Title Different aspect of control theorie for Schrodinger equation.

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As described in the key-note address of **Herschel Rabitz**, the first ideas of applying engineering control concepts to the micro-world go back to the period 1985-1990. They were preceded by largely unsuccessful intuition- driven attempts that followed the invention of the the laser in the early 1960 s. In the same lecture he gives a list of 18 applications that have been demonstrated experimentally in the last decade, which leads him to wonder why this is proving to be so easy.? In the meantime, progress was made by the mathematical community in understanding the relation between different aspects of control: exact and approximate controllability, stabilisation, feedback control and optimisation both at the level of finite dimensional systems (described by ordinary differ- ential equations) and distributed systems (described by partial differential equations).

Describe new and older results for control of Schrödinger equation.

Explore the use of kinetic equation for understanding and proofs.

- Linear control

- Bilinear control

  1. Non exact controllability

  2. Approximate controllability

- Use of Wigner transform ? and ideas of proofs ??

Linear Control

$$i\hbar\partial_t\Psi(x,t) = -\frac{\hbar^2}{2}\Delta\Psi(x,t) + V(x)\Psi + f_c(x,t) = \mathcal{H}\Psi + f_c(x,t)\,,$$
$$x \in \Omega\,,\ \Psi(x,0) \equiv 0\,, \Psi(x,t) = 0 \text{ on } \partial\Omega$$
$$\mathcal{R} = \{\Psi(x,T)\backslash f_c \in L^2([0,T]\times\Omega) \text{ support} f(x,t) \subset \omega\}\times]0,T_*[\,, 0 < T_* < T.$$

Issue Range$\Psi(x,T)$: Duality (HUM) method:

$$i\hbar\partial_t\phi(x,t) = -\frac{\hbar^2}{2}\Delta\phi(x,t) + V(x)\phi\,, \Psi(x,t) = 0 \text{ on } \partial\Omega\,,$$
$$(\Psi(.,T),\phi(.,T)) = \frac{1}{\hbar}\int_0^{T_*}\int_\omega f(x,t)\phi(x,t)dxdt$$
$$\mathcal{L}\phi(.,T) = \Psi(.,T) : i\hbar\partial_t\Psi(x,t) = -\frac{\hbar^2}{2}\Delta\Psi(x,t) + V(x)\Psi + \chi_{[0,T_*]\times\omega}\phi(x,t)\,.$$

- *Approximate controllability :* $[0, T_*] \times \omega$ Uniqueness set for Schrödinger equation $\Rightarrow \mathcal{R}$ dense in $L^2(\Omega)$ K. D. Phung Observability and control of Schrödinger equations SIAM J. Control and Optim. Vol. 40 1 . pp 211-230.

- *Exact controllability* For some $T_* > 0$, $[0, T_*] \times \omega$ satisfies the BLR condition for the flow $H(x,k) = \frac{|k|^2}{2}$ $\Rightarrow \mathcal{R} = L^2(\Omega)$ G. Lebeau Contrôle de l'équation de Schrödinger , J. Math. Pures Appl. 9 , 71 (1992) , 267-291.

$$\|\Psi(x,0)\|_{L^2(\Omega)} \leq e^{C(\omega,T_*)\frac{\|\Delta\Psi(x,0)\|_{L^2(\Omega)}}{\|\Psi(x,0)\|_{L^2(\Omega)}}} \|\Psi(x,0)\|_{L^2(\Omega)}\,,\ \text{Phung}\,,$$

$$\|\phi(x,T)\|_{L^2(\Omega)} \leq C(\omega,T_*)\int_{\omega\times[0,T_*]} |\phi(x,t)|^2 dxdt\,,\ \text{Lebeau}\,.$$

Bilinear Control

Adapted to functional analysis and closer to reality

For wave functions $\Psi(x) \in \mathcal{S}_n \Leftrightarrow \Psi \in L^2(\mathbb{R}^n)\,, \|\Psi\| = 1$ and densities $\rho(x,y) \in \mathcal{D} = \{$ selfadjoint positive trace class operators with trace equal to $1\}$

$$i\hbar\partial_t\Psi(x,t) = -\frac{\hbar^2}{2}\Delta\Psi(x,t) + U\Psi + \sum_{1\le i\le m} u_i(t)V_i(x)\Psi = \mathcal{H}_{u(t)}\Psi \quad (1)$$

$$x \in \Omega\, \Psi(x,T_0) \text{ given }, \Psi(x,t) = 0 \text{ on } \partial\Omega \quad (2)$$

$$\mathcal{R} = \{\Psi(x,T_1), u = (u_1(t), u_2(t)\ldots, u_m(t)) \in \mathcal{C}\} \quad (3)$$

- *Approximate controllability* is generically true for the wave function and for the density matrix.

- *Exact controllability* Exact controllability never holds

$m = 1$, $V \in L^{\infty}$ (for sake of simplicity)

$$i\hbar\partial_t\Psi(x,t) = -\frac{\hbar^2}{2}\Delta\Psi(x,t) + U\Psi + u(t)V(x)\Psi$$

$$\partial_t\Psi = iA\Psi + u(t)iB\Psi\,,A,B \text{ self adjoint and } B \text{ bounded}$$

$$\Psi_u(T_1)(\Psi(T_0)) = e^{it_k(A+u_kB)}e^{it_{k-1}(A+u_{k-1}B)} \circ \ldots e^{it_1(A+u_1B)}$$

$$= \mathbf{V}_u\Psi(T_0)\,,u \text{ piecewise constant}$$

For $\Phi(T_0) \in \mathcal{S}\,,\,,u(t) \in \mathcal{C} \subset L^p(0,T)$, *Ball Marsden Slemrod and Turinici* $\{\Psi(.,T)\}$ given by (1), (2) and (3) is a countable union of closed compact set of $\mathcal{S} \subset L^2(\Omega)$ therefore of closed set with empty interior and therefore it has also an empty interior.

Under “generic conditions ” (cf. Sigalotti)

1. The set $\{\Psi_u(T_1) = \mathbf{V}_u(\Psi(T_0)), \Psi(T_0) \in \mathcal{S}\,, |u| \leq \delta\}$ is dense in $\mathcal{S} \subset L^2(\Omega)$.

2. Lower bound on the steering time is given.

$$T_u \geq \frac{1}{\delta} \sup_{k \in \mathbf{N}} \frac{(\phi_k, \Psi_1 - \Psi_0) - \epsilon}{\|V\phi_k\|}$$

3. For two unitary equivalent density matrices ρ_0 and ρ_1 there exists a control u such that one has:

$$\|\rho_1 - V_u \rho_0 V_u^*\| \leq \epsilon \text{ In trace norm} \tag{4}$$

Chambrion, Mason, Sigalotti and Boscain. http: \\ www. arxiv.org\abs\0801.4893

Generic conditions

The spectra of $-\frac{\hbar^2}{2}\Delta + V(x)$ is discrete $\{\lambda_k\}$ are simple eigenvalues. The number $(\lambda_{k+1} - \lambda_k)$ are $\mathbf{Q}$ linearly independent.

For the eigenvector ϕ_k one has:

$$\forall k \in \mathbf{N} \ \int_\Omega V(x)\phi_k(x)\phi_{k+1}(x)dx \neq 0$$

Proof

- Time reparametrisation

$$\partial_t \Psi = A\Psi + uB\Psi \Leftrightarrow \partial_t \Psi = u(\frac{1}{u}A\Psi + B\Psi) \Leftrightarrow \partial_t \Psi = uA\Psi + B\Psi \,, u \in [\frac{1}{\delta} \infty[$$

- Galerkine approximation and Agrachev version of Rasheski(1938)-Chow (1939) theorem Discretise with the eigenvectors of A A discrete system in $\mathbf{S}^N$ the unit sphere of $\mathbf{C}^N$

$$\partial_t \Psi_d = uA_d\Psi + B_d$$
$$\Rightarrow A_d = i\{\lambda_k\}, \text{diagonal}\,, B_d = i\{(V\phi_i, \phi_j)\}\text{anti-Hermitian}$$
$$\Psi_d(T_0)\,, \Psi_d(T_1) \in \mathbf{S}^N$$

With the hypothesis on A_d and B_d one has $\mathbf{su}(N) \subset$ Lie(A_d, B_b); Agrachev version of Rasheski-Chow theorem$\Rightarrow$ exact controllability of the discrete system.

- Estimates for convergence of the Galerkine approximation.

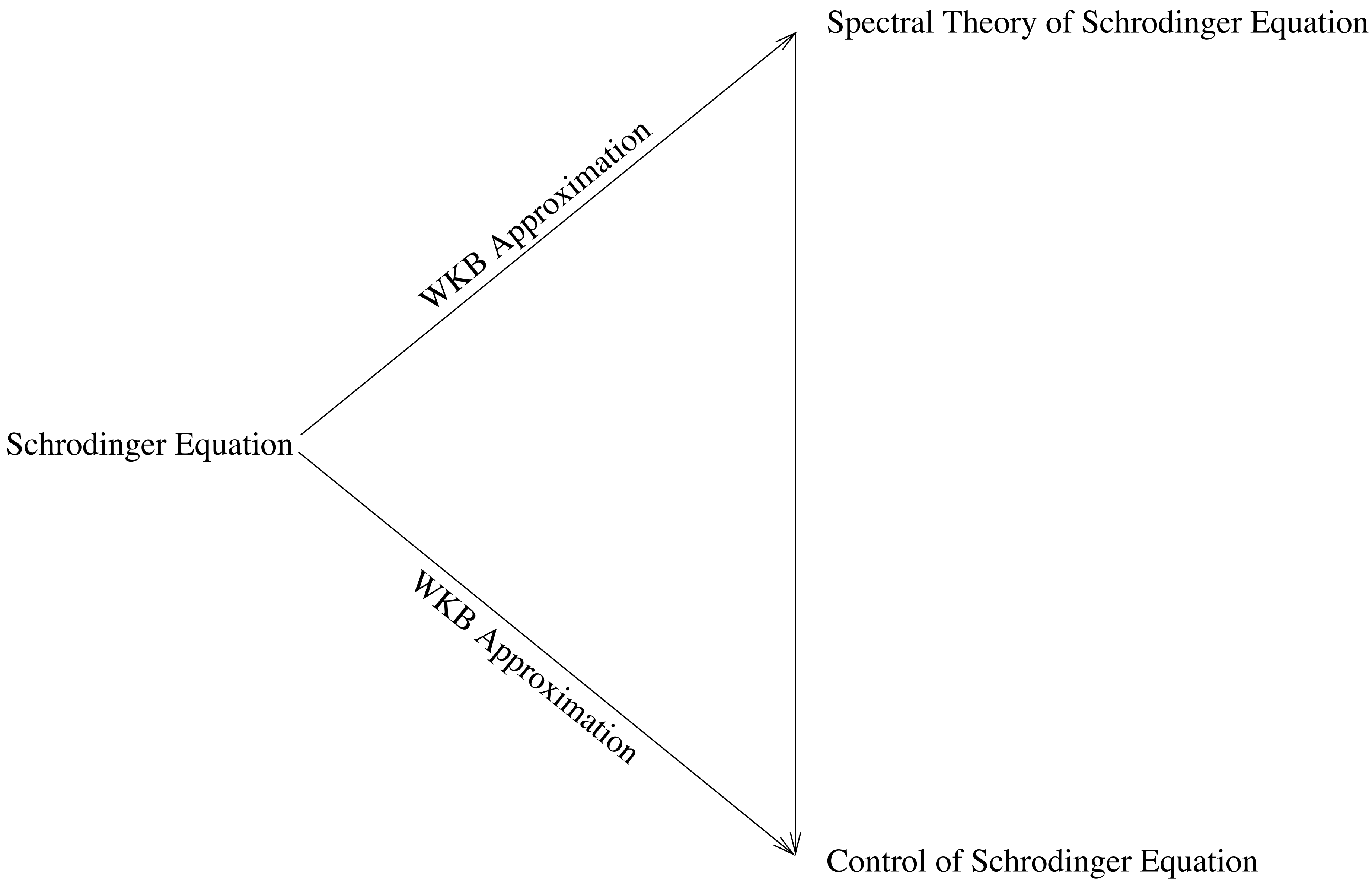
For the density

$$\rho_0 = \sum_{1 \le j < \infty} \mu_j \Psi_j(0) \otimes \overline{\Psi_j(0)}$$
$$\rho_T = \sum_{1 \le j < \infty} \mu_j \Psi_j(T) \otimes \overline{\Psi_j(T)}$$

truncate the above expansion and realise a simultaneous approximate control $\Psi_j(0) \mapsto \Psi_j(T)\,, 1 \le j \le P < \infty$

Still a finite dimensional problem.

Use of Wigner transform ? and ideas of proofs ??



- The most relevant object is the density matrix. The thing to be controlled are the observable Tr($A\rho$)

- Present results are based on spectral hypothesis.

- Spectral properties can be described by the $\hbar \rightarrow 0$ limit. So there is some hope of recovering directly control properties by looking at the limit. This turned out to be true for the linear control of solutions of wave and Schrödinger equations from BLR to present time.

- The limit equation is a Liouville equation for the Wigner transform of the density matrix or its kernel in term of Weyl calculus.

- Several (may be not fully complete) results do exist on the controllability of solutions of the Liouville equation the rely on the vector field counter part of the Rasheski-Chow theorem. Agrachev-Caponigro or Brockett.

Density Matrix, Wigner Transform

$$H_u = -\frac{\hbar^2}{2}\Delta\Psi(x,t) + U\Psi + \sum_{1\le j\le m} u_j(t)V_j(x)$$

$i\hbar\partial_t\rho(x,y,t) = [H_u,\rho]$ Von Neumann equation

$$w_\hbar(x,k,t) = \frac{1}{(2\pi)^d}\int e^{-iky}\rho(x+\frac{\hbar y}{2}, y-\frac{\hbar y}{2})dy$$

$\rho_\hbar(x,y,t) = \int e^{i\frac{k}{\hbar}(x-y)} w_\hbar(\frac{x+y}{2},k,t)dk$ Weyl symbol

$$\rho_\hbar(x,x,t) = \int w_\hbar(x,k,t)dk$$

$$\mathrm{Tr}(A\rho) = \int e^{i\frac{k}{\hbar}(x-y)} A(x,y) w_\hbar(\frac{x+y}{2},k,t)dydx$$

The principal symbol is transported by the Liouville equation (Egoroff Theorem):

$$w_\hbar(x,k,t) = w_0(x,k,t) + O(\hbar)$$
$$\partial_t w_\hbar(x,k,t) + k\cdot\nabla_x w_\hbar(x,k,t) -$$
$$-(\nabla_x U)\nabla_k w_\hbar(x,k,t) - \sum_{1\le j\le m} u_j(t)\nabla_x V_j(x)\nabla_k w_\hbar(x,k,t) = 0(\hbar) \quad (5)$$
$$\partial_t w_0(x,k,t) + k\cdot\nabla_x w_0(x,k,t) -$$
$$-(\nabla_x U)\nabla_k w_0(x,k,t) - \sum_{1\le j\le m} u_j(t)\nabla_x V_j(x)\nabla_k w_0(x,k,t) = 0 \quad (6)$$

Conjectures

1. Exact controllability for the Liouville equation should imply approximate controllability of the bilinear Schrödinger equation...

2. With a convenient bracket hypothesis on the vector fields $k\cdot\nabla_x - (\nabla_x U)\nabla_k$ and $\nabla_x V_j(x)$ exact controllability holds for the Liouville equation (6)

Hint for 1 With change of scale in time and in the size of the $u_i(t)$ one can always assume that $\hbar$ is small!!!

Hint for 2 Consider a general linearly controlled Liouville equation in $\mathbb{R}^m$ and the associated group of transformations:

$$\partial_t w + \sum_i u_i(t) F_i(x) \nabla_x w = 0 \tag{7}$$

$$\dot{X}(s) = \sum_i u_i(t) F_i(X(s)) X(0) = X\,, \Phi_u(T)(X) = X(T) \tag{8}$$

Rasheski(1938)-Chow(1939) Theorem. The following assertions are equivalents

1. Exact controllability $\forall \{X_1\,, X_2\,, T\}$ There exist a control u such that $\Phi_u(T)(X_2) = X_1$

2. Completeness $\forall X \in \mathbb{R}^m$ the Lie Algebra spanned by the bracket of the vectors fields $[F_i(x)\nabla.\,, F_j(x)\nabla.]$ is complete (of dimension m)

Remark

1. The idea is to show that one can move in any direction by a succession of move in the direction $F_i(X)\nabla$ with the transformation
$$e^{u_{i_1}F_{i_1}}e^{u_{i_2}F_{i_1}}\dots e^{u_{i_k}F_{i_k}}, u_I \text{ piece-wise constants}$$

2. This is the same observation which is used in the Hörmander (1967) hypoellipticity theorem.
$$\sum_i (F_i\nabla)^2 u = f \in L^2 \Rightarrow u \in H^{\frac{2}{k}} k \text{number of brackets needed to generate } \mathbb{R}^m$$

3. When the fields are linear $F_i(X) = A_i X$ it is equivalent to say that the bracket $[A_i, A_j]$ is complete.

4. It has several *"Manifolds"* versions as used above in the Galerkine reduction by Chambrion, Mason, Sigalotti and Boscain. $M = \mathbf{su}(\mathbb{R}^m)$ Liealgebra$(\mathbf{su}) \subset$ antihermitian matrices.

From control of Dynamical systems to control of the Liouville equation

$$\partial_t w + \sum_i u_i(t) F_i(x) \nabla_x w = 0 \tag{9}$$

$$\dot{X}(s) = \sum_i u_i(t) F_i(X(s)) X(0) = X \,, \Phi_u(T)(X) = X(T) \tag{10}$$

$$F(x,T) = F_0(\Phi_u^{-1}(T)(X)) = F_0(G(X)) \; G \in \mathrm{Diff}_0 \,. \tag{11}$$

Generate Diff_0 **by controlled versions of (9)**

Agrachev Caponigro Theorem arXiv:0804.4403v1 [math.DG] 28 Apr 2008 For any complete family $\mathcal{F}$ of vector fields on a compact (without boundary) manifold M one has

$$\mathrm{Gr} = \{af : a \in C^\infty(M), f \in F\} = \mathrm{Diff}_0 M$$

$$\Rightarrow \text{controlled , equation : } \partial_t w + \sum_i u_i(t) a(x) F_i(x) \nabla_x w = 0$$

Khesin-Lee Mauser Type Theorem arXiv:0802.1551v2 [math.DG] 12 Mar 2008† ; Connection with Hörmander Criteria (1967) : $M \subset \mathbb{R}^m$ with smooth boundary ∂M $G(t) : M \mapsto M$ a family of smooth diffeomorphisms $w(x,t) = w_0(G^{-1}(t)((x))$ *which preserve the total mass*, then there exist $V(x,t)$ such that

$$\partial_t w + \sum_i \nabla \cdot \left(F_i[wF_i \cdot \nabla V] \right) = 0 \leftrightarrow \partial_t w + \sum_i \nabla \cdot \left(u_i(x,t) F_i w \right) = 0 \qquad (12)$$

Proof Let $w(x,t)$ given then solve for V the equation

$$-\sum_i \nabla \cdot \left(F_i [w F_i \cdot \nabla V] \right) = \partial_t w$$

in the variationnal formulation:

$$\forall W \; \int_M \sum_i w(F_i \cdot \nabla V)(F_i \cdot \nabla W) dx = \int_M \partial_t w W(x) dx \tag{13}$$

Compactness comes from Hörmander theorem; Fredholm alternative from total mass conservation and boundary condition is included in the variationnal formulation (M. Derridj (1969)!)