\frac1r}\\ &\le c(N)\rho^{1+\frac{N}{r'}}t^{-\frac{N}{r}}\Big(\int_{t(B-x)}|\nabla u(x+z)|^r\,dz\Big)^{\frac1r}\\ &\le c(N)\rho^{1+\frac{N}{r'}}t^{-\frac{N}{r}}\|\nabla u\|_{L^r},\end{aligned}$$

we deduce

$$\frac{1}{|B|}\Big|\int_0^1\int_B (y-x)\cdot\nabla u(x+t(y-x))\,dy\,dt\Big| \le c(N)\frac{r}{r-N}\rho^{1-\frac{N}{r}}\|\nabla u\|_{L^r}.$$

It follows that if $B = B(z,\rho)$ and $x \in B$, then

$$\Big|\frac{1}{|B|}\int_B u(y)\,dy - u(x)\Big| \le c(N)\frac{r}{r-N}\rho^{1-\frac{N}{r}}\|\nabla u\|_{L^r}. \tag{B.3.18}$$

Let now $x_1, x_2 \in \mathbb{R}^N$, $x_1 \ne x_2$ and let $z = (x_1+x_2)/2$ and $\rho = |x_1 - x_2|$. Applying (B.3.18) successively with $x = x_1$ and $x = x_2$ and making the sum, we obtain

$$|u(x_1) - u(x_2)| \le c(N)\frac{r}{r-N}|x_1-x_2|^{1-\frac{N}{r}}\|\nabla u\|_{L^r},$$

which proves (B.3.16).

Consider now $1 \le q \le \infty$. We have

$$\Big|\int_B u(y)\,dy\Big| \le |B|^{\frac{1}{q'}}\|u\|_{L^q};$$

and so,

$$\begin{aligned}\frac{1}{|B|}\Big|\int_B u(y)\,dy\Big| &\le |B|^{-\frac1q}\|u\|_{L^q} = N^{\frac1q}\gamma_N^{-\frac1q}\rho^{-\frac{N}{q}}\|u\|_{L^q}\\ &\le c(N)\rho^{-\frac{N}{q}}\|u\|_{L^q}.\end{aligned}$$

Therefore, we deduce from (B.3.18) that

$$|u(x)| \le c(N)\rho^{-\frac{N}{q}}\|u\|_{L^q} + c(N)\frac{r}{r-N}\rho^{1-\frac{N}{r}}\|\nabla u\|_{L^r}.$$

We now choose $\rho = \|u\|_{L^q}^\alpha\|\nabla u\|_{L^r}^{-\alpha}$, with $1 = \alpha(1 - N/r + N/q)$, and we obtain

$$|u(x)| \le c(N)\frac{r}{r-N}\|\nabla u\|_{L^r}^a\|u\|_{L^q}^{1-a}.$$

Since $x \in \mathbb{R}^N$ is arbitrary, this proves (B.3.17). □

For the proof of Theorem B.3.2, we will use the following (first-order) Gagliardo-Nirenberg's inequality, which is a consequence of Sobolev and Morrey's inequalities.

THEOREM B.3.9. *Let $1 \le p, q, r \le \infty$ and assume*
$$\frac{1}{p} = a\Big(\frac{1}{r} - \frac{1}{N}\Big) + \frac{1-a}{q}, \tag{B.3.19}$$
for some $a \in [0,1]$ ($a < 1$ if $r = N \ge 2$). It follows that there exists a constant $C = C(N, p, q, r, a)$ such that
$$\|u\|_{L^p} \le C |u|_{1,r}^a \|u\|_{L^q}^{1-a}, \tag{B.3.20}$$
for every $u \in C_c^1(\mathbb{R}^N)$.

PROOF. We consider separately several cases.

THE CASE $r > N$. Note that in this case, $p \ge q$, so that by Hölder's inequality,
$$\|u\|_{L^p} \le \|u\|_{L^\infty}^{\frac{p-q}{p}} \|u\|_{L^q}^{\frac{q}{p}}.$$
Estimating $\|u\|_{L^\infty}$ by (B.3.17), we deduce (B.3.20).

THE CASE $r < N$ (THUS $N \ge 2$). Let $r^* = Nr/(N-r)$. It follows from Hölder's inequality that
$$\|u\|_{L^p} \le \|u\|_{L^{r^*}}^a \|u\|_{L^q}^{1-a},$$
with a given by (B.3.19). (B.3.20) follows, estimating $\|u\|_{L^{r^*}}$ by (B.3.7).

THE CASE $r = N$. Suppose first $N = 1$. Then by Hölder's inequality,
$$\|u\|_{L^p} \le \|u\|_{L^\infty}^a \|u\|_{L^q}^{1-a},$$
and the result follows from (B.3.4). In the case $N \ge 2$ (thus $a < 1$) we cannot use the same argument since $\|u\|_{L^\infty}$ is *not* estimated in terms of $\|\nabla u\|_{L^N}$. Instead, we apply (B.3.4) with u replaced by $|u|^{t-1}u$ for some $t \ge 1$. As in the proof of (B.3.9), we obtain
$$\|u\|_{L^{\frac{tN}{N-1}}}^t \le (2N)^{-1} t \|u\|_{L^{\frac{(t-1)N}{N-1}}}^{t-1} |u|_{1,r}. \tag{B.3.21}$$
Suppose first that $p \ge q + N/(N-1)$, and let $t \ge 1$ be defined by
$$\frac{tN}{N-1} = p.$$
It follows that $(t-1)N/(N-1) \ge q$. By Hölder's inequality,
$$\|u\|_{L^{\frac{(t-1)N}{N-1}}} \le \|u\|_{L^p}^\alpha \|u\|_{L^q}^{1-\alpha}, \tag{B.3.22}$$
with
$$\frac{N-1}{(t-1)N} = \frac{\alpha(N-1)}{tN} + \frac{1-\alpha}{q}.$$
It follows from (B.3.21)-(B.3.22) that
$$\|u\|_{L^p}^t \le (2N)^{-1} t \|u\|_{L^p}^{(t-1)\alpha} \|u\|_{L^q}^{(t-1)(1-\alpha)} |u|_{1,r};$$
and so,
$$\|u\|_{L^p} \le (t/2N)^{\frac{1}{t-(t-1)\alpha}} \|u\|_{L^q}^{\frac{(t-1)(1-\alpha)}{t-(t-1)\alpha}} |u|_{1,r}^{\frac{1}{t-(t-1)\alpha}}.$$
Since one verifies easily that
$$\frac{1}{t-(t-1)\alpha} = \frac{p-q}{p} = a, \qquad \frac{(t-1)(1-\alpha)}{t-(t-1)\alpha} = \frac{q}{p} = 1-a,$$
this yields (B.3.20), since $t/2N \le p$. For $p < q + N/(N-1)$, we apply Hölder's inequality
$$\|u\|_{L^p} \le \|u\|_{L^{3q}}^{\frac{3(p-q)}{2p}} \|u\|_{L^q}^{\frac{3q-p}{2p}}.$$
(Note that $3q \ge q + 2 \ge p$.) We estimate $\|u\|_{L^{3q}}$ by applying (B.3.20) with $p = 3q$, and the result follows. □

We now study interpolation inequalities for intermediate derivatives.

THEOREM B.3.10. *Given an integer $m \ge 1$, there exists a constant C_m with the following property. If $0 \le j \le m$ and if $1 \le p, q, r \le \infty$ satisfy*

$$\frac{m}{p} = \frac{j}{r} + \frac{m-1}{q}, \tag{B.3.23}$$

then for every $i \in \{1, \dots, N\}$,

$$\|\partial_i^j u\|_{L^p} \le C_m \|u\|_{L^q}^{\frac{m-j}{m}} \|\partial_i^m u\|_{L^r}^{\frac{j}{m}}, \tag{B.3.24}$$

for all $u \in C_c^m(\mathbb{R}^N)$. Moreover,

$$|u|_{j,p} \le C_m \|u\|_{L^q}^{\frac{m-j}{m}} |u|_{m,r}^{\frac{j}{m}}, \tag{B.3.25}$$

for all $u \in C_c^m(\mathbb{R}^N)$.

The proof of Theorem B.3.10 is based on the following lemma.

LEMMA B.3.11. *If $1 \le p, q, r \le \infty$ satisfy*

$$\frac{2}{p} = \frac{1}{q} + \frac{1}{r}, \tag{B.3.26}$$

then

$$\|u'\|_{L^p} \le 8 \|u\|_{L^q}^{\frac{1}{2}} \|u''\|_{L^r}^{\frac{1}{2}}, \tag{B.3.27}$$

for all $u \in C_c^2(\mathbb{R})$.

PROOF. We first observe that we need only prove (B.3.27) for $r > 1$ and $p < \infty$, since the general case can then be obtained by letting $p \uparrow \infty$ or $r \downarrow 1$. Thus we now assume $p < \infty$ and $r > 1$. Let $0 \le \gamma \le 2$ be defined by

$$\gamma = 1 + \frac{1}{p} - \frac{1}{r}. \tag{B.3.28}$$

so that by (B.3.26),

$$-\gamma = -1 - \frac{1}{q} + \frac{1}{p}. \tag{B.3.29}$$

We observe that $p \le 2r$ by (B.3.26), so that

$$\gamma \ge 1/2. \tag{B.3.30}$$

We now fix $u \in C_c^2(\mathbb{R})$ and, given any interval $I \subset \mathbb{R}$, we set

$$f(I) = |I|^{-\gamma p} \|u\|_{L^q(I)}^p, \tag{B.3.31}$$

$$g(I) = |I|^{\gamma p} \|u''\|_{L^r(I)}^p. \tag{B.3.32}$$

We now proceed in five steps.

STEP 1. The estimate

$$\|v'\|_{L^p(0,1)} \le 4\|v\|_{L^q(0,1)} + 2\|v''\|_{L^r(0,1)}, \tag{B.3.33}$$

holds for all $v \in C^2([0,1])$. Let $\xi(x) = 1 - 2x^2$ for $0 \le x \le 1/2$, $\xi(x) = 2(1-x)^2$ for $1/2 \le x \le 1$. It follows that $\xi \in C^1([0,1]) \cap C^2([0,1] \setminus \{1/2\})$ and $0 \le \xi \le 1$, $\xi(0) = 1$, $\xi'(0) = \xi(1) = \xi'(1) = 0$. Moreover, $\xi''(x) = -4$ for $0 \le x \le 1/2$, $\xi''(x) = 4$ for $1/2 \le x \le 1$. An integration by parts yields

$$\int_0^1 \xi v'' = -v'(0) - 4\int_0^{1/2} v + 4\int_{1/2}^1 v;$$

and so,

$$|v'(0)| \le 4\|v\|_{L^1(0,1)} + \|v''\|_{L^1(0,1)}.$$

Given $0 \le x \le 1$, we deduce that

$$|v'(x)| \le |v'(0)| + \int_0^x |v''| \le 4\|v\|_{L^1(0,1)} + 2\|v''\|_{L^1(0,1)}.$$

$x \in [0,1]$ being arbitrary, we conclude that

$$\|v'\|_{L^\infty(0,1)} \le 4\|v\|_{L^1(0,1)} + 2\|v''\|_{L^1(0,1)}.$$

The estimate (B.3.33) follows by Hölder's inequality.

STEP 2. The estimate

$$\|u'\|_{L^p(a,b)} \le 4(b-a)^{-\gamma}\|u\|_{L^q(a,b)} + 2(b-a)^{\gamma}\|u''\|_{L^r(a,b)}, \tag{B.3.34}$$

holds for all $-\infty < a < b < \infty$. Set $v(x) = u(a + (b-a)x)$, so that $v \in C^2([0,1])$. The estimate (B.3.34) follows by applying estimate (B.3.33) to v, then by using (B.3.28)-(B.3.29).

STEP 3. If f and g are defined by (B.3.31)-(B.3.32), then the estimate

$$\int_I |u'|^p \le 2^{2p-1}(2^p f(I) + g(I)), \tag{B.3.35}$$

holds for all finite interval $I \subset \mathbb{R}$. This follows from (B.3.34) and the elementary inequality $(x+y)^p \le 2^{p-1}(x^p + y^p)$.

STEP 4. Given any $\delta > 0$, there exist a positive integer ℓ and disjoints intervals $I_1, \dots, I_\ell$ such that $\cup_{1\le j\le \ell} \overline{I_j} \supset \operatorname{supp} u$ and with the following properties.

$$\ell \le 1 + |\operatorname{supp} u|/\delta, \tag{B.3.36}$$

$$\begin{cases} \text{either } |I_j| = \delta \text{ and } f(I_j) \le g(I_j) \\ \text{or else } |I_j| > \delta \text{ and } f(I_j) = g(I_j), \end{cases} \tag{B.3.37}$$

for all $1 \le j \le \ell$. Indeed, set $x_0 = \inf \operatorname{supp} u$ and let $I = (x_0, x_0 + \delta)$. If $f(I) \le g(I)$, we let $I_1 = I$. If $f(I) > g(I)$, we observe that the functions $\varphi(t) = f(x_0, x_0 + \delta + t)$, $\psi(t) = g(x_0, x_0 + \delta + t)$ satisfy $\varphi(0) > \psi(0)$ and $\varphi(t) \to 0$, $\psi(t) \to \infty$ as $t \to \infty$ (we use (B.3.30)). Thus there exists $t > 0$ such that $\varphi(t) = \psi(t)$ and we let $I_1 = (x_0, x_0 + \delta + t)$. We then see that I_1 satisfies (B.3.37). If $\operatorname{supp} u \not\subset I_1$, we can repeat this construction. Since $\operatorname{supp} u$ is compact and $|I_j| \ge \delta$, we obtain in a finite number of steps, say ℓ, a collection of disjoint open intervals I_j that all satisfy (B.3.37) and such that

$$\underset{1\le j\le \ell-1}{\cup} I_j \subset \operatorname{supp} u \subset \underset{1\le j\le \ell}{\cup} \overline{I_j},$$

which clearly imply (B.3.36).

STEP 5. Conclusion. Fix $\delta > 0$. It follows from Step 4 and (B.3.35) that

$$\int_{\mathbb{R}} |u'|^p \le 2^{2p-1} \sum_{j=1}^{\ell} [2^p f(I_j) + g(I_j)], \tag{B.3.38}$$

We let

$$A_1 = \{j \in \{1, \cdots, \ell\};\ |I_j| = \delta\},$$
$$A_2 = \{j \in \{1, \cdots, \ell\};\ |I_j| > \delta\},$$

so that by (B.3.37)

$$\{1, \cdots, \ell\} = A_1 \cup A_2. \tag{B.3.39}$$

If $j \in A_1$, then $f(I_j) \le g(I_j)$ by (B.3.37), so that

$$\begin{aligned} 2^p f(I_j) + g(I_j) &\le (2^p + 1) g(I_j) \le (2^p+1)|I_j|^{\gamma p} \|u''\|^p_{L^r(I)} \\ &\le (2^p+1)\delta^{\gamma p}\|u''\|^p_{L^r(\mathbb{R})}, \end{aligned} \tag{B.3.40}$$

where we applied (B.3.32). We deduce from (B.3.40) and (B.3.36) that

$$\sum_{j\in A_1}[2^p f(I_j)+g(I_j)]\le (2^p+1)(1+|\operatorname{supp}u|/\delta)\delta^{\gamma p}\|u''\|^p_{L^r(\mathbb{R})}. \tag{B.3.41}$$

If $j\in A_2$, then $f(I_j)=g(I_j)$ by (B.3.37). Since

$$f(I_j)g(I_j)=\|u\|^p_{L^q(I_j)}\|u''\|^p_{L^r(I_j)},$$

we see that

$$f(I_j)=g(I_j)=\|u\|^{\frac{p}{2}}_{L^q(I_j)}\|u''\|^{\frac{p}{2}}_{L^r(I_j)}, \tag{B.3.42}$$

for all $j\in A_2$. It follows from (B.3.42) that

$$\sum_{j\in A_2}[2^p f(I_j)+g(I_j)]\le (2^p+1)\sum_{j\in A_2}\|u\|^{\frac{p}{2}}_{L^q(I_j)}\|u''\|^{\frac{p}{2}}_{L^r(I_j)}. \tag{B.3.43}$$

Using (B.3.26) and applying Hölder's inequality for the sum in the right-hand side of (B.3.43), we deduce that

$$\sum_{j\in A_2}[2^p f(I_j)+g(I_j)]\le (2^p+1)\Big(\sum_{j\in A_2}\|u\|^q_{L^q(I_j)}\Big)^{\frac{p}{2q}}\Big(\sum_{j\in A_2}\|u''\|^r_{L^r(I_j)}\Big)^{\frac{p}{2r}},$$

which implies

$$\sum_{j\in A_2}[2^p f(I_j)+g(I_j)]\le (2^p+1)\|u\|^{\frac{p}{2}}_{L^q(\mathbb{R})}\|u''\|^{\frac{p}{2}}_{L^r(\mathbb{R})}. \tag{B.3.44}$$

We now deduce from (B.3.38), (B.3.39), (B.3.41) and (B.3.44) that

$$\int_{\mathbb{R}}|u'|^p\le 2^{2p-1}(2^p+1)\times \\ [\|u\|^{\frac{p}{2}}_{L^q(\mathbb{R})}\|u''\|^{\frac{p}{2}}_{L^r(\mathbb{R})}+(1+|\operatorname{supp}u|/\delta)\delta^{\gamma p}\|u''\|^p_{L^r(\mathbb{R})}]. \tag{B.3.45}$$

Note that by (B.3.28)

$$\gamma p=1+p-\frac{p}{r}>1,$$

since $r>1$. Letting $\delta\downarrow 0$ in (B.3.45) we obtain

$$\int_{\mathbb{R}}|u'|^p\le 2^{2p-1}(2^p+1)\|u\|^{\frac{p}{2}}_{L^q(\mathbb{R})}\|u''\|^{\frac{p}{2}}_{L^r(\mathbb{R})}.$$

Since $2^p+1\le 2^{p+1}$, we see that $2^{2p-1}(2^p+1)\le 2^{3p}$ and the estimate (B.3.27) follows by taking the p^{th} root of the above inequality. □

Remark B.3.12. The proof of Lemma B.3.11 is fairly technical. Note, however, that some special cases of the inequality (B.3.27) can be established very easily. For example, if $p=q=r$, then setting $f=-u''+u$, we see that $u=(1/2)e^{-|\cdot|}\star f$, so that $u'=\phi\star f$, with $\phi(x)=(x/2|x|)e^{-|x|}$. By Young's inequality, $\|u'\|_{L^p}\le\|\phi\|_{L^1}\|f\|_{L^p}=\|f\|_{L^p}$. Since $\|f\|_{L^p}\le\|u''\|_{L^p}+\|u\|_{L^p}$, (B.3.27) follows. Another easy case is $p=2$ (so that $r=q'$). Indeed, $u'^2=(uu')'-uu''$, so that

$$\int u'^2=-\int uu''\le\|u''\|_{L^r}\|u\|_{L^q},$$

by Hölder's inequality, which shows (B.3.27). Note that in both these simple cases, one obtains (B.3.27) with the (better) constant 1.

Proof of Theorem B.3.10. The cases $j=0$ and $j=m$ being trivial, we assume $1\le j\le m-1$ and we proceed in four steps.

Step 1. If $1\le p,q,r\le\infty$ satisfy (B.3.26) and $i\in\{1,\dots,N\}$, then

$$\|\partial_i u\|_{L^p(\mathbb{R}^N)}\le 8\|u\|^{\frac12}_{L^q(\mathbb{R}^N)}\|\partial_i^2 u\|^{\frac12}_{L^r(\mathbb{R}^N)}, \tag{B.3.46}$$

for all $u \in C_c^2(\mathbb{R}^N)$. Indeed, assume first $p < \infty$ and let $x = (x_1, \dots, x_N) \in \mathbb{R}^N$. We apply (B.3.27) to the function

$$v(t) = u(x_1, \dots, x_{i-1}, t, x_{i+1}, \dots, x_N),$$

and we deduce that

$$\int_{\mathbb{R}} |v'(t)|^p \le 8^p \Big(\int_{\mathbb{R}} |v(t)|^q \Big)^{\frac{p}{2q}} \Big(\int_{\mathbb{R}} |v''(t)|^r \Big)^{\frac{p}{2r}}.$$

Integrating on $\mathbb{R}^{N-1}$ in the variables $(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_N)$ and applying Hölder's inequality to the right-hand side (note that $2q/p + 2r/p = 1$), we deduce (B.3.46). The case $p = \infty$ follows by letting $p \uparrow \infty$ in (B.3.46).

STEP 2. If $m \ge 2$ and if $1 \le p, q, r \le \infty$ satisfy

$$\frac{m}{p} = \frac{1}{r} + \frac{m-1}{q}, \tag{B.3.47}$$

and $i \in \{1, \dots, N\}$, then

$$\|\partial_i u\|_{L^p} \le 8^{2m-3} \|u\|_{L^q}^{\frac{m-1}{m}} \|\partial_i^m u\|_{L^r}^{\frac{1}{m}}, \tag{B.3.48}$$

for all $u \in C_c^m(\mathbb{R}^N)$. We argue by induction on m. By Step 1, (B.3.48) holds for $m = 2$. Suppose it holds up to some $m \ge 2$. Assume

$$\frac{m+1}{p} = \frac{1}{r} + \frac{m}{q}, \tag{B.3.49}$$

and let t be defined by

$$\frac{m}{t} = \frac{1}{r} + \frac{m-1}{p}. \tag{B.3.50}$$

In particular, $\min\{p, r\} \le t \le \max\{p, r\}$, so that $1 \le t \le \infty$. Applying (B.3.48) to $\partial_i u$, we obtain

$$\|\partial_i^2 u\|_{L^t} \le 8^{2m-3} \|\partial_i^{m+1} u\|_{L^r}^{\frac{1}{m}} \|\partial_i u\|_{L^p}^{\frac{m-1}{m}}. \tag{B.3.51}$$

Now, we observe that by (B.3.49) and (B.3.50), $2/p = 1/q + 1/t$, so it follows from (B.3.46) that

$$\|\partial_i u\|_{L^p} \le 8 \|\partial_i^2 u\|_{L^t}^{\frac{1}{2}} \|u\|_{L^q}^{\frac{1}{2}}. \tag{B.3.52}$$

(B.3.51) and (B.3.52) now yield (B.3.48) at the level $m + 1$.

STEP 3. Proof of (B.3.24). We argue by induction on $m \ge 2$. For $m = 2$, the result follows from Step 1. Suppose now that up to some $m \ge 2$, (B.3.24) holds for all $1 \le j \le m - 1$. Assume $1 \le j \le m$,

$$\frac{m+1}{p} = \frac{j}{r} + \frac{m+1-j}{q}, \tag{B.3.53}$$

and let t be defined by

$$\frac{m}{p} = \frac{j-1}{r} + \frac{m+1-j}{t}. \tag{B.3.54}$$

We first note that by (B.3.54) and (B.3.53),

$$\begin{aligned} \frac{m+1-j}{t} &= \frac{m}{m+1} \frac{m+1}{p} - \frac{j-1}{r} \\ &= \frac{m}{m+1} \frac{m+1-j}{q} + \frac{m+1-j}{(m+1)r} \ge 0, \end{aligned}$$

so that $0 \le t \le \infty$. Also, by the above identity, and since $q, r \ge 1$,

$$\begin{aligned} \frac{m+1-j}{t} &= \frac{m}{m+1} \frac{m+1-j}{q} + \frac{m+1-j}{(m+1)r} \\ &\le \frac{m(m+1-j)}{m+1} + \frac{m+1-j}{m+1} = m+1-j, \end{aligned}$$

so that $t \ge 1$. Applying (B.3.24) (with j replaced by $j-1$) to $\partial_i u$, we obtain

$$\|\partial_i^j u\|_{L^p} \le C_m \|\partial_i^{m+1} u\|_{L^r}^{\frac{j-1}{m}} \|\partial_i u\|_{L^t}^{\frac{m-j+1}{m}}. \tag{B.3.55}$$

Now, we observe that by (B.3.53) and (B.3.54), $(m+1)/t = 1/r + m/q$, so it follows from (B.3.48) (applied with m replaced by $m+1$) that

$$\|\partial_i u\|_{L^t} \le 8^{2m-1} \|\partial_i^{m+1} u\|_{L^r}^{\frac{1}{m+1}} \|u\|_{L^q}^{\frac{m}{m+1}}. \tag{B.3.56}$$

(B.3.55) and (B.3.56) now yield (B.3.24) at the level $m+1$.

STEP 4. Proof of (B.3.25). We note that by (B.3.48),

$$|u|_{1,p} \le 8^{2m-3} \|u\|_{L^q}^{\frac{m-1}{m}} |u|_{m,r}^{\frac{1}{m}},$$

whenever (B.3.47) holds. The proof is now parallel to the proof of estimate (B.3.24) in Step 3 above. □

PROOF OF THEOREM B.3.2. We proceed in three steps.

STEP 1. The case $(m-j)r < N$. Let t be defined by

$$\frac{1}{t} = \frac{1}{r} - \frac{m-j}{N}, \tag{B.3.57}$$

so that $r < t < \infty$. It follows from Sobolev's inequality (B.3.11) applied to j^{th} derivatives of u that

$$|u|_{j,t} \le C|u|_{m,r}. \tag{B.3.58}$$

Next, let s be defined by

$$\frac{m}{s} = \frac{j}{r} + \frac{m-1}{q}, \tag{B.3.59}$$

so that $\min\{q,r\} \le s \le \max\{q,r\}$. It follows from the interpolation inequality (B.3.25) that

$$|u|_{j,s} \le C\|u\|_{L^q}^{\frac{m-j}{m}} |u|_{m,r}^{\frac{j}{m}}. \tag{B.3.60}$$

It follows from (B.3.1), (B.3.57) and (B.3.59) that

$$\frac{1}{p} = \frac{\theta}{t} + \frac{1-\theta}{s},$$

with

$$\theta = \frac{ma-j}{m-j}.$$

Since $j/m \le a \le 1$, we see that $0 \le \theta \le 1$, and we deduce from Hölder's inequality that

$$|u|_{j,p} \le |u|_{j,t}^{\theta} |u|_{j,s}^{1-\theta} \le C|u|_{m,r}^{\theta} (\|u\|_{L^q}^{\frac{m-j}{m}} |u|_{m,r}^{\frac{j}{m}})^{1-\theta},$$

where we used (B.3.58) and (B.3.60). The estimate (B.3.2) follows.

STEP 2. The case $(m-j)r \ge N$ and $a = 1$. Note that if $a = 1$, then by (B.3.1),

$$\frac{1}{p} = \frac{1}{r} - \frac{m-j}{N} \le 0.$$

The only possibility is $(m-j)r = N$ and $p = \infty$. This is allowed only if $r = 1$, and the result is then a consequence of Theorem B.3.7.

STEP 3. The case $(m-j)r \ge N$ and $a < 1$. Let t be defined by

$$\frac{m}{t} = \frac{j}{r} + \frac{m-1}{q}, \tag{B.3.61}$$

so that $\min\{q,r\} \le t \le \max\{q,r\}$. It follows from the interpolation inequality (B.3.25) that

$$|u|_{j,t} \le C\|u\|_{L^q}^{\frac{m-j}{m}} |u|_{m,r}^{\frac{j}{m}}. \tag{B.3.62}$$

Next, let s be defined by

$$\frac{m-j}{s} = \frac{1}{r} + \frac{m-j-1}{p}, \tag{B.3.63}$$

so that $\min\{p,r\} \le s \le \max\{p,r\}$. It follows from the interpolation inequality (B.3.25) applied to j^{th} order derivatives of u that

$$|u|_{j+1,s} \le C|u|_{m,r}^{\frac{1}{m-j}} |u|_{j,p}^{\frac{m-j-1}{m-j}}. \tag{B.3.64}$$

Next, let $\alpha \in [0,1)$ be defined by

$$\alpha = \frac{(m-j)(a-j/m)}{1-a+(m-j)(a-j/m)}, \tag{B.3.65}$$

so that by (B.3.61), (B.3.63) and (B.3.1)

$$\frac{1}{p} = \alpha\Big(\frac{1}{s} - \frac{1}{N}\Big) + \frac{1-\alpha}{t}.$$

It follows from Theorem B.3.9 applied to j^{th} order derivatives of u that

$$|u|_{j,p} \le C|u|_{j+1,s}^{\alpha} |u|_{j,t}^{1-\alpha}. \tag{B.3.66}$$

We deduce from (B.3.66), (B.3.64) and (B.3.62) that

$$|u|_{j,p} \le C|u|_{m,r}^{\frac{\alpha+(j/m)(1-\alpha)(m-j)}{\alpha+(1-\alpha)(m-j)}} \|u\|_{L^q}^{\frac{(1-j/m)(1-\alpha)(m-j)}{\alpha+(1-\alpha)(m-j)}}.$$

Since by (B.3.65),

$$a = \frac{\alpha+(j/m)(1-\alpha)(m-j)}{\alpha+(1-\alpha)(m-j)},$$

this yields (B.3.2). □

Corollary B.3.13 (Gagliardo-Nirenberg's inequality). *Let $\Omega \subset \mathbb{R}^N$ be an open subset. Let $1 \le p,q,r \le \infty$ and let j,m be two integers, $0 \le j < m$. Assume that (B.3.1) holds for some $a \in [j/m, 1]$ ($a < 1$ if $r = N/(m-j) > 1$), and suppose further that $r < \infty$. It follows that $D^\alpha u \in L^p(\Omega)$ for all $u \in W_0^{m,r}(\Omega) \cap L^q(\Omega)$ if $|\alpha| = j$. Moreover, the inequality (B.3.2) holds for all $u \in W_0^{m,r}(\Omega) \cap L^q(\Omega)$.*

Proof. We first consider the case $\Omega = \mathbb{R}^N$. Let $u \in W^{m,r}(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$ and let $(u_n)_{n\ge 0} \subset C_c^\infty(\mathbb{R}^N)$ be the sequence constructed by regularization and truncation in the proof of Theorem B.1.8, so that

$$u_n \underset{n\to\infty}{\longrightarrow} u \text{ in } W^{m,r}(\mathbb{R}^N) \quad \text{and} \quad \|u_n\|_{L^q} \le \|u\|_{L^q}. \tag{B.3.67}$$

Applying (B.3.2) to $u_n - u_\ell$, we obtain

$$|u_n - u_\ell|_{j,p} \le C|u_n - u_\ell|_{m,r}^{a} \|u_n - u_\ell\|_{L^q}^{1-a}. \tag{B.3.68}$$

Let α be a multi-index with $|\alpha| = j$. It follows from (B.3.67)-(B.3.68) that $D^\alpha u_n$ is a Cauchy sequence in $L^p(\mathbb{R}^N)$. Thus $D^\alpha u_n$ has a limit v in $L^p(\mathbb{R}^N)$. In particular,

$$\int_{\mathbb{R}^N} D^\alpha u_n \varphi \underset{n\to\infty}{\longrightarrow} \int_{\mathbb{R}^N} v\varphi,$$

for all $\varphi \in C_c^j(\mathbb{R}^N)$. Since

$$\int_{\mathbb{R}^N} D^\alpha u_n \varphi = (-1)^j \int_{\mathbb{R}^N} u_n D^\alpha \varphi \underset{n\to\infty}{\longrightarrow} (-1)^j \int_{\mathbb{R}^N} u D^\alpha \varphi,$$

by (B.3.67), we see that $D^\alpha u = v \in L^p(\mathbb{R}^N)$ and

$$|u_n - u|_{j,p} \underset{n\to\infty}{\longrightarrow} 0, \tag{B.3.69}$$

which proves the first part of the result. Finally, we apply the inequality (B.3.2) to u_n. Letting $n \to \infty$ and using (B.3.67) and (B.3.69), we deduce that (B.3.2) holds for u.

In the general case $\Omega \subset \mathbb{R}^N$, the result follows by extending $u \in W_0^{m,p}(\Omega)$ by 0 outside Ω (see Remark B.1.10 (v)) and applying the result in $\mathbb{R}^N$. □

We are now in a position to state and prove the Sobolev embedding theorems. We restrict ourselves to functions of $W_0^{m,p}(\Omega)$. Similar statements hold for functions of $W^{m,p}(\Omega)$, but they are obtained by using extension operators, so they require a certain amount of regularity of the domain. For functions of $W_0^{m,p}(\Omega)$, instead, no regularity assumption on Ω is necessary. Furthermore, these results are sufficient for our purpose. Our first result in this direction is the following.

THEOREM B.3.14. *Let $\Omega \subset \mathbb{R}^N$ be an open subset, let $1 \le r < \infty$ and let $m \in \mathbb{N}$, $m \ge 1$.*

(i) *If $mr < N$, then $W_0^{m,r}(\Omega) \hookrightarrow L^p(\Omega)$ for all p such that $r \le p \le Nr/(N - mr)$.*
(ii) *If $m = N$ and $r = 1$, then $W_0^{m,r}(\Omega) \hookrightarrow L^p(\Omega)$ for all p such that $r \le p \le \infty$. Moreover, $W_0^{m,r}(\Omega) \hookrightarrow C_0(\Omega)$.*
(iii) *If $mr = N$ and $r > 1$, then $W_0^{m,r}(\Omega) \hookrightarrow L^p(\Omega)$ for all p such that $r \le p < \infty$.*
(iv) *If $mr > N$, then $W_0^{m,r}(\Omega) \hookrightarrow L^p(\Omega)$ for all p such that $r \le p \le \infty$. Moreover, $W_0^{m,r}(\Omega) \hookrightarrow C_0(\Omega)$.*

PROOF. The result follows from Corollary B.3.13 by taking $j = 0$, $q = r$ and $a = N(p - r)/mpr$, except for the embeddings $W_0^{m,r}(\Omega) \hookrightarrow C_0(\Omega)$ in (ii) and (iv). These last embeddings follow from the density of $C_c^\infty(\Omega)$ in $W_0^{m,r}(\Omega)$ and the embedding $W_0^{m,r}(\Omega) \hookrightarrow L^\infty(\Omega)$. □

The next result is the general case of Sobolev's embedding for functions of $W_0^{m,p}(\Omega)$.

THEOREM B.3.15. *Let $\Omega \subset \mathbb{R}^N$ be an open subset, let $1 \le r < \infty$ and let $m, j \in \mathbb{N}$, $m \ge 1$.*

(i) *If $mr < N$, then $W_0^{m+j,r}(\Omega) \hookrightarrow W_0^{j,p}(\Omega)$ for all p such that $r \le p \le Nr/(N - mr)$.*
(ii) *If $m = N$ and $r = 1$, then $W_0^{m+j,r}(\Omega) \hookrightarrow W_0^{j,p}(\Omega) \cap W^{j,\infty}(\Omega)$ for all p such that $r \le p < \infty$. Moreover, $W_0^{m+j,r}(\Omega) \hookrightarrow C_0^j(\Omega)$.*
(iii) *If $mr = N$ and $r > 1$, then $W_0^{m+j,r}(\Omega) \hookrightarrow W_0^{j,p}(\Omega)$ for all p such that $r \le p < \infty$.*
(iv) *If $mr > N$, then $W_0^{m+j,r}(\Omega) \hookrightarrow W_0^{j,p}(\Omega) \cap W^{j,\infty}(\Omega)$ for all p such that $r \le p < \infty$. Moreover, $W_0^{m+j,r}(\Omega) \hookrightarrow C_0^j(\Omega)$.*

PROOF. We first prove (iv). Applying Theorem B.3.14 (iv) to $D^\alpha u$ with $|\alpha| \le j$, we deduce that $W_0^{m+j,r}(\Omega) \hookrightarrow W^{j,p}(\Omega)$ for all $r \le p \le \infty$. The embedding $W_0^{m+j,r}(\Omega) \hookrightarrow W_0^{j,p}(\Omega)$ if $r \le p < \infty$ follows from the density of $C_c^\infty(\Omega)$ in $W_0^{m+j,r}(\Omega)$ and the embedding $W_0^{m+j,r}(\Omega) \hookrightarrow W^{j,p}(\Omega)$. Next, the embedding $W_0^{m+j,r}(\Omega) \hookrightarrow C_0^j(\Omega)$ follows from the density of $C_c^\infty(\Omega)$ in $W_0^{m+j,r}(\Omega)$ and the embedding $W_0^{m+j,r}(\Omega) \hookrightarrow W^{j,\infty}(\Omega)$. The proofs of (i), (ii) and (iii) are similar, using properties (i), (ii) and (iii) of Theorem B.3.14, respectively. □

We now apply Morrey's inequality to obtain embedings in spaces of the type $C^{j,\alpha}(\overline{\Omega})$.

THEOREM B.3.16. *Let $\Omega \subset \mathbb{R}^N$ be an open subset, let $1 \le r < \infty$. Let $m \ge 1$ be the smallest integer such that $mr > N$. It follows that for all integers $j \ge 0$, $W_0^{m+j,r}(\Omega) \hookrightarrow C_0^j(\Omega) \cap C^{j,\alpha}(\overline{\Omega})$ with $\alpha = m - (N/r)$ if $(m-1)r < N$, α any number in $(0, 1)$ if $(m - 1)r = N$.*

Proof. Let $u \in C_c^\infty(\Omega)$. It follows from Theorem B.3.15 (iv) that

$$\|u\|_{W^{j,\infty}} \le C\|u\|_{W^{m+j,r}}. \tag{B.3.70}$$

Let α be a multi-index with $|\alpha| = j$. Setting $v = D^\alpha u$, we see that

$$\|v\|_{W^{m,r}} \le \|u\|_{W^{m+j,r}}. \tag{B.3.71}$$

Let

$$\begin{cases} p = \frac{Nr}{N-(m-1)r} & \text{if } (m-1)r < N, \\ \max\{r, N\} < p < \infty & \text{if } (m-1)r = N, \end{cases}$$

so that $p > N$. It follows from Theorem B.3.15 (i) and (B.3.71) that

$$\|v\|_{W^{1,p}} \le C\|v\|_{W^{m,r}} \le C\|u\|_{W^{m+j,r}}. \tag{B.3.72}$$

Finally, we deduce from (B.3.72) and Morrey's inequality (B.3.16) that

$$|v(x) - v(y)| \le C|x-y|^{1-\frac{N}{p}}\|u\|_{W^{m+j,r}},$$

for all $x, y \in \mathbb{R}^N$. Applying (B.3.70), we conclude that $\|u\|_{C^{j,\alpha}} \le C\|u\|_{W^{m+j,r}}$, with $\alpha = 1 - (N/p)$, which is the desired estimate. □

The following two results are applications of Sobolev's embedding theorems.

Corollary B.3.17. *Given any* $1 \le r \le \infty$, $\bigcap_{m\ge 0} W^{m,r}_{\mathrm{loc}}(\Omega) = C^\infty(\Omega)$.

Proof. It is clear that $C^\infty(\Omega) \subset W^{m,r}_{\mathrm{loc}}(\Omega)$ for all $m \ge 0$. Conversely, suppose $u \in W^{m,r}_{\mathrm{loc}}(\Omega)$ for all $m \ge 0$. Let $\omega \subset\subset \Omega$ and let $\varphi \in C_c^\infty(\Omega)$ satisfy $\varphi(x) = 1$ for $x \in \omega$. It follows from Proposition B.1.14 that $v = \varphi u \in W_0^{m,1}(\Omega)$ for all $m \ge 0$, so that $v \in C^\infty(\Omega)$ by Theorem B.3.15. Thus $u \in C^\infty(\omega)$ and the result follows, since ω is arbitrary. □

Proposition B.3.18. *Let* $1 \le p \le \infty$, $m \in \mathbb{N}$, $m \ge 1$ *and* $u \in W^{m,p}_{\mathrm{loc}}(\Omega)$. *If* $D^\alpha u \in C(\Omega)$ *for all multi-index* α *with* $|\alpha| = m$, *then* $u \in C^m(\Omega)$.

Proof. We proceed in three steps.

Step 1. $u \in C(\Omega)$. Suppose $u \in L^{q_0}_{\mathrm{loc}}(\Omega)$ for some $q_0 \le N$ and let $q_0 \le q_1 < \infty$ satisfy

$$\frac{1}{q_1} \ge \frac{1}{q_0} - \frac{1}{N}.$$

Let $\varphi \in C_c^\infty(\Omega)$ and set $v = \varphi u$, so that $v \in L^{q_0}(\Omega)$. Since $\nabla v = \varphi\nabla u + u\nabla\varphi$, we deduce that $\nabla v \in L^{q_0}(\Omega)$. v being compactly supported in Ω, it follows that $v \in W_0^{1,q_0}(\Omega)$ (see Remark B.1.10 (i)). Applying Theorem B.3.14, we see that $v \in L^{q_1}(\Omega)$ and, since φ is arbitrary, we deduce that $u \in L^{q_1}_{\mathrm{loc}}(\Omega)$. We now iterate the above argument and, starting from $q_0 = 1$, we construct $q_0 < \cdots < q_{k-1} \le N < q_k$ such that $u \in L^{q_j}_{\mathrm{loc}}(\Omega)$ for $0 \le j \le k$. Finally, let $\varphi \in C_c^\infty(\Omega)$ and set $v = \varphi u$. We see as above that $v \in W^{1,q_k}(\Omega)$, and it follows from Theorem B.3.14 that $v \in C(\Omega)$. Since φ is arbitrary, we conclude that $u \in C(\Omega)$.

Step 2. The case $m = 1$. It follows from Step 1 that $u \in C(\Omega)$. Since $\nabla u \in C(\Omega)$ by assumption, we have in particular $u \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$. Let $\omega \subset\subset \Omega$ and let $\varphi \in C_c^\infty(\Omega)$ satisfy $\varphi(x) = 1$ for $x \in \omega$. Set $v = \varphi u$, so that $v \in W^{1,\infty}(\Omega)$ by Proposition B.1.14. Since v is supported in a compact subset of Ω, it follows that if

$$w(x) = \begin{cases} v(x) & \text{if } x \in \Omega, \\ & \text{if } x \in \mathbb{R}^N \setminus \Omega, \end{cases}$$

then $w \in W^{1,\infty}(\mathbb{R}^N) \cap C(\mathbb{R}^N)$. Moreover, one verifies easily that

$$\nabla w = \begin{cases} u\nabla\varphi + \varphi\nabla u & \text{in } \Omega, \\ 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

so that $\nabla w \in C(\mathbb{R}^N)$. Since w and ∇w have compact support, we see that $w, \nabla w \in C_{b,u}(\mathbb{R}^N)$. Applying Proposition B.1.12, we deduce that $w \in C^1(\mathbb{R}^N)$, and since $w = u$ in ω, it follows that $u \in C^1(\omega)$. Hence the result, since ω is arbitrary.

STEP 3. The case $m \geq 2$. We proceed by induction on m. By Step 2, the result holds for $m = 1$. Suppose it holds up to some $m \geq 1$. Let $u \in W^{m+1,p}_{\text{loc}}(\Omega)$ satisfy $D^\alpha u \in C(\Omega)$ for all multi-index α with $|\alpha| = m + 1$. Consider $1 \leq j \leq N$ and set $v = \partial_j u$. It follows that $v \in W^{m,p}_{\text{loc}}(\Omega)$ and $D^\alpha u \in C(\Omega)$ for all multi-index α with $|\alpha| = m$. Applying the result at the level m, we deduce that $v \in C^m(\Omega)$. Thus $\nabla u \in C^m(\Omega)$. In particular, $\nabla u \in C(\Omega)$ and we deduce from Step 2 that $u \in C^1(\Omega)$. Since $\nabla u \in C^m(\Omega)$, we conclude that $u \in C^{m+1}(\Omega)$. □

If $|\Omega| < \infty$, then $\|u\|_{L^r}$ is dominated only in terms of $\|\nabla u\|_{L^r}$ for functions of $W^{1,r}_0(\Omega)$. This is the object of the following result.

THEOREM B.3.19 (Poincaré's inequality). *If $|\Omega| < \infty$ and $1 \leq r < \infty$, then there exists a constant $C = C(N, r)$ (independent of u and Ω) such that*

$$\|u\|_{L^r} \leq C|\Omega|^{\frac{1}{N}} \|\nabla u\|_{L^r}, \tag{B.3.73}$$

for every $u \in W^{1,r}_0(\Omega)$.

PROOF. Let $p = r(N+r)/N$, so that by (B.3.20), $\|u\|_{L^p} \leq C\|\nabla u\|_{L^r}^{\frac{N}{N+r}} \|u\|_{L^r}^{\frac{r}{N+r}}$. Since $\|u\|_{L^r} \leq |\Omega|^{\frac{1}{N+r}} \|u\|_{L^p}$ by Hölder's inequality, the result follows. □

COROLLARY B.3.20. *Let $1 \leq r < \infty$, and suppose $|\Omega| < \infty$. Then $\|u\| = \|\nabla u\|_{L^r}$ defines an equivalent norm on $W^{1,r}_0(\Omega)$.*

REMARK B.3.21. Note that inequality (B.3.73) means that

$$\inf\{\|\nabla u\|_{L^p};\, u \in W^{1,p}_0(\Omega), \|u\|_{L^p} = 1\} > 0.$$

Note that inequality (B.3.73) may fail if $|\Omega| = \infty$. For instance, if $\Omega = \mathbb{R}^N$, then the above infimum is 0, so that (B.3.73) fails. To see that, let $u \in W^{1,p}(\mathbb{R}^N)$, $\|u\|_{L^p} = 1$. Given $\lambda > 0$, let $u_\lambda \in W^{1,p}(\mathbb{R}^N)$ be defined by $u_\lambda(x) = \lambda^{\frac{N}{p}} u(\lambda x)$. Elementary calculations show that $\|u_\lambda\|_{L^p} = 1$ and $\|\nabla u_\lambda\|_{L^p} = \lambda\|\nabla u\|_{L^p}$. Letting $\lambda \downarrow 0$, we see that the above infimum is 0.

We end this section with a result concerning the embedding of L^p spaces in negative order Sobolev spaces.

COROLLARY B.3.22. *Suppose $\Omega \subset \mathbb{R}^N$ is an open set. Let $1 < r < \infty$ and let $m \geq 1$ be an integer. If*

$$\overline{p} = \begin{cases} \infty & \text{if} \quad mr \geq N, \\ \frac{Nr}{N-mr} & \text{if} \quad mr < N, \end{cases}$$

then $L^{p'}(\Omega) \hookrightarrow W^{-m,r'}(\Omega)$ with dense embedding for all $r \leq p \leq \overline{p}$ (and $p < \infty$ if $mr = N$). If, in addition, $|\Omega| < \infty$, then the same property also holds for $1 \leq p < r$.

PROOF. The last part of the result (the case $|\Omega| < \infty$) follows from the dense embedding $L^p(\Omega) \hookrightarrow L^q(\Omega)$ if $1 \leq p \leq q < \infty$. The first part of the result follows from Theorem B.3.14 and Proposition A.1.5, except for the case $p = \infty$ (thus

$mr > N$), since $L^1(\Omega)$ is not the dual of $L^\infty(\Omega)$. In this case, we argue directly as follows. It follows from Theorem B.3.14 that $W_0^{m,r}(\Omega) \hookrightarrow L^\infty(\Omega)$. Define

$$eu(\varphi) = \int_\Omega u\varphi,$$

for all $u \in L^1(\Omega)$ and $\varphi \in W_0^{m,r}(\Omega)$. We have

$$|eu(\varphi)| \le \|u\|_{L^1} \|\varphi\|_{L^\infty} \le \|u\|_{L^1} \|\varphi\|_{W_0^{m,r}},$$

so that e defines a mapping $L^1(\Omega) \to W^{-m,r'}(\Omega)$. This mapping is injective, because if $(eu, \varphi)_{W^{-m,r'}, W_0^{m,r}} = 0$ for all $\varphi \in W_0^{m,r}(\Omega)$, then in particular $\int_\Omega u\varphi = 0$ for all $\varphi \in C_c^\infty(\Omega)$, which implies $u = 0$. It remains to show that the embedding $e : L^1(\Omega) \to W^{-m,r'}(\Omega)$ is dense. To prove this, we observe that by the density of $C_c^\infty(\Omega)$ in $L^{r'}(\Omega)$ and of $L^{r'}(\Omega)$ in $W^{-m,r'}(\Omega)$ (see just above), it follows that $C_c^\infty(\Omega)$ is dense in in $W^{-m,r'}(\Omega)$. The result follows, since $C_c^\infty(\Omega) \subset L^1(\Omega)$. □

B.4. Compactness properties

We now study the compact embeddings of $W_0^{1,r}(\Omega)$. We begin with a local compactness result in $\mathbb{R}^N$.

PROPOSITION B.4.1. *Let $1 \le r < \infty$, and suppose K be a bounded subset of $W^{1,r}(\mathbb{R}^N)$. It follows that for every $R < \infty$, the set $K_R := \{u_{|B_R}; u \in K\}$ is relatively compact in $L^r(B_R)$, where $B_R = B(0,R)$.*

PROOF. We proceed in three steps.

STEP 1. If $(\rho_n)_{n\ge1}$ is a smoothing sequence, then

$$\|u - \rho_n \star u\|_{L^r} \le \frac{C}{n} \|\nabla u\|_{L^r}, \tag{B.4.1}$$

for all $u \in W^{1,r}(\mathbb{R}^N)$, where $C = \big(\int_{\mathbb{R}^N} |y|^r \rho(y)\,dy\big)^{\frac{1}{r}}$. By density, we need only show (B.4.1) for $u \in C_c^\infty(\mathbb{R}^N)$. We claim that

$$\int_{\mathbb{R}^N} |u(x-y) - u(x)|^r \,dx \le |y|^r \|\nabla u\|_{L^r}^r. \tag{B.4.2}$$

Indeed,

$$u(x-y) - u(x) = \int_0^1 \frac{d}{dt} u(x-ty)\,dt = \int_0^1 y \cdot \nabla u(x-ty)\,dt;$$

and so,

$$|u(x-y) - u(x)| \le |y| \int_0^1 |\nabla u(x-ty)|\,dt \le |y| \Big(\int_0^1 |\nabla u(x-ty)|^r \,dt\Big)^{\frac{1}{r}}.$$

(B.4.2) follows after integration in x. Next, since $\|\rho_n\|_{L^1} = 1$,

$$\begin{aligned}\rho_n \star u(x) - u(x) &= \int_{\mathbb{R}^N} \rho_n(y)(u(x-y) - u(x))\,dy \\ &= \int_{\mathbb{R}^N} \rho_n(y)^{\frac{r-1}{r}} [\rho_n(y)^{\frac{1}{r}} (u(x-y) - u(x))]\,dy.\end{aligned}$$

By Hölder's inequality, we deduce

$$|\rho_n \star u(x) - u(x)|^r \le \int_{\mathbb{R}^N} \rho_n(y) |u(x-y) - u(x)|^r \,dy.$$

Integrating the above inequality on $\mathbb{R}^N$ and applying (B.4.2), we find

$$\|\rho_n \star u - u\|_{L^r}^r \le \|\nabla u\|_{L^r}^r \int_{\mathbb{R}^N} |y|^r \rho_n(y)\,dy.$$

Hence (B.4.1).

STEP 2. If $(\rho_n)_{n\ge1}$ is as in Step 1, then

$$\|\rho_n \star u\|_{W^{1,\infty}} \le n^{\frac{N}{r}} \|\rho\|_{L^{r'}} \|u\|_{W^{1,r}}, \tag{B.4.3}$$

for all $u \in W^{1,r}(\mathbb{R}^N)$. Since $\nabla(\rho_n \star u) = \rho_n \star \nabla u$ by Lemma B.1.9, it follows from Young's inequality that

$$\|\rho_n \star u\|_{W^{1,\infty}} \le \|\rho_n\|_{L^{r'}} \|u\|_{W^{1,r}},$$

and the result follows.

STEP 3. Conclusion. Let $R > 0$, let K_R be as in the statement of the proposition, and let $\varepsilon > 0$. Given $n \ge 1$, set $K^n = \{\rho_n \star u;\, u \in K\}$ and $K^n_R = \{u_{|B_R};\, u \in K^n\}$. Fix n large enough so that

$$\sup_{u\in K} \|u - \rho_n \star u\|_{L^r} \le \frac{\varepsilon}{2}. \tag{B.4.4}$$

Such a n exists by (B.4.1). It follows from (B.4.3) that K^n is a set of uniformly Lipschitz continuous functions on $\mathbb{R}^N$. By Ascoli's theorem, K^n_R is relatively compact in $L^\infty(B_R)$, thus in $L^r(B_R)$. Therefore, K^n_R can be covered by a finite number of balls of radius $\varepsilon/2$ in $L^r(B_R)$. By (B.4.4), we see that K_R can be covered by a finite number of balls of radius ε. Since $\varepsilon > 0$ is arbitrary, this shows compactness. □

COROLLARY B.4.2. *Let $\Omega \subset \mathbb{R}^N$ be an open subset, let $1 \le r < \infty$ and let $(u_n)_{n\ge0}$ be a bounded sequence of $W^{1,r}_0(\Omega)$. There exist $u \in L^r(\Omega)$ and a subsequence $(u_{n_k})_{k\ge0}$ such that $u_{n_k} \to u$ a.e. in Ω and in $L^r(\Omega \cap B_R)$, for any $R < \infty$, as $k \to \infty$.*

PROOF. We first consider the case $\Omega = \mathbb{R}^N$. It follows from Proposition B.4.1 applied with $R = 1$ that there exist $n(1,k) \to \infty$ as $k \to \infty$ and $w_1 \in L^r(B_1)$ such that $u_{n(1,k)} \to w_1$ in $L^r(B_1)$ and a.e. in B_1. We now apply Proposition B.4.1 with $R = 2$ to the sequence $(u_{n(1,k)})_{k\ge0}$. It follows that there exist a subsequence $n(2,k) \to \infty$ as $k \to \infty$ and $w_2 \in L^r(B_2)$ such that $u_{n(2,k)} \to w_2$ in $L^r(B_2)$ and a.e. in B_2. By recurrence, we construct $n(\ell,k) \to \infty$ as $k \to \infty$ and $(w_\ell)_{\ell\ge1}$ with $w_\ell \in L^r(B_\ell)$ such that $u_{n(\ell,k)} \to w_\ell$ in $L^r(B_\ell)$ and a.e. in B_ℓ. Moreover, $(n(\ell,k))_{k\ge0}$ is a subsequence of $(n(m,k))_{k\ge0}$ for $\ell > m$, i.e. for every $k \ge 0$, there exists $k' \ge k$ such that $n(\ell,k) = n(m,k')$. We set $n_k = n(k,k)$. Since $n(k,k)$ is a subsequence of $n(\ell,k)$ for any $\ell \ge 1$, we see that $u_{n_k} \to w_\ell$ in $L^r(B_\ell)$ and a.e. in B_ℓ. In particular, $w_\ell \equiv w_m$ on B_m if $\ell \ge m$. We now set $u \equiv w_\ell$ on B_m, for $\ell \ge m$. We have $u_{n_k} \to u$ in $L^r(B_R)$ and a.e. in B_R, for any $R < \infty$. In particular,

$$\|u\|_{L^r(B_R)} = \lim_{k\to\infty} \|u_{n_k}\|_{L^r(B_R)} \le \limsup_{n\to\infty} \|u_n\|_{L^r(\mathbb{R}^N)}.$$

We deduce that $u \in L^r(\mathbb{R}^N)$. Since $u_{n_k} \to u$ a.e. in B_R for any $R < \infty$, we conclude that $u_{n_k} \to u$ a.e. in $\mathbb{R}^N$.

We now consider the case of an arbitrary domain $\Omega \subset \mathbb{R}^N$. Let $(u_n)_{n\ge0}$ be as above and set

$$\widetilde{u}_n(x) = \begin{cases} u_n(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \in \mathbb{R}^N \setminus \Omega, \end{cases}$$

so that the sequence $(\widetilde{u}_n)_{n\ge0}$ is bounded in $W^{1,r}(\mathbb{R}^N)$. (See Remark B.1.10 (v).) It follows from what precedes that there exist $\widetilde{u} \in L^r(\mathbb{R}^N)$, supported in Ω, and a subsequence $(\widetilde{u}_{n_k})_{k\ge0}$ such that $\widetilde{u}_{n_k} \to \widetilde{u}$ as $k \to \infty$ in $L^r(B_R)$ for any $R < \infty$ and a.e. in $\mathbb{R}^N$. The result now follows by setting $u = \widetilde{u}_{|\Omega}$. This completes the proof. □

LEMMA B.4.3. *Let Ω be an open subset of $\mathbb{R}^N$. Let $1 \le r \le \infty$, let $(u_n)_{n\ge 0}$ be a bounded sequence of $L^r(\Omega)$ and let $u \in L^1_{\rm loc}(\Omega)$. Suppose that*

$$\int_\Omega u_n\varphi \underset{n\to\infty}{\longrightarrow} \int_\Omega u\varphi, \tag{B.4.5}$$

for all $\varphi \in C^\infty_{\rm c}(\Omega)$ (which is satisfied in particular if $u_n \to u$ in $L^1(\omega)$ for every $\omega \subset\subset \Omega$). Then $u \in L^r(\Omega)$ and

$$\|u\|_{L^r} \le \liminf_{n\to\infty} \|u_n\|_{L^r}. \tag{B.4.6}$$

Moreover, if $r > 1$ then (B.4.5) holds for all $\varphi \in L^{r'}(\Omega)$. In addition, if $1 < r < \infty$ and if $\|u_n\|_{L^r} \to \|u\|_{L^r}$ as $n \to \infty$, then $u_n \to u$ in $L^r(\Omega)$.

Suppose further that $1 < r < \infty$ and that $(u_n)_{n\ge0}$ is a bounded sequence of $W^{1,r}_0(\Omega)$. It follows that $u \in W^{1,r}_0(\Omega)$,

$$\int_\Omega \nabla u_n\varphi \underset{n\to\infty}{\longrightarrow} \int_\Omega \nabla u\varphi, \tag{B.4.7}$$

for all $\varphi \in L^{r'}(\Omega)$ and

$$\|\nabla u\|_{L^r} \le \liminf_{n\to\infty} \|\nabla u_n\|_{L^r}. \tag{B.4.8}$$

If in addition $\|\nabla u_n\|_{L^r} \to \|\nabla u\|_{L^r}$ as $n \to \infty$, then $\nabla u_n \to \nabla u$ in $L^r(\Omega)$.

PROOF. We claim that for all $u \in L^1_{\rm loc}(\Omega)$,

$$\|u\|_{L^r} = \sup\Big\{\Big|\int_\Omega u\varphi\Big|;\ \varphi \in C^\infty_{\rm c}(\Omega), \|\varphi\|_{L^{r'}} = 1\Big\}. \tag{B.4.9}$$

If $r > 1$, this is immediate because $L^{r'}(\Omega)^\star = L^r(\Omega)$ and $C^\infty_{\rm c}(\Omega)$ is dense in $L^{r'}(\Omega)$. Suppose now $r = 1$ and suppose $u \ne 0$ (the case $u = 0$ is immediate). Fix $0 < M < \|u\|_{L^1} \le \infty$. There exists a compact set $K \subset \Omega$ such that

$$\int_K |u| > M.$$

Let

$$h(x) = \begin{cases} \frac{u(x)}{|u(x)|} & \text{if } u(x) \ne 0 \text{ and } x \in K,\\ 0 & \text{if } u(x) = 0 \text{ or } x \notin K.\end{cases}$$

We have $h \in L^\infty(\Omega)$, $\|h\|_{L^\infty} = 1$. Moreover, h has compact support in Ω and

$$\int_\Omega uh = \int_K |u| > M.$$

Let $(\rho_n)_{n\ge0}$ be a smoothing sequence and set $h_n = (\rho_n \star h)_{|\Omega}$. For n large enough, we have $h_n \in C^\infty_{\rm c}(\Omega)$. Moreover, up to a subsequence, $h_n \to h$ a.e. In addition, $\|h_n\|_{L^\infty} \le \|h\|_{L^\infty} = 1$. By dominated convergence, we deduce

$$\int_\Omega uh_n \underset{n\to\infty}{\longrightarrow} \int_\Omega uh > M;$$

and so,

$$\sup\Big\{\Big|\int_\Omega u\varphi\Big|;\ \varphi \in C^\infty_{\rm c}(\Omega), \|\varphi\|_{L^{p'}} = 1\Big\} \ge M.$$

Since $M < \|u\|_{L^1}$ is arbitrary, we deduce

$$\sup\Big\{\Big|\int_\Omega u\varphi\Big|;\ \varphi \in C^\infty_{\rm c}(\Omega), \|\varphi\|_{L^{p'}} = 1\Big\} \ge \|u\|_{L^1}.$$

The converse inequality being immediate, (B.4.9) follows. Now, since

$$\Big|\int_\Omega u_n\varphi\Big| \le \|u_n\|_{L^r}\|\varphi\|_{L^{r'}},$$

(B.4.6) follows from (B.4.5) and (B.4.9). The fact that if $r > 1$, then (B.4.5) holds for all $\varphi \in L^{r'}(\Omega)$ follows by density of $C_c^\infty(\Omega)$ in $L^{r'}(\Omega)$.

Suppose now that $1 < r < \infty$, that $\|u_n\|_{L^r} \to \|u\|_{L^r}$ as $n \to \infty$ and let us show that $u_n \to u$ in $L^r(\Omega)$. If $\|u\|_{L^r} = 0$, then the result is immediate. Therefore, we may assume that $\|u\|_{L^r} \neq 0$, so that also $\|u_n\|_{L^r} \neq 0$ for n large. Let then $\widetilde{u} = \|u\|_{L^r}^{-1} u$ and $\widetilde{u}_n = \|u_n\|_{L^r}^{-1} u_n$. It follows that

$$\|\widetilde{u}_n\|_{L^r} = \|\widetilde{u}\|_{L^r} = 1.$$

Furthermore, (B.4.5) is satisfied with u and u_n replaced by $\widetilde{u}$ and $\widetilde{u}_n$. Setting $w = 2\widetilde{u}$ and $w_n = \widetilde{u} + \widetilde{u}_n$, we deduce that (B.4.5) is satisfied with u and u_n replaced by w and w_n. In particular, it follows from what precedes that $\|w\|_{L^r} \leq \liminf \|w_n\|_{L^r}$. Since $\|w\|_{L^r} = 2$ and $\|w_n\|_{L^r} \leq \|\widetilde{u}\|_{L^r} + \|\widetilde{u}_n\|_{L^r} = 2$, it follows that

$$\|w_n\|_{L^r} \underset{n\to\infty}{\longrightarrow} 2.$$

If $r \geq 2$, we have Clarkson's inequality (see e.g. [17])

$$\|\widetilde{u}_n - \widetilde{u}\|_{L^r}^r \leq 2^{r-1}(\|\widetilde{u}\|_{L^r}^r + \|\widetilde{u}_n\|_{L^r}^r) - \|\widetilde{u} + \widetilde{u}_n\|_{L^r}^r.$$

Therefore, $\|\widetilde{u}_n - \widetilde{u}\|_{L^r} \to 0$, from which it follows that $u_n \to u$ in $L^r(\Omega)$. In the case $r \leq 2$, the conclusion is the same by using Clarkson's inequality (see e.g. [17])

$$\|\widetilde{u}_n - \widetilde{u}\|_{L^r}^{\frac{r}{r-1}} \leq 2(\|\widetilde{u}\|_{L^r}^r + \|\widetilde{u}_n\|_{L^r}^r)^{\frac{1}{r-1}} - \|\widetilde{u} + \widetilde{u}_n\|_{L^r}^{\frac{r}{r-1}}.$$

Suppose finally that $1 < r < \infty$ and that $(u_n)_{n\geq 0}$ is a bounded sequence of $W_0^{1,r}(\Omega)$. If $\varphi \in C_c^\infty(\Omega)$, then for all $j \in \{1, \dots, N\}$

$$-\int_\Omega \frac{\partial u_n}{\partial x_j}\varphi = \int_\Omega u_n \frac{\partial \varphi}{\partial x_j} \underset{n\to\infty}{\longrightarrow} \int_\Omega u \frac{\partial \varphi}{\partial x_j}, \tag{B.4.10}$$

by (B.4.5). Set now

$$f_j(\varphi) = -\int_\Omega u \frac{\partial \varphi}{\partial x_j},$$

for $\varphi \in C_c^\infty(\Omega)$. It follows from (B.4.10) and the boundedness of the sequence $(u_n)_{n\geq 0}$ in $W^{1,r}(\Omega)$ that

$$|f_j(\varphi)| \leq C\|\varphi\|_{L^{r'}}.$$

Therefore, f can be extended by continuity and density to a linear, continuous functional on $L^{r'}(\Omega)$. Since $L^{r'}(\Omega)^\star = L^r(\Omega)$, there exists $g_j \in L^r(\Omega)$ such that

$$f_j(\varphi) = \int_\Omega g_j \varphi,$$

for all $\varphi \in C_c^\infty(\Omega)$ and by density, for all $\varphi \in L^{r'}(\Omega)$. This implies that

$$\int_\Omega u \frac{\partial \varphi}{\partial x_j} = -\int_\Omega g_j \varphi,$$

for all $\varphi \in C_c^\infty(\Omega)$. Thus $u \in W^{1,r}(\Omega)$. (B.4.7) follows from (B.4.10) and the above identity. The last properties follow by using (B.4.7) and applying the first part of the result to ∇u_n instead of u_n. □

We can now establish the compact sobolev embeddings.

THEOREM B.4.4 (Rellich-Kondrachov). *Let $\Omega \subset \mathbb{R}^N$ be a bounded, open set, and let $1 \leq r < \infty$. It follows that the embedding $W_0^{1,r}(\Omega) \hookrightarrow L^r(\Omega)$ is compact.*

PROOF. Let $(u_n)_{n\geq 0}$ be a bounded sequence of $W_0^{1,r}(\Omega)$. It follows from Corollary B.4.2 that there exist $u \in L^r(\Omega)$ and a subsequence $(u_{n_k})_{k\geq 0}$ such that $u_{n_k} \to u$ in $L^r(\Omega)$ as $k \to \infty$. This completes the proof. □

In fact, we have the following stronger result.

THEOREM B.4.5. *Let $\Omega \subset \mathbb{R}^N$ be a bounded, open set, let $1 \le r < \infty$ and set*

$$\overline{r} = \begin{cases} \infty & \text{if } r \ge N, \\ \frac{Nr}{N-r} & \text{if } r < N. \end{cases}$$

If $(u_n)_{n\ge 0}$ is a bounded sequence of $W_0^{1,r}(\Omega)$, then there is a subsequence $(u_{n_k})_{k\ge 0}$ and $u \in L^r(\Omega)$ such that $u_{n_k} \to u$ in $L^r(\Omega)$ as $k \to \infty$. Moreover, the following properties hold.

(i) *$u \in L^{\overline{r}}(\Omega)$ ($u \in L^\rho(\Omega)$ for all $1 \le \rho < \overline{r}$ if $r = N \ge 2$) and $u_n \to u$ in $L^p(\Omega)$ for all $1 \le p < \overline{r}$.*

(ii) *If $r > 1$, then $u \in W_0^{1,r}(\Omega)$ and*

$$\int_\Omega \nabla u_{n_k} \varphi \underset{k\to\infty}{\longrightarrow} \int_\Omega \nabla u \varphi,$$

for all $\varphi \in L^{r'}(\Omega)$. In particular,

$$\|\nabla u\|_{L^r} \le \liminf_{k\to\infty} \|\nabla u_{n_k}\|_{L^r}.$$

If, in addition, $\|\nabla u\|_{L^r} = \lim \|\nabla u_{n_k}\|_{L^r}$ as $k \to \infty$, then $u_{n_k} \to u$ in $W_0^{1,r}(\Omega)$.

PROOF. The first part of the result follows from Theorem B.4.4. Next, except in the case $r = N \ge 2$, it follows from Theorem B.3.14 that $(u_n)_{n\ge 0}$ is bounded in $L^{\overline{r}}(\Omega)$, from which we deduce $u \in L^{\overline{r}}(\Omega)$. (See Lemma B.4.3.) In the case $r = N \ge 2$, it follows from Theorem B.3.14 that $(u_n)_{n\ge 0}$ is bounded in $L^p(\Omega)$ for any $p < \infty$, from which we deduce $u \in L^p(\Omega)$ for all $p < \infty$. Property (i) now follows from the $L^{\overline{r}}$ bound (or L^p bound for all $p < \infty$ if $r = N \ge 2$) and the L^r convergence by applying Hölder's inequality to $u_{n_k} - u$. Finally, property (ii) follows from Lemma B.4.3. □

REMARK B.4.6. If Ω is not bounded, we still have a local compactness result. Given $R > 0$, set $\Omega_R = \{x \in \Omega;\ |x < R|\}$. Given any bounded sequence $(u_n)_{n\ge 0}$ of $W_0^{1,r}(\Omega)$, there exist a subsequence $(u_{n_k})_{k\ge 0}$ and $u \in L^r(\Omega)$ such that $u_{n_k} \to u$ as $k \to \infty$, a.e. in Ω and in $L^r(\Omega_R)$ for every $R < \infty$. Moreover, the following properties hold.

(i) If $r = N = 1$, then $u \in L^\infty(\Omega)$ and $u_{n_k} \to u$ in $L^p(\Omega_R)$ for every $p < \infty$ and every $R < \infty$. In addition, $\|u\|_{L^p} \le \liminf \|u_n\|_{L^p}$ as $n \to \infty$ for every $1 \le p \le \infty$.

(ii) If $N \ge 2$ and $1 \le r < N$, then $u \in L^{\frac{Nr}{N-r}}(\Omega)$ and $u_{n_k} \to u$ in $L^p(\Omega_R)$ for every $p < Nr/(N-r)$ and every $R < \infty$. In addition, $\|u\|_{L^p} \le \liminf \|u_n\|_{L^p}$ as $n \to \infty$.

(iii) If $N \ge 2$ and $r = N$, then $u \in L^p(\Omega)$ and $u_{n_k} \to u$ in $L^p(\Omega_R)$ for every $p < \infty$ and every $R < \infty$. In addition, $\|u\|_{L^p} \le \liminf \|u_n\|_{L^p}$ as $n \to \infty$.

(iv) If $r > N$, then $u \in L^\infty(\Omega)$ and $u_{n_k} \to u$ in $L^\infty(\Omega_R)$ for every $R < \infty$. In addition, $\|u\|_{L^p} \le \liminf \|u_n\|_{L^p}$ as $n \to \infty$ for every $r \le p \le \infty$.

(v) If $r > 1$, then $u \in W_0^{1,r}\Omega)$ and

$$\int_\Omega \nabla u_{n_k} \varphi \underset{k\to\infty}{\longrightarrow} \int_\Omega \nabla u \varphi,$$

for all $\varphi \in L^{r'}(\Omega)$. In particular,

$$\|\nabla u\|_{L^r} \le \liminf_{k\to\infty} \|\nabla u_{n_k}\|_{L^r}.$$

If moreover $\|\nabla u\|_{L^r} = \lim \|\nabla u_{n_k}\|_{L^r}$ and $\|u\|_{L^r} = \lim \|u_{n_k}\|_{L^r}$ as $k \to \infty$, then $u_{n_k} \to u$ in $W_0^{1,r}(\Omega)$.

Those properties are proved like Theorem B.4.5, except for the local convergence in (i)–(iv). This follows by applying Theorem B.4.5 to the sequence $(\xi u_n)_{n\geq 0}$, where $\xi \in C_c^\infty(\mathbb{R}^N)$ is such that $\xi(x) = 1$ for $|x| \leq R$.

COROLLARY B.4.7. *Suppose $\Omega \subset \mathbb{R}^N$ is a bounded open set. Let $1 < r < \infty$ and let $m \geq 1$ be an integer. If*

$$\overline{p} = \begin{cases} \infty & \text{if } mr \geq N, \\ \frac{Nr}{N-mr} & \text{if } mr < N, \end{cases}$$

then the embeddings $W_0^{m,r}(\Omega) \hookrightarrow L^p(\Omega)$ and $L^{p'}(\Omega) \hookrightarrow W^{-m,r'}(\Omega)$ are compact for all $1 \leq p < \overline{p}$.

PROOF. We first observe that by Theorem B.4.4, the embedding $W_0^{1,r}(\Omega) \hookrightarrow L^r(\Omega)$ is compact, hence the embedding $W_0^{m,r}(\Omega) \hookrightarrow L^1(\Omega)$ is also compact. Applying Theorem B.3.14 and Hölder's inequality, we deduce that if $1 \leq p < \overline{p}$, then the embedding $W_0^{m,r}(\Omega) \hookrightarrow L^p(\Omega)$ is compact. This proves the first part of the result, and the second part follows from the abstract duality property of Proposition A.1.5 (iii). □

B.5. The case of complex-valued functions

So far in this section, we considered real-valued functions but the same theory can be developped for complex-valued functions, with obvious modifications which we describe below.

One has to consider the spaces $\mathcal{D}(\Omega, \mathbb{C})$ and $L^p(\Omega, \mathbb{C})$ instead of the spaces $\mathcal{D}(\Omega, \mathbb{R})$ and $L^p(\Omega, \mathbb{R})$. In particular, a function $f \in L^1_{\text{loc}}(\Omega, \mathbb{C})$ defines a distribution $T_f \in \mathcal{D}'(\Omega, \mathbb{C})$ by the formula

$$\langle T_f, \varphi \rangle = \int_\Omega \operatorname{Re}\left(f(x)\overline{\varphi(x)}\right) dx, \text{ for all } \varphi \in \mathcal{D}(\Omega, \mathbb{C}).$$

In particular, $W^{m,p}(\Omega, \mathbb{C}) \approx W^{m,p}(\Omega, \mathbb{R}) \times W^{m,p}(\Omega, \mathbb{R})$. In other words, a complex-valued function u belongs to $W^{m,p}(\Omega, \mathbb{C})$ if, and only if $\operatorname{Re}(u) \in W^{m,p}(\Omega, \mathbb{R})$ and $\operatorname{Im(u)} \in W^{m,p}(\Omega, \mathbb{R})$. As well, $W_0^{m,p}(\Omega, \mathbb{C}) \approx W_0^{m,p}(\Omega, \mathbb{R}) \times W_0^{m,p}(\Omega, \mathbb{R})$, and it follows in particular that $W^{-m,p'}(\Omega, \mathbb{C}) \approx W^{-m,p'}(\Omega, \mathbb{R}) \times W^{-m,p'}(\Omega, \mathbb{R})$. The scalar product on $H^m(\Omega, \mathbb{C})$ is defined by

$$(u, v)_{H^m} = \sum_{0\leq|\alpha|\leq m} \int_\Omega \operatorname{Re}\left(D^\alpha u(x)\overline{D^\alpha v(x)}\right) dx. \tag{B.5.1}$$

Formula (B.1.6) becomes

$$\langle u, v\rangle_{W^{-m,p}, W_0^{m,p'}} = \operatorname{Re} \int_\Omega u(x)\overline{v(x)}\, dx, \tag{B.5.2}$$

and formula (B.1.8) becomes

$$\langle \Delta u, v\rangle_{W^{-1,p}, W_0^{1,p'}} = -\operatorname{Re} \int_\Omega \nabla u(x) \cdot \overline{\nabla v(x)}\, dx, \text{ for } v \in W_0^{1,p'}(\Omega, \mathbb{C}). \tag{B.5.3}$$

Therefore, most of the results that we established for real-valued functions still hold for complex-valued functions, and are proved by considering separately the real and imaginary parts. Notable exceptions are Proposition B.2.3, Remark B.2.4, and Corollaries B.2.6 and B.2.8 that do not make sense anymore. (See, however, Remark B.5.3 below.) Here is a chain rule that applies to complex-valued functions.

THEOREM B.5.1. *If $F : \mathbb{C} \to \mathbb{C}$ is a Lipschitz continuous function such that $F(0) = 0$ and if $1 \leq p \leq \infty$, then the following properties hold.*

(i) *$F(u) \in W^{1,p}(\Omega, \mathbb{C})$, for every $u \in W^{1,p}(\Omega, \mathbb{C})$.*

(ii) *If* $|F(z_1) - F(z_2)| \le L(z_1, z_2)|z_1 - z_2|$ *for all* $z_1, z_2 \in \mathbb{C}$, *where* $L : \mathbb{C} \times \mathbb{C} \to [0, \infty)$ *is some continous function, then* $|\nabla F(u)| \le L(u, u)|\nabla u|$ *a.e. for every* $u \in W^{1,p}(\Omega, \mathbb{C})$.

(iii) *If* F *is* C^1 *(considered as a function* $\mathbb{R}^2 \to \mathbb{R}^2$*) except at a finite number of points, then* $\nabla F(u) = DF(u)\nabla u$ *a.e. for every* $u \in W^{1,p}(\Omega, \mathbb{C})$. *If moreover* $p < \infty$, *then the mapping* $u \mapsto F(u)$ *is continous* $W^{1,p}(\Omega, \mathbb{C}) \to W^{1,p}(\Omega, \mathbb{C})$.

(iv) *If* $p < \infty$, *then in properties* (i) *and* (iii) *above, one may replace* $W^{1,p}(\Omega, \mathbb{C})$ *by* $W_0^{1,p}(\Omega, \mathbb{C})$.

Proof. We proceed in five steps.

Step 1. Suppose F is C^1 (considered as a function $\mathbb{R}^2 \to \mathbb{R}^2$), then $F(u) \in W^{1,p}(\Omega, \mathbb{C})$ and $\nabla F(u) = DF(u)\nabla u$ a.e. for every $u \in W^{1,p}(\Omega, \mathbb{C})$. This is established as in [5, Proposition 9.5, p. 270]. The idea of the proof is to approximate u by a sequence $(u_n)_{n\in\mathbb{N}} \subset W^{1,p}(\Omega, \mathbb{C}) \cap C^\infty(\Omega, \mathbb{C})$.

Step 2. Proof of Property (i) . Consider a sequence of mollifiers $(\rho_j)_{j\in\mathbb{N}} \subset \mathcal{D}(\mathbb{R}^2)$ and set $F_j = \rho_j \star F$. It follows that $F_j \to F$ uniformly on $\mathbb{C}$ as $j \to \infty$. Moreover, $|F_j(z_1) - F_j(z_2)| \le L|z_1 - z_2|$ where L is the Lipschitz constant of F. Given $u \in W^{1,p}(\Omega, \mathbb{C})$, it follows from Step 1 that $F_j(u) \in W^{1,p}(\Omega, \mathbb{C})$ and that

$$\nabla F_j(u) = DF_j(u)\nabla u.$$

In particular, $|\nabla F_j(u)| \le L|\nabla u|$. This implies that (up to a subsequence) $\nabla F_j(u)$ converges in L^p weak (weak* if $p = \infty$) to some function ψ (apply Dunford-Pettis' theorem if $p = 1$). Since $F_j(u) \to F(u)$ in $L^p(\Omega, \mathbb{C})$, it follows that $\psi = \nabla F(u)$; and so, $F(u) \in W^{1,p}(\Omega)$.

Step 3. Proof of Property (ii). Let F_j be as in Step 2. We have $|DF_j(z)| \le L(z, z)$; and so, $|\nabla F_j(u)| \le L(u, u)|\nabla u|$. Since $\nabla F_j(u)$ converges in L^p weak (weak* if $p = \infty$) to $\nabla F(u)$ (see Step 2), we deduce that $|\nabla F(u)| \le L(u, u)|\nabla u|$. To see this, we need only show that if a sequence $(f_n)_{n\ge0} \subset L^p(\Omega)$ satisfies $f_n \to f$ as $n \to \infty$ and $|f_n| \le g$ a.e., then $|f| \le g$ a.e. Let $\varphi \in L^{p'}(\Omega)$, $\varphi \ge 0$. We have

$$\int_\Omega g\varphi \ge \int_\Omega f_n\varphi \underset{n\to\infty}{\longrightarrow} \int_\Omega f\varphi;$$

and so,

$$\int_\Omega (g - f)\varphi \ge 0,$$

which implies that $g \ge f$ a.e.

Step 4. Proof of Property (iii). Let $E = (x_i)_{1\le i\le k}$ be such that $F \in C^1(\mathbb{C} \setminus E, \mathbb{C})$, and let again F_j be as in Step 2. Note that $DF \in L^\infty(\mathbb{C}, \mathbb{C}^2)$, so that $F_j' = \rho_j \star F'$ (see [5, Lemme 9.1, p. 266]). It follows that $F_j' \to F'$ on $\mathbb{C} \setminus E$. Since $\nabla u = 0$ a.e. on $\omega = \{x \in \Omega;\, u(x) \in E\}$, we see that $DF_j(u)\nabla u$ converges to $DF(u)\nabla u$ a.e. It follows that $\nabla F(u) = DF(u)\nabla u$. Suppose now $u_n \to u$ in $W^{1,p}(\Omega, \mathbb{C})$. We have

$$\nabla F(u_n) - \nabla F(u) = (DF(u_n) - DF(u))\nabla u + DF(u)(\nabla u_n - \nabla u).$$

Since $DF(u_n(x)) \to DF(u(x))$ if $x \notin \omega$ and $\nabla u = 0$ a.e. in ω, we see that $\nabla F(u_n) \to \nabla F(u)$ a.e. If $p < \infty$, then it follows that $\nabla F(u_n) \to \nabla F(u)$ in $L^p(\Omega, \mathbb{C})$ by dominated convergence.

Step 5. Proof of Property (iv). Let $u \in W_0^{1,p}(\Omega, \mathbb{C})$ and let $(u_n)_{n\ge0} \subset \mathcal{D}(\Omega, \mathbb{C})$ be such that $u_n \to u$ in $W^{1,p}(\Omega, \mathbb{C})$ as $n \to \infty$. Up to a subsequence, we may assume that there exists $\psi \in L^p(\Omega)$ such that $|\nabla u_n| \le \psi$ a.e. It follows Property (ii) that $|\nabla F(u_n)| \le L\psi$ a.e., where L is the Lipschitz constant of F. We deduce as in Step 1 that $F(u_n) \to F(u)$ in $W^{1,p}(\Omega, \mathbb{C})$ weak; and so $F(u) \in W_0^{1,p}(\Omega, \mathbb{C})$. □

REMARK B.5.2. When F does not satisfy the assumption of Theorem B.5.1 (iii), we do not know if the mapping $u \mapsto F(u)$ is continuous $W^{1,p}(\Omega, \mathbb{C}) \to W^{1,p}(\Omega, \mathbb{C})$. Note that the formula "$\nabla F(u) = DF(u)\nabla u$" does not hold in general, even for smooth functions u. Indeed, take for example

$$F(u) = \begin{cases} u & \text{if} \quad |u| \le 1, \\ \frac{u}{|u|} & \text{if} \quad |u| \ge 1. \end{cases}$$

F is C^∞, except on the set $\{|u| = 1\}$ where DF is not defined. Taking for example $u(x) = e^{ia\cdot x}$, with $a \in \mathbb{R}^N$, we see that $F(u) = u$, so that $\nabla F(u) = iae^{ia\cdot x}$, but $DF(u)\nabla u$ is not defined a.e. What happens is that (as opposed to the real-valued case) if $E \subset \mathbb{C}$ is a set of measure 0, then ∇u need not vanish a.e. on the set $\{u \in E\}$. Take for example u as above and $E = \{|z| = 1\}$.

REMARK B.5.3. If $u \in W^{1,p}(\Omega, \mathbb{C})$, it follows that $\operatorname{Re} u, \operatorname{Im} u, |u| \in W^{1,p}(\Omega, \mathbb{R})$. In addition, one has a.e. $\nabla \operatorname{Re} = \operatorname{Re} \nabla u$, $\nabla \operatorname{Im} = \operatorname{Im} \nabla u$ and

$$|\nabla |u|\,|^2 = \begin{cases} 0 & \text{if } u = 0, \\ |\nabla u|^2 - \left| \operatorname{Im}\left(\frac{\overline{u}\nabla u}{|u|} \right) \right|^2 & \text{if } u \neq 0. \end{cases}$$

(In particular, one has $|\nabla |u|\,| \le |\nabla u|$, but in general $|\nabla |u|\,| \not\equiv |\nabla u|$. Note that this is in contrast with the real-valued case.) If $p < \infty$, then the mappings $u \mapsto \operatorname{Re} u$, $u \mapsto \operatorname{Im} u$ and $u \mapsto |u|$ are continous $W^{1,p}(\Omega, \mathbb{C}) \to W^{1,p}(\Omega, \mathbb{R})$. Moreover, if $u \in W^{1,p}_0(\Omega, \mathbb{C})$, then $\operatorname{Re} u, \operatorname{Im} u, |u| \in W^{1,p}_0(\Omega, \mathbb{R})$. This follows from properties (iii) and (iv) of Theorem B.5.1.

B.6. The Fourier transform and Sobolev spaces

See for instance [**27**, Chapter 7] and [**2**, Section 1.2] for details. Other useful references on the subject include [**18**, Chapter VII], [**21**, Chapter 5], [**22**, Chapter 1], [**31**, Chapter I], [**33**, Chapter VI]. Throughout this section, we consider complex-valued functions. Unless otherwise specified, all integrals are over $\mathbb{R}^N$. It is convenient to introduce the following notation

$$\widetilde{u}(x) = u(-x) \tag{B.6.1}$$

$$[\tau_y u](x) = u(x - y) \tag{B.6.2}$$

B.6.1. The Fourier transform on $\mathbb{R}^N$. The Fourier transform on $\mathbb{R}^N$ is defined by

$$\mathcal{F}u(\xi) = \widehat{u}(\xi) = \int_{\mathbb{R}^N} e^{-2\pi i x\cdot\xi} u(x)\, dx \tag{B.6.3}$$

for $u \in L^1(\mathbb{R}^N)$.

PROPOSITION B.6.1. *The following properties hold.*

(i) $\mathcal{F}u \in C_0(\mathbb{R}^N)$ *and*
$$\|\mathcal{F}u\|_{L^\infty} \le \|u\|_{L^1} \tag{B.6.4}$$
for all $u \in L^1(\mathbb{R}^N)$.

(ii) *If* $u, v \in L^1(\mathbb{R}^N)$, *then*
$$\int u(x)\widehat{v}(x)\, dx = \int \widehat{u}(x) v(x)\, dx. \tag{B.6.5}$$

(iii) *If* $u, v \in L^1(\mathbb{R}^N)$, *then*
$$\mathcal{F}(u \star v) = \mathcal{F}u\mathcal{F}v. \tag{B.6.6}$$

(iv) *If* $u \in L^1(\mathbb{R}^N)$ *and* $\lambda > 0$, *then* $v(x) = u(x/\lambda)$ *satisfies* $\widehat{v}(\xi) = \lambda^N \widehat{u}(\lambda\xi)$ *for all* $\xi \in \mathbb{R}^N$.

(v) *If* $u \in L^1(\mathbb{R}^N)$ *and* $y \in \mathbb{R}^N$, *then*

$$\mathcal{F}[\tau_y u](\xi) = e^{-2\pi i y\cdot\xi}\widehat{u}(\xi) \tag{B.6.7}$$

with the notation (B.6.2).

PROOF. Property (i) follows easily by dominated convergence; and Properties (ii) and (iii) are immediate applications of Fubini's theorem. Finally, Properties (iv) and (v) follow from elementary calculations. □

REMARK B.6.2. Here are some comments on the Fourier transform of radial functions.

(i) Suppose $u \in L^1(\mathbb{R}^N)$ is radially symmetric, and write $u(x) = u(|x|) = u(r)$ by abuse of notation. It follows that $\widehat{u}$ is also radially symmetric (so we write $\widehat{u}(\xi) = \widehat{u}(|\xi|) = \widehat{u}(\rho)$ by abuse of notation). Indeed, let R is an orthogonal transformation of $\mathbb{R}^N$. Note that $R^\star = R^{-1}$ and that the Jacobian determinant of R is 1. Therefore,

$$\begin{aligned}\mathcal{F}[u(R\cdot)](\xi) &= \int_{\mathbb{R}^N} e^{-2\pi i x\cdot\xi}u(Rx)\,dx = \int_{\mathbb{R}^N} e^{-2\pi i R^{-1}y\cdot\xi}u(y)\,dy\\ &= \int_{\mathbb{R}^N} e^{-2\pi i y\cdot R\xi}u(y)\,dy = \mathcal{F}u(R\xi).\end{aligned}$$

If u is radially symmetric, then $u(R\cdot) = u(\cdot)$ for all orthogonal transformation R, so that $\mathcal{F}u(R\cdot) = \mathcal{F}u(\cdot)$. Thus $\mathcal{F}u$ is radially symmetric.

(ii) Suppose $u \in L^1(\mathbb{R}^N)$ is radially symmetric, so that $\widehat{u}$ is also radially symmetric by (i) above. It follows that

$$\widehat{u}(\rho) = 2\pi r^{-\frac{N-2}{2}}\int_0^\infty u(r)J_{\frac{N-2}{2}}(2\pi r\rho)r^{\frac{N}{2}}\,dr \tag{B.6.8}$$

where the J_μ are the Bessel functions of the first kind. See [31, Chapter IV, Theorem 3.3, p. 155].

(iii) Since the Bessel functions J_μ are real valued, it follows from formula (B.6.8) that if u is radially symmetric and real valued, then $\widehat{u}$ is also radially symmetric and real valued.

REMARK B.6.3. It is useful to calculate the Fourier transform of a Gaussian. It is given by

$$\mathcal{F}[e^{-\pi a|\cdot|^2}](x) = a^{-\frac{N}{2}}e^{-\frac{\pi|x|^2}{a}} \tag{B.6.9}$$

for all $a > 0$ and $x \in \mathbb{R}^N$. Indeed, setting $\rho = \mathcal{F}[e^{-\pi a|\xi|^2}]$, we see that

$$\rho(x) = \int_{\mathbb{R}^N} e^{-2\pi i x\cdot\xi}e^{-\pi a|\xi|^2}\,d\xi$$

and so

$$x\cdot\nabla\rho = \int_{\mathbb{R}^N} x\cdot\nabla_x[e^{-2\pi i x\cdot\xi}e^{-\pi a|\xi|^2}]\,d\xi = -\int_{\mathbb{R}^N} 2\pi i x\cdot\xi e^{-2\pi i x\cdot\xi}e^{-\pi a|\xi|^2}\,d\xi$$

On the other hand

$$\nabla_\xi\cdot(e^{-2\pi i x\cdot\xi}e^{-\pi a|\xi|^2}x) = ia\Big(-2\pi\frac{|x|^2}{a} + 2\pi i x\cdot\xi\Big)e^{-2\pi i x\cdot\xi}e^{-\pi a|\xi|^2}$$

Integrating in ξ, we obtain

$$0 = \int_{\mathbb{R}^N}\Big(-2\pi\frac{|x|^2}{a} + 2\pi i x\cdot\xi\Big)e^{2\pi i x\cdot\xi}e^{-\pi a|\xi|^2}\,d\xi = -2\pi\frac{|x|^2}{a}\rho - x\cdot\nabla\rho$$

so that

$$x\cdot\nabla\rho(x) = -2\pi\frac{|x|^2}{a}\rho(x)$$

Therefore, given any $x \in \mathbb{R}^N$, if we set

$$f(s) = \rho(sx) \quad s \geq 0$$

then

$$f'(s) = x \cdot \nabla\rho(sx) = \frac{1}{s} sx \cdot \nabla\rho(sx) = -\frac{2\pi}{s}\frac{|sx|^2}{a}\rho(sx) = -2\pi s\frac{|x|^2}{a}f(s)$$

which yields

$$f(s) = e^{-\frac{\pi s^2|x|^2}{a}}f(0)$$

Letting $s = 1$, we obtain

$$\rho(x) = e^{-\frac{\pi|x|^2}{a}}\rho(0) = e^{-\frac{\pi|x|^2}{a}}\int_{\mathbb{R}^N} e^{-\pi a|\xi|^2}\,d\xi = e^{-\frac{\pi|x|^2}{a}}a^{-\frac{N}{2}}$$

from which (B.6.9) follows.

B.6.2. The Schwartz space $\mathcal{S}(\mathbb{R}^N)$. The Schwartz space $\mathcal{S}(\mathbb{R}^N)$ is the space of $\varphi \in C^\infty(\mathbb{R}^N, \mathbb{C})$ such that for every nonnegative integer m

$$p_m(u) = \sup_{|\beta|\leq m}\sup_{x\in\mathbb{R}^N}(1+|x|^2)^{\frac{m}{2}}|D^\beta\varphi(x)| < \infty.$$

$\mathcal{S}(\mathbb{R}^N)$ is a Fréchet space when equipped with the seminorms p_m. In particular, if $(\varphi_n)_{n\geq1} \subset \mathcal{S}(\mathbb{R}^N)$, then $\varphi_n \to 0$ in $\mathcal{S}(\mathbb{R}^N)$ iff $p_m(\varphi_n) \to 0$ for all $m \in \mathbb{N}$, i.e.

$$\sup_{|\beta|\leq m}\sup_{x\in\mathbb{R}^N}(1+|x|^2)^{\frac{m}{2}}|D^\beta\varphi_n(x)| \underset{n\to\infty}{\longrightarrow} 0 \tag{B.6.10}$$

for all $m \in \mathbb{N}$.

REMARK B.6.4. Here are some simple consequences of the definition of $\mathcal{S}(\mathbb{R}^N)$.

(i) If $\psi \in C^\infty(\mathbb{R}^N)$ is such that for all integer $m \geq 0$ there exist an integer $k_m \geq 0$ and a constant C_m such that

$$\sup_{|\beta|\leq m}|\partial^\beta\psi(x)| \leq C_m(1+|x|^2)^{\frac{k_m}{2}} \tag{B.6.11}$$

then $\psi\varphi \in \mathcal{S}(\mathbb{R}^N)$ for all $\varphi \in \mathcal{S}(\mathbb{R}^N)$ and the map $\varphi \mapsto \psi\varphi$ is continuous $\mathcal{S}(\mathbb{R}^N) \to \mathcal{S}(\mathbb{R}^N)$. Indeed, $|\partial^\beta(\psi\varphi)|$ is estimated, by Leibniz's formula, by a sum of terms of the form $\partial^{\beta_1}\psi\partial^{\beta_2}\varphi$ where $\beta_1 + \beta_2 = \beta$. Therefore we deduce from (B.6.11) that

$$\begin{aligned} p_m(\psi\varphi) &\leq C\sup_{x\in\mathbb{R}^N}(1+|x|^2)^{\frac{m}{2}}C_m(1+|x|^2)^{\frac{k_m}{2}}(1+|x|^2)^{-\frac{m+k_m}{2}}p_{m+k_m}(\varphi) \\ &\leq CC_m p_{m+k_m}(\varphi) \end{aligned} \tag{B.6.12}$$

and the result follows since m is arbitrary. Note that (B.6.11) is satisfied for instance by $\psi(x) = (1+|x|^2)^{\frac{s}{2}}$ with $s \in \mathbb{R}$.

(ii) It follows from property (i) above that if $\varphi, \psi \in \mathcal{S}(\mathbb{R}^N)$, then $\varphi\psi \in \mathcal{S}(\mathbb{R}^N)$. Moreover, it follows from calculations similar to (B.6.12) that $p_m(\varphi\psi) \leq Cp_m(\varphi)p_m(\psi)$ with a constant C depending on m.

(iii) The space $C_c^\infty(\mathbb{R}^N)$ is dense in $\mathcal{S}(\mathbb{R}^N)$. This follows easily by applying the calculations of property (i) above to appropriate cut-off functions ψ.

(iv) If $\varphi \in \mathcal{S}(\mathbb{R}^N)$, then $x^\alpha D^\beta\varphi \in \mathcal{S}(\mathbb{R}^N)$ for all multi-indices α, β. Moreover, the map $\varphi \mapsto x^\alpha D^\beta\varphi$ is continuous $\mathcal{S}(\mathbb{R}^N) \to \mathcal{S}(\mathbb{R}^N)$.

(v) Let $\varphi, \psi \in \mathcal{S}(\mathbb{R}^N)$. Given $m \in \mathbb{N}$ and a multi-index β with $|\beta| \leq m$,

$$(1+|x|^2)^{\frac{m}{2}}D^\beta\varphi \star \psi(x) = \int_{\mathbb{R}^N}(1+|x|^2)^{\frac{m}{2}}\varphi(y)D^\beta\psi(x-y)\,dy$$

Since $(1+|x|^2)^{\frac{m}{2}} \le C(1+|x-y|^2)^{\frac{m}{2}}(1+|y|^2)^{\frac{m}{2}}$, we deduce that

$$(1+|x|^2)^{\frac{m}{2}}|D^\beta\varphi\star\psi(x)| \le Cp_m(\psi)\int_{\mathbb{R}^N}(1+|y|^2)^{\frac{m}{2}}\varphi(y)\,dy$$
$$\le Cp_m(\psi)p_{m+N+1}(\varphi)$$

Thus we see that the map $(\varphi,\psi)\mapsto\varphi\star\psi$ is continuous $\mathcal{S}(\mathbb{R}^N)\times\mathcal{S}(\mathbb{R}^N)\to\mathcal{S}(\mathbb{R}^N)$.

PROPOSITION B.6.5. *Given $\varphi\in\mathcal{S}(\mathbb{R}^N)$, it follows that $\widehat{\varphi}\in C^\infty(\mathbb{R}^N)$. Moreover*

$$\xi^\gamma D^\beta\widehat{\varphi} = (-1)^{|\beta|}(2\pi i)^{|\beta|-|\gamma|}\mathcal{F}[D^\gamma(x^\beta\varphi(x))] \tag{B.6.13}$$

for all multi-indices β,γ. In addition,

$$D^\beta(\xi^\gamma\widehat{\varphi}) = (-1)^{|\beta|}(2\pi i)^{|\beta|-|\gamma|}\mathcal{F}(x^\beta D^\gamma\varphi(x)) \tag{B.6.14}$$

for all multi-indices β,γ.

PROOF. By dominated convergence, one can take derivatives with respect to ξ in formula (B.6.3). One deduces the property $\widehat{\varphi}\in C^\infty(\mathbb{R}^N)$ and the formula

$$D^\beta\widehat{\varphi}(\xi) = (-1)^{|\beta|}(2\pi i)^{|\beta|}\int_{\mathbb{R}^N}e^{-2\pi i x\cdot\xi}x^\beta\varphi(x)\,dx \tag{B.6.15}$$

for every multi-index β. Let now γ be another multi-index, and note that

$$\xi^\gamma e^{-2\pi i x\cdot\xi} = (-1)^{|\gamma|}(2\pi i)^{-|\gamma|}D^\gamma(e^{-2\pi i x\cdot\xi}) \tag{B.6.16}$$

where the derivative D^γ is with respect to x. Multiplying equation (B.6.15) by ξ^γ and integrating by parts, we obtain (B.6.13). Similarly, applying first (B.6.16) and integrating by parts in (B.6.3) yields

$$\xi^\gamma\widehat{\varphi}(\xi) = (2\pi i)^{-|\gamma|}\int_{\mathbb{R}^N}e^{-2\pi i x\cdot\xi}D^\gamma\varphi(x)\,dx.$$

Taking derivatives with respect to ξ, we obtain (B.6.14). □

COROLLARY B.6.6. *Given $\varphi\in\mathcal{S}(\mathbb{R}^N)$, it follows that $\widehat{\varphi}\in\mathcal{S}(\mathbb{R}^N)$. Moreover, $\mathcal{F}:\mathcal{S}(\mathbb{R}^N)\to\mathcal{S}(\mathbb{R}^N)$ is continuous.*

PROOF. This follows from formula (B.6.13). □

THEOREM B.6.7 (The inversion formula). *Given $\psi\in\mathcal{S}(\mathbb{R}^N)$, let $\overline{\mathcal{F}}\psi\in\mathcal{S}(\mathbb{R}^N)$ be defined by*

$$\overline{\mathcal{F}}\psi(x) = \int_{\mathbb{R}^N}e^{2\pi i\xi\cdot x}\psi(\xi)\,d\xi. \tag{B.6.17}$$

It follows that

$$\varphi = \overline{\mathcal{F}}[\mathcal{F}\varphi] = \mathcal{F}[\overline{\mathcal{F}}\varphi] \tag{B.6.18}$$

for all $\varphi\in\mathcal{S}(\mathbb{R}^N)$.

PROOF. Let $\phi,\theta\in\mathcal{S}(\mathbb{R}^N)$, $\lambda>0$, and let $f\in\mathcal{S}(\mathbb{R}^N)$ be defined by

$$f(x) = \phi\Big(\frac{x}{\lambda}\Big). \tag{B.6.19}$$

It follows from Proposition B.6.1 (iv) that

$$\widehat{f}(\xi) = \lambda^N\widehat{\phi}(\lambda\xi). \tag{B.6.20}$$

Applying (B.6.5) with $u=\theta$ and $v=f$, and using (B.6.20), we obtain

$$\int\theta(x)\lambda^N\widehat{\phi}(\lambda x)\,dx = \int\widehat{\theta}(x)\phi\Big(\frac{x}{\lambda}\Big)\,dx,$$

and, after a change of variables in the integral on the left

$$\int \theta\Big(\frac{x}{\lambda}\Big)\widehat{\phi}(x)\,dx = \int \widehat{\theta}(x)\phi\Big(\frac{x}{\lambda}\Big)\,dx.$$

Letting $\lambda \to \infty$, we deduce that

$$\theta(0)\int \widehat{\phi}(x)\,dx = \phi(0)\int \widehat{\theta}(x)\,dx.$$

We now let $\phi(x) = e^{-\pi|x|^2}$, so that $\widehat{\phi}(x) = e^{-\pi|x|^2}$ by (B.6.9), hence $\int \widehat{\phi}(x)\,dx = 1$, and we obtain

$$\theta(0) = \int \widehat{\theta}(\xi)\,d\xi \tag{B.6.21}$$

for all $\theta \in \mathcal{S}(\mathbb{R}^N)$. Given now $\varphi \in \mathcal{S}(\mathbb{R}^N)$ and $x \in \mathbb{R}$, we let $\theta(\cdot) = \varphi(\cdot + x)$, and we deduce from Proposition B.6.1 (v) that $\widehat{\theta}(\xi) = e^{2\pi i x\cdot\xi}\widehat{\varphi}(\xi)$. Therefore, (B.6.21) yields

$$\varphi(x) = \int e^{2\pi i x\cdot\xi}\widehat{\varphi}(\xi)\,d\xi.$$

Since $x \in \mathbb{R}^N$ is arbitrary, this is the first identity in (B.6.17). The second identity follows by an obvious change of variable. □

Remark B.6.8. Here are some comments on Theorem B.6.7.

(i) Formula (B.6.17) can be written in the form

$$\overline{\mathcal{F}}\psi(x) = \widehat{\psi}(-x). \tag{B.6.22}$$

Thus we see that $\overline{\mathcal{F}}$ and $\mathcal{F}$ have similar properties. In particular, $\overline{\mathcal{F}} : \mathcal{S}(\mathbb{R}^N) \to \mathcal{S}(\mathbb{R}^N)$ is continuous.

(ii) Formula (B.6.17) can be written in the form

$$\overline{\mathcal{F}}\psi = \overline{\mathcal{F}\overline{\psi}} \tag{B.6.23}$$

where the bars on the right-hand side refer to complex conjugation.

(iii) It follows from Theorem B.6.7 and Property (i) above that $\mathcal{F}$ is an homeomorphism of $\mathcal{S}(\mathbb{R}^N)$, with inverse

$$\mathcal{F}^{-1} = \overline{\mathcal{F}}. \tag{B.6.24}$$

Theorem B.6.9. *The Parseval identity*

$$\int_{\mathbb{R}^N} u(x)\overline{v(x)}\,dx = \int_{\mathbb{R}^N} \widehat{u}(\xi)\overline{\widehat{v}(\xi)}\,d\xi \tag{B.6.25}$$

holds for all $u, v \in \mathcal{S}(\mathbb{R}^N)$, and the Plancherel formula

$$\|\widehat{u}\|_{L^2} = \|u\|_{L^2} \tag{B.6.26}$$

holds for all $u \in \mathcal{S}(\mathbb{R}^N)$.

Proof. Let $u, \psi \in \mathcal{S}(\mathbb{R}^N)$. Letting $v = \mathcal{F}^{-1}\overline{\psi}$, we have $\widehat{v} = \overline{\psi}$. Moreover, by (B.6.24) and (B.6.23), $v = \overline{\mathcal{F}}(\overline{\psi}) = \overline{\widehat{\psi}}$. Therefore, applying (B.6.5),

$$\int u(x)\overline{\psi}(x)\,dx = \int \widehat{u}(\xi)\overline{\widehat{\psi}(\xi)}\,d\xi$$

for all $u, \psi \in \mathcal{S}(\mathbb{R}^N)$, which is (B.6.25). Finally, (B.6.26) follows by letting $v = u$ in (B.6.25). □

B.6.3. The Fourier transform on $L^2(\mathbb{R}^N)$. So far, we have defined the Fourier transform on $L^1(\mathbb{R}^N)$. Using Plancherel's formula (B.6.26), we may extend $\mathcal{F}$ to an isometry of $L^2(\mathbb{R}^N)$, which we still denote by $\mathcal{F}$. (By density of $L^2(\mathbb{R}^N)\cap L^1(\mathbb{R}^N)$ in $L^2(\mathbb{R}^N)$.) In particular, Plancherel's formula (B.6.26) holds for all $u\in L^2(\mathbb{R}^N)$, and Parseval's identity (B.6.25) as well as formula (B.6.5) hold for all $u,v\in L^2(\mathbb{R}^N)$.

It follows in particular from (B.6.22) that $\overline{\mathcal{F}}$, which is in principle defined on $\mathcal{S}(\mathbb{R}^N)$, can be extended (by density of $\mathcal{S}(\mathbb{R}^N)$ in $L^2(\mathbb{R}^N)$) to an isometry of $L^2(\mathbb{R}^N)$. Therefore, the inversion formula (B.6.18) holds for all $\varphi\in L^2(\mathbb{R}^N)$. Thus we see that the Fourier transform is an homeomorphism of $L^2(\mathbb{R}^N)$, and that (B.6.24) holds on $L^2(\mathbb{R}^N)$, as well as formulas (B.6.22) and (B.6.23).

Since $\mathcal{F}$ is continuous $L^1(\mathbb{R}^N)\to L^\infty(\mathbb{R}^N)$ with norm ≤ 1 by (B.6.4), and continuous $L^2(\mathbb{R}^N)\to L^2(\mathbb{R}^N)$ with norm 1 by (B.6.26), it follows from the Riesz-Thorin interpolation theorem (see e.g. [4, Theorem 1.1.1]) that $\mathcal{F}$ can be extended to a continuous operator $L^p(\mathbb{R}^N)\to L^{p'}(\mathbb{R}^N)$ with norm ≤ 1, for all $1\le p\le 2$, i.e.

$$\|\mathcal{F}u\|_{L^{p'}}\le\|u\|_{L^p} \tag{B.6.27}$$

for all $u\in L^p(\mathbb{R}^N)$.

B.6.4. Tempered distributions: the space $\mathcal{S}'(\mathbb{R}^N)$. The space $\mathcal{S}'(\mathbb{R}^N)$ of tempered distributions on $\mathbb{R}^N$ is the topological dual of $\mathcal{S}(\mathbb{R}^N)$, and we denote by $\langle\cdot,\cdot\rangle$ the corresponding duality bracket. Since $C_{\mathrm{c}}^\infty(\mathbb{R}^N)$ is a dense subset of $\mathcal{S}(\mathbb{R}^N)$ (see Remark B.6.4 (iii)), it follows that $\mathcal{S}'(\mathbb{R}^N)$ is a subspace of $\mathcal{D}'(\mathbb{R}^N)$. Note that a map $u:\mathcal{S}(\mathbb{R}^N)\to\mathbb{C}$ defines an element $u\in\mathcal{S}'(\mathbb{R}^N)$ iff the following two properties hold

$$u:\begin{cases}\mathcal{S}(\mathbb{R}^N)\to\mathbb{C}\\ \varphi\mapsto u(\varphi)\end{cases}\quad\text{is a }\mathbb{C}\text{-linear map} \tag{B.6.28}$$

$$\text{if } (\varphi_n)_{n\ge1}\subset\mathcal{S}(\mathbb{R}^N)\text{ and }\varphi_n\to0\text{ in }\mathcal{S}(\mathbb{R}^N),\text{ then } u(\varphi_n)\underset{n\to\infty}{\longrightarrow}0 \tag{B.6.29}$$

Moreover, given a sequence $(u_n)_{n\ge1}\subset\mathcal{S}'(\mathbb{R}^N)$,

$$u_n\underset{n\to\infty}{\longrightarrow}0\text{ in }\mathcal{S}'(\mathbb{R}^N)\Longleftrightarrow\langle u_n,\varphi\rangle\underset{n\to\infty}{\longrightarrow}0\text{ for all }\varphi\in\mathcal{S}(\mathbb{R}^N) \tag{B.6.30}$$

Remark B.6.10. Here are some comments on the definition of $\mathcal{S}'(\mathbb{R}^N)$.

(i) If $u_n\to u$ in $\mathcal{S}'(\mathbb{R}^N)$ and $\varphi_n\to\varphi$ in $\mathcal{S}(\mathbb{R}^N)$ as $n\to\infty$, then $\langle u_n,\varphi_n\rangle\to\langle u,\varphi\rangle$. (See [27, Theorem 2.17, p. 51].)

(ii) If $u\in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ and $(1+|\cdot|^2)^{-\frac{m}{2}}u\in L^1(\mathbb{R}^N)$ for some $m\in\mathbb{N}$, then u defines a tempered distribution, via the formula

$$\langle u,\varphi\rangle=\int_{\mathbb{R}^N}u(x)\varphi(x)\,dx \tag{B.6.31}$$

for all $\varphi\in\mathcal{S}(\mathbb{R}^N)$. Indeed,

$$|u(x)\varphi(x)|\le p_m(\varphi)(1+|x|^2)^{-\frac{m}{2}}u(x)\in L^1(\mathbb{R}^N)$$

Therefore, the integral is well defined, and

$$|\langle u,\varphi\rangle|\le p_m(\varphi)\|(1+|\cdot|^2)^{-\frac{m}{2}}u\|_{L^1}$$

so that $\langle u,\varphi_n\rangle\to0$ if $\varphi_n\to0$ in $\mathcal{S}(\mathbb{R}^N)$.

(iii) In connexion with (ii) above, we say that a tempered distribution $u\in\mathcal{S}'(\mathbb{R}^N)$ belongs to $L^p(\mathbb{R}^N)$, where $1\le p\le\infty$, if there exists a function $f\in L^p(\mathbb{R}^N)$ such that (B.6.31) holds for all $\varphi\in\mathcal{S}(\mathbb{R}^N)$.

(iv) Given $u \in \mathcal{S}'(\mathbb{R}^N)$ and a multi-index β, one defines $D^\beta u \in \mathcal{S}'(\mathbb{R}^N)$ by

$$\langle D^\beta u, \varphi \rangle = (-1)^{|\beta|} \langle u, D^\beta \varphi \rangle \tag{B.6.32}$$

for all $\varphi \in \mathcal{S}(\mathbb{R}^N)$. Note that this defines an element of $\mathcal{S}'(\mathbb{R}^N)$ by Remark B.6.4 (iv), and that this definition coincides with the standard definition of derivatives if $u \in \mathcal{S}(\mathbb{R}^N)$, by formula (B.6.31). Moreover, it follows from Remark B.6.4 (iv) that the map $u \mapsto D^\beta u$ is continuous $\mathcal{S}'(\mathbb{R}^N) \to \mathcal{S}'(\mathbb{R}^N)$.

(v) We can multiply tempered distributions by C^∞ functions whose derivatives have at most a power growth at infinity. Indeed, if $\psi \in C^\infty(\mathbb{R}^N)$ satisfies (B.6.11) for all integer $m \ge 0$, then it follows from Remark B.6.4 (i) that $\psi\varphi \in \mathcal{S}(\mathbb{R}^N)$ for all $\varphi \in \mathcal{S}(\mathbb{R}^N)$. Therefore, we can define $\psi u \in \mathcal{S}'(\mathbb{R}^N)$ by

$$\langle \psi u, \varphi \rangle = \langle u, \psi\varphi \rangle \tag{B.6.33}$$

for all $\varphi \in \mathcal{S}(\mathbb{R}^N)$. The fact that this defines an element of $\mathcal{S}'(\mathbb{R}^N)$ follows from (B.6.12). Note that this is consistent with standard multiplication of functions, by formula (B.6.31). Moreover, it follows from Remark B.6.4 (i) that the map $u \mapsto \psi u$ is continuous $\mathcal{S}'(\mathbb{R}^N) \to \mathcal{S}'(\mathbb{R}^N)$.

(vi) It follows in particular from (v) above that if γ is a multi-index, then for every $u \in \mathcal{S}'(\mathbb{R}^N)$, $x^\gamma u$ defined by

$$\langle x^\gamma u, \varphi \rangle = \langle u, x^\gamma \varphi \rangle \tag{B.6.34}$$

is a tempered distribution and the map $u \mapsto x^\gamma u$ is continuous $\mathcal{S}'(\mathbb{R}^N) \to \mathcal{S}'(\mathbb{R}^N)$.

Definition B.6.11. In analogy with (B.6.5), the Fourier transform $\widehat{u} = \mathcal{F}u$ of a tempered distribution $u \in \mathcal{S}'(\mathbb{R}^N)$ is defined by

$$\langle \widehat{u}, \varphi \rangle = \langle u, \widehat{\varphi} \rangle \tag{B.6.35}$$

for all $\varphi \in \mathcal{S}(\mathbb{R}^N)$.

Remark B.6.12. Here are some consequences of Definition B.6.11.

(i) It follows from Corollary B.6.6 that for every $u \in \mathcal{S}'(\mathbb{R}^N)$, $\widehat{u}$ defined by (B.6.35) is a tempered distribution, i.e. $\widehat{u} \in \mathcal{S}'(\mathbb{R}^N)$; and that $\mathcal{F} : \mathcal{S}'(\mathbb{R}^N) \to \mathcal{S}'(\mathbb{R}^N)$ is continuous.

(ii) $\mathcal{F}$ is a bijection $\mathcal{S}'(\mathbb{R}^N) \to \mathcal{S}'(\mathbb{R}^N)$, and

$$\mathcal{F}^{-1} = \overline{\mathcal{F}} \tag{B.6.36}$$

where

$$\langle \overline{\mathcal{F}} u, \varphi \rangle = \langle u, \overline{\mathcal{F}} \varphi \rangle \tag{B.6.37}$$

with the notation (B.6.17). Indeed, it follows from identities (B.6.37), (B.6.35) and (B.6.24) that

$$\langle \overline{\mathcal{F}} \mathcal{F} u, \varphi \rangle = \langle u, \mathcal{F} \overline{\mathcal{F}} \varphi \rangle = \langle u, \varphi \rangle$$

so that $\overline{\mathcal{F}} \mathcal{F} = I$. Similarly, by (B.6.35), (B.6.37) and (B.6.24)

$$\langle \mathcal{F} \overline{\mathcal{F}} u, \varphi \rangle = \langle \overline{\mathcal{F}} u, \widehat{\varphi} \rangle = \langle u, \overline{\mathcal{F}} \widehat{\varphi} \rangle = \langle u, \varphi \rangle$$

so that $\mathcal{F} \overline{\mathcal{F}} = I$. Moreover, it follows from Remark B.6.8 that $\mathcal{F}^{-1}$ is continuous $\mathcal{S}'(\mathbb{R}^N) \to \mathcal{S}'(\mathbb{R}^N)$.

(iii) We define the complex conjugate of $u \in \mathcal{S}'(\mathbb{R}^N)$ by

$$\langle \overline{u}, \varphi \rangle = \overline{\langle u, \overline{\varphi} \rangle} \tag{B.6.38}$$

for all $\varphi \in \mathcal{S}(\mathbb{R}^N)$. (Note that this is consistent with the standard complex conjugation of functions, by (B.6.31).) It follows from (B.6.36) and (B.6.23) that

$$\mathcal{F}^{-1} u = \overline{\mathcal{F} \overline{u}} \tag{B.6.39}$$

for all $u \in \mathcal{S}'(\mathbb{R}^N)$. Indeed, applying identities (B.6.36), (B.6.37), (B.6.23) and (B.6.38), we see that

$$\langle \mathcal{F}^{-1}u, \varphi\rangle = \langle \overline{\mathcal{F}}u, \varphi\rangle = \langle u, \overline{\mathcal{F}}\varphi\rangle = \langle u, \overline{\mathcal{F}\overline{\varphi}}\rangle = \overline{\langle \overline{u}, \mathcal{F}\overline{\varphi}\rangle} = \overline{\langle \mathcal{F}\overline{u}, \overline{\varphi}\rangle} = \langle \overline{\mathcal{F}\overline{u}}, \varphi\rangle$$

for all $\varphi \in \mathcal{S}(\mathbb{R}^N)$.

(iv) Given $u \in \mathcal{S}'(\mathbb{R}^N)$ and multi-indices β, γ, we have

$$\xi^\gamma D^\beta \widehat{\varphi} = (-1)^{|\beta|}(2\pi i)^{|\beta|-|\gamma|}\mathcal{F}[D^\gamma(x^\beta\varphi)] \tag{B.6.40}$$

and

$$D^\beta(\xi^\gamma\widehat{\varphi}) = (-1)^{|\beta|}(2\pi i)^{|\beta|-|\gamma|}\mathcal{F}(x^\beta D^\gamma\varphi). \tag{B.6.41}$$

In particular,

$$\mathcal{F}(D^\beta u) = (2\pi i)^{|\beta|}\xi^\beta\widehat{u} \tag{B.6.42}$$

and

$$D^\beta\widehat{u} = (-2\pi i)^{|\beta|}\mathcal{F}(x^\beta u). \tag{B.6.43}$$

This follows easily by duality from formulas (B.6.13) and (B.6.14). As a particular case, we see that

$$\mathcal{F}(\Delta u) = -4\pi^2|\xi|^2\widehat{u} \tag{B.6.44}$$

for all $u \in \mathcal{S}'(\mathbb{R}^N)$.

We note that if $u \in L^1(\mathbb{R}^N)$ and $\varphi \in \mathcal{S}(\mathbb{R}^N)$, then

$$\begin{aligned} u \star \varphi(x) &= \int u(y)\varphi(x-y)\,dy = \int u(y)\widetilde{\varphi}(y-x)\,dy \\ &= \int u(y)\tau_x\widetilde{\varphi}(y)\,dy = \langle u, \tau_x\widetilde{\varphi}\rangle \end{aligned} \tag{B.6.45}$$

with the notation (B.6.1), (B.6.2) and (B.6.31).

Definition B.6.13. In analogy with identity (B.6.45), we define the convolution of an element of $\mathcal{S}'(\mathbb{R}^N)$ with an element of $\mathcal{S}(\mathbb{R}^N)$ by

$$u \star \varphi(x) = \langle u, \tau_x\widetilde{\varphi}\rangle \tag{B.6.46}$$

for all $u \in \mathcal{S}'(\mathbb{R}^N)$ and $\varphi \in \mathcal{S}(\mathbb{R}^N)$.

We have the following result. (See [27, Theorem 7.19, p. 195].)

Theorem B.6.14. *Let $u \in \mathcal{S}'(\mathbb{R}^N)$ and $\varphi \in \mathcal{S}(\mathbb{R}^N)$, and let $u \star \varphi$ be defined by* (B.6.46).

(i) *$u \star \varphi \in C^\infty(\mathbb{R}^N)$ and*

$$D^\beta(u \star \varphi) = u \star (D^\beta\varphi) = (D^\beta u) \star \varphi \tag{B.6.47}$$

for every multi-index β.

(ii) *$u\star\varphi$ grows at most as a power of $|x|$ as $|x| \to \infty$. In particular, $u\star\varphi \in \mathcal{S}'(\mathbb{R}^N)$ by Remark* B.6.10 (ii)*, and*

$$\langle u \star \varphi, \psi\rangle = \langle u, \varphi \star \psi\rangle \tag{B.6.48}$$

for all $\psi \in \mathcal{S}(\mathbb{R}^N)$.

(iii) *$(u \star \varphi) \star \psi = u \star (\varphi \star \psi)$ for all $\psi \in \mathcal{S}(\mathbb{R}^N)$.*

(iv) *$\mathcal{F}(u \star \varphi) = \widehat{\varphi}\widehat{u}$.*

(v) *$\widehat{u} \star \widehat{\varphi} = \mathcal{F}(\varphi u)$.*

We will also use the following property.

Proposition B.6.15. *Given an interval $I \subset \mathbb{R}$ and $u : I \to \mathcal{S}'(\mathbb{R}^N)$, the following properties hold.*

(i) *$u \in C(I, \mathcal{S}'(\mathbb{R}^N))$ if and only if the map $t \mapsto \langle u(t), \varphi\rangle$ is continuous on I for all $\varphi \in \mathcal{S}(\mathbb{R}^N)$.*

(ii) *$u \in C^1(I, \mathcal{S}'(\mathbb{R}^N))$ if and only if the map $t \mapsto \langle u(t), \varphi \rangle$ is C^1 on I for all $\varphi \in \mathcal{S}(\mathbb{R}^N)$. In this case,*

$$\langle u'(t), \varphi \rangle = \frac{d}{dt} \langle u(t), \varphi \rangle \tag{B.6.49}$$

for all $\varphi \in \mathcal{S}(\mathbb{R}^N)$ and $t \in I$.

(iii) *If $u \in C^1(I, \mathcal{S}'(\mathbb{R}^N))$ and $\varphi \in C^1(I, \mathcal{S}(\mathbb{R}^N))$, then the map $t \mapsto \langle u(t), \varphi(t) \rangle$ is in $C^1(I)$, and*

$$\frac{d}{dt} \langle u(t), \varphi(t) \rangle = \langle u'(t), \varphi(t) \rangle + \langle u(t), \varphi'(t) \rangle \tag{B.6.50}$$

for all $i \in I$.

Proof. We consider the case where $I = (0, T)$ with $T > 0$, the other cases being similar. Property (i) follows from (B.6.30).

To prove Property (ii), suppose first $u \in C^1((0,T), \mathcal{S}'(\mathbb{R}^N))$ and let $0 < t < T$. It follows that $\frac{1}{h}(u(t+h) - u(t)) - u'(t) \to 0$ in $\mathcal{S}'(\mathbb{R}^N)$ as $h \to 0$. This means that for every $\varphi \in \mathcal{S}(\mathbb{R}^N)$, $\frac{1}{h}(\langle u(t+h), \varphi \rangle - \langle u(t)), \varphi \rangle - \langle u'(t), \varphi \rangle \to 0$ as $h \to 0$. Thus we see that the map $t \mapsto \langle u(t), \varphi \rangle$ is C^1 on $(0,T)$ for all $\varphi \in \mathcal{S}(\mathbb{R}^N)$, and that (B.6.49) holds. Conversely, suppose the map $t \mapsto \langle u(t), \varphi \rangle$ is C^1 on $(0,T)$ for all $\varphi \in \mathcal{S}(\mathbb{R}^N)$. Given $\varphi \in \mathcal{S}(\mathbb{R}^N)$ and $0 < t < T$, let $L_t(\varphi) = \frac{d}{dt} \langle u(t), \varphi \rangle$. We have

$$L_t(\varphi) = \lim_{h \to 0} L_{t,h}(\varphi)$$

where

$$L_{t,h}(\varphi) = \frac{1}{h} \langle u(t+h) - u(t), \varphi \rangle$$

L_t is clearly linear. Moreover, for all $t \in 0, T)$ and $h \neq 0$, $u(t+h) - u(t) \in \mathcal{S}'(\mathbb{R}^N)$, so that $L_{t,h} : \mathcal{S}(\mathbb{R}^N) \to \mathbb{C}$ is continuous. Since $\mathcal{S}(\mathbb{R}^N)$ is a Fréchet space, it follows from the Banach-Steinhaus theorem (see [27, Theorem 2.8]) that L_t is continuous $\mathcal{S}(\mathbb{R}^N) \to \mathbb{C}$. Therefore, $L_t \in \mathcal{S}'(\mathbb{R}^N)$, so that $u \in C^1((0,T), \mathcal{S}'(\mathbb{R}^N))$ and $u'(t) = L_t$.

Finally, we prove Property (iii), and we set $f(t) = \langle u(t), \varphi(t) \rangle$. Given $t \in I$ and $h \neq 0$ such that $t + h \in I$, we have

$$f(t+h) - f(t) = \langle u(t+h) - u(t), \varphi(t) \rangle + \langle u(t+h), \varphi(t+h) - \varphi(t) \rangle$$

Dividing by h, letting $h \to 0$, and applying Remark B.6.10 (i), we obtain the identity (B.6.50). □

Remark B.6.16. Given $\nu > 0$, set

$$f^\nu(\xi) = (1 + 4\pi^2 |\xi|^2)^{-\nu} \tag{B.6.51}$$

so that $f^\nu \in \mathcal{S}'(\mathbb{R}^N)$ by Remark B.6.10 (ii). (Note that $f^\nu \in L^2(\mathbb{R}^N)$ if $\nu > \frac{N}{2}$, but $f^\nu \notin L^2(\mathbb{R}^N)$ if $\nu \le \frac{N}{2}$.) The inverse Fourier transform $F^\nu = \mathcal{F}^{-1} f^\nu$ of f^ν is in principle an element of $\mathcal{S}'(\mathbb{R}^N)$, but it turns out that F^ν is in fact a function. More precisely,

$$F^\nu(x) = \frac{1}{(4\pi)^\nu \Gamma(\nu)} \int_0^\infty e^{-\frac{\pi |x|^2}{\rho}} e^{-\frac{\rho}{4\pi}} \rho^{-1-\frac{N}{2}+\nu} d\rho \tag{B.6.52}$$

where Γ is the Gamma function defined by

$$\Gamma(s) = \int_0^\infty e^{-\sigma} \sigma^{s-1} d\sigma \tag{B.6.53}$$

In particular, we see from formula (B.6.52) that F^ν is positive, radially symmetric and radially decreasing. Moreover,

$$\|F^\nu\|_{L^1} = \int_{\mathbb{R}^N} F^\nu(x)\, dx = 1. \tag{B.6.54}$$

(See [30, Chapter V, Proposition 2].) To see this, let $s = \nu$ in (B.6.53), and make the change of variable $\sigma = \frac{\mu}{4\pi}\rho$, with $\mu > 0$, to obtain

$$(4\pi)^{\nu}\Gamma(\nu)\mu^{-\nu} = \int_0^{\infty} e^{-\frac{\mu\rho}{4\pi}}\rho^{\nu-1}d\rho. \tag{B.6.55}$$

For $\mu = 1 + 4\pi^2|\xi|^2$, this yields

$$(4\pi)^{\nu}\Gamma(\nu)f^{\nu}(\xi) = \int_0^{\infty} e^{-\frac{\rho}{4\pi}(1+4\pi^2|\xi|^2)}\rho^{\nu-1}d\rho \tag{B.6.56}$$

On the other hand, given $\varphi \in \mathcal{S}(\mathbb{R}^N)$ and $\rho > 0$, we deduce from (B.6.5) and (B.6.9) (with $a = \rho$) that

$$\int_{\mathbb{R}^N} e^{-\rho\pi|\xi|^2}\widehat{\varphi}(\xi)\,d\xi = \rho^{-\frac{N}{2}}\int_{\mathbb{R}^N} e^{-\frac{\pi|x|^2}{\rho}}\varphi(x)\,dx \tag{B.6.57}$$

so that, multiplying by $e^{-\frac{\rho}{4\pi}}\rho^{\nu-1}$

$$\int_{\mathbb{R}^N} e^{-\frac{\rho}{4\pi}(1+4\pi^2|\xi|^2)}\rho^{\nu-1}\widehat{\varphi}(\xi)\,d\xi = e^{-\frac{\rho}{4\pi}}\rho^{1-\frac{N}{2}+\nu}\int_{\mathbb{R}^N} e^{-\frac{\pi|x|^2}{\rho}}\varphi(x)\,dx \tag{B.6.58}$$

Integrating in ρ and applying Fubini, we obtain

$$\int_{\mathbb{R}^N}\Big(\int_0^{\infty} e^{-\frac{\rho}{4\pi}(1+4\pi^2|\xi|^2)}\rho^{\nu-1}d\rho\Big)\widehat{\varphi}(\xi)\,d\xi$$
$$= \int_{\mathbb{R}^N}\Big(\int_0^{\infty} e^{-\frac{\pi|x|^2}{\rho}}e^{-\frac{\rho}{4\pi}}\rho^{-1-\frac{N}{2}+\nu}d\rho\Big)\varphi(x)\,dx$$

hence, applying (B.6.56),

$$(4\pi)^{\nu}\Gamma(\nu)\int_{\mathbb{R}^N} f^{\nu}(\xi)\widehat{\varphi}(\xi)\,d\xi \;= \int_{\mathbb{R}^N}\Big(\int_0^{\infty} e^{-\frac{\pi|x|^2}{\rho}}e^{-\frac{\rho}{4\pi}}\rho^{-1-\frac{N}{2}+\nu}d\rho\Big)\varphi(x)\,dx$$

Since $\varphi \in \mathcal{S}(\mathbb{R}^N)$ is arbitrary, the above identity, together with formula (B.6.5), yields (B.6.52). To prove (B.6.54), we integrate (B.6.52) over $\mathbb{R}^N$ and apply Fubini, to obtain

$$\int_{\mathbb{R}^N} F^{\nu}(x)\,dx = \frac{1}{(4\pi)^{\nu}\Gamma(\nu)}\int_0^{\infty} e^{-\frac{\rho}{4\pi}}\rho^{-1-\frac{N}{2}+\nu}\Big(\int_{\mathbb{R}^N} e^{-\frac{\pi|x|^2}{\rho}}dx\Big)d\rho$$

Since

$$\int_{\mathbb{R}^N} e^{-\frac{\pi|x|^2}{\rho}}dx = \rho^{\frac{N}{2}}$$

we obtain

$$\int_{\mathbb{R}^N} F^{\nu}(x)\,dx = \frac{1}{(4\pi)^{\nu}\Gamma(\nu)}\int_0^{\infty} e^{-\frac{\rho}{4\pi}}\rho^{-1+\nu}d\rho = 1$$

where the last identity follows from (B.6.55) with $\mu = 1$. Hence (B.6.54). Note also that, since f^{ν} is real-valued, then

$$\widehat{f^{\nu}} = F^{\nu} \tag{B.6.59}$$

by (B.6.39). More generally, given $\lambda > 0$, let

$$f_{\lambda}^{\nu}(\xi) = (\lambda + 4\pi^2|\xi|^2)^{-\nu} \tag{B.6.60}$$

and

$$F_{\lambda}^{\nu} = \widehat{f_{\lambda}^{\nu}} = \mathcal{F}^{-1}f_{\lambda}^{\nu} \tag{B.6.61}$$

Since $f_{\lambda}^{\nu}(\xi) = \lambda^{-\frac{\varepsilon}{2}}f^{\nu}(\xi\lambda^{-\frac{1}{2}})$, an obvious change of variables yields

$$F_{\lambda}^{\nu}(x) = \lambda^{\frac{N-\varepsilon}{2}}F^{\nu}(x\lambda^{\frac{1}{2}}) \tag{B.6.62}$$

In particular, F_λ^ν is positive, radially symmetric and radially decreasing and, applying (B.6.62) and (B.6.54),

$$\|F_\lambda^\nu\|_{L^1} = \int_{\mathbb{R}^N} F_\lambda^\nu(x)\,dx = \lambda^{-\nu}. \tag{B.6.63}$$

B.6.5. Fractional order Sobolev spaces. The L^2 case. It follows from Definitions B.1.2 and B.1.15 that, given $m \in \mathbb{N}$, $H^m(\mathbb{R}^N)$ is the set of $u \in L^2(\mathbb{R}^N)$ such that $D^\alpha u \in L^2(\mathbb{R}^N)$ for all multi-indices β with $|\beta| \le m$, and $H^{-m}(\mathbb{R}^N) = (H^m(\mathbb{R}^N))^\star$. Here is an equivalent definition, based on the Fourier transform and Plancherel's formula.

PROPOSITION B.6.17. *Let $m \in \mathbb{Z}$, and let the Sobolev space $H^m(\mathbb{R}^N)$ be given by Definitions B.1.2 and B.1.15. It follows that*

$$H^m(\mathbb{R}^N) = \{u \in \mathcal{S}'(\mathbb{R}^N);\ (1 + 4\pi^2|\cdot|^2)^{\frac{m}{2}}\widehat{u} \in L^2(\mathbb{R}^N)\} \tag{B.6.64}$$

and

$$\|u\|_{H^m} \sim \|(1 + 4\pi^2|\cdot|^2)^{\frac{s}{2}}\widehat{u}\|_{L^2}. \tag{B.6.65}$$

PROOF. Set

$$\mathcal{H}^m = \{u \in \mathcal{S}'(\mathbb{R}^N);\ (1 + 4\pi^2|\cdot|^2)^{\frac{m}{2}}\widehat{u} \in L^2(\mathbb{R}^N)\}$$

and

$$\|u\|_{\mathcal{H}^m} = \|(1 + 4\pi^2|\cdot|^2)^{\frac{s}{2}}\widehat{u}\|_{L^2}.$$

We first consider the case $m \ge 0$. Let $u \in H^m(\mathbb{R}^N)$, $j \in \{1, \cdots, N\}$, and $k \in \{0, \cdots, m\}$. It follows from formula (B.6.42) that

$$|\xi_j|^k|\widehat{u}| = (2\pi)^{-k}|\mathcal{F}(\partial_j^k u)|$$

so that by Plancherel's formula (B.6.26)

$$\||\xi_j|^k|\widehat{u}|\|_{L^2} \le (2\pi)^{-k}\|\partial_j^k u\|_{L^2} \le \|u\|_{H^m}.$$

It follows easily that $u \in \mathcal{H}^m$ and $\|u\|_{\mathcal{H}^m} \le C\|u\|_{H^m}$ with C independent of u. Conversely, suppose $u \in \mathcal{H}^m$ and let β be a multi-index with $|\beta| \le m$. The preceding calculations show that $D^\beta u \in L^2(\mathbb{R}^N)$ and

$$\|D^\beta u\|_{L^2} \le C\|u\|_{\mathcal{H}^m}$$

with a constant C independent of m. The result in the case $m \ge 0$ easily follows.

We now consider the case $m < 0$. Since $H^m(\mathbb{R}^N) = (H^{-m}(\mathbb{R}^N))^\star$, we need only show that $\mathcal{H}^m = (\mathcal{H}^{-m})^\star$. Any $u \in \mathcal{H}^m$ defines an element $F_u \in (\mathcal{H}^{-m})^\star$ by

$$\langle F_u, v\rangle_{(\mathcal{H}^{-m})^\star, \mathcal{H}^{-m}} = ((1 + 4\pi^2|\cdot|^2)^{\frac{m}{2}}\widehat{u}, (1 + 4\pi^2|\cdot|^2)^{-\frac{m}{2}}\widehat{v})_{L^2} \tag{B.6.66}$$

for all $v \in \mathcal{H}^{-m}$, and it is easy to verify that $\|F(u)\|_{(\mathcal{H}^{-m})^\star} = \|u\|_{\mathcal{H}^m}$. This shows that $\mathcal{H}^m \hookrightarrow (\mathcal{H}^{-m})^\star$. Conversely, let $F \in (\mathcal{H}^{-m})^\star$, and define

$$G(w) = \langle F, \mathcal{F}^{-1}[(1 + 4\pi^2|\cdot|^2)^{\frac{m}{2}}\widehat{w}]\rangle_{(\mathcal{H}^{-m})^\star, \mathcal{H}^{-m}} \tag{B.6.67}$$

for $w \in L^2(\mathbb{R}^N)$. Note that this makes sense. Indeed, if $w \in L^2(\mathbb{R}^N)$, then $\mathcal{F}^{-1}[(1 + 4\pi^2|\cdot|^2)^{\frac{m}{2}}\widehat{w}] \in \mathcal{H}^{-m}$ and $\|w\|_{\mathcal{H}^{-m}} = \|\mathcal{F}^{-1}[(1 + 4\pi^2|\cdot|^2)^{\frac{m}{2}}\widehat{w}]\|_{L^2}$. It follows that $G(w) \in (L^2(\mathbb{R}^N))^\star$, so that by Riesz's representation theorem, there exists $z \in L^2(\mathbb{R}^N)$ such that

$$G(w) = (z, w)_{L^2} = (\widehat{z}, \widehat{w})_{L^2} \tag{B.6.68}$$

for all $w \in L^2(\mathbb{R}^N)$, where the last identity follows from Parseval's identity (B.6.25). Set now $u = \mathcal{F}^{-1}[(1 + 4\pi^2|\cdot|^2)^{-\frac{m}{2}}\widehat{z}] \in \mathcal{H}^m$. It follows from (B.6.67), (B.6.68) and (B.6.66) that, given any $v \in \mathcal{H}^{-m}$,

$$\begin{aligned}\langle F, v\rangle_{(\mathcal{H}^{-m})^\star, \mathcal{H}^{-m}} &= G(\mathcal{F}^{-1}[(1+4\pi^2|\cdot|^2)^{-\frac{m}{2}}\widehat{v}]) \\ &= (\widehat{z}, (1+4\pi^2|\cdot|^2)^{-\frac{m}{2}}\widehat{v})_{L^2} \\ &= ((1+4\pi^2|\cdot|^2)^{\frac{m}{2}}\widehat{u}, (1+4\pi^2|\cdot|^2)^{-\frac{m}{2}}\widehat{v})_{L^2} \\ &= \langle F_u, v\rangle_{(\mathcal{H}^{-m})^\star, \mathcal{H}^{-m}}.\end{aligned}$$

Therefore, $F = F_u$, which shows that $(\mathcal{H}^{-m})^\star \hookrightarrow \mathcal{H}^m$ and completes the proof. □

We now extend the definition of the Sobolev space $H^m(\mathbb{R}^N)$ to non-integer indices. Given $s \in \mathbb{R}$, we define

$$H^s(\mathbb{R}^N) = \{u \in \mathcal{S}'(\mathbb{R}^N);\ (1+4\pi^2|\cdot|^2)^{\frac{s}{2}}\widehat{u} \in L^2(\mathbb{R}^N)\} \tag{B.6.69}$$

and

$$\|u\|_{H^s} = \|(1+4\pi^2|\cdot|^2)^{\frac{s}{2}}\widehat{u}\|_{L^2} \tag{B.6.70}$$

for $u \in H^s(\mathbb{R}^N)$. It follows from Proposition B.6.17 that this definition is consistent with Definitions B.1.2 and B.1.15 for $s \in \mathbb{Z}$. We note that (B.6.69) makes sense. Indeed, given $u \in \mathcal{S}'(\mathbb{R}^N)$, it follows that $\widehat{u} \in \mathcal{S}'(\mathbb{R}^N)$. Therefore, $(1+4\pi^2|\cdot|^2)^{\frac{s}{2}}\widehat{u} \in \mathcal{S}'(\mathbb{R}^N)$ by Remark B.6.10 (v), so that it makes sense to say that $(1+4\pi^2|\xi|^2)^{\frac{s}{2}}\widehat{u} \in L^p(\mathbb{R}^N)$ by Remark B.6.10 (iii).

REMARK B.6.18. Here are some comments on the above definition.

(i) Given any $s \in \mathbb{R}$, $H^s(\mathbb{R}^N)$ is a Hilbert space, with the scalar product

$$(u, v)_{H^s} = ((1+4\pi^2|\cdot|^2)^{\frac{s}{2}}\widehat{u}, (1+4\pi^2|\cdot|^2)^{\frac{s}{2}}\widehat{v})_{L^2}. \tag{B.6.71}$$

for all $u, v \in H^s(\mathbb{R}^N)$.

(ii) Let $s \in \mathbb{R}$ and $1 \le p \le \infty$. The operators $\mathcal{F}$ and $\mathcal{F}^{-1}$, and the multiplication by $(1+|\xi|^2)^{\frac{s}{2}}$ are all continuous $\mathcal{S}(\mathbb{R}^N) \to \mathcal{S}(\mathbb{R}^N)$. (See Remark B.6.4 (i) and Remark B.6.8.) Since $\mathcal{S}(\mathbb{R}^N) \hookrightarrow L^2(\mathbb{R}^N)$, it follows that $\mathcal{S}(\mathbb{R}^N) \hookrightarrow H^s(\mathbb{R}^N)$. Moreover, the embedding $\mathcal{S}(\mathbb{R}^N) \hookrightarrow H^s(\mathbb{R}^N)$ is dense. Indeed, let $u \in H^s(\mathbb{R}^N)$, let $v = \mathcal{F}^{-1}[(1+4\pi^2|\cdot|^2)^{\frac{s}{2}}\widehat{u}] \in L^2(\mathbb{R}^N)$, and let $(v_n)_{n\ge1} \subset \mathcal{S}(\mathbb{R}^N)$ satisfy $v_n \to v$ in L^2. It follows that $u_n \to u$ in $H^{s,p}$, where $u_n = \mathcal{F}^{-1}[(1+4\pi^2|\cdot|^2)^{-\frac{s}{2}}\widehat{v_n}]$. Moreover, $u_n \in \mathcal{S}(\mathbb{R}^N)$, which completes the proof.

(iii) Given $\gamma > 0$, we see that $1+4\pi^2|\xi|^2 \sim 1+\gamma|\xi|^2$. It follows that

$$H^s(\mathbb{R}^N) = \{u \in \mathcal{S}'(\mathbb{R}^N);\ (1+\gamma|\cdot|^2)^{\frac{s}{2}}\widehat{u} \in L^2(\mathbb{R}^N)\}$$

and

$$\|u\|_{H^s} \sim \|(1+\gamma|\cdot|^2)^{\frac{s}{2}}\widehat{u}\|_{L^2}.$$

(iv) Let $s, s' \in \mathbb{R}$, and suppose $s \ge s'$. It follows immediately that $H^s(\mathbb{R}^N) \hookrightarrow H^{s'}(\mathbb{R}^N)$ and, more precisely,

$$\|u\|_{H^{s'}} \le \|u\|_{H^s} \tag{B.6.72}$$

for all $u \in H^s(\mathbb{R}^N)$.

PROPOSITION B.6.19. *Let $s \in \mathbb{R}$ and β a multi-index. It follows that $D^\beta u \in H^{s-|\beta|}(\mathbb{R}^N)$ for all $u \in H^{s,p}(\mathbb{R}^N)$. Moreover, there exists a constant C such that*

$$\|D^\beta u\|_{H^{s-|\beta|}} \le C\|u\|_{H^s} \tag{B.6.73}$$

for all $u \in H^s(\mathbb{R}^N)$. In particular, $\|\Delta u\|_{H^{s-2}} \le C\|u\|_{H^s}$ for all $u \in H^s(\mathbb{R}^N)$.

PROOF. We have $\mathcal{F}[D^\beta u] = (2\pi i\xi)^\beta\widehat{u}$ by (B.6.42), so that

$$|\mathcal{F}[D^\beta u]| \le (2\pi\xi)^{|\beta|}|\widehat{u}|. \tag{B.6.74}$$

The result easily follows. □

PROPOSITION B.6.20. *Let $s \in \mathbb{R}$, $m \in \mathbb{N}$, and $u \in \mathcal{S}'(\mathbb{R}^N)$. It follows that $u \in H^{s+m}(\mathbb{R}^N)$ iff $D^\beta u \in H^s(\mathbb{R}^N)$ for all multi-indices β such that $|\beta| \le m$. Moreover, there exist constants $0 < c \le C < \infty$ such that*

$$c\|u\|_{H^{s+m}} \le \sum_{|\beta|\le m} \|D^\beta u\|_{H^s} \le C\|u\|_{H^{s+m}} \tag{B.6.75}$$

for all $u \in H^{s+m}(\mathbb{R}^N)$.

PROOF. The result easily follows from formula (B.6.74). □

REMARK B.6.21. The fractional order Sobolev spaces $H^s(\mathbb{R}^N)$ are complex interpolation spaces of the standard Sobolev spaces $H^m(\mathbb{R}^N)$. More precisely, given $\ell, m \in \mathbb{Z}$ such that $\ell < m$, and $s \in (\ell, m)$,

$$H^s(\mathbb{R}^N) = [H^\ell(\mathbb{R}^N), H^m(\mathbb{R}^N)]_{[\frac{s-\ell}{m-\ell}]} \tag{B.6.76}$$

See e.g. [**4**, Theorem 6.4.5 (7)]. Moreover, the interpolation inequality

$$\|u\|_{H^s} \le \|u\|_{H^\ell}^{\frac{m-s}{m-\ell}} \|u\|_{H^m}^{\frac{s-\ell}{m-\ell}} \tag{B.6.77}$$

holds for all $u \in H^m(\mathbb{R}^N)$. Indeed,

$$(1 + 4\pi^2|\cdot|^2)^{\frac{s}{2}}|\widehat{u}| = [(1 + 4\pi^2|\cdot|^2)^{\frac{\ell}{2}}|\widehat{u}|]^{\frac{m-s}{m-\ell}}[(1 + 4\pi^2|\cdot|^2)^{\frac{m}{2}}|\widehat{u}|]^{\frac{s-\ell}{m-\ell}}$$

and $\frac{1}{2} = \frac{m-s}{2(m-\ell)} + \frac{s-\ell}{2(m-\ell)}$, so it follows from Hölder's inequality that

$$\begin{aligned}\|u\|_{H^s} &= \|(1 + 4\pi^2|\cdot|^2)^{\frac{s}{2}}\widehat{u}\|_{L^2} \\ &\le \|(1 + 4\pi^2|\cdot|^2)^{\frac{\ell}{2}}\widehat{u}\|_{L^2}^{\frac{m-s}{m-\ell}} \|(1 + 4\pi^2|\cdot|^2)^{\frac{m}{2}}\widehat{u}\|_{L^2}^{\frac{s-\ell}{m-\ell}} \\ &= \|u\|_{H^\ell}^{\frac{m-s}{m-\ell}} \|u\|_{H^m}^{\frac{s-\ell}{m-\ell}}\end{aligned}$$

REMARK B.6.22. Let $s \in \mathbb{R}$ and $m \in \mathbb{N}$, $m \ge |s|$. If $\theta \in C^m(\mathbb{R}^N) \cap W^{m,\infty}(\mathbb{R}^N)$, then the map $u \mapsto \theta u$ is continuous $H^s(\mathbb{R}^N) \to H^s(\mathbb{R}^N)$, and

$$\|\theta u\|_{H^s} \le C\|\theta\|_{W^{m,\infty}}\|u\|_{H^s}$$

This property is easy to prove if $s = m \in \mathbb{Z}$, see Remark B.1.10 (ii) for the case $m \ge 0$ and Remark B.1.16 (ii) for the case $m < 0$. See [**2**, Theorem 1.62] for a proof in the case $s \notin \mathbb{Z}$.

B.6.6. Fractional order Sobolev spaces. The general case. Given $s \in \mathbb{R}$ and $1 \le p \le \infty$, we define

$$H^{s,p}(\mathbb{R}^N) = \{u \in \mathcal{S}'(\mathbb{R}^N);\ \mathcal{F}^{-1}[(1 + 4\pi^2|\cdot|^2)^{\frac{s}{2}}\widehat{u}] \in L^p(\mathbb{R}^N)\} \tag{B.6.78}$$

and

$$\|u\|_{H^{s,p}} = \|\mathcal{F}^{-1}[(1 + 4\pi^2|\cdot|^2)^{\frac{s}{2}}\widehat{u}]\|_{L^p} \tag{B.6.79}$$

for $u \in H^{s,p}(\mathbb{R}^N)$. We note that (B.6.78) makes sense. Indeed, given $u \in \mathcal{S}'(\mathbb{R}^N)$, it follows that $\widehat{u} \in \mathcal{S}'(\mathbb{R}^N)$. Therefore, $(1+4\pi^2|\cdot|^2)^{\frac{s}{2}}\widehat{u} \in \mathcal{S}'(\mathbb{R}^N)$ by Remark B.6.10 (v). Hence $\mathcal{F}^{-1}[(1+4\pi^2|\cdot|^2)^{\frac{s}{2}}\widehat{u}] \in \mathcal{S}'(\mathbb{R}^N)$, so that it makes sense to say that $\mathcal{F}^{-1}[(1+4\pi^2|\cdot|^2)^{\frac{s}{2}}\widehat{u}] \in L^p(\mathbb{R}^N)$ by Remark B.6.10 (iii).

REMARK B.6.23. Here are some comments on the above definition.

(i) It follows from (B.6.78) and (B.6.79) that $H^{0,p}(\mathbb{R}^N) = L^p(\mathbb{R}^N)$ with equal norms for all $1 \le p \le \infty$.

(ii) Given $s \in \mathbb{R}$, define $J_s u$ for $u \in \mathcal{S}'(\mathbb{R}^N)$ by

$$J_s u = \mathcal{F}^{-1}[(1 + 4\pi^2 |\cdot|^2)^{\frac{s}{2}} \widehat{u}] \tag{B.6.80}$$

(J_s are the Bessel potentials.) It follows from Remark B.6.12 (i) and (ii), and Remark B.6.10 (v) that $J_s \in \mathcal{L}(\mathcal{S}'(\mathbb{R}^N))$ for all $s \in \mathbb{R}$. Furthermore, given $s, \sigma \in \mathbb{R}$,

$$\begin{aligned} J_\sigma[J_s u] &= \mathcal{F}^{-1}[(1 + 4\pi^2 |\cdot|^2)^{\frac{\sigma}{2}} \mathcal{F}(J_s u)] \\ &= \mathcal{F}^{-1}[(1 + 4\pi^2 |\cdot|^2)^{\frac{\sigma}{2}} (1 + 4\pi^2 |\cdot|^2)^{\frac{s}{2}} \widehat{u}] = J_{s+\sigma} u \end{aligned}$$

for all $u \in \mathcal{S}'(\mathbb{R}^N)$, so that

$$J_\sigma J_s = J_{s+\sigma} \tag{B.6.81}$$

Since $J_0 = I$, it follows that $J_s : \mathcal{S}'(\mathbb{R}^N) \to \mathcal{S}'(\mathbb{R}^N)$ is a bijection, and

$$J_s^{-1} = J_{-s} \tag{B.6.82}$$

for all $s \in \mathbb{R}$. Note that, given $u \in \mathcal{S}'(\mathbb{R}^N)$ and $1 \le p \le \infty$, we have $u \in H^{s,p}(\mathbb{R}^N)$ iff $J_s u \in L^p(\mathbb{R}^N)$, and

$$\|u\|_{H^{s,p}} = \|J_s u\|_{L^p} \tag{B.6.83}$$

Therefore, $H^{s,p}(\mathbb{R}^N) = J_s^{-1}(L^p(\mathbb{R}^N)) = J_{-s}(L^p(\mathbb{R}^N))$.

(iii) Let $s \in \mathbb{R}$, β a multi-index, and $u \in \mathcal{S}'(\mathbb{R}^N)$. If J_s is defined by (B.6.80), it follows from (B.6.42) (where u is replaced by $J_s u$) that

$$D^\beta J_s u = (2\pi i)^{|\beta|} \mathcal{F}^{-1}[\xi^\beta \mathcal{F}(J_s u)] = (2\pi i)^{|\beta|} \mathcal{F}^{-1}[(1 + 4\pi^2 |\xi|^2)^{\frac{s}{2}} \xi^\beta \widehat{u}]$$

Applying once more (B.6.42), we obtain $D^\beta J_s u = \mathcal{F}^{-1}[(1+4\pi^2|\xi|^2)^{\frac{s}{2}} \mathcal{F}(D^\beta u)]$. Thus we see that

$$D^\beta J_s u = J_s D^\beta u \tag{B.6.84}$$

(iv) Let $s, \sigma \in \mathbb{R}$, $1 \le p \le \infty$ and $u \in \mathcal{S}'(\mathbb{R}^N)$. It follows from (ii) above that $u \in H^{s+\sigma,p}(\mathbb{R}^N)$ iff $J_s \in u H^{\sigma,p}(\mathbb{R}^N)$, and in this case

$$\|u\|_{H^{\sigma+s,p}} = \|J_s u\|_{H^{\sigma,p}} \tag{B.6.85}$$

(v) It follows from (B.6.44) that, with the notation (B.6.80), $J_2 u = -\Delta u + u$ for all $u \in \mathcal{S}'(\mathbb{R}^N)$. In particular, we see that, given $1 \le p \le \infty$, $u \in H^{2,p}(\mathbb{R}^N)$ iff $-\Delta u + u \in L^p(\mathbb{R}^N)$. In this case,

$$\|u\|_{H^{2,p}} = \| - \Delta u + u\|_{L^p} \tag{B.6.86}$$

(vi) It follows from (ii) above that $H^{s,p}(\mathbb{R}^N)$ is a Banach space. Indeed, if $(u_n)_{n\ge1}$ is a Cauchy sequence in $H^{s,p}(\mathbb{R}^N)$, then $(J_s u_n)_{n\ge0}$ is a Cauchy sequence in $L^p(\mathbb{R}^N)$. Therefore, there exists $v \in L^p(\mathbb{R}^N)$ such that $J_s u_n \to v$ in L^p. This means that $u_n \to u$ in $H^{s,p}$, where $u = J_{-s} v \in H^{s,p}(\mathbb{R}^N)$.

(vii) Let $s \in \mathbb{R}$ and $1 \le p \le \infty$. The operators $\mathcal{F}$ and $\mathcal{F}^{-1}$, and the multiplication by $(1+|\xi|^2)^{\frac{s}{2}}$ are all continuous $\mathcal{S}(\mathbb{R}^N) \to \mathcal{S}(\mathbb{R}^N)$. (See Remark B.6.4 (i) and Remark B.6.8.) Since $\mathcal{S}(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$, it follows that $\mathcal{S}(\mathbb{R}^N) \hookrightarrow H^{s,p}(\mathbb{R}^N)$. Moreover, if $1 \le p < \infty$, then the embedding $\mathcal{S}(\mathbb{R}^N) \hookrightarrow H^{s,p}(\mathbb{R}^N)$ is dense. Indeed, let $u \in H^{s,p}(\mathbb{R}^N)$ and let $v = J_s u \in L^p(\mathbb{R}^N)$. Since $p < \infty$, there exists $(v_n)_{n\ge1} \subset \mathcal{S}(\mathbb{R}^N)$ such that $v_n \to v$ in L^p. Therefore, $u_n \to u$ in $H^{s,p}$, where $u_n = J_{-s} v_n$. Moreover, $v_n \in \mathcal{S}(\mathbb{R}^N)$, so that $\mathcal{F} v_n \in \mathcal{S}(\mathbb{R}^N)$; and so $(1 + 4\pi^2 |\cdot|^2)^{-\frac{s}{2}} \mathcal{F} v_n \in \mathcal{S}(\mathbb{R}^N)$. Therefore, $u_n = J_{-s} v_n \in \mathcal{S}(\mathbb{R}^N)$, which completes the proof.

(viii) If $1 \le p < \infty$ and $s \in \mathbb{R}$, then $(H^{s,p}(\mathbb{R}^N))^\star = H^{-s,p'}(\mathbb{R}^N)$ with equivalent norms. Indeed, any $f \in H^{-s,p'}(\mathbb{R}^N)$ defines an element $F \in (H^{s,p}(\mathbb{R}^N))^\star$ by

$$F(u) = \langle J_{-s} f, J_s u \rangle_{L^{p'}, L^p} \tag{B.6.87}$$

as follows easily from (ii) above. Applying (B.6.83), we see that $\|F\|_{(H^{s,p})^\star} \le \|f\|_{H^{-s,p'}}$. Thus $H^{-s,p'}(\mathbb{R}^N) \hookrightarrow (H^{s,p}(\mathbb{R}^N))^\star$. Conversely, let F be an element of $(H^{s,p}(\mathbb{R}^N))^\star$. It follows from (ii) that the map $w \mapsto F(J_{-s}w)$ is a linear, continuous form on $L^p(\mathbb{R}^N)$ with norm $\|F\|_{(H^{s,p})^\star}$. Thus there exists $g \in L^{p'}(\mathbb{R}^N)$ such that $\|g\|_{L^{p'}} = \|F\|_{(H^{s,p})^\star}$ and

$$F(J_{-s}w) = \langle g, w \rangle_{L^{p'},L^p}$$

for all $w \in L^p(\mathbb{R}^N)$. Given $u \in H^{s,p}(\mathbb{R}^N)$, applying the above formula with $w = J_s u \in L^p(\mathbb{R}^N)$ and using (B.6.82), we obtain

$$F(u) = \langle g, J_s u \rangle_{L^{p'},L^p}$$

Letting $f = J_s g \in H^{-s,p'}(\mathbb{R}^N)$, this means that F has the form (B.6.87). Since $\|f\|_{H^{-s,p'}} = \|g\|_{L^{p'}} = \|F\|_{(H^{s,p})^\star}$, we conclude that $(H^{s,p}(\mathbb{R}^N))^\star \hookrightarrow H^{-s,p'}(\mathbb{R}^N)$.

PROPOSITION B.6.24. *Suppose $1 \le p \le \infty$ and $s, s' \in \mathbb{R}$. If $s \ge s'$, then $H^{s,p}(\mathbb{R}^N) \hookrightarrow H^{s',p}(\mathbb{R}^N)$. More precisely,*

$$\|u\|_{H^{s',p}} \le \|u\|_{H^{s,p}} \tag{B.6.88}$$

for all $u \in H^{s,p}(\mathbb{R}^N)$.

PROOF. Let $u \in H^{s,p}(\mathbb{R}^N)$. We have

$$\begin{aligned}\mathcal{F}^{-1}[(1+4\pi^2|\xi|^2)^{\frac{s'}{2}}\widehat{u}] &= \mathcal{F}^{-1}[(1+4\pi^2|\xi|^2)^{-\frac{s-s'}{2}}(1+4\pi^2|\xi|^2)^{\frac{s}{2}}\widehat{u}] \\ &= \mathcal{F}^{-1}[(1+4\pi^2|\xi|^2)^{-\frac{s-s'}{2}}] \star \mathcal{F}^{-1}[(1+4\pi^2|\xi|^2)^{\frac{s}{2}}\widehat{u}]\end{aligned} \tag{B.6.89}$$

Moreover, it follows from (B.6.54) that

$$\|\mathcal{F}^{-1}[(1+4\pi^2|\xi|^2)^{-\frac{s-s'}{2}}]\|_{L^1} = 1. \tag{B.6.90}$$

Estimate (B.6.88) then follows from (B.6.89), (B.6.90) and Young's inequality. □

PROPOSITION B.6.25. *Let $\theta \in \mathcal{S}(\mathbb{R}^N)$ and define $L : \mathcal{S}'(\mathbb{R}^N) \to \mathcal{S}'(\mathbb{R}^N)$ by $Lu = \theta \star u$ for all $u \in \mathcal{S}'(\mathbb{R}^N)$. Let $1 \le p \le q \le \infty$, $s, \sigma \in \mathbb{R}$, and β a multi-index. If $u \in H^{s,p}(\mathbb{R}^N)$, then $D^\beta Lu \in H^{\sigma,q}(\mathbb{R}^N)$ and*

$$\|D^\beta Lu\|_{H^{\sigma,q}} \le \|D^\beta \theta\|_{H^{\sigma-s,r}} \|u\|_{H^{s,p}} \tag{B.6.91}$$

where $1 \le r \le \infty$ is given by $\frac{1}{r} = \frac{1}{q} - \frac{1}{p} + 1$. In particular (with $s = \sigma$)

$$\|D^\beta Lu\|_{H^{s,q}} \le \|D^\beta \theta\|_{L^r} \|u\|_{H^{s,p}} \tag{B.6.92}$$

and (with $\sigma = s$ and $q = p$)

$$\|D^\beta Lu\|_{H^{s,p}} \le \|D^\beta \theta\|_{L^1} \|u\|_{H^{s,p}} \tag{B.6.93}$$

PROOF. By Theorem B.6.14 (i) and (iv), $D^\beta Lu = (D^\beta\theta) \star u = \mathcal{F}^{-1}(\widehat{D^\beta\theta}\widehat{u})$, so that

$$(1+4\pi^2|\cdot|^2)^{\frac{\sigma}{2}}\mathcal{F}(Lu) = (1+4\pi^2|\cdot|^2)^{\frac{\sigma}{2}}\widehat{\theta}\widehat{u} = [(1+4\pi^2|\cdot|^2)^{\frac{\sigma-s}{2}}\widehat{\theta}][(1+4\pi^2|\cdot|^2)^{\frac{s}{2}}\widehat{u}]$$

Therefore, by Theorem B.6.14 (iv),

$$\mathcal{F}^{-1}[(1+4\pi^2|\cdot|^2)^{\frac{\sigma}{2}}\mathcal{F}(Lu)] = \mathcal{F}^{-1}[\widehat{\theta}(1+4\pi^2|\cdot|^2)^{\frac{\sigma-s}{2}}] \star \mathcal{F}^{-1}[(1+4\pi^2|\cdot|^2)^{\frac{s}{2}}\widehat{u}]$$

Taking the L^r norm and applying Young's inequality to the right-hand side yields estimate (B.6.91). □

It turns out that for any $1 < p < \infty$, then the spaces $W^{m,p}(\mathbb{R}^N)$ and $H^{m,p}(\mathbb{R}^N)$ coincide, with equivalent norms.

THEOREM B.6.26. *Given any $m \in \mathbb{Z}$ and $1 < p < \infty$, $W^{m,p}(\mathbb{R}^N) = H^{m,p}(\mathbb{R}^N)$, and $\|u\|_{W^{m,p}} \approx \|u\|_{H^{m,p}}$.*

The proof of Theorem B.6.26 in the case $p = 2$ is rather elementary, see Proposition B.6.17. When $p \neq 2$, the proof is much more delicate. It is based on the following result.

THEOREM B.6.27 (The Mihlin multiplier theorem). *Let $\rho \in L^\infty(\mathbb{R}^N)$ and let $\ell > N/2$ be an integer. Suppose $\rho \in W^{\ell,\infty}_{\mathrm{loc}}(\mathbb{R}^N \setminus \{0\})$ and*

$$\sup_{|\alpha| \le \ell} \operatorname*{ess\,sup}_{\xi \in \mathbb{R}^N} |\xi|^{|\alpha|} |\partial^\alpha \rho(\xi)| < \infty.$$

It follows that for every $1 < p < \infty$, there exists a constant C_p such that

$$\|\mathcal{F}^{-1}(\rho \widehat{v})\|_{L^p} \le C_p \|v\|_{L^p} \tag{B.6.94}$$

for all $v \in \mathcal{S}(\mathbb{R}^N)$.

Theorem B.6.27 is a deep theorem in harmonic analysis. This is for instance Theorem 6.1.6, p. 135 in [4]. The proof is delicate, and an essential ingredient is the Marcinkiewicz interpolation theorem. In fact, only a simplified form of this theorem is needed, namely the form stated in [30], §4.2, Theorem 5, p. 21. A simple proof of this (simplified version of the) Marcinkiewicz interpolation theorem is given in [30], pp. 21–22.

PROOF OF THEOREM B.6.26. We fix $m \in \mathbb{Z}$ and $1 < p < \infty$, and we proceed in five steps.

STEP 1. $\mathcal{S}(\mathbb{R}^N)$ is dense in $H^{m,p}(\mathbb{R}^N)$. Let $u \in H^{m,p}(\mathbb{R}^N)$ and set $w = \mathcal{F}^{-1}[(1 + 4\pi^2 |\cdot|^2)^{\frac{m}{2}} \widehat{u}] \in L^p(\mathbb{R}^N)$. $\mathcal{S}(\mathbb{R}^N)$ being dense in $L^p(\mathbb{R}^N)$, there exists $(w_n)_{n \ge 0} \subset \mathcal{S}(\mathbb{R}^N)$ such that $w_n \to w$ in $L^p(\mathbb{R}^N)$. Setting $u_n = \mathcal{F}^{-1}[(1 + 4\pi^2|\cdot|^2)^{-\frac{m}{2}} \widehat{w_n}] \in \mathcal{S}(\mathbb{R}^N)$, this means that $u_n \to u$ in $H^{m,p}(\mathbb{R}^N)$.

STEP 2. If $m \ge 0$, then $H^{m,p}(\mathbb{R}^N) \hookrightarrow W^{m,p}(\mathbb{R}^N)$. By Step 1, it suffices to show that $\|u\|_{W^{m,p}} \le C \|u\|_{H^{m,p}}$ for all $u \in \mathcal{S}(\mathbb{R}^N)$. Let α be a multi-index with $|\alpha| \le m$ and let

$$\rho(\xi) = \xi^\alpha (1 + 4\pi^2 |\xi|^2)^{-\frac{m}{2}}$$

One checks easily that ρ satisfies the assumptions of Theorem B.6.27. Applying estimate (B.6.94) with $v = \mathcal{F}^{-1}[(1+4\pi^2|\cdot|^2)^{\frac{m}{2}} \widehat{u}]$, we deduce that $\|\mathcal{F}^{-1}(\xi^\alpha \widehat{u})\|_{L^p} \le C\|u\|_{H^{m,p}}$. Since $\mathcal{F}^{-1}(\xi^\alpha \widehat{u}) = (2\pi i)^{-|\alpha|} D^\alpha u$, we obtain $\|D^\alpha u\|_{L^p} \le C \|u\|_{H^{m,p}}$. The result follows, since α with $|\alpha| \le m$ is arbitrary.

STEP 3. If $m \ge 0$, then $W^{m,p}(\mathbb{R}^N) \hookrightarrow H^{m,p}(\mathbb{R}^N)$. By density of $\mathcal{S}(\mathbb{R}^N)$ in $W^{m,p}(\mathbb{R}^N)$, it suffices to show that $\|u\|_{H^{m,p}} \le C \|u\|_{W^{m,p}}$ for all $u \in \mathcal{S}(\mathbb{R}^N)$. Fix a function $\theta \in C^\infty(\mathbb{R})$, $\theta \ge 0$, such that $\theta(t) = 0$ for $|t| \le 1$ and $\theta(t) = 1$ for $|t| \ge 2$. Set

$$\rho(\xi) = (1 + 4\pi^2 |\xi|^2)^{\frac{m}{2}} \Bigl(1 + \sum_{j=1}^N \theta(\xi_j) |\xi_j|^m\Bigr)^{-1}.$$

It is not difficult to show that ρ satisfies the assumptions of Theorem B.6.27. Applying (B.6.94) with $v = \mathcal{F}^{-1}[(1 + \sum_{j=1}^N \theta(\xi_j)|\xi_j|^m)\widehat{u}]$, we deduce that

$$\begin{aligned} \|u\|_{H^{m,p}} &\le C \Bigl\| \mathcal{F}^{-1}\Bigl[\Bigl(1 + \sum_{j=1}^N \theta(\xi_j)|\xi_j|^m\Bigr)\widehat{u}\Bigr]\Bigr\|_{L^p} \\ &\le C\Bigl(\|u\|_{L^p} + \sum_{j=1}^N \|\mathcal{F}^{-1}(\theta(\xi_j)|\xi_j|^m \widehat{u})\|_{L^p}\Bigr). \end{aligned} \tag{B.6.95}$$

Next, we observe that $\rho_j(\xi) = \theta(\xi_j)|\xi_j|^m \xi_j^{-m}$ satisfies the assumptions of Theorem B.6.27. Applying (B.6.94) with $\rho = \rho_j$ and $v = u$, successively for $j = 1, \cdots, N$, we deduce from (B.6.95) that

$$\begin{aligned}\|u\|_{H^{m,p}} &\le C\Big(\|u\|_{L^p} + \sum_{j=1}^{N} \|\mathcal{F}^{-1}(\xi_j^m \widehat{u})\|_{L^p}\Big) \\ &= C\Big(\|u\|_{L^p} + (2\pi)^{-m} \sum_{j=1}^{N} \|\partial_j^m u\|_{L^p}\Big) \le C\|u\|_{W^{m,p}},\end{aligned}$$

which is the desired property.

STEP 4. The conclusion of the theorem holds for all $m \ge 0$. This follows from Steps 2 and 3.

STEP 5. The conclusion of the theorem holds for all $m < 0$. Indeed, it follows from Step 4 that $W^{-m,p'}(\mathbb{R}^N) = H^{-m,p'}(\mathbb{R}^N)$ with equivalent norms. Since, by definition, $(W^{-m,p'}(\mathbb{R}^N))^\star = W^{m,p}(\mathbb{R}^N)$, and $(H^{-m,p'}(\mathbb{R}^N))^\star = H^{m,p}(\mathbb{R}^N)$ by Remark B.6.23 (viii), the result follows. □

Below are some other useful applications of the Mihlin multiplier theorem.

PROPOSITION B.6.28. *Let $1 < p < \infty$, $s \in \mathbb{R}$, and $a, b > 0$. It follows that*

$$H^{s,p}(\mathbb{R}^N) = \{u \in \mathcal{S}'(\mathbb{R}^N);\ \mathcal{F}^{-1}[(a + b|\cdot|^2)^{\frac{s}{2}}\widehat{u}] \in L^p(\mathbb{R}^N)\} \tag{B.6.96}$$

and that there exists constants $0 < c \le C < \infty$ such that

$$c\|u\|_{H^{s,p}} \le \|\mathcal{F}^{-1}[(a + b|\cdot|^2)^{\frac{s}{2}}\widehat{u}]\|_{L^p} \le C\|u\|_{H^{s,p}} \tag{B.6.97}$$

for $u \in H^{s,p}(\mathbb{R}^N)$.

PROOF. By density of $\mathcal{S}(\mathbb{R}^N)$ in $H^{s,p}(\mathbb{R}^N)$ (see Remark B.6.23 (vii)), it suffices to prove (B.6.97) for all $u \in \mathcal{S}(\mathbb{R}^N)$. We first consider

$$\rho(\xi) = \frac{(1 + 4\pi^2|\xi|^2)^{\frac{s}{2}}}{(a + b|\xi|^2)^{\frac{s}{2}}}$$

It is not difficult to check that ρ satisfies the assumptions of Theorem B.6.27. Applying (B.6.94) with $v = \mathcal{F}^{-1}[(a + b|\cdot|^2)^{\frac{s}{2}}\widehat{u}]$, we deduce the first inequality in (B.6.97). The second inequality follows similarly by considering

$$\rho(\xi) = \frac{(a + b|\xi|^2)^{\frac{s}{2}}}{(1 + 4\pi^2|\xi|^2)^{\frac{s}{2}}}$$

and applying (B.6.94) with $v = \mathcal{F}^{-1}[(1 + 4\pi^2|\cdot|^2)^{\frac{s}{2}}\widehat{u}]$. □

PROPOSITION B.6.29. *Let $1 < p < \infty$, $s \in \mathbb{R}$, and β a multi-index. It follows that $D^\beta u \in H^{s-|\beta|,p}(\mathbb{R}^N)$ for all $u \in H^{s,p}(\mathbb{R}^N)$. Moreover, there exists a constant C such that*

$$\|D^\beta u\|_{H^{s-|\beta|,p}} \le C\|u\|_{H^{s,p}} \tag{B.6.98}$$

for all $u \in H^{s,p}(\mathbb{R}^N)$. In particular, $\|\Delta u\|_{H^{s-2,p}} \le C\|u\|_{H^{s,p}}$ for all $u \in H^{s,p}(\mathbb{R}^N)$.

PROOF. By density of $\mathcal{S}(\mathbb{R}^N)$ in $H^{s,p}(\mathbb{R}^N)$ (see Remark B.6.23 (vii)), it suffices to prove (B.6.98) for all $u \in \mathcal{S}(\mathbb{R}^N)$. This easily follows by applying (B.6.94) with

$$\rho(\xi) = \frac{(2\pi i)^{|\beta|}\xi^\beta}{(1 + 4\pi^2|\xi|^2)^{\frac{|\beta|}{2}}}$$

and $v = \mathcal{F}^{-1}[(1 + 4\pi^2|\cdot|^2)^{\frac{s}{2}}\widehat{u}]$. □

Proposition B.6.30. *Let $1 < p < \infty$, $s \in \mathbb{R}$, $m \in \mathbb{N}$, and $u \in \mathcal{S}'(\mathbb{R}^N)$. It follows that $u \in H^{s+m,p}(\mathbb{R}^N)$ iff $D^\beta u \in H^{s,p}(\mathbb{R}^N)$ for all multi-indices β such that $|\beta| \le m$. Moreover, there exist constants $0 < c \le C < \infty$ such that*

$$c\|u\|_{H^{s+m,p}} \le \sum_{|\beta|\le m} \|D^\beta u\|_{H^{s,p}} \le C\|u\|_{H^{s+m,p}} \tag{B.6.99}$$

for all $u \in H^{s+m,p}(\mathbb{R}^N)$.

Proof. It follows from Remark B.6.23 (iv) that $u \in H^{s+m,p}(\mathbb{R}^N)$ iff $J_s u \in H^{m,p}(\mathbb{R}^N)$. By Theorem B.6.26, this is equivalent to $D^\beta J_s u \in L^p(\mathbb{R}^N)$ for all $|\beta| \le m$, and

$$\|u\|_{H^{s+m,p}} \approx \sum_{|\beta|\le m} \|D^\beta J_s u\|_{L^p}$$

Since $D^\beta J_s u = J_s D^\beta u$ by (B.6.84), we deduce from (B.6.85) that $\|D^\beta J_s u\|_{L^p} = \|D^\beta u\|_{H^{s,p}}$, and the result follows. □

Remark B.6.31. The fractional order Sobolev spaces $H^{s,p}(\mathbb{R}^N)$ are complex interpolation spaces of the standard Sobolev spaces $W^{m,p}(\mathbb{R}^N)$. More precisely, given $1 < p < \infty$, $\ell, m \in \mathbb{Z}$ such that $\ell < m$, and $s \in (\ell, m)$,

$$H^{s,p}(\mathbb{R}^N) = [W^{\ell,p}(\mathbb{R}^N), W^{m,p}(\mathbb{R}^N)]_{[\frac{s-\ell}{m-\ell}]} \tag{B.6.100}$$

See e.g. [4, Theorem 6.4.5 (7)]. In particular, the interpolation inequality

$$\|u\|_{H^{s,p}} \le C\|u\|_{W^{\ell,p}}^{\frac{m-s}{m-\ell}} \|u\|_{W^{m,p}}^{\frac{s-\ell}{m-\ell}} \tag{B.6.101}$$

holds for all $u \in W^{m,p}(\mathbb{R}^N)$. (Inequality (B.6.101) is immediate if $p = 2$, see Remark B.6.21.)

Remark B.6.32. Let $s \in \mathbb{R}$ and $m \in \mathbb{N}$, $m \ge |s|$. If $\theta \in C^m(\mathbb{R}^N) \cap W^{m,\infty}(\mathbb{R}^N)$, then the map $u \mapsto \theta u$ is continuous $H^{s,p}(\mathbb{R}^N) \to H^{s,p}(\mathbb{R}^N)$ for all $1 < p < \infty$. Moreover,

$$\|\theta u\|_{H^{s,p}} \le C\|\theta\|_{W^{m,\infty}} \|u\|_{H^{s,p}}$$

This property is easy to prove if $s = m \in \mathbb{Z}$, see Remark B.1.10 (ii) for the case $m \ge 0$ and Remark B.1.16 (ii) for the case $m < 0$. It seems that there is no really simple proof of the property if $s \notin \mathbb{Z}$ and $p \ne 2$. One of the several possible proofs consists in considering two consecutive integers $m_1 < m_2$ such that $m_1 \le s \le m_2$, and using interpolation (see e.g. [4, Theorems 4.1.2]) together with (B.6.100). See [2, Theorem 1.62] for a direct proof in the case $p = 2$.

APPENDIX C

Vector integration

Vector integration is an important tool in the study of evolution equation. Even though most existence and regularity results are stated in terms of continuous functions, weaker regularity classes often appear in intermediate steps.

Throughout this chapter, X is a Banach space with the norm $\|\cdot\|$ and I is an open interval of $\mathbb{R}$ (bounded or unbounded) equipped with the Lebesgue measure. We will use the basic theorems of real valued integration (Fatou's lemma, the monotone convergence theorem, the dominated convergence theorem, Egorov's theorem in particular).

C.1. Measurable functions

Definition C.1.1. A function $f : I \to X$ is measurable if there exists a set $N \subset I$ of measure 0 and a sequence $(f_n)_{n\in\mathbb{N}} \subset C_c(I,X)$ such that $\lim_{n\to\infty} f_n(t) = f(t)$, for all $t \in I \setminus N$.

Remark C.1.2. It follows easily from Definition C.1.1 that if $f : I \to X$ is measurable, then $\|f\| : I \to \mathbb{R}$ is also measurable. Many properties of vector valued measurable functions follow either immediately from the definition or else from the properties of real valued measurable functions applied to $\|f - f_n\|$. In particular, one can show easily the following results.

(i) If $f : I \to X$ is measurable and if Y is a Banach space such that $X \hookrightarrow Y$, then $f : I \to Y$ is measurable.
(ii) If a sequence $(f_n)_{n\in\mathbb{N}}$ of measurable functions $I \to X$ converges a.a. (in the X topology) to a function $f : I \to X$, then f is measurable.
(iii) If $f : I \to X$ and $\varphi : I \to \mathbb{R}$ are measurable, then $f\varphi : I \to X$ is measurable. In particular, if $f : I \to X$ is measurable and if $J \subset I$ is an open interval, then $f_{|J} : J \to X$ is measurable (take $\varphi = 1_J$).
(iv) If $(x_n)_{n\in\mathbb{N}}$ is a family of elements of X and if $(\omega_n)_{n\in\mathbb{N}}$ is a family of measurable subsets of I such that $\omega_i \cap \omega_j = \emptyset$ for $i \neq j$, then $\sum_{n=0}^{\infty} x_n 1_{\omega_n} : I \to X$ is measurable.

In Definition C.1.1 and Remark C.1.2, the strong topology of X is involved. However, in many applications, one needs to prove measurability of a function which is only the limit in the weak topology of X of a sequence of measurable functions. For that purpose, a most useful tool is the following result.

Theorem C.1.3 (Pettis' Theorem). *Consider $f : I \to X$. Then f is measurable if and only if it satisfies the following two conditions:*

(i) *f is weakly measurable (i.e. for every $x' \in X^\star$, the function $t \mapsto \langle x', f(t)\rangle$ is measurable $I \to \mathbb{R}$);*
(ii) *there exists a set $N \subset I$ of measure 0 such that $f(I \setminus N)$ is separable.*

Proof. It is clear that measurability implies weak measurability; and so (i) is necessary. If f is measurable and if $(f_n)_{n\in\mathbb{N}} \subset C_c(I,X)$ converges to f on $I\setminus N$ with $|N| = 0$, then $f_n(I\setminus N)$ is separable; and so $f(I\setminus N)$ is also separable. Therefore (ii) is also necessary.

Let now f satisfy (i) and (ii). By possibly replacing X by the smallest closed subspace of X containing $f(I \setminus N)$, we may assume that X is separable. We first establish that for every $x \in X$, the function $t \mapsto \|f(t) - x\|$ is measurable. Indeed, for $a \geq 0$, we have

$$\{t \in I,\ \|f(t) - x\| \leq a\} = \bigcap_{x' \in S'} \{t \in I,\ |\langle x', f(t) - x\rangle| \leq a\},$$

where S' is the unit ball of $X^\star$. It follows from Lemma A.1.7 that there exists a sequence $(x'_n)_{n\in\mathbb{N}} \subset S'$such that

$$\{t \in I,\ \|f(t) - x\| \leq a\} = \bigcap_{n\in\mathbb{N}} \{t \in I,\ |\langle x'_n, f(t) - x\rangle| \leq a\}.$$

The set on the right-hand side of the above identity is clearly measurable by assumption (i); and so the function $t \mapsto \|f(t) - x\|$ is measurable.

Consider now $n \in \mathbb{N}$. The set $f(I)$ being separable, it can be covered by a countable union of balls B^n_j of center x^n_j and radius $1/n$. Let $f_n : I \to X$ defined by $f_n = \sum_{j=0}^\infty x^n_j 1_{\omega^n_j}$, where $\omega^n_0 = \{t \in I,\ f(t) \in B^n_0\}$ and $\omega^n_j = \{t \in I,\ f(t) \in B^n_j \setminus \cap_{k=0}^{j-1} B^n_k\}$, for $j \geq 1$. It is immediate that $\|f(t) - f_n(t)\| \leq 1/n$, for all $t \in I$. Furthermore, since the function $t \mapsto \|f(t)-x\|$ is measurable for all $x \in X$, it follows that the sets ω^n_j are measurable; and so, by Remark C.1.2 (iv), f_n is measurable. Therefore, by Remark C.1.2 (ii) f is measurable. □

Corollary C.1.4. *If $f : I \to X$ is weakly continuous (i.e. $t \mapsto \langle x', f(t)\rangle_{X^\star,X}$ is continuous for every $x' \in X^\star$), then f is measurable.*

Proof. f is clearly weakly measurable; and so by Theorem C.1.3, it is sufficient to prove that $f(I)$ is separable. It follows from the weak continuity of f that $f(I) \subset E$, where E is the weak closure of the convex hull of $f(I\cap\mathbb{Q})$. On the other hand, $E = \overline{f(I \cap \mathbb{Q})}$; and so E is separable. Hence the result. □

Corollary C.1.5. *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of measurable functions $I \to X$ and let $f : I \to X$. If, for a.a. $t \in I$, $f_n(t) \rightharpoonup f(t)$ in X as $n \to \infty$, then f is measurable.*

Proof. Let $x' \in X^\star$. Since $\langle x', f_n(t)\rangle \to \langle x', f(t)\rangle$ a.a., it follows that the function $t \mapsto \langle x', f(t)\rangle$ is measurable; and so f is weakly measurable.

On the other hand, it follows from Theorem C.1.3 that for every $n \in \mathbb{N}$, there exists a set N_n of measure 0 such that $f_n(I \setminus N_n)$ is separable. Consider the set $N = \cup_{n=0}^\infty N_n$, which is also of measure 0, and let C be the convex hull of $\cup_{n=0}^\infty f_n(I\setminus E)$. Clearly $f(I\setminus E) \subset \tilde{C}$, where $\tilde{C}$ is the weak closure of C. Furthermore, $\tilde{C} = \overline{C}$; and so $\tilde{C}$ is separable. Hence the result, by Theorem C.1.3. □

Corollary C.1.6. *Let $X \hookrightarrow Y$ be two Banach spaces and let $u : I \to Y$ be weakly continuous. Assume that there exists a dense subset E of I such that*

(i) $u(t) \in X$, *for all* $t \in E$,
(ii) $\sup\{\|u(t)\|_X, t \in E\} = K < \infty$.

If X is reflexive, then $u(t) \in X$, for all $t \in I$, and $u : I \to X$ is measurable.

Proof. The result follows from Corollaries A.1.11 and C.1.4. □

Remark C.1.7. Consider two Banach spaces $X \hookrightarrow Y$, and a measurable function $f : I \to Y$. Assume that $f(t) \in X$, for a.a. $t \in I$. It is natural to ask whether $f : I \to X$ is measurable. In general, the answer is negative, as shows the following example. Let $I = \Omega = (0, 1)$ and consider the function $u : I \to L^\infty(\Omega)$ given by $u(t) = 1_{(0,t)}$, for $0 < t < 1$. One verifies easily that $u \in C^{0,1/p}(\overline{I}, L^p(\Omega))$, for every

$p \in [1, \infty)$. In particular, $u : I \to L^p(\Omega)$ is measurable, for every $p \in [1, \infty)$. Furthermore, $u(t) \in L^\infty(\Omega)$ for all $t \in I$. However, $u : I \to L^\infty(\Omega)$ is not measurable. To see this, observe that $\|u(t) - u(s)\|_{L^\infty} = 1$, if $t \neq s$. Therefore, $u(I)$ is a discrete subset of $L^\infty(\Omega)$; and so, given any non-countable subset A of I, $u(A) \subset L^\infty(\Omega)$ is discrete and non-countable, hence non-separable. In particular, given a subset N of I of measure 0, $u(I \setminus N)$ is not a separable subset of $L^\infty(\Omega)$. Therefore, by Theorem C.1.3, $u : I \to L^\infty(\Omega)$ is not measurable. Note that u is an elementary example of a non-measurable function. However, one can obtain measurability results under additional assumptions. This is the object of the following result.

Proposition C.1.8. *Let $X \hookrightarrow Y$ be two Banach spaces and let $f : I \to Y$ be a measurable function. If $f(t) \in X$ for a.a. $t \in I$ and if X is reflexive, then $f : I \to X$ is measurable.*

Proof. By applying Theorem C.1.3 and by modifying f on a set of measure 0, we may assume that $f(I) \subset X$ and that $f(I)$ is a separable subset of Y. By replacing X by the smallest closed subspace of X containing $f(I)$, then by replacing Y by the closure of X in Y, we may assume that Y is separable and that the embedding $X \hookrightarrow Y$ is dense. By applying Lemma A.1.10, it follows that X is separable. Therefore, by applying again Theorem C.1.3, we need only check that f is weakly measurable $I \to X$. To see this, consider $x' \in X^\star$. It follows from Theorem A.1.5 that there exists $(y'_n)_{n \in \mathbb{N}} \subset Y^\star$ such that $y'_n \underset{n\to\infty}{\longrightarrow} x'$ in $X^\star$. In particular,

$$\langle y'_n, f(t) \rangle_{X^\star, X} \underset{n\to\infty}{\longrightarrow} \langle x', f(t) \rangle_{X^\star, X}, \text{ for all } t \in I.$$

On the other hand, we have $\langle y'_n, f(t) \rangle_{X^\star, X} = \langle y'_n, f(t) \rangle_{Y^\star, Y}$ by Theorem A.1.5. Therefore, the mapping $t \mapsto \langle y'_n, f(t) \rangle_{X^\star, X}$ is measurable; and so, the mapping $t \mapsto \langle x', f(t) \rangle_{X^\star, X}$ is measurable. Hence the result. □

C.2. Integrable functions

Definition C.2.1. A measurable function $f : I \to X$ is integrable if there exists a sequence $(f_n)_{n \in \mathbb{N}} \subset C_c(I, X)$ such that

$$\lim_{n\to\infty} \int_I \|f_n(t) - f(t)\| \, dt = 0.$$

(Note that by Remark C.1.2, $\|f_n - f\| : I \to \mathbb{R}$ is a nonnegative measurable function, so that $\int_I \|f_n(t) - f(t)\| \, dt$ makes sense.)

Lemma C.2.2. *Let $f : I \to X$ be integrable. There exists $i(f) \in X$ such that for any sequence $(f_n)_{n \in \mathbb{N}} \subset C_c(I, X)$ satisfying*

$$\lim_{n\to\infty} \int_I \|f_n(t) - f(t)\| \, dt = 0,$$

one has

$$\lim_{n\to\infty} \int_I f_n(t) \, dt = i(f),$$

the above limit being for the strong topology of X.

Proof. Let $(f_n)_{n \in \mathbb{N}} \subset C_c(I, X)$ satisfy the assumption of the lemma. We have

$$\begin{aligned} \Big\| \int_I f_n(t) \, dt - \int_I f_p(t) \, dt \Big\| &\le \int_I \|f_n(t) - f_p(t)\| \, dt \\ &\le \int_I \|f_n(t) - f(t)\| \, dt + \int_I \|f_p(t) - f(t)\| \, dt. \end{aligned}$$

Therefore, $\int_I f_n(t)\,dt$ is a Cauchy sequence, that converges to some element $x \in X$. Consider another sequence $(g_n)_{n\in\mathbb{N}} \subset C_c(I,X)$. We have

$$\begin{aligned}\Big\|\int_I g_n(t)\,dt - x\Big\| &\\ \le \Big\|\int_I g_n(t) - f(t)\,dt\Big\| &+ \Big\|\int_I f(t) - f_n(t)\,dt\Big\| + \Big\|\int_I f_n(t)\,dt - x\Big\| \\ \le \int_I \|g_n(t) - f(t)\|\,dt &+ \int_I \|f_n(t) - f(t)\|\,dt + \Big\|\int_I f_n(t)\,dt - x\Big\|\end{aligned}$$

Therefore, $\displaystyle\int_I g_n(t)\,dt$ converges also to x, as $n \to \infty$. Hence the result, with $i(f) = x$. □

Definition C.2.3. The element $i(f)$ defined by Lemma C.2.2 is called the integral of f on I. We note

$$i(f) = \int f = \int_I f = \int_I f(t)\,dt.$$

If $I = (a,b)$, we also note

$$i(f) = \int_a^b f = \int_a^b f(t)\,dt.$$

As for real-valued functions, it is convenient to note

$$\int_\alpha^\beta f(t)\,dt = -\int_\beta^\alpha f(t)\,dt,$$

if $\beta < \alpha$.

Theorem C.2.4 (Bochner's Theorem). *Let $f : I \to X$ be measurable. Then f is integrable if and only if $\|f\| : I \to \mathbb{R}$ is integrable. In addition,*

$$\Big\|\int_I f(t)\,dt\Big\| \le \int_I \|f(t)\|\,dt,$$

for all integrable functions $f : I \to X$.

Proof. Assume that f is integrable, and consider a sequence $(f_n)_{n\in\mathbb{N}} \subset C_c(I,X)$ such that

$$\lim_{n\to\infty} \int_I \|f_n(t) - f(t)\|\,dt = 0.$$

We have

$$\|f\| \le \|f_n\| + \|f_n - f\|;$$

and so $\|f\|$ is integrable.

Conversely, suppose that f is measurable and that $\|f\|$ is integrable. Let $(g_n)_{n\in\mathbb{N}} \subset C_c(I,\mathbb{R})$ be a sequence such that $g_n \to \|f\|$ in $L^1(I)$ and a.a., and such that $|g_n| \le g$ a.a., for some $g \in L^1(I)$. Let $(f_n)_{n\in\mathbb{N}} \subset C_c(I,X)$ be a sequence such that $f_n \to f$ a.e. Finally, let

$$h_n = \frac{f_n |g_n|}{\|f_n\| + 1/n}.$$

It is clear that $h_n \in C_c(I,X)$, that $\|h_n\| \le g$ a.a. and that $h_n \to f$ in X a.a., as $n \to \infty$. It follows from the dominated convergence theorem that

$$\lim_{n\to\infty} \int \|h_n(t) - f(t)\|\,dt = 0;$$

and so f is integrable. Finally,

$$\Big\|\int f(t)\,dt\Big\| = \lim_{n\to\infty}\Big\|\int h_n(t)\,dt\Big\| \le \lim_{n\to\infty}\int \|h_n(t)\|\,dt \le \int \|f(t)\|\,dt,$$

where the last inequality follows from the dominated convergence theorem. This completes the proof. □

REMARK C.2.5. Theorem C.2.4 allows one to deal with vector valued integrable functions like one deals with real valued integrable functions. It suffices in general to apply the usual convergence theorems to $\|f\|$. For example, one can easily establish the following results.

(i) If $f: I \to X$ is integrable and $\varphi \in L^\infty(I)$, then $f\varphi : I \to X$ is integrable. In particular, if $f: I \to X$ is integrable and if $J \subset I$ is an open interval, then $f_{|J} : J \to X$ is integrable (take $\varphi = 1_J$).

(ii) (The dominated convergence theorem) Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of integrable functions $I \to X$, let $f: I \to X$ and let $g \in L^1(I)$. If

$$\|f_n(t)\| \le g(t) \text{ for a.a. } t \in I \text{ and all } n \in \mathbb{N},$$
$$\lim_{n\to\infty} f_n(t) = f(t) \text{ for a.a. } t \in I,$$

then f is integrable and $\displaystyle\int_I f(t)\,dt = \lim_{n\to\infty}\int_I f_n(t)\,dt$.

(iii) If Y is a Banach space, if $A \in \mathcal{L}(X,Y)$, and if $f: I \to X$ is integrable, then $Af : I \to Y$ is integrable and

$$\int_I Af(t)\,dt = A\Big(\int_I f(t)\,dt\Big).$$

In particular, if $X \hookrightarrow Y$ and if $f: I \to X$ is integrable, then the integral of f in the sense of X coincides with the integral of f in the sense of Y.

Finally, we have the following important geometric property of integrable functions.

PROPOSITION C.2.6. *Suppose $|I| < \infty$, let $K \subset X$ be a closed convex set, let $f: I \to X$ be integrable and let*

$$y = \frac{1}{|I|}\int_I f(t)\,dt.$$

If $f(t) \in K$ for a.a. $t \in I$, then $y \in K$.

PROOF. We argue by contradiction and we assume that $y \notin K$. It follows from Hahn-Banach's theorem (see [5, Theorem 1.7 p.7]) that there exists $x' \in X^\star$ and $\varepsilon > 0$ such that $\langle x', x\rangle_{X^\star,X} \le \langle x', y\rangle_{X^\star,X} - \varepsilon$, for all $x \in K$. In particular,

$$\langle x', f(t)\rangle_{X^\star,X} \le \langle x', y\rangle_{X^\star,X} - \varepsilon,$$

for a.a. $t \in I$. Integrating that above inequality and applying Remark C.2.5 (iii), we obtain

$$\begin{aligned}\langle x', y\rangle_{X^\star,X} &= \frac{1}{|I|}\langle x', \int_I f(t)\,dt\rangle_{X^\star,X} = \frac{1}{|I|}\int_I \langle x', f(t)\rangle_{X^\star,X}\,dt \\ &\le \langle x', y\rangle_{X^\star,X} - \varepsilon,\end{aligned}$$

which is a contradiction. Hence the result. □

C.3. The space $L^p(I,X)$

DEFINITION C.3.1. Let $p \in [1,\infty]$. One denotes by $L^p(I,X)$ the set of (classes of) measurable functions $f : I \to X$ such that the function $t \mapsto \|f(t)\|$ belongs to $L^p(I)$. For $f \in L^p(I,X)$, one defines

$$\|f\|_{L^p(I,X)} = \begin{cases} \Big\{ \int_I \|f(t)\|^p \, dt \Big\}^{\frac{1}{p}} & \text{if } p < \infty, \\ \operatorname{ess\,sup}_{t \in I} \|f(t)\| & \text{if } p = \infty. \end{cases}$$

When there is no risk of confusion, we denote $\|\cdot\|_{L^p(I,X)}$ by $\|\cdot\|_{L^p(I)}$ or $\|\cdot\|_{L^p}$ or $\|\cdot\|_p$. One denotes by $L^p_{\mathrm{loc}}(I,X)$ the set of $f : I \to X$ such that $f_{|J} \in L^p(J,X)$, for every bounded sub-interval J of I.

REMARK C.3.2. The space $L^p(I,X)$ enjoys most of the properties of the space $L^p(I) = L^p(I,\mathbb{R})$, by the same proofs or by applying the classical results to the function $t \mapsto \|f(t)\|$. In particular, one obtains easily the following results.

(i) $\|\cdot\|_{L^p(I,X)}$ is a norm on the space $L^p(I,X)$. $L^p(I,X)$ equipped with that norm is a Banach space. If $p < \infty$, then $C_0^\infty(I,X)$ is dense in $L^p(I,X)$. Indeed, it is not difficult to prove using the argument in second step of the proof of Bochner's Theorem C.2.4 that $C_{\mathrm{c}}(I,X)$ is dense in $L^p(I,X)$. Then the density of $C_0^\infty(I,X)$ follows by the classical procedure of convolution with a smoothing sequence. In particular, if Y is a Banach space such that $Y \hookrightarrow X$ with dense embedding, then $C_0^\infty(I,Y)$ is dense in $L^p(I,X)$ (since $C_0^\infty(I,Y)$ is dense in $C_{\mathrm{c}}(I,X)$ for the norm of $C_{\mathrm{b}}(I,X)$).

(ii) A measurable function $f : I \to X$ belongs to $L^p(I,X)$ if and only if there exists a function $g \in L^p(I)$ such that $\|f\| \le g$ a.a. on I.

(iii) If $f \in L^p(I,X)$ and $\varphi \in L^q(I)$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} \le 1$, then $\varphi f \in L^r(I,X)$ and

$$\|\varphi f\|_{L^r(I,X)} \le \|f\|_{L^p(I,X)} \|\varphi\|_{L^q(I)}.$$

In particular, if $f \in L^p(I,X)$ and if J is an open sub-interval of I, then $f_{|J} \in L^p(J,X)$.

(iv) If $f \in L^p(I,X)$ and $g \in L^q(I,X^\star)$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} \le 1$, then the function h defined by $h(t) = \langle g(t), f(t) \rangle_{X^\star,X}$ belongs to $L^r(I)$ and $\|h\|_{L^r} \le \|f\|_{L^p(I,X)} \, \|g\|_{L^q(I,X^\star)}$.

(v) If $f \in L^p(I,X) \cap L^q(I,X)$ with $p < q$, then $f \in L^r(I,X)$, for every $r \in [p,q]$, and

$$\|f\|_{L^r(I,X)} \le \|f\|^{\theta}_{L^p(I,X)} \|f\|^{1-\theta}_{L^q(I,X)},$$

where $\frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{q}$.

(vi) If I is bounded and $p \le q$, then $L^q(I,X) \hookrightarrow L^p(I,X)$ and

$$\|f\|_{L^p(I,X)} \le |I|^{\frac{q-p}{pq}} \|f\|_{L^q(I,X)},$$

for all $f \in L^q(I,X)$.

(vii) Suppose $f : I \to X$ is measurable. If $f \in L^p(J,X)$ for all $J \subset\subset I$ and if $\|f\|_{L^p(J,X)} \le C$ for some C independent of J, then $f \in L^p(I,X)$ and $\|f\|_{L^p(I,X)} \le C$.

(viii) If Y is a Banach space and if $A \in \mathcal{L}(X,Y)$, then for every $f \in L^p(I,X)$ we have $Af \in L^p(I,Y)$ and

$$\|Af\|_{L^p(I,Y)} \le \|A\|_{\mathcal{L}(X,Y)} \, \|f\|_{L^p(I,X)}.$$

In particular, if $X \hookrightarrow Y$ and if $f \in L^p(I,X)$, then $f \in L^p(I,Y)$.

(ix) (The dominated convergence theorem) Let $(f_n)_{n\in\mathbb{N}} \subset L^p(I, X)$ and let $g \in L^p(I)$. If $p < \infty$ and

$$\|f_n(t)\| \le g(t), \text{ for a.a. } t \in I \text{ and all } n \in \mathbb{N},$$
$$\lim_{n\to\infty} f_n(t) \text{ exists for a.a. } t \in I,$$

then $f := \lim_{n\to\infty} f_n \in L^p(I, X)$ and $\lim_{n\to\infty} f_n = f$ in $L^p(I, X)$.

(x) Let $(f_n)_{n\in\mathbb{N}} \subset L^p(I, X)$ and let $f \in L^p(I, X)$. If $f_n \to f$ in $L^p(I, X)$ as $n \to \infty$, then there exists $g \in L^p(I)$ and a subsequence $(n_k)_{n\in\mathbb{N}}$ such that $\|f_{n_k}(t)\| \le g(t)$ for a.a. $t \in I$ and for every $k \in \mathbb{N}$.

REMARK C.3.3. Duality theorems for the spaces $L^p(I, X)$ are much more difficult to obtain than for the spaces $L^p(I)$. However, if X is reflexive and if $1 < p < \infty$, then it known that $L^p(I, X)$ is reflexive and that $(L^p(I, X))^\star \approx L^{p'}(I, X^\star)$ (see [9, Chapter 13, Corollary 1 of Theorem 8, p. 252]. If $1 \le p < \infty$ and if X is reflexive or if $X^\star$ is separable, then $(L^p(I, X))^\star \approx L^{p'}(I, X^\star)$ (see [9]). Below are some special cases, in which such results are easily obtained.

(i) If X is a Hilbert space with the scalar product $(\cdot, \cdot)$, then it follows that $L^2(I, X)$ is a Hilbert space, for the scalar product

$$\langle\langle f, g\rangle\rangle = \int_I (f(t), g(t))\, dt, \text{ for } f, g \in L^2(I, X).$$

Therefore, $L^2(I, X)$ is reflexive and by the Riesz representation theorem, we see that $(L^2(I, X))^\star \approx L^2(I, X)$ (or $(L^2(I, X))^\star \approx L^2(I, X^\star)$ if one does not identify $X^\star$ with X).

(ii) Let Ω be an open subset of $\mathbb{R}^N$ and let $1 \le p < \infty$. It follows easily from Fubini's Theorem and a density argument that the operator T defined on $L^p(I, L^p(\Omega))$ by $Tu(t, x) = u(t)(x)$ is an isometry from $L^p(I, L^p(\Omega))$ onto $L^p(I \times \Omega)$; and so, $L^p(I, L^p(\Omega))$ is reflexive and $(L^p(I, L^p(\Omega)))^\star \approx L^{p'}(I, L^{p'}(\Omega))$ for every $1 < p < \infty$.

(iii) The results of (ii) above are not anymore valid for $p = \infty$. For example, let $I = \Omega = (0, 1)$ and consider the function $u : I \to L^\infty(\Omega)$ given by $u(t) = 1_{(0,t)}$, for $0 < t < 1$. Evidently $Tu \in L^\infty(I \times \Omega)$, but $u \notin L^\infty(I, L^\infty(\Omega))$. In fact, $u : I \to L^\infty(\Omega)$ is not even measurable, as follows from Remark C.1.7. (However, observe that $u \in C^{0,1/p}(\overline{I}, L^p(\Omega))$, for every $p \in [1, \infty)$.) It follows in particular that $(L^1(I, L^1(\Omega)))^\star \not\approx L^\infty(I, L^\infty(\Omega))$ since the linear form f on $L^1(I, L^1(\Omega))$ defined by

$$f(v) = \int_I \int_\Omega v(t)u(t)\, dx\, dt$$

is continuous but cannot be written as

$$f(v) = \int_I \int_\Omega v(t)z(t)\, dx\, dt$$

for some $z \in L^\infty(I, L^\infty(\Omega))$. Indeed, the definition of f makes sense, since if $v \in L^1(I, L^1(\Omega))$, then $vu \in L^1(I, L^1(\Omega))$; and on the other hand, if z would exist, we would obtain easily that $Tz = Tu$, hence $z = u$.

THEOREM C.3.4. *Let $1 \le p \le \infty$ and let $(f_n)_{n\in\mathbb{N}}$ be a bounded sequence in $L^p(I, X)$. If there exists $f : I \to X$ such that $f_n(t) \rightharpoonup f(t)$ in X as $n \to \infty$, for a.a. $t \in I$, then the following properties hold:*

(i) *$f \in L^p(I, X)$ and $\|f\|_{L^p(I,X)} \le \liminf_{n\to\infty} \|f_n\|_{L^p(I,X)}$;*

(ii) *if $p > 1$, then $\int_I f_n(t)\varphi(t)\, dt \rightharpoonup \int_I f(t)\varphi(t)\, dt$ as $n \to \infty$, for every $\varphi \in L^{p'}(I)$.*

PROOF. By Corollary C.1.5, f is measurable. If $p < \infty$, it follows from Fatou's lemma that

$$\int_I \liminf_{n\to\infty} \|f_n(t)\|^p \, dt \le \liminf_{n\to\infty} \int_I \|f_n(t)\|^p \, dt.$$

By weak lower semicontinuity of the norm, we have

$$\int_I \|f(t)\|^p \, dt \le \int_I \liminf_{n\to\infty} \|f_n(t)\|^p \, dt;$$

and so,

$$\int_I \|f(t)\|^p \, dt \le \liminf_{n\to\infty} \int_I \|f_n(t)\|^p \, dt,$$

from which (i) follows. The case $p = \infty$ follows from an obvious adaptation of this argument. Hence property (i).

We now prove (ii). Consider first $\varphi \in C_{\rm c}(I)$. Let $x' \in X^\star$ and set

$$h_n(t) = \langle x', f_n(t) - f(t)\rangle_{X^\star,X} \varphi(t),$$

for a.a. $t \in I$. It follows that $h_n(t) \to 0$ as $n \to \infty$ for a.a. $t \in I$ and that h_n is bounded in $L^p(I)$, as $n \to \infty$. Since h_n is supported in a compact interval, we deduce (see Lemma A.1.15 below) that $h_n \to 0$ in $L^1(I)$. In particular,

$$\langle x', \int_I f_n(t)\varphi(t)\, dt\rangle_{X^\star,X} \underset{n\to\infty}{\longrightarrow} \langle x', \int_I f(t)\varphi(t)\, dt\rangle_{X^\star,X},$$

from which property (ii) follows, since x' is arbitrary. In the general case $\varphi \in L^{p'}(I)$, let $(\varphi_\ell)_{\ell\ge 0} \subset C_{\rm c}(I)$ be such that $\varphi_\ell \to \varphi$ in $L^{p'}(I)$ as $\ell \to \infty$. Given $x' \in X^\star$, we have

$$\begin{aligned}\Big|\langle x', \int_I (f_n(t) - f(t))\varphi(t)\, dt\rangle_{X^\star,X}\Big| \\ \le \Big|\langle x', \int_I (f_n(t) - f(t))(\varphi(t) - \varphi_\ell(t)\, dt\rangle_{X^\star,X}\Big| \\ + \Big|\langle x', \int_I (f_n(t) - f(t))\varphi_\ell(t)\, dt\rangle_{X^\star,X}\Big|.\end{aligned}$$

Given any $\varepsilon > 0$, we estimate the first term on the right-hand side of the above inequality by $\|x'\|_{X^\star}(\|f_n\|_{L^p(I,X)}+\|f\|_{L^p(I,X)})\|\varphi_\ell-\varphi\|_{L^{p'}} \le \varepsilon/2$ if ℓ is large enough. Given such a ℓ, the second term on the right-hand side is smaller than $\varepsilon/2$ for n large enough by what precedes. Since $\varepsilon > 0$ and $x' \in X^\star$ are arbitrary, the result follows. □

LEMMA C.3.5. *Let $(f_n)_{n\in\mathbb{N}} \subset L^p(I,X)$ and $f \in L^p(I,X)$, where $1 \le p \le \infty$. If $f_n \rightharpoonup f$ in $L^p(I,X)$ as $n \to \infty$, then*

$$\int_I f_n(t)\varphi(t)\, dt \rightharpoonup \int_I f(t)\varphi(t)\, dt,$$

as $n \to \infty$ for every $\varphi \in C_{\rm c}(I)$.

PROOF. Without loss of generality, we may assume that $f = 0$. Consider $\varphi \in C_0^\infty(I)$ and $x' \in X^\star$ and define the linear functional F on $L^p(I,X)$ by

$$F(g) = \langle x, \int_I g(t)\varphi(t)\, dt\rangle_{X^\star,X},$$

for every $g \in L^p(I,X)$. It follows from Remark C.3.2 (iii) that F is continuous. Therefore, $F \in (L^p(I,X))^\star$, which implies that $F(f_n) \to F(0)$ as $n \to \infty$. Since x' is arbitrary, this implies the result. □

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