Analyticity and instabilities of interfaces from Kelvin Helmholtz to water waves.

Compare the properties of these different problems

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Weak solutions of Euler equations, $2d$, in particular with vorticity and density jump. No viscosity, no surface tension.

$$\begin{aligned} \partial_t(\rho u) + \nabla(\rho u \otimes u) + \nabla p &= \rho \underline{g}\,, \\ \nabla u = 0\,, \\ \partial_t \rho + \nabla(\rho u) = 0\,. \end{aligned}$$

The emphasis is on planar flows but most of the present considerations are valid for real $3d$ flows.

Conference in Honor of Basile Nicolaenko.

The $2d$ incompressible flow

Well known simplifications in $2d$ when $\rho \equiv$constant :

$$\partial_t \Omega + \nabla (u\Omega) = 0\,, \nabla u = 0\,, \Omega = \nabla \wedge u \qquad (1)$$

The vorticity is transported by the fluid and it is the only case where some global theorems are available.

- Existence and uniqueness for $\Omega(x,0) \in L^\infty$ (Yudovitch) .

- Existence of a weak solution for $\Omega(x,0) \in L^p (p > 1)$

- Existence of a weak solution when $\Omega(x,0)$ is a Radon measure with distinguished sign (Delors) or with controlled change of sign (Lopes Filho, Nussenzveig Lopes and Xin)

- Non uniqueness ... Existence of a solution with $u \in L^2(\mathbb{R}^2_x \times \mathbb{R}_t)$ with compact support in space-time (Scheffer and Shnirelman).

The density $\rho = \rho_\pm \geq 0$ is constant away from a space time "smooth " surface. The vorticity is a bounded Radon measure with support contained in this same surface.

$$a = \frac{|\rho_+ - \rho_-|}{\rho_+ + \rho_-} \text{ Atwood number}$$
$$0 \leq a \leq 1 \,, a = 1 \text{ if and only if } \rho_+ \text{ or } \rho_- = 0 \,. \tag{2}$$

Kelvin Helmoltz, Rayleigh Taylor, Water Waves.

The Rayleigh-Taylor Instability is responsible for fingers of color formed as water tinted with food coloring flows down through ordinary tap water, which is slightly less dense that the dyed water.

Water waves

$\Sigma_t\,,\ t \in (-T,T)$ a (for the time being) smooth curve $\Sigma = \cup_{-T<t<T}\Sigma_t$
Different parametrizations $x = r(t,\lambda)$ of Σ_t. s the arc lenght :

Averages and jumps :

$$\langle f\rangle = \frac{f_+ + f_-}{2}\,, [f] = f_+ - f_-\,,\ [fg] =< f > [g] + [f] < g >$$

The velocity u of an incompressible fluid with vorticity being a bounded radon measure with support contained in $\Sigma_t = r(t,\lambda)$:

$$\begin{aligned} u(x,t) &= \frac{1}{2\pi} R_{\frac{\pi}{2}} \int \frac{x-r'}{|x-r'|^2} \Omega(t, r(t,\lambda')) ds' \\ &= \frac{1}{2\pi} R_{\frac{\pi}{2}} vp \int \frac{x - r(t,\lambda')}{|x - r(t,\lambda')|^2} \Omega(t, r(t,\lambda')) \frac{\partial s(\lambda',t)}{\partial \lambda'} d\lambda' \end{aligned} \quad (3)$$

$$\tau = \frac{\partial_\lambda (r(t,\lambda))}{\|\partial_\lambda (r(t,\lambda))\|} \text{ and } \nu = R_{\frac{\pi}{2}} \tau$$

$$u_- \cdot \nu = u_+ \cdot \nu = u_\nu \,, \; v =< u >= \frac{1}{2\pi} \text{p.v.} R_{\frac{\pi}{2}} \int \frac{x-r'}{|x-r'|^2} \Omega(t, r(t,\lambda')) ds' \quad (4)$$

- The operator K_Σ defined by

$$K_\Sigma(\Omega) = v(\Omega) \cdot \tau = \left(\frac{1}{2\pi} R_{\frac{\pi}{2}} \int \frac{x-r'}{|x-r'|^2} \Omega(t, r(t,\lambda')) ds' \right) \cdot \tau \quad (5)$$

is (for $r(t,.) \in C^{1+\beta}(\mathbb{R})$) continuous in the space $C^\beta \cap L^2$ (P.L.Sulem and C.Sulem) and compact (V. Kamotski and G. Lebeau) whenever Σ_t is a closed curve with spectra contained in $-1/2, 1/2$.

Distributions calculus shows that whenever the interface Σ is smooth the fact for u to be a weak solution

$$\partial_t(\rho u) + \nabla(\rho u \otimes u) + \nabla p = \rho \underline{g}\,,$$
$$\nabla u = 0\,,$$
$$\partial_t \rho + \nabla(\rho u) = 0\,.$$

is equivalent to the fact that the vorticity density and the curve satisfy the system of equations

$$[\rho u \cdot \tau]_t - \partial_s\left([\rho u \cdot \tau]\partial_t r \cdot \tau + \frac{1}{2}[\rho(|u_\nu|^2 - |u_\tau|^2]) - [\rho] g \cdot r\right) = 0 \tag{6}$$

$$(r_t - v)\cdot \nu = 0 \tag{7}$$

$$v(t,r) = \langle u\rangle(t,r) = R_{\pi/2}\frac{1}{2\pi}p.v.\int \frac{r - r(t,s')}{\|r - r(t,s')\|^2}\Omega(t,s')ds' \tag{8}$$

which describes the evolution of the Rayleigh Taylor, the Kelvin Helmoltz ($[\rho] = 0$) and the water waves interfaces ($\rho_+ = 0$).

Remark 1 *The equation (7) does not completely determine $r(t,\lambda)$ this is due to the freedom in the choice of the parametrization of the interface.*

When the interface is a graph $(y = y(x,t))\,,\langle v\rangle = (v_1, v_2)$ (6) and (7) :

$$y_t + y_x v_1 = v_2 \tag{9}$$

$$\partial_t(\frac{\Omega}{2} + a(v_1 + y_x v_2)) + \partial_x(v_1(\frac{\Omega}{2} + a(v_1 + y_x v_2)) + a(\frac{\Omega^2}{8(1+y_x^2} - \frac{|v|^2}{8} - gy) = 0\,. \tag{10}$$

Or in the case of the Kelvin Helmholtz interface

$$y_t + y_x v_1 = v_2\,, \quad \partial_t(\Omega) + \partial_x(v_1\Omega) = 0\,. \tag{11}$$

For the water waves with $v = \nabla\phi\,.$

$$-\Delta\phi = 0 \text{ for } x < y(x,t) \tag{12}$$

$$y_t + y_x\partial_x\phi = \partial_y\phi\,, \quad \partial_t\phi + \frac{1}{2}|\nabla\phi|^2 = gy\,. \tag{13}$$

Remark 2 *The above formulation show that the Cauchy problem is determined by two data $(y(x,0),\Omega(x,0))$ for RaleighTaylor and $(y(x,0),\phi(x,0))$ for water waves.*

Birkhoff Rott Equation for R.T. and K.H

Start from the identity :

$$[\rho u\cdot\tau]\partial_t r\cdot\tau+\frac{1}{2}[\rho(|u_\nu|^2-|u_\tau|^2]+[\rho]g\cdot r=[\rho u\cdot\tau](\partial_t r-v)\cdot\tau+[\rho](\frac{1}{2}|v|^2-\frac{1}{8}\Omega^2+gr)$$

use the :

$$w=[\rho u\cdot\tau] \text{ and } U=(\frac{1}{2}|v|^2-\frac{1}{8}\Omega^2+gr)$$

With the Schwarz relation (from the dynamic equation) :

$$\partial_t w=\partial_s\left(w(\partial_t r-v)\cdot\tau)+[\rho]U\right)$$

using $w=[\rho u\cdot\tau]\neq 0$ introduce $\lambda(t,s)$ and a new parametrization $t'=t\,,\ \lambda=\lambda(s,t)$ keeping in mind that $\partial_\lambda r=c(\lambda,t')\tau\,,\ w\partial_\lambda r=\tau$:

$$\begin{aligned}
\partial_s\lambda &= w\,, \quad \partial_t\lambda=w((\partial_t r-v)\cdot\tau)+[\rho]U\\
\partial_t r &= \partial_{t'}r+\partial_\lambda(w((\partial_t r-v)\cdot\tau)+[\rho]U)\\
\partial_{t'}r\cdot\tau &= \partial_t r\cdot\tau-\partial_\lambda rw\cdot\tau((\partial_t r-v)\cdot\tau)+\partial_\lambda[\rho]U\cdot\tau\\
\partial_{t'}r &= v-[\rho]U\partial_\lambda r
\end{aligned}$$

In term of the new variables $t' = t$ and $\lambda = \lambda(s,t)$:

$$\partial_t r + [\rho] U \partial_\lambda r = R_{\pi/2} \frac{1}{2\pi} p.v. \int \frac{r(t,\lambda) - r(t,\lambda')}{\|r(t,\lambda) - r(t,\lambda')\|^2} \Omega(t,\lambda') \frac{1}{w} d\lambda' \qquad (14)$$

In the "Kelvin Helmholtz " case $[\rho] = 0$ and $w = \Omega$ the formula (14) becomes the classical Birkhoff Rott equation :

$$\partial_t r = R_{\pi/2} \frac{1}{2\pi} p.v. \int \frac{r(t,\lambda) - r(t,\lambda')}{\|r(t,\lambda) - r(t,\lambda')\|^2} d\lambda' \qquad (15)$$

or with the introduction of the complex variable $z(t,\lambda) = r_1(t,\lambda) + i r_2(t,\lambda)$ the equation :

$$\partial_t \overline{z} = \frac{1}{2\pi} p.v. \int \frac{d\lambda'}{z(t,\lambda) - z(t,\lambda')} \qquad (16)$$

Remark 3 *The above derivation provides a complete equivalence beetwen the solution of the Euler equation and the solution of the "interface problem" whenever smoothness assumption are made. The situation is much more complex (see the discussion below) for weak solution of the Euler equation (say "à la Delors"). The non uniqueness theorem of Scheffer and Shnirelman (in spite of the fact that the vorticity of such solutions are far from being a distinguished signed Radon measure and have non decaying energy) contributes to the difficulty of the problem ?*

On the other one can construct (relaxing the regularity of the vorticity) examples of solutions of the Birkhoff Rott equation which fail to be weak solution of the Euler equation.

Example the Prandlt-Munk vortex sheet with initial vorticity :

$$\Omega_0(x_1,x_2) = \frac{x_1}{\sqrt{1-x_1^2}}\Xi(-1,1)(x_1)\otimes\delta(x_2) \qquad (17)$$

$\Xi(-1,1)$ *the characteristic function of* $(-1,1)$. *With the Biot Savard law the velocity is constant* $(0,-\frac{1}{2})$. *The solution of the Kelvin Helmholtz problem is the vorticity given by the formula*

$$\Omega(x_1,x_2,t) = \Omega_0(x_1,x_2+\frac{t}{2}) \qquad (18)$$

On the other hand it was observed by [Lopes Filho, Nussenzveig Lopes and Souza] that the velocity u *associated to this vorticity is* **not** *even a weak solution of the Euler equation. In fact one has*

$$\nabla\cdot u = 0 \qquad \text{and } \partial_t u + \nabla_x(u\otimes u) + \nabla p = F$$

With F *given by*

$$F = \frac{\pi}{8}[(\delta(x_1+1,x_2+\frac{t}{2}) - \delta(x_1-1,x_2+\frac{t}{2})),0]$$

Pathology is due to the "bad " behaviour of Ω *near the points* $(\pm 1,0)$.

Observe that in the previous case $\Omega(t,s)$ is in $L^\infty(\mathbb{R}_t; L^1(\Sigma_t))$ but not in $L^\infty(\mathbb{R}_t; L^2(\Sigma_t)$ in a following article [Lopes Filho, Nussenzveig Lopes and Sochet] the notion of regular curve is used. A curved C is said to be regular if it is rectifiable and if there exists a constant $A < \infty$ such that for any ball B_r of radius $r$one has

$$|C \cap B_r| < Ar$$

Then in [Lopes Filho, Nussenzveig Lopes and Sochet] a definition of weak solution of the Birkhoff Rott equation is introduced and it is proven that $\Omega = \gamma \delta_{\Sigma_t}$ provides a weak solution of the Euler equation if and only if

$$\gamma \in L^\infty(\mathbb{R}_t; L^2(\Sigma_t)) \quad (19)$$

CLASSICAL THEOREMS

- The water wave problem is well posed for finite time and with analytic initial data Ovsjannikov (1974). With initial data $\Sigma_0 \in H^s$ Nalimov (1974), Craig (1985), Yosihara (1982) for small perturbations of rest and Lannes (2003). Here the Taylor stability condition appears on the bottom. It appears also in Sijue Wu (1997). Infinite depth up to non self intersection :

$$(x_{tt}, y_{tt}) \cdot n - (0, -g) \cdot n \geq 0$$

- For R.T and K.H. solution (for a finite positive or negative time) of the Cauchy only under the assumption that the initial data are *analytic.* C. Sulem, P.L. Sulem, U. Frisch and B. (1981), C. Sulem and P.L. Sulem (1985) with a small enough :

$$a = \frac{|\rho_+ - \rho_-|}{|\rho_+ + \rho_-|}$$

For any $0 \leq a < 1$ ($a = 1$: water waves excluded) valid for initial curve closed and finite. The operator K is compact and $\sigma[K] \subset [-\frac{1}{2}, \frac{1}{2}]$.

$$K_\Sigma(\Omega) = v(\Omega) \cdot \tau = \left(\frac{1}{2\pi} R_{\frac{\pi}{2}} \int \frac{x - r'}{|x - r'|^2} \Omega(t, r(t, s')) ds' \right) \cdot \tau$$

SINGULARITIES

- Numerical experiments show the appearance of singularities in the shape of a cusp and after the cusp a spiral Moore (1979), Meiron, Baker Orsag (1982) and Krasny (1981).

- Mathematical proof of the appearance of singularities have already been given for the Kelvin Helmholtz by Duchon and Robert (1988) and Caflish and Orellana (1989).

From solutions analytic for $t > 0$ and singular at $t = 0$ the full reversibility of the Kelvin Helmholtz problem provides solutions analytic at $t = 0$ and with singularities at a positive time which in some case can be estimated in term of the size of the initial data.

Example of Caflish and Orellana for the Birkhoff equation :

$$z(t,\lambda) = \lambda + \epsilon s_0 + r(\lambda,t) \tag{20}$$

with s_0 given by :

$$s_0(\lambda,t) = (1-i)\{(1-e^{-\frac{t}{2}-i\lambda})^{1+\nu} - (1-e^{-\frac{t}{2}+i\lambda})^{1+\nu}\} \tag{21}$$

$\epsilon > 0$ small enough , $r(\lambda,t)$ proven to be analytic for $t > 0$ and much smaller ($O(\epsilon^2)$) in $C^2(\lambda)$ Since $s_0(\lambda,0) \sim \lambda^{1+\nu}$:

$$z(\lambda,0) \in C^{1+\nu}, z(\lambda,0) \notin C^{1+\nu'} \text{ for } \nu' > \nu \tag{22}$$

Example of Duchon and Robert (a forerunner of this theory) : relax the initial condition on the vorticity at $t = 0$ and assume that the function $y(x,t)$ goes to zero with $t \to \infty$ in the system :

$$y_t + y_x v_1 = v_2 \tag{23}$$

$$\partial_t(\Omega) + \partial_x(v_1\Omega) = 0 \tag{24}$$

$$r(t,x) = (x, y(x,t)) \quad , \quad v(t,r) = R_{\pi/2}\frac{1}{2\pi}p.v.\int \frac{r - r(t,s')}{\|r - r(t,s')\|^2}\Omega(t,s')ds' \tag{25}$$

For any ϵ small enough a an analytic solution on $t > 0$.

$$y(x,0) = \int e^{ix\xi}g(\xi)d\xi/,, \int |g(\xi)|d\xi \leq \epsilon$$

Ellipticity⇒ Analyticity of solutions

Theorem 1 *Kamotski–Lebeau Let* $u \in C(-T,T;L^2(\mathbb{R}^2))$ *is a weak solution of the system*

$$\begin{aligned} \partial_t(\rho u) + \nabla(\rho u \otimes u) + \nabla p &= \rho \underline{g}\,, \\ \nabla u &= 0\,, \\ \partial_t \rho + \nabla(\rho u) &= 0\,. \end{aligned}$$

with $\nabla \wedge u \in L^\infty(0,T;(BRM)(\mathbb{R}^2))$. *Assume the existence of a neighbourhood* $\mathcal{U}$ *of a point* $(t_0\,, r_0 = r(t_0,\lambda_0)$ *with the following properties :*

$$(\text{ support } \nabla \wedge u) \cap \mathcal{U} \subset \cup_{\{-\epsilon+t_0<t<\epsilon+t_0\}} \Sigma_t\,, \nabla \wedge u = \Omega(t,s)\delta_\Sigma$$
$$\text{with } t \mapsto \Sigma(t,s) \in C^\alpha(\mathbb{R}_t; C^{1+\beta}(\mathbb{R}_s))\,, s \text{ arclength}$$

and $\rho = (\rho_+\,,\rho_-)$ *constant in* $\mathcal{U} - \Sigma$. *Assume furthermore that* Ω *and* $w = [\rho u_\tau](t_0,r_0)$ *are both non zero then in a neighbourhood* $\mathcal{U}' \subset \mathcal{U}$ *of* (t_0,r_0) $r(t,s)$ *and* $\Omega(t,s)$ *are analytic.*

Idea of proof Introduce an open set $\mathcal{U}' \subset\subset \mathcal{U}$ and using the formula $\nabla \cdot u = 0\,, \nabla \wedge u = \Omega$ with the chord arc hypothesis write the equation (14) in the form

$$\partial_t r + [\rho] U \partial_\lambda r = \frac{1}{2\pi} R_{\frac{\pi}{2}} p.v. \int_{r(t,\lambda)\in\mathcal{U}} \frac{r(t,\lambda) - r(t,\lambda')}{\|r(t,\lambda) - r(t,\lambda')\|^2} \Omega(t,\lambda') \frac{1}{w} d\lambda' + E(t, r(t,\lambda))$$

with $E(t,r)$ analytic in $\mathcal{U}'$. By a galilean transformation near $r(t_0,\lambda_0)$ assume (with the complex representation $r = r_1 + i r_2$)

$$\Omega(0,0) = 1\,, r(\lambda,t) = (\alpha t + \beta(\lambda + \epsilon f(t,\lambda)) \text{ for } \sup\{|\lambda|, |t|\} \le M, f(0,0) = \nabla f(0,0) = 0$$

Then to "eliminate" $\frac{\Omega}{w}$ use the jump condition :

$$\Omega = \frac{1}{\langle\rho\rangle}(w - [\rho] v_s) = \frac{1}{\langle\rho\rangle}(w - [\rho] v(w \partial_\lambda r)) = \frac{w}{\langle\rho\rangle}(1 - [\rho] Re(r_\lambda \overline{v}))$$

which gives

$$\frac{\Omega}{w} + \frac{[\rho]}{\langle\rho\rangle} Re\left(\frac{1}{2\pi i} p.v. \int_{r(t,\lambda')\in\mathcal{U}} \frac{(1 + \epsilon \partial_\lambda f(t,\lambda)) \frac{\Omega}{w} d\lambda'}{(\lambda - \lambda')(1 - \epsilon \frac{f(t,\lambda) - f(t,\lambda')}{\lambda - \lambda'})}\right) = E(t, r(t,\lambda))\,. \quad (26)$$

We recall that U is given by

$$U = (\frac{1}{2}|v|^2 - \frac{1}{8}\Omega^2 + gr)$$

we replace everywhere Ω by $\frac{\Omega}{w}$. In particular we have :

$$\Omega_0 = \Omega(0,0) = \frac{1}{\langle\rho\rangle}, U_0 = -\frac{1}{8\langle\rho\rangle^2}$$

and finally we obtain, extending $f(t,\lambda)$ and $\Omega(t,\lambda)$ by zero outside $\mathcal{U}$ the system

$$\epsilon|\beta|^2(\partial_t\overline{f} + [\rho]U\partial_\lambda\overline{f}) = \frac{1}{2\pi i}p.v.\int_{r(t,\lambda')\in\mathcal{U}} \frac{\Omega(t,\lambda')d\lambda'}{(\lambda-\lambda')(1-\epsilon\frac{f(t,\lambda)-f(t,\lambda')}{\lambda-\lambda'})} + E(t,r(t\lambda)) \quad (27)$$

$$\frac{\Omega}{w} + \frac{[\rho]}{\langle\rho\rangle}Re(\frac{1}{2\pi i}p.v.\int_{r(t,\lambda')\in\mathcal{U}} \frac{(1+\epsilon\partial_\lambda f(t,\lambda))\Omega(t,\lambda')d\lambda'}{(\lambda-\lambda')(1-\epsilon\frac{f(t,\lambda)-f(t,\lambda')}{\lambda-\lambda'})}) + E(t,r(t,\lambda)) \quad (28)$$

$r \mapsto E(t,r)$ denoting analytic functions in $\mathcal{U}$.

Next one use the expansion

$$\frac{1}{2\pi}pv\int\frac{d\lambda'}{(\lambda-\lambda')(1+\epsilon\frac{f(\lambda,t)-f(\lambda',t)}{\lambda-\lambda'})}d\lambda'$$

$$=\frac{\epsilon}{2\pi}\int\frac{f(\lambda,t)-f(\lambda',t)}{(\lambda-\lambda')^2}d\lambda'+\sum_{n\geq 2}\frac{\epsilon^n}{2\pi}\int\frac{(f(\lambda,t)-f(\lambda',t))^n}{(\lambda-\lambda')^{(n+1)}}d\lambda'\,.$$

and the formulas (Hilbert transform)

$$\frac{1}{2\pi}\int\frac{f(\lambda,t)-f(\lambda',t)}{(\lambda-\lambda')^2}d\lambda'=-\frac{i}{2}sign(D)f\,,\ \frac{1}{2\pi}vp\int\frac{f(\lambda,t)-f(\lambda',t)}{(\lambda-\lambda')^2}d\lambda'=|D|f$$

and deduce from (27) and (28) that the real and imaginary part of $f(t,\lambda)=X(t,\lambda)+iY(t,\lambda)$ are in ($\mathcal{U}'\subset\subset\mathcal{U}$) solutions of the system :

$$\partial_t X+[\rho](\frac{\Omega_0^2}{4\beta^2}+U_0)\partial_\lambda X=\frac{\Omega_0^2}{2\beta^2}|D_\lambda|Y+\epsilon R_1(X,Y)+E_1(t,\lambda)\quad(29)$$

$$\partial_t Y+[\rho](\frac{\Omega_0^2}{4\beta^2}+U_0)\partial_\lambda Y=\frac{\Omega_0^2}{2\beta^2}(1-\frac{[\rho]^2}{4\langle\rho\rangle^2})|D_\lambda|X+\epsilon R_2(X,Y)+E_2(t,\lambda)\quad(30)$$

E_1 and E_2 are analytic in $\mathcal{U}'$. The functions $(X,Y)\mapsto(R_1(X,Y),R_2(X,Y))$ non linear but continuous from $C^{1+\beta}$ to C^β $\quad r=\alpha t+\beta\lambda+\epsilon(X+iY)$ also to be used. Eventually this gives :

$$\left((\partial_t+[\rho](\frac{\Omega_0^2}{4\beta^2}+U_0)\partial_\lambda)^2+(\frac{\Omega_0^2}{2\beta^2})^2(1-\frac{[\rho]^2}{4\langle\rho\rangle^2})\partial_\lambda^2\right)\begin{bmatrix}X\\Y\end{bmatrix}=\epsilon R(X,Y)+E(\lambda,t) \quad (31)$$

The left hand side of (31) is a second order elliptic operator with constant coefficients and right hand side a second order non linear perturbation.. The analyticity follows.

Remark 4 • *The complex representation is used only to simplify the presentation the same type of results (mutatis mutandis) would be valid in 3 dimensions.*

• *The sign of the gravity g or more generally the Taylor stability criteria, which are essential for the water waves equation (cf the nice discussion in [Lannes] or [Wu]) are not present it the above derivation they contribute to a first order term which becomes important when the system is no more elliptic.*

$$(1-\frac{[\rho]^2}{4\langle\rho\rangle^2})=0 \Leftrightarrow \text{ Atwood number } <1 \Leftrightarrow \rho^+\rho^-=0 \qquad (32)$$

The fact that the Atwood number is striclty less 1 *appeared previously in the proof (cf (26)).*

- *Above I have described the spirit of the proof without going into the details of the analysis. In fact this details are "easy " to do if one start with higher regularity hypothesis say* $(r,\Omega) \in C^2(t,\lambda)$*. This is a standard observation for the analysis of free boundary problems of all kinds. The first steps in the proof of regularity near a "threshold" are much more difficult to obtain than later refinements of these results. In fact the Holder hypothesis requires deeper harmonic analysis and the proof of Kamotski and Lebeau makes a substential use of Paley Littlewood decomposition and Bony para product.*

- *This Holder setting is perfectly adapted to the Caflish Orellana example. What that means is that if the solution exists as an interface after the formation of the cusp then it cannot have Holder regularity and this is an hint for the validity of the spiral like curves.*

Improving the regularity threshold

Experiments and numerical simulation, done mostly for the Kelvin Helmholtz problem show the existence and the stability of vortex sheet after the singularity. Furthermore these vortex sheet roll up and seem to lead to rectifiable curves but of infinite length. Therefore the "threshold of regularity" should be above and may include spirals with finite length. In fact the best (to the best of my knowledge) result is due to Sijue Wu. The hypothesis $C^{\alpha}_{loc}(\mathbb{R}_t; C^{1+\beta}_{loc}(\mathbb{R}_\lambda)$ is replaced by $H^1_{loc}(\mathbb{R}_t \times \mathbb{R}_\lambda)$. Estimates are done explicitly using theorems of G. David saying that for all chord arc curves $\Gamma : s \mapsto \xi(s), s$ the arc length the Cauchy integral operator

$$C_\Gamma(f) = pv \int \frac{f(s')}{\xi(s) - \xi(s')} d\xi(s')$$

is bounded in $L^2(ds)$. It is interesting to notice that these results will apply to logarithmic spirals $r = e^{\theta}$ but not to infinite length algebraic spirals.

Conclusion and open problems.

In spite of the fact that the equations for water waves, Rayleigh Taylor and Kelvin Helmholtz are over simplified models discarding effects that would stabilize the fluid (viscosity and surface tension) they are considered as good models and this may be due to the instabilities that their solution exhibit and therefore progress in the understanding is valuable for applications. This motivates the following list of open problems.

- Extend to the Rayleigh Taylor problem the approached proposed by Sijue Wu for the Birkhoff Rott equation. An issue may be technical but most probably feasible

- Prove even for a finite time the existence of “Delors ” type weak solutions for the Rayleigh Taylor problem.

- Define a class of weak solutions for the interface that would be compatible with the loss of regularity. Such solutions will *not* satisfy the hypothesis of [S. Wu]. Sijue Wu proposes for the Birkhoff Rott equation the following defintion : A weak solution is a function from $\mathbb{R}$ to $\mathcal{C} : \alpha \mapsto z(\alpha,t)$ for which the following relation holds :

$$\partial_t(\int \overline{z}(\alpha)\eta(\alpha)d\alpha) = \frac{1}{4\pi i}\int\int \frac{\eta(\alpha)-\eta(\beta)}{z(\alpha)-z(\beta)}d\alpha d\beta\,, \forall \eta \in C_0^\infty\,.$$

In fact [Lopes, Nussenweig and Sochet] propose a weaker definition which contains more freedom with respect to the parameter and may be more adapted. (It is in this formulation that they show that the fact that a vortex density is in L^2 is equivalent to a weak solution in distribution from). May be since the problem is ill posed one could also consider hyperfunctions (dual of analytic functions ? ?).

- As a first step for the justification of frequency of appearance of vortex sheets prove that a regular solution (which exists during a finite time) is the limit of several types of approximations, with viscosity, with smooth initial data etc... A result of this type has been previously obtained by [Caflisch and Lowenguth] for the computation by vortex blob or point vortex.

• A much more difficult problem would be the justification of the persistence of vortex sheets. Such solution should be stationary and the classical stability criteria of Arnold for stationary solutions of the Euler equation cannot be used here (they concern stability in enstrophy norm). At variance one may believe that a stationary vortex sheet will be in non regular class of functions like the algebraic spirals of [Kaden] and [Pullin]. Stability theorem in such class cannot be reached by existing methods.

• Eventually explaining the frequent apparitions (for instance behind an airplane wing) of such vortex sheet is probably even more difficult. Even in the case of smooth stationary solutions of the $2d$ Euler equation only very partial results exist and they are mostly inspired by the approach of statistical mechanic as done for instance in [Lions Marchioro and Pulvirenti].