The shear flow and the $3d$ Euler equation

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More than 250 years after the Euler equation have been written our knowledge of their mathematical structure and their relevance to describe complicated phenomena as turbulence is still at least very incomplete. Both in $2d$ and $3d$ challenging problems remain open.

No idea if in $3d$ solutions which start with smooth initial data will remain smooth all the time or may become singular. In presence of singularity rely on weak solution. However there no construction of weak solution.

Defining an optimal functional space in which the problem is well posed in the sense of Hadamard is also an important issue.

The Kolmogorov Oboukow law lead to the idea that non conservation of energy would be related to loss of regularity. Onsager conjectured the existence of a threshold in the regularity that would discriminate between solutions which conserve energy and solution which dissipate this energy.

Configuration where the vorticity being a measure is concentrated on a curve (in $2d$) or on a surface in $3d$ are called Kelvin Helmotz flow. They seem to play a rôle in numerical simulations and in the description of turbulent phenomena. However mathematical analysis and experiment show that these configurations are extremely unstable.

Detailed study of explicit examples : the shear flow

Defined in the all space $\mathbb{R}^3$ or on the periodic box $(\mathbb{R}/Z)^3$.

$$u(x,t) = (u_1(x_2), 0, u_3(x_1 - tu_1(x_2))) \tag{1}$$

$$\nabla \cdot u = 0 \quad \text{and } \partial_t u + \nabla(u \otimes u) = -\nabla p \text{ with } p \equiv 0 \tag{2}$$

Weak Limit of Oscillating Initial Data : Di Perna Majda

$$u^\epsilon(x,t) = (u_1(\frac{x_2}{\epsilon}), 0, u_3(x_1 - tu_1(\frac{x_2}{\epsilon})) \int_0^1 u_1(s)ds = 0$$

$$\lim_{\epsilon \to 0} weak\ u\epsilon = (0, 0, \overline{u_3}) \,, \overline{u_3} = \int_0^1 u_3(x_1 - tu_1(s))ds$$

$$\lim_{\epsilon \to 0} \left(\partial_t u_3^\epsilon + \nabla \cdot u^\epsilon \otimes u_3^\epsilon \right) = \partial_{x_1} \int_0^1 u_1(s)u_3(x_1 - tu_1(s))ds \neq 0$$

Instability of Cauchy Problem, Loss of regularity

1. For initial data in $C^{1,\alpha}$ the Euler equation has a unique local in time solution in $C^{1,\alpha}$ Same result for initial data in $H^s, s > \frac{5}{2}$.
2. Beale, Kato, Majda : For initial data in $C^{1,\alpha}$ or in $H^s, s > \frac{5}{2}$ the solution exists and depends continuously on the initial data as long as the vorticity remains bounded in L^∞
3. De Lellis and L. Székelyhidi Jr : A residual set (in $L^2(\Omega)$) of initial data u_0 for which the Cauchy problem has an infinite family of solutions in $C_w(\mathbb{R}_t; L^2(\Omega))$.
4. One does not know the existence of $3d$ regular (say in $C^{1,\alpha}$) solution of the Euler equation that become singular in a finite time (blow up problem).

Basic tool : compare with

$$\begin{aligned}
&\partial_t\omega + u\cdot\nabla\omega = \omega\cdot\nabla u \\
&\nabla\cdot u = 0\,, \nabla\wedge u = \omega \;\Rightarrow \nabla u = K(\omega) \\
&\|\omega\cdot\nabla u\| \le C\|\omega\|^2 \|\,.\,\| \text{ Convenient norm}
\end{aligned} \tag{3}$$

Theorem Di Perna - Lions For every $p \geq 1$, $T > 0$ and $M > 0$ there exists a smooth shear flow solution for which $\|u(x, 0)\|_{W^{1,p}} = 1$ and $\|u(x, T)\|_{W^{1,p}} > M$.

Theorem In the Holder spaces C^1 is the *critical space for local in time well posedness*

1. For $(u_1(x), u_3(x)) \in C^{1,\alpha}$, $0 \leq \alpha < 1$ the shear flow solution is always in $C^{1,\alpha}$,
2. For $(u_1(x), u_3(x)) \in C^{0,\alpha}$ the shear flow solution is always in C^{0,α^2},

   There exists shear flow solutions which for $t = 0$ belong to $C^{0,\alpha}$ and wich for $t \neq 0$ are not in $C^{0,\beta}$ for $\beta > \alpha^2$.

Proof Regularity results concern only the component u_3

$$\frac{|u_3(x_1 - tu_1(x_2+h)) - u_3(x_1 - tu_1(x_2))|}{h^{\alpha^2}}$$
$$= \frac{|u_3(x_1 - tu_1(x_2+h)) - u_3(x_1 - tu_1(x_2))|}{|tu_1(x_2+h) - tu_1(x_2)|^\alpha}\left(\frac{|tu_1(x_2+h) - tu_1(x_2)|}{h^\alpha}\right)^\alpha$$
$$\le |t|^\alpha \|u_3\|^{0,\alpha}(\|u_1\|^{0,\alpha})^\alpha\,.$$

Introduces two periodic functions u_1 and u_3 which near the point $x=0$ coincide with $|x|^\alpha$ then the for t given and x_1 and x_2 small enough $u_3(x_1 - tu_3(x_2))$ coincides with

$$|x_1 - t|x_2|^\alpha|^\alpha$$

For $(x_1, x_2, x_3) = (0, x_2, x_3)$ one has

$$u_3(x_1 - tu_3(x_2)) = |t|^\alpha |x_2|^{\alpha^2}$$

and the conclusion follows.

The Kelvin Helmholtz Problem and the Shear flow

Vorticity concentrated on a oriented curve $(2d)$ or surface $(3d)$

$$u(x,t) = \begin{cases} \frac{1}{2\pi} R_{\frac{\pi}{2}} \int \frac{x-r(t,\lambda')}{|x-r(t,\lambda')|^2} \tilde{\omega}(t, r(t,\lambda'))|\partial_\lambda(r(t,\lambda'))| d\lambda' \text{ in } 2d\,, \\ -\frac{1}{4\pi} \int \frac{x-r(t,\lambda',\mu')}{|x-r(t,\lambda',\mu')|^3} \wedge \tilde{\omega}(t, r(t,\lambda',\mu'))|\partial_\lambda(r(t,\lambda')) \wedge \partial_\mu(r(t,\lambda'))| d\lambda' d\mu' \text{ in } 3d \end{cases}$$

When x converges to $r \in \Gamma(t)$ the velocity $u(x,t)$ converges to two different values $u_\pm(r)$

$$u_+(r)\cdot\vec{n} = u_-(r)\cdot\vec{n} \quad , \quad \omega = (u_+(r) - u_-(r)) \wedge \vec{n} \otimes \delta_{\Gamma(t)}$$

The vorticity density :

$$\omega = (u_+(r) - u_-(r)) \wedge \vec{n} \otimes \delta_{\Gamma(t} = \begin{cases} 2d\ \tilde{\omega}(t, r(t,\lambda'))|\partial_\lambda(r(t,\lambda'))| d\lambda' \ , \\ 3d\ \tilde{\omega}(t, r(t,\lambda',\mu'))|\partial_\lambda(r(t,\lambda')) \wedge \partial_\mu(r(t,\lambda'))| d\lambda' d\mu' \end{cases}$$

On the interface

$$v = \frac{u_+ + u_-}{2}$$

$$(\partial_t r - v) \cdot \vec{n} = 0\,,$$

In $2d$

$$\partial_t \tilde{\omega} + \frac{\partial}{\partial \lambda}\left(\frac{\tilde{\omega}}{|r_\lambda|^2}(v - r_\lambda) \cdot r_\lambda\right) = 0$$

In $3d$ with $N = \partial_\lambda r(t,\lambda,\mu) \wedge \partial_\mu r(t,\lambda,\mu)$

$$\partial_t \tilde{\omega} + \frac{\partial}{\partial \lambda}\left(\frac{\tilde{\omega}}{\|N\|^2}(v - \partial_t r, \partial_\mu r, N)\right) - \frac{\partial}{\partial \mu}\left(\frac{\tilde{\omega}}{\|N\|^2}(v - \partial_t r, \partial_\lambda r, N)\right)$$
$$= \frac{1}{\|N\|^2}(\partial_\mu r, N, \tilde{\omega})\partial_\lambda v - \frac{1}{\|N\|^2}(\partial_\lambda r, N, \tilde{\omega})\partial_\mu v = 0\,.$$

Classical results

1. The initial value problem is locally in time well posed in $2d$ and $3d$ the class of analytic data.
2. There exist in $2d$ in $3d$?? analytic solutions which after a finite time exhibit singularities. Ex : Caflisch and O. Orellana analytic solutions for $0 < t < T$ with a cusp for $t \to T$ (with $0 < \nu < 1$) :

$$\lim_{t \to T} (\Gamma(t), \tilde{\omega}(t)) = (\Gamma(T), \tilde{\omega}(T)) \begin{cases} \notin C^{1+\nu} \times C^{\nu}, \\ \in C^{1+\nu'} \times C^{\nu'} \quad \forall\, 0 < \nu' < \nu\,. \end{cases}$$

3. In $2d$ If in the neighbourhood of a point $r(t_0, \lambda_0), \tilde{\omega}(t_0, \lambda_0)$ the density of vorticity does not vanishes and if the functions $r(t, \lambda), \tilde{\omega}(t, \lambda)$ have some *limited regularity* then in fact they are analytic in this neighbourhood.

Limited regularity :

$$(r(t,\lambda), \tilde{\omega}(t,\lambda)) \in C^{1+\alpha} \times C^{\alpha}$$
$$|\lambda - \lambda'| \le C |r(t,\lambda) - r(t,\lambda')| \text{ with } C < \infty$$

The catch in the above $2d$ results :

$$\partial_t y - v_2 = -(v_1 \partial_x y)\,, \tag{4}$$

$$\partial_t \tilde{\omega} + \partial_x (v_1 \Omega_0) = -\epsilon \partial_x (v_1 \tilde{\omega})\,, \tag{5}$$

$$v_1(x,t) = -\frac{1}{2\pi} P.V. \int \frac{y(x,t) - y(x',t)}{(x-x')^2 + \epsilon^2 (y(x,t) - y(x',t))^2} (\Omega_0 + \epsilon \tilde{\omega}) dx'\,, \tag{6}$$

$$v_2(x,t) = \frac{1}{2\pi} P.V. \int \frac{x - x'}{(x-x')^2 + \epsilon^2 (y(x,t) - y(x',t))^2} (\Omega_0 + \epsilon \tilde{\omega}) dx'\,. \tag{7}$$

This system describes perturbations in $\mathbb{R}^2$ of the stationary solution

$$y(x,0) = 0\,, u_- = \frac{\Omega_0}{2}\,, u_+ = -\frac{\Omega_0}{2}\,.$$

The expansion

$$\frac{1}{\pi}P.V.\int\frac{f(x)-f(x')}{(x-x')^2+\epsilon^2(y(x,t)-y(x',t))^2}dx' =$$
$$\frac{1}{\pi}P.V.\int\frac{f(x)-f(x')}{(x-x')^2+\epsilon^2(y(x,t)-y(x',t))^2}dx' =$$
$$\frac{1}{\pi}P.V.\int\frac{f(x)-f(x')}{(x-x')^2}\left(1+\sum_{n\geq 1}(-1)^n\epsilon^{2n}\left(\frac{y(x)-y(x'))}{x-x'}\right)^2\right)dx' \tag{8}$$

leads to the introduction of the operators (Hilbert transform) :

$$Hf(x) = \frac{1}{\pi}\int\frac{1}{x-x'}f(x')dx' = F^{-1}(-i\mathrm{sgn}(\xi)\hat{f}(\xi)) \tag{9}$$
$$|D|f(x) = \frac{1}{\pi}P.V.\int\frac{f(x)-f(x')}{(x-x')^2} = \partial_x(Hf(x)) = F^{-1}(|\xi|)\hat{f}(\xi)). \tag{10}$$

This gives, with the formula (9), (10) , for the perturbation of the stationary solution :

$$\partial_t y_x - \Omega_0 |D| \omega = \epsilon F(y_x, \omega)_x$$
$$\partial_t \omega - |D| y_x = \epsilon G(y_x, \omega)_x$$

In the right hand side F and G are first order operators. Eventually with the introduction of the “Laplacian" one has :

$$\partial_{tt}(y_x) + \Omega_0^2 \partial_{xx}(y_x) = \epsilon(\partial_t(F(y_x, \omega)_x) + |D|(\epsilon G(y_x, \omega)_x) \,, \tag{11}$$
$$\partial_{tt}(\omega) + \Omega_0^2 \partial_{xx}(\omega) = \epsilon(|D|(F(y_x, \omega)_x) + \partial_t(\epsilon G(y_x, \omega)_x) \,. \tag{12}$$

Shear flow with interface

Proposition In $3d$ with the following configuration

$$u_3(s) = \begin{cases} 1 & \text{for } x < 0\,, \\ 0 & \text{for } x > 0 \end{cases}$$

and $y = u_2(s)$ a C^1 curve the shear flow :

$$u(x) = (u_1(x_2), 0, u_3(x_1 - tu_2(x_2)))$$

is a solution of the $3d$ Euler equation with a vorticity, singular on the surface

$$\Gamma(t) = \{(x_1, x_2, x_3)/x_1 = tu_1(x_2), \}$$

and given by :

$$\omega(x,t) = (-t\frac{\partial_{x_2}u_1}{(|t\partial_{x_2}u_1|^2+1)^{\frac{1}{2}}} \otimes \delta_{\Gamma(t)}, \frac{1}{(|t\partial_{x_2}u_1|^2+1)^{\frac{1}{2}}} \otimes \delta_{\Gamma(t)}, -\partial_{x_2}u_1(x_2))\,,$$

The above example is not a genuine solution of the Kelvin Helmholtz problem because of the presence of the distributed vorticity $-\partial_{x_2}u_1(x_2)$.

Not the dominant effect. For regularity consider $\Gamma(t) = \{x_3 = x(x_1, x_2, t)\}$, and *small* perturbation of the stationnary state $x_3 = 0, \tilde{\omega}^0(x_1, x_2) = (\tilde{\omega}_1^0, \tilde{\omega}_2^0, 0)$.

$$\text{Leading part of the perturbed equation } \partial_t \begin{pmatrix} \widehat{x_3} \\ \widehat{\omega_1} \\ \widehat{\omega_2} \\ \widehat{\omega_3} \end{pmatrix} = \mathcal{A} \begin{pmatrix} \widehat{x_3} \\ \widehat{\omega_1} \\ \widehat{\omega_2} \\ \widehat{\omega_3} \end{pmatrix} \tag{13}$$

with $k = |k|(\cos\theta, \sin\theta)$ and

$$\mathcal{A} = \begin{pmatrix} 0 & \frac{i}{2}\sin\theta & -\frac{i}{2}\cos\theta & 0 \\ -\frac{i}{2}|k|^2|\omega^0|^2\sin\theta & 0 & 0 & \frac{1}{2}(k\cdot\omega^0)\sin\theta \\ \frac{i}{2}|k|^2|\omega^0|^2\cos\theta & 0 & 0 & -\frac{1}{2}(k\cdot\omega^0)\cos\theta \\ 0 & -\frac{1}{2}(k\cdot\omega^0)\sin\theta & \frac{1}{2}(k\cdot\omega^0)\cos\theta & 0 \end{pmatrix}.$$

$$\text{eigenvalues of the matrix } \mathcal{A} = (0, 0, -\frac{1}{2}|k\wedge\omega^0|, \frac{1}{2}|k\wedge\omega^0|)\,.$$

$$\partial_t - \mathcal{A}$$

is no more elliptic a basic reason why a smooth (with limited regularity) singular support may persist without being in fact analytic. $3d$ more stable than $2d$!!!

The Shear Flow and the Energy conservation

Onsager conjecture : for weak solutions of the $3d$ Euler equation the Energy decay for weak solutions of $3dEuler \Leftrightarrow$ loss of regularity. Critical value $1/3$.

Constantin E and Titi : $u \in \mathcal{B}^{\frac{1}{3}}_{3+\epsilon,\infty} \Rightarrow$Energy conservation .

DeLellis and Szekelyhidi No conservation for some solutions in $C_{\text{weak}}(\mathbb{R}_t L^2(\mathbb{R}^3))$

Eyink : a function $u_0(x) \in C^{0,\frac{1}{3}}$ which cannot be the initial data of any weak solution which conserves the energy. This is not however a *full* counter example because the existence of solutions with such initial data is an open problem.

Belief for solutions slightly weaker than $\mathcal{B}^{\frac{1}{3}}_{3,\infty}$ there would be no conservation of energy.

No hope for general theorem stating that conservation of energy implies some type of regularity.

$$\text{In a periodic box } u(x,t) = (u_1(x_2), 0, u_3(x_1 - tu_1(x_2))$$

Energy conservation remains true under the only assumption $u \in L^2(\Omega)$.

Lemma For $u_1(x_2), u_3(x_1) \in L^2((\mathbb{R}/Z)) \times L^2((\mathbb{R}/Z))$ and any test functions $\phi_i \,, i = 1, 2$ the following standard formula

$$\iiint_\Omega |u_3(x_1 - tu_1(x_2))|^2 \phi_1(x_1)\phi_2(x_2) dx_1 dx_2$$
$$= \iiint_\Omega |u_3(x_1)|^2 \phi_1(x_1 + tu_1(x_2))\phi_2(x_2) dx_1 dx_2$$

remains valid.

Theorem For any functions $u_1(x_1), u_3(x_3) \in L^2((\mathbb{R}/Z)) \times L^2((\mathbb{R}/Z))$ the shear flow is a solution with constant energy :

$$\frac{d}{dt}\iiint_\Omega \|u(x,t)\|^2 dx = \frac{d}{dt}\iiint_\Omega (|u_1(x_2)|^2 + |u_3(x_1 - tu_1(x_2))|^2) dx_1 dx_2 dx_3 = 0 \,.$$

The hypothesis on the initial data are much weaker than for which the Onsager conjecture is stated in Shvydkoy or Constantin, E and Titi and Cheskidov, Constantin and Friedlander.

Shvydkoy considers the energy conservation for weak solutions the Euler equation with singularities on a curve (in $2d$) and on a surface (in $3d$). This class of solutions includes the Kelvin Helmholtz. More relevant results in dimension 3 than in dimension 2.

In agreement with this observation we propose the following example. Consider for $u_1(s)$ a periodic function which coincide near 0 with the function $\sin \frac{1}{s}$ and for $u_3(s)$ a periodic function which near 0 coincides with $\mathrm{sgn}(s)$ the shear flow

$$u(x,t) = (u_1(x_1), 0, u_3(x_1 - tu_1(x_2))$$

is a weak solution which conserves the energy and which does not satisfies the hypothesis given by Shvydkoy.

In the class of Hölder spaces C^1 is the critical space for the initial value problem to be locally in time well posed $3d$ Euler equations are well posed in $C^{1,\alpha}$ not in C^α for any $0 < \alpha < 1$.

The Kelvin-Helmholtz refers to a free boundary problem were in $2d$ minimal regularity implies analyticity. This result is false in $3d$ and we have shown the existence of solutions which are close to this problem in $3d$ with certain degree of singularity which persists over time.

The relation between dissipation of energy and loss of regularity is an essential issue in the statistical theory of turbulence in relation with the Kolmogorov Obukhov law. It has been shown in the deterministic framework that a regularity of this type implies conservation of energy. With the shear flow we have shown that there is no hope for a converse statement even in the case of solutions singular on a *slit* as Shvydkoy.

This observation may not invalidate the physical belief because the Kolmogorov Oboukov law belongs to the statistical theory of turbulence were results are true in some *averaged sense* . On the other hand our family of examples are particular enough to be of measure zero with respect to any ensemble measure compatible with the statistical theory of turbulence (such measure has not been constructed, up to now, with full mathematical rigor).

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HAPPY BIRTHDAY JOHN.