LECTURE 1

FROM DYNAMIC TO TURBULENCE MODELLING

WHY STUDY EULER NAVIER STOKES BOLTZMANN EQUATIONS....

- Whe are paid for that and there is the CLAY Prize.

- Concerns fundamental physic. By the time of Maxwell and Boltzmann was related to the molecular and atomic hypothesis.

- Progress in the field are compulsory for applications ranging from engineering problems in rarefied media (re-entry in the atmosphere of space vehicles, electric currents in semiconductors small enough to prevent thermodynamic equilibrium, reading head so close to a compact disk that the molecular regime is important etc...) to computation in turbulence.

- It contains most of the classical issues of the analysis of non linear pde.

- Here appears relations or differences between ergodicity and randomness should be understood.

- I believe that progress in the field will be related to a good understanding of the connection between different equations. At present the main obstacle is the lack of understanding of turbulence.

Hamiltonian Dynamic and Liouville equation

$$X_N = (x_1, x_2 \dots x_N)\,,\ V_N = (v_1, v_2 \dots v_N)\,,$$

$$\dot X_N = V_N \quad \dot V_N = -\nabla_{X_N} \sum_{1 \le i < j \le N} \Phi(|x_i - x_j|)$$

$$H_N \equiv \frac{1}{2} \sum_{1 \le i \le N} |v_i|^2 + \sum_{1 \le i < j \le N} \Phi(|x_i - x_j|)$$

$$\partial_t F_N + \{H_N, F_N\} = 0$$

Boltzmann Equation

With indistinguishability

$$\mu_N^k(X_k, V_k) = \int F_N(X_k, V_k, X_k^N, V_k^N) dX_k^N, V_k^N \ .$$

Boltzmann Equation

With indistinguishability

$$\mu_N^k(X_k, V_k) = \int F_N(X_k, V_k, X_k^N, V_k^N) dX_k^N, V_k^N \,.$$

$$f(x, v, t) = \lim_{N \to \infty} \mu_N^1(x, v, t)$$

$$\partial_t f + v \nabla_x f = \frac{1}{\pi \mathrm{Kn}} \mathcal{B}(f, f) \,.$$

Macroscopic Equation

Macroscopic quantities (observables) are computed by averaging the corresponding quantity for a single particle w.r.t. the measure $F(t,x,v)dv$:

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Macroscopic quantities (observables) are computed by averaging the corresponding quantity for a single particle w.r.t. the measure $F(t,x,v)dv$:

density $\rho(x,t) = \int F(t,x,v)dv,$

momentum $\rho(x,t)u(x,t) = \int vF(t,x,v)dv,$

energy $E(x,t) = \int \frac{1}{2}|v|^2 F(t,x,v)dv$

$$\partial_t \rho_\epsilon + \nabla(\rho_\epsilon u_\epsilon) = 0\,,$$

$$\sigma(u) = \frac{1}{2}\left(\nabla_x u + (\nabla_x u)^T - \frac{2}{3}\,\mathrm{div}_x\, uI\right)$$

$$\rho_\epsilon\Big(\partial_t + u_\epsilon\nabla\Big)u_\epsilon + \nabla p_\epsilon = \epsilon\nabla[\nu\sigma(u_\epsilon)]\,, p_\epsilon = \rho_\epsilon\theta_\epsilon\,,$$

$$\frac{3}{2}\rho_\epsilon\Big(\partial_t + u_\epsilon\nabla\Big)\theta_\epsilon + \rho_\epsilon\theta_\epsilon\nabla u_\epsilon = \epsilon\frac{1}{2}\nu\sigma(u_\epsilon) : \sigma(u_\epsilon) + \epsilon\nabla[\kappa\nabla\theta_\epsilon]\,.$$

Turbulence Modelling

Two types of very similar configurations.

- $\overline{u}$ is the weak limit of a sequence of oscillating solutions u_ν
- $u(x, t, \omega)$ is a random variable depending on a parameter random variable ω and one considers equations for its mean value $\overline{u}$

Turbulence Modelling

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$$
\begin{aligned}
&\partial_t u + u\nabla_x u + \nabla_x p - \nu\Delta_x u - \nabla_x\Big(c_\nu[\frac{k^2}{\epsilon}(\nabla_x u +^t \nabla_x u)]\Big) = 0\,,\\
&d_t k + u\nabla_x k - \frac{c_\nu k^2}{2\epsilon}|\nabla_x u + (\nabla_x u)^t|^2 - \nabla_x\cdot[c_\nu\frac{k^2}{\epsilon}\nabla_x k] + \epsilon = 0\,,\\
&\partial_t\epsilon + U\nabla_x\epsilon - \frac{c_1 k}{2}|\nabla_x U + (\nabla_x U)^t|^2 - \nabla_x\cdot[c_3\frac{k^2}{\epsilon}\nabla_x\epsilon] + c_2\frac{\epsilon^2}{k} = 0\,.
\end{aligned}
$$

The Scaling Parameters and Formal limits

For a monatomic gas at room temperature and atmospheric pressure, about $\mathcal{N} \simeq 10^{20}$ gas molecules with radius $\sigma \simeq 10^{-8}$cm are to be found in any volume of 1cm^3

Packed volume (i.e. the total volume occupied by the gas molecules if tightly packed): $10^{20} \times \frac{4\pi}{3} \times \sigma^{-3} \simeq 5 \cdot 10^{-4}\text{cm}^3 \ll 1\text{cm}^3$.

The mean-free path is roughly speaking, the average distance between two successive collisions for any given molecule in the gas. Intuitively, the higher the gas density, the smaller the mean-free path the bigger the molecules, the smaller the mean-free path:

$$\text{mean-free path } \approx \frac{1}{\mathcal{N} \times \pi\sigma^2}, \text{Kn} := \frac{\text{mean free path}}{\text{macroscopic length scale}}$$

The limit $\mathcal{N}\sigma^2 \to \lambda < \infty$ corresponds to a rarefied gases and was proven by Landford in 1973 for the hard spheresmodel under a chaos hypothesis:

$$\mu_N^k(X_k, V_k, 0) = \int F_N(X_k, V_k, X_k^N, V_k^N) dX_k^N, V_k^N \to \prod_{1 \le i \le k} F(x_i, v_i, t)$$

Proof Involves a Duhamel formula to compute what will happen at time t for the marginal μ_N^k in terms of the history of the process for $0 < s < t$ for μ_N^{k+1}. Rarefied gases $\Rightarrow$ only binary collisions important, Uniqueness shows:

$$\mu_N^k(t) = \prod_{1 \le i \le k} F(x_i, v_i, t)$$

with F solution of the Boltzmann equation.

$$\partial_t F + v \nabla_x F = \frac{1}{\mathrm{Kn}} \mathcal{B}(F, F)$$

Boltzmann and Euler equations nonlinear $\Rightarrow$ proof for a very small time of order of the Knudsen number.

ENTROPY

$$\int_{\mathbf{R}^3} \mathcal{B}(F,F)\log F \le 0 \; = 0 \Leftrightarrow F = \frac{\rho(x,t)}{(2\pi\theta(x,t))^{\frac{3}{2}}} e^{-\frac{|v-u(x,t)|^2}{2\theta(x,t)}}$$

$$H(\mu_N^k(X_k,V_k) = \frac{1}{k}\int \mu_N^k(X_k,V_k)\log(\mu_N^k(X_k,V_k))dX_k dV_k$$

$$\text{Liouville equation} \Rightarrow H(\mu_N^N(t)) = H(\mu_N^N(0)$$

$$f(x,v,t) = \lim \mu_N^1(X_k,V_k)(t)\,,\; H(f)(t) < H(f)(0)$$

No obstruction to strong convergence.

Reason is $H_N^k \le H_N^N$ with equality if and only if H_N^N is factorised. This holds for initial data but not for H_N^N because the Hamiltonian flow breaks the factorisation.

The Compressible Euler limit Kn $= \epsilon \longrightarrow 0$

The collision operator satisfies the moment conservation, the entropy decay and relaxation toward a Maxwellian.

$$\int_{\mathbf{R}^3} \mathcal{B}(F,F)dv = \int_{\mathbf{R}^3} v_k \mathcal{B}(F,F)dv = \int_{\mathbf{R}^3} \frac{1}{2}|v|^2 \mathcal{B}(F,F)dv = 0$$

$$\int_{\mathbf{R}^3} \mathcal{B}(F,F) \log F \leq 0 \;= 0 \Leftrightarrow F = \frac{\rho(x,t)}{(2\pi\theta(x,t))^{\frac{3}{2}}} e^{-\frac{|v-u(x,t)|^2}{2\theta(x,t)}}$$

$$\partial_t \int_{\mathbf{R}^3} F_\epsilon dv + \mathrm{div}_x \int_{\mathbf{R}^3} v F_\epsilon dv = 0\,, \text{ (mass)}$$

$$\partial_t \int_{\mathbf{R}^3} v F_\epsilon dv + \mathrm{div}_x \int_{\mathbf{R}^3} v^{\otimes 2} F_\epsilon dv = 0\,, \text{ (momentum)}$$

$$\partial_t \int_{\mathbf{R}^3} \frac{1}{2}|v|^2 F_\epsilon dv + \mathrm{div}_x \int_{\mathbf{R}^3} v \frac{1}{2}|v|^2 F_\epsilon dv = 0\,, \text{ (energy)}$$

$$\partial_t \int_{\mathbf{R}^3} F_\epsilon \log F_\epsilon dv + \mathrm{div}_x \int_{\mathbf{R}^3} v F_\epsilon \log F_\epsilon dv$$

$$-\frac{1}{\epsilon} \int \mathcal{B}(F_\epsilon, F_\epsilon) \log(F_\epsilon), \leq 0 \quad \text{(entropy)}$$

The last equation forces the limit to be a Maxwellian

$$F = \frac{\rho(x,t)}{(2\pi\theta(x,t))^{\frac{3}{2}}} e^{-\frac{|v-u(x,t)|^2}{2\theta(x,t)}}$$

Replacing above gives the compressible Euler equation with decay of entropy.

C. Morrey derivation CPAM 1955

- The above derivation provides only the law of perfect gases:

$$p = \rho\theta\,, \quad E = \rho(\frac{1}{2}|u|^2 + \frac{3}{2}\theta)$$

Reason is the construction is done for **rarefied gases**.

- Start again from

$$H_N \equiv \frac{1}{2}\sum_{1\le i\le N}|v_i|^2 + \sum_{1\le i<j\le N}\Phi(|x_i - x_j|)$$

- Rescale the potential

$$\Phi(r) = U(\frac{r}{\sigma})$$

Indistinguishability

$$\partial_t F_N + V_N \nabla_{X_N} F_N = \frac{1}{2} \sum_{1 \le i \le N} \left(\sum_{1 \le j \ne i \le N} \frac{1}{\sigma} U'(\frac{|x_i - x_j|}{\sigma}) \frac{x_i - x_j}{|x_i - x_j|} \right) \cdot \nabla_{v_i} F_N$$

- Write the equations for the first marginal

$$\partial_t \mu_N^1 + v_1 \nabla_{x_1} \mu_N^1 =$$
$$\frac{N-1}{2} \nabla_{v_1} \int \frac{1}{\sigma} U'(\frac{|x_1 - x_2|}{\sigma}) \frac{x_1 - x_2}{|x_1 - x_2|} \mu_N^2(t, x_1, v_1, x_2, v_2) dx_2 dv_2$$

- Multiply by $(1, v_1, \frac{1}{2}|v_1|^2)$ integrate with respect to v_1 local conservation laws.

- introduce a fast variable $\xi = \frac{(x_2 - x_1)}{\sigma}$ and define

$$G_N^2(t, x_1, v_1, \xi, v_2) = \mu_N^2(t, x_1, v_1, x_1 + \sigma\xi, v_2)$$

Assume the convergence of G^2 and convergence to a dense gase.

$$\mathrm{lim}(\frac{4}{3}\pi\sigma^3 N) = R^*(\text{Volume Macroscopic}) \quad R^* \in]0, 1]$$

$$\partial_t \int \mu^1 dv_1 + \nabla_{x_1} \int v_1 \mu^1 dv_1 = 0$$

$$\partial_t \int v_1 \mu^1 dv_1 + \nabla_{x_1} \int v_1 \otimes v_1 \mu^1 dv_1 =$$

$$\frac{3}{16\pi} R^* \nabla_{x_1} \iiint U'(|\xi|) \frac{\xi \otimes \xi}{|\xi|} \mu^2(t, x_1, v_1, \xi, v_2) d\xi dv_2 dv_1$$

$$\partial_t \left(\int \frac{|v_1|^2}{2} dv_1 + \frac{3}{16\pi} R^* \iiint U'(|\xi|) \frac{\xi \otimes \xi}{|\xi|} \mu^2(t, x_1, v_1, \xi, v_2) d\xi dv_2 dv_1 \right) +$$

$$\nabla_{x_1} \left(\int v_1 \frac{|v_1|^2}{2} dv_1 + \frac{3}{16\pi} R^* \iiint v_1 U'(|\xi|) \frac{\xi \otimes \xi}{|\xi|} \mu^2(t, x_1, v_1, \xi, v_2) d\xi dv_2 dv_1 \right)$$

$$= -\frac{3}{32\pi} R^* \nabla_{x_1} \iiint U'(|\xi|) \frac{\xi \otimes \xi}{|\xi|} (v_1 + v_2) \mu^2(t, x_1, v_1, \xi, v_2) d\xi dv_2 dv_1$$

For μ^1 and μ^2 a closure argument is needed

$$\mu^1 = \frac{\rho(x,t)}{(2\pi\theta(x,t))^{\frac{3}{2}}} e^{-\frac{|v-u(x,t)|^2}{2\theta(x,t)}},$$

$$\mu^2 = \frac{\rho(x,t)}{(2\pi\theta(x,t))^{\frac{3}{2}}} C(|\xi|,\theta) e^{-\frac{|v_1-u(x,t)|^2+|v_2-u(x,t)|^2+2U(|\xi|)}{2\theta(x,t)}}.$$

The Von Karman Relation and the incompressible limit

• The compressible Euler equation gives an approximation of order ϵ of the moments of the Boltzmann equation.

• Chapman Enskog approximation consist in finding $(\rho_\epsilon, u_\epsilon, \theta_\epsilon)$ such that the corresponding moments of

$$\frac{\rho_\epsilon(x,t)}{(2\pi\theta_\epsilon(x,t))^{\frac{3}{2}}} e^{-\frac{|v-u_\epsilon(x,t)|^2}{2\theta_\epsilon(x,t)}}$$

provides an ϵ^2 approximation of the moments of the Boltzmann equation. For this purpose cns $(\rho_\epsilon, u_\epsilon, \theta_\epsilon)$ solution of a compressible Navier Stokes equation with viscosity $\nu(\epsilon)$ and thermal diffusivity $\kappa(\epsilon)$ of the order of ϵ. In a dimensionnal form viscosity is

$$\frac{1}{\text{Reynolds}} = \frac{\nu(\epsilon)}{UL}, \epsilon \simeq \frac{\text{Mach}}{\text{Reynolds}}$$

Analysis of ϵorder fluctuations: perturbation of some uniform Maxwellian

$$M = M_{1,0,1} = \frac{1}{(2\pi)^{-\frac{3}{2}}} e^{-\frac{|v|^2}{2}}$$

Ma=O(ϵ) , $F_\epsilon = M(1 + \epsilon g_\epsilon(x, v, t))\ \epsilon\partial_t F_\epsilon + v\nabla_x F_\epsilon = \epsilon^{-q}\mathcal{B}(F_\epsilon, F_\epsilon)$

Second order correction $\Rightarrow$ the properties of the derivative of $\mathcal{B}$ play an important role.

$$2M^{-1}(\mathcal{B}(Mg, M) = \mathcal{L}(g)\,, M^{-1}(\mathcal{B}(Mg, Mg) = \mathcal{Q}(g, g)$$

$\mathcal{L}$ self adjoint and Fredholm for the scalar product

$$\langle f, g\rangle = \int Mf(v)g(v)dv$$

$$Ker(\mathcal{L}) = \{1, v, \frac{|v|^2}{2} - \frac{3}{2}\}\,, A(v)\ < A(v) >= 0\,, \mathcal{L}(A(v)) = v \otimes v - \frac{1}{3}|v|^2 I$$

$$\epsilon\partial_t g_\epsilon + v\cdot\nabla_x g_\epsilon = \frac{1}{\epsilon^q}\mathcal{L}(g_\epsilon) + \epsilon\mathcal{Q}(g_\epsilon, g_\epsilon)\; q \geq 1\,.$$

$$g_\epsilon \to \rho(x,t) + u(x,t)\cdot v + \theta(\frac{|v|^2 - 3}{2})$$

$$\epsilon\langle v, \partial_t g_\epsilon\rangle + \nabla_x\langle v\otimes v_\epsilon\rangle = 0 \Rightarrow \nabla u \equiv 0$$

$$\partial_t\langle v g_\epsilon\rangle + \nabla_x\langle\frac{v\otimes v\cdot - \frac{1}{3}|v|^2 I}{\epsilon} g_\epsilon\rangle + \nabla p_\epsilon = 0$$

Compute

$$\lim_{\epsilon\to 0}\langle\frac{v\otimes v\cdot-\frac{1}{3}|v|^2 I}{\epsilon}g_\epsilon\rangle$$

$$\langle\frac{v\otimes v\cdot-\frac{1}{3}|v|^2 I}{\epsilon}g_\epsilon\rangle=\langle\frac{\mathcal{L}(A(v))}{\epsilon}g_\epsilon\rangle=\langle A(v),\frac{\mathcal{L}(g_\epsilon)}{\epsilon}\rangle$$

$$=\epsilon^q\partial_t\langle A(v)g_\epsilon\rangle+\epsilon^{q-1}\nabla_x\langle A(v):(v\otimes v\cdot-\frac{1}{3}|v|^2 I)g_\epsilon\rangle-\langle A(v)\mathcal{Q}(g_\epsilon,g_\epsilon)\rangle$$

$$\lim\nabla_x\langle A(v):(v\otimes v\cdot-\frac{1}{3}|v|^2 I)g_\epsilon\rangle=\nu(\nabla u+\nabla^\perp u)$$

$$\lim\langle A(v):\mathcal{Q}(g_\epsilon,g_\epsilon)\rangle=\nabla(u\otimes u)$$

Incompressible Navier Stokes for $q=1$ and incompressible Euler equation for $q>1$.

Remarks

- The form of the limits of

$$\nabla_x \langle A(v) : (v \otimes v \cdot -\frac{1}{3}|v|^2 I) g_\epsilon \rangle \text{ and } \langle A(v) : \mathcal{Q}(g_\epsilon, g_\epsilon) \rangle$$

comes from the galilean invariance (not mentionned or used before) of collision operator.

- The heat equation and the Boussinesq relation between fluctuation of density and temperature are obtained in the same way.

- The above results are proven under convenient convergence hypothesis

- Rigourous results are in full agreement with the state of the art for macroscopic equation.

- Convergence is proven for H^s fluctuations for a finite time or all time if the fluctuation are small with respect to the initial data (Bardos Ukai)

- Convergence to the Euler limit is proven (L. Saint Raymond) as long as a smooth solution of the Euler equation exists.

- Convergence of any Lions-DiPerna renormalised solution of the Boltzmann equation to a Leray solutionof the Navier Stokes equation has been recently proven by F. Golse and L. Saint Raymond.

Mathematical approach to Turbulence

With the introduction of $\overline{\nu}$ the viscosity, U the characteristic velocity of the fluid and L the characteristic size of the vessel (quantities which may be “local") the incompressible Navier Stokes system is

$$\partial_t u + u\nabla u + \nabla p - \frac{1}{\mathrm{Re}}\Delta u = 0 \quad \nabla p = 0\,. \quad \nu^{-1} = \mathrm{Re} = \frac{UL}{\overline{\nu}}$$

Some values of Re: Industrial fluids (pipes ships...) 10^4, Wings of airplanes 10^6, Space Shuttle10^8, Weather Forcast, Oceanography 10^{10}, Astrophysic 10^{12}. In a fluid structures of the order less than $(\mathrm{Re})^{-\frac{3}{4}}$ are damped by the viscosity. Others persist. In a numerical computation this implies a mesh size h of the same order and therefore for $3d$ codes in aircraft industry a number N of degree of freedom $N = 10^{13.5}$. At present computers handle at most N^9 degree of freedom.

Two approaches Statistic and Deterministic

- Assume that the fluid can be described as a random variable

$$u(x,t,\omega) = \overline{u}(x,t,\omega) + u_f(x,t,\omega)\ \langle u \rangle = \overline{u}$$

- Consider limit of solutions

$$\overline{u} = \lim_{\nu \to 0}$$

In both cases

$$\nu \int_0^t \int |\nabla u_\nu|^2 dx ddt \le C < \infty$$

and **closure problem**

$$\overline{u \otimes u} = \overline{u} \otimes \overline{u} - RT \text{ with } RT \text{ in general } \neq 0$$

Spectra

$$R_{stat}(x,t,k) = (\frac{1}{2\pi})^n \int e^{-ik\cdot y} < u(x+\frac{y}{2},t) \otimes u(x-\frac{y}{2},t) > dy\,,$$

$$< u >= 0\; RT(x,t) = \int R_{stat}(x,t,k)dk$$

$$R_\nu(x,t,k) = (\frac{1}{2\pi})^n \int e^{-ik\cdot y} < \tilde{u}_\nu(x+\frac{\sqrt{\nu}y}{2},t) \otimes \tilde{u}_\nu(x-\frac{\sqrt{\nu}y}{2},t) > dy$$

with $\tilde{u} = u - \lim_{\nu\to 0} u$

$$\lim_{\nu\to 0} \tilde{u}_\nu(x,t) \otimes \tilde{u}_\nu(x,t) = \int \lim_{\nu\to 0} R_\nu(x,t,k)dk\,.$$

• $\lim_{\nu\to 0} \tilde{u}_\nu(x,t) \otimes \tilde{u}_\nu(x,t)$ is a positive tensor, a “defect measure" in the sense of Tartar Gérard and others. The last formula is a consequence of a theorem of Gerard Markowich Mauser and Poupaud.

- The “spectra" R is local in (x,t) . It depends only on the high frequencies and on the two points correlation. It should be isotropic as this is observed in statistical theory of turbulence by M. Cherkov A. Pumir and B. Shraiman.

$$RT(x,t,k) = R(|k|,x,t)(I - \frac{k \otimes k}{|k|^2})\,.$$

- Assuming that it is due to the macroscopic effect of the fluid and depends only of ∇u it takes by Galilean invariance the form

$$RT = \nu_{\text{turbulent}}(\nabla \overline{u} + \nabla \overline{u}^T)$$

- The k ϵmodel gives rules for the computation of $\nu_{\text{turbulent}}$

- One should observe that the derivation is even much less complete than any of the previous step.

No intrinsic parameters no relaxation to equilibrium however I believe that any proof in any configuration of the validity of such ansatz would provide huge contribution to the rest of the program.

LECTURE 2

PROOFS FOR THE EULER EQUATION

Various form of the Euler equation

$$\partial_t u + \nabla(u \otimes u) + \nabla p = 0\,; \quad \nabla u = 0\,,$$

$$\partial_t u + (\nabla \wedge u) \wedge u + \nabla(p + \frac{|u|^2}{2}) = 0\,; \quad \nabla u = 0$$

$$\partial_t \omega + u \nabla \omega - \omega \nabla u = 0\,; \quad \nabla \wedge u = \omega \quad \nabla u = 0$$

plus boundary conditions $u \cdot \vec{n} = 0$ on $\partial\Omega$

In $2d$

$$\partial_t \omega + u \nabla \omega = 0\,; \quad \nabla \wedge u = \omega \quad \nabla u = 0$$

And

$$\int |\omega(x,t)|^p dx = \int |\omega(x,t)|^p dx \text{ for } 1 < p \leq \infty \text{ formally for } p = 1$$

- I as said I believe that serious progress may only come from a better understanding of the limits for $\nu \to 0$. In practical problems scaled $\nu \sim 10^{-8}$.

- When $\lim(u_\nu \otimes u_\nu) = (\lim u_\nu) \otimes (\lim u_\nu)$ one obtains the Euler Equation. For $\lim(u_\nu \otimes u_\nu)(\neq (\lim u_\nu) \otimes (\lim u_\nu)) = (\lim u_\nu) \otimes (\lim u_\nu) + RT.$ This is what I call **Turbulence**

- Most of the talks are devoted to estimates. From functionnal analysis we have learned that with relevant estimates we are in good shape to complete the analysis (limits perturbation finite dimensional approximation and so on)

- One of the basic ingredient is the elliptic equation:

$$\text{in } \Omega\,, \nabla_x \cdot u = 0, \nabla_x \wedge u = \omega, u \cdot n = 0 \text{ on } \partial\Omega\,,$$

$$u = \Psi\,, -\Delta\Psi = \omega, \Psi = c \text{ on } \partial\Omega\,,$$

$$u = c_d \int_\Omega (\frac{x-y}{|x-y|^d} + \gamma(x,y)) \wedge \omega(y) dy$$

THE $2d$ EULER EQUATION

The existence theorem of Delors

Theorem. *Assume for the initial data* $u_0(x_1, x_2)$ *the following property:*

$$u_0 \in L^2(\mathbf{R}^2) \quad \nabla \wedge u_0 = \omega'_0 + \omega''_0$$

with $\omega''_0 \in L^p(\mathbf{R}^2)\,, \quad p > 1$ *and* ω'_0 *a finite positive measure. Then there exists at least one "weak" solution*

$$u \in L^\infty_{loc}(\mathbf{R}_t; L^2_{loc}(\mathbf{R}^2)\,.$$

Proof With $\omega'_0 = 0$ and $p = 2$ the proof is easy. It has been extended by Di-Perna-Majda to the case $p > 1$. The case $p = 1$ or equivalently $\omega'_0 \neq 0$ not so easy.

Basic ingredients (Delors) for $\omega = \omega_0' \geq 0$. The unique obstacle is the concentration that may appears in expressions of the form:

$$u^\epsilon \otimes u^\epsilon$$

with u^ϵ any convenient (with L^1 bounded vorticity) approximation and which may lead to

$$\lim(u^\epsilon \otimes u^\epsilon) \neq (\lim u^\epsilon) \otimes (\lim u^\epsilon)$$

Step 1 With $\nabla \cdot u = 0$

$$\nabla(u \otimes u) = \begin{pmatrix} \partial_{x_1}(\frac{1}{2}(u_1^2 - u_2^2)) + \partial_{x_2}(u_1 u_2) \\ \partial_{x_1}(u_1 u_2) - \partial_{x_2}(\frac{1}{2}(u_1^2 - u_2^2)) \end{pmatrix} + \nabla u$$

Therefore by Galilean invariance consider only:

$$\lim \int u_1^\epsilon(z) u_2^\epsilon(z) \phi(z) dz$$

Step 2 introduction of the Green function

$$\int u_1^\epsilon(z)u_2^\epsilon(z)\phi(z) = \iint \omega^\epsilon(x)\omega^\epsilon(y)H_\phi(x,y)dxdy$$

with

$$H_\phi(x,y,z) = \frac{1}{4\pi^2}\partial_{x_1}\partial_{y_2}\int \log\frac{1}{|x-z|}\log\frac{1}{|y-z|}\phi(z)dz$$

Now H_ϕ is smooth away from $x=y$ and uniformly bounded on $\mathbf{R}^2$.

Step 3 To prove convergence show that the contribution of

$$|\iint_{|x-y|\le a}\omega^\epsilon(x)\omega^\epsilon(y)H_\phi(x,y)dxdy| \le C\iint_{|x-y|\le a}\omega^\epsilon(x)\omega^\epsilon(y)dxdy$$

can be made arbitrarily small for a small enough. This comes from the fact that $0<\omega\in H^{-1}$ is a diffuse measure. (Delors lemma 1.2.5)

Youdovich uniqueness Theorem of , Wolibner regularity Theorem

$$\partial_t u + \nabla(u \otimes u) + \nabla p = 0\,, \nabla \cdot u = 0\,, \quad u \cdot n = 0 \text{ on } \partial\Omega$$
$$\partial_t v + \nabla(v \otimes v) + \nabla p = 0\,, \nabla \cdot v = 0\,, \quad v \cdot n = 0 \text{ on } \partial\Omega$$
$$\frac{1}{2}\frac{d|u-v|^2}{dt} + \int (d(u)(u-v),(u-v)) = 0 \quad d(u)_{ij} = \frac{1}{2}(\partial_{x_i} u_j + \partial_{x_j} u_i)\,,$$
$$\frac{1}{2}\frac{d|u-v|^2}{dt} \le |\nabla u|_\infty |u-v|^2\,.$$

Would provide uniqueness and regularity with control on $|\nabla u|_\infty$. This does not follows from $\nabla \wedge u \in L^\infty$!!

Theorem. *For $\nabla \wedge u \in L^\infty(\Omega)$ the $2d$ Euler equation has a unique solution in*

$$L^\infty(\mathbf{R}_t; H^1(\Omega)) \cap C(\mathbf{R}_t; (L^2(\Omega)^2)$$

Proof Assume also that $\nabla \wedge v \in L^\infty$ (hypothesis removed by Kato) and Ω bounded.

$$\forall p\,,2 \leq p < \infty \quad \|d(u)\|_p \leq cp\|\omega(t)\|_\infty = \|\omega(0)\|_\infty$$

with $\dfrac{1}{p} + \dfrac{1}{p'} = 1\,, z = |u-v|^2\,, \dfrac{dz}{dt} \leq cp\left(\int |u-v|^{p'}\right)^{\frac{1}{p'}} \leq cp(|u-v|)^{\frac{2p'-2}{p'}} z^{p'}$

$$\frac{dz}{dt} \leq cpz^{1-\frac{1}{p}}\,, z(t) \leq (cMt)^p.$$

Uniqueness propagates with $p \to \infty$.

The pair dispersion property of Wolibner

Theorem. *Let $u(x,t)$ be a divergence vector field defined in a bounded open setp $\Omega \times \mathbf{R}_t$ with diameter D, tangent to the boundary, with*

$$|\nabla \wedge u(.,t)|_{L^\infty(\Omega)} \leq C < \infty$$

then the solutions of

$$\dot{x}(t) = u(x(t),t)\,, x(0) = x_0$$

are uniquely defined and one has the estimate.

$$\left(\frac{|x(0)-y(0)|}{D}\right)^{e^{-Ct|\nabla\times u|_{L^\infty(\Omega)}}} \geq \frac{|x(t)-y(t)|}{D} \geq \left(\frac{|x(0)-y(0)|}{D}\right)^{e^{Ct|\nabla\times u|_{L^\infty(\Omega)}}}$$

Proof

$$u(x,t) = \frac{1}{\pi}\int_\Omega \left(\frac{x-y}{|x-y|^2} + \gamma(x,y)\right) \wedge \omega(y)dy$$

$$|\nabla u(x) - \nabla u(y)| \leq C|\nabla \times u|_{L^\infty(\Omega)}|x-y| \log\left(\frac{|x-y|}{D}\right)$$

For two trajectories of articles $\dot{x}(t) = u(x(t),t)$, $\dot{y}(t) = u(y(t),t)$ with $\rho(t) = |x(t) - y(t)|$ one has:

$$\left|\frac{d\rho(t)}{dt}\right| \leq \left|\frac{d(x(t)-y(t))}{dt}\right| \leq C|\nabla \times u|_{L^\infty(\Omega)}\rho(t)\log\left(\frac{D}{\rho(t)}\right)$$

Comparison with the solutions of $\rho' = \pm\rho\log(\frac{D}{\rho})$ gives the estimate.

Corollaire. *Assume that the initial data vorticity is bounded is* $C^{0,\alpha}$ *then one has the following uniform (in time) estimate:*

$$\|\nabla \times u(.,t)\|_{C^{0,\alpha(t)}} \leq C\|\nabla \times u(.,0)\|_{C^{0,\alpha}}$$

with $\alpha(t) = \alpha\,\mathrm{exp}\{-Ct\|\nabla \times u(.,0)\|_{L^\infty(\Omega)}\}$.

Proof Just use the fact that the vorticity is conserved on the trajectories of the particles.

Remarks

- The theorem of Delors is essential for the construction of a solution with initial vorticity concentrated on a curve (Kelvin Helmholtz instabilities)

- There is a huge gap for uniqueness between the Youdovich result ($\omega_0 \in L^\infty$) and the existence results $\omega_0 = \omega'' \in L^p +$ bounded measure.

- There is a non uniquenes result (Schaeffer Shnirleman) existence of a solution $u \in L^2(\mathbf{R}_t \times \mathbf{R}^2)$ space time compact support.

- According to physical experiments and partial results of Shnirleman the Wolibner estimates seems to be sharp. With $\omega_0 \in C^{0,\alpha}$ one know by the corollary that for any finite time the norm of the gradient remains bounded. However it is bounded by a quantity which grows like e^{Ct} for $t \to \infty$. Therefore no compactness results are available for the ωlimit set of the function $\omega(x,t), t \to \infty$ even for very regular initial data.

- Start with an initial vorticity ω_0 piecewise constant (Vortex Blob)

$$x \in Q_i \Rightarrow \omega(x, 0) = c_i\,.$$

The flow is uniquely defined and constant on $Q_i(t)$. The above analysis implies that the regularity of the curve propagates (Result proven by Chemin and Constantin). On the other hand this regularity may deteriorate fast enough to let people (Majda) believe that singularities may form.

• Consider a family $u_\epsilon(x,0)$ of smooth initial data such that $\nabla \wedge u_\epsilon(x,0)$ remains uniformly bounded in L^1. What would be the behaviour of the ωlimit set of the corresponding solution. This scenario is believed to lead to coherent structures. Coherent structures are assumed to be stationary solutions of the incompressible Euler equation. With the stream function ψ and from the relation:

$$0 = u\nabla\omega = 0 = (\nabla\psi)^\perp \cdot \nabla\omega$$

one deduces that $-\Delta\phi = F(\phi)$. There is to the best of my knowledge no “mathematical" determination of the function F arguments are based on constructions borrowed from statistical mechanic. To make these argument rigourous the first thing would be to have a well defined invariant measure on the set of solution with finite energy and at this point the result of Shnirelman is a real bad new.

- In spite of the fact that existence and uniqueness of smooth solutions u_ν are well established both for the Navier Stokes equation with viscosity $\nu > 0$ and for the incompressible Euler equation there almost no theorem concerning the convergence (even weak) of u_ν to the corresponding solution of the Euler equation for $\nu \to 0$ when **Dirichlet boundary condition** are inforced at the Navier Stokes level. In fact this is related to the creation of turbulence inside the media say along a vortex street. Even more so one could say that the “Dirichlet boundary condition" is a very good model for the generation of this type of turbulence. In the absence of such pathology the boundary layer would be described by the Prandlt equation.

Theorem. *Consider in a $2d$ bounded open set Ω with smooth fixed initial data u_0 a family of solution of the incompressible Navier Stokes equation with Dirichlet boundary condition fix $T > 0$ and denote by u the solution of the Euler equation with the same boundary condition. Then with $d(x) =$ (distance $x, \partial\Omega)$, $\Gamma(\nu) = \{x \in \Omega, d(x) < \Omega\}$ the following conditions are equivalent*

$$(i) \quad u_\nu(t) \to u(t) \text{ in } C(0,T;(L^2(\Omega))$$

$$(ii) u_\nu(t) \to u(t) \text{ weakly in } L^2(\Omega) \text{ae for} 0 < t < T$$

$$(iii) \nu \int_0^T \int_\Omega |\nabla u|^2 dx dt \to 0$$

$$(iv) \nu \int_0^T \int_{\Gamma(\nu)} |\nabla u|^2 dx dt \to 0$$

Proof The energy estimates:

$$\frac{1}{2}|u_\nu(t)|^2 + \nu \int_0^t |\nabla u_\nu|^2 dt = \frac{1}{2}|u_\nu(0)|^2 = \frac{1}{2}|u(0)|^2 = \frac{1}{2}|u(t)|^2$$

implies that the limit of $|u_\nu(.t)|$ is less or equal to $|u_0(.0)| = |u_0(.t)|$ this proves the implications $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv)$

The converse is more involved.

Step 1 Assuming that the solution u of the Euler equation is smooth one shows the existence of a family of corrector v_ν with the following properties.

$$\text{support of } v_\nu \subset \Gamma(\nu)\,,$$
$$\|v_\nu\|_\infty \le K\,, (|v_\nu| + |\partial_t v_\nu|) \le K\nu^{\frac{1}{2}},$$
$$\|\nabla v_\nu\|_\infty \le K\nu^{-1}\,, |\nabla v_\nu| \le K\nu^{-\frac{1}{2}}$$
$$\nabla \cdot v_\nu = 0 \text{ in } \Omega \text{ and } v_\nu = u \text{ on } \partial\Omega$$

Step 2 write

$$
\begin{aligned}
|u_\nu(t)-u(t)|^2 &= |u_\nu(t)|^2+|u(t)|^2-2(u_\nu,u)\\
&\le 2|u_0|^2-2(u_\nu,u-v_\nu)+o(\nu)\\
&\le -2\int_0^t(\partial_t u_\nu,u-v_\nu)+(u_\nu,\partial_t(u-v_\nu))dt+o(\nu)
\end{aligned}
$$

Step 3 Using the Euler and Navier Stokes equations for u and u_ν one has:

$$
\begin{aligned}
&(\partial_t u_\nu,u-v_\nu)+(u_\nu,\partial_t(u-v_\nu))=-((u_\nu-u)\otimes(u_\nu-u)\nabla u)\\
&+(u_\nu\otimes u_\nu,\nabla v_\nu)+\nu(\nabla u_\nu,\nabla(u-v_\nu))+o(\nu)
\end{aligned}
$$

The term

$$
|((u_\nu-u)\otimes(u_\nu-u)\nabla u)|\le \|\nabla u\|_\infty|(u_\nu-u)|^2
$$

will appears in the right hand side of a gronwall lemma. The two other terms go to zero.

The Hardy estimate takes care of the divergence:

$$\|\nabla v_\nu\|_\infty \le K\nu^{-1}$$
$$|(u_\nu \otimes u_\nu, \nabla v_\nu)| \le |d(x)^{-1}u_\nu|^2_{L^2(\Gamma(\nu))}|d(x)^2\nabla v_\nu|_{L^\infty} \le K\nu|\nabla u_\nu|^2_{L^2(\Gamma(\nu)}$$

$$|\nu(\nabla u_\nu, \nabla(u - v_\nu))| \le \nu|\nabla u_\nu||\nabla u| + \nu|\nabla u_\nu|_{L^2(\Gamma(\nu))}|\nabla v_\nu|_{L^2(\Gamma(\nu))}$$
$$\le K\nu|\nabla u_\nu| + K\nu^{\frac{1}{2}}|\nabla u_\nu|_{L^2(\Gamma(\nu))}$$

To use the Gronwall lemma to obtain

$$\lim_{\nu\to 0} |u_\nu(t) - u(t)|^2 = 0$$

it is enough to show that

$$\lim_{\nu\to 0}\int_0^t (\nu|\nabla u_\nu| + \nu^{\frac{1}{2}}|\nabla u_\nu|_{L^2(\Gamma(\nu))})dt = 0\,.$$

In fact one has

$$\int_0^t \nu|\nabla u_\nu| + \nu^{\frac{1}{2}}|\nabla u_\nu|_{L^2(\Gamma(\nu))} \leq$$
$$C\nu(\int_{\Omega\times]0,t[} |\nabla u_\nu|^2 dxdt)^{\frac{1}{2}} + C(\nu \int_{\Gamma(\nu)\times]0,t[} |\nabla u_\nu|^2 dxdt)^{\frac{1}{2}}$$

and the proof is completed.

Remark

- For a parabolic linear problem and even formally for the Navier Stokes system the boundary layer would be contained in $\Gamma(\nu^{\frac{1}{2}})$ and would lead to the formula:

$$\lim_{\nu\to 0} \nu \int_{\Gamma(\nu^{\frac{1}{2}})\times]0,t[} |\nabla u_\nu|^2 dxdt = c > 0$$

With the relation

$$\lim_{\nu\to 0} \nu \int_{\Gamma(\nu)\times]0,t[} |\nabla u_\nu|^2 dxdt \le \lim_{\nu\to 0} \nu \int_{\Gamma(\nu^{\frac{1}{2}})\times]0,t[} |\nabla u_\nu|^2 dxdt = c > 0$$

one observes that the Kato criteria is not in contradiction with the hypothesis made to justify the boundary layer analysis and the Prandlt equation. It also indicate that to generate turbulence it is in region of the order of ν that the vorticity has to be big.

PROOFS FOR THE $3d$ EULER EQUATION

In $3d$ one has for the equation of the vorticity

$$\partial_t \omega + u\nabla\omega - \omega\nabla u = 0$$

An other object which plays an important role is the deformation tensor

$$D(u) = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$$

No uniform norm control on the vorticity. In this lecture I will mostly ignore difficulties coming from the boundary if the domain has a boundary or from the fact that it may be unbounded. In particular with the boundary condition $u \cdot n = 0$ on $\partial\Omega$ one has:

$$\int_\Omega u\nabla(D^p u)(D^p u)dx = 0 \; s \geq 3 \Rightarrow \frac{d\|u\|_{H^3}^2}{dt} \leq C\|u\|_{H^3}^3$$

Theorem. *With initial data in* $u_0 \in H^3$ *the* $3d$ *Euler equation has for*

$$|t| < T = \frac{C}{\|u_0\|_{H^3}}$$

a unique solution in $C(-T, T; H^3)$ *and the regularity is preserved.*

In some sense this is an “optimal" theorem

Theorem. *Di Perna Lions*

There no p *no* $0 < t$ *and no constant* C *such that the estimate*

$$\|u(t)\|_{W^{1,p}} \leq C\|u(0)\|_{W^{1,p}}$$

would be true for any solution of $3d$ *Euler equation.*

Proof Use a family of $2 + \frac{1}{2}$ pressure less flow.

$$u_\epsilon(x_2), 0, w_\epsilon(x_1 - tu_\epsilon(x_2), x_2)$$

which is a solution of $3d$ Euler with $p = 0$ and choose $(u_\epsilon(x_2), 0, w_\epsilon(x_1, x_2))$ to construct counter examples.

- Other more sophisticated and more turbulent examples can be obtained by rotating flows. (Bardos Golse Nicolaenko).

- The above considerations rule out the project of proving singularities by numerical experiments. Numerical experiments do not discriminate between vorticity with blow up near $t = T^*$ or vorticity behaving near T^* like $\exp(\exp Ct)$. One should speak of quasi singularities.

With the above analysis one can show that the initial regularity including analytic regularity (Bardos Benachour) persist up to $t = T$ if

$$\int_0^T \|\nabla_x u(x,t)\|_\infty dt < \infty\,.$$

The fact that there is no constant such that

$$\|\nabla_x u(x,t)\|_\infty \leq C\|\nabla \wedge u(x,t)\|_\infty$$

justify the interest of the following theorems

Theorem. *With smooth initial data the solution of the Euler equation remains smooth as long as*

$$(1) \quad \int_0^T \|\nabla \wedge u(x,t)\|_\infty dt < \infty$$

$$(2) \quad \int_0^T \|\nabla \wedge u(x,t)\|_{\mathit{BMO}} dt < \infty$$

$$(3) \quad \int_0^T \|Du(x,t)\|_{\mathit{BMO}} dt < \infty\,.$$

Remark • (1) is the well known Beale Kato Majda Theorem, (2) has been obtained by Kozono and in fact (1) is a consequence of (2). Both have physical interpretation. The first one corresponds to observations in the physical space. BMO is a “Fourier" space and therefore estimates make explicit the role of high frequencies or small scale structures which corresponds to intuition in turbulence. Since BMO is a Fourier space (3) is equivalent to (2).

Proof As in Wolibner for the pair dispersion problem, the proof relies on the comparison with solutions of the ode.

$$y' = (y \log y) m(t) \, y > 1 \,,$$

which remain bounded as long as

$$\int_0^t m(s) ds < \infty$$

$$\frac{d\|u\|_{H^3}^2}{dt} \leq C|\nabla u|_\infty \|u\|_{H^3}^2$$

For $\|u\|_{H^3} > 1$ by a direct (Beale Kato Majda) or Paley Wiener (Kozono) decomposition:

$$|\nabla u|_\infty \leq |\nabla \wedge u|_\infty \log(\|u\|_{H^3})$$

or even better

$$|\nabla u|_\infty \leq |\nabla \wedge u|_{BMO} \log(\|u\|_{H^3})$$

In fact the variation of the Deformation tensor $D(u)$ is related to the variation or the direction of the vorticity. In fact as long as the direction of the vorticity remains smooth singularities should not appear

Theorem. *Let u be in $Q = \Omega \times]0, T[$ a solution of the Euler equation with smooth initial data. Assume*

1 That the velocity u is bounded in $L^1([0,T];(L^\infty))$

$$\sup_{x \in \Omega} |u(x,t)| \leq k_1(t) < \infty \,, \int_0^T k_1(t)dt < \infty$$

2 That the direction of the vorticity $\omega = \nabla u$ is bounded in $L^1([0,T];(lip(\Omega)))$

$$|\xi(x,t) - \xi(y,t)| = |\frac{\omega(x,t)}{|\omega(x,t)|} - \frac{\omega(y,t)}{|\omega(y,t)|}| \leq k_2(t)|x-y| \,, \quad \int_0^T k_2(t)dt < \infty \,, \int_0^T k$$

then the solution exists and is as smooth as the initial data up to a time $T + \delta$ with $\delta > 0$

Proof Introduce

$$D(u) = \frac{1}{2}\{\nabla_x u + (\nabla_x u)^T\}(x,t) = S(\omega)(x,t)$$

From

$$\frac{1}{2}(\partial_t|\omega|^2 + u\nabla|\omega|^2) = (\omega\nabla u, \omega) = (S(\omega)\omega, \omega)$$

deduce

$$\frac{d|\omega|_\infty}{dt} \leq \sup_x(|S(\omega)|)|\omega|_\infty$$

Next introduce two smooth non negative functions β_1^δ and β_2^δ with $\beta_1^\delta + \beta_2^\delta = 1$, $\beta_\delta^1 = 0$ for $|x| > 2\delta$ and $\beta_\delta^2 = 0$ for $|x| < \delta$. One has

$$|S(\omega)| \leq \int (\frac{y}{|y|}\cdot\xi(x))(D_{det}(\frac{y}{|y|}, \xi(x+y), \xi(x))\beta_\delta^1(|y|)|\omega(x+y)|\frac{dy}{|y|^3}$$

$$\int (\frac{y}{|y|}\cdot\xi(x))(D_{det}(\frac{y}{|y|}, \xi(x+y), \xi(x))\beta_\delta^2(|y|))|\omega(x+y)|\frac{dy}{|y|^3}$$

For the first term use the bound

$$(D_{det}(\frac{y}{|y|}, \xi(x+y), \xi(x))\beta_\delta^1(|y|)) \leq k_1(t)$$

Write the second as

$$\int (\frac{y}{|y|} \cdot \xi(x))(D_{det}(\frac{y}{|y|}, \xi(x+y), \xi(x))\beta_\delta^2(|y|)) \cdot (\xi(x+y)\nabla_y \wedge u(x+y))\frac{dy}{|y|^3}$$

and integrate by part. The first term gives a bound like $Ck_1(t)\delta|\omega|_\infty$ and the second like $Ck_1(t)k_2(t)\log\delta$ and with $|\omega|_\infty > 1$ and $\delta = |\omega|_\infty^{-1}$ one eventually obtains:

$$\frac{d|\omega|_\infty}{dt} \leq Ck_1(t)(1 + k_2(t)) \log|\omega|_\infty |\omega|_\infty$$

and the conclusion follows as in Beale Kato Majda.

- There is a big “gap" not in the proof but in the strategy the “positive" effect of a large (in modulus) velocity that should stabilize the flow is not taken into account. In much simpler cases (Babin Nicolaenko and others) this effect can be observed.

THE DISSIPATIVE SOLUTION OF PL LIONS

Let $u(x,t)$ be a smooth solution of the Euler equation in Ω and $v(x,t)$. Both u and v are divergence free and tangent to the boundary $\partial\Omega$. Denote by P the Leray projector and by

$$E(v) = \partial_t v + P(v\nabla v)$$

(when v is a solution one has $E(v) = 0$.

One has

$$\begin{aligned}
&\partial_t u + \nabla_x(u \otimes u) + \nabla_x p = 0 \\
&\partial_t w + \nabla_x(w \otimes w) + \nabla_x q = E(w) \\
&\frac{d|u-w|^2}{dt} + 2(D(w)(u-w),(u-w)) = 2(E(w), u-w)
\end{aligned}$$

Or with by integration:

$$|u(t) - w(t)|^2 \leq e^{\int_0^t 2\|D(w)\|_\infty(s)ds}|u(0) - w(0)|^2$$
$$+ 2\int_0^t e^{\int_s^t 2\|D(w)\|_\infty(\tau)d\tau}(E(w), u - w)(s)ds \qquad (1)$$

Definition. *A divergence free, tangent to the boundary function*

$$u \in C(\mathbf{R}_t; (L^2(\Omega))^d - w)$$

is called a dissipative solution if for any smooth test function it satisfies (1).

• Any classical solution is a dissipative solution.

• With $w = 0$ one obtains for the dissipative solution the relation $|u(t)|^2 \leq |u(0)|^2$ this is not in contradiction with an some non conservation of energy coming from lack of regularity and justifies the name dissipative.

• Assume that w is a classical solution and u a dissipative solution then one has

$$|u(t) - w(t)|^2 \leq e^{\int_0^t 2\|D(w)\|_\infty(s)ds}|u(0) - w(0)|^2$$

in particular if there exists a classical solution any dissipative solution with the same initial data coincide with it.

• When Ω has **no boundary** (periodic or whole space) any sequence u_ν Leray solutions of Navier Stokes with fixed initial data and $\nu \to 0$ converge to a dissipative solution.

LECTURE 3

INCOMPRESSIBLE LIMIT OF RENORMALISED SOLUTIONS

The collision integral (hard sphere gas)

- For a gas of hard spheres with radius r, Boltzmann's collision integral is

$$\mathcal{B}(F,F)(v) = \iint_{\mathbf{R}^3\times\mathbf{S}^2} \Big(F(v')F(v'_*) - F(v)F(v_*) \Big) \, r^2 |(v-v_*)\cdot\omega| \, d\omega \, dv_*$$

where the velocities v' and v'_* are defined in terms of v, v_* and ω by

$$v' \equiv v'(v,v_*,\omega) = v - (v-v_*)\cdot\omega\omega$$
$$v'_* \equiv v'_*(v,v_*,\omega) = v_* + (v-v_*)\cdot\omega\omega$$

- Usual notation: F_*, F' and F'_* designate resp. $F(v_*)$, $F(v')$ and $F(v'_*)$

Boltzmann's H Theorem

- Assume that $0 < F \in L^1(\mathbf{R}^3)$ is rapidly decaying and such that $\ln F$ has polynomial growth at infinity. Then

$$\int_{\mathbf{R}^3} \mathcal{B}(F,F) \ln F dv = -\frac{1}{4} \iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} (F'F'_* - FF_*) \ln\left(\frac{F'F'_*}{FF_*}\right) |(v-v_*)\cdot\omega| d\omega dv dv_* \leq 0$$

- The following conditions are equivalent:

$$\int_{\mathbf{R}^3} \mathcal{B}(F,F) \ln F dv = 0 \;\Leftrightarrow\; \mathcal{B}(F,F) = 0 \text{ a.e.} \;\Leftrightarrow\; F \text{ is a Maxwellian}$$

i.e. $F(v)$ is of the form

$$F(v) = M_{\rho,u,\theta}(v) := \frac{\rho}{(2\pi\theta)^{3/2}} e^{-\frac{|v-u|^2}{2\theta}} \text{ for some } \rho, \theta > 0 \text{ and } u \in \mathbf{R}^3$$

Periodic renormalized solution

$$H(f|g) = \iint_{\mathbf{R}^3\times\mathbf{R}^3} \left(f \ln\left(\frac{f}{g}\right) - f + g \right) dxdv \qquad \text{(r.e.)}$$

$$d(f) = \tfrac{1}{4}(f'f'_* - ff_*) \ln\left(\frac{f'f'_*}{ff_*}\right) = \tfrac{1}{4} ff_*\left(\frac{f'f'}{ff_*} - 1\right) \ln\left(\frac{f'f'_*}{ff_*}\right) \quad \text{(di. e.)}$$

Definition. *A sequence of renormalized solutions*

$$F_\epsilon \in C(\mathbf{R}_+; L^1(\mathbf{T}^3 \times \mathbf{R}^3))$$

is a family of functions which satisfies the following properties:

$$\forall \Gamma : z \to \Gamma(z) \geq 0\,, \Gamma(0) = 0, \sup(1+z)\Gamma'(z) < \infty\,,$$
$$\Gamma(F_\epsilon) \in L^1_{loc}(\mathbf{R}_+; L^1(\mathbf{T}^3 \times \mathbf{R}^3))\,,$$
$$\epsilon\partial_t \Gamma(F_\epsilon) + v\nabla_x \Gamma((F_\epsilon)) = \frac{1}{\epsilon^q}\Gamma'(F_\epsilon)\mathcal{B}(F_\epsilon, F_\epsilon)\,.$$

Theorem. *Di Perna, Lions, Masmoudi*

For all initial data F_ϵ^{in} *there exists a renormalized solution which satisfies the following relations:*

$$\partial_t \int F_\epsilon dv + \nabla_x \frac{1}{\epsilon} \int v F_\epsilon dv = 0$$

$$\partial_t \int v F_\epsilon dv + \nabla_x \frac{1}{\epsilon} \int v \otimes v F_\epsilon dv + \nabla_x m_\epsilon = 0\,,$$

$$\int\int |v|^2 F_\epsilon dv dx + \epsilon \int\int \mathit{tr}(m_\epsilon)(t) = \int\int |v|^2 F_\epsilon^{\mathit{int}} dv dx$$

$$H(F_\epsilon|M)(t) + \epsilon \int \mathit{tr}(m_\epsilon)(t) +$$
$$\frac{1}{\epsilon^{q+1}} \int_0^t \int \iiint_{\mathbf{T}\times\mathbf{R}^3\times\mathbf{S}^2} d(F_\epsilon)|(v - v_*)\cdot\omega| dv dv_* d\omega dx ds$$
$$\leq H(F_\epsilon^{in}|M)$$

Mach $\sim$ Knudsen Fluctuations

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon} \int (F_\epsilon - M) v dv = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int v F_\epsilon dv$$

$$H(F_\epsilon | M) = \iint [F_\epsilon \log\left(\frac{F_\epsilon}{M}\right) - F_\epsilon + M] dv dx =$$

$$\iint F_\epsilon \log(F_\epsilon) dx dv + \iint \frac{|v|^2}{2} F_\epsilon dx dv - \iint F_\epsilon dx dv + 1$$

$$H(M(1+g_\epsilon)|M) = \iint ((1+\epsilon g_\epsilon)\log(1+\epsilon g_\epsilon) - \epsilon g_\epsilon) M$$

$$h(z) = (1+z)\log(1+z) - z \sim \frac{|z|^2}{2}$$

Convergence and convexity

$$M_{(1,\epsilon u,1)} = \frac{1}{(2\pi)^{\frac{3}{2}}} e^{-\frac{|v-\epsilon u|^2}{2}},$$

$$\frac{1}{\epsilon}\int M_{(1,\epsilon u,1)} v dv = u, \lim_{\epsilon \longrightarrow 0} \frac{1}{\epsilon^2} H(M_{(1,\epsilon u,1)}|M) = \frac{1}{2}\int |u(x)|^2 dx$$

Theorem.

$$g_\epsilon \rightarrow g \quad w - L^1(Mdvdx) \ \frac{1}{\epsilon^2} H(M(1+g_\epsilon)|M) \leq C$$

Then

$$g \in L^2(Mdxdv) \textit{and} \ \frac{1}{2}\int g^2 M dxdv \leq C$$

The energy is the linearized entropy

For entropy and and entropy dissipation two strictly positive and strictly convex functions

$$h(z) = (1+z)\log(1+z) - z\,, \quad r(z) = z\log(1+z)\,, \quad z > -1$$
$$h(|z|) \le h(z)\,, r(|z|) \le r(z)\,, h(z) \le r(z).$$

Legendre transform.

$$f^*(p) = \sup_z (pz - f(z))\,, (pz) \le f(z) + f^*(p)$$
$$h^*(p) = e^p - p - 1 \le e^p \; r^*(p) = \frac{z^2}{1+z}, \quad \log(1+z) + \frac{z}{1+z} = p$$

CONVERGENCE TO THE EULER DISSIPATIVE SOLUTION

Theorem. *Laure Saint -Raymond. Let F_ϵ be a family of renormalized solutions of*

$$\epsilon\partial_t F_\epsilon + v\nabla_x F_\epsilon = \frac{1}{\epsilon^q}\mathcal{B}(F_\epsilon, F_\epsilon)\,, \quad q > 1$$

Assume the existence of a divergence free vector field $u^{in} \in L^2(\mathbf{T}^3)$ for which

$$\frac{1}{\epsilon^2}H(F_\epsilon^{in}|M_{1,\epsilon u^{in},1}) \to 0 \text{ as } \epsilon \to 0\,, \quad M_{1,\epsilon u^{in},1} = \frac{1}{(2\pi)^{\frac{3}{2}}}e^{-\frac{|v-\epsilon u^{in}|^2}{2}}$$

Then the family

$$(\frac{1}{\epsilon}\int vF_\epsilon dv)$$

is relatively compact in $w - L^\infty(\mathbf{R}_+; L^1(\mathbf{T}^3))$ and each of its limit points is a dissipative solution of the $3d$ Euler equation.

Proof

Step 1 For any test function w one has

$$
\begin{aligned}
&\frac{1}{\epsilon^2}H(F_\epsilon|M_{(1,\epsilon w,1)})(t) - \frac{1}{\epsilon^2}H(F_\epsilon^{\text{in}}|M_{(1,\epsilon w^{\text{in}},1)})(t) + \frac{1}{\epsilon}\int \mathrm{tr}(m_\epsilon)(t) \\
&\leq -\frac{1}{\epsilon}\int_0^t E(w)\int (v-\epsilon w)F_\epsilon(s,x,v)\,dxdvds \\
&- \int_0^t \int D(w) : \frac{1}{\epsilon^2}\int (v-\epsilon w)\otimes(v-\epsilon w)F_\epsilon(s,x,v)\,dxdvds \\
&- \int_0^t D(w) : \frac{1}{\epsilon} m_\epsilon(s,x)\,dsdx
\end{aligned}
$$

$$
E(w) = \partial_t w + P(w\nabla w)\,, \quad D(w) = \frac{1}{2}(\partial_{x_i} w_j + \partial_{x_j} w_i)
$$

Start from the entropy relation and the explicit formula:

$$\frac{1}{\epsilon^2}H(F_\epsilon(t)|M)+\frac{1}{\epsilon}\int \mathrm{tr}(m_\epsilon)(t)\le \frac{1}{\epsilon^2}H(F_\epsilon{}^{\mathrm{in}}|M)$$

$$H(F_\epsilon|M_{(1,\epsilon w,1)})(t)=H(F_\epsilon|M)(t)$$

$$+\frac{1}{2}\iint(\epsilon^2 w^2-2\epsilon v w)F_\epsilon(t,x,v)dxdv$$

to obtain

$$\frac{1}{\epsilon^2}H(F_\epsilon|M_{(1,\epsilon w,1)})(t)-\frac{1}{\epsilon^2}H(F_\epsilon^{\mathrm{in}}|M_{(1,\epsilon w^{\mathrm{in}},1)})(t)$$

$$+\frac{1}{\epsilon}\int \mathrm{tr}(m_\epsilon)(t)\le$$

$$\frac{1}{2\epsilon^2}\int_0^t ds\frac{d}{ds}\iint(\epsilon^2 w^2-2\epsilon v w)F_\epsilon(s,x,v)dxdvds$$

Compute the derivative of the right hand side by the formulas of moment conservations (plus defect measure)

No hope for strong convergence! Step 2

Theorem.

$$\int_0^t \iint D(w) : \frac{1}{\epsilon^2} \int (v - \epsilon w) \otimes (v - \epsilon w) F_\epsilon(s,x,v) dv dx ds$$
$$\leq \frac{C}{\epsilon^2} \int_0^t \|D(w)\|_\infty H(F_\epsilon | M_{(1,\epsilon w,1)})(s) ds + o(1) \text{ as } \epsilon \to 0$$

Hard. Based on two ingredients.

- A proper decomposition of $(F_\epsilon - M_{(1,\epsilon w,1)})$ to use the control of dissipation of entropy.

- With $q > 1$

$$\epsilon \partial_t F_\epsilon + v \nabla_x F_\epsilon = \frac{1}{\epsilon^q} \mathcal{B}(F_\epsilon, F_\epsilon) \Rightarrow \epsilon^{-2} \mathcal{B}(F_\epsilon, F_\epsilon) \Gamma'(F_\epsilon) \to 0$$

Step 3 Combine Step 1 and 2 to obtain

$$
\begin{aligned}
&\frac{1}{\epsilon^2} H(F_\epsilon | M_{(1,\epsilon w,1)})(t) + \frac{1}{\epsilon}\int \mathrm{tr}(m_\epsilon)(t) \\
&\leq -\frac{1}{\epsilon^2} H(F_\epsilon^{\mathrm{in}} | M_{(1,\epsilon w^{\mathrm{in}},1)})(t) \\
&- \frac{1}{\epsilon}\int_0^t E(w)\int (v-\epsilon w) F_\epsilon(s,x,v)\, dx dv ds \\
&C\int_0^t \|D(w)\|_\infty \left(\frac{1}{\epsilon^2} H(F_\epsilon | M_{(1,\epsilon w,1)})(s) + \frac{1}{\epsilon}\int \mathrm{tr}(m_\epsilon)(s)\right) ds + o(1)
\end{aligned}
$$

Apply a Gronwall argument and deduce:

$$|u(t)-w(t)|^2 \leq e^{\int_0^t 2\|D(w)\|_\infty(s)ds}|u(0)-w(0)|^2$$
$$+2\int_0^t e^{\int_s^t 2\|D(w)\|_\infty(\tau)d\tau}(E(w),u-w)(s)ds$$

No strong convergence. Weak convergence :

$$\int F_\epsilon dv \to 0\,, \frac{1}{\epsilon}\int F_\epsilon v dv \to u$$
$$\partial_t \int F_\epsilon dv + \nabla_x(\frac{1}{\epsilon}\int F_\epsilon v dv) = 0 \to \nabla_x \cdot u = 0\,,$$
$$|u-w|_{L^2} \leq \underline{\lim}(\frac{1}{\epsilon^2}H(F_\epsilon|M_{(1,\epsilon w,1)})$$

CONVERGENCE TO THE LERAY SOLUTION

The BGL CPAM 1993 $\longrightarrow$ Golse Saint-Raymond 2003

• Let $F_\epsilon^{in} \geq 0$ be any sequence of measurable functions satisfying the entropy bound $H(F_\epsilon^{in}|M) \leq C^{in}\epsilon^2$, and let F_ϵ be a renormalized solution of the scaled Boltzmann equation

$$\epsilon\partial_t F_\epsilon + v \cdot \nabla_x F_\epsilon = \frac{1}{\epsilon}\mathcal{B}(F_\epsilon, F_\epsilon)\,, \quad F_\epsilon\Big|_{t=0} = F_\epsilon^{in}$$

• Let $g_\epsilon \equiv g_\epsilon(x,v)$ be such that $G_\epsilon := 1 + \epsilon g_\epsilon \geq 0$ a.e.. We say that $g_\epsilon \longrightarrow g$ entropically at rate ϵ as $\epsilon \longrightarrow 0$ iff

$$g_\epsilon \longrightarrow g \text{ in } w - L^1_{loc}(Mdvdx)\,, \text{ and } \frac{1}{\epsilon^2}H(MG_\epsilon|M) \longrightarrow \tfrac{1}{2}\iint g^2 M dvdx$$

Theorem. *Assume that*

$$\frac{F_\epsilon^{in}(x,v) - M(v)}{\epsilon M(v)} \to u^{in}(x) \cdot v$$

entropically at rate ϵ*. Then the family of bulk velocity fluctuations*

$$\frac{1}{\epsilon} \int_{\mathbf{R}^3} v F_\epsilon dv$$

is relatively compact in $w - L^1_{loc}(dtdx)$ *and each of its limit points as* $\epsilon \to 0$ *is a Leray solution of*

$$\partial_t u + \mathrm{div}_x(u \otimes u) + \nabla_x p = \nu \Delta_x u\,, \quad \mathrm{div}_x\, u = 0\,, \quad u\Big|_{t=0} = u^{in}$$

with viscosity given by the formula

$$\nu = \tfrac{1}{10} \int (v \otimes v - \frac{1}{3}|v|^2 I) : AMdv$$

where $A = \mathcal{L}^{-1}(v \otimes v - \frac{1}{3}|v|^2 I)$

Method of proof

- Renormalization: pick $\gamma \in C^{\infty}(\mathbf{R}_{+})$ a nonincreasing function such that

$$\gamma\Big|_{[0,3/2]} \equiv 1\,, \quad \gamma\Big|_{[2,+\infty)} \equiv 0\,; \qquad \text{set } \widehat{\gamma}(z) = \frac{d}{dz}((z-1)\gamma(z))$$

- The Boltzmann equation is renormalized (relatively to M) as follows:

$$\partial_t(g_\epsilon\gamma_\epsilon) + \frac{1}{\epsilon} v \cdot \nabla_x(g_\epsilon\gamma_\epsilon) = \frac{1}{\epsilon^3}\widehat{\gamma}_\epsilon \mathcal{Q}(G_\epsilon, G_\epsilon) \qquad F_\epsilon = MG_\epsilon$$

where $\gamma_\epsilon := \gamma(G_\epsilon)$, $\widehat{\gamma}_\epsilon = \widehat{\gamma}(G_\epsilon)$ and $\mathcal{Q}(G,G) = M^{-1}\mathcal{B}(MG, MG)$

- Continuity equation Renormalized solutions of the Boltzmann equation satisfy the local conservation of mass:

$$\epsilon\partial_t\langle g_\epsilon\rangle + \mathrm{div}_x\langle vg_\epsilon\rangle = 0$$

- The entropy bound implies that

$$(1+|v|^2)g_\epsilon \text{ is relatively compact in } w-L^1_{loc}(dtdx; L^1(Mdv))$$

Modulo extraction of a subsequence

$$g_\epsilon \to g \text{ in } w-L^1_{loc}(dtdx; L^1(Mdv))$$

and hence $\langle vg_\epsilon\rangle \to \langle vg\rangle =: u$ in $w-L^1_{loc}(dtdx)$; passing to the limit in the continuity equation leads to the incompressibility condition

$$\mathrm{div}_x\, u = 0$$

ONE INGREDIENT:THE AVERAGING LEMMA

Theorem. *Let* μ *be a bounded positive measure on* $\mathbf{R}^d$ *such that for any hyperplane* $H \subset \mathbf{R}^d$ *with* $0 \in H$ *one has* $\mu(H) = 0$*. Let* F *such that*

$$\sup_{f \in F}(|f|_{L^2} + |v\nabla_x f|_{L^2}) \leq C$$

then the mapping

$$f \mapsto \int f(x,v)d\mu(v)\,,f \in F$$

is relatively compact in $L^2_{loc}(\mathbf{R}^d)$.

Proof

$$\phi(\xi, v) = \frac{1}{(2\pi)^d} \int e^{-x\cdot\xi} f(x, v) dv$$

$$|\int \phi(\xi, v) d\mu(v)|^2 \leq$$

$$2|\int_{|v\cdot\xi|\leq\frac{|\xi|}{n}} \phi(\xi, v) d\mu(v)|^2 + 2|\int_{|v\cdot\xi|\geq\frac{|\xi|}{n}} \phi(\xi, v) d\mu(v)|^2$$

Cauchy Schwartz

$$|\int \phi(\xi, v) d\mu(v)|^2 d\xi \leq$$

$$2 \sup_{\xi\in\mathbf{R}^d} \mu(\{v \in \mathbf{R}^d | v \cdot \xi| \leq \frac{|\xi|}{n}\}) \int |\phi(\xi, v)|^2 d\mu(v)$$

$$+ 2\mu(\mathbf{R}^d) \frac{n^2}{|\xi|^2} \int |v \cdot \xi|^2 |\phi(\xi, v)|^2 d\mu(v)$$

Compactness follows.

Counter example in L^1

$$g_\epsilon(x,v) \quad \|g_\epsilon\|_{L^1(\mathbf{R}^d\times\mathbf{R}^d} = 1 \quad g_\epsilon \to \delta_{x=0}\otimes\delta_{v=\xi_0}$$

$$f_\epsilon + v\nabla_x f_\epsilon = g_\epsilon$$

$$\int\left(\int f_\epsilon(x,v)\phi(v)dv\right)\theta(x)dx = \int\int(\int_0^\infty e^{-s}g_\epsilon(x-sv,v)ds)\phi(v)\theta(x))dxdv$$

$$= \int\int(\int_0^\infty e^{-s}g_\epsilon(x,v)ds)\phi(v)\theta(x+sv)dxdv \to \int_0^\infty e^{-s}\phi(\xi_0)\theta(s\xi_0,\xi_0)ds$$

Theorem. *Let* $f_n \equiv f_n(x,v)$ *be a bounded sequence in* $L^1_{loc}(dxdv)$ *such that* $v \cdot \nabla_x f_n$ *is also bounded in* $L^1_{loc}(dxdv)$. *Assume that* f_n *is locally uniformly integrable in* v. *Then*

- f_n *is locally uniformly integrable (in* x, v*)*

- *for each test function* $\phi \in L^\infty_{comp}(\mathbf{R}^D_v)$, *the sequence of averages*

$$\rho_n^\phi(x) = \int f_n(x,v)\phi(v)dv$$

is relatively compact in $L^1_{loc}(dx)$.

- Let's prove that the sequence of averages ρ_n^ϕ is locally uniformly integrable assume that f_n and $\phi \geq 0$.

- Let $\chi \equiv \chi(t, x, v)$ be the solution to

$$\partial_t \chi + v \cdot \nabla_x \chi = 0\,, \qquad \chi(0, x, v) = \mathbf{1}_A(x)$$

Clearly, $\chi(t, x, v) = \mathbf{1}_{A_x(t)}(v)$ (χ takes the values 0 and 1 only). On the other hand,

$$|A_x(t)| = \int \chi(t, x, v)dv = \int \mathbf{1}_A(x - tv)dv = \frac{|A|}{t^D}$$

- <u>Remark:</u> this is the basic dispersion estimate for the free transport equation.

- Set $g_n(x,v) = f_n(x,v)\phi(v)$, and $v\cdot\nabla_x g_n(x,v) = \phi(v)(v\cdot\nabla_x f_n(x,v))$: $h_n(x,v) =$: g_n and h_n are bounded in $L^1_{x,v}$ and g_n is uniformly integrable in v.

- Observe that (hint: integrate by parts the 2nd integral in the r.h.s.)

$$\int_A \int g_n dv dx = \int \int_{A_x(t)} g_n dv dx - \int_0^t \iint h_n(x,v)\chi(s,x,v) dx dv ds$$

The second integral on the r.h.s. is $O(t) \sup \|h_n\|_{L^1_{x,v}} < \epsilon$ by choosing $t > 0$ small enough. For that value of t, $|A_x(t)| \longrightarrow 0$ as $|A| \longrightarrow 0$, hence the first integral on the r.h.s. vanishes by uniform integrability in v.

REMARKS AND CONCLUSION

- Many thanks to the Scuola Normale and the De Giorgi center for their hospitality.

- I tried to emphasise the importance of the chain of equations say from molecules to turbulence modelling. and observed that at each step the theorem that have been established are in complete agreement with the ones which concern the next step. Along this line I believe that the most important obstacles for further progresses concern the incompressible Euler equation and the mathematical understanding of turbulence modelling.

- The averaging lemma is a basic step in the construction of solutions of the Boltzmann equation and in the study of their incompressible limit. One may think that at the macroscopic level this effect would be due to a self averaging mechanism of the fluid. Write:

$$u\nabla u = \omega \wedge u + \nabla \frac{1}{2}|u|^2$$

and observe that a “big" vorticity may stabilise the fluid. In some simple configurations this result has been proven by Babin Nicolaenko and others. On the other hand in more realistic situations no one has been able to use this effect. For instance the serie of results of Constantin Fefferman and Majda balance the effect of the magnitude of the vorticity by the fact that its direction is smoothly varying but completely ignore any positive stablizing effect due to this large vorticity.

- The appearance of irreversibility which is related to the non conservation of entropy is also a phenomena badly understood. The Newton equations are reversible but the notion of entropy devised at this level does not seems to be well connected with the notion of entropy which is in the Boltzmann equation. The Boltzmann equation is not reversible while the compressible Euler equation is before the appearance of shocks reversible. Irreversibility comes after the first singularity and is present in the viscous limit.

• In practical problems the effect of the boundary is of paramount importance. In fact the no slip ($u = 0$ on the boundary) can be deduced as a first order approximation. Slip boundary condition comes in the picture as a ϵ order correction due to the Knudsen layer. On the other hand when the Reynolds number goes to infinity the no slip boundary condition generates a boundary layer. In some stable (laminar) regime this boundary layer can be described by the Prandlt equations. These equations are very unstable and this is related to the fact that the non linearity may generate the propagation of high vorticity inside the fluid itself. Giving a mathematical description of such phenomena seems to be of the same order of magnitude of difficulties as any other “big" problem of the theory.

• In these talks I did not touched any probabilistic . I think that at some point the introduction of probability and randomness in the description of the phenomena is both useful and unavoidable. Or the other hand. 1) It is not in my range of expertise. 2) It seems really important in the use of probability to discriminate between cases where it is just an artefact and when it is really compulsory. For exemple the Landford derivation uses ergodicity and not randomness. Irreversibility appears 1) Because one decide to evaluate the macroscopic state at time t with the only knowledge of the macroscopic state at time 0 and not the converse (the converse would produce a change of sign in front of the Boltzmann collision operator) and 2) Because and this is ergodicity one shows that the initial configurations leading to other things than finite number of binary collision are of measure 0.

• The understanding of the constitutive laws for real gases (Wan der Wal laws) need a direct derivation of the macroscopic equation without the intermediate step of a kinetic equation. To complete their derivation a closure principle is needed and to justify this principle it seems that at this point a probabilist approach is compulsory. The same remarks seem to hold for the determination of large coherent structures in $2d$ fluid mechanic.

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