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# Blow up and scattering in the nonlinear Schrödinger equation

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*Eduardo Siqueira*
*Elaine Machtyngier*
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## 1. Introduction and notation.

In these notes, we describe some results concerning the blow up phenomenon and the scattering theory for the nonlinear Schrödinger equation. For simplicity, we consider the model case

$$\begin{cases} iu_t + \triangle u + \lambda |u|^\alpha u = 0, & (t,x) \in [0,T) \times \mathbf{R}^N, \\ u(0,x) = \varphi(x), & x \in \mathbf{R}^N, \end{cases} \tag{1.1}$$

where $\alpha > 0$ and $\lambda \in \mathbf{R}$, and $\varphi$ is a given initial value.

It is well-known that if $\lambda \leq 0$ or if $\alpha < 4/N$, then all solutions of (1.1) are global and bounded in the energy space. Then a natural question is the asymptotic behavior of the solutions. Since the solutions of the linear equation converge locally to 0 as $t \to \pm\infty$, one expects that the "small" solutions of (1.1) behave asymptotically like solutions of the linear equation, at least for $\alpha$ large enough. In the repulsive case $\lambda \leq 0$, one expects the same property to hold also for "large" solutions. Indeed, a scattering theory was developped for (1.1), for both low energy and large energy. Of course, the possible values of $\alpha$ for which the scattering operator can be constructed depend on the space in which one works. The most natural spaces are $H^1$, $L^2$ and $H^1 \cap L^2(|x|^2 dx)$. Some satisfactory results are available, though it seems that some of them are not optimal with respect to the range of $\alpha$'s.

On the other hand, if $\alpha \geq 4/N$ and $\lambda > 0$, then some solutions of (1.1) blow up in finite time. Several questions naturally arise. What norms of $u$ actually blow up? What is the blow up rate? What is the asymptotic profile of $u$? For the heat equation, there are satisfactory answers to these questions (see [12,13,14,15,16,22,23,24,25,33,42,43,44,45, 49,50,51]), while for the Schrödinger equation, they are essentially open. More precisely, besides the $H^1$ norm, there is a family of $L^p$ norms which blows up. However, the optimality is in general unknown, since there does not exist any pointwise upper bound on any $L^p$ norm for $p > 2$. As for the blow up rate, there is a lower bound which is a trivial consequence of the local existence theory, but there is no pointwise upper bound. As a matter of fact, in the so-called pseudo-conformal case, there exist explicit solutions that blow up twice as fast as the lower bound. This shows at least that the lower bound is not always optimal. The asymptotic profile is completely unknown, except for the explicit example mentioned above and some related examples.

In Section 2, we recall some basic properties of the equation (1.1), and we present the general results concerning the blow up: the blow up theorems, the lower estimate and the blow up of a family of $L^p$ norms.

In Section 3, we consider the so-called “critical” or “pseudo-conformal” case $\alpha = 4/N$. We first establish sharp existence results concerning the initial value problem in $H^1(\mathbf{R}^N)$ and $L^2(\mathbf{R}^N)$. Next, we describe some properties of the blowing up solutions that are known only for the critical nonlinearity.

In Section 4, we still consider the “pseudo-conformal” case $\alpha = 4/N$, and we introduce the pseudo-conformal transformation. We deduce some results concerning the blow up of solutions and the scattering theory. These results may appear as miracles, in that they are obtained by merely applying the pseudo-conformal transformation. (Note, however, that some of these results are recovered in the following section by apparently more natural methods.) The fact that one obtains by the same method both results on the blow up phenomenon and results on the asymptotic behavior of global solutions may look surprising. This is, however, natural since the pseudo-conformal transformation closely relates the asymptotic behavior of the global solutions and the behavior near the blow up time of the nonglobal solutions.

In Section 5, we extend certain of the results of Section 4 to the non-conformal case $\alpha \neq 4/N$. They are obtained by a fixed point argument based on certain Strichartz type inequalities.

In Section 6, we present some results concerning the scattering theory in the non-conformal case (i.e. $\alpha \neq 4/N$). Even though the equation is no longer invariant by the pseudo-conformal transformation, we still apply the transformation. This leads to a nonautonomous nonlinear Schrödinger equation. The local (in time) study of that equation corresponds, via the transformation, to the study of the asymptotic behavior of the solutions of (1.1). Finally, Section 7 is devoted to the study of the initial value problem for the nonautonomous equation cited above.

We conclude this section by introducing the notation that will be used throughout these notes. For $m \in \mathbf{Z}$, we denote by $H^m(\mathbf{R}^N)$ the Sobolev space $H^m(\mathbf{R}^N, \mathbf{C})$, equipped with its usual norm $\| \cdot \|_{H^m}$. For $1 \leq p \leq \infty$, we denote by $L^p(\mathbf{R}^N)$ the Lebesgue space $L^p(\mathbf{R}^N, \mathbf{C})$, equipped with its usual norm $\| \cdot \|_{L^p}$, and for $m \in \mathbf{N}$, we denote by $W^{m,p}(\mathbf{R}^N)$ the Sobolev space space $W^{m,p}(\mathbf{R}^N, \mathbf{C})$, equipped with its usual norm $\| \cdot \|_{W^{m,p}}$. Given an interval $I \subset \mathbf{R}$, a Banach space $X$ and $1 \le p \le \infty$, we denote by $L^p(I,X)$ the Banach space of measurable functions $u : I \to X$ such that the function $t \mapsto \|u(t)\|_X$ belongs to $L^p(I)$. Its norm is denoted by $\|\cdot\|_{L^p(I,X)}$, or $\|\cdot\|_{L^p(I)}$, or even $\|\cdot\|_{L^p}$ when there is no risk of confusion. Given $1 \le p \le \infty$, the conjugate exponent $p'$ of $p$ is defined by

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

We define the space $Y \hookrightarrow L^2(\mathbf{R}^N)$ by $Y = \{u \in L^2(\mathbf{R}^N); \, |\cdot|u(\cdot) \in L^2(\mathbf{R}^N)\}$ endowed with the norm $\|u\|_Y = \|u\|_{L^2} + \|xu\|_{L^2}$, and the space $X \hookrightarrow H^1(\mathbf{R}^N)$ by

$$X = H^1(\mathbf{R}^N) \cap Y,$$

endowed with the norm $\|u\|_X = \|u\|_{H^1} + \|u\|_Y$.

We say that a pair $(q,r)$ of real numbers is *admissible* if $2 \le r < \dfrac{2N}{N-2}$ $(2 \le r \le \infty$ if $N = 1)$ and if

$$\frac{2}{q} = N\left(\frac{1}{2} - \frac{1}{r}\right).$$

Note that in particular $2 < q \le \infty$.

We denote by $(T(t))_{t\in\mathbf{R}}$ the group of isometries generated by $i\triangle$ in $L^2(\mathbf{R}^N)$, i.e. $T(t) = e^{it\triangle}$.

Finally, we recall Strichartz' estimates. Let $I \ni 0$ be any open interval of $\mathbf{R}$ (bounded or not) and let $J = \overline{I}$, let $(\gamma,\rho)$ be an admissible pair, let $\varphi \in L^2(\mathbf{R}^N)$ and let $f \in L^{\gamma'}(I, L^{\rho'}(\mathbf{R}^N))$. Then, for any admissible pair $(q,r)$, the function

$$u(t) = T(t)\varphi + \int_0^t T(t-s)f(s)\,ds$$

belongs to $C(J, L^2(\mathbf{R}^N)) \cap L^q(I, L^r(\mathbf{R}^N))$, and

$$\|u\|_{L^q(I,L^r)} \le C\|\varphi\|_{L^2} + C\|f\|_{L^{\gamma'}(I,L^{\rho'})}, \tag{1.2}$$

for some constant $C$ independent of $I$. (See [5], Theorem 3.2.5.)

**2. Some general results.** We recall that under fairly general assumptions, and in particular for all the solutions that we consider in these notes, $u$ is a solution of (1.1) on some interval $I \ni 0$ if and only if

$$u(t) = T(t)\varphi + i\lambda \int_0^t T(t-s)|u(s)|^\alpha u(s)\, ds, \tag{2.1}$$

for all $t \in I$. See for example [5], Remark 2.5.1. We first recall the existence results.

**Theorem 2.1.** *Assume $\lambda \in \mathbf{R}$ and $0 \le \alpha < \dfrac{4}{N-2}$ ($0 \le \alpha < \infty$ if $N = 1$). For every $\varphi \in H^1(\mathbf{R}^N)$, there exists a unique solution $u$ of (1.1) defined on a maximal interval $(-T_*, T^*)$ with $T_*, T^* > 0$, $u \in C((-T_*, T^*), H^1(\mathbf{R}^N)) \cap C^1((-T_*, T^*), H^{-1}(\mathbf{R}^N))$.*

*In addition, we have the following blow up alternative. Either $T^* = \infty$ (respectively, $T_* = \infty$), or else $T^* < \infty$ and $\lim_{t \uparrow T^*} \|u(t)\|_{H^1} = +\infty$ (respectively, $T_* < \infty$ and $\lim_{t \downarrow -T^*} \|u(t)\|_{H^1} = +\infty$).*

*Moreover, there is conservation of charge and energy, i.e.*

$$\|u(t)\|_{L^2} = \|\varphi\|_{L^2}, \tag{2.2}$$

*and*

$$E(u(t)) =: \int_{\mathbf{R}^N} \left( \frac{1}{2}|\nabla u(t,x)|^2 - \frac{\lambda}{\alpha+2}|u(t,x)|^{\alpha+2} \right) dx = E(\varphi), \tag{2.3}$$

*for all $t \in (-T_*, T^*)$.*

*Finally, if $\varphi \in X$, then $u \in C((-T_*, T^*), X)$ and we have the pseudo-conformal conservation law*

$$\begin{aligned} \|(x + 2it\nabla)u(t)\|_{L^2}^2 &- \frac{8\lambda t^2}{\alpha+2}\|u(t)\|_{L^{\alpha+2}}^{\alpha+2} \\ &= \|x\varphi\|_{L^2}^2 - 4\lambda\frac{4 - N\alpha}{\alpha+2}\int_0^t s \int_{\mathbf{R}^N} |u(s,x)|^{\alpha+2}\, dx\, ds, \end{aligned} \tag{2.4}$$

*for all $t \in (-T_*, T^*)$.*

See [5], Theorem 4.3.1 and Remark 4.3.2 for the existence and the conservation of charge and energy; Lemma 6.4.3 for the $X$ regularity; Theorem 7.1.1 for the pseudo-conformal conservation law.

**Remark 2.2.** The solution $u$ has some further properties, which we recall below.

(i) For any admissible pair $(q, r)$, we have $u \in L^q_{loc}((-T_1, T_2), W^{1,r}(\mathbf{R}^N))$ for $-T_* < -T_1 < T_2 < T^*$.

(ii) The solution $u$ depends continuously on the initial value $\varphi$ in the following sense. The mappings $\varphi \mapsto T_*$ and $\varphi \mapsto T^*$ are lower-semicontinuous $H^1(\mathbf{R}^N) \to \mathbf{R}$. Moreover, if $\varphi_n \underset{n\to\infty}{\longrightarrow} \varphi$ in $H^1(\mathbf{R}^N)$ and if $u_n$ is the corresponding solution of (1.1), then $u_n \underset{n\to\infty}{\longrightarrow} u$ in $C([-T_1, T_2], H^1(\mathbf{R}^N))$ for any bounded interval $[-T_1, T_2] \subset (-T_*, T^*)$.

(iii) Moreover, if $\varphi_n \underset{n\to\infty}{\longrightarrow} \varphi$ in $X$, then $u_n \underset{n\to\infty}{\longrightarrow} u$ in $C([-T_1, T_2], X)$.

(iv) If $\varphi \in H^2(\mathbf{R}^N)$, then $u \in C((-T_*, T^*), H^2(\mathbf{R}^N)) \cap C^1((-T_*, T^*), L^2(\mathbf{R}^N))$.

See [5], Theorem 5.3.1 and Remark 5.3.5 for the $W^{1,r}$ smoothing effect; Theorem 4.3.1 for the continuous dependence in $H^1(\mathbf{R}^N)$; Corollary 6.4.4 for the continuous dependence in $X$; Theorem 5.2.1 and Remark 5.2.9 for the $H^2$ regularity.

**Remark 2.3.** If $\alpha < 4/N$, then the Cauchy problem for (1.1) is globally well posed in $L^2(\mathbf{R}^N)$ (for any $\lambda \in \mathbf{R}$). More precisely, we have the following result. (See [5], Theorem 6.3.1.)

For every $\varphi \in L^2(\mathbf{R}^N)$, there exists a unique solution $u$ of (1.1), $u \in C(\mathbf{R}, L^2(\mathbf{R}^N)) \cap L^\theta_{loc}(\mathbf{R}, L^{\alpha+2}(\mathbf{R}^N))$ with $\theta = \dfrac{4(\alpha+2)}{N\alpha}$. Moreover, $u_t \in L^\theta_{loc}(\mathbf{R}, H^{-2}(\mathbf{R}^N))$. In addition, $u \in L^q_{loc}(\mathbf{R}, L^r(\mathbf{R}^N))$ for every admissible pair $(q, r)$.

There is conservation of charge, i.e. (2.2) holds for all $t \in \mathbf{R}$.

The solution $u$ depends continuously on the initial value $\varphi$ in the sense that if $\varphi_n \underset{n\to\infty}{\longrightarrow} \varphi$ in $L^2(\mathbf{R}^N)$, then $u_n \underset{n\to\infty}{\longrightarrow} u$ in $C([-T, T], L^2(\mathbf{R}^N))$ for any $T < \infty$, where $u_n$ is the solution of (1.1) with initial value $\varphi_n$.

**Remark 2.4.** Here are some sufficient conditions for global existence.

(i) It follows immediately from the blow up alternative and the conservations of charge and energy that if $\lambda \le 0$, then for every $\varphi \in H^1(\mathbf{R}^N)$ the solution is global and bounded, i.e. $\sup\limits_{t\in\mathbf{R}} \|u(t)\|_{H^1} < \infty$.

(ii) The same properties hold for $\lambda > 0$ provided $\alpha < 4/N$. Indeed, it follows from Gagliardo-Nirenberg's inequality that

$$\|u\|^{\alpha+2}_{L^{\alpha+2}} \le C \|\nabla u\|^{\frac{N\alpha}{2}}_{L^2} \|u\|^{\alpha+2-\frac{N\alpha}{2}}_{L^2}.$$

Therefore, by conservation of charge and energy

$$\|u(t)\|_{H^1}^2 \le C + C\|u(t)\|_{H^1}^{\frac{N\alpha}{2}} \|\varphi\|_{L^2}^{\alpha+2-\frac{N\alpha}{2}},$$

and the conclusion follows, since $\dfrac{N\alpha}{2} < 2$.

(iii) By applying the same argument in the case $\lambda > 0$ and $\alpha = 4/N$, one obtains easily that there exists $\varepsilon > 0$ such that if $\|\varphi\|_{L^2} \le \varepsilon$, then $u$ is global and bounded. (See Section 3.)

(iv) If $\lambda > 0$ and $\alpha > 4/N$, then there exists $\varepsilon > 0$ such that if $\|\varphi\|_{H^1} \le \varepsilon$, then $u$ is global and bounded. See [5], Theorem 6.2.1.

If $\lambda > 0$ and $\alpha \ge 4/N$, then some solutions of (1.1) blow up in finite time. We recall the first result in this direction. (See Glassey [21].)

**Theorem 2.5.** *Assume $\lambda > 0$ and $\dfrac{4}{N} \le \alpha < \dfrac{4}{N-2}$ ($4/N \le \alpha < \infty$ if $N = 1$). If $\varphi \in X$ is such that $E(\varphi) < 0$, then $T_* < \infty$ and $T^* < \infty$, i.e. the solution $u$ of (1.1) blows up for both $t > 0$ and $t < 0$.*

**Proof.** We recall the idea of the proof. (See for example [5], Theorem 6.4.7 for the complete proof.) Since $\varphi \in X$, it follows that $u \in C((-T_*, T^*), X)$, and in particular $xu \in C((-T_*, T^*), L^2(\mathbf{R}^N))$. One shows, by an approximation argument, that the mapping $t \mapsto \|xu(t,x)\|_{L^2}^2$ belongs to $C^2(-T_*, T^*)$, and that

$$\frac{d^2}{dt^2} \int_{\mathbf{R}^N} |x|^2 |u(t,x)|^2 \, dx = 4N\alpha E(\varphi) - 2(N\alpha - 4) \int_{\mathbf{R}^N} |\nabla u(t,x)|^2 \, dx \le 4N\alpha E(\varphi). \quad (2.5)$$

In particular, the nonnegative function $\|xu(t,x)\|_{L^2}^2$ has a second derivative which is bounded by a negative constant, which implies that $T_* < \infty$ and $T^* < \infty$. □

**Remark 2.6.** The assumption $E(\varphi) < 0$ means in some sense that $\varphi$ is “large”. Indeed, if $\varphi$ is small enough, then the solution is global; and if $\varphi \in X$ is not identically zero, then $E(k\varphi) < 0$ for $|k|$ large enough.

The above proof of blow up is interesting. The condition for blow up is $E(\varphi) < 0$. However, the argument is based on the study of $\|xu(t,x)\|_{L^2}^2$, and this quantity is defined for $\varphi \in X$, but not for a general $\varphi \in H^1(\mathbf{R}^N)$. The question as to whether or not negative energy implies blow up for general $H^1$ solutions is open. There is, however, a recent progress in this direction. Ogawa and Tsutsumi [34] have shown (under the extra assumptions $N \geq 2$ and $\alpha \leq 4$) that if $\varphi \in H^1(\mathbf{R}^N)$ is spherically symmetric and if $E(\varphi) < 0$, then $u$ blows up in finite time. The result is the following.

**Theorem 2.7.** *Assume $N \geq 2$, $\lambda > 0$ and $\frac{4}{N} \leq \alpha < \frac{4}{N-2}$ ($2 \leq \alpha \leq 4$ if $N = 2$). If $\varphi \in H^1(\mathbf{R}^N)$ is such that $E(\varphi) < 0$ and if $\varphi$ is spherically symmetric, then $T_* < \infty$ and $T^* < \infty$, i.e. the solution $u$ of (1.1) blows up for both $t > 0$ and $t < 0$.*

The proof is in some way an adaptation of the proof of Theorem 2.5. Roughly speaking, instead of calculating $\|xu(t,x)\|_{L^2}^2$, we evaluate $\|M(x)u(t,x)\|_{L^2}^2$ where $M : \mathbf{R}^N \to \mathbf{R}$ is a function such that $M(x) = |x|$ for $|x| \leq R$ and $M$ is constant for $|x|$ large. Then, we use the decay properties of the spherically symmetric functions of $H^1(\mathbf{R}^N)$ to estimate certain integrals for $|x|$ large that appear in the calculation of $\|M(x)u(t,x)\|_{L^2}^2$. Note that, as opposed to the case $\varphi \in X$, the appropriate function $M(x)$ depends on the initial value $\varphi$.

The proof makes use of the following lemma.

**Lemma 2.8.** *Let $N \geq 1$ and let $k \in C^1([0,\infty))$ be a nonnegative function such that $r^{-(N-1)}k(r) \in L^\infty(0,\infty)$ and $r^{-(N-1)}(k'(r))^- \in L^\infty(0,\infty)$. There exists a constant $C$ such that*

$$\|k^{\frac{1}{2}}u\|_{L^\infty(\mathbf{R}^N)} \leq C\|u\|_{L^2(\mathbf{R}^N)}^{\frac{1}{2}} \left( \|r^{-(N-1)}ku_r\|_{L^2(\mathbf{R}^N)}^{\frac{1}{2}} + \|r^{-(N-1)}(k')^-\|_{L^\infty}^{\frac{1}{2}} \|u\|_{L^2(\mathbf{R}^N)}^{\frac{1}{2}} \right),$$

*for all spherically symmetric functions $u \in H^1(\mathbf{R}^N)$.*

**Proof.** By density, we may assume that $u \in \mathcal{D}(\mathbf{R}^N)$. For $s > 0$, we have

$$\begin{aligned} k(s)|u(s)|^2 &= -\int_s^\infty \frac{d}{ds}\left(k(s)|u(s)|^2\right) ds \\ &= -\int_s^\infty k'(s)|u(s)|^2\, ds - 2\int_s^\infty k(s)\mathrm{Re}\,(u(s)\overline{u}_r(s))\, ds \\ &\leq \int_s^\infty (k'(s))^-|u(s)|^2\, ds + 2\int_s^\infty k(s)|u(s)|\,|u_r(s)|\, ds \\ &\leq C\|r^{-(N-1)}(k')^-\|_{L^\infty}\|u\|_{L^2(\mathbf{R}^N)}^2 + C\|u\|_{L^2(\mathbf{R}^N)}\|r^{-(N-1)}ku_r\|_{L^2(\mathbf{R}^N)}. \end{aligned}$$

Hence the result. □

**Proof of Theorem 2.7.** By scaling, we may assume $\lambda = 1$. Let $u$ be as in the statement of the theorem. Consider a function $\Psi \in W^{4,\infty}(\mathbf{R}^N)$, and set

$$V(t) = \frac{1}{2} \int_{\mathbf{R}^N} \Psi(x) |u(t,x)|^2 \, dx,$$

for all $t \in (-T_*, T^*)$. We claim that

$$\frac{d^2}{dt^2} V(t) = 2 \int_{\mathbf{R}^N} (H(\Psi) \nabla u, \nabla u) \, dx - \frac{\alpha}{\alpha+2} \int_{\mathbf{R}^N} \triangle \Psi |u|^{\alpha+2} \, dx - \frac{1}{2} \int_{\mathbf{R}^N} \triangle^2 \Psi |u|^2 \, dx, \quad (2.6)$$

for all $t \in (-T_*, T^*)$, where the Hessian matrix $H(\Psi)$ is given by $H(\Psi) = (\partial_j \partial_k \Psi)_{1 \le j,k \le N}$. If $\varphi \in H^2(\mathbf{R}^N)$, then $u$ is an $H^2$ solution (see Remark 2.2 (iv)) and (2.6) follows from elementary calculations (see Kavian [27]). The general case follows by approximating $\varphi$ in $H^1(\mathbf{R}^N)$ by a sequence $(\varphi_n)_{n \ge 0} \subset H^2(\mathbf{R}^N)$ and using the continuous dependance (see Remark 2.2 (ii)). Next, we rewrite (2.6) as follows.

$$\begin{aligned} \frac{d^2}{dt^2} V(t) = 2N\alpha E(u(t)) &- 2 \int_{\mathbf{R}^N} \left\{ \frac{N\alpha}{2} |\nabla u|^2 - (H(\Psi) \nabla u, \nabla u) \right\} dx \\ &+ \frac{\alpha}{\alpha+2} \int_{\mathbf{R}^N} (2N - \triangle \Psi) |u|^{\alpha+2} \, dx - \frac{1}{2} \int_{\mathbf{R}^N} \triangle^2 \Psi |u|^2 \, dx. \quad (2.7) \end{aligned}$$

Let now $\rho \in \mathcal{D}(\mathbf{R})$ be such that $\rho(x) \equiv \rho(4-x)$, $\rho \ge 0$, $\int_{\mathbf{R}} \rho = 1$, $\operatorname{supp}(\rho) \subset (1,3)$ and $\rho' \ge 0$ on $(-\infty, 2)$. We define the function $\theta$ by

$$\theta(r) = r - \int_0^r (r - \sigma) \rho(\sigma) \, d\sigma,$$

for $r \ge 0$. We consider $\varepsilon \in (0,1)$, to be specified later, and we set

$$\Psi(x) \equiv \Psi(r) = \frac{1}{\varepsilon} \theta(\varepsilon r^2),$$

and

$$\gamma(x) \equiv \gamma(r) = 1 - \theta'(\varepsilon r^2) - 2\varepsilon r^2 \theta''(\varepsilon r^2) = \int_0^{\varepsilon r^2} \rho(s) \, ds + 2\varepsilon r^2 \rho(\varepsilon r^2),$$

for $x \in \mathbf{R}^N$ and $r = |x|$. Elementary calculations show that

$$\begin{cases} (H(\Psi) \nabla u, \nabla u) = 2(1 - \gamma(r)) |u_r|^2, \\ \triangle \Psi = 2N(1 - \gamma(r)) + 4(1-N) \varepsilon r^2 \theta''(\varepsilon r^2), \end{cases} \quad (2.8)$$

and that

$$\triangle^2 \Psi = \varepsilon \left( 4N(N+2) \theta''(\varepsilon r^2) + 16(N+2) \varepsilon r^2 \theta'''(\varepsilon r^2) + 16 (\varepsilon r^2)^2 \theta''''(\varepsilon r^2) \right).$$

In particular, there exists a constant $a$ such that

$$\|\triangle^2\Psi\|_{L^\infty} \le 2a\varepsilon. \tag{2.9}$$

It now follows from (2.7), (2.8) and (2.9) that

$$\frac{d^2}{dt^2}V(t) \le 2N\alpha E(\varphi) - 4\int \gamma(r)|u_r|^2 + \frac{2N\alpha}{\alpha+2}\int \gamma(r)|u|^{\alpha+2} + a\varepsilon\|u\|_{L^2}^2, \tag{2.10}$$

where we have used the above relations and the properties $\dfrac{N\alpha}{4} \ge 1$ and $\theta'' \le 0$. We claim that there exists $b$ and $c$ such that

$$\frac{2N\alpha}{\alpha+2}\int \gamma(r)|u|^{\alpha+2} \le b\varepsilon^{\frac{(N-1)\alpha}{4}}\|u\|_{L^2}^{\frac{\alpha+4}{2}}\left(\int \gamma|u_r|^2\right)^{\frac{\alpha}{4}} + c\varepsilon^{\frac{N\alpha}{4}}\|u\|_{L^2}^{\alpha+2}. \tag{2.11}$$

Indeed, we first observe that $\gamma(r) \le 1 + 2\sup_{s\ge 0} s\rho(s)$, so that

$$\begin{aligned}\int \gamma(r)|u|^{\alpha+2} &\le C\|\gamma^{\frac{1}{4}}u\|_{L^\infty}^{\alpha}\|u\|_{L^2}^2\\ &\le C\|u\|_{L^2}^{\frac{\alpha+4}{2}}\|r^{-(N-1)}\gamma^{\frac{1}{2}}u_r\|_{L^2}^{\frac{\alpha}{2}} + C\|u\|_{L^2}^{\alpha+2}\|r^{-(N-1)}\gamma^{-\frac{1}{2}}(\gamma')^-\|_{L^\infty}^{\frac{\alpha}{2}}.\end{aligned}$$

The first inequality follows from the property $\alpha \le 4$ and the second from Lemma 2.8 (one verifies easily that $\gamma^{\frac{1}{2}} \in C^1([0,\infty))$). Observe that $\gamma(r) \equiv 0$ for $r \le \varepsilon^{-\frac{1}{2}}$, so that $\|r^{-(N-1)}\gamma^{\frac{1}{2}}u_r\|_{L^2} \le \varepsilon^{\frac{N-1}{2}}\|\gamma^{\frac{1}{2}}u_r\|_{L^2}$. Next, note that $\gamma \ge 1/2$ for $\varepsilon r^2 \ge 2$. Furthermore,

$$\gamma'(r) = 6\varepsilon r\rho(\varepsilon r^2) + 4\varepsilon^2 r^3\rho'(\varepsilon r^2),$$

so that $\gamma'(r) \ge 0$ for $\varepsilon r^2 \le 2$ and

$$\gamma'(r) \ge -4\varepsilon^2 r^3|\rho'(\varepsilon r^2)| \ge -4\varepsilon^{\frac{1}{2}}\|s^{\frac{3}{2}}\rho'(s)\|_{L^\infty(0,\infty)}.$$

Therefore, $\|r^{-(N-1)}\gamma^{-\frac{1}{2}}(\gamma')^-\|_{L^\infty} \le C\varepsilon^{\frac{N}{2}}$, and (2.11) follows. It now follows from (2.10), (2.11) and conservation of charge that

$$\begin{aligned}\frac{d^2}{dt^2}V(t) \le 2N\alpha E(\varphi) - 4\int \gamma(r)|u_r|^2 + b\varepsilon^{\frac{(N-1)\alpha}{4}}\|\varphi\|_{L^2}^{\frac{\alpha+4}{2}}\left(\int \gamma|u_r|^2\right)^{\frac{\alpha}{4}}\\ + c\varepsilon^{\frac{N\alpha}{4}}\|\varphi\|_{L^2}^{\alpha+2} + a\varepsilon\|\varphi\|_{L^2}^2.\end{aligned}$$

Finally, since $\alpha \le 4$, we may apply the inequality $x^{\frac{\alpha}{4}} \le x + 1$ to obtain

$$\begin{aligned}\frac{d^2}{dt^2}V(t) \le 2N\alpha E(\varphi) - (4 - b\varepsilon^{\frac{(N-1)\alpha}{4}}\|\varphi\|_{L^2}^{\frac{\alpha+4}{2}})\int \gamma(r)|u_r|^2\\ + b\varepsilon^{\frac{(N-1)\alpha}{4}}\|\varphi\|_{L^2}^{\frac{\alpha+4}{2}} + c\varepsilon^{\frac{N\alpha}{2}}\|\varphi\|_{L^2}^{\alpha+2} + a\varepsilon\|\varphi\|_{L^2}^2.\end{aligned} \tag{2.12}$$

We note that the constants $a$, $b$ and $c$ do not depend on $\varphi$ and $\varepsilon$. Since $E(\varphi) \le 0$ and $\alpha \le 4$, it follows immediately from (2.12) that one can choose $\varepsilon > 0$ depending only on $\varphi$ through $\|\varphi\|_{L^2}$ and $E(\varphi)$ such that

$$\frac{d^2}{dt^2}V(t) \le N\alpha E(\varphi), \tag{2.13}$$

for all $t \in (-T_*, T^*)$. Since $V(t) \ge 0$, (2.13) implies that $T_* < \infty$ and $T^* < \infty$. □

**Remark 2.9.** There are two limitations in the above proof. The first one is $\alpha \le 4$. If $\alpha > 4$, there appear powers of $\|\gamma^{\frac{1}{2}} u_r\|_{L^2}$ larger than 2 with positive coefficients in (2.12). This is due to the homogeneity in Lemma 2.8. The other limitation is $N \ge 2$, since if $N = 1$ the power of $\varepsilon$ in the second and third terms of the right hand side of (2.12) vanish. This is due to the fact that the radially symmetric functions in dimension 1 do not have any decay property.

However, in the critical case $N = 1$, $\alpha = 4$, Ogawa and Tsutsumi [35] have proved that all negative energy $H^1$ solutions blow up in finite time without any symmetry assumption. Their method is a more sophisticated version of the above argument.

We now give a lower estimate for blowing up solutions (see [9]).

**Theorem 2.10.** *Assume $\lambda > 0$ and $\frac{4}{N} \le \alpha < \frac{4}{N-2}$ ($4/N \le \alpha < \infty$ if $N = 1$). If $\varphi \in H^1(\mathbf{R}^N)$ is such that $T^* < \infty$, then there exists $\delta > 0$ such that*

$$\|\nabla u(t)\|_{L^2} \ge \frac{\delta}{(T^* - t)^{\frac{1}{\alpha} - \frac{N-2}{4}}}, \tag{2.14}$$

*for $0 \le t < T^*$. A similar estimate holds near $-T_*$ if $T_* < \infty$.*

**Proof.** Generally speaking, every time a local existence result is obtained for an evolution equation, the proof also gives a lower estimate of blow up. Here, we do not go through the entire local existence argument, but instead we give a direct proof. Set $r = \alpha + 2$ and let $q$ be such that $(q, r)$ is an admissible pair. Let $\varphi$ be as above, and let $u$ be the corresponding solution of (1.1). It follows from Hölder's inequality that

$$\|\nabla(|u|^\alpha u)\|_{L^{r'}} \le C\| \, |u|^\alpha |\nabla u| \, \|_{L^{r'}} \le C\|u\|_{L^r}^\alpha \|\nabla u\|_{L^r}. \tag{2.15}$$

By conservation of energy, we have

$$\lambda \|u\|_{L^r}^r = -rE(\varphi) + \frac{r}{2}\|\nabla u\|_{L^2}^2.$$

It follows that

$$\lambda \|u\|_{L^r}^\alpha \le C(1 + \|\nabla u\|_{L^2}^2)^{\frac{\alpha}{r}} \le C(1 + \|\nabla u\|_{L^2})^{\frac{2\alpha}{r}}.$$

It follows from (2.15) and the above inequality that for any $0 < t < \tau < T^*$,

$$\begin{aligned} \|\nabla(|u|^\alpha u)\|_{L^{q'}((t,\tau),L^{r'})} &\le C(1 + \|\nabla u\|_{L^\infty((t,\tau),L^2)})^{\frac{2\alpha}{r}} \|\nabla u\|_{L^{q'}((t,\tau),L^r)} \\ &\le C(\tau - t)^{\frac{q-q'}{qq'}} (1 + \|\nabla u\|_{L^\infty((t,\tau),L^2)})^{\frac{2\alpha}{r}} \|\nabla u\|_{L^q((t,\tau),L^r)}. \end{aligned}$$

Set now

$$f_t(\tau) = 1 + \|\nabla u\|_{L^\infty((t,\tau),L^2)} + \|\nabla u\|_{L^q((t,\tau),L^r)},$$

so that by the above inequality,

$$\|\nabla(|u|^\alpha u)\|_{L^{q'}((t,\tau),L^{r'})} \le C(\tau - t)^{\frac{q-q'}{qq'}} f_t(\tau)^{1+\frac{2\alpha}{r}}. \tag{2.16}$$

On the other hand, it follows from the equation (2.1) and the Strichartz' estimates (1.2) that

$$\|\nabla u\|_{L^\infty((t,\tau),L^2)} + \|\nabla u\|_{L^q((t,\tau),L^r)} \le C\|\nabla u(t)\|_{L^2} + C\|\nabla(|u|^\alpha u)\|_{L^{q'}((t,\tau),L^{r'})},$$

for $0 < t < \tau < T^*$. By (2.16), this implies that

$$f_t(\tau) \le C(1 + \|\nabla u(t)\|_{L^2}) + C(\tau - t)^{\frac{q-q'}{qq'}} f_t(\tau)^{1+\frac{2\alpha}{r}}, \tag{2.17}$$

for $0 < t < \tau < T^*$.

Consider now $t \in (0, T^*)$. Note that if $T^* < \infty$, it follows from the blow up alternative that $f_t(\tau) \to \infty$ as $\tau \uparrow T^*$. Note also that $f_t$ is continuous and nondecreasing on $(t, T^*)$ and that

$$f_t(\tau) \underset{\tau \downarrow t}{\longrightarrow} 1 + \|\nabla u(t)\|_{L^2}.$$

Therefore, there exists $\tau_0 \in (t, T^*)$ such that $f_t(\tau_0) = (C + 1)(1 + \|\nabla u(t)\|_{L^2})$, where $C$ is the constant in (2.17). Choosing $\tau = \tau_0$ in (2.17) yields

$$\begin{aligned} 1 + \|\nabla u(t)\|_{L^2} &\le C(1 + C)^{1+\frac{2\alpha}{r}} (\tau - t)^{\frac{q-q'}{qq'}} (1 + \|\nabla u(t)\|_{L^2})^{1+\frac{2\alpha}{r}} \\ &\le (1 + C)^{2+\frac{2\alpha}{r}} (T^* - t)^{\frac{q-q'}{qq'}} (1 + \|\nabla u(t)\|_{L^2})^{1+\frac{2\alpha}{r}}; \end{aligned}$$

and so,

$$1+\|\nabla u(t)\|_{L^2} \geq \frac{1}{(1+C)^{1+\frac{r}{\alpha}}(T^*-t)^{\frac{r(q-q')}{2\alpha qq'}}}.$$

Hence the result, since $t \in [0,T^*)$ is arbitrary and $\dfrac{r(q-q')}{2\alpha qq'} = \dfrac{1}{\alpha} - \dfrac{N-2}{4}$. □

Before proceeding further, we establish an immediate consequence of the above result concerning the blowing up of certain $L^p$ norms of the solution.

**Corollary 2.11.** *Assume $\lambda > 0$ and $\dfrac{4}{N} \leq \alpha < \dfrac{4}{N-2}$ ($4/N \leq \alpha < \infty$ if $N=1$). If $\varphi \in H^1(\mathbf{R}^N)$ is such that $T^* < \infty$, then $\|u(t)\|_{L^p} \underset{t\uparrow T^*}{\longrightarrow} \infty$ for all $p > \dfrac{N\alpha}{2}$. Moreover,*

$$\|u(t)\|_{L^p} \geq \frac{\delta}{(T^*-t)^{\frac{1}{\alpha}-\frac{N}{2p}}}, \tag{2.18}$$

*for $0 < t < T^*$ if $\dfrac{N\alpha}{2} < p \leq \alpha+2$ and*

$$\|u(t)\|_{L^p} \geq \frac{\delta}{(T^*-t)^{\frac{4-(N-2)\alpha}{\alpha^2}\left(\frac{1}{2}-\frac{1}{p}\right)}}, \tag{2.19}$$

*for $0 < t < T^*$ if $p \geq \alpha+2$. A similar estimate holds near $-T_*$ if $T_* < \infty$.*

**Remark.** Note that if $N \geq 3$ and $p > \dfrac{2N}{N-2}$ or if $N=2$ and $p=\infty$, then it may happen that $\|u(t)\|_{L^p} = \infty$ for some (or all) $t \in (-T_*, T^*)$. Clearly, this does not contradict the above estimates. Note, however, that $u \in L^q_{loc}((-T_*,T^*), W^{1,r}(\mathbf{R}^N))$ for every admissible pair $(q,r)$, so that by Sobolev's embedding theorem, $\|u(t)\|_{L^p} < \infty$ for almost all $t \in (-T_*, T^*)$ provided $N \leq 3$ or $N \geq 4$ and $p < \dfrac{2N}{N-4}$.

**Proof of Corollary 2.11.** Suppose first $\dfrac{N\alpha}{2} < p \leq \alpha+2$. It follows from Gagliardo-Nirenberg's inequality that

$$\|u\|_{L^{\alpha+2}}^{\alpha+2} \leq C\|\nabla u\|_{L^2}^{2-\mu}\|u\|_{L^p}^{\alpha+\mu},$$

with $\mu = \dfrac{4p-2N\alpha}{2N-(N-2)p}$. By conservation of energy and the above inequality, we obtain

$$\begin{aligned}\|\nabla u(t)\|_{L^2}^2 &\leq 2E(\varphi) + \frac{2\lambda}{\alpha+2}\|u(t)\|_{L^{\alpha+2}}^{\alpha+2} \\ &\leq C + C\|\nabla u(t)\|_{L^2}^{2-\mu}\|u(t)\|_{L^p}^{\alpha+\mu},\end{aligned}$$

for all $0 < t < T^*$. Since $\|\nabla u(t)\|_{L^2} \underset{t\uparrow T^*}{\longrightarrow} \infty$, it follows that

$$\|\nabla u(t)\|_{L^2}^{\mu} \le C\|u(t)\|_{L^p}^{\alpha+\mu}.$$

By Theorem 2.10, this implies

$$\|u(t)\|_{L^p} \ge \frac{\varepsilon}{(T^* - t)^{\frac{\mu}{\alpha+\mu}\left(\frac{1}{\alpha} - \frac{N-2}{4}\right)}}.$$

(2.18) follows, since $\dfrac{\mu}{\alpha+\mu}\left(\dfrac{1}{\alpha} - \dfrac{N-2}{4}\right) = \dfrac{1}{\alpha} - \dfrac{N}{2p}$.

Suppose now $p \ge \alpha + 2$. It follows from Hölder's inequality that

$$\|u\|_{L^{\alpha+2}}^{\alpha+2} \le \|u\|_{L^p}^{\frac{\alpha p}{p-2}} \|u\|_{L^2}^{\frac{2(p-(\alpha+2))}{p-2}}.$$

Therefore, by conservation of charge and energy,

$$\begin{aligned}\|\nabla u(t)\|_{L^2}^2 &\le 2E(\varphi) + \frac{2\lambda}{\alpha+2}\|u(t)\|_{L^{\alpha+2}}^{\alpha+2} \\ &\le C + C\|u(t)\|_{L^p}^{\frac{\alpha p}{p-2}} \|\varphi\|_{L^2}^{\frac{2(p-(\alpha+2))}{p-2}},\end{aligned}$$

for all $0 < t < T^*$. (2.19) now follows from Theorem 2.10 and the above inequality. □

**Remark 2.12.** Theorem 2.10 and Corollary 2.11 give lower estimates of $\|\nabla u\|_{L^2}$ and $\|u\|_{L^p}$ near blow up. They do not give any upper estimate. It is interesting to compare these results to the corresponding ones for the heat equation. If one considers the equation $u_t - \triangle u = |u|^{p-1}u$ with Dirichlet boundary condition, then a simple argument (even simpler than the proof of Theorem 2.10) gives the lower estimate $\|u\|_{L^\infty} \ge (T^* - t)^{-\frac{1}{\alpha}}$. If $\alpha < \dfrac{4}{N-2}$, then it is known that this is the actual blow up rate of the solutions (see [49, 15, 22, 44]). However for larger $\alpha$'s, some solutions blow up faster (see [25]). A lower estimate is obtained as well for $\|u\|_{L^p}$, $p > \dfrac{N\alpha}{2}$. In some cases it is known that $\|u\|_{L^p}$ also blows up for $p = \dfrac{N\alpha}{2}$ (see [51]) and that $\|u\|_{L^p}$ remains bounded for $p < \dfrac{N\alpha}{2}$ (see [12]).

**Remark 2.13.** In the case $\alpha > 4/N$, one does not know the exact blow up rate of any blowing up solution. In addition, one does not know whether or not $\|u\|_{L^p}$ blows up for $2 < p \le \dfrac{N\alpha}{2}$. On the other hand, there is an upper estimate of integral form (see Merle [28]). More precisely, for $\varphi \in X$ it follows from (2.5) that

$$\frac{d^2}{dt^2}\|xu(t)\|_{L^2}^2 \le a - b\|\nabla u(t)\|_{L^2}^2,$$

for some constants $a, b > 0$. Since $\|xu(t)\|_{L^2}^2 \geq 0$, this implies that

$$\int_0^{T^*} \int_0^t \|\nabla u(s)\|_{L^2}^2 \, ds \, dt < \infty.$$

Since

$$\int_0^{T^*} \int_0^t \|\nabla u(s)\|_{L^2}^2 \, ds \, dt = \int_0^{T^*} (T^* - t) \|\nabla u(t)\|_{L^2}^2 \, dt,$$

it follows immediately from Hölder's inequality that

$$\int_0^{T^*} \|\nabla u(t)\|_{L^2}^{\mu} \, dt < \infty,$$

for $0 \leq \mu < 1$. If $\varphi \in H^1(\mathbf{R}^N)$ and $\varphi$ is spherically symmetric, then one obtains the same estimate. Indeed, by using the fact that $\alpha > 4/N$ one can improve (2.13) as follows

$$\frac{d^2}{dt^2} V(t) \leq N\alpha E(\varphi) - (N\alpha - 4) \|\nabla u\|_{L^2}^2,$$

and the conclusion is the same.

**Remark 2.14.** In the case $\alpha = 4/N$, then (2.14) becomes

$$\|\nabla u(t)\|_{L^2} \geq \frac{\delta}{(T^* - t)^{\frac{1}{2}}}, \tag{2.20}$$

and (2.18) and (2.19) become

$$\|u(t)\|_{L^p} \geq \frac{\delta}{(T^* - t)^{\frac{N}{2}\left(\frac{1}{2} - \frac{1}{p}\right)}}, \tag{2.21}$$

for $p > 2$. In particular, $\|u\|_{L^p}$ blows up for $p > 2$. Since $\|u\|_{L^2}$ is constant, estimate (2.21) is optimal with respect to $p$. On the other hand, it is known that the blow up rate given by (2.20) and (2.21) is not always optimal, since some solutions blow up twice as fast (see Section 4, and in particular Remark 4.3). Whether or not there exist solutions that blow up at rates different from the rate of these explicit solutions is an open question. Note also that (2.14) gives a lower estimate of blow up in the case $\alpha < 4/N$, but in this case there is no blow up.

**Remark 2.15.** In the proof of Theorem 2.5, one evaluates $\frac{d^2}{dt^2} \|xu(t)\|_{L^2}^2$ in order to show that if $T^*$ were infinite, then $\|xu(t)\|_{L^2}^2$ would become negative in finite time. Note, however that this does not imply that $\|xu(t)\|_{L^2}^2 \to 0$ as $t \uparrow T^*$. This is sometimes the case (see Section 4 and in particular Remark 4.3), but not always. First, observe that by the invariance of the equation under space translation, one constructs easily a solution such that $\|xu(t)\|_{L^2}^2 \not\to 0$ as $t \uparrow T^*$. Indeed, it follows from the conservation of momentum (i.e. $\displaystyle\int \overline{u}\nabla u = \int \overline{\varphi}\nabla\varphi$) that, given $x_0 \in \mathbf{R}^N$,

$$\int |x-x_0|^2|u|^2 = \int |x|^2|u|^2 + |x_0|^2 \int |\varphi|^2 + 2\int x\cdot x_0|\varphi|^2 + 4t\mathrm{Im}\int \overline{\varphi}x_0\cdot\nabla\varphi,$$

so that $\|(x-x_0)u\|_{L^2}$ will not converge to 0 for $x_0$ large enough. Moreover, in general

$$\inf\{\|(x-x_0)u(t)\|_{L^2};\, t\in[0,T^*), x_0\in\mathbf{R}^N\} > 0.$$

To see this, we follow an argument of Merle [31]. Assume $\alpha = 4/N$ and consider a real valued, spherically symmetric function $\varphi \in H^1(\mathbf{R}^N)$ such that $x\varphi(x) \notin L^2(\mathbf{R}^N)$ and $E(\varphi) < 0$. Let $(\varphi_n)_{n\geq 0} \subset X$ be a sequence of real valued, spherically symmetric functions such that $\varphi_n \underset{n\to\infty}{\longrightarrow} \varphi$ in $H^1(\mathbf{R}^N)$ and let $u_n$ be the corresponding solutions of (1.1). In particular, $E(\varphi_n) \underset{n\to\infty}{\longrightarrow} E(\varphi)$ and $\|\varphi_n\|_{L^2} \underset{n\to\infty}{\longrightarrow} \|\varphi\|_{L^2}$. Therefore, it follows from the proof of Theorem 2.7 (see in particular formula (2.13)) that there exists a function $\Psi \in W^{4,\infty}(\mathbf{R}^N)$, $\Psi \geq 0$ such that

$$\frac{d^2}{dt^2}\int \Psi|u_n|^2 \leq 2E(\varphi) < 0,$$

for $0 \leq t < T^*(\varphi_n)$. On the other hand, since $\varphi_n$ is real valued, one verifies easily that

$$\frac{d}{dt}\int \Psi|u_n|^2\Big|_{t=0} = 0,$$

so that

$$\int \Psi|u_n|^2 \leq 2\int \Psi|\varphi|^2 + t^2E(\varphi),$$

for $0 \leq t < T^*(\varphi_n)$ and for $n$ large enough. This implies that there exists $T^0 < \infty$ such that

$$T^*(\varphi_n) \leq T^0, \tag{2.22}$$

for $n$ large enough. On the other hand, for every $x_0 \in \mathbf{R}^N$, we have (see the proof of Theorem 2.5)

$$\|(x-x_0)u_n(t)\|_{L^2}^2 = \|(x-x_0)\varphi_n\|_{L^2}^2 + 8E(\varphi_n)t^2,$$

for $0 \le t < T^*(\varphi_n)$. In particular, for $n$ large enough,

$$\|(x-x_0)u_n(t)\|_{L^2}^2 \ge \|(x-x_0)\varphi_n\|_{L^2}^2 + 16E(\varphi)t^2,$$

for $0 \le t < T^*(\varphi_n)$. Since $\inf\limits_{x_0\in\mathbf{R}^N} \|(x-x_0)\varphi_n\|_{L^2}^2 \underset{n\to\infty}{\longrightarrow} \infty$, it now follows from (2.22) that

$$\inf\{\|(x-x_0)u_n(t)\|_{L^2};\ 0 \le t < T^*(\varphi_n), x_0 \in \mathbf{R}^N\} \underset{n\to\infty}{\longrightarrow} \infty,$$

which proves the claim.

**Remark 2.16.** There is an abundant literature devoted to the determination of the blow up rate by means of numerical computations. See [39] for a survey of some of this literature. These experiments lead to the utmost confusion. For a given equation, the computed blow up rates vary according to the authors, and not by an epsilon. For example, in the case $\alpha = N = 2$, the rates (as of 1993) range from $(T_m - t)^{-\frac{1}{2}}$ to $(T_m - t)^{-\frac{3}{2}}$ (see in particular the fascinating Table 1 of [39], p. 409). Even more confusing, for a given author, the rate may vary from one paper to another. On the top of this, it seems that the only blow up rate which is known to be achieved (by the pseudo-conformally self-similar solutions in the critical case $\alpha = 4/N$) was never observed, or at least never reported. This last point does not seem to worry too much the authors, the usual interpretation being that this blow up rate is “exceptional”.

There might be several reasons to explain this situation. First of all when computing numerically a blowing up solution, no matter how fine is the scheme and how close is the computed solution to the actual solution, one has to stop the computation at some point. Without any *a priori* information on the blow up rate, one cannot be sure that the asymptotics of the solution will not change drastically between the time when one stops the computation and the actual blow up time. Another major difficulty is the following. It seems that most of the numerical computations were made by truncating the solution for $|x|$ large. However the solution, and in particular the blow up time, is quite sensitive to small variations of the phase of the initial value for $|x|$ large (see Lemma 4.7 and Theorem 5.1). An interesting experiment was made to compare several methods for the linear and nonlinear Schrödinger equations, in some cases where an explicit solution exists. In particular, it is seen that the method of truncation of the domain can lead to large errors. It seems that a spectral method that takes into account the values of the initial datum for $|x|$ large is much more accurate. (See Ahjaou [1], Ahjaou and Kavian [2].)

**3. The pseudo-conformally invariant case. Sharp existence and blow up results.** In this section, we assume $\alpha = 4/N$. We recall that if $\lambda \leq 0$, then all the solutions of (1.1) with initial value $\varphi \in H^1(\mathbf{R}^N)$ are global and bounded in $H^1(\mathbf{R}^N)$ (see Remark 2.4 (i)). If $\lambda > 0$, then the solutions are global provided $\varphi \in H^1(\mathbf{R}^N)$ verifies $\|\varphi\|_{L^2} < \delta$, for some $\delta > 0$ (see Remark 2.4 (iii)). In fact, one can determine the optimal $\delta$. Let $R$ be the (unique) spherically symmetric, positive ground state of the elliptic equation

$$-\triangle R + R = |R|^{\frac{4}{N}} R, \tag{3.1}$$

in $\mathbf{R}^N$ (see for example [5], Definition 8.1.14 and Theorems 8.1.5 , 8.1.6 and 8.1.7). Note that any ground state of (3.1) is of the form $e^{i\theta}R(x-y)$ for some $\theta \in \mathbf{R}$ and $y \in \mathbf{R}^N$. We have the following result of Weinstein [46].

**Theorem 3.1.** *Assume $\lambda > 0$. If $\varphi \in H^1(\mathbf{R}^N)$ is such that $\lambda^{\frac{1}{\alpha}} \|\varphi\|_{L^2} < \|R\|_{L^2}$, then the solution $u$ of (1.1) is global and $\sup\limits_{t \in \mathbf{R}} \|u(t)\|_{H^1} < \infty$.*

**Remark 3.2.** The condition $\|\varphi\|_{L^2} < \lambda^{-\frac{1}{\alpha}} \|R\|_{L^2}$ is sharp, in the sense that for any $\rho > \lambda^{-\frac{1}{\alpha}} \|R\|_{L^2}$ (in fact, even for $\rho = \lambda^{-\frac{1}{\alpha}} \|R\|_{L^2}$, see Remark 4.3) there exists $\varphi \in H^1(\mathbf{R}^N)$ such that $\|\varphi\|_{L^2} = \rho$ and such that $u$ blows up in finite time for both $t < 0$ and $t > 0$. Indeed, let $\psi(x) = R(\sqrt{\lambda} x)$, so that $\|\psi\|_{L^2} = \lambda^{-\frac{1}{\alpha}} \|R\|_{L^2}$ and $\psi$ is a solution of

$$-\triangle \psi + \lambda \psi = \lambda |\psi|^{\alpha} \psi.$$

It follows that $E(\psi) = 0$ (see [5], formula (8.1.20)). Let $\rho > \lambda^{-\frac{1}{\alpha}} \|R\|_{L^2}$, set $\gamma = \lambda^{\frac{1}{\alpha}} \dfrac{\rho}{\|R\|_{L^2}} > 1$, and consider $\varphi_\rho = \gamma \psi$. We have $\|\varphi_\rho\|_{L^2} = \rho$ and

$$E(\varphi_\rho) = \gamma^{\alpha+2} E(\psi) - \frac{\gamma^{\alpha+2} - \gamma^2}{2} \|\nabla \psi\|_{L^2}^2 < 0;$$

and so, the corresponding solution $u_\rho$ of (1.1) blows up in finite time (see Theorem 2.5).

**Remark 3.3.** In dimension 1, an elementary calculation shows that

$$R(x) = \frac{3^{\frac{1}{4}}}{\sqrt{\cosh(2x)}}.$$

In particular,

$$\|R\|_{L^2}^2 = \pi \sqrt{3}.$$

Therefore, it follows from Theorem 3.1 that if $\varphi \in H^1(\mathbf{R}^N)$ is such that $\lambda^{\frac{1}{\alpha}} \|\varphi\|_{L^2} < 3^{\frac{1}{4}}\sqrt{\pi}$, then the solution $u$ of (1.1) is global and $\sup_{t\in\mathbf{R}} \|u(t)\|_{H^1} < \infty$.

The proof of Theorem 3.1 relies on the following lemma.

**Lemma 3.4.** *The best constant in the Gagliardo-Nirenberg inequality*

$$\frac{1}{\alpha+2}\|\psi\|_{L^{\alpha+2}}^{\alpha+2} \le \frac{C}{2}\|\nabla\psi\|_{L^2}^2 \|\psi\|_{L^2}^{\alpha},$$

*is* $C = \|R\|_{L^2}^{-\alpha}$.

**Proof.** We follow the argument of [46]. We have to show that

$$\inf_{u\in H^1, u\neq 0} J(u) = \frac{2\|R\|_{L^2}^{\alpha}}{\alpha+2}, \tag{3.2}$$

where

$$J(u) = \frac{\|\nabla u\|_{L^2}^2 \|u\|_{L^2}^{\alpha}}{\|u\|_{L^{\alpha+2}}^{\alpha+2}}.$$

We set

$$\sigma = \inf_{u\in H^1, u\neq 0} J(u),$$

and we consider a minimizing sequence $(u_n)_{n\ge 0}$. We observe that by Gagliardo-Nirenberg's inequality, $\sigma > 0$. We consider $v_n$ defined by $v_n(x) = \mu_n u_n(\lambda_n x)$ with

$$\lambda_n = \frac{\|u_n\|_{L^2}}{\|\nabla u_n\|_{L^2}} \quad \text{and} \quad \mu_n = \frac{\|u_n\|_{L^2}^{\frac{N-2}{2}}}{\|\nabla u_n\|_{L^2}^{\frac{N}{2}}},$$

so that $\|v_n\|_{L^2} = \|\nabla v_n\|_{L^2} = 1$ and

$$\|v_n\|_{L^{\alpha+2}}^{-(\alpha+2)} = J(v_n) = J(u_n) \underset{n\to\infty}{\longrightarrow} \sigma > 0.$$

By symmetrization, we may assume that $v_n$ is spherically symmetric; and so, there exists a subsequence, which we still denote by $(v_n)_{n\ge 0}$ and $v \in H^1(\mathbf{R}^N)$ such that $v_n \underset{n\to\infty}{\longrightarrow} v$ in $H^1(\mathbf{R}^N)$ weak and in $L^{\alpha+2}(\mathbf{R}^N)$ strong. Since $\|v\|_{L^{\alpha+2}} = \lim_{n\to\infty} \|v_n\|_{L^{\alpha+2}} = \sigma^{-\frac{1}{\alpha+2}} > 0$, it follows that $v \neq 0$. This implies that

$$J(v) = \sigma, \quad \text{and} \quad \|v\|_{L^2} = \|\nabla v\|_{L^2} = 1. \tag{3.3}$$

In particular, $\frac{d}{dt}J(v+tw)_{|t=0} = 0$ for all $w \in H^1(\mathbf{R}^N)$, and taking in account (3.3), we obtain

$$-\triangle v + \frac{\alpha}{2}v = \sigma\frac{\alpha+2}{2}|v|^\alpha v.$$

Let now $u$ be defined by $v(x) = au(bx)$ with $a = \left(\frac{\alpha}{\sigma(\alpha+2)}\right)^{\frac{1}{\alpha}}$ and $b = \left(\frac{\alpha}{2}\right)^{\frac{1}{2}}$, so that $u$ is a solution of (3.1) and

$$J(u) = J(v) = \sigma.$$

Since $u$ satisfies equation (3.1), it follows from Pohozaev's identity that

$$\frac{1}{2}\|\nabla u\|_{L^2}^2 = \frac{\lambda}{\alpha+2}\|u\|_{L^{\alpha+2}}^{\alpha+2},$$

and that $2\|\nabla u\|_{L^2}^2 = N\|u\|_{L^2}^2$ (see [5], formulas (8.1.20) and (8.1.21)); and so

$$J(u) = \frac{N}{N+2}\|u\|_{L^2}^{\frac{4}{N}} = \frac{2}{\alpha+2}\|u\|_{L^2}^{\alpha}. \tag{3.4}$$

Since $R$ also satisfies equation (3.1), it satisfies the same identity, and since $u$ minimizes $J$, we must have $J(R) \geq J(u)$, which implies that $\|u\|_{L^2} \leq \|R\|_{L^2}$. On the other hand, $R$ being the ground state of (3.1), it is also the solution of (3.1) of minimal $L^2$ norm, so that $\|R\|_{L^2} \leq \|u\|_{L^2}$. Therefore, $\|R\|_{L^2} = \|u\|_{L^2}$ and the result now follows from (3.4). □

**Proof of Theorem 3.1.** Let $\varphi \in H^1(\mathbf{R}^N)$ verify $\lambda^{\frac{1}{\alpha}}\|\varphi\|_{L^2} < \|R\|_{L^2}$ and let $u$ be the maximal solution of (1.1). By conservation of charge and energy and by Lemma 3.4, we have for all $t \in (-T_*, T^*)$,

$$\begin{aligned}\frac{1}{2}\|\nabla u(t)\|_{L^2}^2 &\leq E(\varphi) + \frac{\lambda}{\alpha+2}\|u(t)\|_{L^{\alpha+2}}^{\alpha+2}\\ &\leq E(\varphi) + \frac{\lambda}{2\|R\|_{L^2}^{\alpha}}\|\nabla u(t)\|_{L^2}^2\|u(t)\|_{L^2}^{\alpha}\\ &\leq E(\varphi) + \frac{\lambda\|\varphi\|_{L^2}^{\alpha}}{2\|R\|_{L^2}^{\alpha}}\|\nabla u(t)\|_{L^2}^2;\end{aligned}$$

and so,

$$\frac{1}{2}\left(1 - \frac{\lambda\|\varphi\|_{L^2}^{\alpha}}{\|R\|_{L^2}^{\alpha}}\right)\|\nabla u(t)\|_{L^2}^2 \leq E(\varphi).$$

Hence the result. □

The case $\alpha = 4/N$ is critical for the $L^2$ theory. However, we have the following result. (See [8,9].)

**Theorem 3.5.** *Assume $\alpha = 4/N$. For any $\varphi \in L^2(\mathbf{R}^N)$, there exists a unique maximal solution $u \in C((-T_*, T^*), L^2(\mathbf{R}^N)) \cap L^{\alpha+2}_{loc}((-T_*, T^*), L^{\alpha+2}(\mathbf{R}^N))$ of (1.1).*

*We have the following blow up alternative. Either $T^* = \infty$ (respectively, $T_* = \infty$), or else $T^* < \infty$ and $\|u\|_{L^q((0,T^*),L^r)} = \infty$ for every admissible pair $(q, r)$ with $r \geq \alpha + 2$ (respectively, $T_* < \infty$ and $\|u\|_{L^q((-T_*,0),L^r)} = \infty$ for every admissible pair $(q, r)$ with $r \geq \alpha + 2$).*

*Moreover $u \in L^q_{loc}((-T_*, T^*), L^r(\mathbf{R}^N))$, for every admissible pair $(q, r)$.*

*If in addition $\varphi \in H^1(\mathbf{R}^N)$, then $u \in C((-T_*, T^*), H^1(\mathbf{R}^N))$.*

*There is conservation of charge, i.e. (2.2) holds for all $t \in (-T_*, T^*)$.*

*Finally, the solution $u$ depends continuously on the initial value $\varphi$ in the following sense. The mappings $\varphi \mapsto T_*$ and $\varphi \mapsto T^*$ are lower-semicontinuous $L^2(\mathbf{R}^N) \to \mathbf{R}$. Moreover, if $\varphi_n \underset{n\to\infty}{\longrightarrow} \varphi$ in $L^2(\mathbf{R}^N)$ and if $u_n$ is the corresponding solution of (1.1), then $u_n \underset{n\to\infty}{\longrightarrow} u$ in $C([-T_1, T_2], L^2(\mathbf{R}^N)) \cap L^{\alpha+2}([-T_1, T_2], L^{\alpha+2}(\mathbf{R}^N))$ for any bounded interval $[-T_1, T_2] \subset (-T_*, T^*)$.*

**Proof.** We argue only for $t \geq 0$, the argument for $t \leq 0$ being the same. (Note that if $u(t)$ satisfies (1.1), then $\overline{u(-t)}$ also satisfies (1.1).) We proceed in five steps.

**Step 1.** We first establish several estimates. Consider $0 < T \leq \infty$, let $I = (0, T)$ and set $J = \overline{I}$. Given $u, v \in L^{\alpha+2}(I, L^{\alpha+2}(\mathbf{R}^N))$, set $f = |u|^\alpha u - |v|^\alpha v$. Then

$$\|f\|_{L^{\frac{\alpha+2}{\alpha+1}}(I,L^{\frac{\alpha+2}{\alpha+1}})} \leq (\alpha + 1)\Big(\|u\|^\alpha_{L^{\alpha+2}(I,L^{\alpha+2})} + \|v\|^\alpha_{L^{\alpha+2}(I,L^{\alpha+2})}\Big) \|u - v\|_{L^{\alpha+2}(I,L^{\alpha+2})}. \quad (3.5)$$

This follows from the estimate $|\,|u|^\alpha u - |v|^\alpha v| \leq (\alpha + 1)(|u|^\alpha + |v|^\alpha)|u - v|$ and Hölder's inequality.

Given $u \in L^{\alpha+2}(I, L^{\alpha+2}(\mathbf{R}^N))$, set

$$\mathcal{F}(u)(t) = \int_0^t T(t - s)|u(s)|^\alpha u(s)\, ds.$$

It follows from (3.5) and Strichartz' estimate (1.2) that

$$\mathcal{F}(u) \in C(J, L^2(\mathbf{R}^N)) \cap L^q(I, L^r(\mathbf{R}^N)),$$

for every admissible pair $(q, r)$ and that

$$\|\mathcal{F}(u)\|_{L^q(I,L^r)} \leq C\|u\|_{L^{\alpha+2}(I,L^{\alpha+2})}^{\alpha+1}, \tag{3.6}$$

for some constant $C$ independent of $T$. In addition, if $v \in L^{\alpha+2}(I, L^{\alpha+2}(\mathbf{R}^N))$, then

$$\|\mathcal{F}(u) - \mathcal{F}(v)\|_{L^q(I,L^r)} \leq C\left(\|u\|_{L^{\alpha+2}(I,L^{\alpha+2})}^{\alpha} + \|v\|_{L^{\alpha+2}(I,L^{\alpha+2})}^{\alpha}\right) \|u - v\|_{L^{\alpha+2}(I,L^{\alpha+2})}. \tag{3.7}$$

**Step 2.** Let $0 < T \leq \infty$, let $I = (0, T)$ and set $J = \overline{I}$. We show that there exists $\delta > 0$ such that if $\varphi \in L^2(\mathbf{R}^N)$ verifies

$$\|T(\cdot)\varphi\|_{L^{\alpha+2}(I,L^{\alpha+2})} < \delta, \tag{3.8}$$

then there exists a unique solution $u \in C(J, L^2(\mathbf{R}^N)) \cap L^{\alpha+2}(I, L^{\alpha+2}(\mathbf{R}^N))$ of (1.1). In addition, $u \in L^q(I, L^r(\mathbf{R}^N))$ for any admissible pair $(q, r)$. Moreover, $u$ depends continuously on $\varphi$, in the sense that if $\varphi, \psi \in L^2(\mathbf{R}^N)$ both verify (3.8) and if $u, v$ denote the corresponding solutions of (1.1), then

$$\|u - v\|_{L^\infty(I,L^2)} + \|u - v\|_{L^{\alpha+2}(I,L^{\alpha+2})} \leq K\|\varphi - \psi\|_{L^2}, \tag{3.9}$$

for some constant $K$ independent of $T$, $u$, $v$.

Let $\delta > 0$, to be chosen later, and let $\varphi$ and $T$ be as above. Consider the set

$$E = \{u \in L^{\alpha+2}(I, L^{\alpha+2}(\mathbf{R}^N));\ \|u\|_{L^{\alpha+2}(I,L^{\alpha+2})} \leq 2\delta\}.$$

For $u \in E$, set

$$\mathcal{H}(u)(t) = T(t)\varphi + i\lambda \int_0^t T(t-s)|u(s)|^\alpha u(s)\, ds,$$

for $t \in I$. It follows easily from (3.6), (3.7) and (3.8) that if $\delta$ is small enough (independently of $\varphi$ and $T$), then $\mathcal{H}$ is a strict contraction on $E$. Thus $\mathcal{H}$ has a fixed point $u$, which is the unique solution of (1.1) in $E$. Applying (3.6) and (1.2), we see that $u \in C(J, L^2(\mathbf{R}^N)) \cap L^q(I, L^r(\mathbf{R}^N))$, for every admissible pair $(q, r)$. (3.9) follows easily from (3.7) and (1.2).

**Step 3.** $u$ being as in Step 2, we show that if $\varphi \in H^1(\mathbf{R}^N)$, then $u \in C(J, H^1(\mathbf{R}^N))$.

It follows from Theorem 2.1 that (1.1) has a solution $v \in C([0, T^*), H^1(\mathbf{R}^N))$. $v$ is in particular an $L^2$ solution, so that by uniqueness $u$ and $v$ coincide as long as they are both defined. Therefore, we need only show that $T^* > T$. Assume to the contrary that $T^* \le T$. Since the equation (1.1) is invariant by space translations and since the gradient is the limit of the finite differences quotient, it follows easily from (3.9) that

$$\|\nabla v\|_{L^\infty((0,T^*),L^2)} \le C\|\nabla\varphi\|_{L^2},$$

which contradicts the blow up alternative for the $H^1$ solutions.

**Step 4.** $u$ being as in Step 2, we show that there is conservation of charge.

Indeed, let $\varphi_n \to \varphi$ in $L^2(\mathbf{R}^N)$, with $\varphi_n \in H^1(\mathbf{R}^N)$. It follows from (1.2) that for $n$ large enough, $\varphi_n$ verifies (3.8), so that by (3.9), we have $u_n \to u$ in $C(J, L^2(\mathbf{R}^N))$, where $u_n$ is the solution associated to $\varphi_n$. On the other hand, it follows from Step 3 that $u_n$ is an $H^1$ solution, so that $\|u_n(t)\|_{L^2} = \|\varphi_n\|_{L^2}$ for all $t \in I$. Passing to the limit as $n \to \infty$, we obtain $\|u(t)\|_{L^2} = \|\varphi\|_{L^2}$ for all $t \in I$.

**Step 5.** Conclusion. Let $\varphi \in L^2(\mathbf{R}^N)$. Since by (1.2), $T(\cdot)\varphi \in L^{\alpha+2}(\mathbf{R}, L^{\alpha+2}(\mathbf{R}^N))$, it follows that $\|T(\cdot)\varphi\|_{L^{\alpha+2}((0,T),L^{\alpha+2})} \to 0$ as $T \downarrow 0$. Therefore, (3.8) is satisfied for $T$ small enough, and we can apply Step 2 to construct a unique local solution. Let $u \in C([0,T^*), L^2(\mathbf{R}^N))$ be the maximal solution. It remains to establish the blow up alternative and the continuous dependence. We show the blow up alternative by contradiction. Suppose that $T^* < \infty$ and that $\|u\|_{L^{\alpha+2}((0,T^*),L^{\alpha+2})} < \infty$. Let $0 \le t \le t+s < T^*$. It follows that

$$T(s)u(t) = u(t+s) - i\lambda \int_0^s T(s-\sigma)|u(t+\sigma)|^\alpha u(t+\sigma)\,d\sigma.$$

By (3.6), there exists $C$ such that

$$\|T(\cdot)u(t)\|_{L^{\alpha+2}((0,T^*-t),L^{\alpha+2})} \le \|u\|_{L^{\alpha+2}((t,T^*),L^{\alpha+2})} + C\|u\|^{\alpha+1}_{L^{\alpha+2}((t,T^*),L^{\alpha+2})}.$$

Therefore, for $t$ close enough to $T^*$, we have

$$\|T(\cdot)u(t)\|_{L^{\alpha+2}((0,T^*-t),L^{\alpha+2})} < \delta/2.$$

It follows from Step 2 that $u$ can be extended after $T^*$, which is a contradiction. This shows that

$$\|u\|_{L^{\alpha+2}((0,T^*),L^{\alpha+2})} = \infty.$$

Let now $(q,r)$ be an admissible pair such that $r > \alpha+2$. It follows from Hölder's inequality that for any $T < T^*$,

$$\|u\|_{L^{\alpha+2}((0,T),L^{\alpha+2})} \leq \|u\|^{\mu}_{L^{\infty}((0,T),L^2)} \|u\|^{1-\mu}_{L^q((0,T),L^r)} \leq \|\varphi\|^{\mu}_{L^2} \|u\|^{1-\mu}_{L^q((0,T),L^r)}, \tag{3.10}$$

with $\mu = \dfrac{2(r-\alpha-2)}{(\alpha+2)(r-2)}$. Letting $T \uparrow T^*$, we obtain $\|u\|_{L^q((0,T^*),L^r)} = \infty$.

Finally, we show the continuous dependence. Consider $T < T^*$. We have $u \in C([0,T], L^2(\mathbf{R}^N))$, so that it follows from (1.2) and an obvious compactness argument that there exists $\tau > 0$ such that

$$\|T(\cdot)u(t)\|_{L^{\alpha+2}((0,\tau),L^{\alpha+2})} \leq \delta/2,$$

for all $t \in [0,T]$, where $\delta$ is as in (3.8). Consider an integer $n$ such that $T \leq n\tau$, let $K \geq 1$ be the constant in (3.9) and let $M$ be such that $\|T(\cdot)v\|_{L^{\alpha+2}(\mathbf{R},L^{\alpha+2})} \leq M\|v\|_{L^2}$ (see (1.2)). Let $\varepsilon > 0$ be small enough so that $MK^{n-1}\varepsilon < \delta/2$. We claim that if $\|\varphi - \psi\|_{L^2} \leq \varepsilon$, then $T^*(\psi) > T$ and $\|u - v\|_{C([0,T],L^2)} + \|u - v\|_{L^{\alpha+2}((0,T),L^{\alpha+2})} \leq nK^n\|\varphi - \psi\|_{L^2}$, where $v$ is the solution corresponding to the initial value $\psi$. Indeed, if $\|\varphi - \psi\|_{L^2} \leq \varepsilon$, then

$$\begin{aligned} \|T(\cdot)\psi\|_{L^{\alpha+2}((0,T/n),L^{\alpha+2})} &\leq \|T(\cdot)\varphi\|_{L^{\alpha+2}((0,T/n),L^{\alpha+2})} \\ &\quad + \|T(\cdot)(\varphi-\psi)\|_{L^{\alpha+2}((0,T/n),L^{\alpha+2})} \\ &\leq \delta/2 + M\varepsilon < \delta. \end{aligned}$$

Therefore, it follows from Step 2 that $T^*(\psi) > T/n$ and that

$$\|u - v\|_{C([0,T/n],L^2)} + \|u - v\|_{L^{\alpha+2}((0,T/n),L^{\alpha+2})} \leq K\|\varphi - \psi\|_{L^2}.$$

In particular, $\|u(T/n) - v(T/n)\|_{L^2} \leq K\varepsilon$. The claim follows by iterating this argument $n$ times. This completes the proof. □

**Remark 3.6.** The blow up alternative in Theorem 3.5 is not very handy. It does not concern the $L^2$ norm of $u$. Indeed, despite of the conservation of charge, $T_*$ and $T^*$ can be finite in some cases. For example, assume $\lambda > 0$ and let $\varphi \in X$ be such that $E(\varphi) < 0$. It follows from Theorem 2.5 that $u$ blows up in $H^1$, for both $t > 0$ and $t < 0$. Therefore, it follows from Theorem 3.5 that $T^* < \infty$ and $T_* < \infty$.

**Remark 3.7.** We conjecture that if $\lambda < 0$, then $T_* = T^* = \infty$ for all $\varphi \in L^2(\mathbf{R}^N)$. However, we only have the following partial result. Assume $\lambda < 0$, and suppose that $\varphi \in L^2(\mathbf{R}^N)$ is such that $x\varphi(x) \in L^2(\mathbf{R}^N)$. Then $T^* = T_* = \infty$ and in addition $u \in L^q(\mathbf{R}, L^r(\mathbf{R}^N))$ for every admissible pair $(q, r)$. Indeed, consider a sequence $(\varphi_n)_{n\geq 0} \in X$, with $\varphi_n \underset{n\to\infty}{\longrightarrow} \varphi$ in $L^2(\mathbf{R}^N)$ and $x\varphi_n(x)$ bounded in $L^2(\mathbf{R}^N)$. The corresponding solutions satisfy $u_n \in C(\mathbf{R}, X)$ (see Remark 2.4), and from the pseudo-conformal conservation law (2.4) we see that $\|u_n(t)\|_{L^{\alpha+2}}^{\alpha+2} \leq Ct^{-2}$ for all $t \in \mathbf{R}$. By continuous dependence, this implies that $\|u(t)\|_{L^{\alpha+2}}^{\alpha+2} \leq Ct^{-2}$ for almost all $t \in (-T_*, T^*)$. In particular, we see that if $T^* < \infty$, then $u \in L^{\alpha+2}((0, T^*), L^{\alpha+2}(\mathbf{R}^N))$, which contradicts the blow up alternative (note that $u \in L^{\alpha+2}((0, T), L^{\alpha+2}(\mathbf{R}^N))$ for all $0 < T < T^*$). We see as well that $T_* = +\infty$. In addition, it is clear that the above estimate implies that $u \in L^{\alpha+2}(\mathbf{R}, L^{\alpha+2}(\mathbf{R}^N))$. The estimate for an arbitrary admissible pair follows from inequality (3.10).

**Remark 3.8.** There exists $\eta > 0$ such that if

$$\|T(\cdot)\varphi\|_{L^{\alpha+2}(\mathbf{R}, L^{\alpha+2})} < \eta, \tag{3.11}$$

then $T_* = T^* = \infty$. Moreover, we have $u \in L^q(\mathbf{R}, L^r(\mathbf{R}^N))$ for every admissible pair $(q, r)$. This follows easily from Step 2 of the proof of Theorem 3.5 (see in particular (3.8)). However, this conclusion does not hold in general for large data. Indeed, if $\lambda > 0$, there exist nontrivial solutions (standing waves) of the form $u(t, x) = e^{i\omega t}\phi(x)$, with $\phi \in H^1(\mathbf{R}^N)$, $\phi \neq 0$ (see [5], Section 8.1). These solutions obviously do not belong to $L^q(\mathbf{R}, L^r(\mathbf{R}^N))$ if $q < \infty$. On the other hand, it follows from Strichartz' estimates (1.2) that (3.11) is satisfied if

$$\|\varphi\|_{L^2} < \mu, \tag{3.12}$$

for $\mu$ small enough.

**Remark 3.9.** Assume $\lambda > 0$ and let $d$ be the supremum of the $\mu$'s such that (3.12) implies global existence. Then it is clear that $d \leq \lambda^{-\frac{1}{\alpha}}\|R\|_{L^2}$, where $R$ is as in Theorem 3.1 (see Remark 3.2). Whether or not $d = \lambda^{-\frac{1}{\alpha}}\|R\|_{L^2}$ is an open question. However, we can show that if $\varphi \in L^2(\mathbf{R}^N)$ verifies $\|\varphi\|_{L^2} < \lambda^{-\frac{1}{\alpha}}\|R\|_{L^2}$, and if in addition $x\varphi \in L^2(\mathbf{R}^N)$, then $T_* = T^* = +\infty$ and $u \in L^q(\mathbf{R}, L^r(\mathbf{R}^N))$ for every admissible pair $(q, r)$. Indeed, consider a sequence $(\varphi_n)_{n\geq 0} \subset X$, with $\varphi_n \underset{n\to\infty}{\longrightarrow} \varphi$ in $L^2(\mathbf{R}^N)$ and $x\varphi_n(x)$ bounded in $L^2(\mathbf{R}^N)$. The corresponding solutions satisfy $u_n \in C(\mathbf{R}, X)$ (see Theorem 3.1), and from the pseudo-conformal conservation law (2.4) we see that (see [5], formula (7.1.6))

$$8t^2 E(v_n(t)) = \|x\varphi_n\|_{L^2}$$

for all $t \in \mathbf{R}$, where $v_n(t) = e^{-\frac{i|x|^2}{4t}} u_n(t)$. In particular, $\|v_n(t)\|_{L^2} = \|u_n(t)\|_{L^2} = \|\varphi_n\|_{L^2}$, so that there exists $\varepsilon > 0$ such that $\|v_n(t)\|_{L^2} \leq \lambda^{-\frac{1}{\alpha}} \|R\|_{L^2} - \varepsilon$ for $n$ large enough. It follows that there exists $C$ such that $\|\nabla v_n(t)\|_{L^2}^2 \leq CE(v_n(t))$ for all $t \in \mathbf{R}$ (see the proof of Theorem 3.1). By Lemma 3.4, this implies that

$$\|v_n(t)\|_{L^{\alpha+2}}^{\alpha+2} \leq CE(v_n(t)) \|\varphi_n\|_{L^2}^{\frac{4}{N}} \leq \frac{C}{t^2},$$

for all $t \in \mathbf{R}$. We conclude as in Remark 3.7 above.

Theorem 3.5 has an immediate application to the study of the blowing up solutions.

**Theorem 3.10.** *Let $\varphi \in L^2(\mathbf{R}^N)$ be such that $T^* < \infty$. If $(t_n)_{n\geq 0}$ is any sequence such that $t_n \uparrow T^*$, then $u(t_n)$ does not have any strong limit in $L^2(\mathbf{R}^N)$. A similar statement holds for $T_*$.*

**Proof.** Assume that $u(t_n) \underset{n\to\infty}{\longrightarrow} w$ in $L^2(\mathbf{R}^N)$. By continuous dependence, we have $T^*(u(t_n)) \geq \frac{1}{2} T^*(w) > 0$ for $n$ large enough. This implies that $T^*(\varphi) \geq t_n + \frac{1}{2} T^*(w)$ for $n$ large. This is absurd, since $t_n \to T^*$. □

In fact, one can prove a stronger result which implies the above theorem (see [10]).

**Theorem 3.11.** *There exists $\rho > 0$ with the following property. Let $\varphi \in L^2(\mathbf{R}^N)$ be such that $T^* < \infty$. If $\mathcal{L}$ is the set of weak $L^2$ limit points of $u(t)$ as $t \uparrow T^*$, then $\|w\|_{L^2}^2 \leq \|\varphi\|_{L^2}^2 - \rho^2$, for all $w \in \mathcal{L}$. A similar statement holds for $T_*$.*

**Proof.** It follows from Step 2 of the proof of Theorem 3.5 (see in particular (3.8)) that there exists $\delta > 0$ such that if

$$\|T(\cdot)\phi\|_{L^{\alpha+2}((0,\tau),L^{\alpha+2})} < \delta,$$

then $T^*(\phi) > \tau$. Taking $\phi = u(t)$, we deduce that

$$\|T(\cdot)u(t)\|_{L^{\alpha+2}((0,T^*-t),L^{\alpha+2})} \geq \delta,$$

for all $t \in [0, T^*)$. Therefore, given any $\psi \in L^2(\mathbf{R}^N)$ and any $t \in [0, T^*)$, we have

$$\begin{aligned}\delta &\leq \|T(\cdot)(u(t)-\psi)\|_{L^{\alpha+2}((0,T^*-t),L^{\alpha+2})} + \|T(\cdot)\psi\|_{L^{\alpha+2}((0,T^*-t),L^{\alpha+2})}\\ &\leq c\|u(t)-\psi\|_{L^2} + \|T(\cdot)\psi\|_{L^{\alpha+2}((0,T^*-t),L^{\alpha+2})},\end{aligned}$$

where $c$ is the constant in the corresponding Strichartz' inequality (1.2). Since

$$\|T(\cdot)\psi\|_{L^{\alpha+2}((0,T^*-t),L^{\alpha+2})} \underset{t\uparrow T^*}{\longrightarrow} 0,$$

it follows that

$$\liminf_{t\uparrow T^*} \|u(t)-\psi\|_{L^2} \geq \frac{\delta}{c}.$$

Therefore, if $u(t_n) \rightharpoonup \psi$ for some sequence $t_n \uparrow T^*$, then

$$\begin{aligned}\frac{\delta^2}{c^2} &\leq \liminf_{n\to\infty} \|u(t_n)-\psi\|_{L^2}^2 = \liminf_{n\to\infty} (u(t_n)-\psi, u(t_n)-\psi)_{L^2}\\ &= \liminf_{n\to\infty} \big(\|u(t_n)\|_{L^2}^2 + \|\psi\|_{L^2}^2 - 2(u(t_n),\psi)_{L^2}\big) = \|u(t_n)\|_{L^2}^2 - \|\psi\|_{L^2}^2\\ &= \|\varphi\|_{L^2}^2 - \|\psi\|_{L^2}^2,\end{aligned}$$

and the result follows. □

For $H^1$ spherically symmetric blowing up solutions in dimension $N \geq 2$, there is a minimal amount of concentration of the $L^2$ norm at the origin, as shows the following result (see Merle and Tsutsumi [32] and Weinstein [48]).

**Theorem 3.12.** *Assume $N \geq 2$ and $\lambda > 0$, and let $R$ be the spherically symmetric, positive ground state of equation (3.1). Let $\gamma : (0,\infty) \to (0,\infty)$ be such that $\gamma(s) \underset{s\downarrow 0}{\longrightarrow} \infty$ and $s^{\frac{1}{2}}\gamma(s) \underset{s\downarrow 0}{\longrightarrow} 0$. If $\varphi \in H^1(\mathbf{R}^N)$ is spherically symmetric and such that $T^* < \infty$, then*

$$\liminf_{t\uparrow T^*} \|u(t)\|_{L^2(\Omega_t)} \geq \lambda^{-\frac{1}{\alpha}} \|R\|_{L^2},$$

*where $\Omega_t = \{x \in \mathbf{R}^N;\ |x| < |T^*-t|^{\frac{1}{2}}\gamma(T^*-t)\}$. A similar statement holds for $T_*$.*

As a consequence of Theorem 3.12, we have the following result.

**Corollary 3.13.** *Assume $N \geq 2$ and $\lambda > 0$, and let $R$ be the spherically symmetric, positive ground state of equation (3.1). Let $\varphi \in H^1(\mathbf{R}^N)$ be spherically symmetric and such that $T^* < \infty$. If $\mathcal{L}$ is the set of weak $L^2$ limit points of $u(t)$ as $t \uparrow T^*$, then*

$$\|w\|_{L^2}^2 \leq \|\varphi\|_{L^2}^2 - \lambda^{-\frac{2}{\alpha}} \|R\|_{L^2}^2,$$

*for all $w \in \mathcal{L}$. A similar statement holds for $T_*$.*

**Remark 3.14.** Note that the minimal loss of $L^2$ norm given by Corollary 3.13 is optimal. Indeed, there exist solutions that blow up in finite time, and for which $\|\varphi\|_{L^2}^2 = \lambda^{-\frac{2}{\alpha}} \|R\|_{L^2}^2$ (see Remark 4.1). Corollary 3.13 improves the conclusion of Theorem 3.11 for $H^1$ spherically symmetric solutions, in the sense that it gives the optimal value of $\rho$.

**Proof of Corollary 3.13.** Assume $t_n \uparrow T^*$ and $u(t_n) \rightharpoonup w$ in $L^2(\mathbf{R}^N)$. Given $\varepsilon > 0$, we have $u(t_n) \rightharpoonup w$ in $L^2(\{|x| > \varepsilon\})$; and so,

$$\|w\|_{L^2(\{|x|>\varepsilon\})}^2 \leq \liminf_{n\to\infty} \|u(t_n)\|_{L^2(\{|x|>\varepsilon\})}^2.$$

On the other hand,

$$\begin{aligned}
\|u(t_n)\|_{L^2(\{|x|>\varepsilon\})}^2 &= \|u(t_n)\|_{L^2}^2 - \|u(t_n)\|_{L^2(\{|x|<\varepsilon\})}^2 \\
&= \|\varphi\|_{L^2}^2 - \|u(t_n)\|_{L^2(\{|x|<\varepsilon\})}^2 \\
&\leq \|\varphi\|_{L^2}^2 - \|u(t_n)\|_{L^2(\Omega_t)}^2,
\end{aligned}$$

and the result follows from Theorem 3.12. □

**Proof of Theorem 3.12.** Set $\omega(t) = \|\nabla u(t)\|_{L^2}^{-1}$, so that $\omega(t) \underset{t\uparrow T^*}{\longrightarrow} 0$. We show that

$$\liminf_{t\uparrow T^*} \|u(t)\|_{L^2(\{|x|<\omega(t)\gamma(T^*-t)\})} \geq \lambda^{-\frac{1}{\alpha}} \|R\|_{L^2}. \tag{3.13}$$

The theorem follows from (3.13) and (2.20), since $\gamma$ is arbitrary. To prove (3.13), we argue by contradiction and we assume that there exists $t_n \uparrow T^*$ such that

$$\lim_{n\to\infty} \|u(t_n)\|_{L^2(\{|x|<\omega(t_n)\gamma(T^*-t_n)\})} < \lambda^{-\frac{1}{\alpha}} \|R\|_{L^2}. \tag{3.14}$$

We set

$$v_n(x) = \omega(t_n)^{\frac{N}{2}} u(t_n, \omega(t_n)x),$$

so that

$$\begin{cases} \|v_n\|_{L^2} = \|u(t_n)\|_{L^2} = \|\varphi\|_{L^2}, \\ \|\nabla v_n\|_{L^2} = 1, \\ E(v_n) = \omega(t_n)^2 E(u(t_n)) = \omega(t_n)^2 E(\varphi) \underset{n\to\infty}{\longrightarrow} 0. \end{cases} \tag{3.15}$$

It follows in particular that

$$E(v_n) = \frac{1}{2} - \frac{\lambda}{\alpha+2}\|v_n\|_{L^{\alpha+2}}^{\alpha+2},$$

so that

$$\|v_n\|_{L^{\alpha+2}}^{\alpha+2} \underset{n\to\infty}{\longrightarrow} \frac{\alpha+2}{2\lambda} \neq 0. \tag{3.16}$$

By (3.15), $(v_n)_{n\geq 0}$ is a bounded sequence in $H^1(\mathbf{R}^N)$, so that there exists a subsequence, which we still denote by $(v_n)_{n\geq 0}$, and $w \in H^1(\mathbf{R}^N)$ such that $v_n \rightharpoonup w$ weakly in $H^1(\mathbf{R}^N)$ as $n \to \infty$. Since the $v_n$'s are spherically symmetric, it follows that $v_n \underset{n\to\infty}{\longrightarrow} w$ in $L^{\alpha+2}(\mathbf{R}^N)$. In particular, we have $E(w) \leq 0$; and by (3.16), $w \neq 0$. By applying Lemma 3.4, we now obtain

$$\lambda^{\frac{1}{\alpha}}\|w\|_{L^2} \geq \|R\|_{L^2}. \tag{3.17}$$

Let $M > 0$. We have in particular

$$\begin{aligned} \|w\|_{L^2(\{|x|<M\})} &= \lim_{n\to\infty} \|v_n\|_{L^2(\{|x|<M\})} \\ &= \lim_{n\to\infty} \|u(t_n)\|_{L^2(\{|x|<M\omega(t_n)\})} \\ &\leq \liminf_{n\to\infty} \|u(t_n)\|_{L^2(\{|x|<\omega(t_n)\gamma(T^*-t_n)\})}, \end{aligned}$$

since $\gamma(s) \to \infty$ as $s \downarrow 0$. Since $M$ is arbitrary, we find by applying (3.17)

$$\liminf_{n\to\infty} \|u(t_n)\|_{L^2(\{|x|<\omega(t_n)\gamma(T^*-t_n)\})} \geq \|w\|_{L^2} \geq \lambda^{-\frac{1}{\alpha}}\|R\|_{L^2},$$

which contradicts (3.14). This completes the proof. □

In fact, Corollary 3.13 can be generalized to nonradial solutions (and also to the dimension 1). More precisely, we have the following result.

**Theorem 3.15.** *Let $R$ be the spherically symmetric, positive ground state of equation (3.1). Let $\varphi \in H^1(\mathbf{R}^N)$ be such that $T^* < \infty$. If $\mathcal{L}$ is the set of weak $L^2$ limit points of $u(t)$ as $t \uparrow T^*$, then*

$$\|w\|_{L^2}^2 \leq \|\varphi\|_{L^2}^2 - \lambda^{-\frac{2}{\alpha}}\|R\|_{L^2}^2,$$

*for all $w \in \mathcal{L}$. A similar statement holds for $T_*$.*

Theorem 3.15 is an immediate consequence of the following proposition.

**Proposition 3.16.** *Let $(u_n)_{n\geq 0} \subset H^1(\mathbf{R}^N) \setminus \{0\}$ and $u \in L^2(\mathbf{R}^N)$ be such that $u_n \rightharpoonup u$ in $L^2(\mathbf{R}^N)$ as $n \to \infty$. If furthermore $\|\nabla u_n\|_{L^2} \underset{n\to\infty}{\longrightarrow} \infty$ and*

$$\limsup_{n\to\infty} \frac{E(u_n)}{\|\nabla u_n\|_{L^2}^2} \leq 0,$$

*then $\|u\|_{L^2}^2 \leq \liminf_{n\to\infty} \|u_n\|_{L^2}^2 - \lambda^{-\frac{2}{\alpha}} \|R\|_{L^2}^2$.*

For the proof of Proposition 3.16, we use the concentration-compactness techniques. Given a bounded sequence $(u_n)_{n\geq 0} \subset L^2(\mathbf{R}^N)$, we set

$$\rho_n(t) = \sup_{y\in\mathbf{R}^N} \int_{\{|x-y|\leq t\}} |u_n(x)|^2\, dx,$$

so that $\rho_n$ is a nondecreasing function of $t$ for all $n \geq 0$, and

$$\rho(t) = \liminf_{n\to\infty} \rho_n(t),$$

so that $\rho$ is also a nondecreasing function of $t$. Finally, we define

$$\mu\left((u_n)_{n\geq 0}\right) = \lim_{t\to\infty} \rho(t).$$

With this notation, we have the following lemma.

**Lemma 3.17.** *If $(u_n)_{n\geq 0} \subset H^1(\mathbf{R}^N)$ is such that*

(i) $\|u_n\|_{L^2}^2 = a > 0$;

(ii) $0 < \inf_{n\geq 0} \|\nabla u_n\|_{L^2} \leq \sup_{n\geq 0} \|\nabla u_n\|_{L^2} < \infty$;

(iii) $\limsup_{n\to\infty} E(u_n) \leq 0$;

*then $\mu\left((u_n)_{n\geq 0}\right) \geq \lambda^{-\frac{2}{\alpha}} \|R\|_{L^2}^2$.*

**Proof of Proposition 3.16 (assuming Lemma 3.17).** Let $(u_n)_{n\geq 0}$ be as in the statement of Proposition 3.16. Set $a = \liminf_{n\to\infty} \|u_n\|_{L^2}^2$. By considering a subsequence, we may assume that

$$\|u_n\|_{L^2}^2 \underset{n\to\infty}{\longrightarrow} a.$$

Set $\omega_n = \|\nabla u_n\|_{L^2}^{-1}$ and define $v_n(x) = \omega_n^{\frac{2}{N}} u_n(\omega_n x)$. It follows that $\|v_n\|_{L^2}^2 = \|u_n\|_{L^2}^2$, $\|\nabla v_n\|_{L^2}^2 = 1$ and

$$\limsup_{n\to\infty} E(v_n) = \limsup_{n\to\infty} \frac{E(u_n)}{\|\nabla u_n\|_{L^2}^2} \leq 0.$$

We first show that $a > 0$. Indeed, it follows from Lemma 3.4 that

$$E(v_n) \geq \frac{1}{2}\left(1 - \lambda \frac{\|v_n\|_{L^2}^\alpha}{\|R\|_{L^2}^\alpha}\right) \|\nabla v_n\|_{L^2}^2,$$

which implies that $\lambda \|v_n\|_{L^2}^\alpha \geq \|R\|_{L^2}^\alpha$; and so, $a \geq \lambda^{-\frac{2}{\alpha}} \|R\|_{L^2}^2$. We now set

$$w_n = \frac{\sqrt{a}}{\|v_n\|_{L^2}} v_n,$$

so that $(w_n)_{n\geq 0}$ verifies the assumptions of Lemma 3.17. Therefore,

$$\mu\left((w_n)_{n\geq 0}\right) \geq \lambda^{-\frac{2}{\alpha}} \|R\|_{L^2}^2.$$

Let $\varepsilon > 0$. It follows from the above inequality that there exists $T$ such that

$$\rho(T) \geq \lambda^{-\frac{2}{\alpha}} \|R\|_{L^2}^2 - \frac{\varepsilon}{4};$$

and so, for $n$ large enough,

$$\rho_n(T) \geq \lambda^{-\frac{2}{\alpha}} \|R\|_{L^2}^2 - \frac{\varepsilon}{2}.$$

Therefore, there exists a sequence $(y_n)_{n\geq 0} \subset \mathbf{R}^N$ such that

$$\int_{\{|x-y_n|\leq T\}} |w_n(x)|^2\,dx \geq \lambda^{-\frac{2}{\alpha}} \|R\|_{L^2}^2 - \varepsilon,$$

for $n$ large, which means that

$$\frac{a}{\|u_n\|_{L^2}^2} \int_{\{|x-z_n|\leq t_n\}} |u_n(x)|^2\,dx \geq \lambda^{-\frac{2}{\alpha}} \|R\|_{L^2}^2 - \varepsilon,$$

with $z_n = \omega_n y_n$ and $t_n = \omega_n T$. Note that $t_n \underset{n\to\infty}{\longrightarrow} 0$. By possibly extracting a subsequence, we may assume that either $|z_n| \underset{n\to\infty}{\longrightarrow} \infty$ or else $z_n \underset{n\to\infty}{\longrightarrow} z$ for some $z \in \mathbf{R}^N$. In the first case, consider $M > 0$. Since $u_n \rightharpoonup u$ in $L^2(\{|x| \leq M\})$, we have

$$\begin{aligned}\|u\|_{L^2(\{|x|\leq M\})}^2 &\leq \liminf_{n\to\infty} \|u_n\|_{L^2(\{|x|\leq M\})}^2 = \liminf_{n\to\infty} \left\{\|u_n\|_{L^2}^2 - \|u_n\|_{L^2(\{|x|\geq M\})}^2\right\} \\ &= a - \limsup_{n\to\infty} \|u_n\|_{L^2(\{|x|\geq M\})}^2 \leq a - \lambda^{-\frac{2}{\alpha}} \|R\|_{L^2}^2 + \varepsilon.\end{aligned}$$

The result follows by letting $M \uparrow \infty$ then $\varepsilon \downarrow 0$. In the second case, consider $\delta > 0$. Since $u_n \rightharpoonup u$ in $L^2(\{|x| \geq \delta\})$, we have

$$\|u\|^2_{L^2(\{|x-z|\geq\delta\})} \leq \liminf_{n\to\infty} \|u_n\|^2_{L^2(\{|x-z|\geq\delta\})} = \liminf_{n\to\infty} \left\{ \|u_n\|^2_{L^2} - \|u_n\|^2_{L^2(\{|x-z|\leq\delta\})} \right\}$$
$$= a - \limsup_{n\to\infty} \|u_n\|^2_{L^2(\{|x-z|\leq\delta\})} \leq a - \lambda^{-\frac{2}{\alpha}} \|R\|^2_{L^2} + \varepsilon.$$

The result follows by letting $\delta \downarrow 0$ then $\varepsilon \downarrow 0$. This completes the proof. □

**Proof of Lemma 3.17.** We claim that there exists $\delta > 0$, depending only on $N$ and $\lambda$ with the following property. If $(u_n)_{n\geq 0} \subset H^1(\mathbf{R}^N)$ is such that

$$\|u_n\|^2_{L^2} = a > 0; \tag{3.18}$$

$$0 < \inf_{n\geq 0} \|\nabla u_n\|_{L^2} \leq \sup_{n\geq 0} \|\nabla u_n\|^2_{L^2} < \infty; \tag{3.19}$$

$$\limsup_{n\to\infty} E(u_n) \leq 0; \tag{3.20}$$

$$\mu\left((u_n)_{n\geq 0}\right) < \lambda^{-\frac{2}{\alpha}} \|R\|^2_{L^2}; \tag{3.21}$$

then there exists a sequence $(\widetilde{u}_n)_{n\geq 0} \subset H^1(\mathbf{R}^N)$ which satisfies (3.19), (3.20) and (3.21), and such that $\|\widetilde{u}_n\|^2_{L^2} = a - \beta$, for some $\beta \geq \delta$. The result follows, since if $(u_n)_{n\geq 0} \subset H^1(\mathbf{R}^N)$ is as in the statement of the lemma, and if $\mu\left((u_n)_{n\geq 0}\right) < \lambda^{-\frac{2}{\alpha}} \|R\|^2_{L^2}$, then one can apply the claim $k$ times with $k$ large enough so that $a - k\delta < 0$ in order to obtain a sequence $(\widetilde{u}_n)_{n\geq 0} \subset H^1(\mathbf{R}^N)$ such that $\|\widetilde{u}_n\|^2_{L^2} < 0$, which is absurd. Therefore, we need only prove the claim, and we consider $(u_n)_{n\geq 0} \subset H^1(\mathbf{R}^N)$ satisfying (3.18)—(3.21). We proceed in several steps.

**Step 1.** There is a constant $K$ such that

$$\|w\|^{\alpha+2}_{L^{\alpha+2}} \leq K \left( \sup_{y\in\mathbf{R}^N} \int_{\{|x-y|\leq t\}} |w(x)|^2\, dx \right)^{\frac{\alpha}{2}} \left( \int_{\mathbf{R}^N} |\nabla w|^2 + \frac{1}{t^2} \int_{\mathbf{R}^N} |w|^2 \right), \tag{3.22}$$

for all $w \in H^1(\mathbf{R}^N)$ and all $t > 0$. Indeed, it follows from [5], Lemma 8.3.7 that there is a constant $K$ such that

$$\|w\|^{\alpha+2}_{L^{\alpha+2}} \leq K \left( \sup_{y\in\mathbf{R}^N} \int_{\{|x-y|\leq 1\}} |w(x)|^2\, dx \right)^{\frac{\alpha}{2}} \|w\|^2_{H^1},$$

for all $w \in H^1(\mathbf{R}^N)$. By changing $w(x)$ to $w(tx)$, we obtain (3.22).

We now set

$$\delta = \left(\frac{\alpha+2}{2K}\right)^{\frac{2}{\alpha}} > 0, \tag{3.23}$$

we set

$$\mu = \mu\left((u_n)_{n\geq 0}\right),$$

and we consider the functions $\rho_n(t)$ introduced for the definition of $\mu$.

**Step 2.** There exists a sequence $n_k \to \infty$ and $t_{n_k} \to \infty$ such that $\mu = \lim_{k\to\infty} \rho_{n_k}(t_{n_k})$, as follows immediately from the definition of $\mu$. Therefore,

$$\mu = \lim_{n\to\infty} \rho_n(t_n). \tag{3.24}$$

with $t_n \to \infty$, after renaming the sequence $(u_n)_{n\geq 0}$.

**Step 3.** There exists a subsequence $(\rho_{n_k})_{k\geq 0}$ and a nondecreasing function $\gamma$ such that $\rho_{n_k}(t) \underset{k\to\infty}{\longrightarrow} \gamma(t)$, uniformly on bounded intervals.

Indeed, fix $n \geq 0$ and $t > 0$. We first show that for all $t > 0$ there exists $y_n(t) \in \mathbf{R}^N$ such that

$$\rho_n(t) = \int_{\{|x-y_n(t)|\leq t\}} |u_n(x)|^2\,dx. \tag{3.25}$$

Indeed, fix $t > 0$. There is a sequence $(y_k^n)_{k\geq 0} \subset \mathbf{R}^N$ such that

$$\rho_n(t) = \lim_{k\to\infty} \int_{\{|x-y_k^n|\leq t\}} |u_n(x)|^2\,dx > 0.$$

We claim that the sequence $(y_k^n)_{k\geq 0}$ is bounded. Otherwise, there exists a subsequence $(y_{k_j}^n)_{j\geq 0}$ such that $\{|x-y_{k_j}^n| \leq t\} \cap \{|x-y_{k_\ell}^n| \leq t\} = \emptyset$ for $j \neq \ell$; and so,

$$\int_{\mathbf{R}^N} |u_n|^2 \geq \sum_{j\geq 0} \int_{\{|x-y_{k_j}^n|\leq t\}} |u_n(x)|^2\,dx = +\infty,$$

which is absurd. Therefore, $(y_k^n)_{k\geq 0}$ has a convergent subsequence, and its limit $y_n(t)$ verifies (3.25). We now claim that the sequence $(\rho_n)_{n\geq 0}$ is uniformly Hölder continuous on bounded intervals. Indeed, let $r > 2$ be such that $H^1(\mathbf{R}^N) \hookrightarrow L^r(\mathbf{R}^N)$ and consider $0 \leq s \leq t < \infty$. We have

$$\begin{aligned}
|\rho_n(t) - \rho_n(s)| = \rho_n(t) - \rho_n(s) &= \int_{\{|x-y_n(t)|\leq t\}} |u_n|^2 - \int_{\{|x-y_n(s)|\leq s\}} |u_n|^2 \\
&= \int_{\{s\leq|x-y_n(t)|\leq t\}} |u_n|^2 + \int_{\{|x-y_n(t)|\leq s\}} |u_n|^2 - \int_{\{|x-y_n(s)|\leq s\}} |u_n|^2 \\
&\leq \int_{\{s\leq|x-y_n(t)|\leq t\}} |u_n|^2,
\end{aligned}$$

by (3.25) and the definition of $\rho_n(s)$. Therefore,

$$|\rho_n(t) - \rho_n(s)| \le C\|u_n\|_{H^1}^2 |\{s \le |x - y_n(t)| \le t\}|^{\frac{r-2}{r}},$$

and the result follows, since $u_n$ is bounded in $H^1(\mathbf{R}^N)$. By Ascoli's theorem, there exists a subsequence, which we still denote by $(\rho_n)_{n\ge 0}$ and a nondecreasing function $\gamma$ such that

$$\rho_n(t) \underset{n\to\infty}{\longrightarrow} \gamma(t), \tag{3.26}$$

uniformly on bounded intervals. Note that by passing to a subsequence, we do not lose property (3.24).

**Step 4.** We have

$$\mu = \lim_{t\to\infty} \gamma(t) \ge \delta, \tag{3.27}$$

where $\delta$ is defined by (3.23). Moreover,

$$\mu = \lim_{n\to\infty} \rho_n(t_n/2), \tag{3.28}$$

where $(t_n)_{n\ge 0}$ is as in (3.24).

Indeed, note first that by definition of $\mu$,

$$\mu \le \limsup_{t\to\infty} \liminf_{n\to\infty} \rho_n(t) = \limsup_{t\to\infty} \gamma(t) = \lim_{t\to\infty} \gamma(t).$$

On the other hand, it follows from (3.24) and the nondecreasing character of $\rho_n$ that for all $t > 0$,

$$\mu = \lim_{n\to\infty} \rho_n(t_n) \ge \lim_{n\to\infty} \rho_n(t) = \gamma(t).$$

By letting $t \uparrow \infty$, we obtain $\mu \ge \lim_{t\to\infty} \gamma(t)$, so that $\mu = \lim_{t\to\infty} \gamma(t)$. Finally, it follows from (3.22) and (3.23) that

$$E(u_n) \ge \frac{1}{2}\left(1 - \left(\frac{\rho_n(t_n)}{\delta}\right)^{\frac{\alpha}{2}}\right) \int |\nabla u_n|^2 - \frac{K}{(\alpha+2)t_n^2} \rho_n(t_n)^{\frac{\alpha}{2}}.$$

If $\mu < \delta$, then we obtain by letting $n \to \infty$ and applying (3.24) and (3.19)

$$\limsup_{n\to\infty} E(u_n) \ge \frac{1}{2}\left(1 - \left(\frac{\mu}{\delta}\right)^{\frac{\alpha}{2}}\right) \inf_{n\ge 0} \int |\nabla u_n|^2 > 0,$$

which is absurd. This proves (3.27).

Next, since $\rho_n(t_n/2) \le \rho_n(t_n)$, it follows from (3.24) that $\limsup_{n\to\infty} \rho_n(t_n/2) \le \mu$. Finally, given $t > 0$, we have $t_n/2 \ge t$ for $n$ large, so that

$$\rho_n(t_n/2) \ge \rho_n(t) \underset{n\to\infty}{\longrightarrow} \gamma(t).$$

Therefore, $\liminf_{n\to\infty} \rho_n(t_n/2) \ge \gamma(t)$. (3.28) follows, by letting $t \to \infty$.

**Step 5.** Construction of the sequence $(\widetilde{u}_n)_{n\ge 0}$. Let $\theta \in C^\infty([0,\infty))$ be such that $\theta(t) \equiv 1$ for $0 \le t \le 1/2$, $\theta(t) \equiv 0$ for $t \ge 3/4$ and $0 \le \theta \le 1$. Let $\varphi \in C^\infty([0,\infty))$ be such that $\varphi(t) \equiv 0$ for $0 \le t \le 3/4$, $\varphi(t) \equiv 1$ for $t \ge 1$ and $0 \le \varphi \le 1$. Finally, set $\psi = 1 - \theta - \varphi$. We define

$$\theta_n(t) = \theta\left(\frac{t}{t_n}\right), \quad \varphi_n(t) = \varphi\left(\frac{t}{t_n}\right), \quad \psi_n(t) = \psi\left(\frac{t}{t_n}\right),$$

where $t_n$ is as in (3.24). It follows in particular that $\theta_n + \varphi_n + \psi_n = 1$ and that

$$\|\nabla\theta_n\|_{L^\infty} + \|\nabla\varphi_n\|_{L^\infty} + \|\nabla\psi_n\|_{L^\infty} \le \frac{C}{t_n}. \tag{3.29}$$

Finally, we set

$$\begin{aligned} v_n(x) &= \theta_n(|x - y_n(t_n/2)|)u_n(x), \\ \widetilde{u}_n(x) &= \varphi_n(|x - y_n(t_n/2)|)u_n(x), \\ w_n(x) &= \psi_n(|x - y_n(t_n/2)|)u_n(x), \end{aligned}$$

where $y_n(t)$ is as in (3.25).

**Step 6.** Conclusion. We have

$$\begin{aligned} \rho_n(t_n/2) = \int_{\{|x-y_n(t_n/2)|\le t_n/2\}} |u_n|^2 &\le \int_{\mathbf{R}^N} |v_n|^2 \\ &\le \int_{\{|x-y_n(t_n/2)|\le t_n\}} |u_n|^2 \le \int_{\{|x-y_n(t_n)|\le t_n\}} |u_n|^2 \le \rho_n(t_n), \end{aligned}$$

so that

$$\|v_n\|_{L^2}^2 \underset{n\to\infty}{\longrightarrow} \mu, \tag{3.30}$$

by (3.24) and (3.28). Furthermore,

$$\begin{aligned} \int_{\mathbf{R}^N} |w_n|^2 &\le \int_{\{t_n/2 \le |x-y_n(t_n/2)|\le t_n\}} |u_n|^2 \\ &= \int_{\{|x-y_n(t_n/2)|\le t_n\}} |u_n|^2 - \int_{\{|x-y_n(t_n/2)|\le t_n/2\}} |u_n|^2 \\ &\le \int_{\{|x-y_n(t_n)|\le t_n\}} |u_n|^2 - \int_{\{|x-y_n(t_n/2)|\le t_n/2\}} |u_n|^2 \\ &= \rho_n(t_n) - \rho_n(t_n/2), \end{aligned}$$

so that

$$\|w_n\|_{L^2}^2 \underset{n\to\infty}{\longrightarrow} 0. \tag{3.31}$$

Since $u_n = v_n + w_n + \widetilde{u}_n$ and since $v_n$ and $\widetilde{u}_n$ have disjoint supports, it follows from (3.30), (3.31) and (3.18) that

$$\|\widetilde{u}_n\|_{L^2}^2 \underset{n\to\infty}{\longrightarrow} a - \mu. \tag{3.32}$$

In addition, since $|\widetilde{u}_n| \le |u_n|$, it follows immediately from (3.21) that

$$\mu\left((\widetilde{u}_n)_{n\ge 0}\right) \le \mu\left((u_n)_{n\ge 0}\right) < \lambda^{-\frac{2}{\alpha}} \|R\|_{L^2}^2. \tag{3.33}$$

On the other hand, it follows from Lemma 3.4, (3.21), (3.30) and (3.31) that there exists $\sigma > 0$ such that

$$E(v_n) \ge \sigma \|\nabla v_n\|_{L^2}^2 \quad \text{and} \quad E(w_n) \ge \sigma \|\nabla w_n\|_{L^2}^2, \tag{3.34}$$

for $n$ large. On the other hand, one verifies easily (since $v_n \widetilde{u}_n \equiv 0$) that

$$\big|\, |u_n|^{\alpha+2} - |v_n|^{\alpha+2} - |w_n|^{\alpha+2} - |\widetilde{u}_n|^{\alpha+2} \big| \le C |u_n|^{\alpha+1} |w_n|,$$

so that

$$\left| \int |u_n|^{\alpha+2} - \int |v_n|^{\alpha+2} - \int |w_n|^{\alpha+2} - \int |\widetilde{u}_n|^{\alpha+2} \right| \underset{n\to\infty}{\longrightarrow} 0. \tag{3.35}$$

(Note that $u_n$ is bounded in $H^1(\mathbf{R}^N)$ and that $w_n$ is bounded in $H^1(\mathbf{R}^N)$ and converges to 0 in $L^2(\mathbf{R}^N)$, hence in $L^{\alpha+2}(\mathbf{R}^N)$.) Furthermore, it follows from an easy calculation that

$$\begin{aligned} |\nabla u_n|^2 - |\nabla v_n|^2 - |\nabla w_n|^2 - |\nabla \widetilde{u}_n|^2 &= 2\big(\nabla \psi_n \cdot (\nabla \theta_n + \nabla \varphi_n) |u_n|^2 \\ &\quad + \psi_n (\theta_n + \varphi_n) |\nabla u_n|^2 + \nabla(\psi_n(\theta_n + \varphi_n)) \mathrm{Re}(\overline{u_n} \nabla u_n)\big) \\ &\ge -\frac{C}{t_n^2} |u_n|^2 - \frac{C}{t_n} |u_n|\, |\nabla u_n|, \end{aligned}$$

by (3.29), so that

$$\liminf_{n\to\infty} \left\{ \int |\nabla u_n|^2 - \int |\nabla v_n|^2 - \int |\nabla w_n|^2 - \int |\nabla \widetilde{u}_n|^2 \right\} \ge 0.$$

This, together with (3.35), implies that

$$\liminf_{n\to\infty} \{ E(u_n) - E(v_n) - E(w_n) - E(\widetilde{u}_n) \} \ge 0. \tag{3.36}$$

It now follows from (3.20), (3.34) and (3.36) that

$$\limsup_{n\to\infty} E(\widetilde{u}_n) \le 0. \tag{3.37}$$

Next, it follows immediately from the definition of $\widetilde{u}_n$ that

$$\|\nabla\widetilde{u}_n\|_{L^2}^2 \le C\|u_n\|_{H^1}^2. \tag{3.38}$$

Finally, we show that

$$\liminf_{n\to\infty} \|\nabla\widetilde{u}_n\|_{L^2} > 0. \tag{3.39}$$

To prove this, we argue by contradiction and we assume that there exists a subsequence, which we still denote by $(\widetilde{u}_n)_{n\ge 0}$ such that $\|\nabla\widetilde{u}_n\|_{L^2} \underset{n\to\infty}{\longrightarrow} 0$. It then follows from (3.32) and Lemma 3.4 that $E(\widetilde{u}_n) \underset{n\to\infty}{\longrightarrow} 0$. (3.20), (3.36) and (3.34) now imply that $\|\nabla v_n\|_{L^2} \underset{n\to\infty}{\longrightarrow} 0$ and $\|\nabla w_n\|_{L^2} \underset{n\to\infty}{\longrightarrow} 0$. Therefore, $\|\nabla u_n\|_{L^2} \underset{n\to\infty}{\longrightarrow} 0$, which contradicts (3.19); and so we have proved (3.39).

It now follows from (3.33), (3.37), (3.38) and (3.39) that the sequence $(\widetilde{u}_n)_{n\ge 0}$ verifies (3.19), (3.20) and (3.21). Changing $\widetilde{u}_n$ to $\dfrac{\sqrt{a-\mu}}{\|\widetilde{u}_n\|_{L^2}}\widetilde{u}_n$ and applying (3.32), we see that the new sequence $(\widetilde{u}_n)_{n\ge 0}$ verifies (3.18)—(3.21). This completes the proof. □

**4. Some applications of the pseudo-conformal transformation.** In this section, we still assume $\alpha = 4/N$. In this case, the pseudo-conformal conservation law (2.4), introduced by Ginibre and Velo [20], becomes an exact conservation law. This conservation law is associated to a group of transformations which leaves invariant the set of solutions of (1.1) (see Ginibre and Velo [19]). We describe below this group of transformations (the pseudo-conformal transformation).

Let $b \in \mathbf{R}$. Given $(t,x) \in \mathbf{R} \times \mathbf{R}^N$, we define the conjugate variables $(s,y) \in \mathbf{R} \times \mathbf{R}^N$ by

$$s = \frac{t}{1+bt}, \quad y = \frac{x}{1+bt},$$

or equivalently

$$t = \frac{s}{1-bs}, \quad x = \frac{y}{1-bs}.$$

Given $u$ defined on $(-S_1, S_2) \times \mathbf{R}^N$ with $0 \le S_1, S_2, \le \infty$, we set

$$T_1 = \begin{cases} \infty \text{ if } bS_1 \le -1, \\ \dfrac{S_1}{1+bS_1} \text{ if } bS_1 > -1, \end{cases} \qquad T_2 = \begin{cases} \infty \text{ if } bS_2 \ge 1, \\ \dfrac{S_2}{1-bS_2} \text{ if } bS_2 < 1, \end{cases}$$

and we define $u_b$ on $(-T_1, T_2)$ by

$$u_b(t,x) = (1+bt)^{-\frac{N}{2}} e^{i\frac{b|x|^2}{4(1+bt)}} u\left(\frac{t}{1+bt}, \frac{x}{1+bt}\right),$$

or equivalently

$$u_b(t,x) = (1-bs)^{\frac{N}{2}} e^{i\frac{b|y|^2}{4(1-bs)}} u(s,y). \tag{4.1}$$

Note that

$$\|u_b(t)\|_{L^2} = \|u(s)\|_{L^2}, \tag{4.2}$$

and more generally,

$$\|u_b(t)\|_{L^{\beta+2}} = (1-bs)^{\frac{N\beta}{2(\beta+2)}} \|u(s)\|_{L^{\beta+2}}, \tag{4.3}$$

if $\beta \ge 0$. In particular,

$$\|u_b(t)\|_{L^{\alpha+2}} = (1-bs)^{\frac{2}{\alpha+2}} \|u(s)\|_{L^{\alpha+2}}, \tag{4.4}$$

so that if $bs_1 > -1$ and $bs_2 < 1$, then

$$\|u_b\|_{L^{\alpha+2}((-t_1,t_2)L^{\alpha+2})} = \|u\|_{L^{\alpha+2}((-s_1,s_2)L^{\alpha+2})}, \tag{4.5}$$

with $t_1 = \dfrac{s_1}{1+bs_1}$ and $t_2 = \dfrac{s_2}{1-bs_2}$. Next, if $u \in C((-S_1, S_2), X)$, then it is clear that $u_b \in C((-T_1, T_2), X)$. In addition,

$$\|xu_b(t)\|_{L^2} = (1-bs)^{-1}\|yu(s)\|_{L^2}, \tag{4.6}$$

$$\|\nabla u_b(t)\|_{L^2} = \frac{1}{2}\|(y + 2i(1-bs)\nabla)u(s)\|_{L^2}, \tag{4.7}$$

$$\|\nabla u(s)\|_{L^2} = \frac{1}{2}\|(x + 2i(1+bt)\nabla)u_b(t)\|_{L^2}. \tag{4.8}$$

The interest of the above transformation lies in the following result.

**Theorem 4.1.** *If $u \in C((-S_1, S_2), L^2(\mathbf{R}^N)) \cap L^{\alpha+2}_{loc}((-S_1, S_2), L^{\alpha+2}(\mathbf{R}^N))$ is a solution of (1.1), then $u_b \in C((-T_1, T_2), L^2(\mathbf{R}^N)) \cap L^{\alpha+2}_{loc}((-T_1, T_2), L^{\alpha+2}(\mathbf{R}^N))$ is also a solution of (1.1).*

*Moreover, if $u \in C((-S_1, S_2), X)$, then $u_b \in C((-T_1, T_2), X)$.*

**Proof.** It is clear that $u_b \in C((-T_1, T_2), L^2(\mathbf{R}^N))$. In addition, it follows from (4.5) that $u_b \in L^{\alpha+2}_{loc}((-T_1, T_2), L^{\alpha+2}(\mathbf{R}^N))$.

Furthermore, one shows that if $0 \le S_1, S_2 < \infty$ and if $bS_1 > -1$ and $bS_2 < 1$, then the mapping $u \mapsto u_b$ is continous $C([-S_1, S_2], L^2(\mathbf{R}^N)) \cap L^{\alpha+2}((-S_1, S_2), L^{\alpha+2}(\mathbf{R}^N)) \to C([-T_1, T_2], L^2(\mathbf{R}^N)) \cap L^{\alpha+2}((-T_1, T_2), L^{\alpha+2}(\mathbf{R}^N))$.

Consider now a solution $u \in C([-S_1, S_2], L^2(\mathbf{R}^N)) \cap L^{\alpha+2}((-S_1, S_2), L^{\alpha+2}(\mathbf{R}^N))$ of (1.1), with $S_1$ and $S_2$ as above. Let $\varphi = u(0)$. We have in particular $T_*(\varphi) > S_1$ and $T^*(\varphi) > S_2$. Consider $(\varphi_n)_{n\ge 0} \subset H^2(\mathbf{R}^N)$ such that $\varphi_n \to \varphi$ in $L^2(\mathbf{R}^N)$. By continuous dependence, we have $T_*(\varphi_n) > S_1$ and $T^*(\varphi_n) > S_2$ for $n$ large enough. We denote by $u^n$ the corresponding solutions of (1.1). Note that $u^n$ is an $H^2$ solution. A tedious, but straightforward calculation shows that $(u^n)_b$ satisfies (1.1) almost everywhere on $(-T_1, T_2) \times \mathbf{R}^N$. The conclusion follows from the continuity property mentioned above and the continous dependence. □

**Remark 4.2.** Note that the pseudo-conformal transformation preserves both the space $L^2(\mathbf{R}^N)$ and the space $X$. On the other hand, it **does not** preserve the space $H^1(\mathbf{R}^N)$.

**Remark 4.3.** The pseudo-conformal transformation has a simple application, which yields interesting information on the blow up (see Weinstein [47]). Assume for simplicity that $\lambda = 1$ and let $\psi$ be a nontrivial solution of (3.1). (Note that $\psi(x)$ has exponential decay as $|x| \to \infty$, see for example [5], Theorem 8.1.1.) It follows that $u(t,x) = e^{it}\psi(x)$ is the solution of (1.1) with $\varphi = \psi$, and that $T^*(\varphi) = T_*(\varphi) = +\infty$. We set $v(t,x) = u_{-1}(t,x)$, i.e.

$$v(t,x) = (1-t)^{-\frac{N}{2}} e^{-i\frac{|x|^2}{4(1-t)}} e^{i\frac{t}{1-t}} \psi\left(\frac{x}{1-t}\right), \tag{4.9}$$

for $x \in \mathbf{R}^N$ and $t < 1$. It follows that for any $1 \le p \le \infty$,

$$\|v(t)\|_{L^p} = (1-t)^{-\frac{N(p-2)}{2p}} \|\psi\|_{L^p}, \tag{4.10}$$

for all $t < 1$. Therefore, $\|v\|_{L^{\alpha+2}((0,1),L^{\alpha+2})} = +\infty$, so that $T^* = 1$. Furthermore, it follows from (4.7) that

$$\|\nabla v(t)\|_{L^2}^2 = \frac{1}{4}\int_{\mathbf{R}^N} \left| \left(x + \frac{2i}{1-t}\nabla\right)\psi(x)\right|^2 dx,$$

so that

$$(1-t)\|\nabla v(t)\|_{L^2} \underset{t\uparrow 1}{\longrightarrow} \|\nabla\psi\|_{L^2}. \tag{4.11}$$

It follows in particular from (4.10) and (4.11) that $v$ blows up twice as fast as the lower estimates (2.20) and (2.21). This implies that, at least in the case $\alpha = 4/N$, the lower estimates (2.20) and (2.21) are not optimal for all the blowing up solutions.

It also follows from (4.10) that $\|v(t)\|_{L^p} \underset{t\uparrow 1}{\longrightarrow} 0$ if $1 \le p < 2$, so that

$$v(t) \rightharpoonup 0, \tag{4.12}$$

in $L^2(\mathbf{R}^N)$ as $t \uparrow 1$. In particular, the loss of $L^2$ norm at blow up is equal to $\|R\|_{L^2}$ if $\psi$ is a ground state of (3.1), but it is larger if $\psi$ is an excited state. (Note that excited states exist if $N \ge 2$, see [4].) Therefore, the loss of $L^2$ norm given by Theorem 3.15 is not always optimal.

Note also that by (4.6),

$$\|xv(t)\|_{L^2} = (1-t)\|x\psi\|_{L^2} \underset{t\uparrow 1}{\longrightarrow} 0. \tag{4.13}$$

(c.f. Remark 2.15.) In particular, $v(t) \underset{t\uparrow 1}{\longrightarrow} 0$ in $L^2(\{|x| \ge \varepsilon\})$ for any $\varepsilon > 0$. Furthermore, one verifies easily that $v(t) \underset{t\uparrow 1}{\longrightarrow} 0$ in $H^1(\{|x| \ge \varepsilon\})$ and in $L^\infty(\{|x| \ge \varepsilon\})$ (this last point because $\psi$ has exponential decay). Therefore, $v(t)$ blows up only at $x = 0$. Furthermore, it follows from an easy calculation that $|v(t)|^2 \underset{t\uparrow 1}{\longrightarrow} \|\psi\|_{L^2}^2 \delta$ in $\mathcal{D}'(\mathbf{R}^N)$, where $\delta$ is the Dirac measure at $x = 0$.

Finally, we observe that formula (4.9) also makes sense for $t > 1$, and that $v$ given by (4.9) is also a solution of (1.1) for $t > 1$. As a matter of fact, the properties of $v$ as $t \uparrow 1$ and as $t \downarrow 1$ are similar. Formula (4.9) gives (formally) an extension of the solution $v$ beyond the blow up time $T^* = 1$. We know that $v$ satisfies (1.1) on $(-\infty, 1)$ and on $(1, \infty)$, and we investigate in what sense $v$ may be a solution near $t = 1$. Note first that $v \in C((-\infty, 1) \cup (1, \infty), L^2(\mathbf{R}^N))$ and that $\|v(t)\|_{L^2} = \|\psi\|_{L^2}$ for $t \neq 1$, so that by property (4.12) $v$ is discontinuous in $L^2(\mathbf{R}^N)$ at $t = 1$. On the other hand, $v \in L^\infty(\mathbf{R}, L^2(\mathbf{R}^N))$, so that $\triangle v \in L^\infty(\mathbf{R}, H^{-2}(\mathbf{R}^N))$. Furthermore, it follows from (4.10) that $\| \, |v(t)|^\alpha v(t)\|_{L^1} = c|1-t|^{\frac{N-4}{2}}$. Therefore, if we assume $N \geq 3$, then $|v|^\alpha v \in L^1_{loc}(\mathbf{R}, L^1(\mathbf{R}^N))$. If $m$ is an integer such that $L^1(\mathbf{R}^N) \hookrightarrow H^{-m}(\mathbf{R}^N)$ (so that in particular $m \geq 2$), then it follows that $|v|^\alpha v \in L^1_{loc}(\mathbf{R}, H^{-m}(\mathbf{R}^N))$. Therefore, $u_t \in L^1_{loc}(\mathbf{R}, H^{-m}(\mathbf{R}^N))$, so that $u \in C(\mathbf{R}, H^{-m}(\mathbf{R}^N))$. It follows that $v(t) \to 0$ in $H^{-m}(\mathbf{R}^N)$ as $t \to 1$. This implies easily that $v$ satisfies (1.1) in $\mathcal{D}'(\mathbf{R}, H^{-m}(\mathbf{R}^N))$. Therefore, we see that $v$ can be extended in a reasonable sense beyond the blow up time $T^* = 1$. However, the meaning of this extension is not quite clear. Indeed, if we define

$$\widetilde{v}(t) = \begin{cases} v(t) \text{ if } t < 1, \\ 0, \text{ if } t \geq 1, \end{cases}$$

then the above argument shows that $\widetilde{v}$ is also an extension of $v$ beyond $T^* = 1$, which satisfies (1.1) in $\mathcal{D}'(\mathbf{R}, H^{-m}(\mathbf{R}^N))$. As a matter of fact, one can define many such extensions. For example, since equation (1.1) is invariant by space translation and by multiplication by a constant of modulus 1, it follows easily that for any $y \in \mathbf{R}^N$ and $\omega \in \mathbf{R}$,

$$\widetilde{v}(t) = \begin{cases} v(t) \text{ if } t < 1, \\ e^{i\omega} v(t, \cdot - y), \text{ if } t \geq 1, \end{cases}$$

is also an extension of $v$ beyond $T^* = 1$, which satisfies (1.1) in $\mathcal{D}'(\mathbf{R}, H^{-m}(\mathbf{R}^N))$. It is not clear whether or not there is such an extension which is, in some way, more natural than the others.

**Remark 4.4.** In dimension 1, the solutions considered in the above remark are completely explicit. Indeed,

$$R(x) = \frac{3^{\frac{1}{4}}}{\sqrt{\cosh(2x)}},$$

(see Remark 3.3), so that

$$u(t,x) = e^{it} R(x)$$

is a solution of the Schrödinger equation $iu_t + u_{xx} + |u|^4 u = 0$; and so,

$$v(t,x) = \frac{1}{\sqrt{1-t}} e^{-i\frac{x^2}{4(1-t)}} e^{i\frac{t}{1-t}} \frac{3^{\frac{1}{4}}}{\sqrt{\cosh(\frac{2x}{1-t})}}$$

is also a solution of the equation, that blows up at $t = 1$.

**Remark 4.5.** Let $v(t)$ be as in Remark 4.3. Given $y \in \mathbf{R}^N$, set $v_y(t) = v(t, \cdot - y)$, so that $v_y$ is a solution of (1.1) for which $T^* = 1$, and that blows up at the point $y \in \mathbf{R}^N$. Given $(y_\ell)_{1 \le \ell \le k}$ with $y_j \ne y_\ell$ for $j \ne \ell$,

$$w(t) = \sum_{\ell=1}^{k} v_{y_\ell}(t),$$

is a function that blows up at $t = 1$, and only at the points $y_\ell$. On the other hand, since (1.1) is nonlinear, $w$ is not a solution of (1.1). However, Merle [29] shows that there exists a solution $u$ of (1.1) on $[0,1)$ for which $T^* = 1$ and which is asymptotic as $t \uparrow 1$ to $w$. This shows in a way the stability of the type of blow up displayed by $v$.

**Remark 4.6.** Assume for simplicity $\lambda = 1$ and let $R$ be the positive, spherically symmetric ground state of (3.1). It follows that for any $\gamma \in \mathbf{R}$, $\mu > 0$ and $y \in \mathbf{R}^N$, $v(t,x) = e^{i\gamma} e^{i\mu^2 t} R(\mu(x-y))$ is a solution of (1.1). For any $b < 0$ and $x_1 \in \mathbf{R}^N$, $u(t,x) = v_b(t, x - x_1)$ is therefore also a solution. An easy calculation shows that

$$u(t,x) = \left(\frac{\omega}{T^* - t}\right)^{\frac{N}{2}} e^{i\theta - i\frac{|x-x_1|^2}{4(T^*-t)} + i\frac{\omega^2}{T^*-t}} R\left(\frac{\omega}{T^*-t}((x - x_1) - (T^* - t)x_0)\right), \tag{4.14}$$

with $T^* = -\dfrac{1}{b}$, $\omega = \mu T^*$, $x_0 = \dfrac{y}{T^*}$, $\theta = \gamma - \mu^2 T^*$.

Let now $\varphi \in H^1(\mathbf{R}^N)$ be such that $\|\varphi\|_{L^2} = \|R\|_{L^2}$ and such that $T^* < \infty$. It follows from Merle [30] that there exist $\theta \in \mathbf{R}$, $\omega > 0$, $x_0, x_1 \in \mathbf{R}^N$ such that $u$ is given by (4.14). Similarly, if $T_* < \infty$, then there exist $\theta \in \mathbf{R}$, $\omega > 0$, $x_0, x_1 \in \mathbf{R}^N$ such that $u$ is given by

$$u(t,x) = \left(\frac{\omega}{T_* + t}\right)^{\frac{N}{2}} e^{i\theta + i\frac{|x-x_1|^2}{4(T_*+t)} - i\frac{\omega^2}{T_*+t}} R\left(\frac{\omega}{T_*+t}((x - x_1) - (T_* + t)x_0)\right).$$

In other words, the only solutions that blow up on the critical sphere are those obtained from the ground state by the pseudo-conformal transformation. Note in particular that if $u$ is a solution on the critical $L^2$ sphere, then $T^*$ and $T_*$ cannot both be finite.

Consider now an initial value $\varphi \in L^2(\mathbf{R}^N)$, and let $u \in C((-T_*, T^*), L^2(\mathbf{R}^N)) \cap L^{\alpha+2}_{loc}((-T_*, T^*), L^{\alpha+2}(\mathbf{R}^N))$ be the maximal solution of (1.1). Given $b \in \mathbf{R}$, set

$$\varphi_b(x) = e^{i\frac{b|x|^2}{4}}\varphi(x),$$

and let $\widetilde{u}_b \in C((-T^b_*, T^*_b), L^2(\mathbf{R}^N)) \cap L^{\alpha+2}_{loc}((-T^b_*, T^*_b), L^{\alpha+2}(\mathbf{R}^N))$ be the maximal solution of (1.1) corresponding to the initial value $\varphi_b$. By uniqueness and Theorem 4.1, it follows that $\widetilde{u}_b$ coincides with $u_b$ given by (4.1), as long as $u_b$ is defined. In particular,

$$T^*_b = \infty \text{ if } bT^* \geq 1, \quad T^*_b \geq \frac{T^*}{1 - bT^*} \text{ if } bT^* < 1,$$

and

$$T^b_* = \infty \text{ if } bT_* \leq -1, \quad T^b_* \geq \frac{T^*}{1 + bT_*} \text{ if } bT_* > -1.$$

Before proceeding further, we introduce some notation. We set

$$\begin{aligned}
\mathcal{B}_+ &= \{\varphi \in L^2(\mathbf{R}^N);\, T^* < \infty\},\\
\mathcal{N}_+ &= \{\varphi \in L^2(\mathbf{R}^N);\, T^* = \infty \text{ and } \|u\|_{L^{\alpha+2}((0,\infty),L^{\alpha+2})} = \infty\},\\
\mathcal{R}_+ &= \{\varphi \in L^2(\mathbf{R}^N);\, T^* = \infty \text{ and } \|u\|_{L^{\alpha+2}((0,\infty),L^{\alpha+2})} < \infty\}.
\end{aligned}$$

Similarly, we set

$$\begin{aligned}
\mathcal{B}_- &= \{\varphi \in L^2(\mathbf{R}^N);\, T_* < \infty\},\\
\mathcal{N}_- &= \{\varphi \in L^2(\mathbf{R}^N);\, T_* = \infty \text{ and } \|u\|_{L^{\alpha+2}((-\infty,0),L^{\alpha+2})} = \infty\},\\
\mathcal{R}_- &= \{\varphi \in L^2(\mathbf{R}^N);\, T_* = \infty \text{ and } \|u\|_{L^{\alpha+2}((-\infty,0),L^{\alpha+2})} < \infty\}.
\end{aligned}$$

We observe that if $u$ satisfies (1.1) on $[0, T]$, then $v(t) = \overline{u(-t)}$ satisfies (1.1) on $[-T, 0]$ with the initial value $v(0) = \overline{\varphi}$. This implies that

$$\begin{aligned}
\mathcal{B}_- &= \overline{\mathcal{B}_+} = \{\varphi \in L^2(\mathbf{R}^N);\, \overline{\varphi} \in \mathcal{B}_+\},\\
\mathcal{N}_- &= \overline{\mathcal{N}_+} = \{\varphi \in L^2(\mathbf{R}^N);\, \overline{\varphi} \in \mathcal{N}_+\},\\
\mathcal{R}_- &= \overline{\mathcal{R}_+} = \{\varphi \in L^2(\mathbf{R}^N);\, \overline{\varphi} \in \mathcal{R}_+\}.
\end{aligned}$$

The following result is an immetiate consequence of (4.5) and of the blow up alternative of Theorem 3.5.

**Lemma 4.7.** *The following properties hold.*

(i) *If $\varphi \in \mathcal{B}_+$, then $\varphi_b \in \mathcal{B}_+$ and $T_b^* = \dfrac{T^*}{1 - bT^*}$ for all $b < \dfrac{1}{T^*}$; $\varphi_{\frac{1}{T^*}} \in \mathcal{N}_+$; $\varphi_b \in \mathcal{R}_+$ for all $b > \dfrac{1}{T^*}$.*

(ii) *If $\varphi \in \mathcal{N}_+$, then $\varphi_b \in \mathcal{B}_+$ and $T_b^* = -\dfrac{1}{b}$ for all $b < 0$; $\varphi_b \in \mathcal{R}_+$ for all $b > 0$.*

(iii) *If $\varphi \in \mathcal{R}_+$, then $T_b^* > -\dfrac{1}{b}$ for all $b < 0$; $\varphi_b \in \mathcal{R}_+$ for all $b \geq 0$.*

*Similar properties hold for $t < 0$. More precisely,*

(iv) *If $\varphi \in \mathcal{B}_-$, then $\varphi_b \in \mathcal{B}_-$ and $T_*^b = \dfrac{T_*}{1 + bT_*}$ for all $b > -\dfrac{1}{T_*}$; $\varphi_{-\frac{1}{T_*}} \in \mathcal{N}_-$; $\varphi_b \in \mathcal{R}_-$ for all $b < -\dfrac{1}{T_*}$.*

(v) *If $\varphi \in \mathcal{N}_-$, then $\varphi_b \in \mathcal{B}_-$ and $T_*^b = \dfrac{1}{b}$ for all $b > 0$; $\varphi_b \in \mathcal{R}_-$ for all $b < 0$.*

(vi) *If $\varphi \in \mathcal{R}_-$, then $T_*^b > \dfrac{1}{b}$ for all $b > 0$; $\varphi_b \in \mathcal{R}_-$ for all $b \leq 0$.*

**Corollary 4.8.** *Let $\varphi \in L^2(\mathbf{R}^N)$. Then either $T^*(\varphi_b) = \infty$ for all $b \in \mathbf{R}$, or else there exists $b_0 \in \mathbf{R}$ such that $T^*(\varphi_b) < \infty$ for all $b < b_0$ and $T^*(\varphi_b) = \infty$ for all $b \geq b_0$. In the first case, we have $\varphi_b \in \mathcal{R}_+$ for all $b \in \mathbf{R}$. In the latter case $\varphi_{b_0} \in \mathcal{N}_+$ and $\varphi_b \in \mathcal{R}_+$ for all $b > b_0$.*

*Similarly, either $T_*(\varphi_b) = \infty$ for all $b \in \mathbf{R}$, or else there exists $b_1 \in \mathbf{R}$ such that $T_*(\varphi_b) < \infty$ for all $b > b_1$ and $T^*(\varphi_b) = \infty$ for all $b \leq b_1$. In the first case, we have $\varphi_b \in \mathcal{R}_-$ for all $b \in \mathbf{R}$. In the latter case $\varphi_{b_1} \in \mathcal{N}_-$ and $\varphi_b \in \mathcal{R}_-$ for all $b < b_1$.*

**Proof.** We begin with the remark that $(\varphi_b)_c = \varphi_{b+c}$ and $(u_b)_c = u_{b+c}$ for all $b, c \in \mathbf{R}$.

Suppose first that $T^*(\varphi_b) = \infty$ for all $b \in \mathbf{R}$. If there exists $b \in \mathbf{R}$ such that $\varphi_b \in \mathcal{N}_+$, then $\|u_b\|_{L^{\alpha+2}((0,\infty),L^{\alpha+2})} = \infty$. By property (ii) of Lemma 4.7, this implies that $\varphi_{b-1} = (\varphi_b)_{-1} \in \mathcal{B}_+$, which is absurd; and so, $\varphi_b \in \mathcal{R}_+$ for all $b \in \mathbf{R}$.

Suppose now that there exists $c \in \mathbf{R}$ such that $\varphi_c \notin \mathcal{R}_+$. Then $\varphi_c \in \mathcal{N}_+ \cup \mathcal{B}_+$. Setting $b_0 = \dfrac{1}{T^*(\varphi_c)}$, it follows from properties (i) and (ii) of Lemma 4.7 that $\varphi_b \in \mathcal{B}_+$ for $b < b_0$, $\varphi_{b_0} \in \mathcal{N}_+$ and $\varphi_b \in \mathcal{R}_+$ for $b > b_0$. The properties for $t < 0$ are proved similarly. □

**Corollary 4.9.** *$\mathcal{R}_+$ and $\mathcal{R}_-$ are open subsets of $L^2(\mathbf{R}^N)$. $\mathcal{R}_+ \cap X$ and $\mathcal{R}_- \cap X$ are open subsets of $X$.*

**Proof.** Let $\varphi \in \mathcal{R}_+$. Set $\psi = \varphi_{-1}$. It follows from property (iii) of Lemma 4.7 that $T^*(\psi) > 1$. By continuous dependence (see Theorem 3.5), it follows that there exists $\varepsilon > 0$ such that if $\|\psi - \phi\|_{L^2} \leq \varepsilon$, then $T^*(\phi) > 1$. This implies that $\phi_1 \in \mathcal{R}_+$. Setting $\gamma = \phi_1$, we see that if $\|\varphi - \gamma\|_{L^2} \leq \varepsilon$, then $\gamma \in \mathcal{R}_+$. The result for $\mathcal{R}_-$ is proved similarly, and the results for $\mathcal{R}_+ \cap X$ and $\mathcal{R}_- \cap X$ follow, since $X \hookrightarrow L^2(\mathbf{R}^N)$. □

**Remark 4.10.** The above results call for several comments.

(i) Assume $\lambda < 0$. In this case, we know that if $\varphi \in Y$, with

$$Y = \{u \in L^2(\mathbf{R}^N);\, xu(x) \in L^2(\mathbf{R}^N)\},$$

then $T^*(\varphi) = T_*(\varphi) = +\infty$ (see Remark 3.7). This implies, in view of Corollary 4.8, that $Y \subset \mathcal{R}_+ \cap \mathcal{R}_-$. The question as to whether or not $L^2(\mathbf{R}^N) = \mathcal{R}_+ = \mathcal{R}_-$ is open (see Remark 3.7).

(ii) Assume $\lambda > 0$. In this case, we know that some solutions blow up in finite time, for both $t > 0$ and $t < 0$. However, it follows from Lemma 4.7 that, given any $\varphi \in L^2(\mathbf{R}^N)$, $\varphi_b \in \mathcal{R}_+$ for $b$ large enough. This means that blow up is not related to the size of the initial value only, since changing $\varphi$ to $\varphi_b$ is only a modification of the phase. Note also that the multiplier $e^{i\frac{b|x|^2}{4}}$ which leads to global existence is independent of the initial value, except for the parameter $b$. This property will be generalized to the case $\alpha \neq 4/N$ in the next section.

(iii) Still assume $\lambda > 0$. It follows from the above observation that $\mathcal{R}_+$ is an unbounded subset of $L^2(\mathbf{R}^N)$. As well, $\mathcal{R}_-$ is an unbounded subset of $L^2(\mathbf{R}^N)$. In addition, it follows from Remark 3.8 and Corollary 4.8 that there exists $\mu > 0$ such that if $\|\varphi\|_{L^2} < \mu$, then $\varphi_b \in \mathcal{R}_+ \cap \mathcal{R}_-$ for all $b \in \mathbf{R}$. In particular, the open set $\mathcal{R}_+ \cap \mathcal{R}_-$ contains 0 in its interior. We do not know whether or not $\mathcal{R}_+ \cap \mathcal{R}_-$ is unbounded.

(iv) Still assume $\lambda > 0$. If $N \geq 2$, then equation (3.1) has an unbounded sequence of solutions (see Berestycki and Lions [4]). On the other hand, if $\varphi$ is a solution of (3.1), then $u(t,x) = e^{it}\varphi(x)$ is a solution of (1.1), for which $T^* = T_* = \infty$ and $\|u\|_{L^{\alpha+2}(-\infty,0),L^{\alpha+2})} = \|u\|_{L^{\alpha+2}(0,\infty),L^{\alpha+2})} = \infty$. Therefore, $\varphi \in \mathcal{N}_+ \cap \mathcal{N}_-$; and so, $\mathcal{N}_+ \cap \mathcal{N}_-$ is an unbounded subset of $L^2(\mathbf{R}^N)$.

(v) Still assume $\lambda > 0$. It follows from Remark 3.9 that if $\varphi \in Y$ verifies $\|\varphi\|_{L^2} < \lambda^{-\frac{1}{\alpha}}\|R\|_{L^2}$, then $\varphi \in \mathcal{R}_+ \cap \mathcal{R}_-$. We do not know whether or not the same conclusion holds without the assumption $\varphi \in Y$.

(vi) Assume for simplicity $\lambda = 1$. Set

$$\mathcal{E} = \left\{ a^{\frac{N}{2}} e^{i\gamma} e^{i\mu|x-y_1|^2} R(a(x-y_1)-y_0);\ a>0, \gamma,\mu \in \mathbf{R}, y_0, y_1 \in \mathbf{R}^N \right\}.$$

If $\varphi \in \mathcal{E}$, then

$$u(t,x) = \left(\frac{a}{1+4\mu t}\right)^{\frac{N}{2}} e^{i\gamma} e^{i\frac{\mu|x-y_1|^2}{1+4\mu t}} e^{i\frac{a^2 t}{(1+4\mu t)}} R\left(\frac{a}{1+4\mu t}(x-y_1)-y_0\right),$$

for $-\infty < t < -\dfrac{1}{4\mu}$ if $\mu < 0$, for $-\dfrac{1}{4\mu} < t < \infty$ if $\mu > 0$, and for $-\infty < t < \infty$ if $\mu = 0$ (see Remark 4.6). On the other hand, if $\varphi \in X$, $\varphi \notin \mathcal{E}$, then $\varphi \in \mathcal{R}_+ \cap \mathcal{R}_-$. Indeed, if $\varphi \notin \mathcal{E}$, then $e^{i\frac{b|x|^2}{4}}\varphi \notin \mathcal{E}$ for all $b \in \mathbf{R}$. Therefore, it follows from Remark 4.6 that $T^* = T_* = \infty$. As well, we have $T^*(\varphi_b) = T_*(\varphi_b) = \infty$ for all $b \in \mathbf{R}$. By Corollary 4.8, this implies that $\varphi \in \mathcal{R}_+ \cap \mathcal{R}_-$. If we assume only $\varphi \in H^1(\mathbf{R}^N)$, then it follows as well from Remark 4.6 that $T^* = T_* = \infty$, but we do not know whether or not $T^*(\varphi_b) = T_*(\varphi_b) = \infty$ for all $b \in \mathbf{R}$, since $\varphi_b$ need not be in $H^1(\mathbf{R}^N)$ for $b \neq 0$. In particular, we do not know if $\varphi \in \mathcal{R}_+ \cap \mathcal{R}_-$. If we assume only $\varphi \in L^2(\mathbf{R}^N)$, then we cannot apply Remark 4.6, and we do not even know whether or not $T^* = T_* = \infty$.

**Remark 4.11.** For the nonlinear Klein-Gordon equation, and under appropriate assumptions on the nonlinearity, the set of initial values for which the solution is global, is a closed subset of the energy space (see [6]). The same property holds for a class of nonlinear heat equations (see [7]). For the nonlinear Schrödinger equation, the situation is quite different. In fact, if $\lambda > 0$, then the set of initial values for which the solution is global (for $t > 0$, for $t < 0$, or for both $t > 0$ and $t < 0$) is not closed, in any of the spaces $L^2(\mathbf{R}^N)$, $H^1(\mathbf{R}^N)$ or $X$. Indeed, assume for simplicity $\lambda = 1$, and let $\varphi = R_{-1}$, so that $\|\varphi\|_{L^2} = \|R\|_{L^2}$ and $T^* = 1$. For $0 < \varepsilon < 1$, set $\varphi_\varepsilon = (1-\varepsilon)\varphi$. We have $\|\varphi_\varepsilon\|_{L^2} = (1-\varepsilon)\|R\|_{L^2} < \|R\|_{L^2}$;

and so $\varphi_\varepsilon \in \mathcal{R}_+ \cap \mathcal{R}_-$ (see Remark 4.10 (v)). On the other hand, we have $\varphi_\varepsilon \underset{\varepsilon\downarrow 0}{\longrightarrow} \varphi$ in $X$. Therefore, a limit point (in $X$, hence also in $H^1(\mathbf{R}^N)$ and in $L^2(\mathbf{R}^N)$) of a sequence of $\mathcal{R}_+ \cap \mathcal{R}_-$ can belong to $\mathcal{B}_+$. By considering $\varphi = R_1$, we see as well that a limit point of a sequence of $\mathcal{R}_+ \cap \mathcal{R}_-$ can belong to $\mathcal{B}_-$.

**Remark 4.12.** In the case $\lambda > 0$, the pseudo-conformal transformation relates the behavior at blow up of blowing up solutions to the behavior at infinity of solutions with initial values in $\mathcal{N}_+$ or $\mathcal{N}_-$. To see this, consider $\varphi \in X$ such that $T^*(\varphi) < \infty$, and let $u$ be the corresponding solution of (1.1). Set $\psi = \varphi_b$ with $b = 1/T^*$, and let $v$ be the solution of (1.1) with the initial value $\psi$. It follows from Lemma 4.7 that $\psi \in \mathcal{N}_+$. On the other hand, we have (see (4.4))

$$\|u(t)\|_{L^{\alpha+2}}^{\alpha+2} = \left(\frac{T^*}{T^* - t}\right)^2 \left\| v\left(\frac{T^* t}{T^* - t}\right) \right\|_{L^{\alpha+2}}^{\alpha+2},$$

for all $0 \le t < T^*$. By conservation of energy, $\|u(t)\|_{L^{\alpha+2}}^{\alpha+2} \approx \|\nabla u(t)\|_{L^2}^2$ near $T^*$. Therefore, we see on the above identity that a lower estimate (respectively, an upper estimate) of $\|\nabla u(t)\|_{L^2}$ at blow up corrseponds to a lower estimate (respectively, an upper estimate) of $\|v(t)\|_{L^{\alpha+2}}$ at infinity. In particular, the lower estimate (2.20) implies that

$$\liminf_{t\to\infty} t\|u(t)\|_{L^{\alpha+2}}^{\alpha+2} > 0,$$

for all solutions $u$ of (1.1) with $\varphi \in X \cap \mathcal{N}_+$, $\varphi \neq 0$. On the other hand, if all solutions of (1.1) with initial data in $X \cap \mathcal{N}_+$ were bounded from above in $L^{\alpha+2}(\mathbf{R}^N)$, then we would obtain the upper estimate

$$\|\nabla u(t)\|_{L^2} \le \frac{C}{T^* - t},$$

at blow up, and if all solutions of (1.1) with initial data in $X \cap \mathcal{N}_+$ were bounded from below in $L^{\alpha+2}(\mathbf{R}^N)$, then we would obtain the lower estimate

$$\|\nabla u(t)\|_{L^2} \ge \frac{c}{T^* - t},$$

at blow up.

We now give some applications to the scattering theory. Some of these results will be generalized to the case $\alpha \neq 4/N$ in Section 6. The reader is referred to the beginning of

Section 6 for the definitions of the scattering states $u_\pm$, of the wave operators $\Omega_\pm$ and of the scattering operator $\mathbf{S}$.

Before stating the results, we introduce some notation. We denote by $\mathcal{F}$ the Fourier transform

$$\mathcal{F}u(\xi) = \int_{\mathbf{R}^N} e^{-2\pi i x\cdot\xi} u(x)\, dx,$$

and we denote by $\mathcal{F}^{-1}$ its inverse. We define the dilation $D_\beta$, $\beta > 0$, by

$$D_\beta u(x) = \beta^{\frac{N}{2}} u(\beta x),$$

and the multiplier $M_b$, $b \in \mathbf{R}$ by

$$M_b u(x) = e^{i\frac{b|x|^2}{4}} u(x).$$

With this notation, and using the explicit kernel

$$T(t)u = \frac{1}{(4\pi i t)^{\frac{N}{2}}} \int_{\mathbf{R}^N} e^{i\frac{|x-y|^2}{4t}} u(y)\, dy,$$

elementary calculations show that

$$T(t)D_\beta = D_\beta T(\beta^2 t), \tag{4.15}$$

and that

$$T(t)M_b = M_{\frac{b}{1+bt}} D_{\frac{1}{1+bt}} T\left(\frac{t}{1+bt}\right), \tag{4.16}$$

if $1 + bt > 0$ and

$$T\left(-\frac{1}{b}\right) M_b = i^{-\frac{N}{2}} M_{-b} D_{-\frac{b}{4\pi}} \mathcal{F}. \tag{4.17}$$

We have the following result (see [10]).

**Theorem 4.13.** *The following properties hold.*

(i) *Let $\varphi \in L^2(\mathbf{R}^N)$ with $T^* = \infty$, and let $u$ be the corresponding solution of (1.1). The limit*

$$u_+ = \lim_{t\to\infty} T(-t)u(t) \tag{4.18}$$

*exists in $L^2(\mathbf{R}^N)$ if and only if $\varphi \in \mathcal{R}_+$. The limit (4.18) exists in $X$ if and only if $\varphi \in \mathcal{R}_+ \cap X$. In addition,*

$$u_+ = i^{\frac{N}{2}} \mathcal{F}^{-1} D_{4\pi} M_{-1} v(1), \tag{4.19}$$

*where $v$ is the solution of (1.1) with the initial value $\varphi_{-1}$. (Note that since $\varphi \in \mathcal{R}_+$, we have $T^*(\varphi_{-1}) > 1$.) In addition, the mapping $U_+ : u \mapsto u_+$ is a bicontinuous bijection $\mathcal{R}_+ \to \mathcal{F}^{-1}(\mathcal{R}_-)$ for the $L^2$ topology and $\mathcal{R}_+ \cap X \to \mathcal{F}^{-1}(\mathcal{R}_- \cap X)$ for the $X$ topology.*

(ii) *Let $\varphi \in L^2(\mathbf{R}^N)$ with $T_* = \infty$, and let $u$ be the corresponding solution of (1.1). The limit*

$$u_- = \lim_{t \to -\infty} T(-t)u(t) \tag{4.20}$$

*exists in $L^2(\mathbf{R}^N)$ if and only if $\varphi \in \mathcal{R}_-$. The limit (4.20) exists in $X$ if and only if $\varphi \in \mathcal{R}_- \cap X$. In addition,*

$$u_- = i^{-\frac{N}{2}} \mathcal{F} D_{4\pi} M_1 w(-1), \tag{4.21}$$

*where $w$ is the solution of (1.1) with the initial value $\varphi_1$. (Note that since $\varphi \in \mathcal{R}_-$, we have $T_*(\varphi_1) > 1$.) In addition, the mapping $U_- : u \mapsto u_-$ is a bicontinuous bijection $\mathcal{R}_- \to \mathcal{F}(\mathcal{R}_+)$ for the $L^2$ topology and $\mathcal{R}_- \cap X \to \mathcal{F}(\mathcal{R}_+ \cap X)$ for the $X$ topology.*

(iii) *The wave operators $\Omega_\pm$ are bicontinuous bijections $\mathcal{R}_\pm \to \mathcal{F}^{\mp 1}(\mathcal{R}_\mp)$ for the $L^2$ topology and $\mathcal{R}_\pm \cap X \to \mathcal{F}^{\mp 1}(\mathcal{R}_\mp \cap X)$ for the $X$ topology.*

(iv) *The scattering operator $\mathbf{S} = \Omega_+^{-1}\Omega_- = U_+\Omega_-$ is a bicontinuous bijection $U_-(\mathcal{R}_+ \cap \mathcal{R}_-) \to U_+(\mathcal{R}_+ \cap \mathcal{R}_-)$ for the $L^2$ topology and $U_-(\mathcal{R}_+ \cap \mathcal{R}_- \cap X) \to U_+(\mathcal{R}_+ \cap \mathcal{R}_- \cap X)$ for the $X$ topology.*

**Proof.** Let $v$ be the solution of (1.1) with the initial value $\varphi_{-1}$. Since $T^* = \infty$, we have $T^*(\varphi_{-1}) \geq 1$. On $[0,1)$, we have $v = u_{-1}$, i.e.

$$v(t) = M_{-\frac{1}{1-t}} D_{\frac{1}{1-t}} u\left(\frac{t}{1-t}\right).$$

Therefore, it follows from (4.15) and (4.16) that

$$T(-t)v(t) = M_{-1} T\left(-\frac{t}{1-t}\right) u\left(\frac{t}{1-t}\right), \tag{4.22}$$

for all $t \in [0,1)$. Assume that the limit (4.18) exists in $L^2(\mathbf{R}^N)$. It follows from (4.22) that $v(t)$ has a limit in $L^2(\mathbf{R}^N)$ as $t \uparrow 1$. Thus $T^*(\varphi_{-1}) > 1$; and so $\varphi \in \mathcal{R}_+$. If the limit (4.18) exists in $X$, then we obtain as well $\varphi \in \mathcal{R}_+ \cap X$.

Conversely, if $\varphi \in \mathcal{R}_+$, then $v$ has a limit in $L^2(\mathbf{R}^N)$ as $t \uparrow 1$, and it follows from (4.22) that the limit (4.18) exists in $L^2(\mathbf{R}^N)$. If $\varphi \in \mathcal{R}_+ \cap X$, then $v$ has a limit in $X$ as $t \uparrow 1$, and it follows from (4.22) that the limit (4.18) exists in $X$. This proves the first part of (i).

Suppose now that the limit (4.18) exists in $L^2(\mathbf{R}^N)$, so that $v(1) = \lim_{t\uparrow 1} v(t)$ exists in $L^2(\mathbf{R}^N)$. It follows from (4.22) that

$$T(-1)v(1) = M_{-1}u_+.$$

Therefore, by (4.17),

$$v(1) = T(1)M_{-1}u_+ = i^{-\frac{N}{2}} M_{-1} D_{\frac{1}{4\pi}} \mathcal{F} u_+,$$

from which (4.19) follows.

We now show that $U_+(\mathcal{R}_+) = \mathcal{F}^{-1}(\mathcal{R}_-)$. Given $\varphi \in \mathcal{R}_+$, we have by (4.19)

$$U_+\varphi = i^{\frac{N}{2}} \mathcal{F}^{-1} D_{4\pi} M_{-1} v(1),$$

and we show that $D_{4\pi}M_{-1}v(1) \in \mathcal{R}_-$. Indeed, since $v$ is defined on $[0,1]$, we have $T_*(v(1)) > 1$; and so, $M_{-1}v(1) \in \mathcal{R}_-$. Furthermore, since the equation (1.1) is invariant by the scaling

$$u(t,x) \to \beta^{\frac{N}{2}} u(\beta^2 t, \beta x),$$

for any $\beta > 0$, it is clear that $D_\beta(\mathcal{R}_-) = \mathcal{R}_-$, so that $D_{4\pi}M_{-1}v(1) \in \mathcal{R}_-$; and so, $U_+(\mathcal{R}_+) \subset \mathcal{F}^{-1}(\mathcal{R}_-)$.

Conversely, let $\eta \in \mathcal{F}^{-1}(\mathcal{R}_-)$. We have $\mathcal{F}\eta \in \mathcal{R}_-$, and it follows from the observation just above that $D_{\frac{1}{4\pi}}\mathcal{F}\eta \in \mathcal{R}_-$. Therefore, $T_*(M_1 D_{\frac{1}{4\pi}}\mathcal{F}\eta) > 1$. Let $z(t)$ be the solution of (1.1) with the initial value $M_1 D_{\frac{1}{4\pi}}\mathcal{F}\eta$. Since $z$ is defined on $[-1,0]$, we have $T^*(z(-1)) > 1$; and so, $M_1 z(-1) \in \mathcal{R}_+$, thus $i^{-\frac{N}{2}} M_1 z(-1) \in \mathcal{R}_+$. We set $\varphi = i^{-\frac{N}{2}} M_1 z(-1)$. We have $\varphi_{-1} = i^{-\frac{N}{2}} z(-1)$, so that the solution $v$ of (1.1) with the initial value $\varphi_{-1}$ is $i^{-\frac{N}{2}} z(t-1)$. Therefore, by (4.19),

$$\begin{aligned} U_+\varphi &= i^{\frac{N}{2}} \mathcal{F}^{-1} D_{4\pi} M_{-1} v(1) = i^{\frac{N}{2}} \mathcal{F}^{-1} D_{4\pi} M_{-1} i^{-\frac{N}{2}} z(0) \\ &= i^{\frac{N}{2}} \mathcal{F}^{-1} D_{4\pi} M_{-1} i^{-\frac{N}{2}} M_1 D_{\frac{1}{4\pi}} \mathcal{F}\eta = \eta. \end{aligned}$$

Therefore, $\eta \in U_+(\mathcal{R}_+)$; and so $U_+(\mathcal{R}_+) = \mathcal{F}^{-1}(\mathcal{R}_-)$.

The continuity properties of $U_+$ and its inverse follow from the continuous dependence of the solutions of (1.1) (applied to the solution $v$). This completes the proof of (i).

(ii) is proved by the same argument, or simply by observing that if $u(t)$ satisfies (1.1) on $[0, \infty)$, then $\overline{u(-t)}$ satisfies (1.1) on $(-\infty, 0]$ with the initial value $\overline{\varphi}$. Properties (iii) and (iv) are immediate consequences of (i) and (ii). □

**Remark 4.14.** Here are some comments about Theorem 4.13.

(i) Assume $\lambda < 0$. As observed before (see Remark 4.10), we do not know whether or not $\mathcal{R}_+ = \mathcal{R}_- = L^2(\mathbf{R}^N)$. However, we know that $\mathcal{R}_+ \cap X = \mathcal{R}_- \cap X = X$. This implies that the scattering operator $\mathbf{S}$, as well as the operators $U_\pm$ and $\Omega_\pm$ are bicontinuous bijections $X \to X$. In addition, it follows from Corollary 4.9 (see also Remark 3.8) that these operators are defined on $L^2$ neighborhoods of 0. In addition, with the notation of Remark 4.10, we have $Y \subset \mathcal{R}_+ \cap \mathcal{R}_-$. Since $\mathcal{F}^{-1} Y = H^1(\mathbf{R}^N)$, this implies that $H^1(\mathbf{R}^N) \subset U_+(\mathcal{R}_+) \cap U_-(\mathcal{R}_-)$. We do not know whether or not $\mathbf{S}$ is defined $H^1(\mathbf{R}^N) \to H^1(\mathbf{R}^N)$.

(ii) Assume $\lambda > 0$. It follows from Remark 4.10 that, if $B$ denotes the ball or $L^2(\mathbf{R}^N)$ of radius $\lambda^{-\frac{1}{\alpha}} \|R\|_{L^2}$, $X \cap B \subset \mathcal{R}_+ \cap \mathcal{R}_-$. Since the operators $\mathbf{S}$, $U_\pm$ and $\Omega_\pm$ are isometric in $L^2(\mathbf{R}^N)$, it follows easily that they all are bicontinuous bijections $X \cap B \to X \cap B$ for the $X$ norm. In addition, it follows from Corollary 4.9 (see also Remark 3.8) that these operators are defined on open subsets of $L^2(\mathbf{R}^N)$ containing 0. However, as observed before (see Remark 4.10), we do not know whether or not they are defined on $B$.

**5. A global existence and decay result in the non-invariant case.** In this section, we extend some of the porperties of Lemma 4.7 to the case $\alpha \neq 4/N$ (see also Remark 4.10 (ii)). We define

$$\alpha_0 = \frac{2 - N + \sqrt{N^2 + 12N + 4}}{2N}, \tag{5.1}$$

and we observe that

$$\frac{2}{N} \leq \frac{4}{N+2} < \alpha_0 < \frac{4}{N} < \frac{4}{N-2},$$

if $N \geq 2$ and

$$\frac{2}{N} < \alpha_0 < \frac{4}{N},$$

if $N = 1$. The main result of this section is the following (see [11]).

**Theorem 5.1.** *Assume $\alpha_0 < \alpha < \dfrac{4}{N-2}$, where $\alpha_0$ is defined by (5.1), and let $\varphi \in X$. Given $b \in \mathbf{R}$, set $\varphi_b(x) = e^{i\frac{b|x|^2}{4}}$ and let $\widetilde{u}_b$ be the solution of (1.1) with the initial value $\varphi_b$. There exists $b_0 < \infty$ such that if $b \geq b_0$, then $T^*(\varphi_b) = \infty$. Moreover, we have $\widetilde{u}_b \in L^q((0,\infty), W^{1,r}(\mathbf{R}^N)$ for any admissible pair $(q,r)$. Finally, if*

$$a = \frac{2\alpha(\alpha+2)}{4 - \alpha(N-2)}, \tag{5.2}$$

*then $\widetilde{u}_b \in L^a((0,\infty), L^{\alpha+2}(\mathbf{R}^N))$.*

**Remark 5.2.** Here are some comments about Theorem 5.1.

(i) Let $\beta = \dfrac{4(\alpha+2)}{N\alpha}$, so that $(\beta, \alpha+2)$ is an admissible pair. One verifies easily that if $\alpha > 4/N$, then $\beta < a$, where $a$ is given by (5.2), and that if $\alpha < 4/N$, then $\beta > a$. Next, if $u \in L^q((0,\infty), W^{1,r}(\mathbf{R}^N))$ for every admissible pair $(q,r)$, we have in particular $u \in L^\beta((0,\infty), L^{\alpha+2}(\mathbf{R}^N))$, and also $u \in L^\infty((0,\infty), L^{\alpha+2}(\mathbf{R}^N))$ by Sobolev's embedding theorem. Therefore, we have also $u \in L^a((0,\infty), L^{\alpha+2}(\mathbf{R}^N))$ if $\alpha \geq 4/N$. On the other hand, if $\alpha < 4/N$, then the property $u \in L^a((0,\infty), L^{\alpha+2}(\mathbf{R}^N))$ expresses a better decay at infinity.

(ii) If $\lambda < 0$, then all solutions of (1.1) are global. Therefore, Theorem 5.1 means that all the solutions $\widetilde{u}_b$ have a certain decay as $t \to \infty$ for $b$ large enough. In fact, we will see in the next section that the conclusions of Theorem 5.1 hold with $b_0 = 0$.

(iii) If $\lambda > 0$ and $\alpha < 4/N$, then all solutions of (1.1) are global. Therefore, Theorem 5.1 means that $\widetilde{u}_b$ has a certain decay as $t \to \infty$ if $b$ is large enough. Note that certain solutions do not decay, in particular the standing waves, i.e. the solutions of the form $e^{i\omega t}\varphi(x)$ (see [5], Chapter 8).

(iv) If $\lambda > 0$ and $\alpha \geq 4/N$, then (1.1) posesses solutions that blow up in finite time. Theorem 5.1 means that for any $\varphi \in X$, the initial value $\varphi_b$ gives rise to a solution which is global and which decays as $t \to \infty$ provided $b$ is large enough.

(v) Suppose $\lambda > 0$ and $\alpha \geq 4/N$. If $\varphi \in X$ is such that $E(\varphi) < 0$, then the maximal solution $u$ of (1.1) blows up in finite time, for both $t > 0$ and $t < 0$. Theorem 5.1 implies that if $b$ is large enough, then the maximal solution $\widetilde{u}_b$ of (1.1) with initial value $\varphi_b$ is positively global and decays as $t \to \infty$. Of course, $E(\varphi_b) \geq 0$ for such $b$'s, and one may wonder if $\widetilde{u}_b$ still blows up at a finite negative time. The answer is yes, as the following argument shows. Changing $\varphi_b$ to $\overline{\varphi_b}$ (which changes $\widetilde{u}_b(t)$ to $\overline{\widetilde{u}_b(-t)}$), it suffices to show that if $E(\varphi) < 0$, then for all $b > 0$ the solution $u$ of (1.1) with initial value $\varphi(x)e^{-i\frac{b|x|^2}{4}}$ blows up at a positive finite time. Let $T^*$ be the maximal existence time of $u$, and let $f(t) = \| |\cdot| u(t,\cdot)\|_{L^2}^2$. We have (see Section 2)

$$f'(t) = 4\mathrm{Im} \int_{\mathbf{R}^N} x\overline{u}\nabla u \, dx,$$

and

$$f''(t) = 16E(u(0)) - \lambda \frac{4(N\alpha - 4)}{\alpha + 2} \int_{\mathbf{R}^N} |u|^{\alpha+2}.$$

Therefore,

$$f(t) = f(0) + tf'(0) + 8E(u(0))t^2 - \lambda \frac{4(N\alpha - 4)}{\alpha + 2} \int_0^t \int_0^s \int_{\mathbf{R}^N} |u|^{\alpha+2} \, dx \, d\sigma \, ds,$$

for all $0 \leq t < T^*$. It follows that

$$f(t) \leq f(0) + tf'(0) + 8E(u(0))t^2, \tag{5.3}$$

for all $0 \leq t < T^*$. Setting $P(t) = f(0) + tf'(0) + 8E(u(0))t^2$ for all $t \geq 0$, a straightforward calculation shows that

$$P(t) = \|x\varphi\|_{L^2}^2 + 4t\left(F(\varphi) - \frac{b}{2}\|x\varphi\|_{L^2}^2\right) + 8t^2\left(E(\varphi) + \frac{b^2}{8}\|x\varphi\|_{L^2}^2 - \frac{b}{2}F(\varphi)\right),$$

with

$$F(\varphi) = \operatorname{Im} \int_{\mathbf{R}^N} x\overline{\varphi}\nabla\varphi \, dx.$$

In particular,

$$P\left(\frac{1}{b}\right) = \frac{8}{b^2} E(\varphi) < 0;$$

and it follows easily from (5.3) that $T^* < 1/b$. Hence the result.

(vi) We do not know whether or not Theorem 5.1 has any extension to the case $\alpha \leq \alpha_0$. Note, however, that in the case $N \neq 2$, $\alpha = \alpha_0$ and $\lambda < 0$, it follows from Corollary 6.9 and Theorem 6.15 below that for any nontrivial solution $u$ of (1.1), we have $(1+t)\|u(t)\|^a_{L^{\alpha+2}} \to \ell > 0$ as $t \to \pm\infty$. In particular, no nontrivial of (1.1) belongs to $L^a((0,\infty), L^{\alpha+2}(\mathbf{R}^N))$.

Theorem 5.1 is based on the following Strichartz inequality, which involves non-admissible pairs.

**Lemma 5.3.** *Let $(q,r)$ be an admissible pair with $r > 2$. Fix $a > \frac{q}{2}$ and define $\widetilde{a}$ by*

$$\frac{1}{\widetilde{a}} + \frac{1}{a} = \frac{2}{q}. \tag{5.4}$$

*If $T > 0$ and $f \in L^{\tilde{a}'}((0,T), L^{r'}(\mathbf{R}^N))$, then $u$ defined by*

$$u(t) = \int_0^t T(t-s)f(s)\, ds,$$

*belongs to $L^a((0,T), L^r(\mathbf{R}^N))$. Furthermore, there exists a constant $C$, depending only on $N, r$ and $a$ such that*

$$\|u\|_{L^a((0,T),L^r)} \leq C\|f\|_{L^{\tilde{a}'}((0,T),L^{r'})}, \tag{5.5}$$

*for every $f \in L^{\tilde{a}'}((0,T), L^{r'}(\mathbf{R}^N))$.*

**Proof.** By density, we need only prove estimate (5.5) for $f \in C([0,T], \mathcal{S}(\mathbf{R}^N))$. It is well known (see [5], Proposition 3.2.1) that

$$\|T(t)\|_{\mathcal{L}(L^{r'},L^r)} \leq |4\pi t|^{-N\left(\frac{1}{2}-\frac{1}{r}\right)},$$

whenever $2 \leq r \leq \infty$. Therefore,

$$\|u(t)\|_{L^r} \leq \int_0^t (4\pi(t-s))^{-\frac{2}{q}} \|f(s)\|_{L^{r'}},$$

and so (5.5) is an immediate consequence of the Riesz potential inequalities (Stein [36], Theorem 1, p. 119). $\square$

**Corollary 5.4.** *Let $r = \alpha + 2$, let $(q, r)$ be the corresponding admissible pair, and let $a$ be given by (5.2). Then $a > q/2$ if and only if $\alpha > \alpha_0$, with $\alpha_0$ defined by (5.1). For such values of $\alpha$ and $a$, and for $0 < T \le \infty$, we have the following estimates for $\mathcal{G}$ defined by*

$$\mathcal{G}(f)(t) = \int_0^t T(t-s) f(s)\, ds,$$

*for $0 \le t < T$.*

(i) *If $u \in L^a((0,T), L^r(\mathbf{R}^N))$, then $\mathcal{G}(|u|^\alpha u) \in L^a((0,T), L^r(\mathbf{R}^N))$. Furthermore, there exists $C$ depending only on $N$ and $\alpha$ such that*

$$\|\mathcal{G}(|u|^\alpha u)\|_{L^a((0,T),L^r)} \le C \|u\|^{\alpha+1}_{L^a((0,T),L^r)}, \tag{5.6}$$

*for every $u \in L^a((0,T), L^r(\mathbf{R}^N))$.*

(ii) *If $u \in L^a((0,T), L^r(\mathbf{R}^N)) \cap L^q((0,T), W^{1,r}(\mathbf{R}^N))$ and if $(\gamma, \rho)$ is any admissible pair, then $\mathcal{G}(|u|^\alpha u) \in L^\gamma((0,T), W^{1,\rho}(\mathbf{R}^N))$. Furthermore, there exists $C$ depending only on $N, \alpha$ and $\rho$ such that*

$$\|\mathcal{G}(|u|^\alpha u)\|_{L^\gamma((0,T),W^{1,\rho})} \le C \|u\|^{\alpha}_{L^a((0,T),L^r)} \|u\|_{L^q((0,T),W^{1,r})}, \tag{5.7}$$

*for every $u \in L^a((0,T), L^r(\mathbf{R}^N)) \cap L^q((0,T), W^{1,r}(\mathbf{R}^N))$.*

**Proof.** The first part of the lemma is a simple calculation, which we omit. For assertions (i) and (ii) consider $\widetilde{a}$ defined by (5.4). Since $(\alpha+1)r' = r$, $(\alpha+1)\widetilde{a}' = a$ and

$$\frac{1}{q'} = \frac{1}{q} + \frac{\alpha}{a},$$

we see that

$$\| \, |u|^\alpha u \|_{L^{\tilde{a}'}((0,T),L^{r'})} = \|u\|^{\alpha+1}_{L^a((0,T),L^r)}$$

and (applying Hölder's inequality twice) that

$$\| \, |u|^\alpha u \|_{L^{q'}((0,T),W^{1,r'})} \le C \|u\|^{\alpha}_{L^a((0,T),L^r)} \|u\|_{L^q((0,T),W^{1,r})}.$$

The results now follow from (5.5) and (1.2) respectively. □

By applying Corollary 5.4, we obtain the following estimate for the solutions of (1.1).

**Proposition 5.5.** *Assume that* $\alpha_0 < \alpha < \dfrac{4}{N-2}$*, where* $\alpha_0$ *is defined by (5.1). If* $u \in C([0,\infty), H^1(\mathbf{R}^N))$ *is a solution of (1.1), and if* $u \in L^a((0,\infty), L^{\alpha+2}(\mathbf{R}^N))$ *with* $a$ *defined by (5.2), then* $u \in L^\gamma((0,\infty), W^{1,\rho}(\mathbf{R}^N))$ *for every admissible pair* $(\gamma,\rho)$*.*

**Proof.** We already know (see Remark 2.2) that $u \in L^\gamma((0,T), W^{1,\rho}(\mathbf{R}^N))$, for every $T<\infty$ and every admissible pair $(\gamma,\rho)$. For a fixed $T \geq 0$, we set $v(t) = u(t+T)$; and so

$$v(t) = S(t)u(T) + i\lambda \int_0^t T(t-s)|v|^\alpha v(s)\,ds.$$

As before, set $r = \alpha+2$, and let $q$ be such that $(q,r)$ is an admissible pair. Let $(\gamma,\rho)$ be any admissible pair, and let $a$ be defined by (5.2). It follows from (1.2) and (5.7) that

$$\|v\|_{L^\gamma((0,\tau),W^{1,\rho})} \leq C_1\|u(T)\|_{H^1} + C_2\|u\|^\alpha_{L^a((T,T+\tau),L^r)}\|v\|_{L^q((0,\tau),W^{1,r})}, \tag{5.8}$$

for every $\tau > 0$. Choosing $T$ large enough so that $C_2\|u\|^\alpha_{L^a((T,\infty),L^r)} \leq 1/2$ and letting $(\gamma,\rho) = (q,r)$, we see that

$$\|v\|_{L^q((0,\tau),W^{1,r})} \leq 2C_1\|u(T)\|_{H^1},$$

for every $\tau>0$. Hence $u \in L^q((0,\infty),W^{1,r}(\mathbf{R}^N))$. That $u \in L^\gamma((0,\infty),W^{1,\rho}(\mathbf{R}^N))$ for any admissible pair $(\gamma,\rho)$ now follows from (5.8). □

We are now in a position to prove the following global existence result.

**Theorem 5.6.** *Assume* $\alpha_0 < \alpha < \dfrac{4}{N-2}$*, where* $\alpha_0$ *is defined by (5.1), and let* $a$ *be given by (5.2). There exists* $\varepsilon > 0$ *such that if* $\varphi \in H^1(\mathbf{R}^N)$ *and* $\|T(\cdot)\varphi\|_{L^a((0,\infty),L^{\alpha+2})} \leq \varepsilon$*, then the maximal solution* $u$ *of (1.1) is positively global. Moreover,* $u \in L^a((0,\infty), L^{\alpha+2}(\mathbf{R}^N))$*, and* $u \in L^\gamma((0,\infty), W^{1,\rho}(\mathbf{R}^N))$ *for every admissible pair* $(\gamma,\rho)$*.*

**Proof.** Let $\varepsilon > 0$ and set $r = \alpha+2$; let $\varphi \in H^1(\mathbf{R}^N)$ be such that

$$\|T(\cdot)\varphi\|_{L^a(0,\infty;L^r)} \leq \varepsilon;$$

and let $u$ be the maximal solution of (1.1) defined on $[0, T^*)$, with $0 < T^* \le \infty$. Consider $q$ such that $(q, r)$ is an admissible pair. It follows from (5.6), (1.2) and (5.7) that there exists $K$ independent of $\varphi$ such that

$$\|u\|_{L^a((0,T),L^r)} \le \varepsilon + K\|u\|^{\alpha+1}_{L^a((0,T),L^r)}, \tag{5.9}$$

and

$$\|u\|_{L^q((0,T),W^{1,r})} \le K\|\varphi\|_{H^1} + K\|u\|^{\alpha}_{L^a((0,T),L^r)}\|u\|_{L^q((0,T),W^{1,r})}, \tag{5.10}$$

for every $T < T^*$. Assume that $\varepsilon$ satisfies

$$2^{\alpha+1}K\varepsilon^{\alpha} < 1. \tag{5.11}$$

Since $\|u\|_{L^a(0,T;L^r)}$ depends continuously on $T$, it follows from (5.9) and (5.11) that

$$\|u\|_{L^a(0,T^*;L^r)} \le 2\varepsilon. \tag{5.12}$$

Applying (5.10) and (5.12), we obtain

$$\|u\|_{L^q(0,T^*;W^{1,r})} \le 2K\|\varphi\|_{H^1}. \tag{5.13}$$

Applying now (1.2) and (5.7) with $(\gamma, \rho) = (\infty, 2)$, and using (5.12) and (5.13), we see that

$$\|u\|_{L^\infty(0,T^*;H^1)} < \infty.$$

Therefore $T^* = \infty$, and the result follows from (5.12) and Proposition 5.5. □

**Proof of Theorem 5.1.** Since $\varphi \in X$, we have $\varphi_b \in X$, for every $b \in \mathbf{R}$. Let $(q, r)$ be the admissible pair such that $r = \alpha+2$ and let $a$ be defined by (5.3). Using formulas (4.15) and (4.16), one verifies easily that

$$\|T(\cdot)\varphi_b\|^a_{L^a((0,\infty),L^r)} = \int_0^{1/b} (1 - b\tau)^{\frac{2(a-q)}{q}} \|T(\tau)\varphi\|^a_{L^r}\, d\tau.$$

Since $\|T(\tau)\varphi\|_{L^r} \le C\|T(\tau)\varphi\|_{H^1} \le C\|\varphi\|_{H^1}$ and $\dfrac{2(a-q)}{q} > -1$, it follows that

$$\lim_{b\uparrow\infty} \|T(\cdot)\varphi_b\|_{L^a(0,\infty;L^r)} = 0.$$

The result now follows from Theorem 5.6. □

**Remark 5.7.** Assume that $\alpha_0 < \alpha < \dfrac{4}{N-2}$, where $\alpha_0$ is defined by (5.1), and let $\varphi \in X$. There exists $s_0 < \infty$ such that for every $s \geq s_0$, the maximal solution of (1.1) with initial value $\psi_s = T(s)\varphi$ is positively global and verifies $u \in L^a((0,\infty), L^{\alpha+2}(\mathbf{R}^N))$, and $u \in L^\gamma((0,\infty), W^{1,\rho}(\mathbf{R}^N))$ for every admissible pair $(\gamma, \rho)$. Indeed, since $\varphi \in X$, $\|T(\cdot)\varphi\|_{L^a(0,\infty;L^r)}$ is finite (See [5], Corollary 3.3.4); and so, the result follows from Theorem 5.6.

**6. Some results on the scattering theory in the non-invariant case.** We first recall the basic notions of the scattering theory. For convenience, we consider the scattering in $X$, though in principle the same notions can be introduced in $H^1(\mathbf{R}^N)$ or even in $L^2(\mathbf{R}^N)$ if $\alpha \leq 4/N$.

Let $\varphi \in X$ be such that the solution of (1.1) is defined for all $t \geq 0$, i.e. $T^* = \infty$. If the limit

$$u_+ = \lim_{t \to \infty} T(-t)u(t), \tag{6.1}$$

exists in $X$, we say that $u_+$ is the scattering state of $\varphi$ (at $+\infty$). As well, if $\varphi \in X$ is such that the solution of (1.1) is defined for all $t \leq 0$, i.e. $T_* = \infty$, and if the limit

$$u_- = \lim_{t \to -\infty} T(-t)u(t), \tag{6.2}$$

exists in $X$, we say that $u_-$ is the scattering state of $\varphi$ at $-\infty$. We set

$$\mathcal{R}_+ = \{\varphi \in X;\, T^* = \infty \text{ and the limit (6.1) exists}\},$$

and

$$\mathcal{R}_- = \{\varphi \in X;\, T_* = \infty \text{ and the limit (6.2) exists}\}.$$

Since changing $t$ to $-t$ in equation (1.1) corresponds to changing $u$ to $\overline{u}$, we see that

$$\mathcal{R}_- = \overline{\mathcal{R}_+} = \{\varphi \in X;\, \overline{\varphi} \in \mathcal{R}_+\}.$$

We define the operators

$$U_\pm : \begin{array}{l} \mathcal{R}_\pm \to X \\ \varphi \mapsto u_\pm \end{array}$$

and we set

$$\mathcal{U}_\pm = U_\pm(\mathcal{R}_\pm).$$

We also have

$$\mathcal{U}_- = \overline{\mathcal{U}_+} = \{v \in X;\, \overline{v} \in \mathcal{U}_+\}.$$

If the mappings $U_\pm$ are injective, we set

$$\Omega_\pm = U_\pm^{-1} : \mathcal{U}_\pm \to \mathcal{R}_\pm.$$

The mappings $\Omega_\pm$ are called the wave operators. Finally, we set

$$\mathcal{O}_\pm = U_\pm(\mathcal{R}_+ \cap \mathcal{R}_-), \tag{6.3}$$

so that

$$\mathcal{O}_- = \overline{\mathcal{O}_+} = \{v \in X; \overline{v} \in \mathcal{O}_+\}.$$

Finally, the scattering operator $\mathbf{S}$ is the mapping

$$\mathbf{S} = U_+\Omega_- : \mathcal{O}_- \to \mathcal{O}_+. \tag{6.4}$$

In other words, $u_+ = \mathbf{S}u_-$ if and only if there exists $\varphi \in X$ such that $T^* = T_* = \infty$ and such that $T(-t)u(t) \to u_\pm$ as $t \to \pm\infty$.

These notions are relevant for conservative equations with dispersion, in order to describe the asymptotic behavior of the solutions. For example, for the linear Schrödinger equation, i.e. if $\lambda = 0$, then $u(t)$ does not have any strong limit in $X$ if $\varphi \neq 0$. Indeed, $u(t) \rightharpoonup 0$ in $L^2(\mathbf{R}^N)$ as $t \to \pm\infty$ and $\|u(t)\|_{L^2} \equiv \|\varphi\|_{L^2}$. On the other hand, the operators introduced above are well-defined. In fact, in the linear case, $\Omega_\pm$ and $\mathbf{S}$ all coincide with the identity in $X$. Note, however, that in general these operators are nonlinear.

A scattering theory can be developped in $H^1(\mathbf{R}^N)$ for $\alpha > 4/N$ and $N \geq 3$. If $\lambda < 0$, we have a global theory (i.e. $\mathbf{S}$ is a bijection $H^1(\mathbf{R}^N) \to H^1(\mathbf{R}^N)$), and if $\lambda > 0$, we have low energy scattering (i.e. $\mathbf{S}$ and $\mathbf{S}^{-1}$ are defined on a neighborhood of 0 in $H^1(\mathbf{R}^N)$). These results are due to Ginibre and Velo [18] (see also [5], Chapter 7).

It seems that a scattering theory in $L^2(\mathbf{R}^N)$ exists only for $\alpha = 4/N$, and that the only available results are those of the preceding section (i.e. low energy scattering).

The scattering theory in $X$ was first developped by Ginibre and Velo [20] for $\alpha \geq 4/N$ by using the pseudo-conformal conservation law, and then improved by Tsutsumi [40] for $\alpha > \alpha_0$.

We now apply the pseudo-conformal transformation (4.1) with $b < 0$, and we suppose for convenience that $b = -1$. Moreover, throughout this section we systematically consider the variables $(s, y) \in \mathbf{R} \times \mathbf{R}^N$ defined by

$$s = \frac{t}{1-t}, \quad y = \frac{x}{1-t},$$

or equivalently,

$$t = \frac{s}{1+s}, \quad x = \frac{y}{1+s}.$$

Given $0 \le a < b \le \infty$ and $u$ defined on $(a,b) \times \mathbf{R}^N$, we set

$$v(t,x) = (1-t)^{-\frac{N}{2}} u\left(\frac{t}{1-t}, \frac{x}{1-t}\right) e^{-i\frac{|x|^2}{4(1-t)}}, \tag{6.5}$$

for $x \in \mathbf{R}^N$ and $\frac{a}{1+a} < t < \frac{b}{1+b}$. In particular, if $u$ is defined on $(0,\infty)$, then $v$ is defined on $(0,1)$. Transformation (6.5) reads as well, using the variables $(s,y)$,

$$v(t,x) = (1+s)^{\frac{N}{2}} u(s,y) e^{-i\frac{|y|^2}{4(1+s)}}. \tag{6.6}$$

One verifies easily that, given $0 \le a < b < \infty$, $u \in C([a,b], X) \cap C^1([a,b], H^{-1}(\mathbf{R}^N))$ if and only if $v \in C([\frac{a}{1+a}, \frac{b}{1+b}], X) \cap C^1([\frac{a}{1+a}, \frac{b}{1+b}], H^{-1}(\mathbf{R}^N))$. Furthermore, a straightforward calculation (see Theorem 4.1) shows that $u$ satisfies (1.1) on $(a,b)$ if, and only if $v$ satisfies the equation

$$iv_t + \triangle v + \lambda(1-t)^{\frac{N\alpha-4}{2}} |v|^\alpha v = 0, \tag{6.7}$$

on the interval $\left(\frac{a}{1+a}, \frac{b}{1+b}\right)$. Note that the term $(1-t)^{\frac{N\alpha-4}{2}}$ is regular, except possibly at $t = 1$, where it is singular for $\alpha < 4/N$. Furthermore, we have the following identities (see (4.6)–(4.8)):

$$\|v(t)\|_{L^{\beta+2}}^{\beta+2} = (1+s)^{\frac{N\beta}{2}} \|u(s)\|_{L^{\beta+2}}^{\beta+2}, \quad \beta \ge 0, \tag{6.8}$$

$$\|\nabla v(t)\|_{L^2}^2 = \frac{1}{4} \|(y + 2i(1+s)\nabla) u(s)\|_{L^2}^2, \tag{6.9}$$

$$\|\nabla u(s)\|_{L^2}^2 = \frac{1}{4} \|(x + 2i(1-t)\nabla) v(t)\|_{L^2}^2. \tag{6.10}$$

It follows from (6.8) and conservation of charge for (1.1) that

$$\frac{d}{dt} \|v(t)\|_{L^2} = 0. \tag{6.11}$$

Moreover, if we set

$$E_1(t) = \frac{1}{2} \|\nabla v(t)\|_{L^2}^2 - (1-t)^{\frac{N\alpha-4}{2}} \frac{\lambda}{\alpha+2} \|v(t)\|_{L^{\alpha+2}}^{\alpha+2},$$

$$E_2(t) = (1-t)^{\frac{4-N\alpha}{2}} E_1(t) = (1-t)^{\frac{4-N\alpha}{2}} \frac{1}{2} \|\nabla v(t)\|_{L^2}^2 - \frac{\lambda}{\alpha+2} \|v(t)\|_{L^{\alpha+2}}^{\alpha+2},$$

and

$$E_3(t) = \frac{1}{8}\|(x + 2i(1-t)\nabla)v(t)\|_{L^2}^2 - (1-t)^{\frac{N\alpha}{2}} \frac{\lambda}{\alpha+2}\|v(t)\|_{L^{\alpha+2}}^{\alpha+2}.$$

Then,

$$\frac{d}{dt}E_1(t) = -(1-t)^{\frac{N\alpha-6}{2}} \frac{4-N\alpha}{2} \frac{\lambda}{\alpha+2}\|v(t)\|_{L^{\alpha+2}}^{\alpha+2}, \tag{6.12}$$

$$\frac{d}{dt}E_2(t) = (1-t)^{\frac{2-N\alpha}{2}} \frac{N\alpha-4}{4}\|\nabla v(t)\|_{L^2}^2, \tag{6.13}$$

and

$$\frac{d}{dt}E_3(t) = 0. \tag{6.14}$$

Indeed, (6.12) and (6.13) are equivalent, and both are equivalent to the pseudo-conformal conservation law for $u$, by (6.8) and (6.9). As well, (6.14) is equivalent to the conservation of energy for $u$, by (6.8) and (6.10).

The results that we present in this section are based on the following observation.

**Proposition 6.1.** *Assume that* $0 < \alpha < \dfrac{4}{N-2}$, *let* $u \in C([0,\infty), X)$ *be a solution of equation (1.1), and let* $v \in C([0,1), X)$ *be the corresponding solution of (6.7) defined by (6.5). Then,* $T(-s)u(s)$ *has a strong limit in* $X$ *(respectively, in* $L^2(\mathbf{R}^N)$*) as* $s \to \infty$ *if and only if* $v(t)$ *has a strong limit in* $X$ *(respectively, in* $L^2(\mathbf{R}^N)$*) as* $t \uparrow 1$, *in which case*

$$\lim_{s\to\infty} T(-s)u(s) = e^{i\frac{|x|^2}{4}} T(-1)v(1), \tag{6.15}$$

*in* $X$ *(respectively, in* $L^2(\mathbf{R}^N)$*).*

**Proof.** It follows from (4.22) that

$$T(-s)u(s) = e^{i\frac{|x|^2}{4}} T\left(-\frac{s}{1+s}\right) v\left(\frac{s}{1+s}\right),$$

from which the result follows. □

The following result implies that if $\alpha \le 2/N$, then no scattering theory can be developped for the equation (1.1) (see Barab [3], Strauss [37], [38], Tsutsumi and Yajima [41]).

**Theorem 6.2.** *Assume* $\alpha \le \dfrac{2}{N}$. *If* $N = 1$, *assume further* $\alpha \le 1$. *Let* $\varphi \in L^2(\mathbf{R}^N)$ *and let* $u \in C(\mathbf{R}, L^2(\mathbf{R}^N))$ *be the corresponding solution of (1.1). If* $\varphi \ne 0$, *then* $T(-t)u(t)$

*does not have any strong limit in $L^2(\mathbf{R}^N)$, neither as $t \to \infty$ nor as $t \to -\infty$. In other words, no nontrivial solution of (1.1) has scattering states, even in $L^2(\mathbf{R}^N)$.*

**Proof.** We consider the case $t \to \infty$, the argument for $t \to -\infty$ being the same. We argue by contradiction and we assume $T(-t)u(t) \underset{t\to\infty}{\longrightarrow} u_+$ in $L^2(\mathbf{R}^N)$. In particular, we have

$$\|u_+\|_{L^2} = \|u(t)\|_{L^2} = \|\varphi\|_{L^2} > 0.$$

On the other hand, it follows from (6.15) that $v(t) \underset{t\uparrow 1}{\longrightarrow} w$ in $L^2(\mathbf{R}^N)$, with

$$w = T(1)\left(e^{-i\frac{|x|^2}{4}} u_+\right) \neq 0.$$

Since $\alpha + 1 \le 2$, we have $|v(t)|^\alpha v(t) \underset{t\uparrow 1}{\longrightarrow} |w|^\alpha w \neq 0$ in $L^{\frac{2}{\alpha+1}}(\mathbf{R}^N)$. Let now $\theta \in \mathcal{D}(\mathbf{R}^N)$ be such that

$$\langle i|w|^\alpha w, \theta\rangle = 1. \tag{6.16}$$

It follows from (6.7) that

$$\begin{aligned}\frac{d}{dt}\langle v(t), \theta\rangle &= \langle i\triangle v, \theta\rangle + \lambda(1-t)^{\frac{N\alpha-4}{2}}\langle i|v|^\alpha v, \theta\rangle \\ &= \langle iv, \triangle\theta\rangle + \lambda(1-t)^{\frac{N\alpha-4}{2}}\langle i|v|^\alpha v, \theta\rangle.\end{aligned}$$

Therefore, by (6.16), and since $v$ is bounded in $L^2(\mathbf{R}^N)$,

$$\left|\frac{d}{dt}\langle v(t), \theta\rangle\right| \ge \frac{1}{2}|\lambda|(1-t)^{\frac{N\alpha-4}{2}},$$

for $1-\varepsilon \le t < 1$, $\varepsilon > 0$ small enough. Since $\dfrac{N\alpha - 4}{2} < -1$, it follows that $|\langle v(t), \theta\rangle| \underset{t\uparrow 1}{\longrightarrow} \infty$, which is absurd. □

**Remark 6.3.** In the case $N = 1$ and $1 < \alpha \le 2$, we have the following result. Let $\varphi \in X$ and let $u \in C(\mathbf{R}, X)$ be the corresponding solution of (1.1). If $\varphi \neq 0$, then $T(-t)u(t)$ does not have any strong limit in $X$, neither as $t \to \infty$ nor as $t \to -\infty$. The proof is similar. One needs only observe that, since $v(t)$ is bounded in $X$, hence in $H^1(\mathbf{R})$, we have $v(t) \underset{t\uparrow 1}{\longrightarrow} w$ in $L^p(\mathbf{R})$ for every $2 \le p \le \infty$; and so, $|v(t)|^\alpha v(t) \underset{t\uparrow 1}{\longrightarrow} |w|^\alpha w$ in $L^2(\mathbf{R})$.

On the other hand, if $\alpha > 2/N$ and if $\lambda \le 0$, then every solution of (1.1) in $X$ has scattering states in $L^2(\mathbf{R}^N)$, as shows the following result (see Tsutsumi and Yajima [41]).

**Theorem 6.4.** *Assume* $\frac{2}{N} < \alpha < \frac{4}{N-2}$ *($2 < \alpha < \infty$ if $N = 1$), and $\lambda < 0$. Let $\varphi \in X$ and let $u \in C(\mathbf{R}, X)$ be the corresponding solution of (1.1). There exist $u_\pm \in L^2(\mathbf{R}^N)$ such that*

$$T(-t)u(t) \underset{t\to\pm\infty}{\longrightarrow} u_\pm,$$

*in $L^2(\mathbf{R}^N)$.*

**Remark 6.5.** Here are some comments on Theorem 6.4.

(i) If $\alpha \geq \alpha_0$, then we will show below that $u_\pm \in X$ and that $T(-t)u(t) \underset{t\to\pm\infty}{\longrightarrow} u_\pm$ in $X$ (see Theorems 6.14 and 6.15). The same conclusion holds if $\alpha > \frac{4}{N+2}$ ($\alpha > 2$ if $N = 1$), and if $\|\varphi\|_X$ is small enough (Theorem 6.7). If $\alpha \leq \frac{4}{N+2}$, or if $\alpha < \alpha_0$ and $\|\varphi\|_X$ is large, then we do not know whether or not $u_\pm \in X$.

(ii) Theorem 6.4 does not apply to the case $\lambda > 0$. In fact, if $\alpha < \dfrac{4}{N+2}$ there are arbitrarily small initial values $\varphi \in X$ which do not have a scattering state, even in the sense of $L^2(\mathbf{R}^N)$. To see this, let $\varphi \in X$ be a nontrivial solution of the equation

$$-\triangle\varphi + \varphi = \lambda|\varphi|^\alpha\varphi.$$

(See for example [5], Chapter 8.) Given $a > 0$, set $\varphi_a(x) = a^{\frac{2}{\alpha}}\varphi(ax)$. It follows that $-\triangle\varphi_a + a^2\varphi_a = \lambda|\varphi_a|^\alpha\varphi_a$. Therefore, $u_a(t,x) = e^{ia^2t}\varphi_a(x)$ satisfies (1.1), and $T(-t)u_a(t) = e^{ia^2t}T(-t)\varphi_a$ does not have any strong limit as $t \to \infty$ in $L^2(\mathbf{R}^N)$. On the other hand, one verifies easily that if $\alpha < \dfrac{4}{N+2}$, then $\|u_a\|_X \underset{a\downarrow 0}{\longrightarrow} 0$. However, we will see below (Theorem 6.7) that if $\alpha > \frac{4}{N+2}$, then small initial values in $X$ have scattering states in $X$ at $\pm\infty$.

**Proof of Theorem 6.4.** By Proposition 6.1, we need only show that $v(t)$ has a strong limit in $L^2(\mathbf{R}^N)$ as $t \uparrow 1$. As observed above (Remark 6.5), we have a better result when $\alpha \geq \alpha_0$. Therefore, we may assume that $\alpha < \alpha_0$, and in particular $\alpha < 4/N$. Therefore, it follows from (6.11) and (6.13) that

$$\|v(t)\|_{L^2} \leq C, \tag{6.17}$$

$$\|v(t)\|_{L^{\alpha+2}} \leq C, \tag{6.18}$$

$$\|\nabla v(t)\|_{L^2} \leq C(1-t)^{\frac{N\alpha-4}{4}}, \tag{6.19}$$

for all $t \in [0,1)$. By using the embeddings $L^{\frac{\alpha+2}{\alpha+1}}(\mathbf{R}^N) \hookrightarrow H^{-1}(\mathbf{R}^N) \hookrightarrow H^{-2}(\mathbf{R}^N)$, and the equation (6.7), we obtain

$$\|v_t\|_{H^{-2}} \le \|\triangle u\|_{H^{-2}} + C(1-t)^{\frac{N\alpha-4}{2}} \| \, |v|^\alpha v\|_{H^{-2}} \le C\|v\|_{L^2} + C(1-t)^{\frac{N\alpha-4}{2}} \|v\|_{L^{\alpha+2}}^{\alpha+1}.$$

Therefore,

$$\|v_t\|_{H^{-2}} \le C + C(1-t)^{\frac{N\alpha-4}{2}},$$

by (6.17) and (6.18). It follows that $v_t \in L^1((0,1), H^{-2}(\mathbf{R}^N))$. Therefore, there exists $w \in H^{-2}(\mathbf{R}^N)$ such that $v(t) \underset{t\uparrow 1}{\longrightarrow} w$ in $H^{-2}(\mathbf{R}^N)$. By using again (6.17), we obtain $w \in L^2(\mathbf{R}^N)$ and

$$v(t) \rightharpoonup w, \tag{6.20}$$

in $L^2(\mathbf{R}^N)$ as $t \uparrow 1$. Consider now $\psi \in H^1(\mathbf{R}^N)$, and let $0 \le t \le \tau < 1$. We have by (6.7)

$$\begin{aligned}(v(\tau) - v(t), \psi)_{L^2} &= \int_t^\tau \langle v_t, \psi \rangle_{H^{-1}, H^1} \, ds \\ &= \int_t^\tau (i\nabla v, \nabla \psi)_{L^2} \, ds + \int_t^\tau (1-s)^{\frac{N\alpha-4}{2}} \langle i\lambda |v|^\alpha v, \psi \rangle_{L^{\frac{\alpha+2}{\alpha+1}}, L^{\alpha+2}} \, ds;\end{aligned}$$

and so,

$$\begin{aligned}|(v(\tau) - v(t), \psi)_{L^2}| &\le C\|\nabla \psi\|_{L^2} \int_t^\tau \|\nabla v\|_{L^2} \, ds + C\|\psi\|_{L^{\alpha+2}} \int_t^\tau (1-s)^{\frac{N\alpha-4}{2}} \|v\|_{L^{\alpha+2}}^{\alpha+1} \, ds \\ &\le C\|\nabla \psi\|_{L^2} \int_t^\tau (1-s)^{\frac{N\alpha-4}{4}} \, ds + C\|\psi\|_{L^{\alpha+2}} \int_t^\tau (1-s)^{\frac{N\alpha-4}{2}} \, ds,\end{aligned}$$

by (6.19) and (6.18). Letting $\tau \uparrow 1$ and applying (6.20), we obtain

$$\begin{aligned}|(w - v(t), \psi)_{L^2}| &\le C\|\nabla \psi\|_{L^2} \int_t^1 (1-s)^{\frac{N\alpha-4}{4}} \, ds + C\|\psi\|_{L^{\alpha+2}} \int_t^1 (1-s)^{\frac{N\alpha-4}{2}} \, ds \\ &\le C(1-t)^{\frac{N\alpha}{4}} \|\nabla \psi\|_{L^2} + C(1-t)^{\frac{N\alpha-2}{2}} \|\psi\|_{L^{\alpha+2}}.\end{aligned}$$

We now let $\psi = v(t)$ and we apply again (6.19) and (6.18). It follows that

$$\begin{aligned}|(w - v(t), v(t))_{L^2}| &\le C(1-t)^{\frac{N\alpha}{4}} (1-t)^{\frac{N\alpha-4}{4}} + C(1-t)^{\frac{N\alpha-2}{2}} \\ &\le C(1-t)^{\frac{N\alpha-2}{2}} \underset{t\uparrow 1}{\longrightarrow} 0.\end{aligned} \tag{6.21}$$

Finally,

$$\|v(t) - w\|_{L^2}^2 = -(w - v(t), v(t))_{L^2} + (w - v(t), w)_{L^2} \underset{t\uparrow 1}{\longrightarrow} 0,$$

by (6.21) and (6.20). This completes the proof. □

Besides the fact that Theorem 6.4 does not apply to the case $\lambda > 0$, it does not either allow us to construct the wave and scattering operators, since the initial value $\varphi$ and the scattering states $u_\pm$ do not belong to the same space. We will improve this result under more restrictive assumptions on $\alpha$ by solving the initial value problem for the nonautonomous equation (6.7), and by applying Proposition 6.1 which relates the behavior of $u$ at infinity and the behavior of $v$ at $t = 1$.

However, we want to solve the Cauchy problem for (6.7) starting from any time $t \in [0, 1]$, including $t = 1$ where the nonautonomous term might be singular. In order to do this, we define the function

$$f(t) = \begin{cases} \lambda(1-t)^{\frac{N\alpha-4}{2}} \text{ if } -\infty < t < 1, \\ \lambda \text{ if } t \geq 1, \end{cases} \tag{6.22}$$

and we consider the equation

$$iv_t + \triangle u + f(t)|v|^\alpha v = 0. \tag{6.23}$$

Under appropriate assumptions on $\alpha$, the initial value problem for (6.23) can be solved starting from any time $t \in \mathbf{R}$, and we have the following result.

**Theorem 6.6.** *Assume* $\frac{4}{N+2} < \alpha < \frac{4}{N-2}$ *($2 < \alpha < \infty$, if $N = 1$). Then, for every $t_0 \in \mathbf{R}$ and $\psi \in X$, there exist $T_m(t_0;\psi) < t_0 < T_M(t_0;\psi)$ and a unique, maximal solution $v \in C((T_m, T_M), X) \cap C^1((T_m, T_M), H^{-1}(\mathbf{R}^N))$ of equation (6.23). The solution $v$ is maximal in the sense that if $T_M < \infty$ (respectively $T_m > -\infty$), then $\|u(t)\|_{H^1} \to \infty$, as $t \uparrow T_M$ (respectively $t \downarrow T_m$). In addition, the solution $v$ has the following properties.*

(i) *If $T_M = 1$, then $\liminf_{t\uparrow 1}\{(1-t)^\delta \|v(t)\|_{H^1}\} > 0$ with $\delta = \frac{N+2}{4} - \frac{1}{\alpha}$ if $N \geq 3$, $\delta$ any number larger than $1 - \frac{1}{\alpha}$ if $N = 2$, and $\delta = \frac{1}{2} - \frac{1}{\alpha}$ if $N = 1$.*

(ii) *The solution $v$ depends continuously on $\psi$ in the following way. The mapping $\psi \mapsto T_M$ is lower semicontinuous $X \to (0, \infty]$, and the mapping $\psi \mapsto T_m$ is upper semicontinuous $X \to [-\infty, 0)$. In addition, if $\psi_n \underset{n\to\infty}{\longrightarrow} \psi$ in $X$ and if $[S, T] \in (T_m, T_M)$, then $v_n \to v$ in $C([S, T], X)$, where $v_n$ denotes the solution of (6.23) with initial value $\psi_n$.*

**Proof.** The result follows immediately by applying Theorems 7.1 and 7.2 of Section 7 with $h(t) = f(t - t_0)$. □

We now give some applications to the scattering theory for (1.1).

**Theorem 6.7.** *Assume* $\frac{4}{N+2} < \alpha < \frac{4}{N-2}$ *($2 < \alpha < \infty$, if $N = 1$). Then, the following properties hold.*

(i) *The sets $\mathcal{R}_\pm$ and $\mathcal{U}_\pm$ are open subsets of $X$ containing 0. The operators $U_\pm : \mathcal{R}_\pm \to \mathcal{U}_\pm$ are bicontinuous bijections (for the $X$ topology) and the operators $\Omega_\pm : \mathcal{U}_\pm \to \mathcal{R}_\pm$ are bicontinuous bijections (for the $X$ topology).*

(ii) *The sets $\mathcal{O}_\pm$ are open subsets of $X$ containing 0, and the scattering operator* **S** *is a bicontinuous bijection $\mathcal{O}_- \to \mathcal{O}_+$ (for the $X$ topology).*

**Remark 6.8.** Theorem 6.7 implies that there is a low energy scattering theory in $X$ for the equation (1.1), provided $\frac{4}{N+2} < \alpha < \frac{4}{N-2}$ ($2 < \alpha < \infty$, if $N = 1$). As observed before (see Remark 6.5), if $\alpha < \frac{4}{N+2}$ and $\lambda > 0$, then there is no low energy scattering.

**Proof of Theorem 6.7.** Let $\varphi \in X$ and let $u$ be the corresponding solution of (1.1). Let $v$ be the solution of (6.7) with the initial value $\psi$ defined by $\psi(x) = \varphi(x)e^{-i\frac{|x|^2}{4}}$ (see Theorem 6.6). It follows that $v$ is defined by (6.6), as long as (6.6) makes sense. Therefore, it follows from Proposition 6.1 and Theorem 6.6 that $\varphi \in \mathcal{R}_+$ if and only if $T_M(0;\psi) > 1$, and that in this case $u_+ = e^{i\frac{|x|^2}{4}}T(-1)v(1)$. Therefore, the open character of $\mathcal{R}_+$ and the continuity of the operator $U_+$ follow from the continuous dependence of $v$ on $\psi$ (property (ii) of Theorem 6.6).

Let now $y \in X$, and set $w = T(1)\left(e^{-i\frac{|x|^2}{4}}y\right)$, so that $y = e^{i\frac{|x|^2}{4}}T(-1)w$. It follows from Proposition 6.1 and Theorem 6.6 that $y = U_+\varphi$ for some $\varphi \in X$ (i.e. $y \in \mathcal{U}_+$) if and only if $T_m(1, w) < 0$. In this case, $\varphi = e^{i\frac{|x|^2}{4}}z(0)$, where $z$ is the solution of (6.7) with the initial value $z(1) = w$. Therefore, the open character of $\mathcal{U}_+$ and the continuity of the operator $\Omega_+ = (U_+)^{-1}$ follow as above from the continuous dependence of $z$ on $w$.

As observed before, the similar statements for $\mathcal{R}_-$, $\mathcal{U}_-$, $U_-$ and $\Omega_-$ are equivalent, by changing $t$ to $-t$ and $u(t)$ to $\overline{u}(-t)$. Therefore, we have proved part (i) of the theorem. Part (ii) now follows from part (i) and the definitions of $\mathcal{O}_\pm$ and **S** (formulas (6.3) and (6.4)). □

**Corollary 6.9.** *Assume* $\dfrac{4}{N+2} < \alpha < \dfrac{4}{N-2}$ *(*$2 < \alpha < \infty$*, if* $N = 1$*). If* $\varphi \in \mathcal{R}_+$*,* $\varphi \neq 0$*, then for any* $p \in [2, \infty)$ *such that* $(N-2)p \leq 2N$ *there exists* $\ell > 0$ *such that*

$$(1+s)^{\frac{N(p-2)}{2}} \|u(s)\|_{L^p}^p \underset{s \to +\infty}{\longrightarrow} \ell.$$

*A similar statement holds if* $\varphi \in \mathcal{R}_-$*.*

**Proof.** Let $\varphi \in \mathcal{R}_+$, $\varphi \neq 0$, and let $u$ be the solution of (1.1). We have $T(-t)u(t) \underset{t \to \infty}{\longrightarrow} u_+$ in $X$. Since $U_+(0) = 0$ and $U_+$ is a bijection, we have $u_+ \neq 0$. Let now $v$ be defined by (6.6). It follows from Proposition 6.1 that $v(t) \underset{t \uparrow 1}{\longrightarrow} T(1)\left(e^{-i\frac{|x|^2}{4}} u_+\right) \neq 0$ in $X$. By Sobolev's embedding, for any $p \in [2, \infty)$ such that $(N-2)p \leq 2N$, there exists $\ell > 0$ such that $\|v(t)\|_{L^p} \to \ell$ as $t \uparrow 1$. The result now follows from (6.8). □

Before proceeding fruther, we need the following lemma.

**Lemma 6.10.** *Let* $\alpha_0$ *be defined by (5.1), and let* $a$ *be defined by (5.2). Assume* $\alpha_0 < \alpha < \dfrac{4}{N-2}$ *and let* $v \in C([0,1), X)$ *be a solution of (6.7), and let* $T_M \geq 1$ *be the maximal existence time of* $v$*. If* $(1-t)^{\frac{N\alpha-4}{2\alpha}} v \in L^a((0,1), L^{\alpha+2}(\mathbf{R}^N))$*, then* $T_M > 1$*.*

**Proof.** Suppose $(1-t)^{\frac{N\alpha-4}{2\alpha}} v \in L^a((0,1), L^{\alpha+2}(\mathbf{R}^N))$. Set $r = \alpha + 2$, and let $(q, r)$ be the corresponding admissible pair. Given $0 \leq t_0 \leq t < 1$, it follows from equation (6.7) and estimate (1.2) that

$$\|v\|_{L^\infty((t_0,t),H^1)} + \|v\|_{L^q((t_0,t),W^{1,r})} \leq C\|v(t_0)\|_{H^1} + C\|(1-s)^{\frac{N\alpha-4}{2}} |v|^\alpha v\|_{L^{q'}((t_0,t),W^{1,r'})}.$$

Applying Hölder's inequality, we obtain

$$\|(1-s)^{\frac{N\alpha-4}{2}} |v|^\alpha v\|_{L^{q'}((t_0,t),W^{1,r'})} \leq C\|(1-s)^{\frac{N\alpha-4}{2}} |v|^\alpha\|_{L^{\frac{q}{q-2}}((t_0,t),L^{\frac{\alpha+2}{\alpha}})} \|v\|_{L^q((t_0,t),W^{1,r})}.$$

Since $\dfrac{\alpha q}{q-2} = a$, it follows that

$$\|(1-s)^{\frac{N\alpha-4}{2}} |v|^\alpha v\|_{L^{q'}((t_0,t),W^{1,r'})} \leq C\|(1-s)^{\frac{N\alpha-4}{2\alpha}} v\|_{L^a((t_0,t),L^{\alpha+2})}^\alpha \|v\|_{L^q((t_0,t),W^{1,r})};$$

and so,

$$\|v\|_{L^\infty((t_0,t),H^1)} + \|v\|_{L^q((t_0,t),W^{1,r})} \leq C\|v(t_0)\|_{H^1} + C\|(1-s)^{\frac{N\alpha-4}{2\alpha}} v\|_{L^a((t_0,t),L^{\alpha+2})}^\alpha \|v\|_{L^q((t_0,t),W^{1,r})}.$$

Choosing $t_0$ close enough to 1 so that $C\|(1-s)^{\frac{N\alpha-4}{2\alpha}}v\|^{\alpha}_{L^a((t_0,1),L^{\alpha+2})} \le 1/2$, we get

$$\|v\|_{L^\infty((t_0,t),H^1)} + \|v\|_{L^q((t_0,t),W^{1,r})} \le 2C\|v(t_0)\|_{H^1}.$$

It follows that $v$ remains bounded in $H^1(\mathbf{R}^N)$ as $t \uparrow 1$, and the result follows from Theorem 6.6. □

**Corollary 6.11.** *Let $\alpha_0$ be defined by (5.1), and let $a$ be defined by (5.2). Assume $\alpha_0 < \alpha < \dfrac{4}{N-2}$, let $\varphi \in X$ be such that $T^* = \infty$ and let $u \in C([0,\infty), X)$ be the solution of (1.1). Then $\varphi \in \mathcal{R}_+$ if and only if $u \in L^a((0,\infty), L^{\alpha+2}(\mathbf{R}^N))$.*

**Proof.** Assume $\varphi \in \mathcal{R}_+$. It follows from Corollary 6.9 that $\sup\limits_{s\ge 0}(1+s)^{\frac{N\alpha}{2}}\|u(s)\|^{\alpha+2}_{L^{\alpha+2}} < \infty$. Therefore,

$$\|u(s)\|^a_{L^{\alpha+2}} \le C(1+s)^{-\frac{N\alpha a}{2(\alpha+2)}}.$$

Since $\dfrac{N\alpha a}{2(\alpha+2)} > 1$, we obtain $u \in L^a((0,\infty), L^{\alpha+2}(\mathbf{R}^N))$.

Conversely, if $u \in L^a((0,\infty), L^{\alpha+2}(\mathbf{R}^N))$ and if $v$ is defined by (6.6), then it follows from (6.8) that $(1-t)^{\frac{N\alpha-4}{2\alpha}}v \in L^a((0,1), L^{\alpha+2}(\mathbf{R}^N))$. Lemma 6.10 now implies that $v(t)$ exists beyond $t=1$, and it follows from Proposition 6.1 that $\varphi \in \mathcal{R}_+$. □

**Corollary 6.12.** *Let $\alpha_0$ be defined by (5.1), and let $\varphi \in X$. Then $e^{i\frac{b|x|^2}{4}}\varphi \in \mathcal{R}_+$ if $b$ is large enough, and $e^{-i\frac{b|x|^2}{4}}\varphi \in \mathcal{R}_-$ if $b$ is large enough.*

**Proof.** The result is an immediate consequence of Corollary 6.11 and Theorem 5.1. □

We now establish further properties of the wave operators $\Omega_\pm$.

**Theorem 6.13.** *Assume $\dfrac{4}{N+2} < \alpha < \dfrac{4}{N-2}$ ($2 < \alpha < \infty$, if $N=1$). Then, the following properties hold.*

(i) *If $\lambda < 0$, then $\mathcal{U}_\pm = X$. Therefore, the wave operators $\Omega_\pm$ are bicontinuous bijections $X \to \mathcal{R}_\pm$.*

(ii) *If $\lambda > 0$ and $\alpha < 4/N$, then $\mathcal{U}_\pm = X$. Therefore, the wave operators $\Omega_\pm$ are bicontinuous bijections $X \to \mathcal{R}_\pm$.*

(iii) *If* $\lambda > 0$ *and* $\alpha \geq 4/N$, *then* $\mathcal{U}_{\pm} \neq X$.

**Proof.** Assume $\lambda < 0$, or $\lambda > 0$ and $\alpha < 4/N$. Let $w \in X$ and let $z$ be the solution of (6.7) with the initial value $z(1) = w$. By Theorem 6.6, $z$ is defined on some interval $[1-\varepsilon, 1]$ with $\varepsilon > 0$. Set

$$\phi(y) = e^{\frac{N}{2}} e^{i\frac{\varepsilon |y|^2}{4}} z(1-\varepsilon, \varepsilon y) \in X.$$

Let $u$ be the solution of equation (1.1) with the initial value

$$u\left(\frac{1-\varepsilon}{\varepsilon}\right) = \phi. \tag{6.24}$$

Since $\lambda < 0$, or $\lambda > 0$ and $\alpha < 4/N$, it follows that $u$ is global. Therefore, we may define $\varphi = u(0)$. We claim that $\varphi \in \mathcal{R}_+$ and that $u_+ = e^{i\frac{|x|^2}{4}} T(-1)w$. Indeed, consider $v$ defined by (6.6). We see by applying (6.6) with $t = 1-\varepsilon$ and (6.24) that

$$v(1-\varepsilon) = z(1-\varepsilon),$$

so that by uniqueness $v \equiv z$; and so the claim follows from Proposition 6.1. Thus we have proved (i) and (ii).

Assume now that $\lambda > 0$ and $\alpha \geq 4/N$. Let $\Phi \in X$ be such that $E(\Phi) < 0$, so that $T^*(\Phi) < \infty$ and $T_*(\Phi) < \infty$. Given $b \in \mathbf{R}$, set $\phi(x) = e^{i\frac{b|x|^2}{4}} \Phi(x)$, and let $w$ be the corresponding solution of (1.1). It follows from Theorem 5.1 and Remark 5.2 (v) that if $b$ is large enough, we have $T_*(\phi) < \infty$, $T^*(\phi) = \infty$ and $w \in L^a((0,\infty), L^{\alpha+2}(\mathbf{R}^N))$. By Corollary 6.11, this implies that $\phi \in \mathcal{R}_+$. Let $\omega = U_+(\phi)$. Let now $\tau$ be large enough so that $-T_*(\phi)+\tau > 0$, and set $u(t) = w(t+\tau)$. It follows that $u$ is defined on $(-T_*(\phi)+\tau, \infty)$ and that $u$ blows up as $t \downarrow -T_*(\phi)+\tau > 0$. In particular, $u$ cannot be defined on $(0,\infty)$. On the other hand, we have

$$T(-t)u(t) = T(\tau)T(-(t+\tau)w(t+\tau)) \underset{t\to\infty}{\longrightarrow} T(\tau)\omega.$$

We claim that $T(\tau)\omega \notin \mathcal{U}_+$. Indeed, if $T(\tau)\omega$ were in $\mathcal{U}_+$, then there would exist a solution $z(t)$ of (1.1) such that $T(-t)z(t) \underset{t\to\infty}{\longrightarrow} T(\tau)\omega$. By Proposition 6.1 and the uniqueness property of Theorem 6.6, we would have $z = u$. This is absurd, since $u$ cannot be defined on $(0,\infty)$. Therefore $\mathcal{U}_+ \neq X$, and $\mathcal{U}_- = \overline{\mathcal{U}_+} \neq X$. This completes the proof. □

We now study the asymptotic completeness.

**Theorem 6.14.** *Let $\alpha_0$ be defined by (5.1), and assume $\alpha_0 < \alpha < \dfrac{4}{N-2}$*

(i) *If $\lambda < 0$, then $\mathcal{R}_+ = \mathcal{R}_- = X$. In particular, $U_\pm$, $\Omega_\pm$ and $\mathbf{S}$ are bicontinuous bijections $X \to X$.*

(ii) *If $\lambda > 0$, then $\mathcal{R}_\pm \neq X$. Moreover, $\mathcal{R}_\pm$ are unbounded subsets of $L^2(\mathbf{R}^N)$.*

**Proof.** Assume first $\lambda < 0$. Let $\varphi \in X$, let $u$ be the solution of (1.1) and let $v$ be defined by (6.6). If $\alpha \geq 4/N$, then it follows from (6.12) that $E_1(t)$ is nonincreasing, which implies that $\|\nabla v(t)\|_{L^2}$ is bounded as $t \uparrow 1$. Since $\|v(t)\|_{L^2}$ is also bounded by (6.11), it follows that $\|v(t)\|_{H^1}$ is bounded as $t \uparrow 1$. By Theorem 6.6, this implies that $v(t)$ has a limit as $t \uparrow 1$; and so $\varphi \in \mathcal{R}_+$ by Proposition 6.1. If $\alpha < 4/N$, then it follows from (6.13) that $\|v(t)\|_{L^{\alpha+2}}$ remains bounded as $t \uparrow 1$. This implies that $(1-t)^{\frac{N\alpha-4}{2\alpha}} v \in L^a((0,1), L^{\alpha+2}(\mathbf{R}^N))$; and so, by Lemma 6.10, $v(t)$ has a limit as $t \uparrow 1$. One concludes as above. This shows that $\mathcal{R}_\pm = X$. Property (i) now follows from Theorem 6.13.

Assume now that $\lambda > 0$. The unbounded character of $\mathcal{R}_\pm$ follows from Corollary 6.12. Consider now a standing wave, i.e. a solution of the form $u(t,x) = e^{i\omega t}\phi(x)$, with $\phi \in X$, $\phi \neq 0$ (see [5], Section 8.1). It is clear that $\phi \notin \mathcal{R}_+$ and $\phi \notin \mathcal{R}_-$; and so, $\mathcal{R}_+ \neq X$ and $\mathcal{R}_- \neq X$. □

Finally, we extend the asymptotic completeness result to the case $\alpha = \alpha_0$ (see [11]).

**Theorem 6.15.** *Assume $N = 1$ or $N \geq 3$. Let $\alpha_0$ be defined by (5.1), and assume $\alpha = \alpha_0$. If $\lambda < 0$, then $\mathcal{R}_+ = \mathcal{R}_- = X$. In particular, $U_\pm$, $\Omega_\pm$ and $\mathbf{S}$ are bicontinuous bijections $X \to X$.*

**Proof.** Since $\mathcal{R}_- = \overline{\mathcal{R}_+}$, we need only show that $\mathcal{R}_+ = X$. Let $\varphi \in X$, let $u \in C(\mathbf{R}, X)$ be the solution of (1.1) and let $v$ be defined by (6.6). Note that, since $u$ is defined on $[0,\infty)$, $v$ is defined on $[0,1)$. By Proposition 6.1, $\varphi \in \mathcal{R}_+$ if $v(t)$ has a limit in $X$ as $t \uparrow 1$. Therefore, in view of Theorem 6.6, we need only show that

$$\sup_{t\in[0,1)} \|v(t)\|_{H^1} < \infty. \tag{6.25}$$

We argue by contradiction and we assume that (6.25) does not hold, i.e.

$$\limsup_{t\uparrow 1} \|v(t)\|_{H^1} = \infty. \tag{6.26}$$

We consider separately the cases $N \geq 3$ and $N = 1$.

**Case $N \geq 3$.** By (6.26) and property (i) of Theorem 6.6, we have

$$\|\nabla v(t)\|_{H^1}^2 \geq \frac{a}{(1-t)^{\frac{N+2}{2}-\frac{2}{\alpha}}},$$

for some constant $a > 0$ and all $t \in [0, 1)$. By applying (6.13), we obtain

$$\frac{d}{dt}E_2(t) \leq -\frac{b}{(1-t)^{\frac{N\alpha-2}{2}+\frac{N+2}{2}-\frac{2}{\alpha}}},$$

for some constant $b > 0$. Since $\alpha = \alpha_0$, the above inequality means

$$\frac{d}{dt}E_2(t) \leq -\frac{b}{(1-t)},$$

which implies that $E_2(t) \underset{t\uparrow 1}{\longrightarrow} -\infty$. This is absurd, since $E_2(t) \geq 0$. This completes the proof in the case $N \geq 3$.

**Case $N = 1$.** The argument is the same as above, except that we first need to improve the lower estimate of blow up given by property (i) of Theorem 6.6. We claim that

$$\|v(t)\|_{H^1} \geq \frac{a}{(1-t)^{\frac{(\alpha-2)(\alpha+4)}{4\alpha}}}, \tag{6.27}$$

for some constant $a > 0$ and all $t \in [0, 1)$. Indeed, note first that it follows from (6.13) that

$$\frac{d}{dt}E_2(t) \leq 0;$$

and so,

$$\sup_{t\in[0,1)} \|v(t)\|_{L^{\alpha+2}} < \infty.$$

Fix $t_0 \in [0, 1)$. It follows from the equation (6.23) and the Strichartz estimate (1.2) that

$$\|v\|_{L^\infty((t_0,t),H^1)} \leq C\|v(t_0)\|_{H^1} + C\|f|v|^\alpha v\|_{L^1((t_0,t),H^1)},$$

for all $t \in (t_0, 1)$. On the other hand,

$$\|\,|v|^\alpha v\|_{H^1} \leq C\|v\|_{L^\infty}^\alpha \|v\|_{H^1},$$

and, by Gagliardo-Nirenberg's inequality,

$$\|v\|_{L^\infty} \le C\|v\|_{H^1}^{\frac{2}{\alpha+4}} \|v\|_{L^{\alpha+2}}^{\frac{\alpha+2}{\alpha+4}}.$$

Therefore, it follows from the four above inequalities that there exists a constant $K$ independent of $t_0$ and $t$ such that

$$\|v\|_{L^\infty((t_0,t),H^1)} \le K\|v(t_0)\|_{H^1} + K\|f\|_{L^1(t_0,t)}\|v\|_{L^\infty((t_0,t),H^1)}^{\frac{3\alpha+4}{\alpha+4}}. \tag{6.28}$$

Now by (6.26), there exists $t_1 \in (t_0,1)$ such that $\|v\|_{L^\infty((t_0,t-1),H^1)} = (K+1)\|v(t_0)\|_{H^1}$. Taking $t = t_1$ in (6.28), we obtain

$$\|v(t_0)\|_{H^1} \le K\left((K+1)\|v(t_0)\|_{H^1}\right)^{\frac{3\alpha+4}{\alpha+4}} \|f\|_{L^1(t_0,t)},$$

hence

$$1 \le K\left((K+1)\right)^{\frac{3\alpha+4}{\alpha+4}} \|v(t_0)\|_{H^1}^{\frac{2\alpha}{\alpha+4}} \|f\|_{L^1(t_0,t_1)}.$$

Since $\|f\|_{L^1(t_0,t_1)} \le \|f\|_{L^1(t_0,1)} \le C(1-t)^{\frac{\alpha-2}{2}}$, we obtain (6.27). We now conclude exactly as in the case $N \ge 3$. This completes the proof. □

**Remark 6.16.** Here some comments concerning Theorem 6.15.

(i) We do not know whether or not the conclusion of Theorem 6.15 holds in the case $N = 2$. The argument that we used in the case $N = 1$ does not seem to be applicable, due to the particular homogeneity of the Gagliardo-Nirenberg's inequality in dimension 2.

(ii) If $\dfrac{4}{N+2} < \alpha < \alpha_0$, then we do not know whether or not $\mathcal{R}_+ = \mathcal{R}_- = X$. Showing this property amounts to showing that no solution of (6.7) can blow up at $t = 1$.

**Remark 6.17.** Ginibre and Velo [17] extended the construction of the wave operators $\Omega_\pm$ to a wider range of $\alpha$'s, by working in the space $H^\rho(\mathbf{R}^N) \cap \mathcal{F}(H^\rho(\mathbf{R}^N))$, where $0 < \rho < 2$. The lower bound on $\alpha$ for this method is given by

$$\alpha > \max\left\{\rho - 1, \frac{4}{N+2\rho}, \frac{2}{N}\right\}$$

If $N = 3$, one obtains the lower bound $\alpha > 2/N$ by taking $\rho = 3/2$. If $N \ge 4$, there is still a gap between the admissible values of $\alpha$ and the lower bound $\alpha > 2/N$ for the scattering theory given by Theorem 6.2.

**7. The Cauchy problem for a nonautonomous Schrödinger equation.** In this section, we study the Cauchy problem for equation (6.7) starting from any point $t \in [0,1]$. In fact, we consider the more general Cauchy problem (see [11])

$$\begin{cases} iv_t + \triangle v + h(t)|v|^\alpha v = 0, \\ v(0) = \psi, \end{cases} \tag{7.1}$$

where $h \in L^1_{loc}(\mathbf{R}, \mathbf{R})$. We will study equation (7.1) under the equivalent form

$$v(t) = T(t)\psi + i\int_0^t T(t-s)h(s)|v(s)|^\alpha v(s)\, ds. \tag{7.2}$$

We have the following existence and uniqueness result.

**Theorem 7.1.** *Assume* $0 < \alpha < \dfrac{4}{N-2}$ *(*$0 < \alpha < \infty$ *if* $N = 1$*). Let* $\theta = \dfrac{4}{4 - \alpha(N-2)}$ *(*$\theta = 1$*, if* $N = 1$*;* $\theta > 1$ *and* $(2-\alpha)\theta \le 1$*, if* $N = 2$*), and consider a real valued function* $h \in L^\theta_{loc}(\mathbf{R}, \mathbf{R})$*. Then, for every* $\psi \in H^1(\mathbf{R}^N)$*, there exist* $T^*, T_* > 0$ *and a unique, maximal solution* $v \in C((-T_*, T^*), H^1(\mathbf{R}^N)) \cap W^{1,\theta}_{loc}((-T_*, T^*), H^{-1}(\mathbf{R}^N))$ *of equation (7.2). The solution* $v$ *is maximal in the sense that if* $T^* < \infty$ *(respectively* $T_* < \infty$*), then* $\|u(t)\|_{H^1} \to \infty$*, as* $t \uparrow T^*$ *(respectively* $t \downarrow -T_*$*). In addition, the solution* $v$ *has the following properties.*

(i) *If* $T^* < \infty$*, then* $\liminf_{t \uparrow T^*} \{\|v(t)\|^\alpha_{H^1} \|h\|_{L^\theta(t,T^*)}\} > 0$.

(ii) *If* $T_* < \infty$*, then* $\liminf_{t \downarrow -T_*} \{\|v(t)\|^\alpha_{H^1} \|h\|_{L^\theta(-T_*,t)}\} > 0$.

(iii) $u \in L^q_{loc}((-T_*, T^*), W^{1,r}(\mathbf{R}^N))$*, for every admissible pair* $(q, r)$.

(iv) *There exists* $\delta > 0$*, depending only on* $N, \alpha$ *and* $\theta$ *such that if*

$$\|\psi\|^{\alpha\theta}_{H^1} \int_{-\tau}^{\tau} |h(s)|^\theta\, ds \le \delta,$$

*then* $[-\tau, \tau] \subset (-T_*, T^*)$ *and* $\|v\|_{L^q((-\tau,\tau),W^{1,r})} \le K\|\psi\|_{H^1}$ *for every admissible pair* $(q, r)$*, where* $K$ *depends only on* $N, \alpha, \theta$ *and* $q$*. In addition, if* $\psi'$ *is another initial value satisfying the above condition and if* $v'$ *is the corresponding solution of (7.2), then* $\|v - v'\|_{L^\infty((-\tau,\tau),L^2)} \le K\|\psi - \psi'\|_{L^2}$.

(v) *If* $\psi \in X$*, then* $v \in C((-T_*, T^*), X)$*. In addition,*

$$\|xv(t)\|^2_{L^2} = \|x\psi\|^2_{L^2} + 4Im\int_0^t \int_{\mathbf{R}^N} \overline{v(s)}(x \cdot \nabla v(s))\, dx\, ds,$$

*for all* $t \in (-T_*, T^*)$.

**Proof.** For technical reasons, we suppose first that $N \geq 3$. Afterwards, we will indicate the modifications needed to handle the cases $N = 2$ and $N = 1$. Let $2^* = \dfrac{2N}{N-2}$, and define $r$ by

$$1 - \frac{2}{r} = \frac{\alpha}{2^*}. \tag{7.3}$$

Since $(N-2)\alpha < 4$, it follows that $2 < r < 2^*$. Therefore, there exists $q$ such that $(q, r)$ is an admissible pair. A simple calculation shows that

$$\frac{1}{q'} = \frac{1}{\theta} + \frac{1}{q}. \tag{7.4}$$

By (1.2), there exists $K$ such that

$$\|T(\cdot)\psi\|_{L^\infty(\mathbf{R},H^1)} + \|T(\cdot)\psi\|_{L^q(\mathbf{R},W^{1,r})} \leq K\|\psi\|_{H^1},$$

for every $\psi \in H^1(\mathbf{R}^N)$. Given $M > 0$ and $0 \leq T_1, T_2$ such that $T_1 + T_2 > 0$, let

$$E = \{v \in C([-T_1, T_2], H^1(\mathbf{R}^N)) \cap L^q((-T_1, T_2), W^{1,r}(\mathbf{R}^N)); \\ \|v\|_{L^\infty((-T_1,T_2),H^1)} + \|v\|_{L^q((-T_1,T_2),W^{1,r})} \leq (K+1)M\}.$$

Endowed with the metric $d(u, v) = \|v - u\|_{L^q((-T_1,T_2),L^r)}$, $(E, d)$ is a complete metric space. Given $v \in E$, it follows from (7.3), (7.4), Sobolev's and Hölder's inequalities that $h|v|^\alpha v \in L^{q'}((-T_1, T_2), W^{1,r'}(\mathbf{R}^N))$ and that

$$\begin{aligned}\|h|v|^\alpha v\|_{L^{q'}((-T_1,T_2),W^{1,r'})} &\leq C\|h\|_{L^\theta(-T_1,T_2)}\|v\|^\alpha_{L^\infty((-T_1,T_2),L^{2^*})}\|v\|_{L^q((-T_1,T_2),W^{1,r})} \\ &\leq C_1\|h\|_{L^\theta(-T_1,T_2)}(K+1)^{\alpha+1}M^{\alpha+1}.\end{aligned} \tag{7.5}$$

Furthermore, given $u, v \in E$, one has as well

$$\|h(|v|^\alpha v - |u|^\alpha u)\|_{L^{q'}((-T_1,T_2),L^{r'})} \leq C\|h\|_{L^\theta(-T_1,T_2)}(\|v\|^\alpha_{L^\infty((-T_1,T_2),H^1)} \\ + \|u\|^\alpha_{L^\infty((-T_1,T_2),H^1)})\|v - u\|_{L^q((-T_1,T_2),L^r)}; \tag{7.6}$$

and so,

$$\|h(|v|^\alpha v - |u|^\alpha u)\|_{L^{q'}((-T_1,T_2),L^{r'})} \leq C_2\|h\|_{L^\theta(-T_1,T_2)}(K+1)^\alpha M^\alpha d(u, v). \tag{7.7}$$

Given $v \in E$ and $\psi \in H^1(\mathbf{R}^N)$ such that $\|\psi\|_{H^1} \leq M$, let $\mathcal{G}(v)$ be defined by

$$\mathcal{G}(v)(t) = T(t)\psi + i \int_0^t T(t-s)h(s)|v|^\alpha v(s)\, ds,$$

for $t \in (-T_1, T_2)$. It follows from (1.2) and (7.5) that

$$\mathcal{G}(v) \in C([-T_1, T_2], H^1(\mathbf{R}^N)) \cap L^q((-T_1, T_2), W^{1,r}(\mathbf{R}^N)),$$

and that

$$\|\mathcal{G}(v)\|_{L^\infty((-T_1,T_2),H^1)} + \|\mathcal{G}(v)\|_{L^q((-T_1,T_2),W^{1,r})} \leq KM + C_3(K+1)^{\alpha+1}M^{\alpha+1}\|h\|_{L^\theta(-T_1,T_2)}.$$

Therefore, if $T_1 + T_2$ is small enough so that

$$C_3(K+1)^{\alpha+1}M^\alpha\|h\|_{L^\theta(-T_1,T_2)} \leq 1,$$

then $\mathcal{G}(v) \in E$. Furthermore, (1.2) and (7.7) imply that

$$d(\mathcal{G}(v), \mathcal{G}(u)) \leq C_4(K+1)^\alpha M^\alpha \|h\|_{L^\theta(-T_1,T_2)} d(u, v).$$

Consequently, if $T_2 - T_1$ is small enough so that

$$K_1 M^\alpha \|h\|_{L^\theta(T_1,T_2)} \leq 1/2, \tag{7.8}$$

where $K_1 = (K+1)^{\alpha+1}\max\{C_3, C_4\}$, then $\mathcal{G}$ is Lipschitz continuous $E \to E$ with Lipschitz constant $1/2$. Therefore, $\mathcal{G}$ has a unique fixed point $v \in E$, which satisfies equation (7.2). In addition, the first part of property (iv) follows from (7.8), (7.5) and (1.2), and the second part from (7.6) and (1.2). Uniqueness in the class $C([-T_1, T_2]; H^1(\mathbf{R}^N))$ follows from (7.6) and (1.2). (Note that uniqueness is a local property and needs only to be established for $T_1 + T_2$ small enough.) Now, by uniqueness, $v$ can be extended to a maximal interval $(-T_*, T^*)$, and property (iii) follows from property (iv). Suppose that $T^* < \infty$. Applying the above local existence result to $v(t)$, $t < T^*$ with $T_1 = 0$, we see from (7.8) that if

$$K_1\|v(t)\|_{H^1}^\alpha \|h\|_{L^\theta(t,T^*)} \leq 1/2,$$

then $v$ can be continued up to and beyond $T^*$, which is a contradiction. Therefore, we have

$$K_1\|v(t)\|_{H^1}^\alpha \|h\|_{L^\theta(t,T^*)} > 1/2,$$

which proves property (i). Property (ii) is proved by the same argument. Finally, since $v$ satisfies the equation (7.1) in $L^\theta_{loc}((-T_*,T^*),H^{-1}(\mathbf{R}^N)$ and $h$ is real valued, property (v) is proved by standard arguments. For example, multiply the above equation by $|x|^2e^{-\varepsilon|x|^2}\overline{v}$, take the imaginary part and integrate over $\mathbf{R}^N$, then let $\varepsilon\downarrow 0$ (see Proposition 6.4.2 p.107 in [5] for a similar argument).

If $N=2$, the proof is the same as in the case $N\geq 3$, except that we set $r=2\theta$ and use the embedding $H^1(\mathbf{R}^2)\hookrightarrow L^p(\mathbf{R}^2)$ with $p=\dfrac{\alpha\theta}{\theta-1}$.

If $N=1$, the argument is slightly simpler. We let

$$E=\{v\in C([-T_1,T_2],H^1(\mathbf{R}));\ \|v\|_{L^\infty((-T_1,T_2),H^1)}\leq 2M\},$$

equipped with the metric $d(u,v)=\|v-u\|_{L^\infty(T_1,T_2;L^2)}$, and use the embedding $H^1(\mathbf{R})\hookrightarrow L^\infty(\mathbf{R})$. □

We now study the continuous dependence of the solutions on the initial value. The result is the following.

**Theorem 7.2.** *Under the assumptions of Theorem 7.1, suppose there exists $\theta_1>\theta$ such that $h\in L^{\theta_1}_{loc}(\mathbf{R})$. The solution $v$ of (7.2) given by Theorem 7.1 depends continuously on $\psi$ in the following way.*

(i) *The mappings $\psi\mapsto T^*$ and $\psi\mapsto T_*$ are lower semicontinuous $H^1(\mathbf{R}^N)\to(0,\infty]$.*

(ii) *If $\psi_n \underset{n\to\infty}{\longrightarrow}\psi$ in $H^1(\mathbf{R}^N)$ and if $v_n$ denotes the solution of (7.2) with initial value $\psi_n$, then $v_n\to v$ in $C([-T_1,T_2],H^1(\mathbf{R}^N))$ for any interval $[-T_1,T_2]\in(-T_*,T^*)$. If in addition $\psi_n\to\psi$ in $X$, then $v_n\to v$ in $C([T_1,T_2],X)$.*

**Proof.** We proceed in two steps.

**Step 1.** We show that for every $M>0$, there exists $\tau>0$ such that if $\psi\in H^1(\mathbf{R}^N)$ verifies $\|\psi\|_{H^1}<M$, then $[-\tau,\tau]\subset(-T_*,T^*)$, and $v$ has the following continuity properties.

(a) If $\|\psi\|_{H^1}<M$, $\psi_n\underset{n\to\infty}{\longrightarrow}\psi$ in $H^1(\mathbf{R}^N)$, and if $v_n$ denotes the solution of (7.2) with initial value $\psi_n$, then $v_n\to v$ in $C([-\tau,\tau],H^1(\mathbf{R}^N))$.

(b) If $\|\psi\|_{H^1}<M$, $\psi_n\underset{n\to\infty}{\longrightarrow}\psi$ in $X$, and if $v_n$ denotes the solution of (7.2) with initial value $\psi_n$, then $v_n\to v$ in $C([-\tau,\tau],X)$.

We base our proof on arguments of Kato [26] used in the autonomous case. We only prove the result in the case $N \geq 3$ (see the proof of Theorem 7.1 for the necessary modifications in the cases $N = 1, 2$). Given $M > 0$, we choose $\tau$ so that the inequality in property (iv) of Theorem 7.1 is met whenever $\|\psi\|_{H^1} < M$. In particular, if $\|\psi\|_{H^1} < M$, then $[-\tau, \tau] \subset (-T_*, T^*)$. Next, observe that $\dfrac{1}{\theta} > \dfrac{4 - \alpha N}{4}$; and so we may assume without loss of generality that $\dfrac{1}{\theta_1} > \dfrac{4 - \alpha N}{4}$. Therefore, if we define $\sigma$ by

$$\frac{1}{\sigma} = \frac{2}{N}\left(1 - \frac{1}{\theta_1}\right),$$

then $2 < \alpha\sigma < \dfrac{2N}{N-2}$. Let now $\rho$ be defined by

$$1 - \frac{2}{\rho} = \frac{1}{\sigma}.$$

Since $\dfrac{1}{\sigma} < \dfrac{2}{N}$, it follows that $2 < \rho < \dfrac{2N}{N-2}$. Finally, let $\gamma$ be such that $(\gamma, \rho)$ is an admissible pair. It follows easily from Hölder's inequality that for every $-\infty < a < b < \infty$,

$$\|hwz\|_{L^{\gamma'}((a,b),L^{\rho'})} \leq \left(\int_a^b |h(s)|^{\theta_1} \|w(s)\|_{L^\sigma}^{\theta_1}\right)^{1/\theta_1} \|z\|_{L^\gamma((a,b),L^\rho)}. \tag{7.9}$$

Consider now $\psi$ such that $\|\psi\|_{H^1} \leq M$, and let $\psi_n$ be as in (a). Let $v, v_n$ be the corresponding solutions of (7.2). It follows from (1.2) that there exists $C$, depending only on $\gamma$, such that

$$\begin{aligned} \|v - v_n\|_{L^\gamma((-\tau,\tau),W^{1,\rho})} + \|v - v_n\|_{L^\infty((-\tau,\tau),H^1)} &\leq C\|\psi - \psi_n\|_{H^1} \\ &+ \|h(|v|^\alpha v - |v_n|^\alpha v_n)\|_{L^{\gamma'}((-\tau,\tau),W^{1,\rho'})}. \end{aligned} \tag{7.10}$$

On the other hand, a straightforward calculation shows that

$$|\nabla(|v|^\alpha v - |v_n|^\alpha v_n)| \leq C|v_n|^\alpha |\nabla v - \nabla v_n| + \phi(v, v_n)|\nabla v|, \tag{7.11}$$

where $C$ depends on $\alpha$, and the function $\phi(x, y)$ is bounded by $C(|x|^\alpha + |y|^\alpha)$ and verifies $\phi(x,y) \underset{y \to x}{\longrightarrow} 0$. Therefore, applying (7.9), (7.10) and (7.11), we get

$$\begin{aligned} \|v - v_n\|_{L^\gamma((-\tau,\tau),W^{1,\rho})} + \|v - v_n\|_{L^\infty((-\tau,\tau),H^1)} &\leq C\|\psi - \psi_n\|_{H^1} \\ &+ \|h\|_{L^{\theta_1}(-\tau,\tau)} \|v_n\|_{L^\infty((-\tau,\tau),L^{\alpha\sigma})} \|v - v_n\|_{L^\gamma((-\tau,\tau),W^{1,\rho})} \\ &+ C\left(\int_a^b |h(s)|^{\theta_1} \|\phi(v, v_n)\|_{L^\sigma}^{\theta_1}\right)^{1/\theta_1} \|v\|_{L^\gamma((-\tau,\tau),W^{1,\rho})}. \end{aligned} \tag{7.12}$$

Note that by property (iv) of Theorem 7.1, $v_n$ is bounded in $H^1(\mathbf{R}^N)$, hence in $L^{\alpha\sigma}(\mathbf{R}^N)$, with the bound, for $t \in [-\tau, \tau]$, depending only on $\|\psi_n\|_{H^1}$, hence (for large values of $n$) only on $M$. As well, the bound on $\|v\|_{L^\gamma((-\tau,\tau),W^{1,\rho})}$ depends only on $M$. Therefore, it follows from (7.12) that

$$\|v - v_n\|_{L^\gamma((-\tau,\tau),W^{1,\rho})} + \|v - v_n\|_{L^\infty((-\tau,\tau),H^1)} \le C\|\psi - \psi_n\|_{H^1} + C\|h\|_{L^{\theta_1}(-\tau,\tau)}\|v - v_n\|_{L^\gamma((-\tau,\tau),W^{1,\rho})} + C\left(\int_a^b |h(s)|^{\theta_1}\|\phi(v, v_n)\|_{L^\sigma}^{\theta_1}\right)^{1/\theta_1},$$

where the constant $C$ depends only on $M$. Therefore, if we consider $\tau$ possibly smaller so that $C\|h\|_{L^{\theta_1}(-\tau,\tau)} \le 1/2$ (note that $\tau$ still depends on $M$), it follows that

$$\|v - v_n\|_{L^\gamma((-\tau,\tau),W^{1,\rho})} + \|v - v_n\|_{L^\infty((-\tau,\tau),H^1)} \le C\|\psi - \psi_n\|_{H^1} + C\left(\int_a^b |h(s)|^{\theta_1}\|\phi(v, v_n)\|_{L^\sigma}^{\theta_1}\right)^{1/\theta_1}.$$

Therefore, property (a) follows, provided we show that

$$\left(\int_a^b |h(s)|^{\theta_1}\|\phi(v, v_n)\|_{L^\sigma}^{\theta_1}\right) \underset{n\to\infty}{\longrightarrow} 0.$$

By the dominated convergence theorem, it suffices to verify that

$$\|\phi(v, v_n)\|_{L^\sigma} \underset{n\to\infty}{\longrightarrow} 0,$$

for all $t \in [-\tau, \tau]$. To see this, we argue by contradiction. We assume that there exists $t$ and a subsequence, which we still denote by $v_n(t)$, such that $\|\phi(v(t), v_n(t))\|_{L^\sigma} \ge \mu > 0$. Note that $v_n(t) \to v(t)$ in $L^2(\mathbf{R}^N)$ and $v_n(t)$ is bounded in $H^1(\mathbf{R}^N)$ by property (iv) of Theorem 7.1. Therefore, by Sobolev's and Hölder's inequalities, $v_n(t) \to v(t)$ in $L^{\alpha\sigma}(\mathbf{R}^N)$. It follows that there exists a subsequence, which we still denote by $v_n(t)$, and a function $f \in L^{\alpha\sigma}(\mathbf{R}^N)$ such that $v_n(t) \to v(t)$ almost everywhere in $\mathbf{R}^N$ and $|v_n(t)| \le f$ almost everywhere in $\mathbf{R}^N$. Applying the dominated convergence theorem, it follows that $\|\phi(v(t), v_n(t))\|_{L^\sigma} \to 0$, which is a contradiction. Hence property (a).

Property (b) follows from property (a) and Theorem 7.1 (v). Briefly, use Hölder's inequality on the formula in Theorem 7.1 (v) to obtain a uniform bound in $X$ on the solutions $v_n$. The integral term then converges along subsequences where $xv_n$ converges weakly in $L^2(\mathbf{R}^N)$ to $xv$, and the rest of the proof is standard.

**Step 2.** Conclusion. Let $\psi \in H^1(\mathbf{R}^N)$, let $v$ be the maximal solution of (7.2) given by Theorem 7.1 and let $[-T_1, T_2] \subset (-T_*, T^*)$. Set

$$M = \frac{1}{2} \sup_{-T_1 \le t \le T_2} \|v(t)\|_{H^1},$$

and consider $\tau > 0$ given by Step 1. By applying Step 1 $m$ times, where $(m-1)\tau < T_1 + T_2 \le m\tau$, we see that if $\|\psi - \widetilde{\psi}\|_{H^1}$ is small enough, then the solution of (7.2) with initial value $\widetilde{\psi}$ exists on $[-T_1, T_2]$. Hence property (i). Property (ii) follows easily from the same argument. □

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