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STRUCTURAL PROPERTIES OF THE SET OF GLOBAL SOLUTIONS OF THE NONLINEAR HEAT EQUATION

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Abstract. In this paper we consider the nonlinear heat equation on a bounded domain with Dirichlet boundary conditions, and we study the structure of the sets $\mathcal{G}$ and $\mathcal{B}$ of initial values in C_0 producing global and blowing up solutions, respectively. We discuss in particular the convexity and the connectedness of these sets. We obtain both positive and negative answers, and we state a number of open problems.

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1 Introduction

Let Ω be a bounded, smooth domain of $\mathbb{R}^N$ and $f \in C^1(\mathbb{R},\mathbb{R})$. Let $C_0(\Omega)$ be the Banach space of continous functions on $\overline{\Omega}$ that vanish on $\partial\Omega$, with the sup norm. Given an initial value $u_0 \in C_0(\Omega)$, we consider the nonlinear heat equation

$$\begin{cases} u_t - \Delta u = f(u), \\ u_{|\partial\Omega} = 0, \\ u(0) = u_0. \end{cases} \tag{1.1}$$

It is well known that the above initial value problem is locally well-posed. More precisely, there exists a maximal time $0 < T_{u_0} \le \infty$ and a function $u \in C([0,T_{u_0}), C_0(\Omega)) \cap C((0,T_{u_0}), C^2(\overline{\Omega})) \cap C^1((0,T_{u_0}), C_0(\Omega))$ which is a classical solution of (1.1) on $(0,T_{u_0})$ and such that $u(0) = u_0$. Moreover, u is the unique solution of (1.1) in $L^\infty((0,T)\times\Omega)$ for any $0 < T < T_{u_0}$

$$u(t) = e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta} f(u(s))\, ds,$$

where $(e^{t\Delta})_{t\ge 0}$ is the heat semigroup with Dirichlet boundary conditions. Furthermore, there is the blowup alternative: either $T_{u_0} = \infty$ (i.e. u is a global solution) or else $T_{u_0} < \infty$ and $\lim_{t\uparrow T_{u_0}} \|u(t)\|_{L^\infty} = \infty$ (i.e. u blows up in finite time). In addition, if v is a supersolution of (1.1), i.e $v_t - \Delta v \ge f(v)$, $v_{|\partial\Omega} \ge 0$ and $v(0) \ge u_0$, then $v(t) \ge u(t)$ as long as both u and v are defined. The notion of subsolution is defined with reversed inequalities, yielding the analogous conclusion. (See e.g [18].)

We denote by $\mathcal{G}$ the set of initial values u_0 producing global solutions of (1.1) and by $\mathcal{B}$ the set of initial values u_0 producing blowing up solutions of (1.1), i.e.

$$\begin{aligned} \mathcal{G} &= \{u_0 \in C_0(\Omega);\ T_{u_0} = \infty\}, \\ \mathcal{B} &= \{u_0 \in C_0(\Omega);\ T_{u_0} < \infty\}. \end{aligned}$$

In particular, $\mathcal{G} \cup \mathcal{B} = C_0(\Omega)$ and it is immediate that both the sets $\mathcal{G}$ and $\mathcal{B}$ are invariant under the flow generated by (1.1). This paper is concerned with the structure of the sets $\mathcal{G}$ and $\mathcal{B}$.

It can happen, depending on the nonlinearity f, that one of the two sets $\mathcal{G}$ or $\mathcal{B}$ is empty. For example, if $sf(s) \le C(1+s^2)$ for all u, then it is well known that $\mathcal{G} = C_0(\Omega)$. (See e.g. [7].) On the other hand, if $f(s) \ge \lambda_1 s + h(s)$ where $\lambda_1 > 0$ is the first eigenvalue of $-\Delta$ on Ω with Dirichlet boundary conditions and h is a convex, positive function such that $\int_0^\infty ds/h(s) < \infty$, then it is well known that $\mathcal{G} = \emptyset$. (See [13].)

On the other hand, there are many nonlinear functions f for which both $\mathcal{G}$ and $\mathcal{B}$ are nonempty. For example, if the problem (1.1) has a (classical) stationary solution Ψ, i.e.

$$\begin{cases} -\Delta\Psi = f(\Psi), \\ \Psi_{|\partial\Omega} = 0, \end{cases} \tag{1.2}$$

then $\Psi \in \mathcal{G}$. In particular, if $f(0) = 0$ then $\Psi = 0 \in \mathcal{G}$. Note that if Ψ is a stationary solution of (1.1) for a given nonlinearity f, and if one modifies $f(u)$ for $|u| > \|\Psi\|_{L^\infty}$, then

Ψ is also a stationary solution of (1.1) with the modified nonlinearity. On the other hand, there are conditions on the function f, which concern $f(s)$ only for $|s|$ large, which imply that $\mathcal{B}$ is nonempty. In particular, if $f(s) \geq \eta s^{1+\varepsilon}$ for some $\eta, \varepsilon > 0$ and all sufficiently large $s > 0$, then $\mathcal{B} \neq \emptyset$. (See e.g. [7, 18].) Thus we see that $\mathcal{G}$ and $\mathcal{B}$ can both be nonempty, a typical example being $f(s) = |s|^\alpha s$ with $\alpha > 0$.

The first property of $\mathcal{G}$ we discuss is convexity. It is well known that if f is a convex function, then the set $\mathcal{G}$ is convex. To see this, suppose $u_0, v_0 \in \mathcal{G}$ and let $u(t)$ and $v(t)$ be the corresponding solutions of (1.1). Given $0 < \lambda < 1$, the function $(1-\lambda)u(t) + \lambda v(t)$ has the initial value $(1-\lambda)u_0 + \lambda v_0$ and is a (global) supersolution of the equation. Moreover, since f is bounded from below by a linear function, it is easy to construct a global subsolution. Thus $(1-\lambda)u_0 + \lambda v_0 \in \mathcal{G}$. Similarly, if f is a concave function, then the set $\mathcal{G}$ is convex. This raises the question of whether the set $\mathcal{G}$ is convex for more general functions f. If for example f is concave for $s \geq 0$ and convex for $s \leq 0$, then every solution is global so $\mathcal{G} = C_0(\Omega)$ is convex. If on the other hand f is convex for $s \geq 0$ and concave for $s \leq 0$, for example $f(s) = |s|^\alpha s$, the situation is less clear. It is proved in [5, Corollary 1.5] that when Ω is the unit ball of $\mathbb{R}^N$ and $f(s) = |s|^\alpha s$ with $0 < \alpha < 4/(N-2)^+$, the set $\mathcal{G}$ is not convex. The following theorem extends this property to arbitrary domains and more general nonlinearities.

Theorem 1.1. *Suppose $f \in C^1(\mathbb{R}, \mathbb{R})$ satisfies*

$$s^2 f'(s) \geq (1+\varepsilon) s f(s) \text{ for all } s \in \mathbb{R} \tag{1.3}$$

$$s f(s) > 0 \text{ for } |s| \text{ large,} \tag{1.4}$$

$$|f(s)| \leq C(1 + |s|^\beta) \text{ for all } s \in \mathbb{R}, \tag{1.5}$$

where $\varepsilon > 0$ and $1 \leq \beta < (N+2)/(N-2)^+$. Then the set $\mathcal{G}$ is not convex.

When $f(0) = 0$, $u(t) \equiv 0$ is a solution of (1.1) and one can ask if $\mathcal{G}$ has the weaker property of being star-shaped around 0. It is proved in [4, Theorem 1.1] that when Ω is the unit ball of $\mathbb{R}^N$, $N \geq 3$ and $f(s) = |s|^\alpha s$ with $0 < \alpha < 4/(N-2)$ sufficiently close to $4/(N-2)$, then $\mathcal{G}$ is not even star-shaped around 0. More precisely, if Ψ is any sign changing, radially symmetric stationary solution of (1.1), then there exists $\varepsilon > 0$ such that $\lambda\Psi \in \mathcal{B}$ for all $0 < |\lambda - 1| < \varepsilon$. This last property also holds when Ω is the unit ball of $\mathbb{R}^3$ and $f(s) = |s|^\alpha s$ with $\alpha > 0$ sufficiently small. See [6] and Remark 2.6 below.

The next property we consider is connectedness. If Ψ is a stationary solution of (1.1), then the set of initial values u_0 giving rise to global solutions of (1.1) that converge to Ψ as $t \to \infty$ is clearly path-connected. One can ask if the entire set $\mathcal{G}$ is connected. This question seems delicate since $\mathcal{G}$ is made up of all stationary solutions of (1.1), all global solutions that are attracted by the stationary solutions, and all global, unbounded solutions. (See [10].) Even in the case $f(s) = |s|^\alpha s$ where α is Sobolev subcritical, he question seems to be open. For radially symmetric solutions in a ball, we can use the results in [1] to give the following partial answer to this question.

Proposition 1.2. *Suppose Ω is the unit ball of $\mathbb{R}^N$ and $f(s) = |s|^\alpha s$ with $0 < \alpha < 4/(N-2)^+$. Set*

$$\mathcal{G}_{\mathrm{rad}} = \{u_0 \in \mathcal{G}; u_0 \text{ is radially symmetric}\}.$$

It follows that the set $\mathcal{G}_{\mathrm{rad}}$ is connected.

The set $\mathcal{B}$, being the complement of $\mathcal{G}$, also has a nontrivial structure. However, as stated below, $\mathcal{B}$ is path-connected under very general hypotheses on f.

Proposition 1.3. *If f satisfies* (1.3)*, then the set $\mathcal{B}$ is path-connected.*

The rest of the paper is organized as follows. We prove Theorem 1.1 in the next section. As part of the proof, we obtain certain results interesting in their own right, which give some insight into the stability properties of sign changing stationary solutions. See Remark 2.4. Propositions 1.2 and 1.3 are proved in Section 3, and Section 4 is devoted to further remarks and open problems.

2 Proof of Theorem 1.1

The proof of Theorem 1.1 relies on a local analysis of the instability of sign changing stationary solutions. The first observation is that, under the hypotheses of Theorem 1.1, there does indeed exist a sign changing stationary solution. This is a consequence of results in [2, 3]. More precisely, we apply Theorem 4.2 in the survey article [15]. The following elementary lemma, whose proof is left to the reader, shows that the hypotheses of Theorem 4.2 in [15] are implied by our assumptions on f.

Lemma 2.1. *If $f \in C^1(\mathbb{R},\mathbb{R})$ satisfies* (1.3) *and*

$$F(s) = \int_0^s f(\sigma)\,d\sigma, \tag{2.1}$$

for $s \in \mathbb{R}$, then the following properties hold.

(i) $f(0) = 0$ *and* $f'(0) \le 0$.

(ii) $sf(s) \ge (2+\varepsilon)F(s)$ *for all* $s \in \mathbb{R}$.

(iii) *Either $f(s) \le 0$ for all $s \ge 0$, or else there exists $c, d > 0$ such that $F(s) \ge cs^{2+\varepsilon} - d$ for all $s \ge 0$.*

(iv) *Either $f(s) \ge 0$ for all $s \le 0$, or else there exists $c, d > 0$ such that $F(s) \ge c|s|^{2+\varepsilon} - d$ for all $s \le 0$.*

(v) *If, in addition, f satisfies* (1.4)*, then $F(s) > 0$ and $f'(s) > 0$ for $|s|$ large.*

Next, let Ψ be a nontrivial solution of (1.2) and consider the linearized operator L on $L^2(\Omega)$ defined by

$$\begin{cases} D(L) = H^2(\Omega) \cap H_0^1(\Omega), \\ Lu = -\Delta u - f'(\Psi)u, \qquad u \in D(L). \end{cases} \tag{2.2}$$

It is well known and easy to see by taking Ψ as a test function in the Rayleigh quotient that the lowest eigenvalue of L is negative.

The linearized operator L is used in [4, Proposition B.1] (see also Theorem 10 in [11]) to prove the following result.

Proposition 2.2 ([4]). *Suppose f satisfies (1.3)–(1.5), and in particular $f(0) = 0$. Let $\Psi \in C_0(\Omega)$ be a sign changing stationary solution of (1.1), i.e. a sign changing solution of (1.2). If $u_0 \in C_0(\Omega)$, $u_0 \not\equiv \Psi$ satisfies either $u_0 \geq \Psi$ or $u_0 \leq \Psi$, then the corresponding solution of (1.1) blows up in finite time.*

We apply Proposition 2.2 in the proof of the following theorem, which is the key step to show Theorem 1.1.

Theorem 2.3. *Suppose $f \in C^1(\mathbb{R}, \mathbb{R})$ satisfies (1.3)–(1.5). Let $\Psi \in C_0(\Omega)$ be a sign changing solution of (1.2). Let Φ be a positive eigenvector of the self-adjoint operator L given by (2.2), corresponding to the least eigenvalue. Let $h \in C^1(\mathbb{R}, C_0(\Omega))$ and assume that*

$$h(0) = \Psi \quad \textit{and} \quad \int_\Omega h'(0)\Phi \neq 0.$$

It follows that there exists $\varepsilon > 0$ such that if $0 < |\lambda| \leq \varepsilon$, then the solution of (1.1) with the initial value $u_0 = h(\lambda)$ blows up in finite time.

Proof. We treat the case

$$\int_\Omega h'(0)\Phi > 0, \tag{2.3}$$

the other case being entirely similar. We linearize the equation (1.1) about Ψ. More precisely, given $\lambda \in \mathbb{R}\setminus\{0\}$, we denote by u^λ the solution of (1.1) with the initial condition

$$u^\lambda(0) = h(\lambda),$$

and we set

$$\lambda z^\lambda(t) = u^\lambda(t) - \Psi, \tag{2.4}$$

so that

$$\begin{cases} z_t^\lambda = \Delta z^\lambda + \dfrac{1}{\lambda}[f(\lambda z^\lambda + \Psi) - f(\Psi)], \\ z^\lambda(0) = (h(\lambda) - \Psi)/\lambda. \end{cases}$$

Note that $z(t) = e^{-tL}h'(0)$, where L is the linearized operator defined by (2.2), satisfies

$$\begin{cases} z_t - \Delta z - f'(\Psi)z = 0, \\ z(0) = h'(0). \end{cases} \tag{2.5}$$

It follows from continuous dependence that given any $T > 0$, u^λ is well defined on $[0, T]$ for $|\lambda|$ small enough and that for any $0 < \delta < T < \infty$,

$$z^\lambda \to z, \tag{2.6}$$

in $C([\delta, T], C^1(\overline{\Omega}))$ as $\lambda \to 0$. Since the equation (2.5) is linear and autonomous, the solution z can be expanded in terms of the eigenfunctions of L, so that the sign of z for large time is determined by the sign of the left-hand side of (2.3). (See e.g. Lemma A.1 in [4].) Therefore, it follows from (2.3) that there exists $t_0 > 0$ such that

$$z(t_0) \geq c\mathrm{d}_\Omega, \tag{2.7}$$

where $c > 0$ and d_Ω is the distance to $\partial\Omega$. By applying (2.6), (2.7) and (2.4), we conclude that there exist $\overline{\lambda} > 0$ such that $u^\lambda(t_0) < \Psi$ for $-\overline{\lambda} < \lambda < 0$ and $u^\lambda(t_0) > \Psi$ for $0 < \lambda < \overline{\lambda}$. Since Ψ is sign changing, we deduce in both cases from Proposition 2.2 that u^λ blows up in finite time. This completes the proof. □

Remark 2.4. Theorem 2.3 has an interesting interpretation in terms of the local stable manifold $\mathcal{M}$ of Ψ for the flow determined by (1.1). It is well-known that the tangent space $T_\Psi\mathcal{M}$ of $\mathcal{M}$ at Ψ is generated by the eigenfunctions of L with positive eigenvalues. Thus if $u_0 = \Psi + \varepsilon\eta$, where η has a nontrivial component in the direction of Φ, then for all sufficiently small $\varepsilon > 0$, u_0 cannot be on the stable manifold $\mathcal{M}$. Theorem 2.3 gives the stronger result that if $\varepsilon > 0$ is sufficiently small, then the resulting solution of (1.1) blows up in finite time.

Letting $h(\lambda) = (1-\lambda)\Psi + \lambda v$ in Theorem 2.3, we obtain the following corollary.

Corollary 2.5. *Under the same hypotheses of Theorem* 2.3*, let* $v \in C_0(\Omega)$ *be such that*

$$\int_\Omega (\Psi - v)\Phi \neq 0. \tag{2.8}$$

Then there exists $\varepsilon > 0$ *such that if* $0 < |\lambda| \leq \varepsilon$*, then the solution of* (1.1) *with the initial value* $u_0 = (1-\lambda)\Psi + \lambda v$ *blows up in finite time.*

Remark 2.6. If (2.8) holds with $v = 0$, Corollary 2.5 implies immediately that $\mathcal{G}$ is not star-shaped around 0. This result is a key step of the proofs in [4] and [6].

We are now ready to prove Theorem 1.1. By Lemma 2.1, $f(0) = 0$ and $f'(0) \leq 0$, so that 0 is an asymptotically stable solution of (1.1). (See [12].) It is easy to see that $v = \varepsilon\Phi \in \mathcal{G}$ and (2.8) holds for all sufficiently small $\varepsilon > 0$. It follows from Corollary 2.5 that $\mathcal{G}$ is not convex.

3 Connectedness

In this section, we prove Propositions 1.2 and 1.3.

Proof of Proposition 1.2. We note that 0 is a stationary solution of (1.1) and we denote by $\mathcal{G}_0$ the set of initial values in $\mathcal{G}$ for which the resulting solution of (1.1) converges uniformly to 0 as $t \to \infty$, i.e.

$$\mathcal{G}_0 = \{u_0 \in \mathcal{G};\ u(t) \underset{t\to\infty}{\longrightarrow} 0\}. \tag{3.1}$$

It is well known that 0 is an asymptotically stable stationary solution of (1.1), so that $\mathcal{G}_0$ is a nonempty open subset of $C_0(\Omega)$. Moreover, given any $u_0 \in \mathcal{G}$, the corresponding solution $u(t)$ of (1.1) is uniformly bounded as $t \to \infty$ (see [8]), so it has an ω-limit set $\omega(u_0)$ made up of stationary solutions of (1.1) (see [10]).

To show that $\mathcal{G}_{\text{rad}}$ is connected, we consider a connected component $\mathcal{H}$. If $u_0 \in \mathcal{H}$ and $u(t)$ is the corresponding solution of (1.1), it follows by continuity that $u(t) \in \mathcal{H}$ for all

$t \geq 0$. Since $\mathcal{H}$ is closed, we deduce that $\omega(u_0) \subset \mathcal{H}$. On the other hand, it is proved in [1] that every radially symmetric stationary solution of (1.1) belongs to the closure of $\mathcal{G}_0$. (See the end of Section 1 in [1].) Thus we see that $\mathcal{H} \cap \overline{\mathcal{G}_0} \neq \emptyset$. Since $\mathcal{H}$ is open, it follows that $\mathcal{H} \cap \mathcal{G}_0 \neq \emptyset$. Given $u_0 \in \mathcal{H} \cap \mathcal{G}_0$, we have $\{0\} = \omega(u_0) \subset \mathcal{H}$, so that $0 \in \mathcal{H}$. This shows that there is only one connected component of $\mathcal{G}_{\mathrm{rad}}$. □

Proof of Proposition 1.3. Let the energy functional E be defined by

$$E(w) = \frac{1}{2} \int_\Omega |\nabla w|^2 - \int_\Omega F(w), \tag{3.2}$$

for all $w \in C_0(\Omega) \cap H_0^1(\Omega)$, where F is given by (2.1). Recall that if the initial value $u_0 \in C_0(\Omega) \cap H_0^1(\Omega)$ satisfies $E(u_0) < 0$, then $u_0 \in \mathcal{B}$. (See [14].)

Suppose first $f(s) \leq 0$ for all $s \geq 0$ and $f(s) \geq 0$ for all $s \leq 0$. It follows easily that, given any $u_0 \in C_0(\Omega)$, $e^{t\Delta}|u_0|$ is a supersolution of (1.1) and, similarly, $-e^{t\Delta}|u_0|$ is a subsolution. Thus $u_0 \in \mathcal{G}$, so that $\mathcal{B}$ is empty.

Suppose next that $sf(s)$ takes positive values for both positive and negative s. It follows from Lemma 2.1 that there exists $c, d > 0$ such that

$$F(s) \geq c|s|^{2+\varepsilon} - d, \tag{3.3}$$

for all $s \in \mathbb{R}$.

Let ω_1, ω_2 be two nonempty, disjoint open subsets of Ω. Let $v_1, v_2 \in C_c^\infty(\Omega)$ satisfy $\operatorname{supp} v_j \subset \omega_j$, $v_j \geq 0$ and $v_j \not\equiv 0$ for $j = 1, 2$. We claim that there exists $\lambda > 0$ such that

$$E(\theta\lambda v_1 - (1-\theta)\lambda v_2) < 0, \tag{3.4}$$

for all $0 \leq \theta \leq 1$. Indeed, since v_1 and v_2 have disjoint support,

$$E(\theta\lambda v_1 - (1-\theta)\lambda v_2) = E(\theta\lambda v_1) + E(-(1-\theta)\lambda v_2).$$

It follows from (3.3) that

$$\begin{aligned} E(\theta\lambda v_1) &= \frac{\theta^2\lambda^2}{2} \int_\Omega |\nabla v_1|^2 - \int_\Omega F(\theta\lambda v_1) \\ &\leq \frac{\theta^2\lambda^2}{2} \int_\Omega |\nabla v_1|^2 + d|\omega_1| - c\lambda^{2+\varepsilon}\theta^{2+\varepsilon} \int_\Omega |v_1|^{2+\varepsilon}. \end{aligned}$$

Estimating $E(-(1-\theta)\lambda v_2)$ similarly, we deduce that

$$\begin{aligned} E(\theta\lambda v_1 - (1-\theta)\lambda v_2) \leq d|\Omega| + \lambda^2 \Big[&\frac{\theta^2}{2} \int_\Omega |\nabla v_1|^2 + \frac{(1-\theta)^2}{2} \int_\Omega |\nabla v_2|^2 \\ &- c\lambda^\varepsilon \theta^{2+\varepsilon} \int_\Omega |v_1|^{2+\varepsilon} - c\lambda^\varepsilon (1-\theta)^{2+\varepsilon} \int_\Omega |v_2|^{2+\varepsilon} \Big]; \end{aligned}$$

and so,

$$\begin{aligned} E(\theta\lambda v_1 - (1-\theta)\lambda v_2) \leq d|\Omega| + \lambda^2 \Big[&\frac{1}{2} \max\Big\{ \int_\Omega |\nabla v_1|^2, \int_\Omega |\nabla v_2|^2 \Big\} \\ &- 2^{-1-\varepsilon} c\lambda^\varepsilon \min\Big\{ \int_\Omega |v_1|^{2+\varepsilon}, \int_\Omega |v_2|^{2+\varepsilon} \Big\} \Big]. \end{aligned}$$

The claim (3.4) immediately follows by choosing $\lambda > 0$ sufficiently large. Setting

$$w_1 = \lambda v_1, \quad w_2 = \lambda v_2,$$

we see that w_1 and $-w_2$ are connected in $\mathcal{B}$ by the path $\theta \mapsto \theta w_1 - (1-\theta) w_2$.

Let now $u_0 \in \mathcal{B}$ and u the corresponding solution of (1.1), so that $\|u(t)\|_{L^\infty} \to \infty$ as $t \uparrow T_{u_0}$. It follows that $\limsup \|u(t)^+\|_{L^\infty} = \infty$ or $\limsup \|u(t)^-\|_{L^\infty} = \infty$ as $t \uparrow T_{u_0}$. Suppose first

$$\limsup_{t \uparrow T_{u_0}} \|u(t)^+\|_{L^\infty} = \infty. \tag{3.5}$$

Let $\widetilde{u_0} \in C_0(\Omega)$, $\widetilde{u_0} \geq u_0$, and let $\widetilde{u}$ be the corresponding solution of (1.1). We have $\widetilde{u}(t) \geq u(t)$ for all $0 \leq t < \min\{T_{u_0}, T_{\widetilde{u_0}}\}$, and we deduce from (3.5) that $\widetilde{u_0} \in \mathcal{B}$. In particular, $u_0 \leq u_0^+ - \theta u_0^- \in \mathcal{B}$ for all $0 \leq \theta \leq 1$, so that u_0 is connected in $\mathcal{B}$ to u_0^+. Similarly, u_0^+ connects in $\mathcal{B}$ to $u_0^+ + w_1$. On the other hand, since $\theta u_0^+ + w_1 \geq w_1 \geq 0$ for $\theta \geq 0$, we see that $\theta u_0^+ + w_1 \in \mathcal{B}$. Therefore, $u_0^+ + w_1$ connects in $\mathcal{B}$ to w_1. Thus we see that u_0 is connected to w_1. One shows similarly that if $\limsup \|u(t)^-\|_{L^\infty} = \infty$ as $t \uparrow T_{u_0}$, then u_0 connects to $-w_2$. Since w_1 is connected to $-w_2$, we conclude that $\mathcal{B}$ is path-connected.

Suppose now that $f(s) \geq 0$ for all $s \leq 0$ and $F(s) \geq c|s|^{2+\varepsilon} - d$ for all $s \leq 0$. Let $u_0 \in C_0(\Omega)$ and let u be the corresponding solution of (1.1). It follows that $-e^{t\Delta} u_0^-$ is a subsolution of (1.1), so that $u(t) \geq -e^{t\Delta} u_0^-$ for all $0 \leq t < T_{u_0}$. Thus we see that if $u_0 \in \mathcal{B}$, $u^-(t)$ is bounded, so that (3.5) holds. Thus we deduce as in the preceding case that u_0 connects in $\mathcal{B}$ to w_1, and it follows that $\mathcal{B}$ is path-connected.

Finally, in the last case when $f(s) \leq 0$ for all $s \geq 0$ and $F(s) \geq c|s|^{2+\varepsilon} - d$ for all $s \leq 0$, one shows as in the preceding case that every element of $\mathcal{B}$ connects to $-w_2$, so that $\mathcal{B}$ is path-connected. □

4 Further remarks and open problems

We begin with some comments on the convexity results of Section 1.

Theorem 1.1 shows that the set $\mathcal{G}$ is not convex under the assumptions (1.3)–(1.5). Note that in particular $f(0) = 0$ and $f'(0) \leq 0$ (see Lemma 2.1), so that the set $\mathcal{G}_0$ defined by (3.1) is a nonempty open subset of $C_0(\Omega)$. It is natural to ask the following question.

Open problem 4.1. *Assume* (1.3)–(1.5). *Is the set* $\mathcal{G}_0$ *defined by* (3.1) *convex?*

We recall that $\mathcal{G}_0$ is the interior of $\mathcal{G}$. (See [11, Theorem 8].) If it were known that $\mathcal{G}$ is the closure of $\mathcal{G}_0$, the answer to the open question 4.1 would be no. However, as far as we are aware, it is an open problem as to whether or not $\mathcal{G}$ is the closure of $\mathcal{G}_0$, even in the case where f is a pure power nonlinearity. Nonetheless, we have the following partial result.

Theorem 4.2 ([5])**.** *Suppose* Ω *is the unit ball of* $\mathbb{R}^N$ *and* $f(u) = |u|^\alpha u$ *with* $0 < \alpha < 4/(N-2)^+$. *It follows that the set* $\mathcal{G}_0$ *defined by* (3.1) *is not convex.*

Next, we observe that Theorem 1.1 only applies to Sobolev subcritical nonlinearities f. This raises the following question.

Open problem 4.3. *Consider the equation* (1.1) *with* $N \geq 3$ *and* $f(u) = |u|^\alpha u$ *with* $\alpha \geq 4/(N-2)$. *Is the set* $\mathcal{G}$ *convex?*

One can also ask if the set $\mathcal{B}$, which is the complement of $\mathcal{G}$ in $C_0(\Omega)$, is convex. This is of course the case if $\mathcal{B} = C_0(\Omega)$. However, it seems that under fairly general assumptions the set $\mathcal{B}$ is not convex. For example, if $f(s) = |s|^\alpha s$ with $\alpha > 0$, then 0 is a solution of (1.1), so that $0 \in \mathcal{G}$. On the other hand, if $\varphi \in C_c^\infty(\Omega)$, $\varphi \not\equiv 0$, then $E(\lambda\varphi) < 0$ for $|\lambda|$ large, where the energy E is defined by (3.2). It follows that $\lambda\varphi \in \mathcal{B}$ if $|\lambda|$ is large, and since $0 \in \mathcal{G}$, we see that $\mathcal{B}$ is not convex. The same argument can easily be adapted to the case when f satisfies (1.3) and if there exists $s > 0$ such that $f(s) > 0$ and $f(-s) < 0$. However, this leaves open the following question.

Open problem 4.4. *Is there a nonlinearity* f *such that* $\mathcal{B} \neq C_0(\Omega)$, $\mathcal{B} \neq \emptyset$ *and* $\mathcal{B}$ *is convex?*

We now comment the connectedness issue. Proposition 1.2 has several limitations. First, it says that the set $\mathcal{G}_{\mathrm{rad}}$ is connected, not path-connected. Next, it applies only when Ω is a ball and, finally, only when α is sub-critical. This suggests the following questions.

Open problem 4.5. *Under the assumptions of Proposition* 1.2, *is the set* $\mathcal{G}_{\mathrm{rad}}$ *path-connected?*

Open problem 4.6. *Let* Ω *be a smooth, bounded domain of* $\mathbb{R}^N$ *and* $f(s) = |s|^\alpha s$ *with* $s > 0$. *Is the set* $\mathcal{G}$ *connected? Is the set* $\mathcal{G}$ *path-connected?*

Finally, we make some comments on natural questions which we did not address in this paper. An interesting property that one might consider is the closedness of $\mathcal{G}$ and $\mathcal{B}$. In the case of the model nonlinearity $f(s) = |s|^\alpha s$, it is known that $\mathcal{G}$ is closed (and so $\mathcal{B}$ is open) in the subcritical case $\alpha < 4/(N-2)^+$. This follows from estimates of global solutions of (1.1) that depend only on the L^∞ norm of the initial value, see [17]. On the other hand, in the supercritical case $N \geq 3$ and $\alpha > 4/(N-2)$ and when Ω is convex, then $\mathcal{G}$ is not closed. More precisely, given any initial value $u_0 \in C_0(\Omega)$, $u_0 \geq 0$, $u_0 \not\equiv 0$, there exists $\lambda^\star > 0$ such that $\lambda u_0 \in \mathcal{G}_0$ for all $0 \leq \lambda < \lambda^\star$ and $\lambda^\star u_0 \in \mathcal{B}$. See [9, Theorem B]. See also [16] for an earlier result in the radial case. It follows from this result that $\mathcal{G} \cap \{u_0 \geq 0\} = \mathcal{G}_0 \cap \{u_0 \geq 0\}$, so that $\mathcal{G} \cap \{u_0 \geq 0\}$ is open in $\{u \in C_0(\Omega);\, u \geq 0\}$.

Open problem 4.7. *Assume* $N \geq 3$ *and* $f(s) = |s|^\alpha s$ *with* $\alpha > 4/(N-2)$. *Is the set* $\mathcal{G}$ *open? Does* $\mathcal{G} = \mathcal{G}_0$*?*

Concerning the critical case, we ask the following question.

Open problem 4.8. *Assume* $N \geq 3$ *and* $f(s) = |s|^{\frac{4}{N-2}} s$. *Is the set* $\mathcal{G}$ *closed? Is the set* $\mathcal{G}$ *open? The problem seems to be open even for radially symmetric solutions in a ball.*

Another aspect of the sets $\mathcal{G}$ and $\mathcal{B}$ that one might consider is their boundedness. Of course, when $\mathcal{B}$ is nonempty, it is unbounded (by the blowup alternative) since it is invariant by the flow of (1.1). In the model case $f(s) = |s|^\alpha s$, the set $\mathcal{G}$ is also unbounded. In fact, there exist arbitrarily large, nonnegative initial values in $\mathcal{G}_0$, where $\mathcal{G}_0$ is defined by (3.1), see [18, Remark 19.12]. On the other hand, even though $\mathcal{G}$ is not bounded in the L^∞ norm, there is some form of boundedness of all nonnegative initial values in $\mathcal{G}$. Indeed, it is proved in [19, Theorem 1] (see also [20]) that there exists a constant M such that if $u_0 \in \mathcal{G}$ and $u_0 \geq 0$, then $\sup_{t>0} t^{\frac{1}{\alpha}} \|e^{t\Delta} u_0\|_{L^\infty} \leq M$. Thus we can ask the following question.

Open problem 4.9. *Let Ω be a smooth, bounded domain of $\mathbb{R}^N$ and $f(s) = |s|^\alpha s$ with $s > 0$. Is there a constant M such that $\sup_{t>0} t^{\frac{1}{\alpha}} \|e^{t\Delta} u_0\|_{L^\infty} \leq M$ for all $u_0 \in \mathcal{G}$?*

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