

INSTABILITY OF STATIONARY STATES IN NON-LINEAR SCHRÖDINGER  
AND KLEIN-GORDON EQUATIONS

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ABSTRACT

This paper is concerned with instability results for standing waves of the type  $\phi(t, \mathbf{x}) = e^{i\omega t} u(\mathbf{x})$  in the Schrödinger equation (S):  $i\phi_t - \Delta\phi = g(\phi)$  or the type  $\phi(t, \mathbf{x}) = u(\mathbf{x})$  in the Klein-Gordon equation (KG):  $\phi_{tt} - \Delta\phi + \omega\phi = g(\phi)$ . Here  $u(\mathbf{x})$  is a ground state solution of the nonlinear Euclidean scalar field equation (E):  $-\Delta u + \omega u = g(u)$  in  $\mathbb{R}^N$ ,  $u \in H^1(\mathbb{R}^N)$ ,  $u \neq 0$ . That is,  $u$  is a solution of (E) with minimal action. For instance, suppose  $g(\phi) = |\phi|^{p-1} \phi$  with  $1 + (4/N) \leq p < (N+2)/(N-2)$  in the case of (S) or  $1 < p < (N+2)/(N-2)$  in the case of (KG). Then, we show that there exist initial data arbitrarily close to the ground state for which the solutions of the evolution problems blow up in finite time. Other qualitative properties for these equations and blow-up results for a nonlinear heat equation are also included here.

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1. INTRODUCTION

Consider the nonlinear Schrödinger equation:

$$(S) \quad i\phi_t - \Delta\phi = g(\phi)$$

and the non-linear Klein-Gordon equation

$$(KG) \quad \phi_{tt} - \Delta\phi + \omega\phi = g(\phi)$$

where  $\phi = \phi(t, x): \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{C}$ . These equations are classical non-linear models in quantum mechanics and in field theory. They also arise in a variety of contexts in physics. For instance, when  $N = 2$ , the nonlinear Schrödinger equation (S) describes the propagation of non-linear laser beams. (This last aspect of (S) and the relevance of the results obtained here will be somewhat discussed in the Appendix. See also references [17, 18, 34, 35, 38, 40, 41].)

It will be assumed throughout this paper that

$$\omega > 0 \quad \text{and} \quad g(z) = f(|z|^2)z, \quad \forall z \in \mathbb{C}$$

with  $f: \mathbb{R}_+ \rightarrow \mathbb{R}$  being a continuous function. We denote

$$G(z) = \int_0^{|z|} g(s) ds \quad \forall z \in \mathbb{C};$$

Note that  $G(z) = \frac{1}{2}F(|z|^2)$  where  $F(t) = \int_0^t f(s)ds$  and that  $G$  is real valued.

Associated with the above equations is the elliptic semi-linear problem ("nonlinear Euclidean scalar field equation"):

$$(E) \quad -\Delta u + \omega u = g(u), \quad u \in H^1(\mathbb{R}^N), \quad u \neq 0.$$

If  $u = u(x): \mathbb{R}^N \rightarrow \mathbb{R}$  is a solution of (E) (thus,  $u$  is non-trivial), then  $\phi(t, x) = e^{-i\omega t}u(x)$  is a "standing wave" solution of (S). Note that  $u(x)$  is also a stationary solution of (KG). These solutions are solitary waves.

From the physical viewpoint, an important role is played by the ground state solution(s) of (E). We recall that a solution  $u$  of (E) is termed a ground state if it has minimum action among all solutions of (E). That is, if  $u$  satisfies

$$(1.1) \quad S(u) \leq S(v) \quad \forall v \text{ solution of (E),}$$

where the action  $S$  is defined by

$$S(w) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla w|^2 dx + \frac{\omega}{2} \int_{\mathbb{R}^N} w^2 dx - \int_{\mathbb{R}^N} G(w) dx.$$

The existence and characterization of ground state solutions, in the case  $N \geq 3$ , was established by S. Coleman, V. Glazer and A. Martin [11] and was detailed by H. Berestycki and P.L. Lions [4,5], under quite general hypotheses on the non-linearity  $g$ . The existence of infinitely many solutions (bound states) was shown in H. Berestycki and P.L. Lions [4,5]. The analogous results for the dimension  $N = 2$  (which require a separate treatment) are obtained in H. Berestycki, T. Gallouet and O. Kavian [3]. Some of these results will be recalled later on.

In this paper, we derive instability properties for the standing waves of (S) or (KG) associated with a ground state of (E). Under certain assumptions on  $g$ , we indeed show that there exist initial data which are arbitrarily close to the ground state and such that the corresponding solution of (S) or (KG) blow up in finite time.

In order to illustrate clearly our methods and the type of results that are shown here, we will first consider in the next section the model case of a power non-linearity :

$$(1.2) \quad g(u) = \alpha |u|^{p-1} u$$

where  $\alpha > 0$  is a constant and  $1 < p$ . In section 3 we will prove the instability of the ground state for (S) in this special case, when  $1 + (4/N) < p < (N+2)/(N-2)$ . The precise statement for the general case is given and derived in section 4. Instability results (in the above strong sense of finite time blow up) and proved for the equation (KG) in section 5. Lastly, in section 6 we briefly consider a nonlinear heat equation. Further comments and some open problems are also indicated there.

Most of the results presented in this paper have been announced together with the summaries of the proofs in our Note [2].

## 2. MAIN RESULTS IN THE CASE OF A POWER NON-LINEARITY

When  $g$  is given by (1.2), equation (E) reads

$$(E)^* \quad -\Delta u + \omega u = \alpha |u|^{p-1} u, \quad u \in H^1(\mathbb{R}^N), \quad u \neq 0.$$

(E)\* is known to have a ground state solution if and only if

$$(2.1) \quad 1 < p < (N+2)/(N-2).$$

(Recall that  $\omega > 0$ ,  $\alpha > 0$ .) This ground state  $u$  is positive<sup>(1)</sup>, spherically symmetric, decreasing with  $|x|$  and has exponential fall-off at infinity.

Furthermore, by a recent result of K. McLeod and J. Serrin [23,24] this ground state is unique (up to shifts of the origin) when  $p < N/(N-2)$ .

(The case  $p \geq N/(N-2)$  is open. More general uniqueness results are proved in [23,24].) In this model case, equations (S) and (KG) take the form

$$(S)^* \quad i\phi_t - \Delta\phi = \alpha |\phi|^{p-1} \phi$$

$$(KG)^* \quad \phi_{tt} - \Delta\phi + \omega\phi = \alpha |\phi|^{p-1} \phi$$

When specialized to this case, our main results are the following:

Theorem 2.1. Suppose that  $\omega > 0$  and  $1 + (4/N) \leq p < (N+2)/(N-2)$ . Let  $u$  be a ground state solution of (E)\*. Then, for any  $\epsilon > 0$  there exists  $\phi_0 \in H^1(\mathbb{R}^N)$  with  $\|\phi_0 - u\|_{H^1(\mathbb{R}^N)} < \epsilon$  and such that the solution  $\phi(t,x)$

<sup>(1)</sup> Modulo multiplication by a constant complex number of modulus one.

of the Cauchy problem for (S)\* corresponding to the initial datum  
 $\phi(0, \cdot) = \phi_0$  has the following property. There exists a finite time  
 $T < +\infty$  such that  $\phi$  is defined for  $0 \leq t < T$ ,

$$\phi \in C([0, T], H^1(\mathbb{R}^N))$$

and

$$\lim_{t \nearrow T} \|\phi(t, \cdot)\|_{H^1(\mathbb{R}^N)} = +\infty .$$

Theorem 2.2. Suppose that  $\omega > 0$  and  $1 < p < (N+2)/(N-2)$ . Let  $u$  be a  
ground state solution of (E)\*. Then, for any  $\epsilon > 0$ , there exists  
 $\phi_0 \in H^1(\mathbb{R}^N)$  with  $\|\phi_0 - u\|_{H^1(\mathbb{R}^N)} < \epsilon$  and such that the solution of the  
Cauchy problem for (KG)\* corresponding to the initial data  $\phi(0, \cdot) = \phi_0$ ,  
 $\phi_t(0, \cdot) = 0$  has the following property. There exists a maximum time  
 $T < +\infty$  such that  $\phi$  is defined for  $0 < t \leq T$ ,

$$\phi \in C([0, T], H^1(\mathbb{R}^N))$$

and

$$\lim_{t \nearrow T} \|\phi(t, \cdot)\|_{H^1(\mathbb{R}^N)} = +\infty .$$

The proof of Theorem 2.1 will be detailed in the next section.

Actually, the same argument allows us to show more generally that  $\phi(t, x) = e^{i\omega t} u(x)$ , where  $u$  is a ground state of (E)\*, is an unstable stationary state (in the sense of Theorem 2.1) with respect to the equation

$$(2.2) \quad \pm i\phi_t - \Delta\phi + a\phi = \alpha |\phi|^{p-1} \phi$$



where  $a \mp \mu = \omega$ , and  $p$  is in the same range as above.

The result of Theorem 2.1 is sharp. Indeed, it is known that when  $1 < p < 1 + (4/N)$ , then there is global existence and uniform boundedness in  $H^1$  for the Cauchy problem associated with  $(S)^*$ , whatever the initial datum is (compare J. Ginibre and G. Velo [13], W. Strauss [30]). In fact, it has been shown recently by T. Cazenave and P. L. Lions [10] that in the case  $1 < p < 1 + (4/N)$ , the solution  $e^{-i\omega t} u(x)$  associated with a ground state  $u$  of  $(E)^*$  is, on the contrary, orbitally stable.

Remark 2.3. Many works in the literature deal with finite time blowing-up of solutions to the Cauchy problem associated with  $(S)$  or  $(KG)$ . (Compare J. M. Ball [1], H. Fujita [12], R. T. Glassey [14,15], H. A. Levine [19,20,21], L. E. Payne and D. H. Sattinger [26], M. Tsutsumi [36,37].) To our knowledge, however, all of these works that concern  $(S)$  or  $(KG)$  impose a restriction on the initial data which is not fulfilled in our context. For instance, and more precisely, R. T. Glassey [14] shows that the solution  $\phi(t,x)$  of  $(S)^*$ , corresponding to  $\phi(0,x) = \phi_0$  blows up in finite time under the hypothesis that

$$(2.3) \quad 1 + (4/N) < p < (N+2)/(N-2)$$

$$(2.4) \quad E(\phi) \leq 0 ,$$

where

$$(2.5) \quad E(\phi) \equiv \int_{\mathbb{R}^N} \left\{ \frac{1}{2} |\nabla \phi|^2 - \frac{\alpha}{p+1} |\phi|^{p-1} \right\} dx .$$

Now, in our case, if  $u$  is a ground state of  $(E)^*$ , an easy computation relying on Pohžaev's identity (see section 3) shows that

$$(2.6) \quad E(u) = \left( \frac{1}{2} - \frac{2}{N(p-1)} \right) \int_{\mathbb{R}^N} |\nabla u|^2 dx$$

Therefore, since  $p > 1 + (4/N)$ , we have  $E(u) > 0$ . By taking an initial datum  $\phi_0$  close to  $u$  in  $H^1$  norm, one also has  $E(\phi) > 0$  and (2.4) is contradicted. The present work thus seems to be the first to address the instability of the standing wave solution associated with a ground state of  $(E)$ .<sup>(1)</sup> □

Remark 2.4. When one thinks of  $(S)$  or  $(S)^*$  as a model for non-linear laser propagation, the solutions of the form  $e^{i\omega t}u(x)$  are of special interest. They represent <<self-trapped beams>>. In this framework, the blowing-up solution of Theorem 2.1 (or of its more general version in section 4) produce so called <<self focusing beams>>. (See the Appendix.) □

Remark 2.5. It will be seen in section 6 that a ground state solution  $u$  of  $(E)^*$  is also unstable (in the sense of finite-time blow-up for arbitrarily near initial data) with respect to a nonlinear heat equation:

$$(H)^* \quad \frac{\partial \psi}{\partial \tau} - \Delta \psi + \omega \psi = g(\psi) . \quad \square$$

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<sup>(1)</sup> Independently, M. Weinstein [39] has proved a particular case of Theorem 1.1. It concerns  $(S)^*$  (not  $(S)$ ) and the case  $p = 1 + (4/N)$ . Observe that in this case  $E(u) = 0$ . A more detailed description is achieved in [39] for this case.

### 3. INSTABILITY OF THE GROUND STATE IN THE MODEL CASE

The purpose of this section is to prove Theorem 2.1. Although a more general result is derived in section 6, we single out the power case as we hope that the argument is more transparent there.

Throughout this section it will be assumed that  $\omega > 0$  and that

$$(3.1) \quad 1 + (4/N) < p < (N+2)/(N-2)$$

with the understanding that in dimensions  $N = 1, 2$  this condition reduces to  $1 + (4/N) < p$ .<sup>(1)</sup> For  $p$  in this range and  $u \in H^1(\mathbb{R}^N)$  the following quantities are well defined:

$$E(u) = \frac{1}{2} \int |\nabla u|^2 - \frac{\alpha}{p+1} \int |u|^{p+1}$$

$$S(u) = E(u) + \frac{\omega}{2} \int |u|^2$$

$$Q(u) = \int |\nabla u|^2 - \frac{N}{2} \frac{p-1}{p+1} \alpha \int |u|^{p+1} .$$

Here and thereafter, unless otherwise specified, all integrals are over  $\mathbb{R}^N$  and the Lebesgue measure  $dx$  is understood. We recall that a ground state of

$$(E)^* \quad \begin{cases} -\Delta u + \omega u = \alpha |u|^{p-1} u & \text{in } \mathbb{R}^N \\ u \in H^1(\mathbb{R}^N), \quad u \neq 0, \end{cases}$$

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<sup>(1)</sup>The limiting case  $p = 1 + (4/N)$  is considered in a remark at the end of this section (remark 3.6).

is defined to be a solution of  $(E)^*$  with the property that

$$S(u) \leq S(v) \quad \forall v \text{ solution of } (E)^*.$$

The next result provides a variational characterization of the ground state solutions to  $(E)^*$  that will be useful. (Incidentally, it also yields an alternate proof of the existence of a ground state.)

Proposition 3.1. Let  $M = \{u \in H^1(\mathbb{R}^N) \setminus \{0\}, Q(u) = 0\}$ . For  $p < (N+2)/(N-2)$ , any solution  $u$  of  $(E)^*$  belongs to  $M$ . Under condition (3.1), there exists  $u \in M$  such that

$$(3.2) \quad S(u) = \min_{v \in M} S(v).$$

Such a  $u$  is a ground state solution of  $(E)^*$ . Moreover, any ground state solution of  $(E)^*$  is a solution of the minimization problem (3.2).

The proof of this proposition is divided into several steps.

Step 1: Solutions of  $(E)^*$  belong to  $M$ . Let  $v$  be a solution of  $(E)^*$ .

Then  $v$  verifies

$$(3.3) \quad \int |\nabla v|^2 + \omega \int |v|^2 = \alpha \int |v|^{p+1},$$

$$(3.4) \quad \frac{N-2}{N} \int |\nabla v|^2 + \omega \int |v|^2 = \frac{2\alpha}{p+1} \int |v|^{p+1}.$$

(3.3) obtains multiplying the equation by  $v$  and integrating, while (3.4) is simply the "virial Theorem". It is also often referred to as the Pohožaev

identity (see e.g. [4]). Subtracting (3.4) from (3.3) yields  $Q(v) = 0$ , that is  $v \in M$ .

Step 2: S is bounded below on M. If  $v \in M$ , then

$$(3.5) \quad S(v) = \left( \frac{1}{2} - \frac{2}{N(p-1)} \right) \int |\nabla v|^2 + \frac{\omega}{2} \int |v|^2.$$

Therefore,  $S(v) > 0 \forall v \in M$ , for  $1 + (4/N) < p$ .

Step 3: A technical lemma.

Lemma 3.2. For  $v \in H^1(\mathbb{R}^N)$ ,  $v \neq 0$  and  $\lambda > 0$ , let  $v_\lambda(x) = \lambda^{N/2} v(\lambda x)$ .

There exists a unique  $\mu > 0$  (depending on  $v$ ) such that  $Q(v_\mu) = 0$ .

Then,  $Q(v_\lambda) > 0$  for  $\lambda \in (0, \mu)$  and  $Q(v_\lambda) < 0$  for  $\lambda > \mu$ . Furthermore,

$S(v_\mu) \geq S(v_\lambda)$ ,  $\forall \lambda > 0$ .

Proof. It just suffices to write down the expressions of  $Q(v_\lambda)$  and  $S(v_\lambda)$ :

$$Q(v_\lambda) = \lambda^2 \int |\nabla v|^2 - \frac{N}{2} \frac{p-1}{p+1} \alpha \lambda^{\frac{N}{2}(p-1)} \int |v|^{p+1}$$

$$S(v_\lambda) = \frac{\lambda^2}{2} \int |\nabla v|^2 + \frac{\omega}{2} \int |v|^2 - \frac{\alpha}{p+1} \lambda^{\frac{N}{2}(p-1)} \int |v|^{p+1}.$$

The lemma follows from the observations that  $\frac{N}{2}(p-1) > 2$  and that

$$\frac{d}{d\lambda} S(v_\lambda) = \lambda^{-1} Q(v_\lambda).$$

Step 4: Selection of a minimizing sequence. Let  $(v_n) \subset M$  be a minimizing sequence for (3.4):

$$(3.6) \quad S(v_n) \leq \inf_{v \in M} S(v) \geq 0 .$$

Let  $v^*$  denote the Schwarz spherical rearrangement of a function  $|v|$ . We recall that  $v^*$  is the spherically symmetric nonincreasing (with respect to  $|x|$ ) function having the same distribution function as  $|v|$ . The symmetrization has the following properties:

$$(3.7) \quad \int |\nabla v^*|^2 \leq \int |\nabla v|^2$$

$$(3.8) \quad \int G(v^*) = \int G(|v|) .$$

For any function  $G: \mathbb{R} \rightarrow \mathbb{R}$ . Furthermore, it is straightforward to check that

$$(3.9) \quad (v_\lambda)^* = (v^*)_\lambda$$

where, as in Lemma 3.2,  $w_\lambda(x) = \lambda^{N/2} w(\lambda x)$ .

Now, for the minimizing sequence  $(v_n)$ , we let  $u_n = (v_n^*)_{\mu_n}$ , where  $\mu_n > 0$  is uniquely determined by  $Q(u_n) = Q((v_n^*)_{\mu_n}) = 0$ , (Lemma 3.2). In view of (3.9) one also has  $u_n = [(v_n)_{\mu_n}]^*$  and therefore, by (3.7), (3.8):

$$(3.10) \quad S(u_n) \leq S((v_n)_{\mu_n}) \leq S(v_n) .$$

The right hand side inequality in (3.10) is a consequence of Lemma 3.2.

Indeed,  $Q(v_n) = 0$  implies  $S((v_n)_\lambda) \leq S(v_n) \quad \forall \lambda > 0$ .

Thus,  $u_n$  verifies  $Q(u_n) = 0$ , i.e.  $u_n \in M$  and  $S(u_n) \leq S(v_n)$ . Therefore,  $(u_n)$  itself is a minimizing sequence for (3.2).

Step 5: Passage to the limit. From (3.5) one knows that  $\|u_n\|_{H^1}$  is bounded. Since  $(u_n)$  is a sequence of spherically symmetric nonincreasing functions, it is compact in  $L^{p+1}(\mathbb{R}^N)$  by a compactness lemma in W. Strauss [32] (see also [4]). There is a subsequence of  $(u_n)$  which we relabel  $(u_n)$  such that

$$(3.11) \quad \begin{aligned} u_n &\longrightarrow u_\infty && \text{weakly in } H^1(\mathbb{R}^N) \\ u_n &\longrightarrow u_\infty && \text{a.e. in } \mathbb{R}^N \\ u_n &\longrightarrow u_\infty && \text{strongly in } L^{p+1}(\mathbb{R}^N) . \end{aligned}$$

By the Gagliardo-Nirenberg inequality,  $u_n$  verifies

$$(3.12) \quad \|u_n\|_{L^{p+1}} \leq C \|\nabla u_n\|_{L^2}^\theta \|u_n\|_{L^2}^{1-\theta}$$

where  $\frac{1}{p+1} = \frac{\theta}{2^*} + \frac{1-\theta}{2}$  with  $2^* = 2N(N-2)^{-1}$  if  $N \geq 3$  and  $2^*$  arbitrarily large if  $N = 1, 2$ . Here and thereafter  $C > 0$  denotes various positive constants. Since  $\|u_n\|_{L^2} \leq C$ , (3.12) implies

$$(3.13) \quad \int |u_n|^{p+1} \leq C \left( \int |\nabla u_n|^2 \right)^\rho$$

with  $\rho = (p-1)\frac{N}{4}$  if  $N \geq 3$  and  $\rho$  arbitrarily close to  $\frac{p-1}{2}$  if  $N = 1, 2$ . Thus, in all cases,  $\rho > 2$ . Since  $Q(u_n) = 0$ , (2.13) yields

$$(3.14) \quad \int |\nabla u_n|^2 \leq C \left( \int |\nabla u_n|^2 \right)^\rho$$

which implies that  $\|\nabla u_n\|_{L^2}$  is bounded away from 0.

We claim that this shows  $u_\infty \neq 0$ . Indeed, arguing by contradiction if  $u_\infty = 0$ , then  $u_n \rightarrow 0$  strongly in  $L^{p+1}$  and from  $Q(u_n) = 0$  one would then derive

$$\lim_{n \rightarrow +\infty} \|\nabla u_n\|_{L^2} = 0$$

where is a contradiction to (3.14).

Hence,  $u_\infty \neq 0$ . Now, let  $u$  be defined by  $u = (u_\infty)_\mu$  with  $\mu > 0$  uniquely determined from the condition  $Q(u) = Q[(u_\infty)_\mu] = 0$ . Rewriting (3.11), one gets

$$(3.15) \quad \begin{cases} (u_n)_\mu \longrightarrow u & \text{strongly in } L^{p+1} \\ (u_n)_\mu \longrightarrow u & \text{weakly in } H^1. \end{cases}$$

Since  $Q(u_n) = 0$ , Lemma 3.2 shows that

$$(3.16) \quad S((u_n)_\mu) \leq S(u_n).$$

Hence, using (3.7), (3.8) we have

$$(3.17) \quad S(u) \leq \liminf_{n \rightarrow +\infty} S[(u_n)_\mu] \leq \lim_{n \rightarrow +\infty} S(u_n) = \inf_M S$$

since  $u \neq 0$  and  $Q(u) = 0$ , that is  $u \in M$ , we conclude that  $u$  solves the minimization problem (3.2).



Step 6:  $u$  is a solution of (E)\*. Let us first observe that  $S$  and  $Q$  are  $C^1$  functionals in  $H^1(\mathbb{R}^N)$  (see e.g. [4]). On  $M$ ,  $Q'^{(1)}$  does not vanish for

$$(3.18) \quad \langle Q'(v), v \rangle < 0, \quad \forall v \in M.$$

Indeed,  $\langle Q'(v), v \rangle = 2 \int |\nabla v|^2 - \frac{N}{2}(p-1) \int |v|^{p+1} = [2 - (p+1)] \int |\nabla v|^2 < 0 \quad \forall v \in M.$

Therefore,  $u$  being a solution of (3.2), there exists a Lagrange multiplier  $\Lambda$  such that

$$(3.19) \quad S'(u) = \Lambda Q'(u) .$$

We claim that  $\Lambda = 0$ .

To prove this, define a function  $u^\sigma$  by

$$(3.20) \quad u^\sigma(x) = \sigma^{-\frac{2}{p-1}} u(x/\sigma)$$

It is straightforward to show that

$$(3.21) \quad Q(u^\sigma) = \sigma^{N-2-4(p-1)^{-1}} Q(u)$$

Since  $u \in M$ , (3.21) shows that  $u^\sigma \in M$ ,  $\forall \sigma > 0$ . The function  $\sigma \mapsto S(u^\sigma)$  therefore achieves a minimum at  $\sigma = 1$ . Hence,

$$(3.22) \quad \frac{d}{d\sigma} S(u^\sigma) \Big|_{\sigma=1} = 0 .$$

Now, one has the following expression for  $S(u^\sigma)$ :

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(<sup>1</sup>)  $Q'$  denotes the gradient of a functional  $Q$ .

$$S(u^\sigma) = \frac{1}{2} \sigma^{N-2-4(p-1)^{-1}} \int |\nabla u|^2 + \frac{\omega}{2} \sigma^{N-4(p-1)^{-1}} \int |u|^2 - \\ - \frac{\alpha}{p+1} \sigma^{N-2(p+1)(p-1)^{-1}} \int |u|^{p+1} .$$

Using the fact that  $Q(u) = 0$ , one immediately obtains

$$(3.23) \quad \left. \frac{d}{d\sigma} S(u^\sigma) \right|_{\sigma=1} = \beta \langle S'(u), u \rangle$$

with  $\beta = N - \frac{4}{p-1}$ . Since  $p > 1 + 4/N$ , we have  $\beta > 0$  and (3.22), (3.23) yield

$$(3.24) \quad \langle S'(u), u \rangle = 0 .$$

Combining (3.18), (3.19) and (3.24) obviously implies  $\Lambda = 0$ , whence

$$(3.25) \quad S'(u) = 0 .$$

That is,  $u$  is a solution of  $(E)^*$  as  $(E)^*$  is the Euler-Lagrange equation for the functional  $S$ . Note that  $u$  is in fact a classical solution of  $(E)^*$  (and even a  $C^\infty$  solution since  $u$  is positive, see [4]).

Step 7: Conclusion. We have seen that any solution  $v$  of  $(E)^*$  lies on the manifold  $M$ . Therefore, one has

$$S(u) \leq S(v) .$$

We have therefore shown that  $u$  is a ground state solution of  $(E)^*$  and

furthermore, that (3.2) completely characterizes this type of solutions.

The proof of Proposition 3.1 is thereby complete. □

We now turn to the proof of Theorem 2.1. Consider the Cauchy problem for (S)\*:

$$(3.26) \quad \begin{cases} i\phi_t - \Delta\phi = \alpha |\phi|^{p-1} \phi \\ \phi(0, \cdot) = \phi_0 \end{cases}$$

It is known by a result of J. Ginibre and G. Velo [13] that for any  $\phi_0 \in H^1(\mathbb{R}^N)$ , there exists a unique solution  $\phi$  of (3.26) defined on a maximal time interval  $[0, T_{\max}(\phi_0))$ . For simplicity, we denote  $T = T_{\max}(\phi_0)$ . Moreover,  $\phi \in C([0, T), H^1(\mathbb{R}^N))$  and either  $T = +\infty$  or  $T < +\infty$  and  $\lim_{t \uparrow T} \|\phi(t, \cdot)\|_{H^1} = +\infty$ . It is known that for all  $t \in [0, T)$ ,  $\phi(t) = \phi(t, \cdot)$  verifies the conservation laws

$$(3.27) \quad \int |\phi(t)|^2 = \int |\phi_0|^2$$

$$(3.28) \quad S(\phi(t)) = S(\phi_0).$$

Lastly, if the function  $x \mapsto |x| \phi_0(x)$  is in  $L^2(\mathbb{R}^N)$ , then  $|\cdot| \phi(t, \cdot) \in L^2(\mathbb{R}^N)$  and

$$(3.29) \quad \frac{d^2}{dt^2} \left\| |\cdot| \phi(t, \cdot) \right\|_{L^2(\mathbb{R}^N)}^2 = 8 Q[\phi(t)]$$

for all  $t \in [0, T)$ . (See R. Glassey [14]; compare also H. Brezis [8] for a rigorous and simple derivation of (3.29).)

We will derive Theorem 2.1 from the next result.

Proposition 3.3. Let  $h = \min_M S > 0$  and define

$$K = \{u \in H^1(\mathbb{R}^N), Q(u) < 0, S(u) < h\} .$$

Then, for any  $\phi_0 \in K$ , one has  $\phi(t) \in K, \forall t \in [0, T)$ . Furthermore, if  
 $|\cdot| \phi_0(\cdot) \in L^2(\mathbb{R}^N)$ , then  $T$  is finite and

$$\lim_{t \nearrow T} \|\phi(t)\|_{H^1(\mathbb{R}^N)} = +\infty .$$

Before proving this proposition, let us derive Theorem 2.1. Let  $u$  be a ground state of  $(E)^*$ . Then, by Lemma 3.2, the function  $\phi_0(x) = \lambda^{N/2} u(\lambda x)$  verifies  $Q(\phi_0) < 0, S(\phi_0) < S(u) = h$ , for all  $\lambda > 1$ . Furthermore, since  $u$  has an exponential fall-off at infinity (see e.g. W. Strauss [32] or H. Berestycki and P. L. Lions [4]), it is clear that  $|\cdot| \phi_0(\cdot) \in L^2(\mathbb{R}^N)$ . As  $\lambda \searrow 1$ ,  $\|\phi_0 - u\|_{H^1}$  can be made arbitrarily small. By Proposition 3.3, the corresponding solution of the Cauchy problem (3.26) blows up (in  $H^1$  norm) in finite time.

The proof of Theorem 2.1 is thereby complete upon proving the proposition.

Proof of Proposition 3.3. Let  $\phi_0 \in K$ . Since  $S(\phi(t)) = S(\phi_0) < h$ , it is clear that  $Q(\phi(t)) < 0$ . For if not, by continuity there would exist a  $\bar{t} > 0$  such that  $Q[\phi(\bar{t})] = 0$ , that is  $\phi(\bar{t}) \in M$ . This is impossible for

$S(\phi(\bar{t})) < h$  and  $h = \min_M S$ . Thus,  $K$  is invariant under the flow generated by (3.26).

Actually, one disposes of a sharper upper estimate on  $Q(t)$ . We first require a Lemma which complements Lemma 3.2.

Lemma 3.4. With the notations of Lemma 3.2, let  $v \in H^1(\mathbb{R}^N)$ ,  $v \neq 0$  and  $\mu > 0$  such that  $Q(v_\mu) = 0$ . Suppose that  $\mu < 1$ . Then,

$$(3.30) \quad S(v) - S(v_\mu) \geq \frac{1}{2} Q(v) .$$

Proof of Lemma 3.4. Comparing with the proof of Lemma 3.2 above, we know that

$$Q(v_\lambda) = a \lambda^2 - b \lambda^{2+v}$$

$$S(v_\lambda) = a \frac{\lambda^2}{2} - \frac{b}{2+v} \lambda^{2+v} + c$$

with  $a, b, c > 0$ ,  $2+v = \frac{N}{2}(p-1) > 2$ , ( $v > 0$ ).  $Q(v_\mu) = 0$  thus means

$$(3.31) \quad a \mu^2 = b \mu^{2+v} .$$

Observe that  $v = v_1$  and  $Q(v) = a-b$ . Using (3.31) it is straightforward to show that

$$(3.32) \quad S(v) - S(v_\mu) = \frac{1}{2} Q(v) + \frac{v}{2(2+v)} b(1 - \mu^{2+v})$$

and the Lemma follows from (3.32) since  $\mu < 1$ . □

Proof of Proposition 3.3 continued. For fixed  $t \in [0, T)$  let  $\mu > 0$  be defined by  $Q[(\phi(t))_\mu] = 0$ . Since  $Q[\phi(t)] < 0$ , we know from Lemma 3.2 that  $\mu < 1$ . Since  $S((\phi(t))_\mu) \geq h$  and  $S(\phi(t)) = S(\phi_0)$ , we have by (3.30):

$$(3.33) \quad Q(\phi(t)) \leq S(\phi_0) - h < 0.$$

Let  $\delta = h - S(\phi_0)$ .  $\delta$  is a fixed positive number. By (3.29), (3.33), we derive

$$(3.34) \quad \frac{d^2}{dt^2} \left\| |\cdot| \phi(t, \cdot) \right\|_{L^2(\mathbb{R}^N)}^2 \leq -8\delta < 0.$$

Obviously,  $\left\| |\cdot| \phi(t, \cdot) \right\|_{L^2(\mathbb{R}^N)}^2$  being a positive function cannot verify (3.34) for all time. Therefore, it must be the case that  $T < +\infty$  which implies

$$\lim_{t \nearrow T} \left\| \phi(t, \cdot) \right\|_{H^1} = +\infty.$$

The proofs of Proposition 3.3 and of Theorem 2.1 are thereby complete.



Remark 3.5. If one defines  $\hat{K} = \{u \in H^1(\mathbb{R}^N); S(u) < h, Q(u) \geq 0\}$ , then by the same continuity argument as above,  $\hat{K}$  is invariant under the flow generated by (3.26).

Remark 3.6. The finite time blow up result of Theorem 2.1 still holds in the limiting case  $p = 1 + (4/N)$ . Indeed, as we pointed out earlier, when  $p = 1 + (4/N)$ , any solution  $u$  of  $(E)^*$  verifies  $E(u) = 0$ . Then, it is obvious that  $E(\lambda u) < 0$  for any  $\lambda > 1$ . Therefore, by R. Glassey's result [14], the solution  $\phi(t, x)$  of  $(S)^*$  corresponding to  $\phi(0, \cdot) = \lambda u$  blows up in finite time for any  $\lambda > 1$ . Hence, the result in this case. We want to emphasize that in this special case,  $u$  is just any solution of  $(E)^*$  and not necessarily a ground state. Although it requires a few modifications, the proof given in this section for the ground state can also be adapted to treat the particular case  $p = 1 + (4/N)$ . The details will be omitted here. Compare further M. Weinstein [39] for a more detailed analysis of this case. □

#### 4. MORE GENERAL NON-LINEAR SCHRÖDINGER EQUATIONS

##### 4.1. Main result and comments

In this section we consider more general non-linear terms in the Schrödinger equation :

$$(S) \quad i \phi_t - \Delta \phi = g(\phi)$$

where  $g(\phi) = f(|\phi|^2)\phi$  for some real valued and continuous  $f : \mathbb{R}_+ \rightarrow \mathbb{R}$ . The standing wave  $\phi(t, x) = e^{i\omega t} u(x)$  is a solution of (S) provided  $u$  is a solution of

$$(E) \quad -\Delta u + \omega u = g(u) \quad , \quad u \in H^1(\mathbb{R}^N) \quad , \quad u \neq 0 .$$

In the sequel,  $g$  will be thought of either as a function  $\mathbb{R} \rightarrow \mathbb{R}$  or as a function  $\mathbb{C} \rightarrow \mathbb{C}$ . The following conditions will be imposed on  $g$  throughout

the remaining of the paper.

$$(4.1) \quad g \in C^1(\mathbb{R}, \mathbb{R}), \quad g \text{ is odd}, \quad g'(0) = 0.$$

$$(4.2) \quad g(s) \geq 0, \quad \forall s \geq 0.$$

For  $z \in \mathbb{C}$  set  $G(z) = G(|z|) = \int_0^{|z|} g(s) ds$ . In this section, we further require  $g$  to verify

$$(4.3) \quad sg(s) \geq \sigma G(s) \quad \forall s \geq s_0, \quad \text{with } \sigma > 2 + \frac{4}{N}.$$

Here  $s_0 > 0$  is some constant.

$$(4.4) \quad \left| \begin{array}{l} h(s) = \{sg(s) - 2G(s)\} / s^{2+(4/N)} \text{ is a strictly} \\ \text{increasing function for } s \in (0, +\infty). \end{array} \right.$$

$$(4.5) \quad \lim_{s \searrow 0} h(s) = 0, \quad \lim_{s \nearrow +\infty} h(s) = +\infty.$$

In the sequel, we denote

$$(4.6) \quad \ell = (N+2)/(N-2) \text{ if } N \geq 3 \text{ and } 1 < \ell < \infty \text{ if } N = 1, 2$$

(that is  $\ell$  is arbitrary if  $N \leq 2$ ). For  $p > 1$  we denote  $p'$  the conjugate exponent :  $(1/p) + (1/p') = 1$ . The last assumption is

$$(4.7) \quad \left| \begin{array}{l} \exists p_1, p_2, r_0, r \text{ with } 1 \leq p_1 \leq p_2 < \ell, \quad 2 \leq r_0 \leq r < \ell+1 \\ r/r' \leq p_1 \leq p_2 < r_0 / r_0' \text{ such that} \\ |g'(s)| \leq C(s^{p_1-1} + s^{p_2-1}) \quad \forall s \geq 0. \end{array} \right.$$

Our main result is the following.



Theorem 4.1. Let  $g$  verify conditions (4.1) - (4.7), let  $\omega > 0$  and let  $u$  be a ground state solution of (E). Then,  $e^{i\omega t} u(x)$  is a strongly unstable solution of (S) in the sense below. For any  $\varepsilon > 0$ , there exists  $\phi_0 \in H^1(\mathbb{R}^N)$ ,  $\|u - \phi_0\|_{H^1(\mathbb{R}^N)} < \varepsilon$  such that the solution  $\phi(t) = \phi(t, x)$  of (S) corresponding to  $\phi(0) = \phi_0$  has the following property. There is a finite time  $T < \infty$  such that  $\phi$  is defined for  $0 \leq t < T$ ,  $\phi \in C([0, T), H^1(\mathbb{R}^N))$  and  $\lim_{t \nearrow T} \|\phi(t)\|_{H^1(\mathbb{R}^N)} = +\infty$ .

Let us first somewhat explain the assumptions of this Theorem.

Condition (4.7) is purely technical. It corresponds to the conditions of J. Ginibre and G. Velo [13] for solving the Cauchy problem associated with (S). Indeed, let us recall from [13] that the following holds assuming (4.1) and (4.7). For any  $\phi_0 \in H^1(\mathbb{R}^N)$  there exists a unique solution  $\phi$  of (S) defined on a maximal time interval  $[0, T)$  (to emphasize the dependence on  $\phi_0$  we sometimes write  $T = T_{\max}(\phi_0)$ ),  $0 < T \leq \infty$  such that

$$(4.8) \quad \phi \in C([0, T), H^1(\mathbb{R}^N)) ,$$

where  $\phi(t)$  is the function  $\phi(t, \cdot)$ . Furthermore,

$$(4.9) \quad \text{Either } T = \infty \text{ or, if } T < \infty, \lim_{t \nearrow T} \|\phi(t)\|_{H^1} = \infty .$$

In addition,  $\phi(t)$  verifies the two conservation laws

$$(4.10) \quad \int_{\mathbb{R}^N} |\phi(t)|^2 dx = \int_{\mathbb{R}^N} |\phi_0|^2 dx \quad \forall t \in [0, T)$$

$$(4.11) \quad S(\phi(t)) = S(\phi_0) \quad \forall t \in [0, T)$$

where  $S(w) = \frac{1}{2} \int |\nabla w|^2 + \frac{\omega}{2} \int |w|^2 - \int G(w)$ . (For these results and the details compare J. Ginibre and G. Velo [13]).

Now, concerning assumptions (4.3)-(4.5) consider the pure power case  $g(u) = \alpha |u|^{p-1} u$ , with  $\alpha > 0$ ,  $1 < p$ . Then our assumptions (4.1)-4.7) exactly mean  $1 + \frac{4}{N} < p < \ell$ . This shows Theorem 4.1 to be somewhat sharp. Indeed, if  $p \geq \ell$ , then no ground state exists (compare [4, 32]). While, as we have seen, if  $p < 1 + \frac{4}{N}$ , not only is the conclusion of Theorem 4.1 wrong, but also, by a result of T. Cazenave and P.L. Lions [10],  $e^{i\omega t} u(x)$  is then arbitrally stable.

Remark 4.2. If  $g$  is such that the function  $\gamma(s) = \{sg(s) - \sigma G(s)\}s^{-2}$  is for some  $\sigma > 2 + (4/N)$  increasing and  $\gamma(0) = 0$ , then, as it is easily checked,  $g$  also verifies (4.3) and (4.4). □

Remark 4.3. In the situation of the above theorem, the result of R. Glassey [15] never applies. In [15], finite time blow up is shown under the assumption  $E(\phi_0) < 0$ . Here,

$$E(w) = \frac{1}{2} \int |\nabla w|^2 - \int G(w) .$$

In the present framework, on the contrary,  $u$  is such that  $E(u) > 0$ . This can be shown either by direct computation (compare section 3 above) or just by noticing that for  $\phi_0 = u$  there certainly is no finite time blow up since  $u(x) e^{i\omega t}$  is defined for all time ! Therefore, for  $\phi_0$  near to  $u$ ,  $\phi_0$  still verifies  $E(\phi_0) > 0$ . □

#### 4.2. Variational characterization of a ground state of (E)

The proof of Theorem 4.1, as in section 3, hinges crucially on the next result. For  $w \in H^1(\mathbb{R}^N)$  we let

$$Q(w) = \int_{\mathbb{R}^N} |\nabla w|^2 dx - \frac{N}{2} \int_{\mathbb{R}^N} \{\overline{wg}(w) - 2G(w)\} dx .$$

We also mention an assumption weaker than (4.7) which is sufficient here :

$$(4.12) \quad |g'(s)| \leq C s^{\ell-1}, \quad \forall s \geq s_0; \quad \lim_{s \rightarrow +\infty} g(s)s^{-\ell} = 0.$$

Proposition 4.4. Assume  $g$  satisfies (4.1)-(4.5) and (4.12). Let  $M = \{u \in H^1(\mathbb{R}^N); u \neq 0, Q(u) = 0\}$ . Then,  $u \in H^1(\mathbb{R}^N)$  is a ground state solution of (E) if and only if  $u \in M$  and

$$S(u) = \min_{w \in M} S(w).$$

The proof of Proposition 4.4. parallels the one of Proposition 3.1. above. Computations are just somewhat more intricate here. A few technical lemmas are first required before we derive this Proposition.

Lemma 4.5. For  $u \in H^1(\mathbb{R}^N)$ ,  $u \neq 0$ , let

$$a(\lambda) = \lambda^{-(N+2)} \int \{ \lambda^{N/2} \bar{u} g(\lambda^{N/2} u) - 2 G(\lambda^{N/2} u) \}$$

Then,  $a > 0$  is strictly increasing on  $(0, +\infty)$ . Moreover,  $\lim_{\lambda \searrow 0} a(\lambda) = 0$ ,  
 $\lim_{\lambda \rightarrow +\infty} a(\lambda) = +\infty$ .

Proof of Lemma 4.5. Using the function  $h$  appearing in assumption (4.4), we have

$$(4.13) \quad a(\lambda) = \int h(\lambda^{N/2} |u|) |u|^{2+(4/N)}.$$

Hence, by (4.4) and (4.5),  $a > 0$  and  $a$  is increasing. Furthermore, for all  $\lambda \leq 1$ ,  $h(\lambda^{N/2} |u|) |u|^{2+(4/N)} \leq \bar{u} g(u) - 2 G(u) \in L^1(\mathbb{R}^N)$  (by (4.11) and the Sobolev embedding theorem). Thus, by Lebesgue's convergence theorem and owing to (4.5), we obtain  $\lim_{\lambda \searrow 0} a(\lambda) = 0$ . Lastly, for some  $\epsilon > 0$ ,

$\int_{|u|>\varepsilon} |u|^{2+(4/N)} = \delta > 0$ . This shows that  $a(\lambda) \geq h(\lambda^{N/2} \varepsilon) \delta$  whence,

by (4.5),  $\lim_{\lambda \rightarrow +\infty} a(\lambda) = \infty$ . □

Henceforth, for  $u \in H^1(\mathbb{R}^N)$ ,  $u \neq 0$ , we set

$$(4.14) \quad u^\lambda(x) = \lambda^{N/2} u(\lambda x)$$

$$(4.15) \quad s(\lambda) = S(u^\lambda) \quad , \quad q(\lambda) = \frac{1}{\lambda} Q(u^\lambda) .$$

Lemma 4.6. For any  $u \in H^1(\mathbb{R}^N) \setminus \{0\}$ , there exists a unique  $\lambda^* > 0$  such that  $q > 0$  on  $(0, \lambda^*)$ ,  $q < 0$  on  $(\lambda^*, +\infty)$ ,  $q(\lambda^*) = 0$ ;  $q$  is decreasing on  $(\lambda^*, +\infty)$  and  $q(\lambda) \rightarrow -\infty$  as  $\lambda \rightarrow +\infty$ .

Proof. With the notation of Lemma 4.5, a simple computation shows that

$$(4.16) \quad q(\lambda) \equiv \frac{1}{\lambda} Q(u^\lambda) = \lambda \left\{ |\nabla u|^2 - \frac{N}{2} a(\lambda) \right\} .$$

Hence, the result follows from Lemma 4.6. □

Lemma 4.7. Let  $u \in H^1(\mathbb{R}^N)$ ,  $u \neq 0$ , let  $s(\lambda)$  be defined by (4.14), (4.15) and  $\lambda^*$  be defined by  $\lambda^* > 0$ ,  $q(\lambda^*) = 0$ . Then,  $s(\lambda)$  increases on  $(0, \lambda^*)$ , achieves uniquely its maximum at  $\lambda^*$  and decreases on  $(\lambda^*, +\infty)$ . Moreover,  $s(\lambda)$  is strictly concave on  $(\lambda^*, +\infty)$ . Lastly,

$$\lim_{\lambda \rightarrow 0} s(\lambda) = \frac{\omega}{2} \int |u|^2 \quad , \quad \lim_{\lambda \rightarrow \infty} s(\lambda) = -\infty .$$

Proof of Lemma 4.7. It is straightforward to check that

$$(4.17) \quad s(\lambda) = \frac{\lambda^2}{2} \int |\nabla u|^2 + \frac{\omega}{2} \int |u|^2 - \lambda^{-N} \int G(\lambda^{N/2} u) .$$

Hence, with the exception of the limit of  $s(\lambda)$  as  $\lambda \rightarrow 0$ , all the statements in the above Lemma just follow from Lemma 4.6 and the observation that

$$(4.18) \quad s'(\lambda) = q(\lambda) \equiv \frac{1}{\lambda} Q(u^\lambda) .$$

To compute  $\lim_{\lambda \searrow 0} s(\lambda)$ , we note that by (4.1), (4.4) and (4.5) it easily

follows that  $G(s) s^{-2}$  is an increasing function on  $\mathbb{R}_+$ . Hence, for  $\lambda \leq 1$ ,  $\lambda^{-N} G(\lambda^{N/2} u) \leq G(u) \in L^1(\mathbb{R}^N)$ . Since  $g'(0) = 0$ , Lebesgue's theorem in (4.17) yields

$$\lim_{\lambda \searrow 0} s(\lambda) = \frac{\omega}{2} \int |u|^2. \quad \square$$

Corollary 4.8. For any  $u \in M$ ,  $S(u) \geq \frac{\omega}{2} \int |u|^2$ .

Indeed, when  $u \in M$ ,  $\lambda^* = 1$  and by the above Lemma,  $S(u) = s(1) \geq \lim_{\lambda \searrow 0} s(\lambda)$ .  $\square$

Lemma 4.9. If  $(u_n) \subset M$  is a sequence along which  $S(u_n)$  is bounded from above, then  $u_n$  is bounded in  $H^1(\mathbb{R}^N)$ .

Proof of Lemma 4.9. Clearly, by the corollary,  $\|u_n\|_{L^2}$  is bounded. By (4.3)  $sg(s) - 2G(s) \geq (\sigma-2)G(s)$  for  $|s| \geq s_0$ , where  $\sigma > 2 + \frac{4}{N}$ . Since  $G(s)s^{-2} \rightarrow 0$  as  $s \rightarrow 0$ , one can find a constant  $C > 0$  such that for all  $s \in \mathbb{R}$ :

$$(4.19) \quad (\sigma-2) G(s) \leq \{sg(s) - 2 G(s)\} + C s^2.$$

From  $u_n \in M$ , i.e.  $Q(u_n) = 0$ , and (4.19) we derive

$$\int G(u_n) \leq \left(\frac{1}{2} - \eta\right) \int |\nabla u_n|^2 + C \int |u_n|^2,$$

for some  $\eta > 0$ . Therefore,

$$(4.20) \quad S(u_n) \geq \eta \int |\nabla u_n|^2 - C \int |u_n|^2.$$

It then follows that  $\int |\nabla u_n|^2 \leq C$  whence that  $\|u_n\|_{H^1(\mathbb{R}^N)}$  is bounded.  $\square$

Corollary 4.10. For any  $u \in M$ ,  $\lambda > 0$  one has

$$(4.21) \quad S(u^\lambda) \leq S(u) \quad \forall u \in M, \quad \forall \lambda > 0.$$

This is just a particular case in Lemma 4.7 as  $u \in M$  is equivalent to  $\lambda^* = 1$ . □

We are now ready to prove Proposition 4.4. The argument is divided into several steps.

Proof of Proposition 4.4.

Step 1. Any solution  $v$  of (E) belongs to  $M$ .

Indeed,  $v$  satisfies (compare e.g. [4]):

$$\int |\nabla v|^2 + \omega \int v^2 = \int g(v) v$$

$$\frac{N-2}{2} \int |\nabla v|^2 + \omega \int v^2 = 2 \int G(v)$$

subtraction yields  $Q(v) = 0$  i.e.  $v \in M$ .

Step 2.  $S$  is bounded below on  $M$ .

Compare corollary 4.8.

Step 3. Selection of a minimizing sequence.

Let  $(w_n) \subset M$ ,  $S(w_n) \searrow \inf_M S$ . We denote by  $w^*$  the Schwarz symmetrization

of a given function  $w \in H^1(\mathbb{R}^N)$ . Recall that  $\int |\nabla w^*|^2 \leq \int |\nabla w|^2$  and

$\int F(|w^*|) = \int F(|w|)$  for any function  $F$ ,  $\forall w \in H^1(\mathbb{R}^N)$  (compare e.g.

[4, Appendix]). Let  $\lambda_n$  be the number  $\lambda^*$  associated with the function

$(w_n)^*$  by Lemma 4.6. Noticing that  $(w_n^*)^{\lambda_n} = (w_n^{\lambda_n})^*$ , we have

$$(4.22) \quad S((w_n^*)^{\lambda_n}) = S((w_n^{\lambda_n})^*) \leq S(w_n^{\lambda_n}) \leq S(w_n).$$

The first inequality in (4.22) is a consequence of the general properties of the Schwarz symmetrization aforementioned. The last inequality is an application of Corollary 4.10 as  $w_n \in M$ .

Thus,  $v_n = (w_n^{\lambda_n})^*$  is also a minimizing sequence and is such that  $v_n \geq 0$ ,  $v_n$  is spherically symmetric and non-increasing with respect to  $|x|$ .

Step 4. A priori bounds :  $\|v_n\|_{H^1(\mathbb{R}^N)} \leq C$ , (by Lemma 4.9).

Step 5. Passage to the limit.

Using the compactness lemma of W. Strauss [32] (see also [4, Appendix])

we can find a subsequence of  $v_n$  which we name again  $v_n$  such that :

$v_n \rightharpoonup v$  weakly in  $H^1(\mathbb{R}^N)$  ;  $v_n \rightarrow v$  a.e. in  $\mathbb{R}^N$  ;  $\int G(v_n) \rightarrow \int G(v)$  ;

$\int v_n g(v_n) \rightarrow \int v g(v)$ , as  $n \rightarrow +\infty$ . Furthermore, by our assumptions on  $g$

we derive that for any  $\epsilon > 0$  there is a constant  $C_\epsilon > 0$  such that

$\forall s > 0$  :

$$(4.23) \quad \frac{N}{2} \{sg(s) - 2G(s)\} \leq \epsilon |s|^{2+(4/N)} + C_\epsilon |s|^{\ell+1}.$$

From  $Q(v_n) = 0$  using (4.23) we get

$$(4.24) \quad \int |\nabla v_n|^2 \leq \epsilon \int |v_n|^{2+(4/N)} + C_\epsilon \int |v_n|^{\ell+1}.$$

The right hand side in (4.24) can be estimated by means of the Hölder and

Sobolev inequalities. This shows that  $\forall \epsilon > 0$ ,  $\exists C_\epsilon > 0$  with :

$$(4.25) \quad \int |\nabla v_n|^2 \leq \epsilon \left( \int |v_n|^2 \right)^{2/N} \int |\nabla v_n|^2 + C_\epsilon \left( \int |\nabla v_n|^2 \right)^{(\ell+1)/2}.$$

Now, by step 4, choosing  $\epsilon > 0$  sufficiently small one obtains

$\epsilon \left( \int |v_n|^2 \right)^{2/N} < 1/2 \quad \forall n$ , whence, from (4.25) :

$$(4.26) \quad 1/2 \leq C_\epsilon \left( \int |\nabla v_n|^2 \right)^{(\ell-1)/2}.$$

This shows that  $v \neq 0$ . Indeed, suppose  $v = 0$ . From  $Q(v_n) = 0$  and from

the fact that then  $\int v_n g(v_n) - 2G(v_n) \rightarrow 0$  we obtain  $\int |\nabla v_n|^2 \rightarrow 0$  which

is ruled out by (4.26).

Step 6. Existence of a minimum. At this point of our argument we know by weak lower semi-continuity that  $S(u) \leq \inf_M S$ ; we also know that  $u \neq 0$  and  $Q(u) \leq 0$ .

But it seems difficult to show a priori that  $Q(v) = 0$ , that is that  $v$  lies on  $M$ . Hence we cannot conclude from  $(v_n)$  alone. Thus we construct another minimizing sequence. We know that  $Q(v) \leq 0$ . Hence there exists a unique  $\lambda \in (0,1)$  such that  $Q(v^\lambda) = 0$ , i.e. such that  $u = v^\lambda \in M$ . The sequence  $v_n^\lambda$  obviously has the same convergence properties towards  $u$  as  $v_n$  does towards  $v$ . Since  $v_n \in M$ , by corollary 4.10 we know that  $S(v_n^\lambda) \leq S(v_n)$ . Therefore :

$$(4.27) \quad S(u) \leq \lim_{n \rightarrow \infty} S(v_n^\lambda) \leq \lim_{n \rightarrow \infty} S(v_n) \equiv \inf_M S .$$

Since  $u \in M$ , we obviously have a solution  $u$  of the minimization problem

$$(4.28) \quad S(u) = \min_M S$$

Step 7.  $u$  is a solution of (E). There exists a Lagrange multiplier  $\Lambda \in \mathbb{R}$  such that

$$(4.29) \quad S'(u) = \Lambda Q'(u)$$

We wish to show that  $\Lambda = 0$  (since (E) is precisely the Euler-Lagrange equation of the functional  $S$ ). Let us first give a heuristic argument :  
By (4.29) and by (4.14), (4.15) and (4.18), we have

$$(4.30) \quad \frac{d}{d\lambda} S(u^\lambda) = \Lambda \frac{d}{d\lambda} Q(u^\lambda)$$

$$(4.31) \quad \left. \frac{d}{d\lambda} S(u^\lambda) \right|_{\lambda=1} = Q(u) = 0 .$$



Thus we only need to show that  $\frac{d}{d\lambda} Q(u^\lambda) \neq 0$  in order to conclude that  $\Lambda = 0$ . But using the notation of (4.13) and (4.16) we have, since  $Q(u) = 0$  :

$$\frac{d}{d\lambda} Q(u^\lambda) = -\frac{N}{2} a'(1)$$

and we know by lemma 4.5 that  $a'(1) > 0$ . Hence the result.

This argument is not quite complete as in order to write  $\frac{d}{d\lambda} S(u^\lambda)$  we actually need to know that  $\frac{d}{d\lambda} u^\lambda \in H^1$  which is not true in general. Thus, to give a rigorous argument we write (4.29) in the form :

$$(4.32) \quad -\Delta u + \omega u - g(u) = \Lambda \left\{ -\Delta u - \frac{N}{2} (ug'(u) - g(u)) \right\}$$

Now, to equation (4.32) we may apply the virial theorem on Pohozaev's identity. Indeed, though we do not know a priori how small  $u$  is as  $|x| \rightarrow \infty$ , this identity was derived under very general conditions in [4]<sup>(1)</sup>. This yields :

$$(4.33) \quad \begin{aligned} & \frac{N-2}{N} \int |\nabla u|^2 + \omega \int |u|^2 - 2 \int G(u) = \\ & = \Lambda \left\{ \frac{N-2}{N} \int |\nabla u|^2 - N \int (ug(u) - 2G(u)) \right\}. \end{aligned}$$

Multiplying by  $\bar{u}$  and integrating by parts yields :

$$(4.34) \quad \begin{aligned} & \int |\nabla u|^2 + \omega \int |u|^2 - \int ug(u) = \\ & = \Lambda \left\{ \int |\nabla u|^2 - \frac{N}{2} \int (u^2 g'(u) - ug(u)) \right\} \end{aligned}$$

Subtracting (4.33) from (4.34) then gives :

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(1) To apply the result one needs to know that  $u \in L_{Loc}^\infty$  which is obvious from the equation.

$$(4.35) \quad 0 = \frac{2}{N} Q(u) = \Lambda \left\{ \frac{2}{N} \int |\nabla u|^2 - \frac{N}{2} \int (u^2 g'(u) + ug(u) - 2G(u)) \right\}$$

i.e.

$$(4.36) \quad \Lambda \left\{ \left( \frac{2}{N} - 1 \right) \int |\nabla u|^2 - \frac{N}{2} \int u^2 g'(u) \right\} = 0$$

When  $N \geq 2$  the coefficient of  $\Lambda$  in (4.36) is  $< 0$ . Hence  $\Lambda = 0$ , Q.E.D.

(when  $N = 1$ , (4.36) reads  $\Lambda \left\{ \int |\nabla u|^2 - \frac{1}{2} \int u^2 g'(u) \right\} = 0$ . Using  $Q(u) = 0$ , we then have  $-\frac{\Lambda}{2} \left\{ \int u^2 g'(u) + 2G(u) - ug(u) \right\} = 0$ . It is then easy to check that this also yields  $\Lambda = 0$ ).

### Step 8. Conclusion.

There exists  $u \in M$  such that  $S(u) = \underset{M}{\text{Min}} S$ .  $u$  is a solution of (E).

Now let  $v$  be any other solution of (E). It lies on  $M$  hence  $S(u) \leq S(v)$ .

Thus  $u$  is a ground state solution of (E).

The proof of proposition 4.4 is thereby complete. Notice that  $S(u) > 0$ .  
Indeed  $S(u) = \frac{1}{N} \int |\nabla u|^2$ .

### 4.3. Proof of the instability theorem 4.1.

By the results of J. Ginibre and G. Velo [13] we know that for any  $\phi_0 \in H^1(\mathbb{R}^N)$  the corresponding solution of the Cauchy problem

$$(4.37) \quad (S), \quad \phi(0, x) = \phi_0$$

verifies the properties (4.10), (4.11) and is unique on  $(0, T)$ . In the following we denote  $T = T_{\max}(\phi_0)$  the maximum time length. Theorem 4.1 is a consequence of the following :

Proposition 4.11 : Let  $h = S(u)$  where  $u$  is a ground state solution of (E). Set  $K = \{w \in H^1(\mathbb{R}^N), S(w) < h, Q(w) < 0\}$ . Then  $K$  is invariant under the

flow associated with (S). That is, for any  $\phi_0 \in K$ , the corresponding solution  $\phi(t, x)$  of (4.37) is such that  $\phi(t) \in K$  for  $t \in [0, T_{\max}(\phi_0))$ . Furthermore, if  $|\cdot| \phi_0(\cdot) \in L^2(\mathbb{R}^N)$ , then  $T_{\max}(\phi_0) < \infty$  and (4.9) holds.

Before proving proposition 4.11 let us explain why it gives theorem 4.1.

Let  $u$  be a ground state solution of (E). Choosing  $\phi_0$  of the form  $\phi_0(x) = \lambda^{N/2} u(\lambda x)$  (i.e.  $\phi_0 = u^\lambda$ ) with  $\lambda > 1$ ,  $\lambda$  near to 1, one has  $S(\phi_0) < h$  by corollary 4.10 and  $Q(\phi_0) < 0$  by lemma 4.6. Hence,  $\phi_0 \in K$ . Since  $u$  is a ground state, it has exponential fall off at infinity (e.g. cp [32,4]) whence  $|\cdot| \phi_0(\cdot) \in L^2(\mathbb{R}^N)$ . Thus  $T_{\max}(\phi_0) < \infty$ . Lastly, it is obvious that  $\|\phi_0 - u\|_{H^1}$  is arbitrarily small. One could also make  $\phi_0 - u$  arbitrarily small in stronger norms.

Proof of proposition 4.11. :

Step 1. K is invariant. Let  $\phi_0 \in K$  and let  $\phi(t)$  be the associated solution of the Cauchy problem. We let  $T = T_{\max}(\phi_0)$ . We have  $S(\phi(t)) \equiv S(\phi_0)$  and since  $\phi \in C([0, T], H^1(\mathbb{R}^N))$ ,  $Q(\phi(t))$  is a continuous function of time.

For a given  $t > 0$  such that  $\phi(t) \in K$ , let  $s(\lambda) = S(\phi(t)^\lambda)$  where we use the previous notation (4.14). Since  $Q(\phi(t)) < 0$  it follows from Lemma 4.7 that the unique  $\lambda^*$  such that  $Q(\phi(t)^{\lambda^*}) = 0$  verifies  $\lambda^* \in (0, 1)$ . Since  $s$  is concave on  $[\lambda^*, 1]$ , we have :

$$s(1) \geq s(\lambda^*) + (1 - \lambda^*)s'(1) .$$

Whence (as  $s'(1) < 0$ )

$$s(1) \geq s(\lambda^*) + s'(1) .$$

Using (4.18) we derive

$$(4.38) \quad Q(\phi(t)) = s'(1) \leq s(1) - s(\lambda^*) .$$

Now observe that  $\phi(t)^{\lambda^*} \in M$  whence  $S(\phi(t)^{\lambda^*}) \geq h$  and this yields in (4.38) :

$$(4.39) \quad Q(\phi(t)) \leq - (h - S(\phi_0)) = - \delta < 0 .$$

Now a simple continuity argument shows that  $\phi(t)$  remains in  $K$  for all  $t \in [0, T)$ .

### Step 2. Conclusion.

Assume that  $|\cdot| \phi_0(\cdot) \in L^2(\mathbb{R}^N)$  .

Then  $|\cdot| \phi(t, \cdot) \in C([0, T), L^2(\mathbb{R}^N))$  and  $\gamma(t) = \| |\cdot| \phi(t, \cdot) \|_{L^2}$  satisfies  $\gamma \in C^2([0, T))$ . Furthermore we have :

$$(4.40) \quad \gamma''(t) = 8 Q(\phi(t)) \quad \text{for any } t \in [0, T) .$$

(compare (3.29), and see R. Glassey [14] and H. Brezis [8] for a proof.)

Whence (4.39) and (4.40) yields

$$\gamma''(t) \leq - \delta < 0 \quad \text{for } t \in [0, T) .$$

Since  $\gamma(t) > 0$  , this implies  $T < \infty$  . Q.E.D.

## 5. THE NONLINEAR KLEIN-GORDON EQUATION.

### 5.1. The main result.

We derive here an instability result for the nonlinear Klein-Gordon equation (K.G.). For the sake of simplicity we only consider the case of a space dimension  $N \leq 3$  . Analogous results for  $N \geq 4$  could be proved by applying the characterization of section 5.2 below and the relevant existence theorems for the evolution equation (see [28] , [29] , [31] ).

The method of proof here is inspired from [26] which deals with the case of a bounded domain.

We use the notations of chapter 4. We consider the semilinear elliptic problem

$$(E) \quad -\Delta u + \omega u = g(u) \quad , \quad u \in H^1(\mathbb{R}^N) \quad , \quad u \neq 0 \quad .$$

and the nonlinear Klein-Gordon equation

$$(K.G.) \quad \phi_{tt} - \Delta \phi + \omega \phi = g(\phi) \quad .$$

$g$  will be required to satisfy the following hypothesis, in addition to (4.1), (4.2) :

$$(5.1) \quad \lim_{s \rightarrow +\infty} g(s) s^{-\ell} = 0$$

$$(5.2) \quad \overline{\lim}_{s \rightarrow +\infty} s^{-\ell+1} |g'(s)| < \infty$$

where  $\ell$  is given by (4.6)

$$(5.3) \quad \text{There is } \varepsilon > 0 \text{ such that } \overline{\lim}_{s \rightarrow +\infty} g(s) s^{-\ell+\varepsilon} < \infty$$

$$(5.4) \quad \text{There is } \theta \in ]0, 1/2[ \text{ such that } G(s) \leq \theta s g(s) \\ \text{for any } s \geq 0 \text{ where } G(s) = \int_0^s g(\tau) d\tau \quad .$$

$$(5.5) \quad \text{The function } k \text{ defined by } k(s) = s^{-1} g(s) \text{ for } s > 0 \text{ is strictly increasing.}$$

Our main result is the following :

Theorem 5.1. *Suppose that  $\omega > 0$  ,  $N \leq 3$  and that  $g$  satisfies hypotheses (5.1) to (5.5). Let  $u$  be a ground state solution of (E). Then for any  $\varepsilon > 0$  there exists  $\phi_0 \in H^2(\mathbb{R}^N)$  such that  $\|\phi_0 - u\|_{H^1} \leq \varepsilon$  and with the following property. The solution  $\phi$  of the Cauchy problem for (K.G) corresponding to the initial data  $\phi(0, \cdot) = \phi_0$  ,  $\phi_t(0, \cdot) = 0$  is defined for  $0 \leq t < T < +\infty$  ,  $\phi \in C([0, T], H^1(\mathbb{R}^N))$  and  $\lim_{t \rightarrow T} \|\phi(t)\|_{H^1(\mathbb{R}^N)} = +\infty$  .*

Remark 5.2. It is straightforward to verify that theorem 2.2 is a consequence of theorem 5.1.

Remark 5.3. We may choose  $\phi_0$  with a compact support (see section 5.4). This proves that the instability is a local property rather than a global one.

### 5.2. A variational characterization.

The proof of the instability for the nonlinear Schrödinger equation relies on a variational characterization of the ground state solutions of (E). To prove theorem 5.1 it seems that we need a different characterization. To that end, define a functional  $R$  on  $H^1(\mathbb{R}^N)$  by

$$R(u) = \int_{\mathbb{R}^N} |\nabla u|^2 + \omega \int_{\mathbb{R}^N} |u|^2 - \int_{\mathbb{R}^N} \bar{u}g(u) \quad \text{for } u \in H^1(\mathbb{R}^N)$$

Then we have the following characterization of the ground state solutions of (E).

Proposition 5.4. Assume that  $\omega > 0$  and that  $g$  satisfies (5.1), (5.2), (5.4) and (5.5). Let  $N = \{u \in H^1(\mathbb{R}^N), R(u) = 0, u \neq 0\}$ . Then there is a solution to the problem

$$(5.6) \quad u \in N, \quad S(u) = \min_{v \in N} S(v)$$

In addition the set of solution to (5.6) coincides with the set of ground state solutions of (E).

Since the proof of Proposition 5.4 is very similar to that of Proposition 4.4 we shall not give the full details. The proof proceeds in several steps.

#### Step 1 : Solutions of (E) lie on N.

Just multiply the equation by  $\bar{u}$  and integrate by parts.

Step 2 : S is bounded below on N.

Let  $v \in H^1(\mathbb{R}^N)$ . Then :

$$\begin{aligned} S(v) &\geq \frac{1}{2} \int |\nabla u|^2 + \frac{\omega}{2} \int |v|^2 - \theta \int \bar{v} g(v) \\ &\geq \theta R(v) + \left(\frac{1}{2} - \theta\right) \left\{ \int |\nabla v|^2 + \omega \int |v|^2 \right\} \end{aligned}$$

with  $\theta$  given by (5.4). Hence, if  $v \in N$ , we have with  $\delta = \frac{1}{2} - \theta > 0$

$$(5.7) \quad S(v) \geq \delta \left\{ \int |\nabla v|^2 + \omega \int |v|^2 \right\}$$

Step 3 : A technical lemma.

Lemma 5.5. For any  $v \in H^1(\mathbb{R}^N)$ ,  $v \neq 0$ , there exists a unique  $\lambda^* > 0$  (depending on  $v$ ) such that  $R(\lambda^* v) = 0$ .

In addition,  $R(\lambda v) > 0$  for  $\lambda \in (0, \lambda^*)$ ,  $R(\lambda v) < 0$  for  $\lambda > \lambda^*$  and  $S(\lambda v) < S(\lambda^* v)$  for  $\lambda \neq \lambda^*$ .

Proof. We have  $\frac{d}{d\lambda} S(\lambda v) = \frac{1}{\lambda} R(\lambda v)$ . Since

$$R(\lambda v) = \lambda^2 \left\{ \int |\nabla v|^2 + \omega \int |v|^2 - \int |v|^2 k(\lambda |v|^2) \right\}$$

with  $k$  defined by (5.5), it follows from (5.5) that there is a unique  $\lambda^* > 0$  such that  $R(\lambda v) > 0$  for  $\lambda \in (0, \lambda^*)$ ,  $R(\lambda^* v) = 0$  and  $R(\lambda v)$  is strictly negative and decreasing on  $(\lambda^*, +\infty)$ .

Remark 5.6. Clearly  $S(\lambda v)$  is concave on  $(\lambda^*, +\infty)$ . Whence, if  $v \in H^1(\mathbb{R}^N)$ ,  $v \neq 0$ , and if  $\lambda > \lambda^*$

$$R(\lambda v) < - \frac{S(\lambda^* v) - S(\lambda v)}{\lambda - \lambda^*}.$$

As a consequence, if  $v \in H^1(\mathbb{R}^N)$  is such that  $R(v) < 0$ , the above relation

yields, using the fact that  $\lambda^* v \in N$

$$(5.8) \quad R(v) < - \{ \inf_{w \in N} S(w) - S(v) \}$$

Step 4 : Selection of a minimizing sequence.

We just proceed in the same manner as at step 3 of the proof of proposition 4.4. We find a sequence  $(v_n) \in N$  such that  $S(v_n) \searrow \inf_N S$ ,  $v_n > 0$ ,  $v_n$  is spherically symmetric decreasing with respect to  $|x|$ .

Step 5 : Passage to the limit.

We can assume  $v_n \rightarrow v$  weakly in  $H^1(\mathbb{R}^N)$ . Then we may repeat the argument of the proof of Proposition 4.4 (step 5, 6) provided we can show that  $v \neq 0$ . This will be argued by contradiction. Indeed, assume  $v = 0$ . Then  $v_n \rightarrow 0$  weakly in  $H^1(\mathbb{R}^N)$ . Using the compactness properties of  $v_n$ , we get :

$$\int v_n g(v_n) \xrightarrow{n \rightarrow \infty} 0$$

Since  $R(v_n) = 0$  it follows that  $v_n \rightarrow 0$  strongly in  $H^1(\mathbb{R}^N)$ . Let now  $\ell$  be given by (5.2). We have  $sg(s) \leq \omega s^2 + C s^{\ell+1}$ . Since  $v_n \in N$  we obtain

$$\int |\nabla v_n|^2 \leq C \int |v_n|^{\ell+1}.$$

Applying Gagliardo-Nirenberg's inequality (if  $N \leq 2$ ) or Sobolev's inequality (if  $N \geq 3$ ) the above relation provides a lower bound on  $\int |\nabla v_n|^2$  which leads to a contradiction.

Step 6 : A solution of (5.6) is a solution of (E).

Let us first remark that  $R \in C^1(H^1(\mathbb{R}^N), \mathbb{R})$  and that  $R'$  does not vanish on  $N$ . Indeed, let  $v \in N$ . Then :

$$(R'(v), v) = \int \{ \bar{v} g(v) - |v|^2 g'(|v|) \}.$$



On the other hand, we have (with  $k$  given by (5.5))

$$s g(s) - s^2 g'(s) = -s^3 k'(s)$$

whence  $(R'(v), v) < 0$ .

Now, if  $u$  is a solution of (5.6) there is a Lagrange multiplier  $\Lambda \in \mathbb{R}$  such that  $S'(u) = \Lambda R'(u)$ .

Since  $(S'(u), u) = R(u) = 0$  and  $(R'(u), u) < 0$ , we have  $\Lambda = 0$  and  $u$  is a solution of (E).

Step 7 : Conclusion.

The conclusion is the same as that of Proposition 4.4, step 8.

5.3. The Cauchy problem.

Let us recall some well known facts about the Cauchy problem for (K.G.) (see [1], [8], [20], [28], [31], [33] and references there in).

By the classical semi-groups method of I. Segal [28] (see for instance [29]) it is known that if  $g$  satisfies (5.1), (5.2), (5.3) and if  $N \leq 3$  then for any  $(\phi_0, \phi_1) \in H^2(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$  there is a unique solution  $\phi$  to (K.G.) such that  $\phi(0) = \phi_0$ ,  $\phi_t(0) = \phi_1$  defined on a maximal time interval  $[0, T)$ ,  $\phi \in C([0, T), H^2(\mathbb{R}^N))$ .

In addition, either  $T = +\infty$ , or  $T < \infty$  and  $\lim_{t \rightarrow T} \|\phi(t)\|_{H^1} = +\infty$ .

Furthermore, for  $t \in [0, T)$  we have

$$(5.9) \quad S(\phi(t)) + \frac{1}{2} \int_{\mathbb{R}^N} |\phi_t(t, x)|^2 dx = S(\phi_0) + \frac{1}{2} \int_{\mathbb{R}^N} |\phi_1|^2$$

$$(5.10) \quad \frac{d^2}{dt^2} \int_{\mathbb{R}^N} |\phi(t, x)|^2 dx = 2 \int_{\mathbb{R}^N} |\phi_t(t, x)|^2 dx - 2R(\phi(t)).$$

We are now ready to prove Theorem 5.1.

5.4. Proof of the instability theorem.

Let  $u$  be a ground state solution of (E), hence a solution of (5.6). Applying lemma (5.5) we obtain that for any  $\lambda > 1$  :

$$S(\lambda u) < S(u) \quad \text{and} \quad R(\lambda u) < 0$$

Since  $\lambda u \xrightarrow{\lambda \rightarrow 1} u$  in  $H^1(\mathbb{R}^N)$ , for any  $\varepsilon > 0$  we may choose  $\phi_0 \in H^2(\mathbb{R}^N)$  with a compact support such that  $\|\phi_0 - u\|_{H^1} \leq \varepsilon$  and such that

$$(5.11) \quad R(\phi_0) < 0$$

$$(5.12) \quad S(\phi_0) < S(u)$$

by smoothing  $\lambda u$  for  $\lambda$  close enough to 1 .

Let  $\phi$  be the solution of the Cauchy problem for (K.G.) with initial data  $(\phi_0, 0)$  and let  $[0, T)$  be the maximal existence interval of  $\phi$  . From (59) it follows that for any  $t \in [0, T)$  ,  $S(\phi(t)) \leq S(\phi_0)$  .

On the other hand, we deduce from (5.11) that on some time interval  $[0, \bar{t})$  we have  $R(\phi(t)) < 0$  . Applying (5.8) we get

$$R(\phi(t)) \leq - \{S(u) - S(\phi_0)\} = - \delta < 0 .$$

As a consequence we may choose  $\bar{t} = T$  and (5.10) implies that

$$\frac{d^2}{dt^2} \int_{\mathbb{R}^N} |\phi(t, x)|^2 dx \geq 2\delta \quad \text{for } t \in [0, T) .$$

Applying (5.4), standard computations (see e.g. [ 14 ], [ 19 ], [ 26 ], [ 31 ]) show that necessarily  $T < \infty$  . Q.E.D.

6. FURTHER COMMENTS AND OPEN PROBLEMS.

6.1. The nonlinear Heat equation.

In this section we consider the nonlinear Heat equation

$$(H) \quad \phi_t - \Delta\phi + \omega\phi = g(\phi) .$$

It is known that for general nonlinearities, (H) have unstable stationary solutions (see [ 6 ] ) , but to our knowledge, no result of instability by "blow up" seems to be known. It is possible to prove such results, by applying proposition 4.4 and relevant existence theorems for (H). For example, we give the following :

Theorem 6.1. Assume that  $\omega > 0$  ,  $N \leq 3$  and that  $g$  satisfies hypothesis (5.1) to (5.5). Let  $u$  be a ground state solution of (E).

Then, for any  $\varepsilon > 0$  there exists  $\phi_0 \in H^2(\mathbb{R}^N)$  with  $\|\phi_0 - u\|_{H^1(\mathbb{R}^N)} \leq \varepsilon$  and such that the solution  $\phi$  of the Cauchy problem for (H) corresponding to the initial data  $\phi(0, \cdot) = \phi_0$  has the following property.

There exists a maximum time  $T < +\infty$  such that  $\phi$  is defined for  $0 \leq t < T$  ,  $\phi \in C([0, T), H^1(\mathbb{R}^N))$  and  $\lim_{t \rightarrow T} \|\phi(t, \cdot)\|_{H^1(\mathbb{R}^N)} = +\infty$  .

Proof. It is well known that under the hypothesis above, for any  $\phi_0 \in H^2(\mathbb{R}^N)$  there is a unique solution  $\phi$  to the Cauchy problem for (H) with initial data  $\phi_0$  , defined on a maximal time interval  $[0, T)$  ,  $\phi \in C([0, T), H^2(\mathbb{R}^N))$ . In addition, either  $T = \infty$  or  $T < \infty$  and  $\lim_{t \rightarrow T} \|\phi(t)\|_{H^1} = +\infty$  . Furthermore, for any  $t \in [0, T)$  , we have :

$$(6.1) \quad \int_0^t ds \int_{\mathbb{R}^N} |\phi_t(s, x)|^2 dx + S(\phi(t)) = S(\phi_0)$$

$$(6.2) \quad \frac{d}{dt} \int |\phi(t,x)|^2 = -2R(\phi(t)) .$$

Let now  $u$  be a ground state solution of (E). For  $\epsilon > 0$ , let  $\lambda > 1$  be such that  $\phi_0 = \lambda u$  satisfies  $\|\phi_0 - u\|_{H^1} \leq \epsilon$ .

Let  $\phi$  be the solution of (H) such that  $\phi(0) = \phi_0$ , and let  $[0, T)$  be the maximal existence interval of  $\phi$ .

Applying the techniques of the proof of theorem 5.1, it is easily checked that  $R(\phi(t)) \leq -\delta < 0$  for any  $t \in [0, T)$ .

Classical computations show that this property, together with (6.1) and (6.2), implies that  $T < \infty$  (compare e.g. [26]).

## 6.2. Some open problems.

Out of many, we just mention a few open problems in the context of the non-linear Schrödinger equation (S).

1. The nature of the blowing up. Finite time blow up results are always derived by contradiction arguments. Therefore, very little has been known about the precise behaviour of the solution near the time when it blows up. For instance, we conjecture the following :

a) Conjecture : Let  $\varphi_0(x) = R(x)$  with  $R$  symmetric and decreasing (i.e.  $|x| \leq |y| \rightarrow R(x) \geq R(y)$ ). Suppose that  $1 + \frac{4}{N} < p < \frac{N+2}{N-2}$  and that  $R \in K$  ( $K$  is defined in Proposition 3.3). Then, the corresponding solution  $\varphi(t,x)$  of the Cauchy problem (3,26) satisfies  $\lim_{t \uparrow T} |\varphi(t,0)| = +\infty$ ,  $\lim_{t \uparrow T} |\varphi(t,x)| = 0$   $\forall x \neq 0$ .

b) More generally, can one give a more precise description of the solution. For instance, can one find equivalents to  $|\varphi(t,x)|$  of the form

$(T-t)^{-\alpha} R [x(T-t)^{-\beta}]$  as  $t \nearrow T$  with  $\alpha, \beta > 0$ . Such self-similar (as well as other) types of behaviour have been conjectured long ago by Zakharov - Sobolev - Synakh [41] on the basis of numerical computations.

A first rigorous (and remarkable) proof of such a result (in the case  $N = 1$  and  $p = 5$ , (the so-called "critical exponent" for the power non-linearity) has been announced recently by D. Mc Laughlin, G. Papanicolaou and M. Weinstein [43]. Some numerical techniques for dealing with this type of problem are developed by C. Sulem, P.L. Sulem and A. Patera [45]. We also refer to their work for further numerical investigations. (see also C. Sulem [44]).<sup>(1)</sup>

c) In the framework of Theorem 2.1 with  $\varphi_0 = R$  symmetric decreasing, is it true that  $\lim_{t \nearrow T} \int_{\mathbb{R}^N} |x|^2 |\varphi(t,x)|^2 dx = 0$ ? This would be supported by the very argument given above.

2. Asymptotic weak stability of 0. We conjecture that the following global weak asymptotic stability of 0 holds (we keep on using the notations of Theorem 2.1 and of remark 3.5) : If  $\varphi_0 \in \hat{K}$ , then

$$\varphi(t, \cdot) \rightharpoonup 0 \text{ weakly in } H^1(\mathbb{R}^N) \text{ as } t \nearrow \infty.$$

(Recall that in this case  $\varphi$  is defined for all time ; compare remark 3.5).

3. Bound states. In [4] H. Berestycki and P.L. Lions (and in [3] H. Berestycki, T. Gallouët and O. Kavian, for  $N = 2$ ) have proved that (E) has infinitely many solutions corresponding to higher levels of the action  $S(u)$ . Nothing is known as of now regarding the stability of these. We conjecture that the following two results are true.

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<sup>(1)</sup> Recently, F. Weissler [46] has solved the problem of the nature of the blow up for a nonlinear heat equation (with power non-linearity) in spatial dimension 1).

a) For any  $g(u)$ , the bound states are unstable (in a weaker sense -of course- if the non-linearity does not allow for finite time blow up).

b) In the framework of Theorems 2.1. and 4.1. we conjecture that any bound state is unstable in the strong sense of finite time blow up <sup>(1)</sup> .

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<sup>(1)</sup> After completion of this manuscript, H. Berestycki and P.L. Lions [42] have very recently shown that there are infinitely many bound states -the ones obtained by min-max argument [4,5]- for which one second result holds.

APPENDIX. Self focusing of Laser beams.

We give a brief discussion of how equation (S) can modelize the propagation of a Laser beam. For a more detailed argument we refer to [ 18 ] [ 34 ] [ 35 ] [ 38 ] and references there in.

Let us consider a Laser beam propagating along the z-axis in the 3-dimensional space with coordinates (x,y,z).

We start with equation (see [ 34 ])

$$(A.1) \quad -\frac{1}{c^2} \frac{\partial^2}{\partial t^2} (\epsilon \vec{E}) + \Delta \vec{E} = 0$$

where  $E(x,y,z,t)$  is the electric field,  $\epsilon$  is the dielectric index of the media and  $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ .

Without the effect of the beam, the dielectric index would be a constant  $\epsilon$ , The nonlinear effect is due to significant increase of the index under the action of the beam through physical mechanisms, heating for instance. Since a stationary state (with respect to t) will be established,  $\epsilon$  will be a function of  $|\vec{E}|$  only and we write  $\epsilon = \epsilon_0 + \epsilon_1 f(|\vec{E}|)$ . Now, we make the so-called "quasi optical approximation" that is, we assume that  $\vec{E}$  is a linearly polarized wave of frequency  $\omega$  propagating along the z-axis. In other terms we write

$$\vec{E} = \vec{e} E' \exp\{i(kz - \omega t)\}$$

where  $\vec{e}$  is a unit vector,  $k = \frac{\epsilon_0^{1/2} \omega}{c}$ .

$\exp\{i(kz - \omega t)\}$  is the propagating part of the wave, while  $E'(x,y,z)$  is the "slowly varying" part.

Equation (A.1) becomes then

$$(A.2) \quad 2ik \frac{\partial E'}{\partial z} + \Delta E' + \frac{\omega^2 \epsilon_1}{2c^2} f(|E'|) E' = 0$$

We assume that  $E'$  is slowly varying in the sense that

$$\left| \frac{\partial^2 E'}{\partial z^2} \right| \ll 2k \left| \frac{\partial E'}{\partial z} \right|$$

and we drop the second  $z$ -derivative. Our equation becomes then

$$(A.3) \quad 2ik \frac{\partial E'}{\partial z} + \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) E' + g(E') = 0$$

Making a scale change in the  $z$ -direction we are left with an equation of type (8) where  $z$ -stands for  $t$ .

The nonlinearity  $g$  depends on the media through which the beam propagates. Various models are considered (see [17,18,34,35,38,40,41]).

A classical one is  $g(u) = |u|^{p-1} u$ .

Various types of solutions can be visualized.

So-called "self-trapped" laser beams correspond to stationary state of (A.3), that is to solutions of the form :

$$E'(x,y,z) = \exp(i\lambda z) u(x,y)$$

The electric field has then the form :

$$\vec{E}(x,y,z,t) = \vec{e} \exp\{i(\delta z - \omega t)\} u(x,y)$$

The beams preserves its shape while propagating along the  $z$ -axis. According to the results of this paper, self-trapped beams are unstable for a wide class of nonlinearities.



Another classical situation is "self focusing". The beam collapses at some finite distance  $z_0$ . It corresponds to blowing up solutions of (S). A detailed study of the singularities giving rise to blowing-up would give crucial information on the "self-focusing" phenomenon.

In the case where the power of the beam is not enough to ensure self focusing, the beam can spread out as  $z \rightarrow \infty$ . An important problem is whether the behaviour of a beam of a general form can be recovered from these three types of behaviors.

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