A Hamilton-Jacobi-Bellman approach for the optimal control of an abort landing problem

Mohamed Assellaou*, Olivier Bokanowski**, Anya Desilles***, and Hasnaa Zidani****

Abstract—We study an aircraft abort landing problem modelled by a five dimensional state system with state constraints and a maximum running cost function as introduced by Bulirsch, Montrone and Pesch (J. Optim. Theory Appl., Vol. 70, No 1, pp 1–23, 1991). We propose a Hamilton-Jacobi-Bellman (HJB) approach in order to compute the value function associated to the problem, as well as trajectory reconstruction procedures based on the value function, or on a related exit time function. Some numerical illustrations are included to show the relevance of our approach.

I. INTRODUCTION

The problem of the control of the aircraft flying through the wind shear is one of the most important issues in the aerodynamics of the flight, in particular, in a landing framework. Indeed, several aircraft accidents have been attributed to wind shear [1]. This meteorological phenomenon is defined as the change on speed and direction of the wind over a small distance. This change of the wind affects the aircraft motion relative to the ground and it has more significant effects during the landing case.

As the aircraft passes through the wind shear level, the aircraft suffers a loss of the lift force and the airspeed. The pilot encounters a headwind with transition to the tailwind coupled with a descending air which spreads horizontally near the ground. This generates a significant threat of the resulting inertia wind shear force.

The penetration landing in presence of wind shear is unsafe in a high altitude. The abort landing problem is the best strategy to avoid the failed landing. This procedure consists in steering the aircraft to the maximum altitude that can be reached in order to prevent a crash on the ground. In the references [2] and [3], a Chebyshev-type optimal control is proposed, for which an approximate solution for the problem is given with the associated feedback control. This solution was improved in [1] and [4] by considering the switching structure of the problem that has bang-bang subarcs and singular arcs. In all these papers, the Pontryagin principle has been extensively investigated.

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The present work deals with the application of the Hamilton-Jacobi-Bellman (HJB) approach to solve a similar problem as in [2], [3], [4].

Note that in [5], an HJB approach has been already used on a slightly simplified 4-dimensional model, where the angle of attack $\alpha$ is the control (see (2) below).

Let us point out that one of the main advantages of the HJB theory is that it provides a global optimality information encoded in the value function. Once this function is characterized and computed, it is then possible to reconstruct the (global) optimal trajectory for any initial position of the dynamical system. The presence of state constraints in the definition of the optimal control problem - without any controllability assumption - precludes the usual characterization of the value function as the unique solution of an HJB equation. In order to overcome this difficulty, we will characterize the epigraph of the value function using an auxiliary optimal control problem on which a dynamic programming principle holds and an adequate HJB characterization can be obtained. This characterization leads to a well posed Partial Differential Equation (PDE) with rigorous boundary equations. We next study a numerical approximation of the HJB equation, using a finite difference method, and introduce several optimal trajectory reconstruction algorithms.

In particular, we discuss the regularity issue of the numerical control, and show how a slight modification of a reconstruction algorithm can regularize the feedback control while keeping the convergence result for the optimal trajectories. This article is organized as follows. The general description of the model and formulation of the control problem is given in section II. Section III is devoted to the theoretical study of the HJB approach. In section IV, we introduce some numerical schemes and study their stability, and propose different reconstruction algorithms based on the value function. Then, in section V, the behavior of these algorithms illustrated the abort landing problem of a Boeing 727 aircraft model.

II. STATEMENT OF THE PROBLEM

A. Background for the flight aerodynamic

Consider the flight of an aircraft in a vertical plane over a flat earth where the thrust force, the aerodynamic force and the weight force act on the center of gravity $G$ of the aircraft and lie in the same plane of symmetry. Let $V$ be the velocity vector of the aircraft relative to the atmosphere. To obtain the equations of motion, the following coordinate systems are considered:
(i) the ground axes system $Ex_ey_ez_e$, fixed to the surface of earth at mean sea level.

(ii) the wind axes system denoted by $Ox_0y_0z_0$, moving with the aircraft and the $x_0$ axis coincides with the velocity vector.

The path angle $\gamma$ defines the wing axes orientation with respect to the ground horizon axes.

We write Newton's law as $F = m \frac{d^2x}{dt^2}$, where $V = V + w$ is the resultant velocity of the aircraft relative to the ground axis system, and $w$ denotes the velocity of the atmosphere relative to the ground axis system. The resultant force consists of:

- the thrust force $F_T$, directed along the aircraft. The modulus of the thrust force is of the form $F_T := F_T(v)$ where $v = |V|$ is the modulus of the velocity and, assuming a maximal power thrust,

$$F_T := F_T(v) = A_0 + A_1 v + A_2 v^2.$$  

- the lift and drag forces $F_L$, $F_D$. The norm of these forces are supposed to satisfy the following relations:

$$F_L(v, \alpha) = \frac{1}{2} \rho S v^2 c_l(\alpha), \quad F_D(v, \alpha) = \frac{1}{2} \rho S v^2 c_d(\alpha), \quad (1)$$

where $\rho$ is the air density on altitude, $S$ is the wing area, the coefficients $c_d(\alpha)$ and $c_l(\alpha)$ depend on the angle of attack and the nature of the aircraft.

- the weight force $F_P$: its modulus satisfies $F_P = mg$ where $m$ is the aircraft mass and $g$ the gravitational force per unit mass.

A complete description of the coefficients such as $c_l$, $c_r$, $A_0$, etc. is given in Appendix A.

By Newton's law, the equation of motion are [1]:

$$\dot{x} = v \cos \gamma + w_x \quad (2a)$$

$$\dot{h} = v \sin \gamma + w_h \quad (2b)$$

$$\dot{v} = \frac{F_T}{m} \cos(\alpha) - \frac{F_D}{m} - g \sin \gamma \quad (2c)$$

$$\dot{\gamma} = \frac{1}{v} \left( \frac{F_T}{m} \sin(\alpha) + \frac{F_L}{m} - g \cos \gamma \right) \quad (2d)$$

$$\dot{\alpha} = u \quad (2e)$$

where $w_x$ and $w_h$ are respectively the horizontal and the vertical components of the wind velocity vector $w$, and

$$\dot{w}_x := \frac{\partial w_x}{\partial x} (v \cos \gamma + w_x) + \frac{\partial w_x}{\partial h} (v \sin \gamma + w_h)$$

$$\dot{w}_h := \frac{\partial w_h}{\partial x} (v \cos \gamma + w_x) + \frac{\partial w_h}{\partial h} (v \sin \gamma + w_h).$$

The precise model for $w_x$, $w_h$ as functions of $x$ and $h$ is provided in Appendix A. Furthermore, $u(t)$ denotes the control variable and is such that

$$u(t) \in U := [u_{\text{min}}, u_{\text{max}}], \quad (3)$$

with $u_{\text{min}} \leq u_{\text{max}}$ in $\mathbb{R}$, and the corresponding set of admissible controls is

$$U := \{ u : [0, T] \rightarrow U, u(\cdot) \text{ measurable function} \}.$$

In the sequel, the state variables are represented by the column vector of $\mathbb{R}^5$:

$$y(\cdot) = (x(\cdot), h(\cdot), v(\cdot), \gamma(\cdot), \alpha(\cdot))^T.$$

Hence, the differential system (2) will be simply denoted by:

$$\dot{y}(t) = f(y(t), u(t)) \quad (4)$$

where the dynamics $f$ stands for the right-hand-side of (2).

B. State constraints

The trajectory should remain in a set $K \subset \mathbb{R}^d$ where

$$K := \begin{bmatrix} x_{\text{min}}, x_{\text{max}} \end{bmatrix} \times \begin{bmatrix} h_{\text{min}}, h_{\text{max}} \end{bmatrix} \times \begin{bmatrix} v_{\text{min}}, v_{\text{max}} \end{bmatrix} \times \begin{bmatrix} \gamma_{\text{min}}, \gamma_{\text{max}} \end{bmatrix} \times \begin{bmatrix} \alpha_{\text{min}}, \alpha_{\text{max}} \end{bmatrix}, \quad (5)$$

where for instance $x_{\text{min}}$ defines the lower altitude below which the abort landing is very difficult, $h_{\text{max}}$ is a reference altitude (the cruise altitude for instance), $v_{\text{max}}$ is given by the aircraft constructor, and $v_{\text{min}} > 0$ is the desired minimum velocity value (see also appendix A for the values of $x_{\text{min}}, x_{\text{max}}, \text{etc.}$).

C. Optimality criterion

In the case of wind-shear, the Airport Traffic Control Tower has to choose between two options. The first one is to penetrate inside the wind shear area and try to make a successful landing. If the altitude is high enough, it is safer to choose another option: the abort landing, in order to avoid any unexpected instability of the aircraft. In this article we focus on this second option.

Starting from an initial point $y \in \mathbb{R}^d$, the optimal control problem is to maximize the lower altitude over a given time interval, that is,

$$\max \left( \min_{\theta \in [0, T]} h(\theta) \right)$$

where $h(\theta)$ is the altitude at time $\theta$ corresponding to the second component of the vector $y^u(\theta)$ solution of (4) at time $\theta$ and such that $y^u(0) = y$.

For commodity, the problem is recasted into a minimization problem as follows. Let $H_r > 0$ be a given reference altitude, and set

$$\Phi(y) := H_r - h, \quad (6)$$

where $h$ is the second component of the vector $y$.

The state constrained control problem with a maximum cost related to $\Phi$, hereafter denoted $(F_1)$, is now to minimize

$$\max_{\theta \in [0, T]} \Phi(y^u(\theta))$$

over all controls $u \in U$ and corresponding trajectories such that

$$\forall \theta \in [0, T], \quad y^u(\theta) \in K. \quad (7)$$
III. HAMILTON JACOBI BELLMAN APPROACH

Consider the value function associated to \((P_t)\) and defined, for \(t \in [0, T]\) and \(y \in \mathbb{R}^d\), by:

\[
\vartheta(t, y) := \inf_{u \in \mathcal{U}} \Phi(y^u(\theta)) \text{ where } y^u(\theta) := \left\{ \max_{\theta \in [0, t]} \Phi(y^u(\theta)) \right\},
\]

First, we point out that no controllability assumption is made in this paper, which means that for some values of \((t, y)\), it may happen that there is no admissible trajectory \(y\) that stays in \(\mathcal{K}\) on \([0, t]\). In this case, the value \(\vartheta(t, y)\) is equal to \(+\infty\). However, since \(\mathcal{K}\) is a bounded set, there exists some constants \(m \leq M\) such that

\[
\Phi(y) \in [m, M], \quad \forall y \in \mathcal{K}.
\]

Therefore, for any \((t, y) \in [0, T] \times \mathbb{R}^d\) such that \(\vartheta(t, y) < +\infty\), we have \(\vartheta(t, y) \in [m, M]\). We shall now use some ideas developed recently in [6], [7] in order to characterize the value function \(\vartheta\).

Throughout this paper we denote by \(d(y, \mathcal{K})\) the distance to \(\mathcal{K}\) for the \(\ell^\infty\) norm. Now, let us introduce the signed distance to \(\mathcal{K}\) defined by:

\[
g(y) := \left\{ \begin{array}{ll}
d(y, \mathcal{K}) & \text{if } y \notin \mathcal{K}, \\
d(y, \mathbb{R}^d \setminus \mathcal{K}) & \text{otherwise}.
\end{array} \right.
\]

Let \(\eta > 0\) be a constant. We define an extended set \(\mathcal{K}_\eta\) as follows:

\[
\mathcal{K}_\eta := \mathcal{K} + \eta \mathbb{B}_\infty,
\]

where \(\mathbb{B}_\infty := \{ y \mid \|y\|_{\ell^\infty} \leq 1 \}\) is the unit ball of \(\mathbb{R}^d\) centered at the origin (here \(\|y\|_{\ell^\infty} := \max |y_i|\) is the \(\ell^\infty\) norm of \(\mathbb{R}^d\)). Hence \(g(y) \leq 0 \iff y \in \mathcal{K}\), as well as

\[
g(y) \leq \eta \iff y \in \mathcal{K}_\eta.
\]

Using the usual notations

\[
a \lor b := \max(a, b), \quad \text{and} \quad a \land b := \min(a, b),
\]

we define an auxiliary cost function \(\Psi\) as follows:

\[
\Psi(y, z) := \left( \Phi(y) - z \right) \lor g(y) \land \eta
\]

where \(\Phi\) is as in (6). We then introduce an auxiliary value function \(w\) defined by:

\[
w(t, y, z) := \inf_{y \in \mathcal{S}_{[0,t]}(y)} \max_{\theta \in [0, t]} \Psi(y^u(\theta), z).
\]

where \(\mathcal{S}_{[0,t]}(y)\) denotes the set of admissible trajectories starting from \(y\) along the time interval \([0, t]\).

The advantage of using the value \(w\) is also that it has always a well defined finite value, whereas this is not the case of \(\vartheta(t, x)\) when the problem is not feasible (i.e. no trajectory satisfies the state constraints). A second important property of \(w\) is that it obeys a dynamic programming principle, from which we can deduce an HJB equation.

\[
\text{Theorem 3.1: (i) The value function } w \text{ is the unique Lipschitz continuous solution (in the viscosity sense, see [8]) of the following Hamilton Jacobi equation}
\]

\[
\min \left( \partial_tw + H(y, \nabla_yw), \ w - \Psi(y, z) \right) = 0,
\]

in \([0, T) \times \mathcal{K}_\eta \times [m, M]\),

\[
w(0, y, z) = \Psi(y, z), \quad (y, z) \in \mathcal{K}_\eta \times [m, M],
\]

\[
w(t, y, z) = \eta,
\]

for \(t \in [0, T), \ y \notin \mathcal{K}_\eta, \ z \in [m, M]\),

where the Hamiltonian function \(H\) is defined by

\[
H(y, p) := \sup_{u \in \mathcal{U}} \left( - (f(y, u), p) \right).
\]

(ii) For any \(t \in [0, T], \ y \in \mathbb{R}^d\), and \(z \in \mathbb{R}\), we have: \(\eta \geq w(t, y, z) \geq \Psi(y, z) \geq (g(y) \lor k(z)) \land \eta\).

(iii) Finally, \(w\) describes the epigraph of \(\vartheta\) as follows:

\[
\vartheta(t, y) = \inf \left\{ z \in [m, M], \ w(t, y, z) \leq 0 \right\}
\]

A more detailed mathematical analysis can be found in [7]. Statement (i) of Theorem 3.1 says that it possible to compute \(w\) on the bounded set \([0, T] \times \mathcal{K}_\eta \times [m, M]\), and then by (iii) it is also sufficient to get the value \(\vartheta\).

To complete this section let us define also the exit time function \(T: \mathbb{R}^{d+1} \to[0, T]\), which associates with each starting point \((y, z) \in \mathbb{R}^{d+1}\), the maximum time to remain inside the epigraph of the function \(\Phi(E \pi(\Phi)) := \{(y, z), \Phi(y) \leq z\}\) with an admissible \(y^u(\cdot)\) solution of (4) for an admissible control \(u \in \mathcal{U}\), i.e.

\[
T(y, z) := \sup \left\{ t \in [0, T] \mid \exists u \in \mathcal{U}, \ s.t. \right. \left. \left( y^u(\theta), z \right) \in \Delta_1, \ \forall \theta \in [0, t] \right\},
\]

where

\[
\Delta_1 := \left\{ \tilde{y} = (y, z) \in \mathbb{R}^{d+1} \mid y \in \mathcal{K} \ \text{and} \ \tilde{y} \in E \pi(\Phi) \right\}.
\]

\[
\text{Theorem 3.2: The exit time function } T \text{ satisfies the following relations:}
\]

\[
\begin{align*}
(i) \quad & T(y, z) = t \Rightarrow w(t, y, z) = 0, \\
(ii) \quad & \vartheta(t, y) = \inf \left\{ z \mid T_1(y, z) \geq t \right\}.
\end{align*}
\]

The function \(T\) has one less variable compared to the value function \(w\). Still, it will be interesting in order to propose a different optimal trajectory reconstruction procedure.

IV. NUMERICAL SCHEME AND SOLVER

A. NUMERICAL APPROXIMATION OF THE VALUE FUNCTION AND OF THE EXIT TIME FUNCTION

We consider finite difference approximation methods for the approximation of the Hamilton-Jacobi-Bellman equation (10). For the general approach see the seminal paper [9], introducing monotone and convergent finite difference schemes. In our case we have considered slightly more
precise schemes, in particular an Essentially Non Oscillatory (ENO) scheme of second order of [10]. These schemes have been numerically observed to be very efficient. Notice that we could have also considered other discretization methods such as Semi-Lagrangian methods (see [11], [12]). For the present application we prefer to use the ENO scheme because there is no need of a control discretization in the definition of the numerical Hamiltonian function (see H below).

Let i denote a multi-index \((i_1, \ldots, i_d)\), Let \( \Delta y := (\Delta y_i)_{i \leq i \leq d}, \Delta z, \) and \( \Delta t \) denote non-negative mesh steps, and let \( y_i := i \Delta y \equiv (i \Delta y_k)_{1 \leq k \leq d}, z_j := j \Delta z, \) \( t_n := n \Delta t. \) Define the following grid of \( \mathcal{K} \times [m, M] \):

\[
\mathcal{G} := \{ (y_i, z_j) \in \mathcal{K} \times [m, M] : (y_i, z_j) \in \mathcal{K} \times [m, M] \},
\]

as well as \( \psi_{y, z} := \Psi(y_i, z_j). \) In the following, we denote \( W_i^n \) an approximation of the solution \( w \) at the node \((t_n, y_i, z_j)\).

Given a numerical Hamiltonian \( H: \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R} \) (see Remarks 4.1-4.3 below) the following scheme is considered:

\[
W_{i,j}^{n+1} = \max \left( W_{i,j}^n - \Delta t H(y_i, D^+W_{i,j}^n, D^-W_{i,j}^n), \psi_{y, z} \right) \\
0 \leq n \leq N - 1, (y_i, z_j) \in \mathcal{G}, W_{i,j}^0 = \psi_{y, z}, (y_i, z_j) \in \mathcal{G}.
\]

The monotone (first order) finite difference approximation is obtained when using \( D^\pm W_{i,j}^n := (D^\pm W_{i,j}^n)_{1 \leq i \leq d} \), with

\[
D_k \pm W_{i,k}^n := \pm (W_{i,k}^n - W_{i,k-j}^n)/\Delta y_k.
\]

The second order ENO scheme [10] is used in order to estimate more precisely the terms \( D_k \pm W_{i,k}^n \).

**Remark 4.1:** If the numerical Hamiltonian \( H \) is Lipschitz continuous on all its arguments, consistent with \( H(H(y,p,p) = H(y,p)) \) and monotone (i.e., \( \frac{\partial H}{\partial p_0}(y, p^0, p^0) \leq 0, \frac{\partial H}{\partial p_i}(y, p^0, p^0) \geq 0 \)) together with the following Courant-Friedrichs-Lewy (CFL) condition

\[
\Delta t \leq \frac{1}{\Delta y} \max_{y, p^0, p^0} \left| \Delta y \right| \left| \frac{\partial H}{\partial p_0}(y, p^0, p^0) + \frac{\partial H}{\partial p_i}(y, p^0, p^0) \right| \leq 1, \quad (14)
\]

then the scheme \( W_i^n \) converges to the desired solution (see [13] for more details).

**Remark 4.2:** Since the control variable \( u \) enters linearly and only in 5th component of the dynamics \( f \) in (4) the Hamiltonian \( H(y, p) \) takes the following simple analytical form (here in the case of \( u_{\max} = -u_{\min} \geq 0 \)):

\[
H(y, p) := \sum_{i=1}^4 -f_i(y)p_i + u_{\max}[p_5]
\]

where \((p_i)_{i \leq d} \) are the components of \( p \) and \((f_1(y), \ldots, f_4(y)) \) are the first four components of \( f \).

**Remark 4.3:** A simple numerical Hamiltonian \( H \) is obtained as follows (Lax-Friedrich Hamiltonian): \( H(y, p^-, p^+) = H(y, p^- + \frac{p^+ - p^-}{2} - \sum_{k=1}^4 \frac{p_k}{2}, p_k - p_k) \), with constants \( c_k \geq \frac{\partial H}{\partial p_k} \), can be shown to satisfies all the required conditions of Remark 4.1 for a sufficiently small time step \( \Delta t \) (i.e., satisfying (14)).

**Remark 4.4:** In comparison with the recent work of Botkin and Turova [5], let us mention that in our model we deal with a full 5-dimensional model (as in [1], [4]) and not an approximated one. This is made possible by the use of the ROC-HJ software [14] for solving HJB equations.

### B. Optimal trajectory reconstruction algorithms

Assuming that the value function has been computed, we consider four different reconstruction algorithms for the obtention of the optimal trajectories and control laws.

The first algorithm (algorithm A) is based on the minimization of the value function \( w \). The second one (algorithm B), will involve the exit time function. The third algorithm (algorithm E), will be based again on the minimization of the value function but with a penalization term in order to reduce the control variations. Such an algorithm is supposed to provide an improvement in the quality of the optimal feedback control. The last algorithm (algorithm F), is based on the maximization of the Hamiltonian function.

In all the algorithms we shall consider \( n \in \mathbb{N}, n \geq 1 \), a time step \( h = \frac{t}{n} \) and a uniform partition of \([0, T]\) by \( t_n = nh \). We assume that \( \psi(t_n, y) < \infty \) (otherwise there is no admissible trajectories). We set \( z = \psi(t_n, y) \) and drop \( y^n \) for short. Then we iterate for \( k = 0, \ldots, n - 1 \), knowing the state \( y_k \) at time \( t_k \), as follows.

**Algorithm A (value function).** Step 1. Choose

\[
u_k \in \arg\min_{u \in U} \left\{ w_{t_n-k}, y^k + hf(y^k, u), z \right\} \psi(y^k, z),
\]

where \([w](y, z)\) denotes a polynomial interpolation of \( w(., z) \) in the variable \( y \).

Step 2. Define \( u^n(t) := u^n_k \) as the constant control and \( y^n(t) \), for \( t \in [t_k, t_{k+1}] \), as the solution of

\[
\dot{y}(t) := f(y(t), u^n(t)), \quad t \in [t_k, t_{k+1}], \quad y(t_k) = y^k,
\]

and set \( y^n_{k+1} := y^n(t_{k+1}) \).

**Algorithm B (exit time).** Step 1. Choose

\[
u_k \in \arg\max_{u \in U} \left\{ t \right\} \psi(y^k + hf(y^k, u), z \right\} \psi(y^k, z)
\]

where \([T](y, z)\) denotes a polynomial interpolation of \( T(., z) \) in the variable \( y \).

Step 2. Same as in algorithm A, step 2.

The following algorithm also penalizes the variations of the control.

**Algorithm E (value function and penalization).** Step 1. Let \( \lambda \) be a positive constant. For \( k = 0 \) it is the same as Algorithm (A). For \( k \geq 1 \), choose

\[
u_k \in \arg\min_{u \in U} \left\{ w_{t_n-k}, y^k + hf(y^k, u), z \right\} \psi(y^k, z)
\]

\[
+\lambda[u - u_{k-1}]
\]

Step 2. Same as algorithm A, step 2.

**Algorithm F (Hamiltonian function).** Same as in algorithm A excepted that, denoting \( [DW^n](x_k) \) a numerical approximation of the gradient of \( W^n \) at point \( x_k \), we choose

\[
u_k \in \arg\max_{u \in U} \left( -f(x_k, u) \right) \psi(y^k, z)
\]

\(3633\)
(this choice is related to Eq. (11)).

A convergence analysis of algorithms A and E (when \( n \lambda \to 0 \) as \( n \to \infty \)) is given in [7]. Convergence of algorithms B and F is the subject of ongoing works.

V. NUMERICAL RESULTS

We have first computed the value function as well as the exit time function on a grid of size \( N_G = 30^3 \times 20^2 \times 10 \): 30 points per axis for the first three components (\( x, h \) and \( v \)), 20 points per axis for the angles \( \gamma \) and \( \alpha \), and 10 points for the variable \( z \). For a realistic model with aerodynamic forces close to the real ones, we consider the strong wind with coefficient \( k = 1 \) in (15). Moreover, we start from an altitude close to the lower boundary of the constraints in the altitude, i.e., initial point

\[
y_0 = (0.0 m, 600.0 m, 239.7 m s^{-1}, -2.249 \ deg, 7.373 \ deg).
\]

In Figure 1 we have plotted the optimal trajectories obtained using the algorithms A (in red) and B (in blue). One can observe that these results are quite similar.

![Fig. 1. Comparison of the optimal trajectories obtained by value (red) and exit time (dashed blue) reconstruction methods.](image)

We remark that algorithm A requires more memory and CPU time. Indeed, this algorithm needs the value function \( w \) to be stored at each time step of the numerical scheme. For a refined grid this can be very expensive. On the other hand, algorithm B, based on the exit time function, needs only one final saving at the end of the HJB computation. It is a good alternative in order to use less memory, and is less CPU time consuming for a high-dimensional state space.

Now we focus on the last state variable \( \alpha \), see Figure 1 (middle bottom). One can remark a chattering phenomenon. This may appear in particular when the value function or the exit time function have locally only small variations, leading to numerical difficulties in finding an optimal control (as in the algorithms A and B). We have numerically observed that the shattering phenomena diminishes when the refinement of the grid increases.

On the other hand, by adding a penalty term on the variation of the control variable in the reconstruction algorithm E, we see in Figure 3 (bottom left) that we can diminish the variations of \( \alpha \). In this figure, the angle of attack is also plotted for the different reconstruction algorithms.

In Figure 2, the heights of the different trajectories with respect to the \( x \) (position) variable is shown, for the different reconstruction algorithms.

The minimal altitude obtained for each tested method is reported in Table I. The reconstruction by minimizing the value function (algorithm A) provides the best performance since it gives the maximal minimal altitude.

![Fig. 2. History of the trajectory in the plane \((Oxh)\) for the control problem with maximum cost using different methods of reconstruction.](image)

<table>
<thead>
<tr>
<th>TABLE I</th>
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<td>THE OPTIMALITY CRITERION AT TIME ( t = T ) OBTAINED WITH DIFFERENT ALGORITHMS.</td>
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<tr>
<td>Minimal altitude (m)</td>
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<td><strong>algorithm A</strong></td>
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<td><strong>algorithm B</strong></td>
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<tr>
<td><strong>algorithm E</strong></td>
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</tbody>
</table>

The regularization allows to improve partially the performance. From Figure 3 one can conclude that the algorithm of reconstruction by the exit time function allows to obtain a smoother behaviour of the angle of attack \( \alpha (\cdot) \).

APPENDIX

The data corresponding to a Boeing B 727 aircraft is considered. The wind velocity components relative to the winshere model satisfies the following relations:

\[
w_v(x) = kA(x), \quad w_h(x, h) = \frac{h}{h_*} B(x),
\]

where
and the constants $a$, $b$, $c$, $d$, $e$ are given in Table II.

**TABLE II**

**BOEING 727 AIRCRAFT MODEL DATA.**

<table>
<thead>
<tr>
<th>$\rho$</th>
<th>$2.203 \times 10^{-3}$</th>
<th>Ib sec$^2$ ft$^{-4}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S$</td>
<td>$1.56 \times 10^3$</td>
<td>ft$^2$</td>
</tr>
<tr>
<td>$g$</td>
<td>$32.172$</td>
<td>ft sec$^{-2}$</td>
</tr>
<tr>
<td>$mg$</td>
<td>$1.5 \times 10^4$</td>
<td>lb</td>
</tr>
<tr>
<td>$\delta$</td>
<td>$3.49 \times 10^{-2}$</td>
<td>rad</td>
</tr>
<tr>
<td>$A_0$</td>
<td>$4.456 \times 10^4$</td>
<td>lb</td>
</tr>
<tr>
<td>$A_3$</td>
<td>$1.442 \times 10^{-2}$</td>
<td>lbsec$^{-2}$ ft$^{-2}$</td>
</tr>
<tr>
<td>$A_1$</td>
<td>$-23.98$</td>
<td>lbsec ft$^{-1}$</td>
</tr>
<tr>
<td>$B_0$</td>
<td>$0.1552$</td>
<td>rad$^{-1}$</td>
</tr>
<tr>
<td>$B_1$</td>
<td>$0.1237$</td>
<td>rad$^{-1}$</td>
</tr>
<tr>
<td>$B_2$</td>
<td>$2.4203$</td>
<td>rad$^{-1}$</td>
</tr>
<tr>
<td>$C_0$</td>
<td>$0.7125$</td>
<td>rad$^{-1}$</td>
</tr>
<tr>
<td>$C_1$</td>
<td>$6.0877$</td>
<td>rad$^{-1}$</td>
</tr>
<tr>
<td>$C_2$</td>
<td>$-0.9277$</td>
<td>rad$^{-1}$</td>
</tr>
<tr>
<td>$\alpha_*$</td>
<td>$0.2094$</td>
<td>rad</td>
</tr>
<tr>
<td>$k$</td>
<td>$0.1$</td>
<td></td>
</tr>
<tr>
<td>$h_0$</td>
<td>$1000$</td>
<td>ft</td>
</tr>
<tr>
<td>$a$</td>
<td>$6 \times 10^{-8}$</td>
<td>sec$^{-1}$ ft$^{-2}$</td>
</tr>
<tr>
<td>$b$</td>
<td>$-4 \times 10^{-11}$</td>
<td>sec$^{-1}$ ft$^{-3}$</td>
</tr>
<tr>
<td>$c$</td>
<td>$-ln \left( \frac{3}{20} \right) \times 10^{-12}$</td>
<td>sec$^{-1}$ ft$^{-2}$</td>
</tr>
<tr>
<td>$d$</td>
<td>$-8.02881 \times 10^{-8}$</td>
<td>sec$^{-1}$ ft$^{-3}$</td>
</tr>
<tr>
<td>$e$</td>
<td>$6.28363 \times 10^{-11}$</td>
<td>sec$^{-1}$ ft$^{-3}$</td>
</tr>
</tbody>
</table>

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**REFERENCES**


