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On the initial value problem for a linear model of well-reservoir coupling

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Abstract. We consider a linearized model of well-reservoir coupling for a monophasic flow, with boundary conditions corresponding to oil production at either a given pressure or at a given flow rate. By using the semigroup theory, we show that the initial value problem is well-posed and we study the asymptotic behavior of the solutions.

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1. Introduction

Porous medium flow is usually modelled through Darcy's law, which relates fluid velocity u and pressure p. For a monophasic flow, it is given by the equation

$$u = -\frac{k}{\mu}\nabla(p - gz), \tag{1.1}$$

where k is the permeability tensor, μ is the fluid viscosity, g is the gravity constant and z is the vertical coordinate. Similar relation holds for multiphase flow.

An important field of application of flow in porous media is in oil recovery. Oil exploitation is performed through the drilling of wells, which can be considered as long and thin tubes, into the reservoir. Therefore, oil reservoir models usually treat the wells as Dirac mesures over an interval length. Moreover, the well-reservoir coupling is modelled under some quite simplified assumptions, in particular by considering that the well pressure distribution is hydrostatic.

It is recognized, however, that in many situations, such as those related to non-vertical wells, these simplifications do not take into account some relevant physical aspects of the coupled flow. In order to overcome this difficulty, more complex models have been proposed (see for example [3, 1]). For a monophasic flow, A. Bourgeat [3] has considered the coupling of a non-linear hyperbolic system for the well to a non-linear parabolic equation for the reservoir.

In this work we consider a linearization of the model presented in [3]. Two different boundary conditions, oil production at a given flow rate or at a given pressure, are discussed. For technical reasons, they are treated separately. However, they both lead to evolution equations of the form

$$\begin{cases} U'(t) + AU(t) = F(t), \\ U(0) = U_0, \end{cases} \tag{1.2}$$

in appropriate functional spaces. In Section 2 we introduce the set of equations to be solved and we discuss in Section 3 the properties of the operator A. We show, in particular, that A is m-accretive for the boundary conditions considered. It follows that, under appropriate regularity assumptions, (1.2) is well-posed. The asymptotic behavior of $(T(t))_{t\geq 0}$, the semigroup generated by $-A$, is studied in Section 4. Taking convenient perturbed energy functionals, we show that $T(t)$ exponentially decays to zero. Finally, in Section 5 we prove the main results of existence and asymptotic behavior, Theorems 2.1, 2.2, 2.3 and 2.4 below.

2. **The coupled model**

Let $\Omega = (0,1) \times (0,1)$, $\Gamma_0 = \{0\} \times [0,1]$ and $\Gamma = \partial\Omega \setminus \Gamma_0$. Given $q_0, w_0 : (0,1) \to \mathbb{R}$, $Q_0 : \Omega \to \mathbb{R}$ and $\alpha : (0,T) \to \mathbb{R}$, we consider the problem of finding

$q, w : (0, T) \times (0, 1) \to \mathbb{R}$ and $Q : (0, T) \times \Omega \to \mathbb{R}$ such that:

$$\begin{cases} q_t + w_y + \dfrac{\partial Q}{\partial n}_{|\Gamma_0} = 0 & \text{in } (0, T) \times (0, 1), \\ w_t + q_y = 0 & \text{in } (0, T) \times (0, 1), \\ q(0, y) = q_0(y), \quad w(0, y) = w_0(y) & \text{for } y \in (0, 1), \end{cases} \tag{2.1}$$

and

$$\begin{cases} Q_t - \triangle Q = 0 & \text{in } (0, T) \times \Omega, \\ Q(t, x, y) = q(t, y) & \text{for } (t, x, y) \in (0, T) \times \Gamma_0, \\ \dfrac{\partial Q}{\partial n}(t, x, y) = 0 & \text{for } (t, x, y) \in (0, T) \times \Gamma, \\ Q(0, x, y) = Q_0(x, y) & \text{for } (x, y) \in \Omega. \end{cases} \tag{2.2}$$

q and w satisfy one of the following boundary conditions. Either

$$q(t, 0) = \alpha(t) \quad \text{and} \quad w(t, 1) = 0, \tag{2.3}$$

or

$$w(t, 0) = \alpha(t) \quad \text{and} \quad w(t, 1) = 0. \tag{2.4}$$

We can interpret (2.1)-(2.2) as follows. Ω is the reservoir region, Γ_0 the wellbore, q the well pressure, w the well velocity and Q the reservoir pressure. Considering the fluid state equation relating density and pressure as $\rho(q) = q$, the first equation in (2.1) expresses mass conservation. Then, $\dfrac{\partial Q}{\partial n}_{|\Gamma_0}$ is the well-reservoir mass flow exchange, given by Darcy's law (2.1) with $\dfrac{k}{\mu} = 1$ and $g = 0$. Frequently (see [2, 6]) the diffusion equation has been considered as a simplified model for the reservoir flow. The second equation in (2.2) is a continuity condition for the pressure at the wellbore. No-flow is assumed at the reservoir boundary. We consider production at a given pressure (2.3) or at a given mass flow rate (2.4).

We show under appropriate assumptions on α that the initial value problems (2.1)-(2.2)-(2.3) and (2.1)-(2.2)-(2.4) are well posed for $(q_0, w_0, Q_0) \in L^2(0, 1) \times L^2(0, 1) \times L^2(\Omega)$. We also establish some regularity properties and determine the asymptotic behavior of the solutions.

We first consider the problem (2.1)-(2.2)-(2.3). In order to get homogeneous boundary conditions we make the following change of variables.

$$\begin{cases} p(t, y) = q(t, y) - \alpha(t)\theta(y) & \text{for } (t, y) \in (0, T) \times (0, 1), \\ v(t, y) = w(t, y) & \text{for } (t, y) \in (0, T) \times (0, 1), \\ P(t, x, y) = Q(t, x, y) - \alpha(t)\theta(y) & \text{for } (t, x, y) \in (0, T) \times \Omega. \end{cases}$$

Here, $\theta \in C^\infty([0,1])$ is such that $\theta(0) = 1$ and $\theta(1) = \theta'(0) = \theta'(1) = 0$, for example $\theta(y) = \dfrac{1+\cos(\pi y)}{2}$. System (2.1)-(2.2)-(2.3) is equivalent to

$$\begin{cases} p_t + v_y + \dfrac{\partial P}{\partial n}_{|\Gamma_0} = -\alpha'(t)\theta(y) & \text{in } (0,T)\times(0,1), \\ v_t + p_y = -\alpha(t)\theta'(y) & \text{in } (0,T)\times(0,1), \\ p(0,y) = p_0(y), \quad v(0,y) = v_0(y) & \text{for } y \in (0,1), \\ p(t,0) = v(t,1) = 0 & \text{for } t \in (0,T), \end{cases} \tag{2.5}$$

and

$$\begin{cases} P_t - \triangle P = \alpha(t)\theta''(y) - \alpha'(t)\theta(y) & \text{in } (0,T)\times\Omega, \\ P(t,x,y) = p(t,y) & \text{for } (t,x,y) \in (0,T)\times\Gamma_0, \\ \dfrac{\partial P}{\partial n}(t,x,y) = 0 & \text{for } (t,x,y) \in (0,T)\times\Gamma, \\ P(0,x,y) = P_0(x,y) & \text{for } (x,y) \in \Omega, \end{cases} \tag{2.6}$$

where

$$\begin{cases} p_0(y) = q_0(y) - \alpha(0)\theta(y), \\ v_0(y) = w_0(y), \\ P_0(x,y) = Q_0(x,y) - \alpha(0)\theta(y). \end{cases}$$

We write the system (2.5)-(2.6) in the form

$$\begin{cases} \dfrac{dU}{dt} + AU = F(t), \\ U(0) = U_0, \end{cases} \tag{2.7}$$

with

$$U_0 = (p_0, v_0, P_0).$$

Here, the operator A is defined on the Hilbert space

$$H = L^2(0,1)\times L^2(0,1)\times L^2(\Omega),$$

by

$$D(A) = \Big\{(p,v,P) \in H^1(0,1)^2 \times H^1(\Omega);\ p(0) = v(1) = 0, \\ \triangle P \in L^2(\Omega), \frac{\partial P}{\partial n}_{|\Gamma} = 0,\ P_{|\Gamma_0} = p\Big\} \tag{2.8}$$

and

$$A(p,v,P) = (v_y + \frac{\partial P}{\partial n}_{|\Gamma_0}, p_y, -\triangle P), \tag{2.9}$$

for $(p,v,P) \in D(A)$. $F(t) : (0,T) \to H$ is defined by

$$F(t) = (-\alpha'(t)\theta(y), -\alpha(t)\theta'(y), \alpha(t)\theta''(y) - \alpha'(t)\theta(y)). \tag{2.10}$$

We will show in the next section that A is well defined and that A is m-accretive in H, so that $-A$ generates a semigroup of contractions in H which we denote by $(T(t))_{t\geq 0}$. Therefore, by Duhamel's formula, the solution U of (2.7) is given by

$$U(t) = T(t)U_0 + \int_0^t T(t-s)F(s)\,ds. \tag{2.11}$$

Conversely, U given by (2.11) is called the weak solution of (2.7). We have the following results.

Theorem 2.1. *Let A be defined by (2.8)-(2.9), let $T > 0$ and let $\alpha \in W^{1,1}(0,T)$.*

(i) *Given $U_0 = (p_0, v_0, P_0) \in H$, there exists a unique weak solution $U = (p,v,P) \in C([0,T],H)$ of (2.7). In addition, $P \in L^2((0,T),H^1(\Omega))$.*

(ii) *If $\alpha \in W^{2,1}(0,T)$ and $U_0 \in D(A)$, then $U \in C([0,T],D(A)) \cap W^{1,1}((0,T),H)$ and U solves the equation (2.7) for almost all $t \in (0,T)$. If furthermore $\alpha \in C^2([0,T])$, then $U \in C^1([0,T],H)$.*

Theorem 2.2. *Assume $\alpha \in W^{1,1}_{loc}(0,\infty)$.*

(i) *If $\sup_{t\geq 0} \int_t^{t+1} (|\alpha(s)|+|\alpha'(s)|)\,ds < \infty$, then for every $U_0 \in H$ the solution U of (2.7) is bounded, i.e. $\sup_{t\geq 0} \|U(t)\|_H < \infty$. Moreover, if U_1 and U_2 are two solutions, then $U_1(t) - U_2(t)$ converges exponentially to 0 in H as $t \to \infty$.*

(ii) *If in addition α is τ-periodic for some $\tau > 0$, then there exists a τ-periodic solution U of (2.7).*

We now consider the case of the boundary condition (2.4). In order to get homogeneous boundary conditions we make the following change of variables.

$$\begin{cases} p(t,y) = q(t,y) & \text{for } (t,y) \in (0,T) \times (0,1), \\ v(t,y) = w(t,y) - (1-y)\alpha(t) & \text{for } (t,y) \in (0,T) \times (0,1), \\ P(t,x,y) = Q(t,x,y) & \text{for } (t,x,y) \in (0,T) \times \Omega. \end{cases}$$

System (2.1)-(2.2)-(2.4) is then equivalent to

$$\begin{cases} p_t + v_y + \dfrac{\partial P}{\partial n}_{|\Gamma_0} = \alpha(t) & \text{in } (0,T)\times(0,1), \\ v_t + p_y = -(1-y)\alpha'(t) & \text{in } (0,T)\times(0,1), \\ p(0,y) = p_0(y) \quad v(0,y) = v_0(y) & \text{for } y\in(0,1), \\ v(t,0) = v(t,1) = 0 & \text{for } t\in(0,T), \end{cases} \tag{2.12}$$

and

$$\begin{cases} P_t - \triangle P = 0 & \text{in } (0,T)\times\Omega, \\ P(t,x,y) = p(t,y) & \text{for } (t,x,y)\in(0,T)\times\Gamma_0, \\ \dfrac{\partial P}{\partial n}(t,x,y) = 0 & \text{for } (t,x,y)\in(0,T)\times\Gamma, \\ P(0,x,y) = P_0(x,y) & \text{for } (x,y)\in\Omega, \end{cases} \tag{2.13}$$

where

$$\begin{cases} p_0(y) = q_0(y), \\ v_0(y) = w_0(y) - (1-y)\alpha(0), \\ P_0(x,y) = Q_0(x,y). \end{cases}$$

System (2.12)-(2.13) can be written as (2.7), where now the operator A is defined on the Hilbert space H by

$$D(A) = \Big\{(p,v,P)\in H;\, p\in H^1(0,1), v\in H^1_0(0,1), \\ \triangle P\in L^2(\Omega),\, \frac{\partial P}{\partial n}_{|\Gamma} = 0,\, P_{|\Gamma_0} = p\Big\}, \tag{2.14}$$

and

$$A(p,v,P) = \Big(v_x + \frac{\partial P}{\partial n}_{|\Gamma_0}, p_x, -\triangle P\Big), \tag{2.15}$$

for $(p,v,P)\in D(A)$. $F(t) : (0,T)\to H$ is defined by

$$F(t) = (\alpha(t), -(1-y)\alpha'(t), 0). \tag{2.16}$$

We have the following results.

Theorem 2.3. *Let A be defined by (2.14)-(2.15), let $T>0$ and let $\alpha\in W^{1,1}(0,T)$.*

(i) *Given $U_0 = (p_0,v_0,P_0)\in H$, there exists a unique weak solution $U = (p,v,P)\in C([0,T],H)$ of (2.7). In addition, $P\in L^2((0,T),H^1(\Omega))$.*

(ii) *If* $\alpha \in W^{2,1}(0,T)$ *and* $U_0 \in D(A)$, *then* $U \in C([0,T], D(A)) \cap W^{1,1}((0,T), H)$ *and* U *solves the equation* (2.7) *for almost all* $t \in (0,T)$. *If furthermore* $\alpha \in C^2([0,T])$, *then* $U \in C^1([0,T], H)$.

Theorem 2.4. *Assume* $\alpha \in W^{1,1}_{loc}(0,\infty)$ *and that*

$$\int_0^t \alpha(s)\, ds,$$

is bounded as $t \to \infty$.

(i) *If* $\sup_{t\geq 0} \int_t^{t+1} (|\alpha(s)|+|\alpha'(s)|)\, ds < \infty$, *then for every* $U_0 \in H$ *the solution* U *of* (2.7) *is bounded, i.e.* $\sup_{t\geq 0} \|U(t)\|_H < \infty$. *Moreover, if* U_1 *and* U_2 *are two solutions, then* $U_1(t) - U_2(t)$ *converges to* 0 *in* H *as* $t \to \infty$.

(ii) *If in addition* α *is* τ*-periodic for some* $\tau > 0$, *then there exists a* τ*-periodic solution* U *of* (2.7).

3. **The operator** A

In this section, we study several properties of the operator A defined either by (2.8)–(2.9) or by (2.14)–(2.15). We begin by considering a related elliptic problem.

Lemma 3.1. *Given* $f \in L^2(\Omega)$ *and* $p \in H^1(0,1)$, *there exists a unique solution* $u \in H^1(\Omega)$ *of the equation*

$$\begin{cases} -\triangle u = f & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0 & \text{in } \Gamma, \\ u = p & \text{in } \Gamma_0. \end{cases} \tag{3.1}$$

The mapping $(f,p) \mapsto u$ *is continuous* $L^2(\Omega) \times H^1(0,1) \to H^1(\Omega)$. *Furthermore,* $\frac{\partial u}{\partial n}_{|\Gamma_0}$ *is well defined,* $\frac{\partial u}{\partial n}_{|\Gamma_0} \in L^2(0,1)$, *and the mapping* $(f,p) \mapsto \frac{\partial u}{\partial n}_{|\Gamma_0}$ *is continuous* $L^2(\Omega) \times H^1(0,1) \to L^2(0,1)$. *Moreover,*

$$\int_\Omega \nabla u \cdot \nabla v = \int_\Omega f v + \int_{\Gamma_0} v \frac{\partial u}{\partial n}, \tag{3.2}$$

for all $v \in H^1(\Omega)$.

Proof. We proceed in four steps.

Step 1. Existence. We observe that if $v \in H^1(\Omega)$, then $v \in L^2((0,1), H^1(0,1)) \cap H^1((0,1), L^2(0,1))$. Therefore, $v \in C([0,1], L^2(0,1))$, so that $v_{|\Gamma_0}$ is well defined, $v_{|\Gamma_0} \in L^2(0,1)$. It follows that

$$V = \{v \in H^1(\Omega);\ v_{|\Gamma_0} = 0\},$$

is a closed subspace of $H^1(\Omega)$. Furthermore, $\|v\|_V = \|\nabla v\|_{L^2(\Omega)}$ is a norm on V which is equivalent to the H^1 norm, as follows easily by integrating the identity

$$v(x,y) = \int_0^x \frac{\partial v}{\partial x}(s,y)\,dy.$$

As usual, we write the equation (3.1) in a weak form, i.e. we say that $u \in H^1(\Omega)$ is a solution of (3.1) if $u_{|\Gamma_0} = p$ and

$$\int_\Omega \nabla u \cdot \nabla v = \int_\Omega fv, \quad \text{for all } v \in V. \tag{3.3}$$

Setting

$$u(x,y) = \widetilde{u}(x,y) + p(y),$$

solving (3.1) is then equivalent to solving the following problem

$$\begin{cases} \widetilde{u} \in V, \\ \displaystyle\int_\Omega \nabla\widetilde{u} \cdot \nabla v = \int_\Omega fv - \int_\Omega p_y v_y, \quad \text{for all } v \in V. \end{cases}$$

Existence and uniqueness for this last problem follows immediately from Lax-Milgram's lemma, since the right hand side is clearly a linear continuous functional of $v \in V$. Continuity of the mapping $(f,p) \mapsto u$ from $L^2(\Omega) \times H^1(0,1) \to H^1(\Omega)$ follows.

Step 2. Properties of $\frac{\partial u}{\partial n}_{|\Gamma_0}$ in the case $f = 0$. Suppose that $f = 0$, and consider first the case $p(y) = \cos(m\pi y)$ for some integer $m \geq 0$. Then one verifies by a direct calculation that

$$u(x,y) = \frac{\cosh(m\pi(x-1))}{\cosh(m\pi)} \cos(m\pi y);$$

and so

$$\frac{\partial u}{\partial n}_{|\Gamma_0}(y) = m\pi \tanh(m\pi) \cos(m\pi y).$$

Now if

$$p(y) = \sum_{m=0}^{\ell} a_m \cos(m\pi y),$$

we obtain

$$u(x,y) = \sum_{m=0}^{\ell} a_m \frac{\cosh(m\pi(x-1))}{\cosh(m\pi)} \cos(m\pi y);$$

and so

$$\frac{\partial u}{\partial n}_{|\Gamma_0}(y) = \sum_{m=0}^{\ell} m\pi a_m \tanh(m\pi)\cos(m\pi y).$$

Elementary calculations yield

$$\|p_y\|_{L^2}^2 = \frac{1}{2}\sum_{m=1}^{\ell} m^2\pi^2 a_m^2,$$

and

$$\Big\|\frac{\partial u}{\partial n}\Big\|_{L^2(\Gamma_0)}^2 = \frac{1}{2}\sum_{m=1}^{\ell} m^2\pi^2 a_m^2 (\tanh(m\pi))^2,$$

from which we deduce

$$\Big\|\frac{\partial u}{\partial n}\Big\|_{L^2(\Gamma_0)} \le \|p_y\|_{L^2}^2 \le \|p\|_{H^1}^2.$$

Since $(\cos(m\pi y))_{m\ge 0}$ is the set of eigenvectors of $-\dfrac{d^2}{dy^2}$ in $H^1(0,1)$ with Neumann boundary conditions, it is a complete orthonormal system of $H^1(0,1)$, and the result follows for general $p \in H^1(0,1)$. (Note that when $f = 0$ we have $u \in C^\infty(\overline{\Omega}\backslash\Gamma_0)$. This can be seen either on the Fourier series expansion, or else by extending u to a harmonic function $\widetilde{u}$ on the strip $(0,2)\times\mathbb{R}$ by symmetry, see Step 3.)

Step 3. Properties of $\dfrac{\partial u}{\partial n}_{|\Gamma_0}$ in the case $p = 0$. Suppose that $p = 0$, and define the function $\widetilde{u}$ on $\mathbb{R}^2$ as follows.

$$\begin{cases} \widetilde{u}(x,y) = u(x,y) & \text{for } (x,y)\in\Omega, \\ \widetilde{u}(-x,y) = -\widetilde{u}(x,y) & \text{for } (x,y)\in\Omega, \\ \widetilde{u}(x,-y) = \widetilde{u}(x,y) & \text{for } (x,y)\in\Omega, \\ \widetilde{u}(x+2m, y+2\ell) = (-1)^m\widetilde{u}(x,y) & \text{for } (x,y)\in\mathbb{R}^2 \text{ and } m,\ell\in\mathbb{Z}. \end{cases}$$

Define $\widetilde{f}$ by

$$\begin{cases} \widetilde{f}(x,y) = f(x,y) & \text{for } (x,y) \in \Omega, \\ \widetilde{f}(-x,y) = -\widetilde{f}(x,y) & \text{for } (x,y) \in \Omega, \\ \widetilde{f}(x,-y) = \widetilde{f}(x,y) & \text{for } (x,y) \in \Omega, \\ \widetilde{f}(x+2m, y+2\ell) = (-1)^m \widetilde{f}(x,y) & \text{for } (x,y) \in \mathbb{R}^2 \text{ and } m, \ell \in \mathbb{Z}. \end{cases}$$

It follows easily that $\widetilde{u} \in H^1_{loc}(\mathbb{R}^2)$, $\widetilde{f} \in L^2_{loc}(\mathbb{R}^2)$, and that

$$-\triangle \widetilde{u} = \widetilde{f} \quad \text{in } \mathcal{D}'(\mathbb{R}^2).$$

Therefore, $\widetilde{u} \in H^2_{loc}(\mathbb{R}^2)$. It follows in particular that $u \in H^2(\Omega)$, from which we deduce (see Step 1) $\frac{\partial u}{\partial n}_{|\Gamma_0} \in L^2(\Gamma_0)$ and $\Big\| \frac{\partial u}{\partial n} \Big\|_{L^2(\Gamma_0)} \le C \|f\|_{L^2(\Omega)}$.

Step 4. Conclusion. The properties of $\frac{\partial u}{\partial n}_{|\Gamma_0}$ in the general case $p \in H^1(0,1)$, $f \in L^2(\Omega)$ follow from Steps 2 and 3. Green's formula (3.2) is immediate in the case $p(y) = \sum_{m=0}^{\ell} a_m \cos(m\pi y)$, $\ell < \infty$ since (by Steps 2 and 3) $u \in H^2(\Omega)$. The same result is obtained for $p \in H^1(0,1)$ by density. This completes the proof. □

Remark 3.2. It follows from the proof of Lemma 3.1 (Steps 2 and 3) that

$$\Big\| \frac{\partial u}{\partial n}_{|\Gamma_0} \Big\|_{L^2} \le C(\|f\|_{L^2(\Omega)} + \|p_y\|_{L^2(0,1)}), \tag{3.4}$$

for all $f \in L^2(\Omega)$ and all $p \in H^1(0,1)$.

Let A be the operator on $H = L^2(0,1)^2 \times L^2(\Omega)$ defined by (2.8)–(2.9). From Lemma 3.1 the definition of A makes sense. We describe in the following lemma the main properties of A.

Lemma 3.3. *If A is defined by (2.8)–(2.9), then the following properties hold.*

(i) $(AU,U)_H = \int_\Omega |\nabla P|^2$, *for all $U \in D(A)$.*

(ii) $A \ge 0$ *and* $R(A) = H$.

(iii) *There exists $\delta > 0$ such that $\|AU\|_H \ge \delta(\|p\|_{H^1(0,1)} + \|v\|_{H^1(0,1)} + \|P\|_{H^1(\Omega)})$ for all $U = (p,v,P) \in D(A)$.*

(iv) *A is closed. In particular, $D(A)$ equipped with the norm $\|U\|^2_{D(A)} = \|U\|^2_H + \|AU\|^2_H$ is a Hilbert space. $\|AU\|_H$ is an equivalent norm on $D(A)$. Moreover, the embedding $D(A) \hookrightarrow H$ is compact.*

(v) *A is m-accretive.*

Proof. (i) We have

$$(AU,U)_H = \int_0^1 (pv_y + p_y v) + \int_0^1 p\frac{\partial P}{\partial n}_{|\Gamma_0} - \int_\Omega P\triangle P$$
$$= \int_0^1 p\frac{\partial P}{\partial n}_{|\Gamma_0} - \int_\Omega P\triangle P.$$

On the other hand, (3.2) gives

$$-\int_\Omega P\triangle P = \int_\Omega |\nabla P|^2 - \int_0^1 p\frac{\partial P}{\partial n}_{|\Gamma_0},$$

hence property (i).

(ii) It is clear from (i) that $A \geq 0$. Let now $(\varphi, \psi, f) \in H$. Set

$$p(y) = \int_0^y \psi(s)\,ds \quad \text{for } y \in (0,1).$$

We have $p \in H^1(0,1)$ and $p(0) = 0$. Let $P \in H^1(\Omega)$ be the solution of

$$\begin{cases} -\triangle P = f & \text{in } \Omega, \\ \dfrac{\partial P}{\partial n} = 0 & \text{in } \Gamma, \\ P = p & \text{in } \Gamma_0, \end{cases}$$

given by Lemma 3.1. Since $\dfrac{\partial P}{\partial n}_{|\Gamma_0} \in L^2(0,1)$, if

$$v(y) = \int_y^1 \frac{\partial P}{\partial n}_{|\Gamma_0} + \int_y^1 \varphi,$$

then $v \in H^1(0,1)$ and $v(1) = 0$. Therefore $U = (p,v,P) \in D(A)$, $AU = (\varphi, \psi, f)$ and so, $R(A) = H$.

(iii) Let $U = (p,v,P) \in D(A)$. We have

$$\|AU\|^2_H = \int_0^1 \left|v_y + \frac{\partial P}{\partial n}_{|\Gamma_0}\right|^2 + \int_0^1 p_y^2 + \int_\Omega |\triangle P|^2.$$

Since $p(0) = 0$, we have

$$\|p\|_{H^1} \le C\|p_y\|_{L^2},$$

for some C independent of U. Next, it follows from Lemma 3.1 that

$$\|P\|_{H^1(\Omega)} + \Big\|\frac{\partial P}{\partial n}{}_{|\Gamma_0}\Big\|_{L^2(\Gamma_0)} \le C(\|\triangle P\|_{L^2(\Omega)} + \|p\|_{H^1(0,1)}).$$

Finally, since $v(1) = 0$, we have

$$\|v\|_{H^1} \le C\|v_y\|_{L^2} \le C\Big\|\frac{\partial P}{\partial n}{}_{|\Gamma_0}\Big\|_{L^2(\Gamma_0)} + C\Big\|v_y + \frac{\partial P}{\partial n}{}_{|\Gamma_0}\Big\|_{L^2(\Gamma_0)}.$$

Combining the above inequalities we obtain (iii).

(iv) It follows from (iii) that $\|AU\|_H \ge \delta\|U\|_H$. Since $R(A) = H$, the closedness of A follows immediately. Next, the mapping $U \mapsto (AU, U)$ is an isometry $D(A) \to H \times H$, whose image is $G(A)$. Since $G(A)$ is closed, $G(A)$ is a Hilbert space, and so is $D(A)$. Compactness of the embedding $D(A) \hookrightarrow H$ follows from (iii) and the compactness of the embeddings $H^1(0,1) \hookrightarrow L^2(0,1)$ and $H^1(\Omega) \hookrightarrow L^2(\Omega)$.

(v) Since $A \ge 0$, we need only show that $R(I+A) = H$. Let $V \in R(I+A)^\perp$. We have

$$(V, U + AU)_H = 0 \quad \text{for all } U \in D(A).$$

Let $W \in D(A)$ be such that $V = AW$. Choosing $U = W$ in the above identity, we obtain

$$0 = (AW, W + AW)_H = (AW, W)_H + \|AW\|_H^2 \ge \|AW\|_H^2 \ge \delta\|W\|_H^2;$$

and so $W = 0$, thus $V = 0$. Therefore $\overline{R(I + A)} = H$, and we need only show that $R(I + A)$ is closed. Suppose $Z_n \underset{n\to\infty}{\longrightarrow} Z$ in H, where $Z_n = U_n + AU_n$. Then Z_n is a Cauchy sequence in H. Since A is accretive, U_n is also a Cauchy sequence. Therefore, $U_n \longrightarrow U \in H$. Hence, $AU_n \longrightarrow Z - U$. By the closedness of A we deduce that $Z = U + AU$, which proves the closedness of $R(I+A)$. □

It follows from Lemma 3.3 that $-A$ generates a semigroup of contractions in H, which we denote by $(T(t))_{t\ge 0}$.

We now consider the operator A on H defined by (2.14)–(2.15). It follows easily from Lemma 3.1 that A is well defined $D(A) \to H$. On the other hand, A has a nontrivial nullspace. More precisely, it can be easily seen that

$$N(A) = \{U = (p, v, P) \in H;\ \exists c \in \mathbb{R}, p \equiv c \text{ and } P \equiv c\}.$$

In other words, the nullspace of A is the (one-dimensional) subspace of H spanned by the vector $(1_{(0,1)}, 0, 1_\Omega)$. It is convenient to introduce the restriction of A to $N(A)^\perp$. Therefore, we consider the Hilbert space

$$\widetilde{H} = N(A)^\perp = \left\{(p, v, P) \in L^2(0,1)^2 \times L^2(\Omega);\ \int_0^1 p + \int_\Omega P = 0\right\},$$

and we define the operator $\widetilde{A}$ on $\widetilde{H}$ by

$$D(\widetilde{A}) = \Big\{(p, v, P) \in \widetilde{H};\ p \in H^1(0,1), v \in H_0^1(0,1),$$
$$\triangle P \in L^2(\Omega),\ \frac{\partial P}{\partial n}_{|\Gamma} = 0,\ P_{|\Gamma_0} = p\Big\}, \quad (3.5)$$

and

$$\widetilde{A}(p, v, P) = (v_x + \frac{\partial P}{\partial n}_{|\Gamma_0}, p_x, -\triangle P), \quad (3.6)$$

for $(p, v, P) \in D(\widetilde{A})$. Clearly, if $(p, v, P) \in D(\widetilde{A})$, then

$$\int_0^1 (v_y + \frac{\partial P}{\partial n}_{|\Gamma_0}) + \int_\Omega (-\triangle P) = \int_0^1 \frac{\partial P}{\partial n}_{|\Gamma_0} + \int_\Omega (-\triangle P) = 0,$$

where the last identity follows from (3.2) applied with $u = P$ and $v = 1$. Therefore, $\widetilde{A}U \in \widetilde{H}$; and so, $\widetilde{A}$ is an operator on $\widetilde{H}$. We describe in the following lemma the main properties of $\widetilde{A}$.

Lemma 3.4. *If $\widetilde{A}$ is defined by (3.5)–(3.6), then the following properties hold.*

(i) $(\widetilde{A}U, U)_{\widetilde{H}} = \int_\Omega |\nabla P|^2$, *for all $U \in D(\widetilde{A})$.*

(ii) $\widetilde{A} \geq 0$ *and* $R(\widetilde{A}) = \widetilde{H}$.

(iii) *There exists $\delta > 0$ such that $\|\widetilde{A}U\|_{\widetilde{H}} \geq \delta(\|p\|_{H^1(0,1)} + \|v\|_{H^1(0,1)} + \|P\|_{H^1(\Omega)})$ for all $U \in D(\widetilde{A})$.*

(iv) *$\widetilde{A}$ is closed. In particular, $D(\widetilde{A})$ equipped with the norm $\|U\|^2_{D(\widetilde{A})} = \|U\|^2_{\widetilde{H}} + \|\widetilde{A}U\|^2_{\widetilde{H}}$ is a Hilbert space. $\|\widetilde{A}U\|_{\widetilde{H}}$ is an equivalent norm on $D(\widetilde{A})$. Moreover, the embedding $D(\widetilde{A}) \hookrightarrow \widetilde{H}$ is compact.*

(v) *$\widetilde{A}$ is m-accretive.*

(vi) *There exists a constant $\mu > 0$ such that*

$$(\widetilde{A}U, U)_{\widetilde{H}} \geq \mu\left(\|P\|^2_{L^2(\Omega)} + \|p\|^2_{L^2(0,1)}\right),$$

for all $U \in D(\widetilde{A})$.

Proof. The proof of property (i) is the same as the proof of Lemma 3.3 (i). We next prove (vi), which we will use in the proof of (iii). We argue by contradiction and we assume that there exists $(U_n)_{n\geq 0} \subset D(\widetilde{A})$ such that

$$\|p_n\|_{L^2(0,1)} + \|P_n\|_{L^2(\Omega)} = 1, \quad \text{and} \quad \int_\Omega |\nabla P_n|^2 \underset{n\to\infty}{\longrightarrow} 0.$$

By the compact embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$, there exists a subsequence $(n_k)_{k\geq 0}$ and a constant C such that $P_{n_k} \underset{k\to\infty}{\longrightarrow} C$ in $H^1(\Omega)$. It follows that $p_{n_k} = P_{n_k|\Gamma_0} \underset{k\to\infty}{\longrightarrow} C$ in $L^2(0,1)$. Since $(U_n)_{n\geq 0} \subset \widetilde{H}$, we deduce that $C = 0$; and so $\|p_{n_k}\|_{L^2(0,1)} + \|P_{n_k}\|_{L^2(\Omega)} \underset{k\to\infty}{\longrightarrow} 0$, which is absurd.

The proof of the properties (ii) to (v) is almost the same as the proof of Lemma 3.3. We only indicate the modifications. To show that $R(\widetilde{A}) = \widetilde{H}$, let $(\varphi, \psi, f) \in \widetilde{H}$. Using Lemma 3.3, we find $U = (p, v, P) \in H^1(0,1)^2 \times H^1(\Omega)$ such that

$$\begin{cases} v_y + \dfrac{\partial P}{\partial n}_{\Gamma_0} = \varphi, \\ p_y = \psi, \\ -\triangle P = f, \end{cases}$$

with the boundary conditions

$$\begin{cases} p(0) = v(1) = 0, \\ \dfrac{\partial P}{\partial n}_{|\Gamma} = 0 \quad \text{and} \quad P_{|\Gamma_0} = p. \end{cases}$$

But $(\varphi, \psi, f) \in \widetilde{H}$ implies

$$-v(0) = -\int_0^1 v_y = \int_0^1 \varphi - \int_{\Gamma_0} \frac{\partial P}{\partial n} = \int_0^1 \varphi + \int_\Omega f = 0,$$

that is, $v \in H_0^1(0,1)$. We now consider

$$c = \frac{1}{2}\int_0^1 p + \frac{1}{2}\int_0^1 P,$$

and we set $\widetilde{U} = U - c(1_{(0,1)}, 0, 1_\Omega)$. Then, it is clear that $\widetilde{U} \in D(\widetilde{A})$ and $\widetilde{A}\widetilde{U} = (\varphi, \psi, f)$.

The proof of (iii) is modified as follows. We have

$$\|P\|_{H^1(\Omega)} + \Big\|\frac{\partial P}{\partial n}_{|\Gamma_0}\Big\|_{L^2(\Gamma_0)} \leq C(\|\triangle P\|_{L^2(\Omega)} + \|p\|_{H^1}) \leq C(\|\widetilde{A}U\|_{\widetilde{H}} + \|p\|_{L^2}).$$

Since $v \in H_0^1(0,1)$, we have

$$\begin{aligned}\|v\|_{H^1} \leq C\|v_y\|_{L^2} &\leq C\Big\|\frac{\partial P}{\partial n}_{|\Gamma_0}\Big\|_{L^2(\Gamma_0)} + C\Big\| - v_y + \frac{\partial P}{\partial n}{}_{\Gamma_0}\Big\|_{L^2(\Gamma_0)} \\ &\leq C(\|\widetilde{A}U\|_{\widetilde{H}} + \|p\|_{L^2}).\end{aligned}$$

Therefore,

$$\|p\|_{H^1(0,1)} + \|v\|_{H^1(0,1)} + \|P\|_{H^1(\Omega)} \leq C(\|\widetilde{A}U\|_{\widetilde{H}} + \|p\|_{L^2}).$$

The desired estimate follows by applying estimate (vi) and Cauchy-Schwarz' inequality. □

Corollary 3.5. *Consider the operator A on H defined by (2.14)–(2.15) and the operator $\widetilde{A}$ on $\widetilde{H}$ defined by (3.5)-(3.6).*

(i) *A is m-accretive and $(AU, U)_H = \displaystyle\int_\Omega |\nabla P|^2$.*

(ii) *If $(T(t))_{t\geq 0}$ is the semigroup of contractions in H generated by $-A$ and $(\widetilde{T}(t))_{t\geq 0}$ is the semigroup of contractions in $\widetilde{H}$ generated by $-\widetilde{A}$, then for every $U_0 = (p_0, v_0, P_0) \in H$, we have*

$$T(t)U_0 = c + \widetilde{T}(t)\widetilde{U}_0,$$

where $c = \dfrac{1}{2}\left(\displaystyle\int_0^1 p_0 + \int_\Omega P_0\right)$, and $\widetilde{U}_0 \in \widetilde{H}$ is defined by $\widetilde{U}_0 = (p_0 - c, v_0, P_0 - c)$.

Proof. Property (i) follows from Lemma 3.4 and the property $\widetilde{H} = N(A)^\perp$. Property (ii) is immediate, since $\widetilde{U}_0$ is the orthogonal projection of U_0 on $\widetilde{H}$. □

4. Asymptotic behavior

We consider first the case where the wellhead pressure is known, that is, let $\widetilde{A}$ be given by (3.5)-(3.6) and let $(\widetilde{T}(t))_{t\geq 0}$ be the semigroup of contractions in $\widetilde{H}$ generated by $-\widetilde{A}$. We define $\widetilde{K} : \widetilde{H} \longrightarrow \widetilde{H}$ by

$$\widetilde{K}(U) = \langle \widetilde{A}^{-1}U, U\rangle,$$

where the scalar product $\langle \cdot, \cdot \rangle$ is defined by

$$\langle U_1, U_2 \rangle = \int_0^1 p_1 p_2 + 2 \int_0^1 v_1 v_2 + \int_\Omega P_1 P_2,$$

if $U_j = (p_j, v_j, P_j)$. We have the following results.

Lemma 4.1. *There exists $c_0 > 0$ such that if $\widetilde{U}_0 \in D(\widetilde{A})$ and $\widetilde{U} = \widetilde{T}(t)\widetilde{U}_0$, then*

$$\frac{d}{dt}\widetilde{K}(\widetilde{U}) \leq c_0 \int_0^1 \widetilde{p}^2 - \frac{1}{2}\int_0^1 \widetilde{v}^2 - \int_\Omega \widetilde{P}^2, \tag{4.1}$$

for all $t \geq 0$.

Proof. Indeed,

$$\begin{aligned}\frac{d}{dt}\widetilde{K}(\widetilde{U}(t)) &= \langle \widetilde{A}^{-1}\frac{d\widetilde{U}}{dt}, \widetilde{U}\rangle + \langle \widetilde{A}^{-1}\widetilde{U}, \frac{d\widetilde{U}}{dt}\rangle = -\langle \widetilde{A}^{-1}\widetilde{A}\widetilde{U}, \widetilde{U}\rangle - \langle \widetilde{A}^{-1}\widetilde{U}, \widetilde{A}\widetilde{U}\rangle \\ &= -\langle \widetilde{U}, \widetilde{U}\rangle - \langle \widetilde{A}^{-1}\widetilde{U}, \widetilde{A}\widetilde{U}\rangle.\end{aligned}$$

Set $(\overline{p}, \overline{v}, \overline{P}) = \widetilde{A}^{-1}\widetilde{U}$. We have

$$\begin{cases} \overline{v}_y + \dfrac{\partial \overline{P}}{\partial n}_{|\Gamma_0} = \widetilde{p}, \\ \overline{p}_y = \widetilde{v}, \\ -\triangle \overline{P} = \widetilde{P}. \end{cases} \tag{4.2}$$

Next,

$$\begin{aligned}\langle \widetilde{A}^{-1}\widetilde{U}, \widetilde{A}\widetilde{U}\rangle &= \int_0^1 \overline{p}\left(\widetilde{v}_y + \frac{\partial \widetilde{P}}{\partial n}_{|\Gamma_0}\right) + 2\int_0^1 \overline{v}\widetilde{p}_y - \int_\Omega \overline{P}\triangle\widetilde{P} \\ &= -\int_0^1 \overline{p}_y \widetilde{v} - 2\int_0^1 \overline{v}_y \widetilde{p} - \int_\Omega \widetilde{P}\triangle\overline{P} + \int_0^1 \frac{\partial \overline{P}}{\partial n}_{|\Gamma_0}\widetilde{p} \\ &= -\int_0^1 \widetilde{v}^2 - 2\int_0^1 \widetilde{p}^2 + 3\int_0^1 \widetilde{p}\frac{\partial \overline{P}}{\partial n}_{|\Gamma_0} + \int_\Omega \widetilde{P}^2.\end{aligned}$$

Hence,

$$\frac{d}{dt}\widetilde{K}(\widetilde{U}(t)) = \int_0^1 \widetilde{p}^2 - \int_0^1 \widetilde{v}^2 - 2\int_\Omega \widetilde{P}^2 - 3\int_0^1 \widetilde{p}\frac{\partial \overline{P}}{\partial n}_{|\Gamma_0}. \tag{4.3}$$

On the other hand, it follows from (3.4) that

$$\left\| \frac{\partial \overline{P}}{\partial n}_{|\Gamma_0} \right\|_{L^2} \le C(\|\triangle \overline{P}\|_{L^2} + \|\overline{p}_y\|_{L^2}) = C(\|\widetilde{P}\|_{L^2} + \|\widetilde{v}\|_{L^2}). \tag{4.4}$$

(4.1) now follows from (4.3), (4.4) and Cauchy-Schwarz' inequality. □

Remark 4.2. Let A be the operator in H defined by (2.8)-(2.9), and let $(T(t))_{t\ge 0}$ be the semigroup generated by $-A$. Let $K(U) = \langle A^{-1}U, U\rangle$ for $U \in D(A)$, where $\langle \cdot, \cdot \rangle$ is as above, but in H instead of $\widetilde{H}$. The same argument as in Lemma 4.1 shows that if $U_0 \in D(A)$ and $U = T(t)U_0$, then

$$\frac{d}{dt} K(U(t)) \le c_0 \int_0^1 p^2 - \frac{1}{2}\int_0^1 v^2 - \int_\Omega P^2,$$

for all $t \ge 0$, where c_0 is the same constant as in (4.1).

Lemma 4.3. *If $\widetilde{T}(t))_{t\ge 0}$ is the semigroup generated by $-\widetilde{A}$, then there exists $\delta > 0$ such that*

$$\|\widetilde{T}(t)\|_{\mathcal{L}(\widetilde{H})} \le Ce^{-\delta t},$$

for all $t \ge 0$.

Proof. Let $\widetilde{U}_0 \in D(\widetilde{A})$, and set $\widetilde{U}(t) = (\widetilde{p}(t), \widetilde{v}(t), \widetilde{P}(t)) = \widetilde{T}(t)\widetilde{U}_0$. Then,

$$\frac{d}{dt}\|\widetilde{U}\|^2_{\widetilde{H}} = 2(\widetilde{U}_t, \widetilde{U})_{\widetilde{H}} = -2(\widetilde{A}\widetilde{U}, \widetilde{U})_{\widetilde{H}} = -2\int_\Omega |\nabla \widetilde{P}|^2. \tag{4.5}$$

For $\varepsilon > 0$ small enough

$$|||\widetilde{U}|||^2 = \|\widetilde{U}\|^2_{\widetilde{H}} + \varepsilon \widetilde{K}(\widetilde{U}),$$

defines an equivalent norm on $\widetilde{U}$, since $\widetilde{A}^{-1} \in \mathcal{L}(\widetilde{H})$. It follows from (4.1) and (4.5) that

$$\frac{d}{dt}|||\widetilde{U}|||^2 \le -2\int_\Omega |\nabla \widetilde{P}|^2 - \frac{\varepsilon}{2}\int_0^1 \widetilde{v}^2 - \varepsilon \int_\Omega \widetilde{P}^2 + \varepsilon c_0 \int_0^1 \widetilde{p}^2.$$

Applying properties (i) and (vi) of Lemma 3.4, we obtain by choosing ε small enough,

$$\frac{d}{dt}|||\widetilde{U}|||^2 \le -\int_\Omega |\nabla \widetilde{P}|^2 - \frac{\varepsilon}{2}\int_0^1 \widetilde{v}^2 \le -\alpha_1 \|\widetilde{U}\|^2_{\widetilde{H}} \le -\alpha_2 |||\widetilde{U}|||^2,$$

where α_1 and α_2 are positive constants independent of $\widetilde{U}_0$. Therefore,

$$|||\widetilde{U}(t)||| \le e^{-\alpha_2 t}|||\widetilde{U}_0|||. \tag{4.6}$$

Since α_2 is independent of $\widetilde{U}_0$, (4.6) holds for every $\widetilde{U}_0 \in \widetilde{H}$, and the result follows. □

We now consider again the operator A given by (2.8)-(2.9). For the study of the asymptotic behavior of the semigroup $(T(t))_{t\ge 0}$ generated by $-A$, we will use the following lemma.

Lemma 4.4. *For any $\lambda > 0$, there exists C_λ such that*

$$\|P\|^2_{L^2(\Omega)} + \|p\|^2_{L^2(0,1)} \le C_\lambda \left(\int_\Omega |\nabla P|^2 + \int_\Omega |\nabla \overline{P}|^2 \right) + \lambda \|v\|^2_{L^2(0,1)}, \tag{4.7}$$

for every $U = (p, v, P) \in D(A)$. Here, $\overline{U} = (\overline{p}, \overline{v}, \overline{P}) = A^{-1}(U)$.

Proof. Let $\lambda > 0$. If (4.7) does not hold, then there exists a sequence $(U_n)_{n\ge 0} \subset D(A)$ such that

$$\|P_n\|^2_{L^2(\Omega)} + \|p_n\|^2_{L^2(0,1)} = 1, \tag{4.8}$$

$$||v_n||^2_{L^2(0,1)} \le \frac{1}{\lambda}, \tag{4.9}$$

$$\|\nabla P_n\|^2_{L^2} + \|\nabla \overline{P}_n\|^2_{L^2} \underset{n\to\infty}{\longrightarrow} 0. \tag{4.10}$$

It follows from (4.8) and (4.10) (cf. the proof of Lemma 3.4 (vi)) that there exists a constant C_1 such that $P_n \to C_1$ in $H^1(\Omega)$ and $p_n \to C_1$ in $L^2(0,1)$. Moreover, since A^{-1} is compact, $\overline{P}_n$ has a limit in $L^2(\Omega)$. By (4.10), this limit is a constant C_2, and $\overline{P}_n \to C_2$ in $H^1(\Omega)$. Therefore, $\triangle \overline{P}_n \to 0$ in $H^{-1}(\Omega)$; and so $C_1 = 0$, since $-\triangle \overline{P}_n = P_n$. This contradicts (4.8). □

Lemma 4.5. *If $(T(t))_{t\ge 0}$ is the semigroup generated by $-A$, then there exists $\delta > 0$ such that*

$$\|T(t)\|_{\mathcal{L}(H)} \le Ce^{-\delta t},$$

for all $t \ge 0$.

Proof. With the notation of Remark 4.2, let $\varepsilon > 0$ be small enough so that $|||U||| = (\|U\|^2_H + \varepsilon K(U))^{\frac{1}{2}}$ be equivalent to $\|\cdot\|$. Let $U_0 \in D(A)$ and $U(t) = T(t)U_0$. It follows from Remark 4.2 (see the proof of Lemma 4.3) that

$$\frac{d}{dt}|||U|||^2 \le -2\int_\Omega |\nabla P|^2 - \frac{\varepsilon}{2}\int_0^1 v^2 - \varepsilon \int_\Omega P^2 + \varepsilon c_0 \int_0^1 p^2.$$

Let now $\overline{U} = (\overline{p}, \overline{v}, \overline{P}) = A^{-1}U$. In particular, $\overline{U}(t) = T(t)\overline{U}(0)$; and so

$$\frac{d}{dt}\|\overline{U}\|^2 = -2(A\overline{U}, \overline{U})_H = -2\int_\Omega |\nabla\overline{P}|^2.$$

We define the norm $|||U|||_1 = \left(|||U|||^2 + \|A^{-1}U\|_H^2\right)^{1/2}$, which is equivalent to the $|||\cdot|||$ norm. It follows from the two above relations that

$$\frac{d}{dt}|||U|||_1^2 + 2\|\nabla P\|_{L^2}^2 + 2\|\nabla\overline{P}\|_{L^2}^2 + \frac{\varepsilon}{2}\|v\|_{L^2}^2 + \varepsilon\|P\|_{L^2}^2 - \varepsilon c_0\|p\|_{L^2}^2 \leq 0. \quad (4.11)$$

On the other hand, taking $\lambda = \dfrac{1}{8c_0}$ in (4.7), we find a constant C such that

$$2c_0\|p\|_{L^2}^2 - \frac{1}{4}\|v\|_{L^2}^2 \leq C(\|\nabla P\|_{L^2}^2 + \|\nabla\overline{P}\|_{L^2}^2). \quad (4.12)$$

On multiplying (4.12) by ε and summing up with (4.11), we obtain

$$\frac{d}{dt}|||U|||_1^2 + \|\nabla P\|_{L^2}^2 + 2\|\nabla\overline{P}\|_{L^2}^2 + \frac{\varepsilon}{4}\|v\|_{L^2}^2 + \varepsilon\|P\|_{L^2}^2 + \varepsilon c_0\|p\|_{L^2}^2 \leq \varepsilon C(\|\nabla P\|_{L^2}^2 + \|\nabla\overline{P}\|_{L^2}^2).$$

Now, if we choose ε small enough so that $\varepsilon C \leq 1$, we find $\mu > 0$ such that

$$\frac{d}{dt}|||U|||_1^2 + \mu\,|||U|||_1^2 \leq 0,$$

from which we obtain the desired result. □

5. Proofs of Theorems 2.1, 2.2, 2.3 and 2.4

Theorems 2.1 and 2.3 are immediate consequences of the results of Section 3 and the semigroup theory (see Pazy [5], Chapter 4, Corollaries 2.2, 2.10 and 2.5), except for the property $P \in L^2((0,T), H^1(\Omega))$ for $U_0 \in H$. This last property is a consequence of the identity $(AU, U)_H = \|\nabla U\|_{L^2}^2$. More precisely, we have the following result.

Lemma 5.1. *Let A be the operator in H defined by (2.8)-(2.9) (respectively, by (2.14)-(2.15)) and let $(T(t))_{t\geq 0}$ be the semigroup of contractions in H generated by $-A$. Let $\alpha \in W^{1,1}(0,T)$ and let $F(t)$ be defined by (2.10) (respectively, by (2.16)). If $U_0 = (p_0, v_0, P_0) \in H$ and $U = (p, v, P)$ is given by (2.11), then*

$\nabla P \in L^2((0,T), L^2(\Omega))$ and

$$\|U(t)\|_H^2 + \int_0^t \int_\Omega |\nabla P(t,x,y)|^2 \, dx\, dy\, dt \le C\left(\|U_0\|_H^2 + ||\alpha||^2_{W^{1,1}(0,t)}\right), \quad (5.1)$$

for all $t \in [0,T]$. Here, C is a constant independent of T, α and U_0.

Proof. By density and continuous dependence, we need only show (5.1) when $\alpha \in C^2([0,T])$ and $U_0 \in D(A)$. In this case, $U \in C([0,T], D(A)) \cap C^1([0,T], H)$ is the solution of (2.7). Taking the scalar product of the equation with U and applying property (i) of Lemma 3.3 (respectivey, property (i) of Corollary 3.5), we find

$$\frac{1}{2}\frac{d}{dt}\|U(t)\|_H^2 + \int_\Omega |\nabla P|^2 = (F(t), U(t))_H.$$

Since

$$(F(t), U(t))_H \le \|F(t)\|_H \|U(t)\|_H \le C(|\alpha(t)| + |\alpha'(t)|)\, ||U(t)||_H,$$

it follows that

$$\frac{1}{2}\frac{d}{dt}\left(\|U(t)\|_H^2 + 2\int_0^t \int_\Omega |\nabla P|^2\right) \le C(|\alpha(t)| + |\alpha'(t)|)\, ||U(t)||_H.$$

The above differential inequality yields

$$\left(||U(t)||_H^2 + 2\int_0^t \int_\Omega |\nabla P(t,x,y)|^2 \, dx\, dy\, dt\right)^{\frac{1}{2}} \le ||U_0||_H + C||\alpha||_{W^{1,1}(0,t)},$$

from which (5.1) follows. □

Theorem 2.2 follows from the exponential decay of the semigroup $(T(t))_{t\ge 0}$ (see Section 4) and the results of Haraux [4] (see Theorem 7 p. 154).

For proving Theorem 2.4, we write $U(t) = \widetilde{U}(t) + V(t)$, where $V(t) = (a(t)1_{(0,1)}, 0, a(t)1_\Omega)$ with

$$a(t) = \frac{1}{2}\left(\int_0^1 p_0 + \int_\Omega P_0 + \int_0^t \alpha(s)\right).$$

Note that α is τ-periodic and that $\displaystyle\int_0^t \alpha(s)\, ds$ is bounded, so that α has mean value 0. Therefore V is τ-periodic. Furthermore, we have

$$\widetilde{U}(t) = \widetilde{T}(t)\widetilde{U}_0 + \int_0^t \widetilde{T}(t-s)\widetilde{F}(s)\, ds,$$

where $\widetilde{U}_0 = (p_0 - c, v_0, P_0 - c)$ with

$$c = \frac{1}{2}\left(\int_0^1 p_0 + \int_\Omega P_0\right),$$

so that $\widetilde{U}_0 \in \widetilde{H}$, and

$$\widetilde{F}(t) = \frac{1}{2}(\alpha(t), -(1-y)\alpha'(t), -\alpha(t)) \in \widetilde{H}.$$

Therefore, the result follows from the exponential decay of the semigroup $(\widetilde{T}(t))_{t\geq 0}$ (see Section 4) and the same argument as above.

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References

[1] J. Alvestad, K. Holing, K. Christoffersen, O. Langeland, O. Stava, Interactive Modelling of multiphase inflow performance of horizontal and highly deviated wells, in "European Petroleum Computer Conference, Aberdeen, 1994".

[2] K. Aziz, A. Settari, *Petroleum Reservoir Simulation*, Applied Science Publishers, London, 1979.

[3] A. Bourgeat, Simulating gas-liquid flow in a well-reservoir system, *Numerical Methods in Engeneering and Applied Sciences*, H. Alder, J.C. Heinrich, S. Lavanchy, E. Oñate and B. Suárez (eds.), CIMNE, Barcelona, 1992.

[4] A. Haraux, *Nonlinear evolution equations: Global behavior of solutions*, Lecture Notes in Math. **841**, Springer, New York, 1981.

[5] A. Pazy, *Semi-groups of linear operators and applications to partial differential equations*, Applied Math. Sciences **44**, Springer, New-York, 1983.

[6] D.W. Peaceman, *Fundamentals of Numerical Reservoir Simulation*, Elsevier Science Publishers, Amsterdam, 1977.