ON THE EXISTENCE OF STATIONARY STATES FOR CLASSICAL NONLINEAR DIRAC FIELDS

Thierry Cazenave

Université Pierre et Marie Curie & CNRS
Laboratoire Jacques-Louis Lions
B.C. 187, 4 place Jussieu
75252 Paris Cedex 05, France
email address: thierry.cazenave@upmc.fr
URL: http://www.ljll.math.upmc.fr/cazenave/

1. Introduction

Our purpose in these notes is to describe and give a self-contained proof of the results of Cazenave and Vazquez [5] and Balabane, Cazenave, Douady and Merle [2]. We study the existence of stationary states for the following nonlinear Dirac equation

$$i\sum_{\mu=0}^{3}\gamma^\mu\partial_\mu\psi - m\psi + F(\overline{\psi}\psi)\psi = 0. \tag{1.1}$$

The notation is the following. ψ is defined on $\mathbb{R}^4$ with values in $\mathbb{C}^4$, $\partial_\mu = \frac{\partial}{\partial x_\mu}$, m is a positive constant $\overline{\psi}\psi = (\gamma^0\psi,\psi)$, where $(\cdot,\cdot)$ is the usual scalar product on $\mathbb{C}^4$, and the γ^μ's are the 4×4 matrices of the Pauli-Dirac representation, given by

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \quad \text{and} \quad \gamma^k = \begin{pmatrix} 0 & \sigma^k \\ \sigma^k & 0 \end{pmatrix} \text{ for } k=1,2,3,$$

where

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Finally, $F:\mathbb{R}\to\mathbb{R}$ models the nonlinear interaction.

Nonlinear spinor fields giving rise to equations of the form (1.1) were considered first by D. Ivanenko [15], H. Weyl [35] and by W. Heisenberg [13] in his unified theory of elementary particles. Later, R. Finkelstein, C.F. Fronsdal and P. Kaus [11] considered the case of a spinor field with several types of fourth order self-couplings. But it was M. Soler [26] who was the first to investigate the stationary states of the nonlinear Dirac field with the scalar fourth order self-coupling (corresponding to $F(x) = x$ in (1.1)) proposing them as a model of elementary extended fermions. Subsequently, the electromagnetic interaction was introduced [27, 23, 24] in order to construct a model of extended charged fermion, which in spite of its simplicity describes with a reasonable accuracy the properties of the nucleons [22]. To improve the model, the pseudoscalar fields were introduced in order to represent the cloud of pions[25, 12]. A summary of the above models, with the numerical computations and further developments are described by Rañada [20, 21]. The case $F(x) = x$ was also considered by Rafelski [19], Takahashi [29] and Van der Merwe [30].

We are interested in stationary states, or localized solutions of (1.1), that is solutions ψ of the form $\psi(t,x) = e^{i\omega}\varphi(x)$, where $t = x_0$ and $x = (x_1,x_2,x_3)$. In addition, we seek finite energy solutions; and so we want φ to be at least square integrable. Clearly, the equation for $\varphi : \mathbb{R}^3 \to \mathbb{C}^4$ is

$$i \sum_{k=1}^{3} \gamma^k \partial_k \varphi - m\varphi + \omega\gamma^0\varphi + F(\overline{\varphi}\varphi)\varphi = 0. \tag{1.2}$$

In all the sequel, we assume that F satisfies the following hypotheses.

$$\begin{cases} F \in C^1(\mathbb{R}, \mathbb{R}), \\ F(0) = 0, \\ F(x) \le 0 \text{ for } x \le 0, \\ F \text{ is increasing on } (0, \infty), \\ \lim\limits_{x\to+\infty} F(x) > m + \omega, \\ F'(F^{-1}(m - \omega)) > 0. \end{cases} \tag{H1}$$

We also assume that

$$0 < \omega < m. \tag{H2}$$

Then we have the following result.

Theorem 1.1. *Assume (H1) and (H2) hold. Then (1.2) has infinitely many different solutions. More precisely, for every integer $n \ge 0$, there exists a solution $\varphi_n \in C^1(\mathbb{R}^3, \mathbb{C}^4)$ of (1.2) such that*

(i) *φ_n and $\nabla\varphi_n$ have exponential decay as $|x| \to \infty$;*
(ii) *the function $x \mapsto \overline{\varphi_n}\varphi_n$ is spherically symmetric and has n nodes as a function of r.*

Theorem 1.1 calls for a few comments. First, observe that it is conclusion (ii) that ensures that the solutions φ_n are nontrivial and distinct. Observe also that (i) implies that $\varphi_n \in H^1(\mathbb{R}^3, \mathbb{C}^4)$; and so φ_n has finite energy.

Concerning the assumption (H1), it is interesting to note that we did not impose any restriction on the growth of F as $x \to \infty$. In particular, we can take $F(x) = |x|^{p-1}x$, for any $p > 1$. This is in striking contrast with the Klein-Gordon and Schrödinger equations where finite-energy stationary states exist only for $p < 5$ in dimension $1 + 3$ (compare [3]). Observe also that even though the argument of F takes negative values for the solution φ_n with $n \ge 1$, the only assumption on F for $x < 0$ is $F(x) \le 0$. In particular, we do not impose any oddness condition. Assumption (H1) can be slightly improved with a few modifications in the proof. See the comments of Theorem 1.2 below and Section 6.

Equation (1.2) has a variational structure. More precisely, solutions of (1.2) are (formally) critical points of the Lagrangian $\mathcal{L}$ given by

$$\mathcal{L}(\varphi) = \int_{\mathbb{R}^3} \Big\{ \sum_{k=1}^{3} (i\gamma^0\gamma^k\partial_k\gamma, \varphi) - m\overline{\varphi}\varphi + \omega|\varphi|^2 + G(\overline{\varphi}\varphi) \Big\}\, dx,$$

where G is a primitive of F and $(\cdot,\cdot)$ is the scalar product in $\mathbb{C}^4$. Let us point out that it would be extremely interesting to solve this variational problem since a better knowledge of the structure of the Lagrangian might give relevant information on the initial value problem for (1.1), and in particular on the stability of the stationary states. However, this seems to be delicate, due to the defect of coerciveness of both the term involving derivatives and the non-quadratic term in the Lagrangian.

Here, we do not attempt to solve the variational problem. Instead, and following M. Wakano [33] and M. Soler [26], we seek solutions that are separable in spherical coordinates, of the form

$$\varphi(x) = \begin{pmatrix} v(r)\begin{pmatrix}1\\0\end{pmatrix} \\ iu(r)\begin{pmatrix}\cos\theta\\ \sin\theta e^{i\phi}\end{pmatrix}\end{pmatrix}. \tag{1.3}$$

Here, $r = |x|$, and (θ, ϕ) are the angular parameters. The Dirac equation then turns to a nonautonomous planar differential system, in the r variable, which is

$$\begin{cases} u' = -\dfrac{2u}{r} + v[F(u^2 - v^2) - (m - \omega)], & (1.4)\\ v' = u[F(u^2 - v^2) - (m - \omega)]. & (1.5)\end{cases}$$

In order to avoid solutions with singularity at the origin, due to the term $\frac{2}{r}u$ in (1.4), we impose

$$u(0) = 0; \tag{1.6}$$

and since we are interested in finite energy solutions of (1.2), we seek solutions of (1.4)-(1.5) that fulfill

$$|u(r)| + |v(r)| \underset{r\to\infty}{\longrightarrow} 0. \tag{1.7}$$

For every given x, there exists a local solution (u, v) of (1.4)-(1.6) with the initial condition $v(0) = x$. The problem is to find x such that the corresponding solution is global (i. e. defined for all $r \geq 0$), and satisfies (1.7). We have the following result, from which Theorem 1.1 is an immediate consequence.

Theorem 1.2. *Assume that* (H1) *and* (H2) *hold. There exists an increasing sequence* $(x_n)_{n\geq 0}$ *of positive numbers with the following properties . For every* $n \geq 0$*,*

(i) *the solution* (u_n, v_n) *of* (1.4)-(1.6) *with* $v_n(0) = x_n$ *is global;*
(ii) *both* u_n *and* v_n *have exactly* n *zeroes on* $(0, \infty)$*;*
(iii) (u_n, v_n) *converges exponentially to* $(0, 0)$ *as* $r \to \infty$*.*
(iv) *Furthermore, if* $F(x) \geq \delta(\log x)^\beta$ *for* x *large where* $\delta > 0$ *and* $\beta > 2$*, the sequence* $(x_n)_{n\geq 0}$ *is bounded.*

Several remarks are in order, concerning Theorem 1.2. The first analytical study of system (1.4)-(1.5) was done by L. Vazquez [31], who obtained sufficient conditions for the existence of solutions. The first existence result was obtained by T. Cazenave and L. Vazquez [5]. They proved the existence of a solution without nodes (positive u and v), which is essentially the solution (u_0, v_0) of Theorem 1.2. Later, that result was extended to a wider class of nonlinearities by F. Merle [18]. Theorem 1.2 in the present form is due to M. Balabane, T. Cazenave, A. Douady and F. Merle [2].

We do not know whether or not the assumption (H2) is necessary in Theorem 1.1. However, it is almost necessary in Theorem 1.2. Indeed, it was shown in [31] that when $F(x) = x$ there is no solution of (1.4)-(1.6) such that φ is square-integrable when $|\omega| > m$ or $\omega = 0$. The argument works as well when F satisfies (H1) . On the other hand, elementary calculations show that if $-m \leq \omega < 0$, no solution of (1.4)-(1.5) can converge to $(0, 0)$ as $r \to \infty$. In conclusion, the condition $0 < \omega \leq m$ is necessary in Theorem 1.2. Let us also mention that it does not restrict the generality to consider only the case $v(0) > 0$. Indeed, if (u, v) is a solution of (1.4)-(1.5), then $(-u, -v)$ is also a solution. Our method of proof can be modified in order to weaken hypothesis (H1) . In particular, the conclusions of Theorem 1.2 (and so these of Theorem 1.1) still hold if (H1) is replaced by (H1′) below (see Section 6)

$$\begin{cases} F \in C^1(\mathbb{R}, \mathbb{R}), \\ F(0) = 0, \\ F(x) \le 0 \text{ for } x \le 0, \\ F \text{ is increasing on } (0, \infty), \\ \lim\limits_{x \to +\infty} F(x) > m - \omega. \end{cases} \tag{H1′}$$

The numerical experiments performed on system (1.4)-(1.5) indicate the following (L. Vazquez [32], for $F(x) = x$). First, starting from x larger than some $x^\star$, the solutions blow-up (compare Proposition 3.2). It is also observed that the global solutions converge to one of the rest points of the system, which are $(0, 0)$, $(0, \sqrt{F^{-1}(m - \omega)})$ and $(0, -\sqrt{F^{-1}(m - \omega)})$. The solutions wind around the origin (in the plane (v, u)) before converging to a rest point. Note that the rest points $(0, \pm\sqrt{F^{-1}(m - \omega)})$ are stable while $(0, 0)$ is a saddle point. The set of positive initial data for which the solution turns $n/2$ times (n being an integer) around $(0, 0)$ before converging to $(0, \pm\sqrt{F^{-1}(m - \omega)})$ seems to be an interval of the form (x_{n-1,x_n}). The only solutions with positive initial data converging to $(0, 0)$ appear to be those starting from x_n, and they have n nodes. We prove these properties here, except uniqueness of the n-nodes solution with positive initial datum. The MacMath of J.H. Hubbard and B.H. West [14] was also helpful to have qualitative intuition for the dynamical system.

The proof of Theorem 1.2, that we give here is adapted from [5] and [2]. We consider (1.4)-(1.5) as a non-autonomous planar dynamical system (r being the time variable), and we follow essentially the scheme suggested by the numerical results. For every $n \ge 0$, we construct an open, non-empty set I_n of initial data for which the solution turns $n/2$ times around the origin and then remains trapped near one of the stable rest points. Next, we show that the solutions with initial data in I_n are bounded, uniformly in $r \ge 0$ and in the initial datum. Finally, we show that the solution with initial datum $\sup I_n$ is the expected n-node solution. The boundedness of the sequence $(x_n)_{n\ge0}$ follows from a blow-up result (Proposition 3.2). It will appear to the reader that the proof of Theorem 1.2 given below is very long and tedious. However, this is the only proof available so far. Observe also that we deal with system (1.4)-(1.5) without any sophisticated tool. All the arguments in the proof are absolutely elementary; and since the conclusion is the existence of infinitely many solutions for a system of nonlinear partial differential equations, it is reasonable to expect that there is a lot of such argument to be put together! We included a large number of figures in order to make the arguments clearer.

Theorem 1.2 raises some open questions, apart from the uniqueness problem. For example, the sequence $(x_n)_{n\ge0}$ is bounded, but we do not know whether the corresponding sequence $(u_n, v_n)_{n\ge0}$ of solutions is bounded or not (the numerical experiments indicate that it is unbounded). A related question is the following. In the case where $(x_n)_{n\ge0}$ is bounded, consider the limit, say x_∞ of x_n. Does the solution of (1.4)-(1.6) blow-up in a finite time when $v(0) \ge x_\infty$?

Notice the importance of the term $\frac{2}{r}u$ in (1.4). Indeed, it is the non-autonomous term that allows the existence of infinitely many solutions (the same phenomenon appears in the semilinear elliptic problems, see [3]). More surprisingly, it is also the non-autonomous term (even though it is linear) that makes some solutions blow-up in a finite time (when this term is removed, all the solutions are global solutions).

For completeness, let us indicate that for functions φ of the form (1.3), the Lagrangian $\mathcal{L}$ defined above has the following form.

$$\mathcal{L}(\varphi) = \int_0^\infty \Big\{-u'v - \frac{2uv}{r} + v'u - m(v^2-u^2) + \omega(u^2+v^2) + G(v^2-u^2)\Big\} r^2 dr.$$

Finally, observe that little seems to be known on the Cauchy problem (initial value problem) for (1.1) except global existence for small initial data (see Dias and Figueira [6, 7, 8]) and more recently global existence of weak solutions (see Dias and Figueira [9, 10]). In particular, we do not know whether stationary states are stable or not. For instance, some authors claim that the stationary states are unstable [4, 34, 17], while others find regions of stable behaviour by using numerical [1] or analytic [28] arguments.

The notes are organized as follows. In Section 2, we introduce the notation and we collect some basic properties of system (1.4)-(1.5). In Section 3, we study the blowing-up of solutions and in Section 4, we establish the main boundedness property. Finally in Section 5, we complete the proof of Theorem 1.2, and Section 6 is devoted to a few further comments.

2. Preliminary results on system (1.4)-(1.5)

We begin by introducing some notation. We consider $a, b > 0$ such that

$$F(a^2) = m - \omega, \quad F(b^2) = m + \omega. \tag{2.1}$$

It is clear from (H1) that a and b are uniquely determined. We also define the functions G and H by

$$G(x) = \int_0^x F(s)\, ds \quad x \in \mathbb{R} \tag{2.2}$$

$$H(u,v) = \frac{1}{2}\{G(v^2-u^2) - m(v^2-u^2) + \omega(v^2+u^2)\} \quad (u,v) \in \mathbb{R}^2. \tag{2.3}$$

It follows from (H1) that

$$G(x) \geq 0 \text{ for } x \geq 0, \tag{2.4}$$

and that there exist $x_0 > 0$ and $\varepsilon > 0$ such that

$$G(x) \geq (m - \omega + \varepsilon)x \text{ for } x \geq x_0. \tag{2.5}$$

Putting together (2.4) and (2.5), it follows that there exists C such that

$$G(x) \geq (m - \omega + \varepsilon)x - C \text{ for } x \in \mathbb{R}. \tag{2.6}$$

and so, for $\varepsilon > 0$ possibly smaller,

$$2H(u,v) \geq \varepsilon(u^2+v^2) - C \text{ for } (u,v) \in \mathbb{R}^2. \tag{2.7}$$

2.1. **The associated Hamiltonian system.** We will consider the Hamiltonian system associated with (1.4)-(1.5), which is

$$\begin{cases} u' = v[F(u^2-v^2) - (m-\omega)], & (2.8)\\ v' = u[F(u^2-v^2) - (m-\omega)]. & (2.9)\end{cases}$$

where $'$ denotes the differentiation with respect to the r variable. The corresponding Hamiltonian is H. The basic properties of H are summarized in the following lemma.

Lemma 2.1. *H has the following properties.*

(i) *The critical points of H are $(0,0)$ and $(0,\pm a)$,*
(ii) *the minimum of H is negative and is achieved for $(u,v)=(0,\pm a)$,*
(iii) *$H(u,v)\to\infty$ as $|u|+|v|\to\infty$,*
(iv) *$\{H(u,v)\le H(0,b)\}\subset\{F(v^2-u^2)\le m+\omega\}$, where b is defined by (2.1).*

Proof. (i) is immediate, and (iii) follows from (2.7). Therefore, in order to prove (ii), it remains to verify that the minimum of H is negative. This is clear since

$$\frac{d}{dv}H(0,v)=v[F(v^2)-(m-\omega)]<0 \text{ for } 0<v<a.$$

Note that in particular

$$H(0,v)<0 \text{ for } 0<v<a. \tag{2.10}$$

Let (u,v) be such that $F(v^2-u^2)>m+\omega$. In particular, $v^2-u^2>b^2$ and $F(s)-m>\omega$ for $s\in[b^2,v^2-u^2]$. It follows that

$$\begin{aligned} H(u,v)&=H(0,b)+\frac{1}{2}\int_{b^2}^{v^2-u^2}(F(s)-m)\,ds+\frac{\omega}{2}(v^2+u^2-b^2)\\ &\ge H(0,b)+\frac{\omega}{2}(v^2-u^2-b^2)+\frac{\omega}{2}(v^2+u^2-b^2)\\ &=H(0,b)+\omega(v^2-u^2-b^2)+\omega u^2>H(0,b). \end{aligned}$$

Hence (iv). □

In order to study the solutions of (2.8)-(2.9), we define the curves Γ_C for $C\in[\min H,\infty)$ by

$$\Gamma_C=\{(u,v)\in\mathbb{R}^2;\ H(u,v)=C\}.$$

It is clear from the definition of H that the curves Γ_C are symmetric about both the u and v axes. The properties of Γ_C are described in the following lemma and in Figure 1.

Lemma 2.2. *Let $C\in[\min H,\infty)$. Then*

(i) *if $C=\min H$, then $\Gamma_C=\{0,a\}\cup\{0,-a\}$,*
(ii) *if $C\in[\min H,0)$, then Γ_C is the union of two connected compact curves Γ_C^+ and Γ_C^- with $\Gamma_C^+\subset\{v^2>u^2;\ v>0\}$ and $\Gamma_C^+\subset\{v^2>u^2;\ v<0\}$,*
(iii) *Γ_0 is a connected compact curve and $(0,0)\in\Gamma_0$,*
(iv) *if $C>0$, then Γ_C is a connected compact curve.*

Proof. (i) follows from Lemma 2.1 (ii). Furthermore, it follows from (2.3) and (2.4) that if $v^2\le u^2$, then $H(u,v)\ge 0$. Therefore, if $C\in[\min H,0)$, then $\Gamma_C\subset\{v^2>u^2\}$. It follows from the symmetries of H and Lemma 2.1 (iii) that Γ_C is the union of at least two connected closed curves. The uniqueness of the curves Γ_C^+ and Γ_C^- follows from the fact that F is increasing on $(0,\infty)$. The compactness property is a consequence of Lemma 2.1 (iii). Properties (iii) and (iv) follow from similar considerations. □

We can now describe the solutions of (2.8)-(2.9), which are illustrated by arrows in Figure 1.

Lemma 2.3. *Let $(x,y)\in\mathbb{R}^2$ and let (u,v) be the solution of (2.8)-(2.9) such that $u(0)=x$ and $v(0)=y$. Then*

(i) *if $H(x,y)=0$ or $H(x,y)=\min H$, then $(u,v)\equiv(x,y)$,*

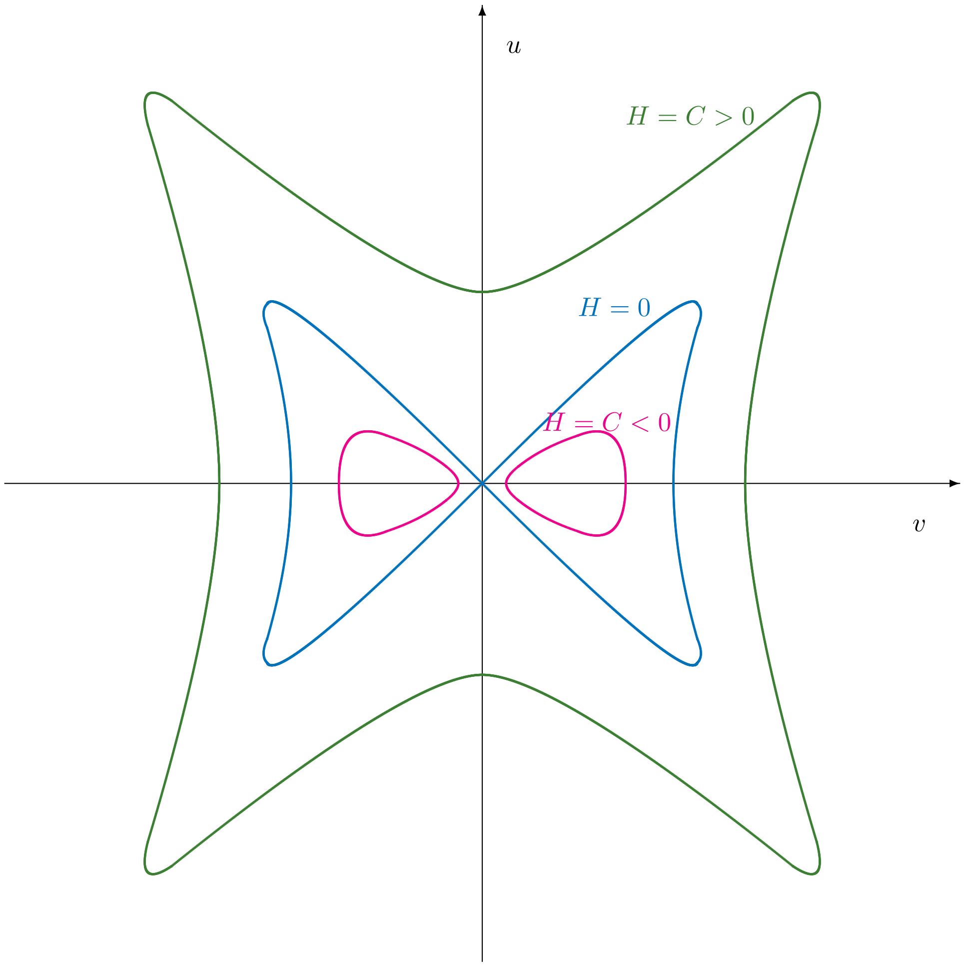

FIGURE 1. The energy levels $\{H = C\}$

(ii) *if* $H(x, y) = C \in (\min H, 0)$ *and* $(x, y) \in \Gamma_C^{\pm}$ *then* (u, v) *is periodic and* $(u, v)(\mathbb{R}) = \Gamma_C^{\pm}$,

(iii) *if* $H(x, y) = C \in (0, \infty)$, *then* (u, v) *is periodic and* $(u, v)(\mathbb{R}) = \Gamma_C$. *Furthermore, if* $xy > 0$, *then the first zero of* uv *is a zero of* v *and the zeroes of* u *and* v *alternate.*

Proof. (i) follows from Lemma 2.1 (i). Next, if $\nabla H(x, y) \neq 0$ then Γ_C is bounded and $|\nabla H(u, v)| \geq \alpha > 0$ for $(u, v) \in \Gamma_C$; and so the solution (u, v) must cover in a finite time the connected component of Γ_C to which (x, y) belongs. Therefore property (ii) and the first part of property (iii) follow from Lemma 2.2. For proving the second part of property (iii) it is sufficient by symmetry to consider the case $x > 0, y > 0$. Since $2H(u, 0) = G(u^2) + (m + \omega)u^2$, it follows that $H(u, 0)$ is an increasing function of u that ranges from 0 to $+\infty$. Thus Γ_C intersects $\{v = 0, u > 0\}$. Since $(u, v)(\mathbb{R}) = \Gamma_C$, uv has to vanish. Let ρ be the first zero of uv. Since $(0, 0) \notin \Gamma_C$, we have either $u(\rho) = 0, v(\rho) > 0$ or $u(\rho) > 0, v(\rho) = 0$. If $u(\rho) = 0$, then $u'(\rho) \leq 0$, which implies by (2.8) that $F(v(\rho)^2) < m - \omega$; and so $0 < v(\rho) < a$.

By (2.10) , we have then $C = H(0, v(\rho)) < 0$, which is a contradiction. Thus, $v(\rho) = 0$ and $u(\rho) > 0$. By a similar argument, one proves that the zeroes of u and v alternate, which completes the proof. □

2.2. **Basic properties of the system (1.4)-(1.5).** We now study the first properties of system (1.4)-(1.5). The local existence properties are described in the following Lemma.

Lemma 2.4. *For every $x \in \mathbb{R}$, there exists a unique, maximal solution (u_x, v_x) of (1.4)-(1.5) such that $(u_x(0), v_x(0)) = (0, x)$. (u_x, v_x) is defined on the maximal interval interval $[0, R_x)$, $(u_x, v_x) \in C^1([0, R_x), \mathbb{R}^2)$, and if $R_x < \infty$ then $|u_x(r)| + |v_x(r)| \to \infty$ as $r \uparrow R_x$. In addition,*

$$(u_x, v_x) \text{ depends continuously on } x \text{ in } C^1([0, R], \mathbb{R}^2), \text{ for every } R < R_x. \tag{2.11}$$

Proof. We write system (1.4)-(1.6) in the form

$$\begin{cases} u(r) = \dfrac{1}{r} \displaystyle\int_0^r s^2 v(s)[F(v^2 - u^2) - (m - \omega)]\, ds, \\ v(r) = \displaystyle\int_0^r u(s)[F(v^2 - u^2) - (m + \omega)]\, ds, \end{cases}$$

Since the integrand is a locally Lipschitz-continuous function of (u, v), existence of a maximal solution follows from the classical contraction mapping argument. The continuous dependence in $C([0, R_x), \mathbb{R}^2)$ is easily seen on the above formula; then the continuous dependence in $C^1([0, R_x), \mathbb{R}^2)$ follows from the equations (1.4)-(1.5). □

We shall also need the following stability result.

Lemma 2.5. *Let $C > \min H$. For every $T > 0$ and $\varepsilon > 0$, there exists $R > 0$ with the following property. Let $(x, y) \in \Gamma_C$ and let $\rho > R$. Let (u, v) be the solution of (2.8)-(2.9) with initial data (x, y) and let $(\upsilon, \varpi) \in C^1([0, \tau(\rho, x, y)), \mathbb{R}^2)$ be the maximal solution of*

$$\begin{cases} \upsilon' = -\dfrac{2\upsilon}{\rho + r} + \varpi[F(\varpi^2 - \upsilon^2) - (m - \omega)], \\ \varpi' = \upsilon[F(\varpi^2 - \upsilon^2) - (m + \omega)], \end{cases} \tag{2.12}$$

such that $(\upsilon(0), \varpi(0)) = (x, y)$. It follows that $\sup\limits_{r \in [0,T]} |u(r) - \upsilon(r)| + |v(r) - \varpi(r)| \le \varepsilon$.

Proof. First, observe that since $\rho > 0$, system (2.12) is not singular; and so the initial value problem for (2.12) is well-posed. Let $M = \sup\{|u| + |v|; (u, v) \in \Gamma_C\} < \infty$. It follows easily from a contraction mapping argument that there exists $\mu > 0$ such that for every $\rho \ge 1$ and$(x, y) \in R^2$ with $|x| + |y| \le 2M$ the maximal solution $(\upsilon, \varpi) \in C^1([0, \tau], \mathbb{R}^2)$ of (2.12) such that $(\upsilon, \varpi)(0) = (x, y)$ satisfies $\tau > \mu$ and $|\upsilon(r)| + |\varpi(r)| \le 4M$ for every $r \in [0, \mu]$. Let $(\upsilon_j, \varpi_j) \in C^1([0, \tau_j], \mathbb{R}^2)$, $j = 1, 2$ be two such solutions of (2.12) corresponding to ρ_j and (x_j, y_j), and let L be the Lipschitz constant of the right hand side of (2.12) on the ball of $\mathbb{R}^2$ of radius $4M$. It follows from (2.12) that

$$\begin{aligned} |\upsilon_2(r) - \upsilon_1(r)| + |\varpi_2(r) - \varpi_1(r)| \le |x_2 - x_1| + |y_2 - y_1| + 8M\mu\Big(\frac{1}{\rho_2} + \frac{1}{\rho_1}\Big) \\ + L\int_0^r [|\upsilon_2(s) - \upsilon_1(s)| + |\varpi_2(s) - \varpi_1(s)|]\, ds, \end{aligned}$$

for $0 \leq r \leq \mu$. On applying Gronwall's lemma, it follows that there exists $K \geq 1$, depending only on M, such that

$$|\upsilon_2(r) - \upsilon_1(r)| + |\varpi_2(r) - \varpi_1(r)| \leq K\Big(|x_2 - x_1| + |y_2 - y_1| + \frac{1}{\rho_2} + \frac{1}{\rho_1}\Big), \quad (2.13)$$

for $0 \leq r \leq \mu$. Given $T > 0$ and $0 < \varepsilon < M$, consider n such that $n\mu < T \leq (n+1)\mu$, and let R be such that $(n+1)Kn + 1 \leq \varepsilon R$. Let (x, y), (u, v) and (υ, ϖ) be as in the statement of Lemma 2.5, and let $\rho > R$. On applying formula (2.13) with $(x_1, y_1) = (x_2, y_2) = (x, y)$, $\rho_1 = \rho$ and $\rho_2 = \infty$, we obtain

$$|u(r) - \upsilon(r)| + |v(r) - \varpi(r)| \leq \frac{K}{R} \leq M,$$

for $0 \leq r \leq \mu$. Therefore, we can iterate formula (2.13) with $(x_1, y_1) = (\upsilon(\mu), \varpi(\mu))$, $(x_2, y_2) = (u(\mu), v(\mu))$, $\rho_1 = \rho + \mu$ and $\rho_2 = \infty$. After $n + 1$ iterations, we get

$$|u(r) - \upsilon(r)| + |v(r) - \varpi(r)| \leq \frac{1}{R}\sum_{j=1}^{n+1} K^j \leq \frac{(n+1)K^{n+1}}{R} \leq \varepsilon,$$

for $r \leq (n+1)\mu$. Hence the result, since $(n+1)\mu \geq T$. □

The following corollary will be important in the sequel.

Corollary 2.6. *Let $C > 0$ and consider an integer $n \geq 1$. There exists $T(n, C) < \infty$ with the following property. If $x \in \mathbb{R}$ is such that $(u_x(r), v_x(r)) \in \Gamma_C$ and $u_x(r)v_x(r) > 0$ for some $r > T(n, C)$, then there exists $R \in (r, R_x)$ such that $u_x v_x$ has n zeroes on (r, R) and these zeroes are alternatively a zero of v_x and a zero of u_x, the first being a zero of v_x.*

Proof. This is an immediate consequence of Lemma 2.3 (iii) and of Lemma 2.5 applied with $(x, y) = (u_x(r), v_x(r))$ and $(\upsilon(t), \varpi(t)) = (u_x(r + t), v_x(r + t))$. □

For convenience, we now define the function H_x, for $x \in \mathbb{R}$, by

$$H_x(r) = H(u_x(r), v_x(r)) \text{ for every } r \in (0, R_x).$$

It follows from straightforward calculations that

$$\frac{d}{dr}H_x(r) = \frac{2}{r}u_x(r)^2[F(v_x(r)^2 - u_x(r)^2) - (m + \omega)], \quad (2.14)$$

for every $x \in \mathbb{R}$ and $r \in (0, R_x)$; and so, the sign of the variation of H_x, at some time $r > 0$, is determined only by the region of the plane where $(u_x, v_x)(r)$ belongs. More precisely, the sign of the variation of H_x depends only of the position of $(u_x, v_x)(r)$ with respect to the hyperbola $\{F(v^2 - u^2) = m + \omega\}$ (see Figure 2).

We shall need the following identities, which are easily obtained from (1.4)-(1.5) and hold for $x \in \mathbb{R}$ and $r \in (0, R_x)$ whenever the right hand side makes sense.

$$(v_x^2 - u_x^2)' = \frac{4}{r}u_x^2 - 4\omega u_x v_x, \quad (2.15)$$

$$\Big(\frac{u_x}{v_x}\Big)' = \frac{1}{v_x^2}\Big[(v_x^2 - u_x^2)[F(v_x^2 - u_x^2) - (m - \omega)] + 2\omega u_x^2 - \frac{2}{r}u_x v_x\Big], \quad (2.16)$$

$$\Big(\frac{v_x}{u_x}\Big)' = \frac{1}{u_x^2}\Big[-(v_x^2 - u_x^2)[F(v_x^2 - u_x^2) - (m - \omega)] - 2\omega u_x^2 + \frac{2}{r}u_x v_x\Big]. \quad (2.17)$$

It will also be useful to keep in mind the velocity field of the dynamical system (1.4)-(1.5) for various values of r, which is displayed in Figures 3 to 5.

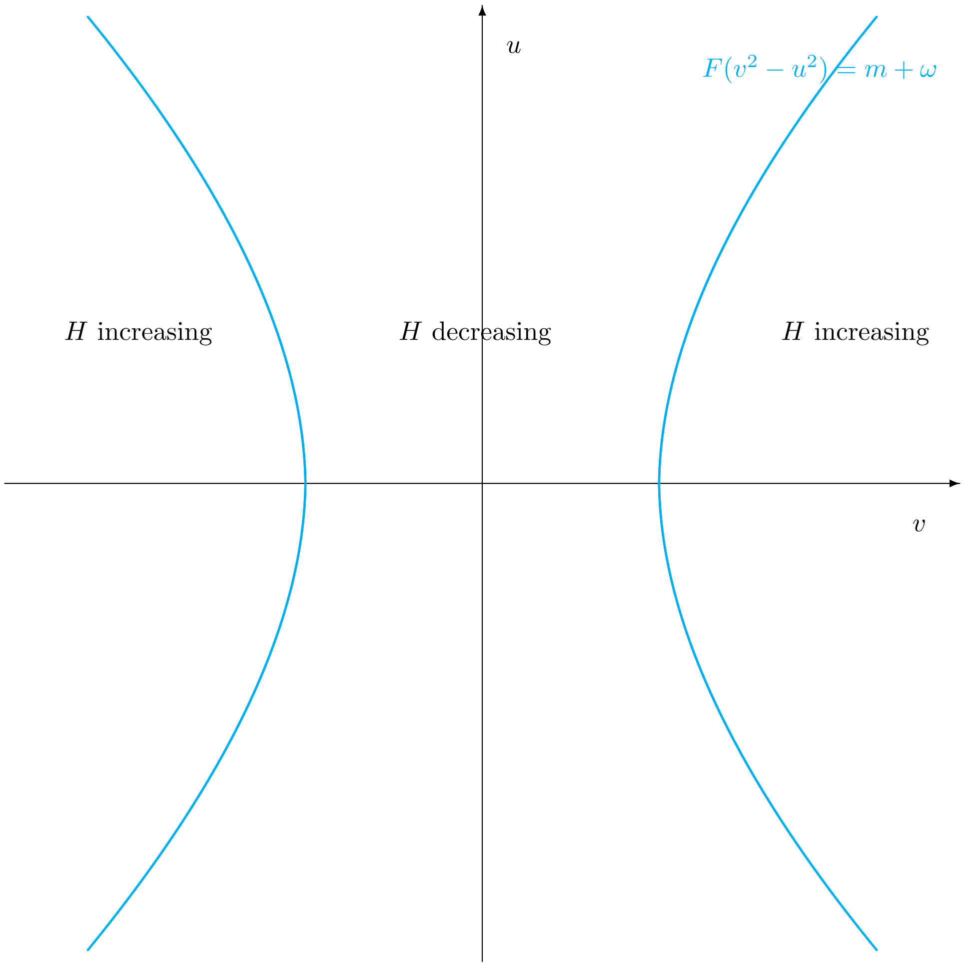

FIGURE 2. The sign of H'_x

Next, in the following lemmas, we collect some basic properties of a geometric nature for solutions of (1.4)-(1.5).

Lemma 2.7. *Let $x \neq 0$, $x \neq \pm a$. If, for some $r_0 > 0$ we have $u_x(r_0) = 0$, then $v_x(r_0) \notin \{0, \pm a\}$ and $u'_x(r_0) \neq 0$. If, for some $r_0 > 0$, we have $v_x(r_0) \in \{0, \pm a\}$, then $u_x(r_0) \neq 0$ and $v'_x(r_0) \neq 0$.*

Proof. First, observe that the rest points of (1.4)-(1.5) are $(0, 0)$ and $(0, \pm a)$. Furthermore, for $r_0 > 0$, the Cauchy problem for (1.4)-(1.5) is locally well-posed for any initial datum $(u_0, v_0) \in \mathbb{R}^2$, for both $r \geq r_0$ and $r \leq r_0$. Thus, a rest point cannot be reached in a finite time. Hence the result. □

Lemma 2.8. *Let $x \in \mathbb{R}$. Assume $R_x = \infty$, and $(u_x, v_x) \to (0, \pm a)$ as $r \to \infty$. Then u_x has infinitely many zeroes.*

Proof. It is equivalent to show that (u_x, v_x) cannot converge to $(0, \pm a)$, while being in one of the half-planes $\{u > 0\}$ or $\{u < 0\}$. Let us first prove that (u_x, v_x) cannot

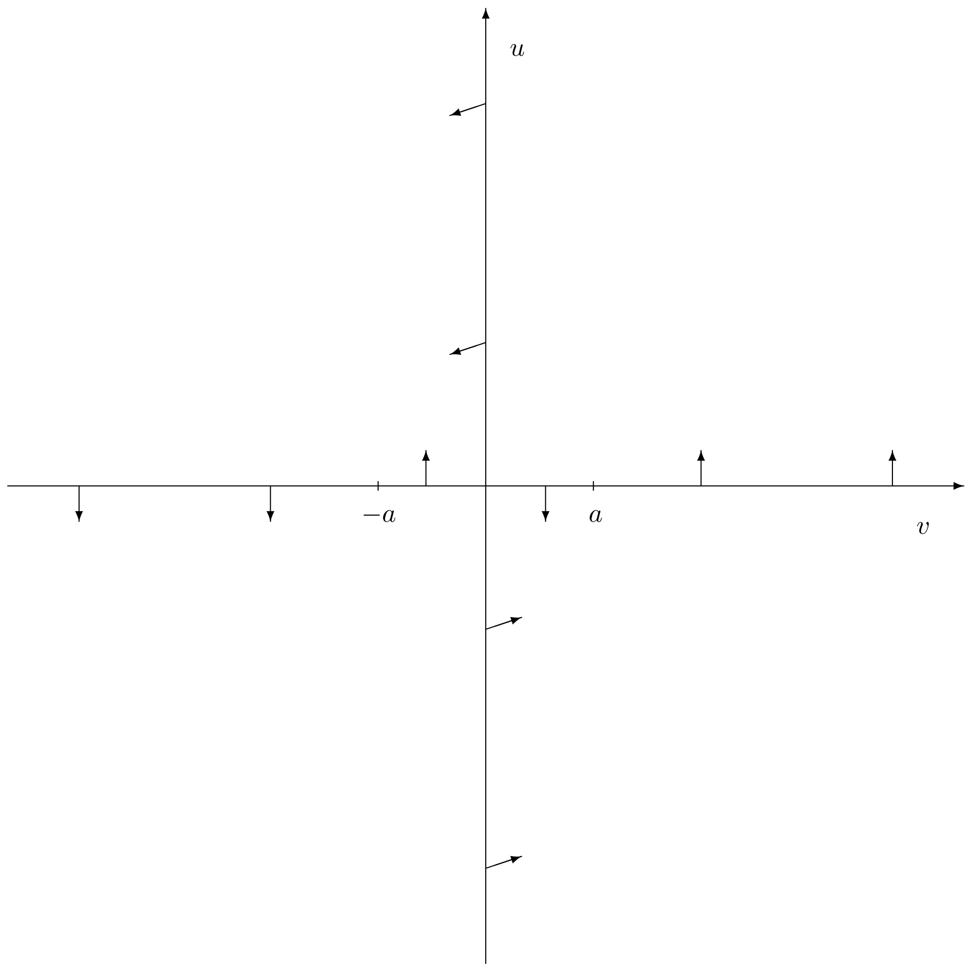

FIGURE 3. The velocity field on the axes for $r \geq 0$

converge to $(0, a)$, while being in the half-plane $\{u > 0\}$. We argue by contradiction and we let $(U, V) = (u_x, v_x - a)$. By assumption $(U, V) \to (0, 0)$ as $r \to \infty$ and $U > 0$ for r large. The equations for U and V are

$$\begin{cases} U' = -\dfrac{2U}{r} + (a+V)[F(a^2 + V^2 + 2aV - U^2) - (m - \omega)], \\ V' = U[F(a^2 + V^2 + 2aV - U^2) - (m + \omega)]. \end{cases}$$

We have $V^2 + 2aV - U^2 \to 0$, and $|V^2 + 2aV - U^2| \leq C(U + |V|)$. Therefore if we set

$$d = F'(a^2) > 0,$$

we have

$$F(a^2 + V^2 + 2aV - U^2) = (m - \omega) + 2adV + o(U + |V|).$$

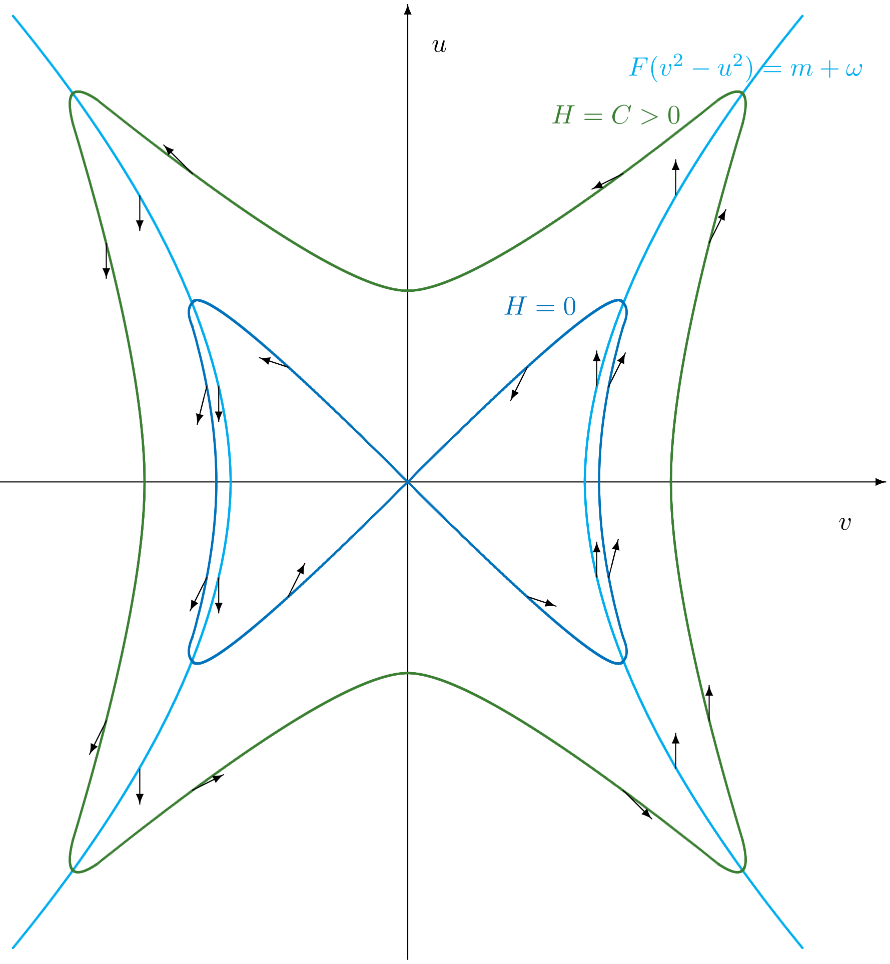

FIGURE 4. The velocity field for $r > 1/\omega$

The equations for U and V become then

$$\begin{cases} U' = -\dfrac{2U}{r} + 2a^2 dV + o(U + |V|), \\ V' = -2\omega U + U o(U + |V|). \end{cases} \tag{2.18}$$

Observe first that from the last equation in (2.18), it follows that $V' < 0$ for r large. Since $V \to 0$, we get $V > 0$, for r large. It follows from (2.18) that

$$(V - U)' = \omega(V - U) - (\omega - \frac{2}{r})U - (\omega + 2a^2 d)V + o(|U| + |V|) \le \omega(V - U),$$

for r large. Therefore $e^{-\omega r}(V - U)$ is nonincreasing for r large. Since $e^{-\omega r}(V - U) \to 0$, as $r \to \infty$, it follows that $V - U \ge 0$ for r large. Thus we can replace $o(|U| + |V|)$ by $o(V)$. We get from the first equation in (2.18)

$$U' = a^2 dV - \frac{2}{r}U + o(V) \ge \frac{a^2 d}{2} V \ge 0,$$

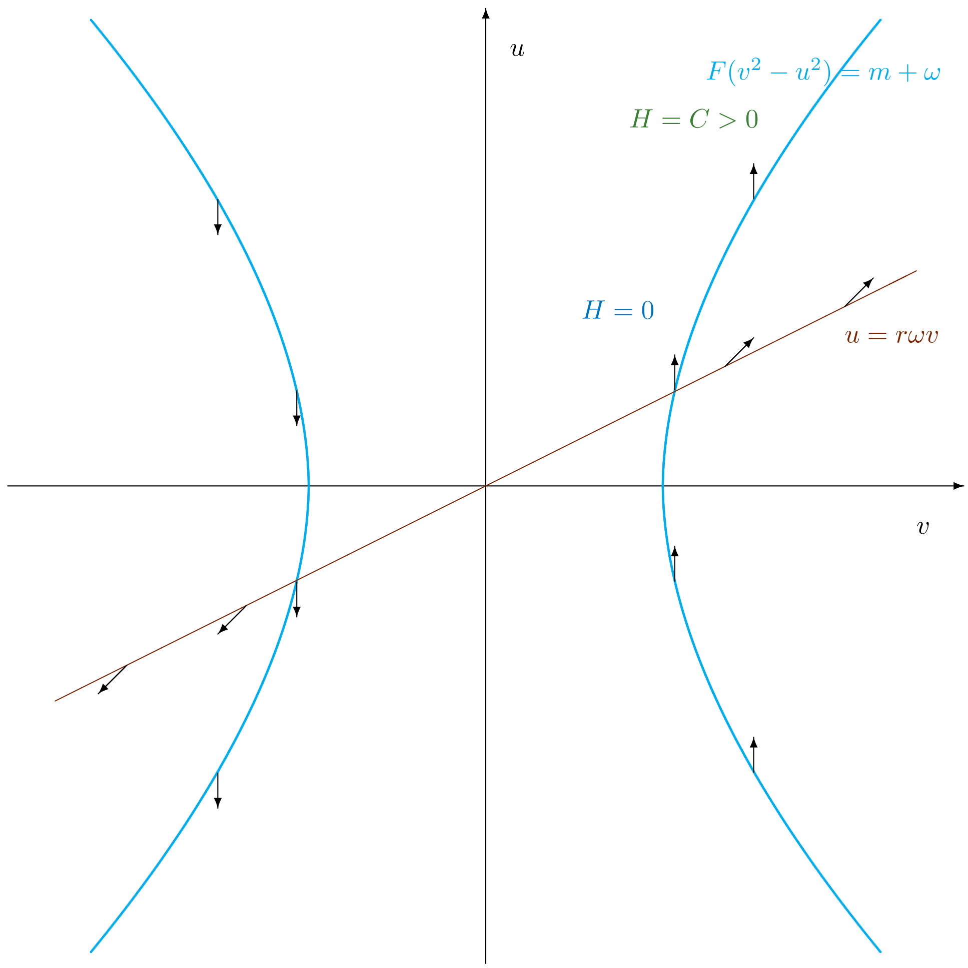

FIGURE 5. The velocity field for $r < 1/\omega$

for r large. Thus U is increasing for r large, which is a contradiction since $U > 0$ and $U \to 0$. By symmetry, (u_x, v_x) cannot converge to $(0, -a)$, while being in the half-plane $\{u < 0\}$. The same proof applies to the two other situations. □

Lemma 2.9. *Let $x \neq 0$, and let $\theta = \min\{R_x, \frac{1}{\omega}\}$. Then $v_x^2(r) - u_x^2(r) \geq e^{-4}x2$, for $0 \leq r < \theta$.*

Proof. Let $r = \sup\{r \in [0,\theta);\, v_x^2 \geq u_x^2 \text{ on } [0,r)\}$. Observe that on $[0,\rho)$ we have $u_x v_x \leq v_x^2$. Thus, for $r \in [0,\rho)$ we have by (2.15)

$$(v_x^2 - u_x^2)'(r) = \frac{4}{r}u_x^2 - 4\omega u_x v_x \geq 4\omega(u_x^2 - u_x v_x) \geq 4\omega(u_x^2 - v_x^2).$$

It follows that on$[0,\rho)$, $e^{4\omega r}(v_x^2 - u_x^2)$ is nondecreasing. Therefore

$$(v_x^2 - u_x^2)(r) \geq e^{-4\omega r}x^2 \geq e^{-4}x^2 > 0,$$

for $r \in [0,\rho)$. Therefore $\rho = \theta$. Hence the result. □

Lemma 2.10. *Let $x \neq 0$. Assume that $(u_x, v_x) \in \{F(v^2 - u^2) \geq m + \omega\}$ on some interval $[r_0, r_1) \subset [0, R_x)$. Then $r_1 - r_0 < \frac{3}{2\omega}$.*

Proof. $\{F(v^2 - u^2) \geq m + \omega\}$ is the union of two disconnected components $\{F(v^2 - u^2) \geq m + \omega, v > 0\}$ and $\{F(v^2 - u^2) \geq m + \omega, v < 0\}$. Since v_x is continuous, we may assume, say, $v_x > 0$ on $[r_0, r_1)$. Thus, we have

$$-1 < \frac{u_x}{v_x} < 1 \tag{2.19}$$

on $[r_0, r_1)$. From (2.16) we get

$$\Big(\frac{u_x}{v_x}\Big)' \geq \frac{1}{v_x^2}\Big[2\omega v_x^2 - \frac{2}{r} u_x v_x\Big]$$

on $[r_0, r_1)$; and so

$$\Big(r^2 \frac{u_x}{v_x}\Big)' \geq 2\omega r^2. \tag{2.20}$$

Integrating (2.20) between r_0 and r_1, and using (2.19), we get

$$r_1^2 + r_0^2 \geq \frac{2\omega}{3}(r_1^3 - r_0^3) = \frac{2\omega}{3}(r_1 - r_0)(r_1^2 + r_1 r_0 + r_0^2) \geq \frac{2\omega}{3}(r_1 - r_0)(r_1^2 + r_0^2).$$

This completes the proof. □

Proposition 2.11. *There exists a function $\Phi \in C(\mathbb{R}, \mathbb{R})$ with the following property. If $x \in \mathbb{R}$ is such that $R_x > \frac{1}{\omega}$ and $(u_x, v_x) \in \{F(v^2 - u^2) \geq m + \omega\}$ on some interval $[r_0, r_1) \subset [\frac{1}{\omega}, R_x)$, then $r_1 < R_x$ and $(|u_x| + |v_x|)(r) \leq \Phi((|u_x| + |v_x|)(r_0))$, for every $r \in [r_0, r_1)$.*

Proof. By continuity and symmetry, we may assume, $(u_x, v_x) \in \{F(v^2 - u^2) \geq m + \omega, v > 0\}$ for $r \in [r_0, r_1)$. Let $D^+ = \{F(v^2 - u^2) \geq m + \omega, v \geq 0, u \geq 0\}$, and $D^- = \{F(v^2 - u^2) \geq m + \omega, v > 0, u \leq 0\}$. Observe that, from (1.5), $u_x' > 0$ on $[r_0, r_1)$; and so, if $(u_x, v_x)(\rho) \in D^+$ for some $\rho \in [r_0, r_1)$, then $(u_x, v_x)(r) \in D^+$, for all $r \in [\rho, r_1)$. Thus, we may assume that $(u_x, v_x)(r) \in D^-$ on some interval $[r_0, r_2]$, and $(u_x, v_x)(r) \in D^+$ on $[r_2, r_1)$. On $(r_0, r_2]$, we have $u_x' > 0$ and $v_x' < 0$. Therefore,

$$(|u_x| + |v_x|)(r) \leq (|u_x| + |v_x|)(r_0), \tag{2.21}$$

for every $r \in [r_0, r_2]$. Next, $v_x^2 - u_x^2$ is nonincreasing on $[r_2, r_1)$ by (2.15). Thus, by (1.4)-(1.5),

$$(u_x + v_x)' \leq (u_x + v_x) F((v_x^2 - u_x^2)(r_2)), \tag{2.22}$$

on $[r_2, r_1)$. Finally, by Lemma 2.10, we have $r_1 - r_2 \leq \frac{3}{2\omega}$. Therefore, (2.21) and (2.22) yield

$$(|u_x| + |v_x|)(r) \leq (|u_x| + |v_x|)(r_0) e^{\frac{3}{2\omega} F([(|u_x| + |v_x|)(r_0)]^2)},$$

for every $r \in [r_0, r_1)$. Hence the bound on $(|u_x| + |v_x|)$; and so $r_1 < R_x$, unless $r_1 = R_x = \infty$. This is ruled out by Lemma 2.10. □

Proposition 2.12. *(See Figure 6.) Let $x \neq 0$. Assume that for some $r_0 > 0$, we have $v_x(r_0) \leq u_x(r_0)$ and $u_x(r_0) > 0$ (respectively, $u_x(r_0) \leq v_x(r_0)$ and $u_x(r_0) < 0$). Then there exists $r_0 < r_1 < R_x$ such that $|u_x| > 0$ on (r_0, r_1), and $u_x(r_1) = 0$, $|v_x(r_1)| > a$. In addition , either $v_x(r_0) > 0$ (respectively , $v_x(r_0) < 0$), and then v_x has exactly one zero in (r_0, r_1), or else $v_x(r_0) \leq 0$ (respectively $v_x(r_0) \geq 0$), and then $|v_x| > 0$, on $(r_0, r_1]$.*

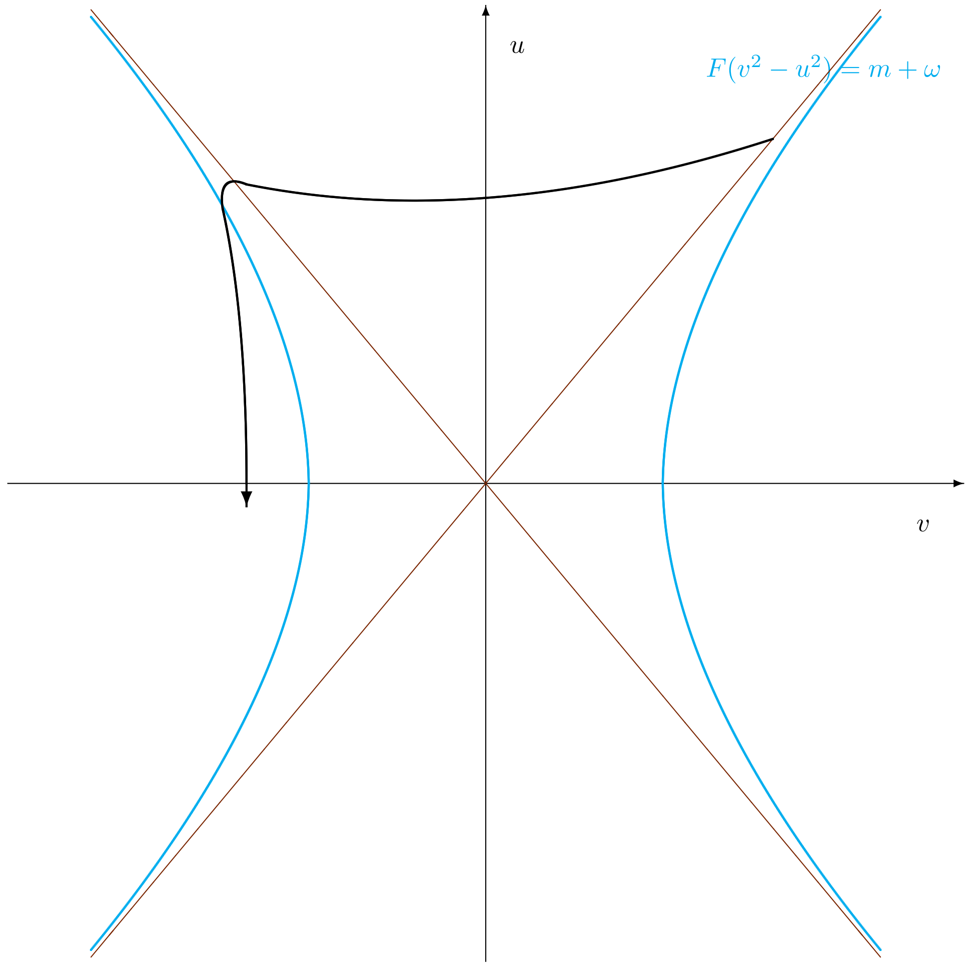

FIGURE 6. Illustration for Proposition 2.12

Proof. Assume for example $v_x(r_0) \le u_x(r_0)$ and $u_x(r_0) > 0$. Suppose first that $v_x(r_0) \ge -u_x(r_0)$. Because the velocity field on $\{v = -u, u > 0\}$ points towards $\{v < -u\}$, (u_x, v_x) can only enter $\{-u \le v \le u, u > 0\}$ by crossing the diagonal $\{u = v\}$; and so, by Lemma 2.9, $r_0 > \frac{1}{\omega}$. For $r \ge r_0$, and while (u_x, v_x) belongs to $\{-u \le v \le u, u > 0\}$, we have by (2.17)

$$\left(\frac{v_x}{u_x}\right)' \le \frac{1}{u_x^2}\left(\frac{2}{r_0}u_x v_x - 2\omega u_x^2\right) \le -2\left(\omega - \frac{1}{r_0}\right) < 0.$$

Thus, (u_x, v_x) exits $\{-u \le v \le u, u > 0\}$ in a finite time, by crossing the u-axis once, if $v_x(r_0) > 0$, and then, by crossing the diagonal $\{v = -u, u > 0\}$ (note that in the region $\{-u \le v \le u, u > 0\}$, the trajectory remains trapped in the set $\{H(u, v) \le H_x(r_0)\}$ due to (2.14)). Therefore, we may assume $v_x(r_0) < -u_x(r_0)$. Suppose now that $F(v_x^2(r_0) - u_x^2(r_0)) < m + \omega$. While (u_x, v_x) belongs to the region $D = \{F(v^2 - u^2) \le m + \omega, u + v \le 0, u \ge 0\}$, we have, by (2.5), $H_x' \le 0$; and so (u_x, v_x) is bounded. We claim that (u_x, v_x) must exit D in a finite time.

Indeed, assume that (u_x, v_x) remains in D for $r \geq r_0$. We have $v_x' \leq 0$, thus v_x has a negative limit as $r \to \infty$. It is not difficult to show that u_x also has a limit (compare [5], proof of Lemma 2.10). The limit (k, h) of (u_x, v_x) is therefore a rest point of (1.4)-(1.5) in the half-plane $\{v < 0\}$. Thus, $(k, h) = (0, -a)$. This is ruled out by Lemma 2.8. Therefore, (u_x, v_x) must exit D in a finite time. Next, (u_x, v_x) cannot exit D by crossing the diagonal $\{u = -v, u > 0\}$, since the velocity field on $\{v = -u, u > 0\}$ points towards $\{v < -u\}$. If (u_x, v_x) exits D by crossing the v-axis, then $u_x' < 0$; and so $v_x < -a$, which is the desired estimate. Therefore, we may suppose that (u_x, v_x) exits D by crossing the hyperbola $\{F(v^2 - u^2) = m + \omega\}$. When (u_x, v_x) belongs to $D' = \{F(v^2 - u^2) \geq m + \omega, v < 0, u \geq 0\}$, we have by (2.15)

$$(v_x^2 - u_x^2)' \leq 0;$$

and so, (u_x, v_x) cannot exit D' by crossing again the hyperbola $\{F(v^2 - u^2) = m + \omega\}$. By Lemma 2.10 and Proposition 2.11, we know that (u_x, v_x) must exit D' in a finite time by crossing the v-axis; and when it does so, we have $v_x \leq -a$. Hence the result. □

Corollary 2.13. *Let $x \neq 0$. Assume that, for some $0 < r_0 < R_x$, we have $v_x(r_0) = 0$. Then there exists $r_1 \in (r_0, R_x)$, such that $|u_x| > 0$, $|v_x| > 0$ on (r_0, r_1), and $u_x(r_1) = 0$, $|v_x(r_1)| > a$.*

Proof. If $u_x(r_0) > 0$, we have $v_x(r_0) \leq u_x(r_0)$, and if $u_x(r_0) < 0$, we have $u_x(r_0) \leq v_x(r_0)$. Thus, we may apply Proposition 2.12. □

Lemma 2.14. *Let $x \neq 0$ be such that $R_x = \infty$. Assume that for some $r_0 > 0$, we have $|u_x| > 0$ on $[r_0, \infty)$. Then , we have the following .*

(i) *$u_x v_x > 0$ on (r_0, ∞),*
(ii) *there exists C such that $0 < |u_x(r)| < |v_x(r)| < Ce^{-\frac{m-\omega}{2} r}$, for $r \in (r_0, \infty)$.*

Proof. Assume, for example, $u_x > 0$ on $[r_0, \infty)$. Then, by Proposition 2.12, we must have

$$0 < u_x < v_x,$$

on $[r_0, \infty)$. This proves (i). By (2.15), and the above inequality, we have

$$(v_x^2 - u_x^2)' < 0,$$

for r large. Therefore, by Lemma 2.10, we have

$$F(v_x^2 - u_x^2) < m + \omega,$$

for r large; and so, $v_x' < 0$, for r large. Thus, v_x has a limit as $r \to \infty$. It is not difficult to show that u_x also has a limit (compare [5], proof of Lemma 2.10). The limit (k, h) of (u_x, v_x) is therefore a rest point of (1.4)-(1.5). By Lemmas 2.1 and 2.8, we have $(k, h) = (0, 0)$. Therefore, for r large, we have

$$0 < F(v_x^2 - u_x^2) < \frac{m - \omega}{2}.$$

It follows from (1.4)-(1.5) that we have then

$$(u_x + v_x)' \leq -\frac{m - \omega}{2}(u_x + v_x);$$

from which the exponential decay follows. □

Lemma 2.15. *Let $x \in \mathbb{R}$ and let $0 \leq r_0 < R_x$ satisfy $u_x(r_0) = 0$ and $F(v_x(r_0)^2) > m + \omega$. Let $\rho \in (r_0, R_x]$ be defined by*

$$\rho = \sup\{r \in [r_0, R_x), u_x v_x > 0 \text{ on } (r_0, r)\}.$$

Then, there exists $r_0 < r_1 \leq R_x$ such that the function $f(r) = F(v_x^2(r) - u_x^2(r)) - (m+\omega)$ satisfies $f(r) > 0$ on $[r_0, r_1)$and $f(r) < 0$ on (r_1, ρ). In addition, if $r_1 = \rho$, then $r_1 = \rho = R_x \leq \frac{1}{\omega}$.

Proof. Note first that $|v_x(r_0)| > a$, and so (see Figure 3) $u_x v_x > 0$ on $(r_0, r_0 + \varepsilon)$, for some $\varepsilon > 0$. Therefore $\rho > r_0$. Furthermore, since $f(r_0) > 0$, we can define the number $r_1 \in (r_0, \rho]$ by

$$r_1 = \sup\{r \in [r_0, \rho];\ f(r) > 0 \text{ on } (r_0, r)\},$$

and we may assume by symmetry that $v_x(r_0) > 0$. Suppose first that $r_1 = \rho$. Then we have $r_1 = \rho = R_x \leq \frac{1}{\omega}$. To see this, observe that on $[r_0, r_1]$, v_x is nondecreasing; and so if we had $\rho < R_x$, we would have

$$v_x(\rho) \geq v_x(r_0) > a. \tag{2.23}$$

By definition of ρ this would imply $u_x(\rho) = 0$; and so we would get $u_x'(\rho) \leq 0$, which is incompatible with (2.23) (see Figure 3). Therefore, $r_1 = \rho = R_x$, and then by Proposition 2.11 we get $R_x \leq \frac{1}{\omega}$. Suppose now that $r_1 < \rho$. Then we have

$$f(r_1) = 0, \quad f'(r_1) \leq 0. \tag{2.24}$$

From (2.24) and (1.5), we get

$$v_x'(r_1) = 0. \tag{2.25}$$

Furthermore, $v_x^2 - u_x^2$ cannot be increasing in a neighborhood of r_1 since otherwise f also would be increasing, which is impossible by definition of r_1. Thus we have

$$(v_x^2 - u_x^2)'(r_1) \leq 0. \tag{2.26}$$

It follows from (2.25) and (2.26) that

$$u_x'(r_1) \geq 0. \tag{2.27}$$

Next, we claim that there exists $\varepsilon > 0$ such that $f(r) < 0$ on $(r_1, r_1 + \varepsilon)$. Indeed if $u_x'(r1) > 0$, then it follows from (2.26) that $(v_x^2 - u_x^2)'(r_1) < 0$, which implies the claim. On the other hand, if $u_x'(r_1) = 0$, then it follows from (2.25) that $(v_x^2 - u_x^2)'(r_1) = 0$. These formulas, together with (1.4) and (1.5) imply that $u_x''(r_1) = \frac{2}{r_1} u_x(r_1) > 0$ and $v_x''(r_1) = 0$; and so

$$(v_x^2 - u_x^2)''(r_1) = 2v_x v_x'(r_1) - 2u_x u_x'(r_1) = -\frac{4u_x(r_1)}{r_1} < 0,$$

which again implies the claim. It remains to prove that $f(r) < 0$ on (r_1, ρ). We argue by contradiction and we assume that there exists $r_2 \in (r_1, \rho)$ such that $f(r) < 0$ on (r_1, r_2) and $f(r_2) = 0$ (see Figure 7). From (2.15) and (2.26) we get

$$r_1 \geq \frac{u_x(r_1)}{2\omega v_x(r_1)}. \tag{2.28}$$

As well, we have $f(r_2) = 0$ and $f'(r_2) \geq 0$; and so

$$r_2 \leq \frac{u_x(r_2)}{2\omega v_x(r_2)}. \tag{2.29}$$

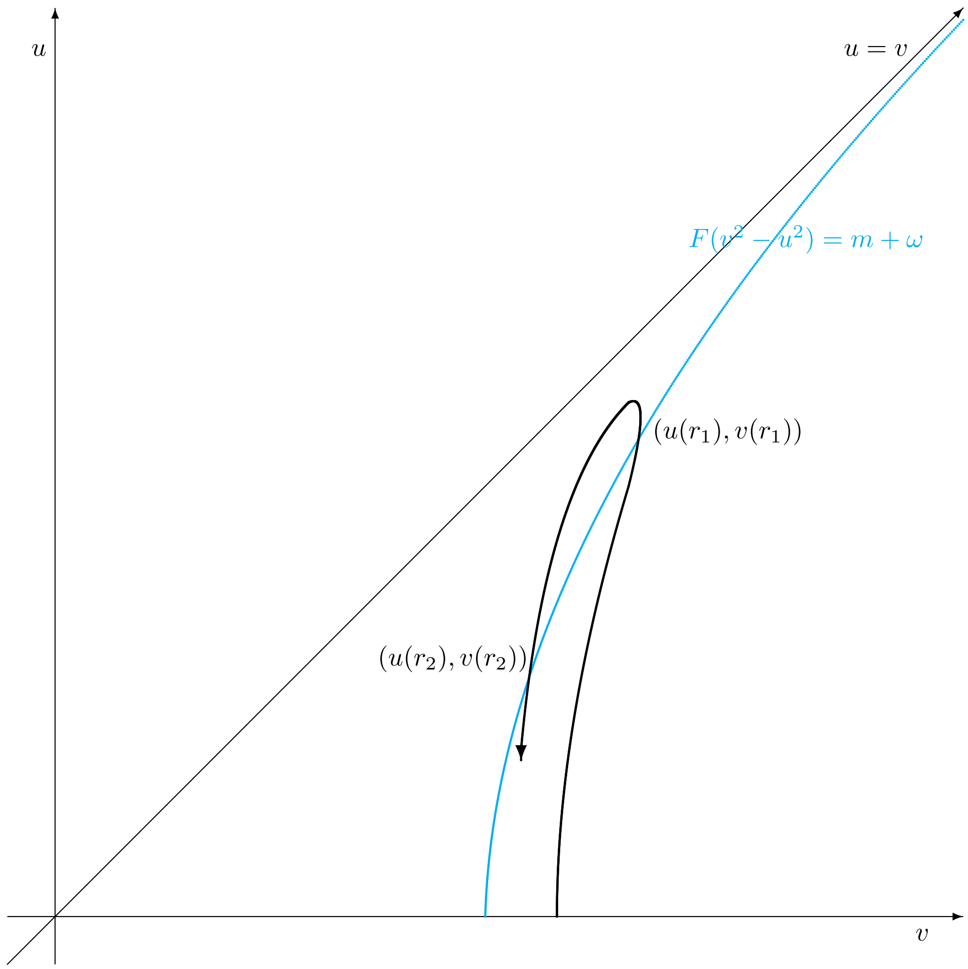

FIGURE 7. Notation for Lemma 2.15

Note that on (r_1, r_2) we have $v_x' < 0$; and so $v_x(r_2) < v_x(r_1)$. Furthermore we have $u_x(r_1) > 0$ and $u_x(r_2) > 0$, and $f(r_1) = f(r_2) = 0$. This implies $\frac{u_x(r_2)}{2\omega v_x(r_2)} < \frac{u_x(r_1)}{2\omega v_x(r_1)}$. Applying (2.28) and (2.29) we get $r_2 < r_1$, which is a contradiction. This completes the proof of the lemma. □

Corollary 2.16. *(See Figure 8.) Let $x \neq 0$, $x \neq \pm a$. Assume that for some $r_0 \in [0, R_x)$ we have $u_x(r_0) = 0$. Then, one of the following properties holds .*

(i) *$|v_x(r_0)| < a$, and there exists $r_1 \in (r_0, R_x)$ such that $u_x v_x < 0$ on (r_0, r_1), $0 < |u_x| < |v_x|$ on (r_0, r_1), $|v_x(r_1)| > a$, and $u_x(r_1) = 0$;*

(ii) *$|v_x(r_0)| > a$, and there exists $r_1 \in (r_0, R_x)$ such that $u_x v_x > 0$ on (r_0, r_1), $0 < |u_x| < |v_x|$ on (r_0, r_1), $|v_x(r_1)| < a$, and $u_x(r_1) = 0$;*

(iii) *$|v_x(r_0)| > a$, $R_x = \infty$, $u_x v_x > 0$ on (r_0, ∞), and $0 < |u_x| < |v_x| < Ce^{-\frac{m-\omega}{2} r}$, on (r_0, ∞);*

(iv) *$|v_x(r_0)| > a$, and there exists $r_1 \in (r_0, R_x)$ such that $u_x v_x > 0$ on (r_0, r_1), and $v_x(r_1) = 0$;*

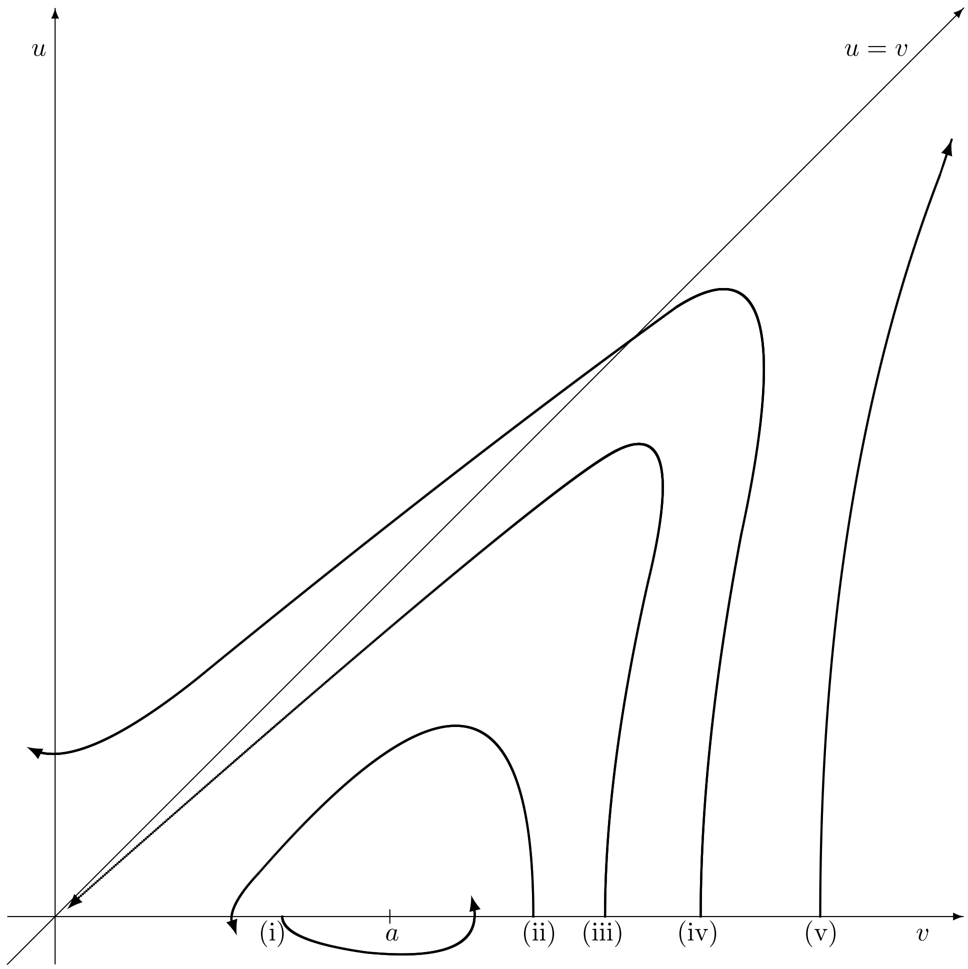

FIGURE 8. Illustration for Corollary 2.16

(v) $|v_x(r_0)| > a$, $R_x \leq \frac{1}{\omega}$, $u_x v_x > 0$ *on* (r_0, R_x) *and* $F(v_x^2 - u_x^2) > m + \omega$ *on* (r_0, R_x).

Proof. Assume for example $v_x(r_0) > 0$. By Lemma 2.7, we have $v_x(r_0) \neq 0, a$. If $v_x(r_0) < a$, then $u_x'(r_0) < 0$. Applying Proposition 2.12, we get (i), except the property $0 < |u_x| < |v_x|$ on (r_0, r_1). Observe that by (2.10) we have $H_x(r_0) < 0$, and that on (r_0, r_1), we have $H_x' \leq 0$ and $v_x' \geq 0$, until possibly (u_x, v_x) crosses the hyperbola $\mathcal{H} = \{F(v^2 - u^2) = m + \omega\}$; and so, until then , we have $H_x < 0$, from which it follows (cf. Lemma 2.2 (ii)) that $0 < |u_x| < |v_x|$. Now, the velocity field on $\mathcal{H} \cap \{u < 0\}$ points towards $\{F(v^2 - u^2) > m + \omega\}$. Thus, if (u_x, v_x) crosses H, it cannot come back in the region $\{F(v^2 - u^2) \leq m + \omega\}$ on (r_0, r_1). Hence (i). Suppose now $v_x(r_0) > a$, and let $\rho = \sup\{r \in (r_0, R_x);\, u_x v_x > 0 \text{ on } (r_0, r)\}$. If $\rho < R_x$, we have either $v_x(\rho) = 0$, in which case we get (iv), or else $u_x(\rho) = 0$. In the last case, we must have $0 < |u_x| < |v_x|$ on (r_0, ρ), by Proposition 2.12; hence (ii). Assume now $\rho = R_x$. By Lemma 2.15 we have either (v) or else there exists $r_0 < r_1 < \rho$ such that the function $f(r) = F(v_x^2(r) - u_x^2(r)) - (m+\omega)$ satisfies $f > 0$ on (r_0, r_1) and $f < 0$ on (r_1, ρ). It follows from (2.14) that $H_x(r) \le H_x(r1)$ for $r_1 \le r \le \rho = R_x$. By Lemma 2.1 (ii) and Lemma 2.4, this implies $R_x = \infty$. Applying Lemma 2.14, we obtain (iii). This completes the proof. □

Lemma 2.17. *Let $x \neq 0$. Assume that v_x has at least two zeroes. Then, between two consecutive zeroes of v_x, u_x has an odd number of zeroes .*

Proof. Assume for example that $v_x > 0$ on (r_0, r_1), and $v_x(r_0) = v_x(r_1) = 0$. Because $v_x' < 0$ when $v_x = 0$, $u_x > 0$, and $v_x' > 0$ when $v_x = 0$, $u_x < 0$, we have $u_x(r_0)u_x(r_1) < 0$. Hence the result. □

Lemma 2.18. *Let $x \neq 0$. Assume that $R_x \ge \frac{1}{\omega}$, and that u_x has a finite number of zeroes . Then $R_x = \infty$, and $|u_x| + |v_x| \to 0$ as $r \to \infty$.*

Proof. In the region $\{F(v^2 - u^2) \le m + \omega\}$, H_x is nonincreasing, by (2.15); and so (u_x, v_x) cannot blow up. Now, the region$\{F(v^2 - u^2) > m + \omega\}$ is the union of two connected, open components D_1 and D_2, where $D_1 = \{F(v^2 - u^2) \le m + \omega, v > 0\}$ and $D_2 = \{F(v^2 - u^2) \le m + \omega, v < 0\}$. Considering the velocity field on the boundary of D_1, and for $r \ge \frac{1}{\omega}$, it is not difficult to show that (u_x, v_x) can enter D_1 only in the half-plane $\{u < 0\}$, and can exit D_1 only in the half-plane $\{u > 0\}$. Thus, when the solution crosses D_1 after $r = \frac{1}{\omega}$, u_x has at least one zero. By symmetry, the same holds for D_2; and so, (u_x, v_x) can only cross D_1 or D_2 a finite number of times. Thus, by Proposition 2.11, we have $R_x = \infty$. Note that v_x cannot vanish after the last zero of u_x, by Corollary 2.13. Therefore, we may apply Lemma 2.14, from which the result follows. □

Lemma 2.19. *Assume that $F(x^2) \le m+\omega$. Then $R_x = \infty$. Furthermore $F(v_x^2(r) - u_x^2(r)) \le m + \omega$ and $H_x(r) \le H(0, x)$, for every $r \ge 0$.*

Proof. It follows from Lemma 2.1 (iv) that $\{H(u, v) \le H(0, x)\} \subset \{F(v^2 - u^2) \le m + \omega\}$. Therefore (2.14), together with an obvious continuity argument, implies that $H_x(r) \le H(0, x)$ for every $r \in [0, R_x)$; and so $F(v_x^2(r) - u_x^2(r)) \le m + \omega$, for every $0 \le r < R_x$. It follows from Lemma 2.1 (iii) and Lemma 2.4 that $R_x = \infty$. □

3. A blow-up result

In this section, we prove that under some additional assumption on F and for $|x|$ large enough, the solution (u_x, v_x) blows up in a finite time, and remains in the region $\{F(v^2 - u^2) > m + \omega\}$. This shows the existence of an a priori bound on admissible initial data for the solutions of (1.4)-(1.7). Let us introduce some notation. We define the function Φ on $[0, \infty)$ by

$$\Phi(x) = F(F^{-1}(m + \omega) + 4\omega e^{-4}x) - (m + \omega) \ge 0.$$

By assumption (H1), Φ is increasing. We set

$$\Psi(x) = \int_0^x s\Phi(s)\, ds,$$

for $x \ge 0$. Ψ is also increasing on $[0, \infty)$ and we assume in this section that

$$\int_0^\infty \frac{ds}{\sqrt{1 + \Psi(s)}} < \infty. \tag{3.1}$$

Remark 3.1. Assumption (3.1) requires that F goes to ∞ fast enough at infinity. For example, it is easy to check that (3.1) is satisfied if there exists $\alpha > 0$ and $\beta > 2$ such that

$$F(x) \geq \alpha(\log x)^{\beta},$$

for x large,

Our main result is the following.

Proposition 3.2. *For every $\tau > 0$, there exists $B(\tau)$ such that if $|x| \geq B(\tau)$, then $R_x < \tau$ and $F(v_x^2 - u_x^2) > m + \omega$ for $r \in (0, R_x)$.*

Proof. We use a slightly simpler argument than that of [2]. Let $\tau \in (0, \frac{1}{2\omega})$ and let $B > 0$ be large enough so that

$$F(e^{-4}B^2) > m + \omega, \tag{3.2}$$

and

$$\int_0^{\infty} \frac{ds}{\sqrt{B^2 + \frac{2\omega}{3}\Psi(s)}} < \tau. \tag{3.3}$$

It is clear from (3.1) that (3.2) and (3.3) can be achieved for B large enough. By symmetry, it is sufficient to consider the case $x > 0$. Let $x \geq B$ and let $\theta = \min\{R_x, \tau\}$. From (2.15), we get for $r \in [0, \theta)$

$$(v_x^2 - u_x^2)'(r) = \frac{4}{r}u_x^2 - 4\omega u_x v_x \geq 8\omega u_x^2 - 4\omega u_x v_x = 8\omega(u_x^2 - v_x^2) + 4\omega(2v_x^2 - u_x v_x).$$

By Lemma 2.9, we have $u_x v_x \leq v_x^2$ on $[0, \theta)$; and so

$$(v_x^2 - u_x^2)'(r) \geq 8\omega(u_x^2 - v_x^2) + 4\omega v_x^2,$$

for $r \in [0, \theta)$. Therefore,

$$[e^{8\omega r}(v_x^2 - u_x^2)]'(r) \geq 4\omega e^{8\omega r} v_x^2 \geq 4\omega v_x^2, \tag{3.4}$$

for $r \in [0, \theta)$. Integrating (3.4) between 0 and $r < \theta < \frac{1}{2\omega}$, and applying (3.2), we get

$$(v_x^2 - u_x^2)(r) \geq F^{-1}(m + \omega) + 4\omega e^{-4} \int_0^r v_x(s)^2 ds,$$

for $r \in [0, \theta)$. Therefore,

$$(v_x^2 - u_x^2) - (m + \omega) \geq \Phi(f(r)), \tag{3.5}$$

for $r \in [0, \theta)$, with

$$f(r) = \int_0^t v_x(s)^2 ds,$$

for $r \in [0, \theta)$. Observe that by (3.5) and (1.5), v_x is increasing on $[0, \theta)$, and so f is convex. On the other hand, it follows from (3.5) that we can apply formula (2.20), which yields

$$u_x(r) \geq \frac{2\omega}{3} r v_x(r), \tag{3.6}$$

for $r \in [0, \theta)$. It follows from (1.5), (3.5) and (3.6) that

$$v_x'(r) \geq \frac{2\omega}{3} r v_x(r)\Phi(f(r)), \tag{3.7}$$

for $r \in [0, \theta)$. On multiplying (3.7) by v_x we get

$$f''(r) \geq \frac{2\omega}{3} r f'(r)\Phi(f(r)), \tag{3.8}$$

for $r \in [0,\theta)$. Since f is convex, positive and increasing, we have $rf'(r) \geq f(r)$; and so (3.8) yields

$$f''(r) \geq \frac{2\omega}{3} f(r)\Phi(f(r)), \tag{3.9}$$

for $r \in [0,\theta)$. We now multiply (3.9) by f' to get

$$(f'^2)'(r) \geq \frac{4\omega}{3} f'(r)f(r)\Phi(f(r)) = \frac{4\omega}{3}\Psi(f(r))', \tag{3.10}$$

for $r \in [0,\theta)$. Integrating (3.10) we obtain

$$(f'^2)(r) \geq x^2 + \frac{4\omega}{3}\Psi(f(r)), \tag{3.11}$$

for $r \in [0,\theta)$. Taking the square root of (3.11) we get

$$f'(r) \geq \sqrt{x^2 + \frac{4\omega}{3}\Psi(f(r))}, \tag{3.12}$$

for $r \in [0,\theta)$. It follows from (3.12) and (3.3) that

$$r \leq \int_0^{f(r)} \frac{ds}{\sqrt{x^2 + \frac{4\omega}{3}\Psi(s)}} \leq \int_0^\infty \frac{ds}{\sqrt{x^2 + \frac{4\omega}{3}\Psi(s)}} < \tau, \tag{3.13}$$

for $r \in [0,\theta)$. By definition of θ, this implies that $\theta = R_x$; and so $R_x < \tau$. This completes the proof. □

4. The main estimate

We introduce the sets I_n, A_n, and E_n (see figure 9) defined for $n \in \mathbb{N}$ by

$$I_n = \{x > a, \exists r_{x,n} \in (0,R_x) \text{ s.t. both } u_x \text{ and } v_x \text{ have exactly } n \text{ zeroes on } (0,r_{x,n}) \text{ and } u_x(r_{x,n}) = 0\},$$

$$A_n = \{x > a, R_x = +\infty \text{ and both } u_x \text{ and } v_x \text{ have exactly } n \text{ zeroes on } (0,\infty) \text{ and } |u_x| + |v_x| \to 0 \text{ as } r \to \infty\},$$

$$E_n = I_n \cup A_n.$$

For $x \in A_n$, we set

$$r_{x,n} = +\infty.$$

Finally, we define the sets I, A and E by

$$I = \bigcup_{n\geq 0} I_n, \quad A = \bigcup_{n\geq 0} A_n, \quad E = \bigcup_{n\geq 0} E_n.$$

In this section, we shall establish a boundedness property for solutions of (1.4)-(1.5), with initial data in E_n . Our main result is the following.

Proposition 4.1. *Let $n \in \mathbb{N}$ and assume $E_n \neq \emptyset$. It follows that*

$$\sup_{x\in E_n} \sup_{r\in[0,r_{x,n})} |u_x(r)| + |v_x(r)| < \infty.$$

Before proceeding to the proof of Proposition 4.1, we need some preliminary results, where we set topological features of the sets E_n and properties of trajectories with initial data in E_n. We begin with

Lemma 4.2. *For every $n \geq 0$, I_n is an open subset of $(a,+\infty)$.*

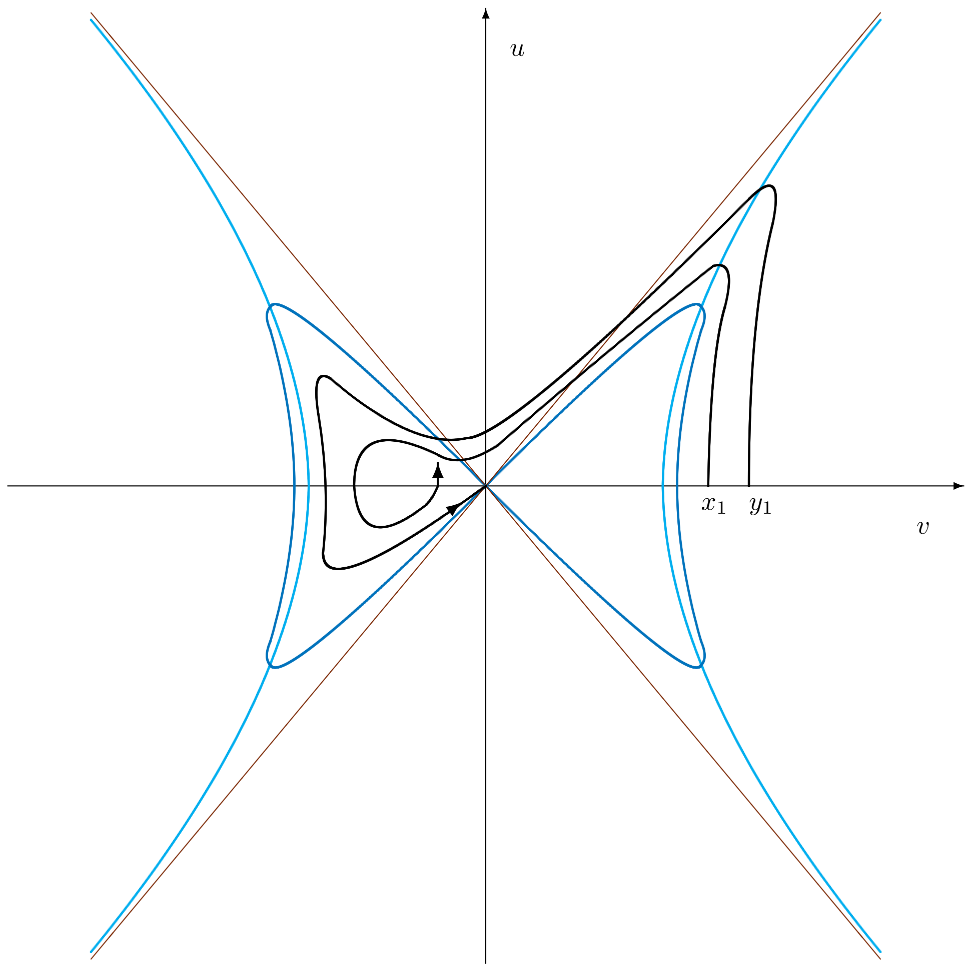

FIGURE 9. Trajectories with $x_1 \in I_1$ and $y_1 \in A_1$

Proof. Observe that if $v_x = 0$ (respectively $u_x = 0$), then by Lemma 2.7, we have $v_x' \neq 0$ (respectively $u_x' \neq 0$). Therefore, the result follows from (2.11). □

Next, we consider the case $n = 0$, for which the proof is different from the general case. We begin with the following result.

Lemma 4.3. *Let $x > a$ and let $\rho = \sup\{r \in (0, R_x), 0 < u_x < v_x$ on $(0, r)\}$. There exists $\theta \in [0, \rho]$ such that*

(i) *H_x and v_x are increasing and $F(v_x^2(r) - u_x^2(r)) > m + \omega$ on $(0, \theta)$;*
(ii) *H_x and v_x are decreasing and $F(v_x^2(r) - u_x^2(r)) < m + \omega$ on (θ, ρ).*

Proof. This follows from Lemma 2.15 and (1.5). □

Corollary 4.4. *Assume $E_0 \neq \emptyset$. It follows that*

$$\sup_{x \in E_0} \sup_{r \in [0, r_{x,0})} |u_x(r)| + |v_x(r)| < \infty.$$

Proof. Consider $C > 0$ and set

$$M = \sup\{|u| + |v|;\, H(u,v) = C\}.$$

If $x \in E_0$ is such that $H_x(r) \le C$ on $(0, r_{x,0})$, then $|u_x| + |v_x| \le M$ on $(0, r_{x,0})$. Consider now $x \in E_0$ such that there exists $r \in (0, r_{x,0})$ with $H_x(r) > C$. Observe that if $x \in I_0$ then $v_x(r_{x,0}) \in (0, a)$ (Corollary 2.16, case (ii)); and so $H_x(r_{x,0}) < 0$ (by (2.10)). On the other hand, if $x \in A_0$ then $H_x(r) \to 0$, as $r \to r_{x,0}$. Thus in both cases there exists $\tau \in (0, r_{x,0})$ such that $H_x(\tau) = C$ and $H_x(r) < C$ for $r \in (\tau, r_{x,0})$. By Corollary 2.6 and the definition of E_0, it follows that there exists $\Theta < \infty$ independent of $x \in E_0$ such that $\tau \le \Theta$. Let us now apply Corollary 2.16 with $r_0 = 0$. By definition of E_0 we are either in case (ii) if $x \in I_0$ or in case (iii) if $x \in A_0$. It follows that $0 < u_x < v_x$ on $(0, r_{x,0})$. Therefore we may apply Lemma 4.3 with $\rho = r_{x,0}$. It follows that

$$0 \le u_x \le v_x \le v_x(\theta) \text{ on } [0, r_{x,0}); \tag{4.1}$$

$$0 \le F(v_x^2 - u_x^2) \le m + \omega \text{ on } [\theta, \tau]. \tag{4.2}$$

It follows from (4.2) and (1.5) that $|v_x'| \le (m+\omega) v_x$ on (θ, τ). This implies that

$$v_x(\theta) \le v_x(\tau) e^{(m+\omega)(\tau-\theta)} \le M e^{(m+\omega)\tau} \le M e^{(m+\omega)\Theta}. \tag{4.3}$$

The result follows from (4.1) and (4.3). □

We now consider the case $n \ge 1$. In the following Lemma, we describe some topological properties of (u_x, v_x), when $x \in I_n$.

Lemma 4.5. *Let $n \ge 1$ and $x \in I_n$. Then ,*

(i) $r_{x,n} > \frac{1}{\omega}$,
(ii) *the first zero of $u_x v_x$ in $(0, r_{x,n})$ is a zero of v_x,*
(iii) *the $2n$ zeroes of $u_x v_x$ in $(0, r_{x,n})$ are alternatively one zero of v_x and one zero of u_x,*
(iv) *the last two zeroes of $u_x v_x$ in $(0, r_{x,n}]$ are zeros of u_x,*
(v) *we have $|v_x(r_{x,n})| < a$.*

Proof. Let r_0 be the first zero of v_x. By Corollary 2.13, u_x has at least n zeroes in $(r_0, r_{x,n}]$. If u_x has a first zero, say, $\rho_0 \in (0, r_0)$, we have $u_x'(\rho_0) < 0$, and so $0 < v_x(\rho_0) < a$. By corollary 2.16, u_x must have another zero in $(0, r_0)$; and so u_x has at least $n + 2$ zeroes in $(0, r_{x,n}]$, which is impossible by definition of E_n. This proves (ii). Therefore, (u_x, v_x) must cross the diagonal $\{u = v\}$ in $(0, r_0)$. Hence (i), by Lemma 2.9. Next, note that, by Corollary 2.13, a zero of u_x follows every zero of v_x; hence (iii) and (iv). Property (v) follows from (iii), (iv), and Corollary 2.16, part (ii). □

A similar result holds for solutions with initial data in A_n. More precisely, we have.

Lemma 4.6. *Let $n \ge 1$, and $x \in A_n$. Then , the first zero of $u_x v_x$ in $(0, r_{x,n})$ is a zero of v_x. The $2n$ zeroes of $u_x v_x$ in $(0, r_{x,n})$ are alternatively one zero of v_x and one zero of u_x. Furthermore , for some $r_0 > 0$, we have $u_x v_x > 0$ on (r_0, ∞), and there exists C such that*

$$(|u_x| + |v_x|)(r) \le C e^{-\frac{m-\omega}{2} r},$$

for r≥ 0.

Proof. The alternate character of the zeroes of u_x and v_x is proved with the argument of the proof of Lemma 4.5. The other properties follow from Lemma 2.14. □

The following corollary is an immediate consequence of Lemmas 4.5 and 4.6.

Corollary 4.7. *We have $I \cap A = \emptyset$. Furthermore, for $k \neq j$, we have $I_k \cap I_j = \emptyset$.*

Finally, we will need the following two results.

Lemma 4.8. *For any $K > 0$ and $n \in \mathbb{N}$, there exists $C(K,n)$ with the following property. Assume $x > 0$ is such that $R_x > \frac{1}{\omega}$. Assume furthermore that there exists $r_0, r_1 \in [\frac{1}{\omega}, R_x]$ such that $H_x(r_0) \leq K$ and u_x has at most n zeroes on (r_0, r_1). Then*

$$(|u_x| + |v_x|)(r) \leq C(K,n),$$

for $r \in (r_0, r_1)$.

Proof. Considering the velocity field on $\{F(v^2 - u^2) = m + \omega\}$ for $r > \frac{1}{\omega}$ (see Figure 4), we observe that (u_x, v_x) can enter $D_1 = \{F(v^2 - u^2) > m + \omega, v > 0\}$ only in the half-plane $\{u < 0\}$. Similarly, (u_x, v_x) can exit D_1 only in the half-plane $\{u > 0\}$. Therefore, when (u_x, v_x) crosses D_1, u_x has at least one zero. The same holds for $D_2 = \{F(v^2 - u^2) > m + \omega, v < 0\}$; and so (u_x, v_x) crosses D_1 or D_2, at most n times on (r_0, r_1). When (u_x, v_x) crosses the set $\{F(v^2 - u^2) \geq m + \omega\}$, we have $H_x' \leq 0$, and so (u_x, v_x) remains bounded by (2.7). Therefore, the result follows from Proposition 2.11. □

Lemma 4.9. *For every $n \geq 1$, and $K > 0$, there exists $\tau(K,n) > 0$, with the following property . Suppose $x \in E_n$ and $H_x(\rho) = K$, for some $\rho \in [0, r_{x,n})$.Then $\rho \leq \tau(K,n)$.*

Proof. This follows from the definition of E_n and Corollary 2.6. □

As a consequence of the previous results, we have the following.

Corollary 4.10. *For every $x > a$, $x \notin I$, one of the following properties is satisfied.*

(i) *$R_x \leq \frac{1}{\omega}$, and u_x, v_x and $[F(v_x^2 - u_x^2) - (m + \omega)]$ are positive on $(0, R_x)$,*
(ii) *$R_x > \frac{1}{\omega}$, both u_x and v_x have infinitely many zeroes on $(0, R_x)$, and they are alternate,*
(iii) *$x \in A_n$, for some $n \in \mathbb{N}$.*

Proof. Let x be as above. We have $u_x'(0) > 0$, and so $u_x > 0$ and $v_x > 0$, for some time. Let

$$\rho = \sup\{r \in (0, R_x), v_x > u_x > 0 \text{ on } (0, r)\}.$$

Suppose first that $\rho = R_x$. Then, either v_x is bounded, and therefore $R_x = \infty$. By Lemma 2.14, we have then $x \in A_0$. Otherwise, v_x is unbounded. In that case, v_x' is positive somewhere. Thus,

$$J = \{r \in [0, R_x);\, F(v_x^2(r) - u_x^2(r)) > m + \omega\} \neq \emptyset.$$

Therefore, by Lemma 2.15, J is either the interval $[0, R_x)$, or else some interval $[0, R)$, with $R < R_x$. In the latter case, v_x' is negative on $[R, R_x)$, and so v_x is bounded, which is a contradiction. Thus, $J = [0, R_x)$. Applying Proposition 2.11, we must have $R_x \leq \frac{1}{\omega}$, which implies property (i). Suppose now that $\rho < R_x$. Then, we have either $u_x(\rho) = 0$, or else $u_x(\rho) = v_x(\rho)$. In the first case, we have $0 < v_x(\rho) < a$ (see Figure 3). This implies $x \in I_0$, (Corollary 2.16 (i)) which is a contradiction. Therefore $u_x(\rho) = v_x(\rho)$; and so, Proposition 2.12 implies that v_x has at least one zero, which is the first zero of $u_x v_x$. Furthermore, by Corollary 2.13, after every zero of v_x, the next zero of $u_x v_x$ is a zero of u_x. A zero of v_x cannot be followed by two zeroes of u_x, since we would have $x \in I$, and this is ruled out by Corollary 4.7. Therefore, if v_x has infinitely many zeroes, then (ii) is satisfied, and if v_x has a finite number of zeroes, then (iii) holds, by Lemma 2.18. Hence the result. □

Proof of Proposition 4.1. From Corollary 4.4, we need only prove the property for $n \ge 1$. Therefore, by Lemma 4.5, we have $r_{x,n} > \frac{1}{\omega}$. Now, we fix $K > 0$. Consider $x \in E_n$. By Lemma 4.5 (v) and (2.10), we have $H_x(r_{x,n}) < 0$, if $x \in I_n$. Furthermore, $H_x(r) \to 0$ as $r \to \infty$, if $x \in A_n$. Therefore, there exists $r \in (\frac{1}{\omega}, r_{x,n})$, with $H_x(r) < K$. We define the number ρ by

$$\rho = \begin{cases} \frac{1}{\omega} & \text{if } H_x(\frac{1}{\omega}) \le K, \\ \inf\{r \in (\frac{1}{\omega}, r_{x,n});\ H_x(r) < K\} & \text{if } H_x(\frac{1}{\omega}) > K. \end{cases}$$

By Lemma 4.8, we have

$$(|u_x| + |v_x|)(r) \le C(K,n) \text{ for } r \in [\rho, r_{x,n}).$$

Therefore, it remains to bound the solution on $(0,\rho)$. Note that, from Lemma 4.9, there exists $\Theta > 0$, depending only on n, such that $\rho \le \Theta$. We shall bound the solution separately on $[0, \frac{1}{\omega}]$ and on $[\frac{1}{\omega}, \rho]$.

STEP 1. A bound on $[0, \frac{1}{\omega}]$. Let us establish the following estimate.

$$(|u_x| + |v_x|)(r) \le \frac{2}{\omega} e^{\frac{m+\omega}{\omega}} (|u_x| + |v_x|)\Big(\frac{1}{\omega}\Big), \quad r \in \Big[0, \frac{1}{\omega}\Big]. \tag{4.4}$$

We first apply Corollary 2.16 with $r_0 = 0$. By definition of E_n we are in case (iv). It follows that $0 < u_x < v_x$ untill the solution crosses the line $\{u = v\}$. By Lemma 2.9, this happens after $\frac{1}{\omega}$. Therefore we may apply Lemma 4.3 with $\rho > \frac{1}{\omega}$. It follows that

$$0 \le u_x \le v_x \le v_x(\theta) \quad \text{on } \Big[0, \frac{1}{\omega}\Big]; \tag{4.5}$$

$$0 \le F(v_x^2 - u_x^2) \le m + \omega \quad \text{on } \Big(\theta, \frac{1}{\omega}\Big); \tag{4.6}$$

for some $\theta \in [0, \frac{1}{\omega}]$. If $\theta = \frac{1}{\omega}$, (4.4) follows from (4.5). If $\theta < \frac{1}{\omega}$, it follows from (4.6) and (1.5) that $|v_x'| \le (m+\omega) v_x$ on $(\theta, \frac{1}{\omega})$. This implies that

$$v_x(\theta) \le e^{(m+\omega)(\frac{1}{\omega} - \theta)} v_x\Big(\frac{1}{\omega}\Big) \le e^{\frac{m+\omega}{\omega}} v_x\Big(\frac{1}{\omega}\Big). \tag{4.7}$$

(4.4) is then a consequence of (4.5) and (4.7). Consequently, in order to estimate the solution on $[0,\rho]$, it is sufficient to estimate it on $[\frac{1}{\omega}, \rho]$. This is the object of Step 2 below.

STEP 2. A bound on $[\frac{1}{\omega}, \rho]$. Let us consider the regions Δ_1, Δ_2, Δ_3, and Δ_4, (see figure 10), defined by

$$\begin{aligned} \Delta_1 &= \{F(v^2 - u^2) > m + \omega;\ H(u,v) > K, v > 0\}, \\ \Delta_2 &= \{F(v^2 - u^2) < m + \omega;\ H(u,v) > K, u > 0\}, \\ \Delta_3 &= \{F(v^2 - u^2) > m + \omega;\ H(u,v) > K, v < 0\}, \\ \Delta_4 &= \{F(v^2 - u^2) < m + \omega;\ H(u,v) > K, u < 0\}. \end{aligned}$$

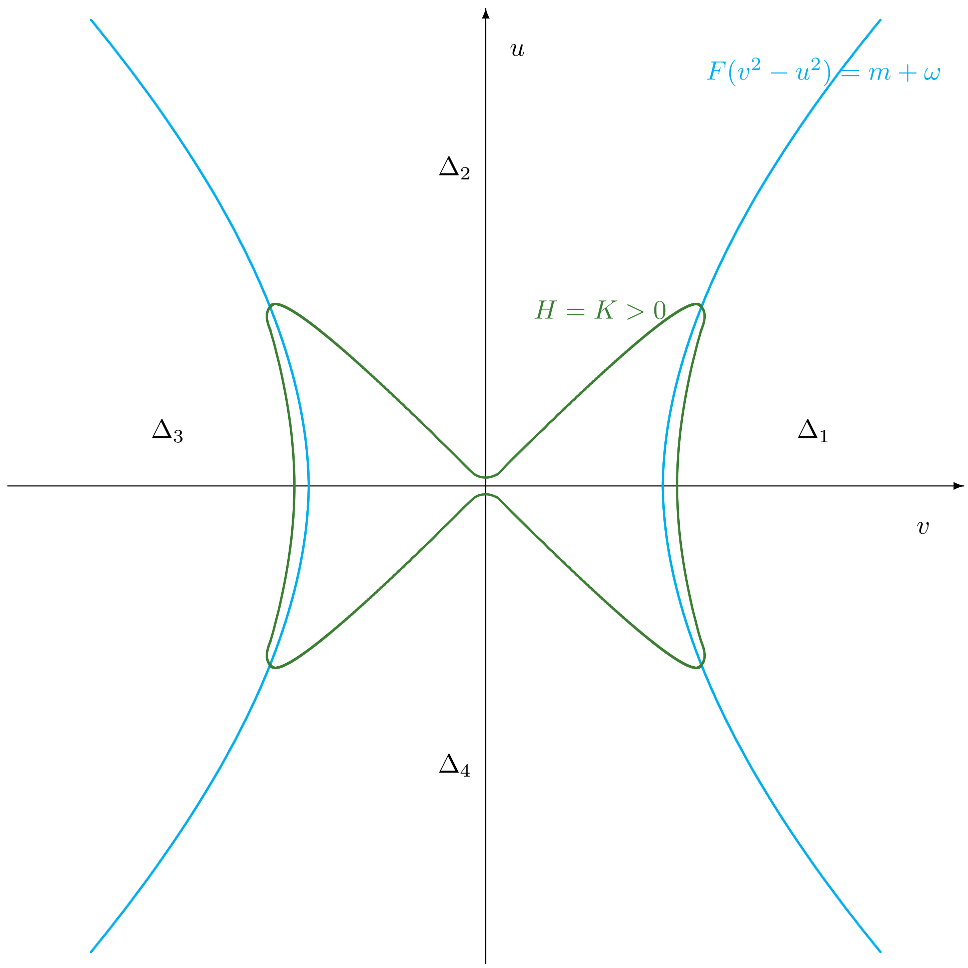

FIGURE 10. The sets Δ_i, $i = 1, 2, 3, 4$

Assume that (u_x, v_x) enters Δ_1 at some time $t \in [1/\omega, \rho]$. Considering the velocity field on the hyperbola $\{F(v^2 - u^2) = m + \omega\}$ and on the set $\{H(u, v) = K\}$ (see Figure 4), we obtain that $u_x(t) < 0$. By Lemma 2.10, (u_x, v_x) must exit Δ_1 in a finite time. The velocity field on the hyperbola $\{F(v^2 - u^2) = m + \omega\}$ and on the set $\{H(u, v) = K\}$ forces (u_x, v_x) to exit Δ_1 in the half-plane $\{u > 0\}$. Therefore, u_x has at least one zero in Δ_1. The same holds for Δ_3, by symmetry. A similar argument shows that, when (u_x, v_x) crosses Δ_2 or Δ_4, v_x has at least one zero (see Figures 3 and 4). Therefore, (u_x, v_x) crosses at most n times each of the sets Δ_i. Thus, the proof of Proposition 4.1 will be complete, provided we show the following lemma. □

Lemma 4.11. *For every $i \in \{1, 2, 3, 4\}$, and for every $\Xi > 0$, $A > 0$, there exists $B > 0$, with the following property. Let $x \in \mathbb{R}$ be such that $R_x > \Xi$. If on some interval $[r_0, r_1] \subset [\frac{1}{\omega}, \Xi]$, we have $(u_x, v_x) \in \Delta_i$, and if $(|u_x| + |v_x|)(r_1) \leq A$, then $(|u_x| + |v_x|)(r) \leq B$, for every $r \in [r_0, r_1]$.*

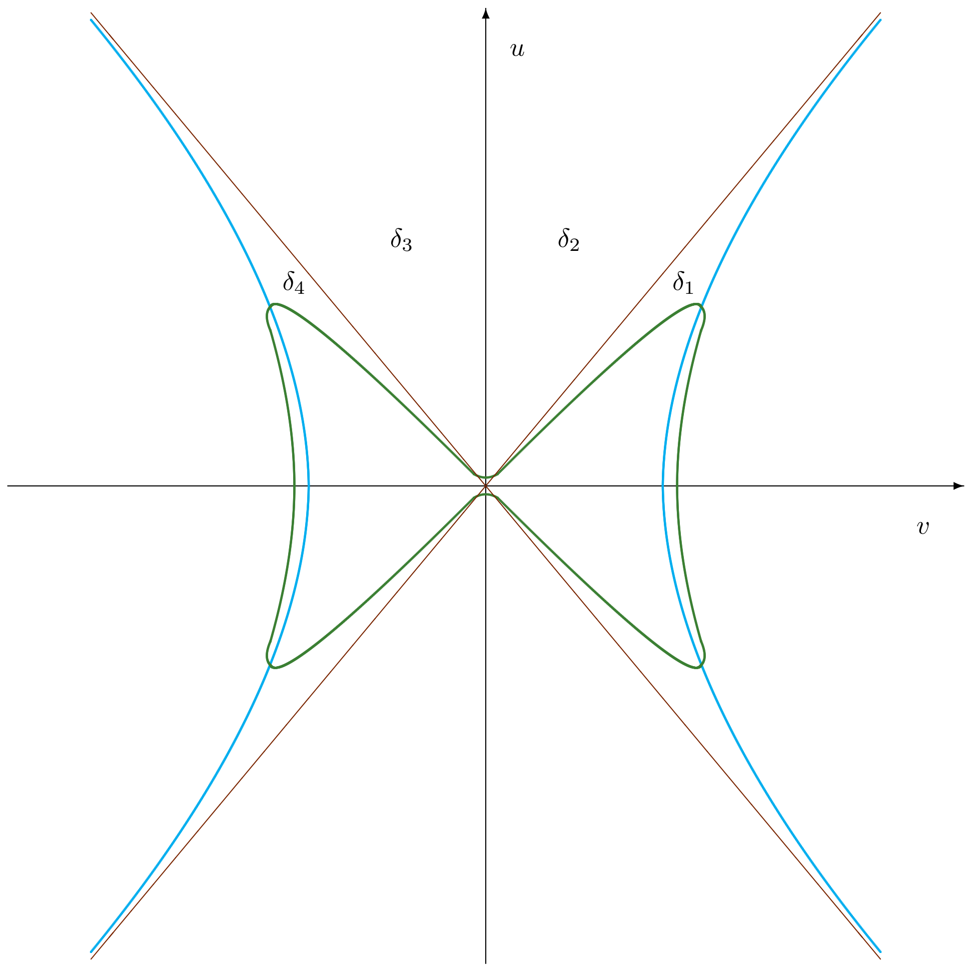

FIGURE 11. The sets δ_i, $i = 1, 2, 3, 4$

Proof. If $i = 1$, or $i = 3$, H_x is nondecreasing on $[r_0, r_1]$; and the result follows from (2.7). Now, assume for example $i = 2$, the case $i = 4$ being symmetric. We split Δ_2 in four subdomains δ_1, δ_2, δ_3, and δ_4, defined by (see figure 11)

$$\delta_1 = \{(u, v) \in \Delta 2;\, u \le v\},$$
$$\delta_2 = \{(u, v) \in \Delta 2;\, 0 \le v \le u\},$$
$$\delta_3 = \{(u, v) \in \Delta 2;\, 0 \le -v \le u\},$$
$$\delta_4 = \{(u, v) \in \Delta 2;\, u \le -v\}.$$

Considering the velocity field on the axes $\{u = v\}$, $\{v = 0\}$ and $\{u = -v\}$, for $r > \frac{1}{\omega}$, we obtain easily that (u_x, v_x) can only move from δ_1 to δ_2, from δ_2 to δ_3, and from δ_3 to δ_4. Therefore, we may assume, without loss of generality, that $(u_x, v_x)(r_0) \in \delta_1$, and $(u_x, v_x)(r_1) \in \delta_4$. We denote by τ_i, for $i = 1, 2, 3$, the time when (u_x, v_x) moves from δ_i to δ_{i+1}. On $[\tau_3, r_1]$, we have $v_x < 0$, $v_x' < 0$, and

$u_x \le -v_x$. Thus,

$$(|u_x| + |v_x|)(r) \le 2A \quad \text{on } [\tau_3, r_1]. \tag{4.8}$$

Next, in δ_3 we set

$$g(r) = u_x(r) - v_x(r) = |u_x(r)| + |v_x(r)| \quad \text{for } r \in [\tau_2, \tau_3].$$

Considering (1.4)-(1.5), we obtain

$$g' = -\frac{2}{r}g - \frac{2}{r}v_x - \omega(v_x + u_x) + mg - gF(v_x^2 - u_x^2) \quad \text{on } [\tau_2, \tau_3];$$

Observe that in δ_4 we have $-\frac{2}{r}g \ge -2\omega g$, $-\frac{2}{r}v_x \ge 0$, $-\omega(v_x + u_x) \ge -\omega g$ and $-gF(v_x^2 - u_x^2) \ge 0$. Therefore

$$g' \ge -2\omega g + (m - \omega)g \ge -2\omega g. \tag{4.9}$$

Integrating (4.9) between $r \in [\tau_2, \tau_3]$ and τ_3 yields

$$g(r) \le g(\tau_3)e^{2\omega\Xi} \quad \text{for every } r \in [\tau_2, \tau_3]. \tag{4.10}$$

Putting together (4.8) and (4.10) , we obtain

$$(|u_x| + |v_x|)(r) \le 2Ae^{2\omega\Xi} \quad \text{on } [\tau_2, r_1]. \tag{4.11}$$

Then, in δ_2, for $r \in [\tau_1, \tau_2]$, we set

$$h(r) = u_x^2(r) - v_x^2(r) \ge 0.$$

Applying (2.6), we obtain

$$h' \ge 4\omega(u_x v_x - u_x^2) \ge -4\omega h \quad \text{on } [\tau_1, \tau_2]. \tag{4.12}$$

Integrating (4.12) between $r \in [\tau_1, \tau_2]$ and τ_2, and applying (4.11), we obtain

$$h(r) \le h(\tau_2)e^{4\omega\Xi} \le (2Ae^{4\omega\Xi})^2 \quad \text{for every } r \in [\tau_1, \tau_2]. \tag{4.13}$$

Considering (1.4) and (4.13), we obtain

$$u_x' \ge -\frac{2}{r}u_x - Mv_x \ge -(M + 2\omega)u_x \quad \text{for every } r \in [\tau_1, \tau_2], \tag{4.14}$$

where $M = \sup\{-F(-x);\, 0 \le x \le (2Ae^{4\omega\Xi})^2\}$. Since $u_x \ge v_x$ on $[\tau_1, \tau_2]$, integrating (4.14) between $r \in [\tau_1, \tau_2]$ and τ_2, and applying (4.11), gives

$$(|u_x| + |v_x|)(r) \le 4Ae^{2\omega\Xi}e^{(M+\omega)\Xi} \quad \text{on } [\tau_1, r_1]. \tag{4.15}$$

Finally, in δ_1 observe that $0 \le F(v_x^2 - u_x^2) \le m + \omega$; and so by (1.5)

$$|v_x'| \le (m + \omega)v_x \quad \text{on } [r_0, \tau_1].$$

It follows that for $r \in [r_0, \tau_1]$ we have

$$v_x(r) \le v_x(\tau_1)e^{(m+\omega)\Xi}. \tag{4.16}$$

Since in δ_1 we have $0 < u_x < v_x$, it follows from (4.15) and (4.16) that

$$(|u_x| + |v_x|)(r) \le 4Ae^{2\omega\Xi}e^{(M+\omega)\Xi}e^{(m+\omega)\Xi}. \tag{4.17}$$

If we define B as the right hand side of (4.17), the result follows. □

5. Proof of Theorem 1.2

We begin with two preliminary observations.

Lemma 5.1. *Let $n \ge 0$. Assume $A_n \ne \emptyset$, and let x belong to the closure of A_n. Then $x \in \bigcup_{0 \le j \le n} A_j$.*

Proof. By Corollary 4.7, we have $I \cap A = \emptyset$. Thus, since I is open (Lemma 4.2), $x \notin I$. By (2.11) and Proposition 4.1, (u_x, v_x) is bounded. This proves $R_x = \infty$. If u_x and v_x have more than n zeroes, then by (2.11) it is the same for x' close to x. This is impossible, since x belongs to the closure of A_n. Therefore, the result follows from Corollary 4.10. □

Lemma 5.2. *Let $n \ge 0$. Assume $I_n \ne \emptyset$, and let $x > a$ belong to the boundary of I_n. Then $x \in \bigcup_{0 \le j \le n} A_j$.*

Proof. By Proposition 4.1, $I_n \ne (a, \infty)$, so its boundary is nonempty. By Corollary 4.7, and since I_n is open, $x \notin I_n$. Assume $x \in I_j$ for some $j \ne n$. Then by Lemma 4.2), we would have $I_n \cap I_j \ne \emptyset$. This is ruled out by Corollary 4.7. Therefore, $x \notin I$. Now, we apply Corollary 4.10. Property (i) is ruled out by Proposition 4.1, while property (ii) is ruled out by (2.11). Thus, x belong to some A_j. By continuous dependence, again, we must have $j \le n$. □

In order to show that I_n is nonempty, we need the following lemmas.

Lemma 5.3. *For every $C > 0$, there exists $T > 0$ with the following property. Let $x \ne 0$ be such that $R_x \ge T$. Assume that for some $\rho \ge T$, we have $v_x(\rho) = 0$ and $|u_x(\rho)| \le Ce^{-\frac{m-\omega}{2}\rho}$. Then, there exists $\theta \in (\rho, R_x)$ such that $|u_x| > 0$ and $|v_x| \le a$ on (ρ, θ), $|v_x(\theta)| = a$ and $H_x(\theta) < 0$.*

Proof. (See figure 12) Let x and ρ be such that $\rho \le R_x$, $v_x(\rho) = 0$, and

$$|u_x(\rho)| \le Ce^{-\frac{m-\omega}{2}\rho}. \tag{5.1}$$

Assume for example $u_x(\rho) > 0$. Then, v_x will become negative before u_x vanishes (see Figure 3). Let $\mu \in (0, a)$ be such that $F(\mu^2) = \frac{m-\omega}{2}$. If ρ is large enough, we have $u_x(\rho)) \le \frac{\mu}{2}$. We define the number $R > 0$, by

$$R = \sup\{r \in (\rho, R_x), u_x + |v_x| \le \mu \text{ on } [\rho, r]\}. \tag{5.2}$$

By Corollary 2.13, we have $R < R_x$. Thus,

$$(u_x + |v_x|)(R) = \mu. \tag{5.3}$$

Furthermore, still by Corollary 2.13 we have $u_x > 0$ and $v_x < 0$ on $(\rho, R]$; and so

$$\begin{aligned}(u_x - v_x)' &= -(u_x - v_x)F(v_x^2 - u_x^2) + (m + \omega)(u_x - v_x) - \frac{2}{r}u_x + 2\omega v_x \\ &\le (u_x - v_x)(m + \omega - F(v_x^2 - u_x^2)).\end{aligned}$$

Note further that on $[\rho, R]$ we have $v_x^2 - u_x^2 \ge -\mu^2$. Thus, if we set

$$c = -\min\{F(x), -\mu^2 \le x \le 0\},$$

we obtain

$$(u_x - v_x)' \le (m + \omega + c)(u_x - v_x), \text{ on } [\rho, R]. \tag{5.4}$$

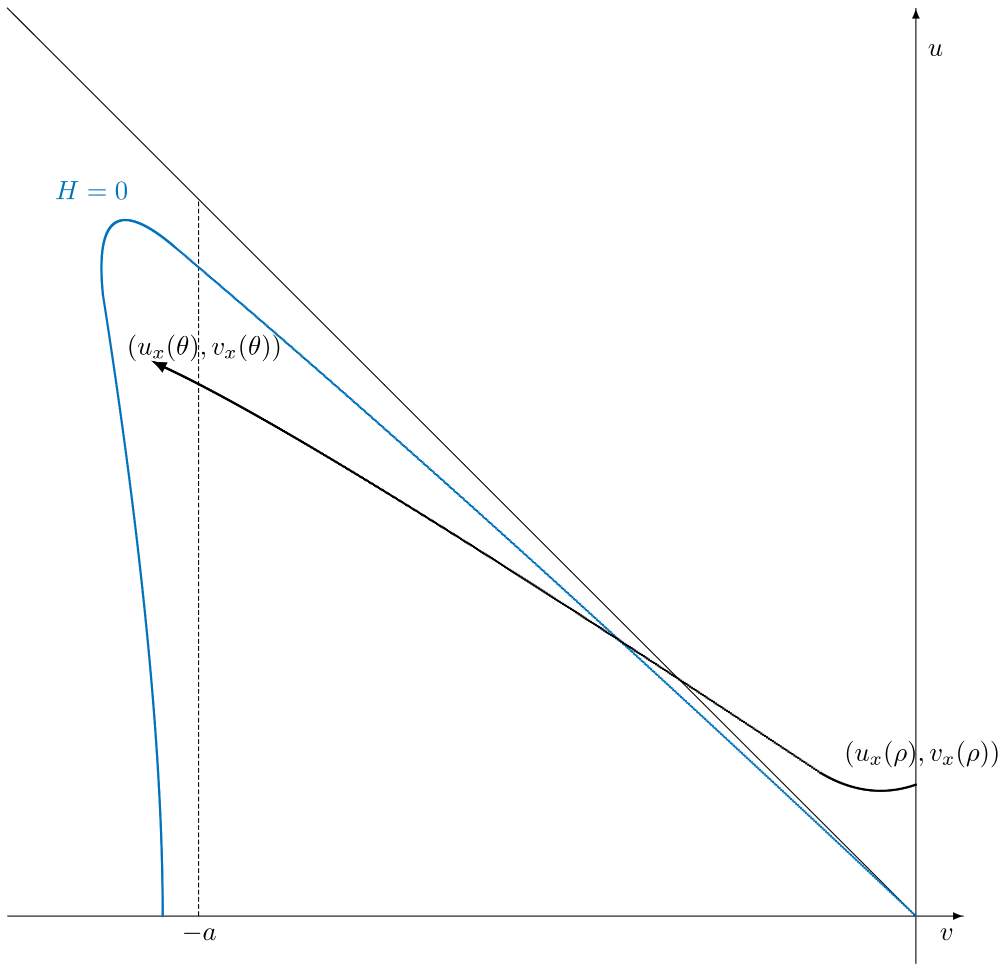

FIGURE 12. Notation for Lemma 5.3

Integrating (5.4) between ρ and R, we obtain

$$(u_x + |v_x|)(R) \leq u_x(\rho)e^{(m+\omega+c)(R-\rho)}. \tag{5.5}$$

Putting together (5.1), (5.3), and (5.5), we obtain

$$(m + \omega + c)(R - \rho) \geq \log\left(\frac{\mu}{C}\right) + \frac{m - \omega}{2}\rho.$$

Hence, for some $\delta > 0$ and ρ large enough

$$R \geq (1 + \delta)\rho. \tag{5.6}$$

We claim that, if ρ is large enough, then $H_x(R) < 0$. To see this, we first make the following observation. Consider x, y such that $x^2 \leq \mu^2$ and $y^2 \leq \frac{m-\omega}{2(m+\omega)}x^2$. It follows from (2.3) that

$$2H(y, x) = G(x^2 - y^2) - (m - \omega)x^2 + (m + \omega)y^2 \leq G(x^2 - y^2) - \frac{m - \omega}{2}x^2.$$

Furthermore $x^2 - y^2 \geq 0$; and so

$$G(x^2 - y^2) \leq G(x^2) \leq x^2 F(x^2) \leq \frac{m-\omega}{2} x^2.$$

Therefore, $H(y, x) \leq 0$. It follows that

$$u^2 \geq \frac{m-\omega}{2(m+\omega)} v^2, \text{ for all } u, v \text{ such that } H(u,v) \geq 0 \text{ and } |u| + |v| \leq \mu. \tag{5.7}$$

Let now

$$\tau = \sup\{r \in [\rho, R];\ H_x \geq 0 \text{ on } [\rho, r]\}. \tag{5.8}$$

From (5.7) and (5.8), we get

$$\Lambda u_x(r) \geq -v_x(r), \text{ for } r \in [\rho, \tau], \tag{5.9}$$

where $\Lambda > 0$ is given by $\Lambda^2 = \frac{2(m+\omega)}{m-\omega}$. Next, on $[\rho, R]$, we have

$$F(v_x^2 - u_x^2) - (m+\omega) \leq -\frac{m+\omega}{2}; \quad F(v_x^2 - u_x^2) - (m-\omega) \leq -\frac{m-\omega}{2}. \tag{5.10}$$

Therefore, we have, by (1.4) and (1.5),

$$(u_x - v_x)' \geq -\frac{2}{r} u_x + \frac{m-\omega}{2}(u_x - v_x). \tag{5.11}$$

Since $-u_x \geq -(u_x - v_x)$, and if $\rho > \frac{8}{m-\omega}$, we get from (5.11)

$$(u_x - v_x)(r) \geq u_x(\rho) e^{\frac{(m-\omega)(r-\rho)}{4}} \text{ for } r \in [\rho, R]. \tag{5.12}$$

Putting together (5.9) and (5.12), we obtain

$$u_x^2(r) \geq k^2 u_x^2(\rho) e^{\frac{(m-\omega)(r-\rho)}{4}} \text{ for } r \in [\rho, \tau], \tag{5.13}$$

where $k = 1/(1+\Lambda)$. Now, from (5.8), (5.10) and (2.5), we have

$$0 \leq H_x(\tau) \leq H_x(\rho) - (m+\omega) \int_\rho^\tau \frac{1}{r} u_x(r)^2 dr. \tag{5.14}$$

Next, observe that if ρ is large enough, we have by (2.3) and (5.1)

$$H_x(\rho) = H(u_x(\rho), 0) \leq \frac{1}{2} G(-u_x^2(\rho)) + \frac{m+\omega}{2} u_x^2(\rho) \leq (m+\omega) u_x^2(\rho). \tag{5.15}$$

Therefore, putting together (5.13), (5.14) and (5.15), we obtain

$$0 \leq 1 - k^2 \int_\rho^\tau \frac{1}{r} e^{\frac{(m-\omega)(r-\rho)}{2}} dr. \tag{5.16}$$

We deduce easily from (5.16) that

$$\frac{k^2}{\tau}[e^{\frac{(m-\omega)(\tau-\rho)}{2}} - 1] \leq m - \omega.$$

Therefore, for any $\alpha > 0$, we get for ρ large enough

$$\tau \leq (1+\alpha)\rho. \tag{5.17}$$

Putting together (5.6) and (5.17), we obtain $\tau < R$. Since by (2.14) we have $H_x' < 0$, on $[\tau, R]$, we get

$$H_x(R) < 0, \text{ if } \rho \text{ is large enough.} \tag{5.18}$$

To conclude, observe that by Corollary 2.13, v_x must decrease to some value less than $-a$, before u_x vanishes. Note also that when $|v_x| \leq a$, we have $H_x' < 0$; and

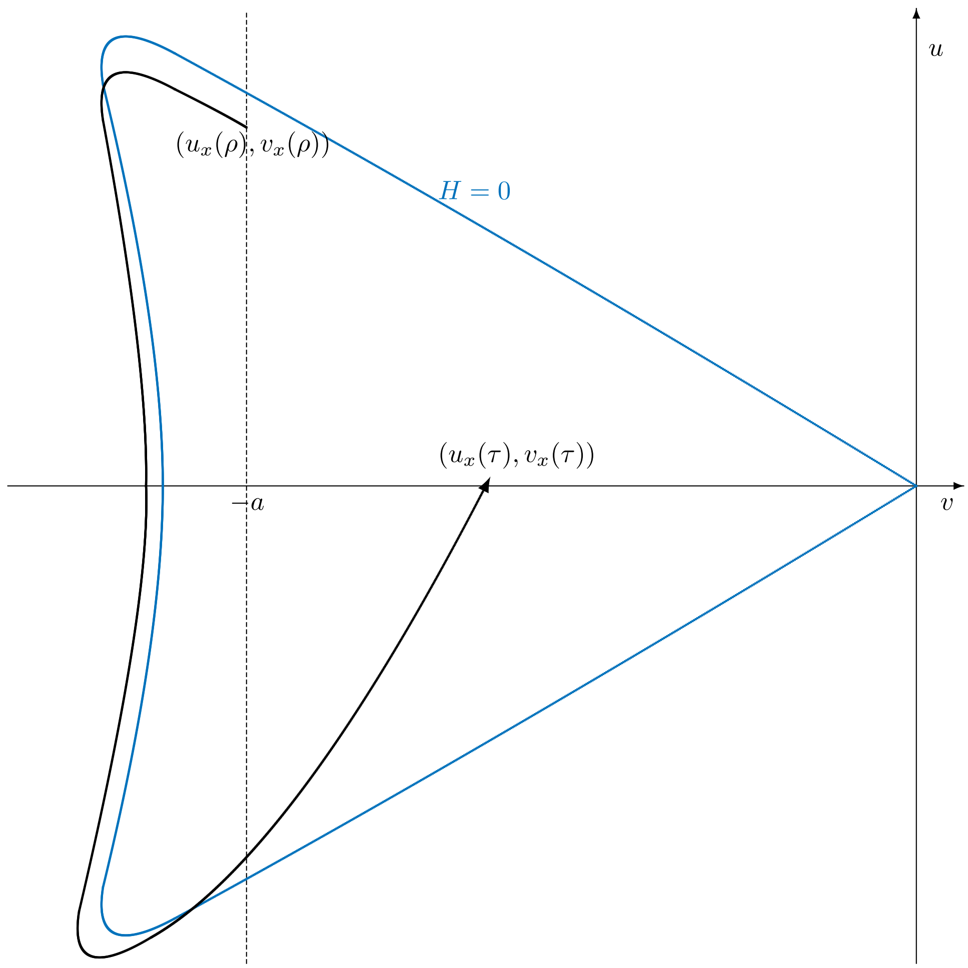

FIGURE 13. Notation for Lemma 5.4

so by (2.14), at the time when $v_x = -a$, we have $H_x < 0$. Thus, the proof of Lemma 5.3 is complete. □

Lemma 5.4. *There exists R with the following property. Assume $x \neq 0$ is such that $R_x \geq R$. Assume further that for some $\rho \geq R$, we have $v_x(\rho) = -a$, $u_x(\rho) > 0$ (respectively $v_x(\rho) = a$, $u_x(\rho) < 0$), and $H_x(\rho) < 0$. Then , there exists $\tau > \rho$ such that $|v_x| > 0$ on (ρ, τ) and such that u_x has two zeroes on (ρ, τ).*

Proof. (See figure 13) Let x be as above, and assume, for example, $u_x(\rho) > 0$. By Proposition 2.12, and Corollary 2.16, there exists $\rho_1 \in (\rho, R_x)$, such that $v_x(\rho_1) = -a$, $v_x < -a$ on (ρ, ρ_1), and u_x has exactly one zero on (ρ, ρ_1). Assume that $H_x(\rho_1) < 0$ and let

$$\Sigma = \{H(u, v) < H_x(\rho_1), 0 > v > -a, u < 0\}.$$

Observe that $H_x'(\rho_1) < 0$, and $v_x'(\rho_1) > 0$. Therefore, (u_x, v_x) will belong to Σ, for $r \in (\rho_1, \rho_1 + \varepsilon]$, where ε is some postive number. Since $H_x(\rho_1) < 0$, Σ is bounded away from the u-axis by Lemma 2.2 (ii). Furthermore, when $(u_x, v_x) \in \Sigma$, we have $H_x' < 0$ and $v_x' > 0$, by (1.5) and (2.14). Thus, (u_x, v_x) cannot exit Σ by crossing the line $\{v = -a\}$ or the curve $\{H(u, v) = H_x(\rho_1)\}$. Finally, since Σ is bounded away from the u-axis, it follows from Corollary 2.16, that (u_x, v_x) exits Σ in a finite time, by crossing the v-axis. This is the desired result.

Therefore, it remains to prove that $H_x(\rho_1) < 0$. Note that for a fixed value of $H_x(\rho)$, and for ρ large enough, this is a consequence of the fact that the trajectory remains close to a trajectory of the Hamiltonian motion (Lemma 2.5). Actually, we need an estimate on ρ that does not depend on the value of $H_x(\rho)$. To prove this, we consider two cases.

If $H(0, b) \geq 0$ (b given by (2.1)), then the sets $\{F(v^2 - u^2) \geq m - \omega\}$ and $\{H(u, v) < 0\}$ are disconnected by Lemma 2.1 (iv); and so, by (2.14), the set $\{H(u, v) < 0\}$ is a trapping region for the solutions of (1.4)-(1.5). Thus, we have $H_x(\rho_1) < 0$.

Suppose now $H(0, b) < 0$. By Lemma 2.19, the set $\{H(u, v) \leq H(0, b)\}$ is a trapping region. Thus, if for some $r \in [\rho, \rho_1]$, we have $H_x(r) \leq H(0, b)$, we get immediately $H_x(\rho_1) < 0$. Therefore, it only remains to consider the case where x is such that $H_x(\rho) < 0$, and

$$H(0, b) < H_x(r) \text{ for every } r \in [\rho, \rho_1]. \tag{5.19}$$

Let Λ be the set of such x's. Consider the set

$$J = \{z > 0, H(0, b) < H(z, -a) < 0\}.$$

Clearly, we have

$$u_x(\rho) \in J \text{ for every } x \in \Lambda. \tag{5.20}$$

Note that, since $H(0, b) < 0$, there exists $\delta \in (0, a)$ such that $H(0, \delta) = H(0, b)$, since $H(0, 0) = 0$ and $H(0, a) < H(0, b)$. It is not too difficult to show that for $z \in J$, there exists $T_z > 0$, such that the solution (u_z, v_z) of (2.8)-(2.9) with $u_z(\rho) = z$, $v_z(\rho) = -a$, will satsify

$$v_z(r) < -\delta \text{ for } t \in (\rho, \rho + Tz), u_z(\rho + T_z) < 0, \text{ and } v_z(\rho + T_z) = -\delta. \tag{5.21}$$

The set $\{0 \geq H(u, v) \geq H(0, b), v < -\delta\}$ is bounded away from the critical points of H, and so, there exists $M > 0$ such that

$$T_z \leq M \text{ for every } z \in J. \tag{5.22}$$

Note also that, since J is bounded,

$$(u_z(r), v_z(r)) \text{ is uniformly bounded, with respect to} r \in R \text{ and } z \in J. \tag{5.23}$$

Next, by Lemma 2.5 we know that for every $x \in \Lambda$, there exists $r_x > 0$, such that if $\rho \geq r_x$, then

$$|(u_x, v_x) - (u_z, v_z)| \leq \frac{a - \delta}{2} \quad \text{on } (\rho, \rho + M); \tag{5.24}$$

where $z = u_x(\rho)$. Note that, since J is bounded, r_x can be chosen to be bounded, uniformly in $x \in \Lambda$. Thus, if R is large enough, we have (5.24), for every $x \in \Lambda$. Therefore, from (5.21), (5.22) and (5.24), we get

$$\rho_1 - \rho \leq M. \tag{5.25}$$

On the other hand, by (5.23), (5.24), (5.25) and (1.4)-(1.5) we have

$$(u_x, v_x) \text{ is bounded in } C^1([\rho, \rho_1], \mathbb{R}^2), \text{uniformly in } x \in \Lambda. \tag{5.26}$$

Let now σ be the area of the set $\{H(u,v) \le H(0,b), v \le -a\}$. It follows from (5.25) and (5.26), that if R is large enough, we have

$$\Big|\int_\rho^{\rho_1}\Big(1-\frac{\rho}{r}\Big)u_x(r)v_x'(r)\,dr\Big| \le \frac{\sigma}{2} \text{ for every } x\in\Lambda. \tag{5.27}$$

Next, observe that the curve $\cup_{r\in[\rho,\rho_1]}\{u_x(r),v_x(r)\}$ has no multiple points, since on (ρ,ρ_1) we have $(v_x^2-u_x^2)'>0$ while $u_x>0$, and $(v_x^2-u_x^2)'<0$ while $u_x<0$. This comes from (2.15), assuming $R>1/\omega$. Let Σ_x be the region contained between the curve and the line $\{v=-a\}$, and let δ_x be its area. By (5.19), we have $\Sigma_x \ni \{H(u,v)\le H(0,(m+\omega)^{1/2}, v\le -a\}$; and so

$$\delta x \ge \sigma. \tag{5.28}$$

Now, applying Green's formula, we have

$$\delta_x = -\int_\rho^{\rho_1} u_x(r)v_x'(r)\,dr. \tag{5.29}$$

Putting together (5.27), (5.28) and (5.29), we get

$$\int_\rho^{\rho_1}\frac{1}{r}u_x(r)v_x'(r)\,dr \le -\frac{\sigma}{2\rho}. \tag{5.30}$$

Integrating (2.14), between ρ and ρ_1, and using (1.5) and (5.30), we obtain the inequality

$$H_x(\rho_1)-H_x(\rho)\le -\sigma/\rho<0;$$

from which the result follows. ☐

Corollary 5.5. *Let $n\ge 0$ be such that $A_n\ne\emptyset$, and let $x_n=\sup A_n$. Assume that $x_n\in A_n$ and that $x_n\ge\sup I_n$. Then , there exists $\varepsilon>0$ such that $(x_n,x_n+\varepsilon)\subset I_{n+1}$.*

Proof. Let us set $(U,V)=(u_{x_n},v_{x_n})$ and let R be the last zero of U. Assume for example that $U>0$ on (R,∞). By Lemma 4.6, there exists $\tau>R$ such that

$$0<U(r)<V(r)\le\frac{a}{4} \text{ for } r\ge\tau. \tag{5.31}$$

Next, observe that for x close to x_n, we have the following. Both u_x and v_x have n zeroes on $(0,\tau)$, which are alternate, and

$$0<u_x(\tau)<v_x(\tau)\le\frac{a}{2}. \tag{5.32}$$

This follows from (2.10) and Lemma 4.6. Now, assume in addition that $x>x_n$. Let

$$\rho_x=\sup\{r\in(\tau,R_x), v_x>0 \text{ and } u_x>0 \text{ on } (\tau,r)\}.$$

Observe that, from (5.32) and (1.4)-(1.5) it follows easily that $u_x'<0$ and $v_x'<0$ on (τ,ρ_x), and so

$$0<u_x\le\frac{a}{2} \text{ and } 0<v_x\le\frac{a}{2} \text{ on } (\tau,\rho_x). \tag{5.33}$$

Next, $x\notin A_n$ by definition of x_n; and so, by Lemma 2.14, ρ_x must be finite. Therefore, $u_x(\rho_x)=0$ or $v_x(\rho_x)=0$. Furthermore, $x\notin I_n$, and so $u_x(\rho_x)\ne 0$. Thus $v_x(\rho_x)=0$. Note that, from (5.31) and (2.10), we have

$$\rho_x \underset{x\to x_n}{\longrightarrow} +\infty. \tag{5.34}$$

Finally, from (1.4)-(1.5) and (5.33), we get

$$(u_x + v_x)' \le -\frac{1}{2}(m-\omega)(u_x+v_x) \text{ on } (\tau, \rho_x). \tag{5.35}$$

Integrating (5.35), we obtain

$$u_x(\rho_x) \le \frac{a}{2} e^{\frac{m-\omega}{2}\tau} e^{-\frac{m-\omega}{2}\rho_x}. \tag{5.36}$$

Putting together (5.34), (5.36), and applying Lemmas 5.3 and 5.4, we obtain that if x is close enough to x_n, then $u_x v_x$ has at least two zeroes after ρ_x, and the first two zeroes are zeroes of u_x; and so $x \in I_{n+1}$. This completes the proof of Corollary 5.5. □

End of the proof of Theorem 1.2. Let us first show that $I_0 \neq \emptyset$. To see this, consider $x > a$ such that $F(x^2) < m + \omega$ and $H(0,x) < 0$ (such an x exists by definition of a and by Lemma 2.1 (ii)). By Lemma 2.19 we have $R_x = \infty$ and $H_x(r) \le H(0,x)$ for r≥ 0. By Lemma 2.2 (ii) we have then $v_x(r) \ge \alpha$ for $r \ge 0$, for some $\alpha > 0$. Applying Corollary 2.16 with $r_0 = 0$, it follows that we are in case (ii); and so $x \in I_0$. Consider now $y_0 = \sup I_0$. y_0 is finite by Proposition 4.1. By Lemma 5.2, $y_0 \in A_0$. Let now $x_0 = \sup A_0$. x_0 is also finite by Proposition 4.1. Applying Lemma 5.1, we get $x_0 \in A_0$. Therefore, by Corollary 5.5, there exists $\varepsilon_0 > 0$, such that $(x_0, x_0 + \varepsilon_0) \subset I_1$. Thus, $I_1 \neq \emptyset$. Let $y_1 = \sup I_1$. y_1 is finite by Proposition 4.1. We have $y_1 > x_0 \ge y_0$; and so, by Lemma 5.2, $y_1 \in A_1$; then by Lemma 5.1, $x_1 := \sup A_1 \in A_1$ (x_1 is also finite by Proposition 4.1). Iterating this argument, we construct an increasing sequence $(x_n)n \ge 0$, with $x_n \in A_n$. The exponential decay follows from Lemma 4.6. This proves Theorem 1.2 (i), (ii) and (iii). Finally, (iv) follows from Proposition 3.2 and Remark 3.1. □

6. Further results

Our goal in this section is to show how to modify the proof of Theorem 1.2 in order to weaken the assumptions on F, and to establish the following result.

Theorem 6.1. *Assume that (H1′) and (H2) hold. There exists an increasing sequence $(x_n)_{n\ge 0}$ of positive numbers with the following properties . For every $n \ge 0$,*

- (i) *the solution (u_n, v_n) of (1.4)-(1.6) with $v_n(0) = x_n$ is a global solution ,*
- (ii) *both u_n and v_n have exactly n zeroes on $(0, +\infty)$,*
- (iii) *(u_n, v_n) converges exponentially to $(0,0)$, as $r \to +\infty$.*
- (iv) *Furthermore, if $F(x) \ge \delta(\log x)^\beta$ for x large, where $\delta > 0$ and $\beta > 2$, the sequence $(x_n)_{n\ge 0}$ is bounded.*

Remark 6.2. It follows from Theorem 6.1 that the conclusions of Theorem 1.1 hold if we replace assumption (H1) by assumption (H1′).

The differences between (H1) and (H1′) are the following. Instead of assuming that $F(x) > m + \omega$ for x large, we assume only that $F(x) > m - \omega$ for x large, and we remove the assumption $F'(F^{-1}(m-\omega)) > 0$. However, note that F is increasing on $(0, \infty)$, so that $F'(F^{-1}(m-\omega)) \ge 0$. Therefore, with respect to Theorem 1.2, the only new situations are the cases where

$$F(x) < m + \omega \quad \text{for all } x \in \mathbb{R}, \tag{6.1}$$

and/or

$$F'(F^{-1}(m-\omega)) = 0. \tag{6.2}$$

We shall explain how to modify the proof of Theorem 1.2 when either (6.1) or (6.2) hold. The case where both (6.1) and (6.2) hold requires both series of modifications.

Modifications under assumption (6.1). The proof of Theorem 1.2 applies as it is to this case. However, it can be simplified a lot, due to the following reason. The set $\{F(v^2-u^2) \geq m+\omega\}$ is empty. In particular, it follows from (2.14) that H_x is always nonincreasing; and so for every $x \in \mathbb{R}$ we have $R_x = \infty$ and the solution (u_x, v_x) is uniformly bounded. Let us now indicate the main simplifications that arise in the proof (the rest of the proof can be leaved unchanged). In Section 2, the number b defined in (2.1) does not exist anymore. Furthermore, Lemma 2.1 (iv), Lemma 2.10, Proposition 2.11, Lemma 2.15, Corollary 2.16 (v) and Lemma 2.19 can be removed. Section 3 can be removed entierely, since F never satisfies (3.1). In Section 4, Lemma 4.3, one always has $\theta = 0$. The proof of Lemma 4.8 is trivial since H_x is always nonincreasing, and property (i) of Corollary 4.10 can be removed. Finally, step 2 of the proof of Proposition 4.1 is simplified since $\Delta_1 = \Delta_3 = \emptyset$. In Section 5, the proof of Lemma 5.4 is trivial since H_x is nonincreasing (we are formally in the case where $H(0,b) \geq 0$).

Modifications under assumption (6.2). In this case, the proof does not apply as it is and must be adapted. Indeed, the assumption $F'(F-1(m-\omega)) > 0$ was used in Lemma 2.8. Therefore, the conclusion of Lemma 2.8 may not hold anymore and we must admit the possibility of solutions converging to $(0, \pm a)$ in one of the half-planes $\{u > 0\}$ or $\{u < 0\}$. Lemma 2.8 was applied several times in the sequel; and so this induces the following modifications. In Section 2, Lemma 2.8 does not hold anymore and must be removed. Therefore, Proposition 2.12 must be replaced by the following result, whose proof is essentially the same.

Proposition 2.12′. (See Figure 14) Let $x \neq 0$. Assume that for some $r_0 > 0$, we have $v_x(r_0) \leq u_x(r_0)$ and $u_x(r_0) > 0$ (respectively , $u_x(r_0) \leq v_x(r_0)$ and $u_x(r_0) < 0$). Then one of the following properties hold.

(i) There exists $r_0 < r_1 < R_x$ such that $|u_x| > 0$ on (r_0, r_1), and $u_x(r_1) = 0$, $|v_x(r_1)| > a$. In addition, either $v_x(r_0) > 0$ (respectively , $v_x(r_0) < 0$), and then v_x has exactly one zero in (r_0, r_1), or else $v_x(r_0) \leq 0$ (respectively $v_x(r_0) \geq 0$), and then $|v_x| > 0$ on $(r_0, r_1]$;

(ii) $R_x = \infty$, $|u_x| > 0$ on (r_0, ∞), $u_x \to 0$ and $v_x \downarrow -a$ as $r \to \infty$ (respectively $v_x \uparrow a$ as $r \to \infty$).

This implies modifications of Corollary 2.13 and Lemma 2.14 as follows.

Corollary 2.13′. Let $x \neq 0$. Assume that, for some $0 < r_0 < R_x$, we have $v_x(r_0) = 0$. Then one of the following properties hold.

(i) There exists $r_1 \in (r_0, R_x)$ such that $|u_x| > 0$, $|v_x| > 0$ on (r_0, r_1), and $u_x(r_1) = 0$, $|v_x(r_1)| > a$.

(ii) $R_x = \infty$, $|u_x| > 0$ on (r_0, ∞), $u_x \to 0$ and $|v_x| \uparrow a$ as $r \to \infty$.

Lemma 2.14′. Let $x \neq 0$ be such that $R_x = \infty$. Assume that for some $r_0 > 0$, we have $|u_x| > 0$ on $[r_0, \infty)$. Then , one of the following properties hold.

(i) $u_x v_x > 0$ on (r_0, ∞), and there exists C such that $0 < |u_x(r)| < |v_x(r)| < Ce^{-(1/2)(m-\omega)r}$ for $r \in (r_0, \infty)$.

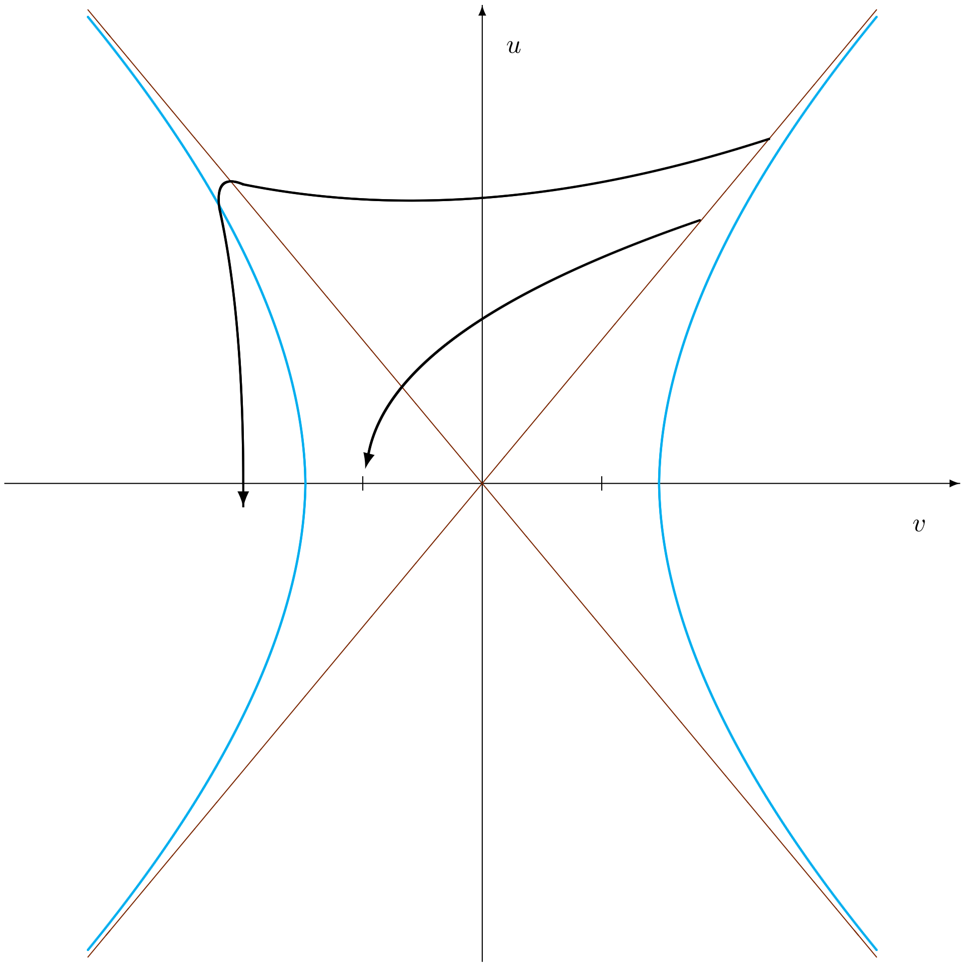

FIGURE 14. Illustration for Proposition 2.12′

(ii) $u_x \to 0$ and $v_x \to \pm a$ as $r \to \infty$.

The necessary changes in the proofs are obvious. Corollary 2.16 also has to be modified as follows (with little change in the proof).

Corollary 16′. (see Figures 8 and 15) Let $x \neq 0$, $x \neq \pm a$. Assume that for some $r_0 \in [0, R_x)$ we have $u_x(r_0) = 0$. Then, one of the following properties holds .

(i) $|v_x(r_0)| < a$, and there exists $r_1 \in (r_0, R_x)$ such that $u_x v_x < 0$ on (r_0, r_1), $0 < |u_x| < |v_x|$ on (r_0, r_1), $|v_x(r_1)| > a$, and $u_x(r_1) = 0$;
(ii) $|v_x(r_0)| > a$, and there exists $r_1 \in (r_0, R_x)$ such that $u_x v_x > 0$ on (r_0, r_1), $0 < |u_x| < |v_x|$ on (r_0, r_1), $|v_x(r_1)| < a$, and $u_x(r_1) = 0$;
(iii) $|v_x(r_0)| > a$, $R_x = \infty$, $u_x v_x > 0$ on (r_0, ∞), and $0 < |u_x| < |v_x| < C \exp(-(1/2)(m - \omega)r)$ on (r_0, ∞);
(iv) $|v_x(r_0)| > a$, and there exists $r_1 \in (r_0, R_x)$ such that $u_x v_x > 0$ on (r_0, r_1), and $v_x(r_1) = 0$;

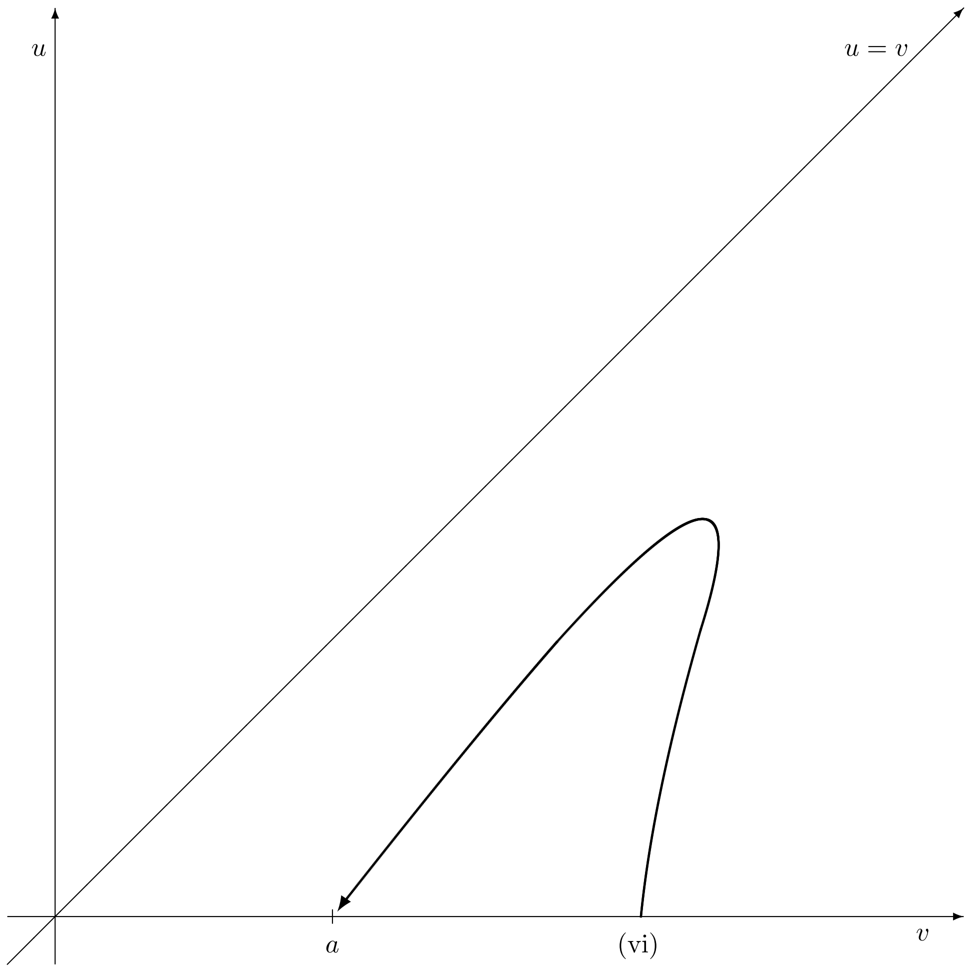

FIGURE 15. Illustration for Corollary 16′ (vi)

(v) $|v_x(r_0)| > a$, $R_x \leq 1/\omega$, $u_x v_x > 0$ on (r_0, R_x), and $F(v_x^2 - u_x^2) > m + \omega$ on (r_0, R_x).
(vi) $|v_x(r_0)| > a$, $R_x = \infty$, $u_x v_x > 0$ on (r_0, ∞), $u_x \to 0$ and $v_x \to a$ as $r \to \infty$.

Finally, the following result is easily obtained.

Lemma 2.18′. Let $x \neq 0$. Assume that $R_x \geq 1/\omega$, and that u_x has a finite number of zeroes . Then $R_x = \infty$, and either $|u_x| + |v_x| \to 0$ as $r \to \infty$, or else $u_x \to 0$ and $v_x \to \pm a$ as $r \to \infty$.

Section 3 in unchanged, since everything takes place in the set $\{F(v^2 - u^2) \geq m + \omega\}$, which does not contain the points $(0, \pm a)$.

Major changes occur in Section 4, concerning the definition of the sets I_n. One has to define

$$I_0 = J_0 \cup U_0,$$

where

$$J_0 = \{x > a; \exists r_{x,0} \in (0, R_x) \text{ s.t. } u_x > 0 \text{ and } v_x > 0 \text{ on } (0, r_{x,0}), \text{ and } u_x(r_{x,0}) = 0\},$$

and

$$U_0 = \{x > a; R_x = \infty, u_x > 0 \text{ and } v_x > 0 \text{ on } (0, R_x), \\ \text{and } u_x \to 0, v_x \to a \text{ as } r \to \infty\}.$$

For $x \in U_0$, we set $r_{x,0} = \infty$. For $n \geq 1$, one must define

$$I_n = J_n \cup U_n \cup O_n,$$

where (see Figure 16)

$$J_n = \{x > a; \exists r_{x,n} \in (0, R_x) \text{ s.t. } u_x \text{ and } v_x \\ \text{have exactly } n \text{ zeroes on } (0, r_{x,n}), \text{ and } u_x(r_{x,n}) = 0\},$$
$$U_n = \{x > a; R_x = \infty, \text{ both } u_x \text{ and } v_x \text{ have exactly} \\ n \text{ zeroes on } (0, R_x), \text{ and } u_x \to 0, u_x \to \pm a \text{ as } r \to \infty\},$$
$$O_n = \{x > a; R_x = \infty, v_x \text{ has exactly } n \text{ zeroes on } (0, R_x), u_x \text{ has exactly} \\ n-1 \text{ zeroes on } (0, R_x), \text{ and } u_x \to 0, v_x \to \pm a \text{ as } r \to \infty\}.$$

With these new definitions of the sets In , the rest of Section 4 is essentially unchanged. Only the proof of Lemma 4.2) is a little bit more delicate, but is still easy due to the fact that the points $(0, \pm a)$ are dynamically stable for system (1.4)-(1.5) (indeed, H_x is nonincreasing in a neighborhood of $(0, \pm a)$).

In Section 5, Lemmas 5.3 and 5.4 have to be modified in the following way, with an obvious adaptation of the proof.

Lemma 5.3′. For every $C > 0$, there exists $T > 0$ with the following property . Let $x \neq 0$ be such that $R_x \geq T$. Assume that for some $\rho \geq T$, we have $v_x(\rho) = 0$ and $|u_x(\rho)| \leq C \exp(-(1/2)(m - \omega)\rho)$. Then,

(i) either there exists $\theta \in (r, R_x)$ such that $|u_x| > 0$ and $|v_x| \leq a$ on (ρ, θ), $|v_x(\theta)| = a$, and $H_x(\theta) < 0$ (see Figure 12);
(ii) or else $R_x = \infty$, $|u_x| > 0$ and $|v_x| \leq a$ on (ρ, ∞), $u_x \to 0$ and $|v_x| \uparrow a$, as $r \to \infty$ (see Figure 17).

Lemma 5.4′. There exists R with the following property. Assume $x \neq 0$ is such that $R_x \geq R$. Assume further that for some $\rho \geq R$, we have $v_x(\rho) = -a$, $u_x(\rho) > 0$ (respectively $v_x(\rho) = a$, $u_x(\rho) < 0$), and $H_x(\rho) < 0$. Then,

(i) either there exists $\tau > \rho$ such that $|v_x| > 0$ on (ρ, τ) and such that u_x has two zeroes on (ρ, τ) (see Figure 13).
(ii) or else $R_x = \infty$, $|v_x| > 0$ on (ρ, ∞), u_x has one zero on (ρ, ∞) and $u_x \to 0$, $|v_x| \to a$ as $r \to \infty$(see Figure 18).

The rest of the proof is essentially unchanged.

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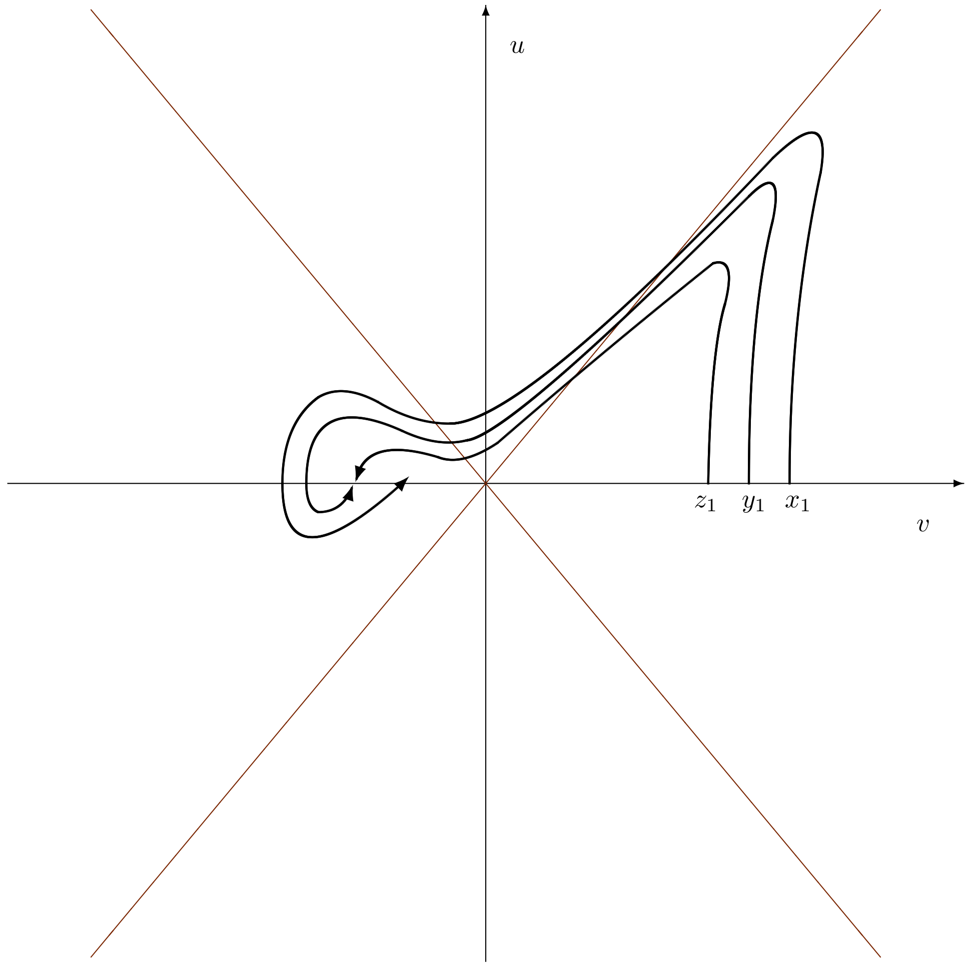

FIGURE 16. Trajectories with $x_1 \in J_1$, $y_1 \in U_1$ and $z_1 \in O_1$

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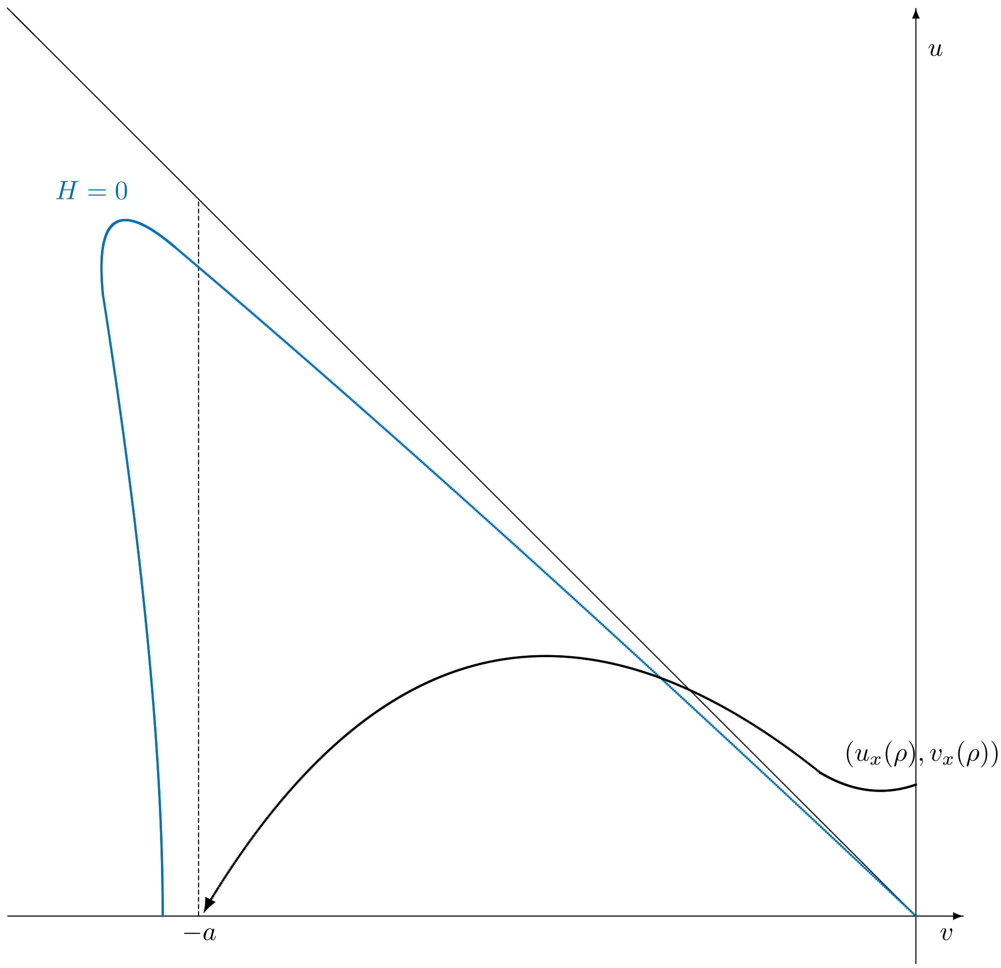

FIGURE 17. Illustration for Lemma 5.3′ (ii)

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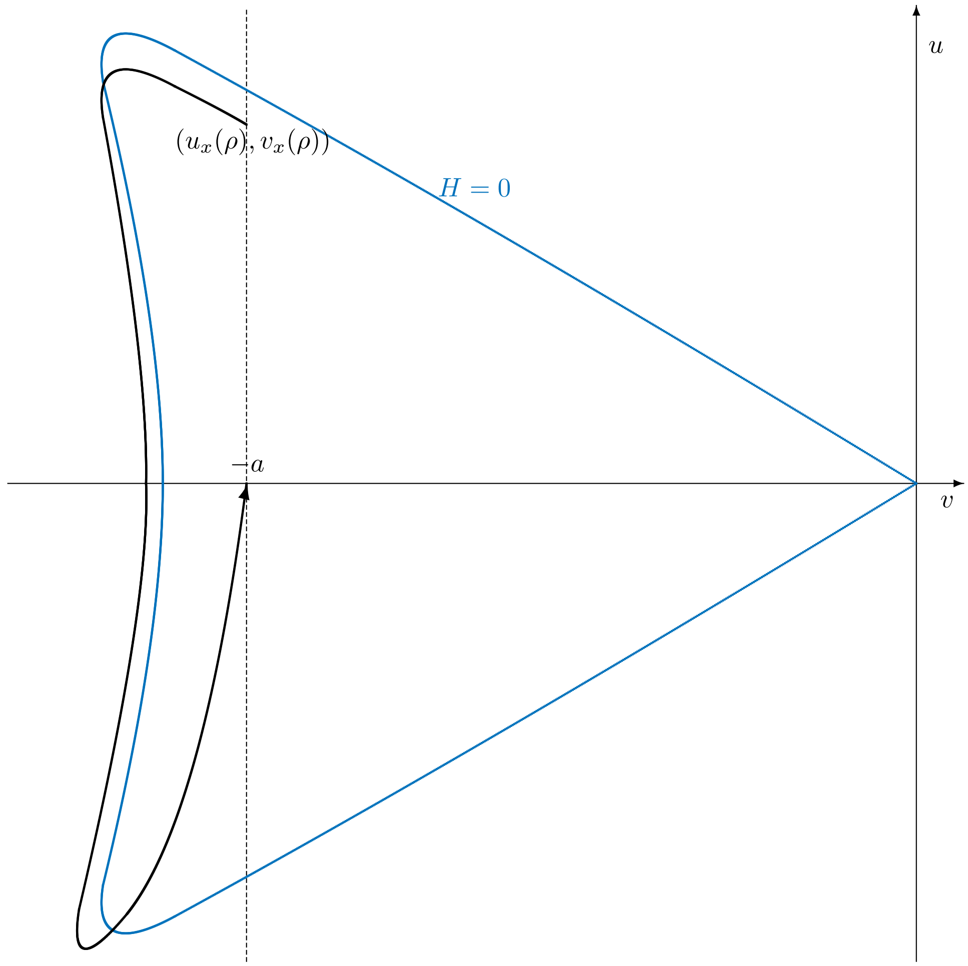

FIGURE 18. Illustration for Lemma 5.4′ (ii)

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