GLOBAL WELL POSEDNESS FOR THE GHOST EFFECT SYSTEM

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1. Introduction. The Boltzmann equation is a fundamental equation in statistical physics for rarefied gas which describes the time evolution of particle distribution. However, the study of the hydrodynamic limit of Boltzmann equation is important and challenging. This is one among the problems introduced by HILBERT. He asked for a full mathematical justification uncover the connection of the Boltzmann equation with the traditional frameworks of fluid dynamics through asymptotic extensions regarding the Knudsen number tends to zero. The Knudsen number being the inverse of the average number of collisions for each particle per unit of time. Until now, there is no complete answer of this problem. Many authors, such a C. BARDOS, F. GOLSE, C. D. LEVERMORE, P.-L. LIONS, etc, respond for particular cases of this big question. The reader at this stage can consult the paper of C. VILLANI [22] for a more details about this problem. Nevertheless, when was expected that the analysis of zero Knudsen number of Boltzmann equation yields to the classical fluid dynamics systems, recent studies in this way by Y. SONE [21], [20] shows that there is an important class of problems where both the Euler equations and the Navier-Stokes equations fail to describe the behavior of a gas in the continuum limit and infinitesimal quantities (or quantities of the order of the Knudsen number) affect the behavior of the gas. In reality, this observation was pointed out firstly by MAXWELL in [18]. He claimed a correction to Navier-Stokes stress tensor that relies on derivatives of the temperature. Nonetheless, he
simply considered regimes in which the effect of this correction entered only through boundary conditions. We refer the reader to [18, 21, 20] and references therein for more details.

According to this observation, the quantity that completes the classical gas dynamic system is given by the flow velocity of the first order of the Knudsen number. In other words, if one lives in the world in the continuum limit, one does not know this velocity. Something that cannot be perceived in the world produces a finite effect on the behavior of a gas. It may be called a ghost effect as Sone was coined.

This inscrutable view advances many authors to go through to realize rigorously what correction can be inspired for Navier-Stokes or Euler equations from Kinetic theory. In [13], C. D. Levermore, W. Sun and K. Trivisa have established a low Mach number of a compressible fluid dynamics system that includes dispersive corrections to Navier-Stokes equations (derived from Kinetic theory). The limiting system is a ghost effect system 1, which is so named also because it cannot be derived from the Navier-Stokes system of gas dynamics, while it can be derived from Kinetic theory.

The ghost effect system derived in [13] describes the evolution of the density \( \rho(t, x) \), velocity \( u(t, x) \), and pressure \( p(t, x) \) as function of time \( t \in \mathbb{R}^+ \) and position \( x \) over a torus domain \( \mathbb{T}^d \) (\( d = 2, 3 \)),

\[
\begin{align*}
\rho T &= 1, \\
\partial_t \rho + \text{div}(\rho u) &= 0, \\
\partial_t (\rho u) + \text{div}(\rho u \otimes u) + \nabla p &= \text{div} \Sigma + \text{div} \tilde{\Sigma}, \\
\frac{5}{2} \text{div} u &= \text{div} (k(T)\nabla T),
\end{align*}
\tag{1}
\]

where \( \kappa(T) \) is the coefficient of thermal conductivity, \( \Sigma \) is the viscous stress and \( \tilde{\Sigma} \) is the thermal stress related to the fluids variables \( \rho, u \) and \( T \) through the constitutive relations

\[
\begin{align*}
\Sigma &= \mu(T)(\nabla u + \nabla^t u - \frac{2}{d} \text{div} u I), \\
\tilde{\Sigma} &= \tau_1(T)(\nabla^2 T - \frac{1}{d} \Delta T I) + \tau_2(T) (\nabla T \otimes \nabla T - \frac{1}{d} |\nabla T|^2 I),
\end{align*}
\]

where \( \mu(T) \) represents the coefficients of shear viscosity \( \tau_1(T), \tau_2(T) \) are transport coefficients that arise from Kinetic theory such that \( \tau_1 > 0 \).

While an existence of local weak solutions is known since the work of C. D. Levermore et al. in [14], there is no result until now, to the author’s knowledge, concerns the global existence of weak solutions to the above system even in special cases, namely, particular case of heat conductivity equation for example, or when the density is assumed to be close to the equilibrium state. At the first point of view, it is not act for an easy problem. Roughly speaking, since apparently we do not have a classical energy estimate associated to System (1), we do not know even in what space we can expect our solution and consequently, how we can proceed to analyze the question of global existence. Furthermore, in the treatment of this system, we need to be careful on the strongly nonlinear third order differential operator and the dispersive structure of the momentum equation. To this purpose, we try firstly to rewrite the term \( \text{div} \Sigma \) in a simplified form. Indeed, taking in account the following equality

\[
\text{div}(\tau_1(\rho, T)\nabla^2 T) = \nabla (\text{div}(\tau_1(\rho, T)\nabla T)) - \text{div}(\tau_1(\rho, T)\nabla T \otimes \nabla T),
\]
we observe that \( \text{div}(\Sigma) \) can be written as

\[
\text{div}(\Sigma) = \nabla(\text{div}(\tilde{\tau}_1(T)\nabla T)) - \nabla(\tilde{\tau}_1(T)\frac{1}{d} \Delta T) - \text{div}(\tilde{\tau}_2(T))\nabla T \otimes \nabla T) + \nabla(\tilde{\tau}_2(T)\frac{1}{d} |\nabla T|^2),
\]

with

\[
\tilde{\tau}_1(T) = \tau_1(\frac{1}{T}, T), \quad \tilde{\tau}_2(T) = \tau_2(\frac{1}{T}, T).
\]

The key observation here is that the gradient terms in Equation (2) can be incorporated into the pressure term to produce a new pressure term \( \nabla \pi \) where

\[
\pi = p - \text{div}(\tilde{\tau}_1 \nabla T) + \frac{1}{d} \tilde{\tau}_1 \Delta T + \frac{1}{d} \tilde{\tau}_2 |\nabla T|^2.
\]

By introducing this new pressure, and replace \( T \) by its value \( (T = 1/\rho) \), System (1) recasts in the following one

\[
\begin{aligned}
\partial_t \rho + \text{div}(\rho u) &= 0, \\
\partial_t (\rho u) + \text{div}(\rho u \otimes u) + \nabla \pi - \text{div} \Sigma &= - \text{div}(K(\rho)\nabla \rho), \\
\text{div} u &= \frac{2}{5} \text{div}(k(\rho^{-1})\nabla(\rho^{-1})),
\end{aligned}
\]

where we denoted by

\[
K(\rho) := \frac{1}{\rho^2}(\tilde{\tau}_1(\frac{1}{\rho}) - \tilde{\tau}_2(\frac{1}{\rho})).
\]

Inspired by the framework developed recently for Euler-Korteweg system by D. Bresch, F. Couderc, P. Noble and J.-P Vila in [2] based on a new functional equality called by generalized Bohm identity, we can rewrite the term in the right hand side of the momentum equation (3) as follows

\[
- \text{div}(K(\rho)\nabla \rho) = \rho \nabla \left( \sqrt{K(\rho)} \Delta \left( \int_0^\rho \sqrt{K(s)} ds \right) \right) - \nabla \left( \rho \text{div}(K(\rho)\nabla \rho) + \frac{1}{2}(K(\rho) - \rho K'(\rho)) |\nabla \rho|^2 \right)
\]

where the first term in the right hand side can be read as following

\[
\rho \nabla \left( \sqrt{K(\rho)} \Delta \left( \int_0^\rho \sqrt{K(s)} ds \right) \right) = \text{div}(F(\rho) \nabla \nabla \psi(\rho)) - \nabla \left( (F(\rho) - \rho F'(\rho)) \Delta \psi(\rho) \right)
\]

with

\[
\sqrt{\rho \psi'(\rho)} = \sqrt{K(\rho)}, \quad F'(\rho) = \sqrt{K(\rho)} \rho.
\]

This functional inequality is one among the key points of this work. Apparently, the study of system (3) is much more simplified than system (1) but the last system remains also difficult. Before that, we focused here in a particular form of heat conductivity coefficient and capillarity tensor. Precisely, we propose to study the following system

\[
\begin{aligned}
\partial_t \rho + \text{div}(\rho u) &= 0, \\
\partial_t (\rho u) + \text{div}(\rho u \otimes u) + \nabla \pi - 2 \text{div}(\mu(\rho) D(u)) &= \epsilon \rho \nabla \left( \sqrt{\rho^{2m-1}} \Delta \left( \int_0^\rho \sqrt{s^{2m-1}} ds \right) \right), \\
\text{div} u &= -2\kappa \Delta \varphi(\rho),
\end{aligned}
\]
where $c$ is a constant which can take both negative or nonnegative value and $\varphi$ is a function depending on $\rho$ to be determined later. The constant $c$ is strongly related to the choice of the transport coefficients given in $\Sigma$ (see for instance Remark (1)).

Now we move to make a little discussion regarding some particular cases of System (8). Firstly, when $c = 0$ in (8), System (8) recasts to low Mach number system namely Kazhikhov-Smagulov type system. However, there is a large literature concerning the existence of solution of this system. At the level of local strong solution, the reader is referred to [7] where the existence of global solutions in homogeneous Besov spaces with critical regularity is proved assuming the initial density to be close to constant and the initial velocity small enough. In the meantime, P.-L. Lions in [17] showed, in two-dimensional case, that for a positive conductivity coefficient and $\varphi = \frac{1}{\rho}$ a small perturbation of a constant density provides a global existence of weak solutions without restriction on the initial velocity. Nevertheless, the first global existence result of such system without smallness assumption was obtained by D. Bresch, E.H. Essoufi and M. Sy in [4] when a certain algebraic relation between $\mu$ and $\kappa \varphi$ is assumed, namely

$$\varphi' = \mu'(s)/s \quad \text{and} \quad \kappa = 1.$$  

This result was also extended recently in [5] to be held with $0 < \kappa < 1$. Later, X. Liao in [15] employed this algebraic condition to show uniqueness of solution in dimension 2 in the critical non-homogeneous Besov spaces.

One of the main tool employed in [5] is the concept of $\kappa$–entropy estimate based on the generalization of the BD entropy. Through this new entropy, the authors introduce a two-velocity hydrodynamical formulation for a low Mach number system. The interested reader is referred to [3] for a generalization of this concept to compressible Navier-Stokes with degenerate viscosities.

On the other hand, when $c > 0$ and $m = 0$ some progress has been made for compressible situation, i.e., without the equation on $\text{div} \ u$ and when $\pi = \rho \gamma$. Firstly, A. Jüngel in [11] proved a global existence result with finite energy initial data for $\mu(\rho) = \mu \rho$, $\sqrt{c} > \mu$ and $\gamma > 3$. He was introduced a special definition of global weak solutions. This result was also extended to be held when $\mu \leq \sqrt{c}$ (see for instance [8, 10]). In [9], M. Gisclon, I. Violet proved existence of global weak solutions without the assumption $\gamma > 3$ if $d = 3$ and with uniform estimates allowing to perform the limit when $c$ tends to zero. For more detail see the recent work by P. Antonelli and S. Spirito for an existence proof of standard global weak solutions [1].

The main goal we are going to prove in this paper is the existence of global weak solutions of System (8) when $\varphi = \log \rho$ and $0 \leq m \leq 1/2$ in (8). Remarkably, when $m = 0$, we obtain $-2\rho \Delta (\frac{1}{\sqrt{\rho}} \Delta \sqrt{\rho})$ as a capillarity tensor studied in [11]. Of course not for the compressible model but we stress here that this result can be also achieved for the compressible model. A first important feature in this paper is that, we can establish the existence of global weak solutions of System (8) when $c$ is negative but close to $0^-$ without assuming any smallness assumption on the initial data. This case is somewhat surprising and not expected in general. Here, it should be emphasize that the procedure developed when $c < 0$ cannot be used for compressible model, especially when the density is close to vacuum. As a matter of fact, the equation on $\text{div} \ u$ plays an important role in absorbing the terms with bad signs. Moreover, since the density is assumed here to be far from the vacuum, we can establish a maximum principle after introducing a new velocity $w = u + 2\kappa \nabla \log \rho$.
RUNNING HEADING WITH FORTY CHARACTERS OR LESS

This fact is also crucial in this procedure. The second important feature here is the proof of a new functional inequality inspired of what proposed proposed by A. Jünge and D. Matthes for Derrida-Lebowitz-Speer-Spohn equation [12]. This new equality will be the key tool of global existence analysis.

In what follows, we suppose that

$$\varphi(\rho) = \log \rho$$

in System (8) and consequently to take advantage of the interesting work in [5], we suppose that the relation (9) holds, whence $$\mu(\rho) = \rho$$.

The rest of this paper is organized as follows. The next section presents the reformulation of System (8) by means of a new velocity $$w$$ and provides some tools to be used in the subsequent sections. Section 3 states the main result. Finally, in the last section we prove Lemma 4.1 which is the key tool of the proof of global existence of solutions, namely Theorem 3.1.

2. Reformulation of the system and definitions of weak solution. In this section we want to reformulate System (8) by means of the so-called effect velocity

$$w = u + 2 \kappa \nabla \log \rho, \quad 0 < \kappa < 1.$$ 

Before that, we present in the next subsection some useful equalities to be used in this reformulation and also in the proof of existence result.

2.1. Useful equalities. We shall now rewrite the capillarity term in (8) using the generalized Bohm identity (6). For reader’s convenience, we give here the proof of this identity in the particular case, namely when $$K(\rho) = \sqrt{\rho^{2m-1}}$$. More precisely, we have the following Lemma

**Lemma 2.1.** For any smooth function $$\rho(x)$$, we have

$$\rho \nabla \left( \sqrt{\rho^{2m-1}} \Delta \left( \int_0^\rho \sqrt{s^{2m-1}} ds \right) \right) = \frac{1}{m(m+1)} \left[ \text{div} (\rho^{m+1} \nabla \rho^m) + m \nabla (\rho^{m+1} \Delta \rho^m) \right].$$

**Proof.** By straightforward computation, we write

$$\rho \partial_j \left( \rho^{m-1/2} \partial^2 \left( \int_0^\rho s^{m-1/2} ds \right) \right)$$

$$= \frac{1}{m} \rho \partial_j (\rho^{m-1/2} \partial_i (\rho^{1/2} \partial_i \rho^m))$$

$$= \frac{1}{m} \rho \partial_j (\rho^m \partial^2_i \rho^m + \frac{1}{2} \rho^{m-1} \partial_i \rho \partial_i \rho^m)$$

$$= \frac{1}{m} \partial_j (\rho^{m+1} \partial_i^2 \rho^m) - \frac{1}{m} \rho^m \partial_j \rho \partial^2_i \rho^m + \frac{1}{2 m^2} \rho \partial_j ((\partial_i \rho^m)^2)$$

$$= \frac{1}{m} \partial_j (\rho^{m+1} \partial_i^2 \rho^m) - \frac{1}{m(m+1)} \partial_j \rho^{m+1} \partial_i^2 \rho^m + \frac{1}{m(m+1)} \partial_j \rho^{m+1} \partial_i \rho^m$$

$$= \frac{1}{m(m+1)} \left[ (m+1) \partial_j (\rho^{m+1} \partial_i^2 \rho^m) - \rho^m \partial_j \rho^{m+1} \partial_i^2 \rho^m + \partial_i (\rho^{m+1} \partial_j \rho^m) \right.$$ 

$$- \rho^{m+1} \partial_i \partial_j \rho^m \right]$$

$$= \frac{1}{m(m+1)} \left[ \partial_i (\rho^{m+1} \partial_j \rho^m) + m \partial_j (\rho^{m+1} \partial_i \rho^m) \right].$$

This finishes the proof. \qed
For a reason that will be signaled later, we want to rewrite Equation (11) in another form.

**Lemma 2.2.** Suppose the function $\rho(x)$ is sufficiently smooth, then we can write

$$
\rho \nabla \left( \sqrt{\rho^{2m-1}} \Delta \left( \int_0^\rho \sqrt{s^{2m-1}} ds \right) \right) = \frac{1}{(2m + 1)} \text{div}(\rho^{2m+1} \nabla \nabla \log \rho)
+ \frac{m}{(m + 1)(2m + 1)^2} \nabla \Delta \rho^{2m+1} + \frac{1}{(m + 1)} \nabla (\rho^{m+1} \Delta \rho^m).
$$

(12)

**Proof.** Firstly, we can verify that we have (taking in mind that $\nabla \log \rho^\alpha = \alpha \nabla \log \rho$)

$$
\text{div}(\rho^{m+1} \nabla \rho^m) = m \text{div}(\rho^{m+1} \nabla (\rho^m \nabla \log \rho))
= m \left[ \text{div}(\rho^{2m+1} \nabla \log \rho) + m \text{div}(\rho^{2m+1} \nabla \log \rho \otimes \nabla \log \rho) \right].
$$

On the other hand we know that the following identity holds

$$
\text{div}(z \nabla \nabla \log z) = \nabla \Delta z - \text{div}(z \nabla \log z \otimes \nabla \log z),
$$

for all $z \in H^2(\Omega)$, which yields after taking $z = \rho^\alpha$ to the following equation

$$
\text{div}(\rho^\alpha \nabla \log \rho \otimes \nabla \log \rho) = \frac{1}{\alpha^2} \nabla \Delta \rho^\alpha - \frac{1}{\alpha} \text{div}(\rho^\alpha \nabla \nabla \log \rho).
$$

Finally, we conclude with

$$
\text{div}(\rho^{m+1} \nabla \rho^m) = m \left[ \text{div}(\rho^{2m+1} \nabla \log \rho) + m \left[ \frac{1}{(2m + 1)^2} \nabla \Delta \rho^{2m+1}
- \frac{1}{(2m + 1)} \text{div}(\rho^{2m+1} \nabla \log \rho) \right] \right]
= \frac{m}{(2m + 1)} \left[ (m + 1) \text{div}(\rho^{2m+1} \nabla \log \rho) + \frac{m}{(2m + 1)} \nabla \Delta \rho^{2m+1} \right].
$$

(13)

**Remark 1.** It seems perhaps not very clear how the system (1) was simplified into System (8), even with a particular case of heat conductivity equation. In order to keep ideas clear and avoid unpleasant technicalities, we shall take here a particular choices of $\tau_1$ and $\tau_2$ leading to System (8). Firstly, consider a particular form of heat-conductivity coefficient, such that

$$
\kappa(\rho^{-1}) = 5 \kappa \rho \quad \kappa > 0,
$$

we infer that the third equation in System (1) becomes equal to log $\rho$. Secondly, if we take

$$
\tau_1(\rho) = \beta \rho^2 \quad \tau_2(\rho) = -\beta \rho^3,
$$

then we can compute the first and the third term in $\text{div}(\nabla)$ (since the other terms can be incorporated into the pressure) to get (see for instance [5], Section 7)

$$
- \beta \text{div}(\rho \nabla \log \rho)
$$

(14)

which corresponds to the case when $m = 0$ in System (3). Noticing that the following equality

$$
\text{div}(\rho \nabla \log \rho) = 2\rho \nabla \left( \frac{1}{\sqrt{\rho}} \Delta \sqrt{\rho} \right),
$$

is a known equality called by Bohm potential arises from the fluid dynamical formulation of the single-state Schrödinger equation.
2.2. Reformulation of the system. To take advantage of the almost divergence-free of the expression \( u + 2\kappa \nabla \log \rho \), we introduce a new solenoidal velocity field \( w \) as follows

\[
w = u + 2\kappa \nabla \log \rho.
\]

We next try to rewrite the terms concerning the original velocity \( u \) in System (8) in light of the newly introduced velocity \( w \). Firstly it is easy to see that the continuity equation becomes

\[
\partial_t \rho + \text{div}(\rho w) - 2\kappa \Delta \rho = 0. \tag{15}
\]

Therefore, we can take advantage here on the parabolic type of Equation (15) and apply the maximum principle to deduce strict positivity of the density \( \rho \) if \( \rho \big|_{t=0} \) is strictly positive and the velocity \( w \) is smooth.

Besides, before rewriting the momentum equation in terms of \( w \), we differentiate the continuity equation with respect to space to obtain

\[
\partial_t (\rho \nabla \log \rho) + \text{div}(\rho \nabla \log \rho \otimes \rho u) + \text{div}(\nabla \log \rho \otimes \rho u) = 0. \tag{16}
\]

Now, we can multiply the above equation by \( 2\kappa \) and add it to the equation of conservation of the momentum (8) to get

\[
\partial_t (\rho w) + \text{div}(\rho u) + \nabla \pi = c \rho \nabla \left( \sqrt{\rho^{2m-1}} \Delta \left( \int_0^\rho \sqrt{s^{2m-1}} \, ds \right) \right).
\]

By virtue of (12), we can define a new pressure \( \pi_1 \) such that

\[
\pi_1 = \pi - \frac{cm}{(m+1)(2m+1)^2} \nabla \Delta \rho^{2m+1} \rho - \frac{c}{(m+1)} \nabla (\rho^{m+1} \Delta \rho^m),
\]

and therefore System (8) recasts in the following system for the new unknown triplet \((\rho, w, \pi)\)

\[
\begin{align*}
\partial_t \rho + \text{div}(\rho u) & = 0, \\
\partial_t (\rho w) + \text{div}(\rho u) + \nabla \pi_1 - 2(1 - \kappa) \text{div}(\rho D(u)) & = c \rho \nabla^{2m+1} \nabla \log \rho, \\
\text{div} w & = 0,
\end{align*} \tag{17}
\]

where we have denoted by

\[
A(u) = \frac{1}{2} (\nabla u - \nabla^t u).
\]

Our purpose in the forthcoming subsection is to introduce the notion of weak solutions associated to this system and then make a link between this definition and the definition of weak solution \((\rho, u)\) of System (8) (take attention of (10)).

2.3. Weak solutions. In this subsection, we introduce several notions of weak solutions to the ghost effect system. We complete System (17)-(8) by the periodic boundary conditions

\[
\Omega = T^3,
\]

and initial conditions

\[
\rho \big|_{t=0} = \rho^0, \quad u \big|_{t=0} = u^0, \quad w \big|_{t=0} = w^0 = u^0 + 2\kappa \nabla \log \rho^0 \text{ in } \Omega. \tag{18}
\]

We introduce the following spaces

\[
H = \{ z \in L^2(\Omega); \, \text{div} z = 0 \}, \quad \text{and} \quad V = \{ z \in W^{1,2}(\Omega); \, \text{div} z = 0 \}.
\]
We assume that the initial conditions satisfy
\[ \rho^0 \in H^1(\Omega), \quad 0 < r \leq \rho^0 \leq R < \infty, \quad w^0 \in H. \] (19)

Now, we can able to introduce our notion of weak solutions.

**Definition of weak solutions in term of \((\rho, w)\).**

**Definition 2.3.** We say that the couple \((\rho, w)\) is a global weak solution to System (17) and (18) if the following regularity properties are satisfied
\[ 0 < r \leq \rho \leq R < \infty, \quad \text{a.e in } (0, T) \times \Omega, \]
\[ \rho \in L_\infty(0, T; H^1(\Omega)) \cap L^2(0, T; H^2(\Omega)), \]
\[ w \in L_\infty(0, T; H) \cap L^2(0, T; V), \]

the k-entropy estimate holds
\[
\sup_{\tau \in [0, T]} \int_{\Omega} \rho \left( \frac{|w|^2}{2} + [(1 - \kappa)\kappa + c \frac{m^2}{4} \rho^{2m}] \frac{|2\nabla \log \rho|^2}{2} \right) (\tau) \, dx \\
+ 2\kappa \int_0^T \int_{\Omega} \rho |A(w)|^2 \, dx \, dt \\
+ 2(1 - \kappa) \int_0^T \int_{\Omega} \rho D(w - 2\kappa \nabla \log \rho) + \frac{2\kappa}{d} \Delta \log \rho \| |^2 \, dx \, dt \\
+ 2(1 - \kappa) \int_0^T \int_{\Omega} \rho \left[ 2\kappa \Delta \log \rho \right]^2 \, dx \, dt + \frac{8c\kappa c_0}{(2m + 1)^2} \int_0^T \int_{\Omega} \left( \Delta \rho^{2m+1} \right)^2 \, dx \\
\leq \int_{\Omega} \rho \left( \frac{|w|^2}{2} + [(1 - \kappa)\kappa + c \frac{m^2}{4} \rho^{2m}] \frac{|2\nabla \log \rho|^2}{2} \right) (0) \, dx,
\]

where \( A(w) = \frac{1}{2}(\nabla w - \nabla^T w), c_0 \) a positive constant (see Proposition (1)) and the equations of system (17) holds in the sense of distributions. Precisely, the continuity equation
\[
\int_0^T \int_{\Omega} \rho \partial_t \phi \, dx \, dt + \int_0^T \int_{\Omega} \rho w \cdot \nabla \phi \, dx \, dt \\
- 2\kappa \int_0^T \int_{\Omega} \rho \nabla \log \rho \cdot \nabla \phi \, dx \, dt = - \int_{\Omega} \rho \phi(0) \, dx
\] (21)

holds for all \( \phi \in C^\infty([0, T] \times \Omega) \), s.t. \( \phi(T) = 0 \).

The momentum equation is satisfied in the following sense
\[
\int_0^T \int_{\Omega} \rho \partial_t \phi_1 \, dx \, dt + \int_0^T \int_{\Omega} \rho \phi_1 \, dx \, dt \\
- 2(1 - \kappa) \int_0^T \int_{\Omega} \rho D(w) : \nabla \phi_1 \, dx \, dt + 4\kappa(1 - \kappa) \int_0^T \int_{\Omega} \rho \nabla \nabla \rho \cdot \nabla \phi_1 \, dx \, dt \\
- 2\kappa \int_0^T \int_{\Omega} A(w) : A(\phi_1) \, dx \, dt - \frac{c}{(2m + 1)} \int_0^T \int_{\Omega} \rho^{2m+1} \nabla \nabla \log \rho \cdot \nabla \phi_1 \, dx \, dt \\
= - \int_{\Omega} \rho \phi_1(0) \, dx
\] (22)

for \( \phi_1 \in C^\infty([0, T] \times \Omega)^{\mathbb{R}}, \) s.t. \( \text{div} \phi_1 = 0 \) and \( \phi_1(T) = 0 \).
The equation for $\nabla \log \rho$ (see (16))
\[
\int_0^T \! \int_\Omega \! \rho \nabla \log \rho \cdot \partial_t \phi_2 \, dx \, dt + \int_0^T \! \int_\Omega \! \rho (w - 2\kappa \nabla \log \rho) \otimes \nabla \log \rho : \nabla \phi_2 \, dx \, dt \\
- 2\kappa \int_0^T \! \int_\Omega \! \rho \nabla^2 \log \rho : \nabla \phi_2 \, dx \, dt + \int_0^T \! \int_\Omega \! \rho \nabla^4 w : \nabla \phi_2 \, dx \, dt \\
= - \int_\Omega \! \rho^0 \nabla \log \rho^0 \cdot \phi_2(0) \, dx
\]  
(23)
holds for all $\phi_2 \in (C^\infty([0, T] \times \Omega))^3$, s.t. $\phi_2(T) = 0$.

Likewise, the notion of a weak solution to the original system (8) in terms of $(\rho, u)$ can be stated as follows.

**Definition of weak solution in term of $(\rho, u)$.

**Definition 2.4.** We called that the couple $(\rho, u)$ is a weak solution to System (8)-(18) if the following regularity holds

\[0 < r \leq \rho \leq R < \infty, \quad \text{a.e. in } (0, T) \times \Omega,
\]
\[\rho \in L^\infty(0, T; H^1(\Omega)) \cap L^2(0, T; H^2(\Omega)),
\]
\[u \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)),
\]
and the equations of System (8) holds in the sense of distributions.

More precisely, the mass equation is satisfied in the following sense
\[
\int_0^T \! \int_\Omega \! \rho \partial_t \phi \, dx \, dt + \int_0^T \! \int_\Omega \! \rho u \cdot \nabla \phi \, dx \, dt = - \int_\Omega \! \rho^0 \phi(0) \, dx
\]
(24)
for $\phi \in C^\infty((0, T] \times \Omega)$ with $\phi(T) = 0$. The momentum equation is satisfied in the following sense
\[
\int_0^T \! \int_\Omega \! \rho u \cdot \partial_t \phi \, dx \, dt + \int_0^T \! \int_\Omega \! \rho u \otimes u : \nabla \phi \, dx \, dt \\
- 2 \int_0^T \! \int_\Omega \! \rho D(u) : \nabla \phi \, dx \, dt - \frac{c}{m(m+1)} \int_0^T \! \int_\Omega \! \rho^{m+1} \Delta \rho^m : \nabla \phi \, dx \\
= - \int_\Omega \! \rho^0 u^0 \cdot \phi(0) \, dx
\]
(25)
for $\phi \in (C^\infty([0, T] \times \Omega))^3$ such that $\text{div} \phi = 0$ and $\phi(T) = 0$.

The constraint
\[\text{div} u = -2\kappa \Delta \log \rho
\]
is satisfied in $L^2(0, T; L^2(\Omega))$.

**Remark 2.** Before showing the link between the two definitions, it sufficient to define $u = w - 2\kappa \nabla \log \rho$ and hence the weak solution in the sense of Definition 2.3 gives a weak solution in the sense of Definition 2.4. Indeed, using the definition of $u$ and the fact that $w$ is a divergence free vector field, the weak formulation of the momentum equation from Definition 2.4 is obtained by choosing $\phi_1 = \phi_2 = \phi$ in Equations (22) and (23), multiplying the second equation by $2\kappa$ and subtracting it from the first one.
Remark 3. Observe that the pressure function $\pi_1$ is not included in the weak formulation (22) for the same reason as in the Navier-Stokes equations theory. But if the couple $(\rho, w)$ satisfies (21)-(22), one can deduce using De Rham theorem the existence of a distribution $\pi_1 \in D'(\Omega)$ such that the triplet $(\rho, w, \pi)$ verifies equation (17) in $D'(\Omega)$ (see for instance [19, 16]).

3. Main theorem. The main result of this paper concerns the global in time existence of weak solutions to System (17).

Theorem 3.1. 1- Assume $c \geq 0$. Let $0 < \kappa < 1$ and $0 \leq m \leq 1/2$. Moreover suppose that the initial data $(\rho^0, w^0)$ satisfies

$$\rho^0 \in H^1(\Omega), \quad 0 < r \leq \rho^0 \leq R < \infty, \quad w^0 \in H,$$

(26)

where

$$H = \{z \in L^2(\Omega); \text{div} z = 0\} \quad \text{and} \quad V = \{z \in W^{1,2}(\Omega); \text{div} z = 0\},$$

then there exists at least one global weak solution $(\rho, w)$ of System (17) with the following properties

$$\|\rho\|_{L^2(0,T;H^1(\Omega))} + ||w||_{L^\infty(0,T;H)} + ||w||_{L^2(0,T,V)} \leq c,$$

$$\|\rho\|_{L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega))} + ||D(u)||_{L^2(0,T;L^2(\Omega))} \leq c.$$

2- Assume $c < 0$ and let $0 < \kappa < 1$. Moreover suppose that the initial data $(\rho^0, w^0)$ satisfies (26) and the following condition holds

$$r < R \leq \left(\frac{8\kappa \sqrt{(1-\kappa)(2m+1)}}{|c|} r\right)^{1/(2m+1)},$$

(27)

then there exists at least one global weak solution $(\rho, w)$ of System (17).

Remark on the constraint (27). We shall make a discussion about the constraint (27).

- The important feature of this condition is that we do not need to assume that the initial density to be close to the equilibrium state. Roughly speaking, given an initial data $\rho^0$ such that $r \leq \rho^0 \leq R$, we can easily found a constant $c$ that satisfies (27).

- The condition (27) ensures that, more than $c$ is small, more than we have large data existence. Take for example $\kappa = \frac{1}{2}$, $m = 0$, $r = 5$, condition (27) recasts to

$$r = 5 < R < \frac{40}{|c|}.$$

Clearly, more than $c$ is small, more than the interval $[r, R]$ is large.

- The condition (27) provides the constant $|c|$ to have an upper bound and this bound depends on the choice of $\kappa, m, r$ and $R$. For example, take $m = 0$, condition (27) remains

$$r < R \leq \frac{8\kappa (1-\kappa)}{|c|} r,$$

and hence $c$ should satisfy

$$|c| < 8\kappa (1-\kappa).$$

Here, we observe also that more than $c$ is small more than we have large data existence.
4. Proof of Theorem 3.1. To establish the existence of global weak solutions, we want to establish a uniform estimates for approximate solutions. The procedure of construction of solution follows the lines given in [5].

4.1. A priori estimates. The main task in this paragraph is the proof of the following inequality which is needed in Proposition (1) to establish the a priori estimates associated to System (17). This inequality can be useful also for other systems related to fluid dynamics.

**Lemma 4.1.** For $\rho \in H^2(\Omega)$ and $0 \leq m \leq 1/2$, there exists a constant $c_0 > 0$ such that

$$I = \int_{\Omega} \rho^{m+1} \nabla \nabla \rho^m : \nabla \nabla \log \rho \, dx + m \int_{\Omega} \rho^{m+1} \Delta \rho^m \Delta \log \rho \, dx$$

$$\geq 4c_0 \frac{m(m+1)}{(2m+1)^2} \int_{\Omega} (\Delta \rho^{2m+1})^2 \, dx.$$  \hspace{1cm} (28)

The constant $c_0$ should satisfy an upper bound, namely

$$0 < c_0 \leq 1 - \frac{(d-1)^2}{d(d+2)} \frac{1}{1+2m}.$$  \hspace{1cm} (29)

**Proof.** Let us denote by $v = \rho^{(m+1)/2}$, then the integral $I$ can be read as $I = J_1 + J_2$ with

$$J_1 = \frac{2}{m+1} \int_{\Omega} v^2 \nabla \nabla v^{\frac{2m}{m+1}} : \nabla \nabla \log v \, dx,$$

$$J_2 = \frac{2m}{m+1} \int_{\Omega} v^2 \Delta v^{\frac{2m}{m+1}} \Delta \log v \, dx.$$  \hspace{1cm} (30)

Now, we may follow the strategy introduced by A. Jüngel and D. Matthes in [12] in order to get a lower bound for the sum of $J_1$ and $J_2$. Before that, we choose $\gamma$ such that $2m/(m+1) = 2(\gamma - 1)$, that’s imply $\gamma = (2m+1)/(m+1)$. With this new variable, the above two integrals can be written as

$$J_1 = 4(2-\gamma)(\gamma - 1) \frac{1}{2(\gamma - 1)} \int_{\Omega} v^2 \nabla \nabla v^{2(\gamma - 1)} \nabla \nabla \log v \, dx,$$

$$J_2 = 4(\gamma - 1)^2 \frac{1}{2(\gamma - 1)} \int_{\Omega} v^2 \Delta v^{2(\gamma - 1)} \Delta \log v \, dx.$$  \hspace{1cm} (31)

Now, we sum $J_1$ and $J_2$ to get

$$J_1 + J_2 = 4(\gamma - 1) \left( (2-\gamma)A + (\gamma - 1)B \right)$$

with

$$A := \frac{1}{2(\gamma - 1)} \int_{\Omega} v^2 \nabla \nabla v^{2(\gamma - 1)} \nabla \nabla \log v \, dx, \quad B := \frac{1}{2(\gamma - 1)} \int_{\Omega} v^2 \Delta v^{2(\gamma - 1)} \Delta \log v \, dx.$$
We compute $A$ in the following manner

$$A = \frac{1}{2(\gamma - 1)} \int_\Omega v^2 \partial^2_{ij} v^{2(\gamma - 1)\partial^2_{ij} \log v} dx$$

$$= \frac{1}{2(\gamma - 1)} \int_\Omega (v \partial^2_{ij} v - \partial_i v \partial_j v) \partial^2_{ij} (v^{2(\gamma - 1)}) dx$$

$$= \int_\Omega (v \partial^2_{ij} v - \partial_i v \partial_j v) v^{2(\gamma - 2)} (v \partial^2_{ij} v + (2\gamma - 3)\partial_i v \partial_j v) dx$$

$$= \int_\Omega v^{2\gamma} \left( \frac{\|\nabla^2 v\|^2}{v^2} - 2(2 - \gamma) \frac{\nabla^2 v}{v} \otimes \frac{\nabla v}{v} + (3 - 2\gamma) \frac{\|\nabla v\|^4}{v^4} \right) dx.$$ 

In the same manner, we can compute $B$ to obtain

$$B = \int_\Omega v^{2\gamma} \left( \left( \frac{\Delta v}{v} \right)^2 - 2(2 - \gamma) \frac{\Delta v}{v} \left( \frac{\nabla v}{v} \right)^2 + (3 - 2\gamma) \frac{\|\nabla v\|^4}{v^4} \right) dx.$$ 

In order to simplify the computations, we introduce the functions $\theta, \lambda$ and $\xi$, respectively, by (recall that $\rho > 0$, hence $v > 0$)

$$\theta = \frac{|\nabla v|}{v}, \quad \lambda = \frac{1}{d} \frac{\Delta v}{v}, \quad (\lambda + \xi)\theta^2 = \frac{1}{v^3} \nabla^2 v : (\nabla v)^2,$$

and $\eta \geq 0$ by

$$\|\nabla^2 v\|^2 = (d\lambda^2 + \frac{d}{d-1} \mu^2 + \eta^2) v^2.$$ 

We express now $A$ and $B$ in terms of the functions $\theta, \lambda, \xi$ and $\eta$ to get

$$A = \int_\Omega v^{2\gamma} \left( d\lambda^2 + \frac{d}{d-1} \mu^2 + \eta^2 - 2(2 - \gamma)(\lambda + \xi)\theta^2 + (3 - 2\gamma)\theta^4 \right) dx,$$

$$B = \int_\Omega v^{2\gamma} \left( d^2 \lambda^2 - 2(2 - \gamma)d\lambda \theta^2 + (3 - 2\gamma)\theta^4 \right) dx.$$ 

Now we need to compare $J = (2 - \gamma)A + (\gamma - 1)B$ to

$$K = \frac{1}{\gamma - 1} \int_\Omega (\Delta v)^2 dx = \int_\Omega v^{2(\gamma - 2)} (v \Delta v + (\gamma - 1)|\nabla v|^2)^2 dx$$

$$= \int_\Omega v^{2\gamma}(d\lambda + (\gamma - 1)\theta^2)^2 dx.$$ 

More precisely, we shall determine a constant $c_0 > 0$ independent of $v$ such that $J - c_0 K \geq 0$ for all positive functions $v$. The strategy presented here is an adaptation of the method developed in [12]. We formally perform integration by parts in the expression $J - c_0 K$ by adding a linear combination of certain dummy integrals which are actually zero and hence do not change the value of the value of $J - c_0 K$. The coefficients in the linear combination are determined in a such a way that makes the resulting integrand pointwise non-negative.

We shall rely on the following two dummy integrals expressions:

$$F_1 = \int_\Omega \text{div}(v^{2\gamma - 2}(\nabla^2 v - \Delta v I) \cdot \nabla v) dx,$$

$$F_2 = \int_\Omega \text{div}(v^{2\gamma - 3}|\nabla v|^2 \nabla v) dx,$$

where $I$ is the unit matrix in $\mathbb{R}^d \times \mathbb{R}^d$. Obviously, in view of the boundary conditions, $F_1 = F_2 = 0$. Our purpose now is to find constants $c_0, c_1$ and $c_2$ such that $J - c_0 K =
$J - c_0 K + c_1 F_1 + c_2 F_2 \geq 0$. The computation in [12] yields to

$$F_1 = \int_{\Omega} v^{2\gamma} \left( -d(d-1)\lambda^2 + \frac{d}{d-1} \xi + \eta^2 + 2(\gamma - 1)(d-1)\lambda \theta^2 + \xi \theta^2 \right) \, dx,$$

$$F_2 = \int_{\Omega} v^{2\gamma} \left( (d+2)\lambda \theta^2 + 2\xi \theta^2 + (2\gamma - 3)\theta^4 \right) \, dx.$$  \hspace{0.5cm} (29)

After simple calculation, we obtain that

$$J - c_0 K = J - c_0 K + c_1 F_1 + c_2 F_2$$

$$= \int_{\Omega} v^{2\gamma} \{d\lambda[2 - \gamma + (\gamma - 1)d - c_0 d - c_1 (d - 1)]$$

$$+ \lambda \theta^2 [2(2 - \gamma)(2 - \gamma + (\gamma - 1)d) - 2d(\gamma - 1)c_0$$

$$- 2c_1(\gamma - 1)(d - 1) + c_2(d + 2)] + Q(\theta, \xi, \eta) \} \, dx,$$

where $Q$ is a polynomial in $\theta, \xi$ and $\eta$ with coefficients depending on $c_0, c_1$ and $c_2$ but not in $\lambda$. As in [12], we choose to eliminate $\lambda$ from the above integrand by defining $c_1$ and $c_2$ appropriately. One can check that the linear system

$$2 - \gamma + (\gamma - 1)d - c_0 d - c_1 (d - 1) = 0,$$

$$-2(2 - \gamma)(2 - \gamma + (\gamma - 1)d) - 2d(\gamma - 1)c_0 - 2c_1(\gamma - 1)(d - 1) + c_2(d + 2) = 0,$$

has the solution

$$c_1 = \frac{d(\gamma - 1 - c_0) + 2 - \gamma}{d - 1}, \hspace{0.5cm} c_2 = \frac{2 - \gamma + d(\gamma - 1)}{d + 2}.$$

With this choice, the polynomial $Q$ in (29) reads as

$$Q(\theta, \xi, \eta) = \frac{1}{(d-1)^2(d+2)} (b_1 \xi^2 + 2b_2 \xi \theta^2 + b_3 \theta^4 + b_4 \eta^2),$$

where

$$b_1 = d^2(d+2)(1-c_0),$$

$$b_2 = d(d-1)[d(4\gamma - 5) - (\gamma - 2) - (\gamma - 1)(d + 2)c_0],$$

$$b_3 = (d-1)^2[d(3 - 2\gamma)^2 + 2(\gamma - 1)(3 - 2\gamma) - c_0(\gamma - 1)^2(d + 2)],$$

$$b_4 = d(d+2)(d-1)(1-c_0).$$

Observe that if $c_0 < 1$, then $b_4 \geq 0$. We want to choose $c_0 < 1$ in such a way that the remaining sum $b_1 \xi^2 + 2b_2 \xi \theta^2 + b_3 \theta^4$ is nonnegative as well, for any $\xi$ and $\theta$. In fact, it remains to have the following two conditions

$$\begin{array}{ll}
(i) \ b_1 > 0 & \hspace{0.5cm} (ii) \ b_1b_3 - b_2^2 \geq 0.
\end{array}$$

(30)

The first condition holds for $c_0 \leq 1$. For the second condition is equivalent to

$$-d(d+2)(1-c_0)(\gamma - 1)(3\gamma - 4)$$

$$- (d-1)^2(3\gamma - 4)^2 - d(d+2)(1-c_0)(3\gamma - 4) \geq 0.$$  \hspace{0.5cm} (31)

We advise the readers to make the change of variable $X = (\gamma - 1)$ and $Y = (1 - c_0)$ in the above computation in order to simplify the calculation. Therefore, if $\gamma > 4/3$ there is no solution to the preceding inequality with $c_0 < 1$. However, for $\gamma \leq 4/3$, condition (ii) is further equivalent to

$$c_0 \leq 1 - \frac{(d-1)^2(4 - 3\gamma)}{d(d+2)} \frac{\gamma}{3}.$$  \hspace{0.5cm} (32)

The best choice for $c_0$ is clearly to make $\gamma$ small as possible. Remember that $\gamma = (2m + 1)/(m + 1)$, and then $\gamma \in [1, 2]$. Thus we have found constants $c_0, c_1$
and \( c_2 \) for which the expression

\[
J - c_0 K + c_1 F + c_2 F_2
\]

is nonnegative such that \( \gamma \in [1, 4/3] \).

Now, we shall assume that \( \rho \) and \( w \) are smooth enough and we will try to derive an original estimate on \((\rho, w, \nabla \log \rho)\) by the help of Lemma 4.1 which is necessary to understand the idea of construction of solution.

**Maximum principle and \( H^1 \) bound on the density.** First, applying the standard maximum principle for the continuity equation

\[
\partial_t \rho + w \cdot \nabla \rho - 2\kappa \Delta \rho = 0, \tag{31}
\]

we deduce that

\[
0 < r \leq \rho \leq R < \infty, \tag{32}
\]

and the basic energy estimate gives

\[
||\rho||_{L^\infty(0,T; L^2(\Omega))} + ||\rho||_{L^2(0,T; H^1(\Omega))} \leq c. \tag{33}
\]

**Proposition 1.** Let \((\rho, w)\) be sufficiently smooth solution to (17), then if \( 0 \leq m \leq 1/2 \) and \( c > 0 \) there exists a constant \( c_0 > 0 \) such that \((\rho, w)\) satisfies the following inequality

\[
\frac{d}{dt} \int_\Omega \rho \left( \frac{|w|^2}{2} + (1 - \kappa)\kappa \frac{|2\nabla \log \rho|^2}{2} \right) dx + c \frac{m^2}{2} \frac{d}{dt} \int_\Omega \rho^{2m+1} |\nabla \log \rho|^2 dx \\
+ 2(1 - \kappa) \int_\Omega \rho |D(u)|^2 dx + 2\kappa \int_\Omega \rho |A(u)|^2 dx \\
+ \frac{8c\kappa c_0}{(2m + 1)^2} \int_\Omega \left( \Delta \rho^{\frac{2m+1}{2}} \right)^2 dx \leq 0. \tag{34}
\]

The constant \( c_0 \) is a positive constant which satisfies the following inequality

\[
c_0 \leq 1 - \frac{(d - 1)^2 (1 - 2m)}{d(d + 2) 1 + 2m}.
\]

The constraint on \( m \) follows from condition (ii) in (30) (remember that \( \gamma = (2m + 1)/(m + 1) \)).

**Proof.** Multiplying the second equation in (17) by \( w \in L^\infty(0,T; H^1(\Omega)) \cap L^2(0,T; V) \), we get (remember \( \text{div} \ w = 0 \))

\[
\frac{1}{2} \frac{d}{dt} \int_\Omega \rho |w|^2 dx + 2(1 - \kappa) \int_\Omega \rho |D(u)|^2 dx \\
+ 2\kappa \int_\Omega \rho |A(u)|^2 dx - \frac{c}{(2m + 1)} \int_\Omega \text{div}(\rho^{2m+1} \nabla^2 \log \rho) \cdot w \, dx \tag{35}
\]

Besides, testing (12) in Lemma 2.2 by \( w \in L^\infty(0,T; H^1(\Omega)) \cap L^2(0,T; V) \), we get (remember \( \text{div} \ w = 0 \))

\[
I := -\frac{c}{(2m + 1)} \int_\Omega \text{div}(\rho^{2m+1} \nabla^2 \log \rho) \cdot w \, dx \\
= -c \int_\Omega \rho \nabla (\sqrt{\rho^{2m-1}} \Delta \left( \int_0^\rho \sqrt{s^{2m-1}} ds \right)) \cdot w \, dx \\
= I_1 + I_2. \tag{36}
\]
where $I_1$ and $I_2$ (remember $w = u + 2\kappa \nabla \log \rho$)

$$I_1 = -c \int_{\Omega} \rho \nabla (\sqrt{\rho^{2m-1}} \Delta (\int_{0}^{\rho} \sqrt{s^{2m-1}} ds)) \cdot u \, dx$$

$$I_2 = -2\kappa c \int_{\Omega} \rho \nabla (\sqrt{\rho^{2m-1}} \Delta (\int_{0}^{\rho} \sqrt{s^{2m-1}} ds)) \cdot \nabla \log \rho \, dx$$

Now after integrating by part the integral $I_1$ and using the continuity equation (17), we obtain

$$I_1 = cm^2 \frac{d}{dt} \int_{\Omega} \rho^{2m-1} |\nabla \rho|^2 \, dx = cm^2 \frac{d}{dt} \int_{\Omega} \rho^{2m+1} |\nabla \log \rho|^2 \, dx.$$

The integration by part of $I_2$ using Equation (11) yields to

$$I_2 = \frac{2\kappa c}{m(m+1)} \left[ \int_{\Omega} \rho^{m+1} \nabla \nabla \rho^m : \nabla \nabla \log \rho \, dx + m \int_{\Omega} \rho^{m+1} \Delta \rho^m \Delta \log \rho \, dx \right] \quad (37)$$

Therefore, one can deduce from Lemma 4.1 that

$$I \geq cm^2 \frac{d}{dt} \int_{\Omega} \rho^{2m+1} |\nabla \log \rho|^2 \, dx + \frac{8c\kappa c_0}{(2m+1)^2} \int_{\Omega} \left( \Delta \rho^{\frac{2m+1}{2}} \right)^2 \, dx. \quad (38)$$

To finish the proof, it remains to estimate the last term in (35) since it does not have a sign. To this purpose, we try to multiply the continuity equation associated to $u$ (17) by $-\Delta \sqrt{\rho}/\sqrt{\rho}$ and we integrate it with respect to $\Omega$ to get

$$\frac{d}{dt} \int_{\Omega} \rho \frac{|\nabla \log \rho|^2}{2} \, dx - \int_{\Omega} \rho \nabla u : \nabla \log \rho \, dx = 0 \quad (39)$$

Next, we multiply (39) by $4(1 - \kappa)\kappa$ and add it to (35), we obtain

$$\frac{d}{dt} \int_{\Omega} \rho \left( \frac{|w|^2}{2} + (1 - \kappa)\kappa \frac{2|\nabla \log \rho|^2}{2} \right) \, dx$$

$$+ 2(1 - \kappa) \int_{\Omega} \rho |D(u)|^2 \, dx + 2\kappa \int_{\Omega} \rho |A(u)|^2 \, dx - I = 0 \quad (40)$$

Gathering the above computation we can easily check that we have

$$\frac{d}{dt} \int_{\Omega} \rho \left( \frac{|w|^2}{2} + (1 - \kappa)\kappa \frac{2|\nabla \log \rho|^2}{2} \right) \, dx + \frac{8c\kappa c_0}{(2m+1)^2} \int_{\Omega} \left( \Delta \rho^{\frac{2m+1}{2}} \right)^2 \, dx \leq 0,$$

which end the proof of proposition (1). \qed

**Remark 4.** We emphasize that from Estimate (34), we can deduce that

$$\Delta \rho^{\frac{2m+1}{2}} \in L^2(0,T;L^2(\Omega)),$$

but thanks to Estimate (32), we can infer with

$$\Delta \rho \in L^2(0,T;L^2(\Omega)),$$

which yields by virtue of estimate $\rho \in L^\infty(0,T;L^2(\Omega))$ (see (33)) and the periodic boundary condition to

$$\rho \in L^2(0,T;H^2(\Omega)).$$
Obviously, when an initial vacuum may exists, this conclusion is not true but the framework developed here can be adapted to deal with the vacuum.

4.2. Construction of solution. The procedure of construction of solution follows the idea developed recently in [5] for low Mach number. Precisely, we construct the approximate solution using an augmented approximate system. We propose to study the following system

\[
\begin{align*}
\partial_t \rho + \text{div}(\rho \mathbf{w}) - 2\kappa \Delta \rho &= 0, \\
\partial_t (\rho \mathbf{w}) + \text{div}((\rho \mathbf{w} - 2\kappa \nabla \rho) \otimes \mathbf{w}) - 2(1 - \kappa) \text{div}(\rho D(\mathbf{w})) - 2\kappa \text{div}(\rho A(\mathbf{w})) + \nabla \pi_1 &= -2\kappa(1 - \kappa) \text{div}(\rho \nabla v) + \frac{c}{2(2m + 1)} \text{div}(\rho^{2m+1} \nabla v), \\
\partial_t (\rho v) + \text{div}((\rho \mathbf{w} - 2\kappa \nabla \rho) \otimes v) - 2\kappa \text{div}(\rho \nabla v) &= -2 \text{div}(\rho \nabla' \mathbf{w}),
\end{align*}
\]

\[\text{div} \mathbf{w} = 0.\]

Remark that if we prove that \(v = 2 \nabla \log \rho\), we can infer a weak solution \((\rho, \mathbf{w})\) of System (17) in the sense of Definition 2.3. Now, compared to [5], the last term depending on \(\rho\) and on \(v\) in the momentum equation (42) is new. Nonetheless, the treatment of this first term is similar to \(-2\kappa(1 - \kappa) \text{div}(\rho \nabla v)\) and this support our attempts in writing the capillarity term as (12) in Lemma 2.2. Besides, to build such a solution to the above augmented system, we need to go through several levels of approximations. Firstly, a smoothing parameter denoting standard mollification with respect to \(x\) is introduced in all the transport terms. Secondly, a smoothing high-order derivative term \(\varepsilon(\Delta^2 \mathbf{w} - \text{div}(1 + |\nabla \mathbf{w}|^2) \nabla \mathbf{w})\) has to be introduced to control large spatial variations of \(\mathbf{w}\) on the one hand, and to guarantee the global in time existence of solution at the level of Galerkin procedure on the other hand. We shall make a discussion of this point later. At this level let us present the approximate system with full regularization terms as

1. The continuity equation is replaced by its regularized version

\[
\begin{align*}
\partial_t \rho + \text{div}(\rho [\mathbf{w}]^{\delta}) - 2\kappa \Delta \rho &= 0, \\
\rho(0, x) &= [\rho]^\delta,
\end{align*}
\]

2. The momentum equation is replaced by its Faedo-Galerkin approximation with additional regularizing term \(\varepsilon(\Delta^2 \mathbf{w} - \text{div}(1 + |\nabla \mathbf{w}|^2) \nabla \mathbf{w})\)

\[
\begin{align*}
\int_0^T \int_\Omega \rho \mathbf{w}(\tau) \cdot \phi \, dx \, dt - \int_0^T \int_\Omega ((\rho [\mathbf{w}]^{\delta} - 2\kappa \nabla \rho) \otimes \mathbf{w}) : \nabla \phi \, dx \, dt \\
+ 2(1 - \kappa) \int_0^T \int_\Omega \rho D(\mathbf{w}) : \nabla \phi \, dx \, dt \\
+ \varepsilon \int_0^T \int_\Omega (\Delta \mathbf{w} \Delta \phi + (1 + |\nabla \mathbf{w}|^2) \nabla \mathbf{w} : \nabla \phi) \, dx \, dt \\
+ 2\kappa \int_0^T \int_\Omega \rho A(\mathbf{w}) : \nabla \phi \, dx \, dt - 2\kappa(1 - \kappa) \int_0^T \int_\Omega \rho \nabla v : \nabla \phi \, dx \, dt \\
+ \frac{c}{2(2m + 1)} \int_0^T \int_\Omega \rho^{2m+1} \nabla v : \nabla \phi \, dx \, dt = \int_\Omega (\rho [\mathbf{w}]^{\delta})^0 \cdot \phi \, dx,
\end{align*}
\]

satisfied for any \(\tau \in [0, T]\) and any test function \(\phi \in X_n = \text{span}\{\phi_i\}_{i=1}^n\) and \(\{\phi_i\}_{i=1}^\infty\) is an orthonormal basis of \(V\) such that \(\phi_i \in (C^\infty(\Omega))^3\) with \(\text{div} \phi_i = 0\) for all \(i \in \mathbb{N}\).
3- The Faedo-Galerkin approximation for the artificial equation

\[
\int_{\Omega} \rho(\tau) \cdot \xi \, dx - \int_{0}^{T} \int_{\Omega} (\rho[w]_{\delta} - 2\kappa \nabla \rho) \otimes v) : \nabla \xi \, dx \, dt \\
+ 2\kappa \int_{0}^{T} \int_{\Omega} \rho \nabla v : \nabla \xi \, dx \, dt - 2 \int_{0}^{T} \int_{\Omega} \rho \nabla' w : \nabla \xi \, dx \, dt = \int_{\Omega} (\rho v)^{0} \cdot \xi \, dx,
\]

satisfied for any \( \tau \in [0, T] \) and any test function \( \xi \in Y_{n} = \text{span}\{\xi_{i}\}_{i=1}^{n} \) and \( \{\xi_{i}\}_{i=1}^{\infty} \) is an orthonormal basis in \( W^{1,2}(\Omega) \) such that \( \xi \in (C^{\infty}(\Omega))^{3} \) for all \( i \in \mathbb{N} \).

4.3. Outline of construction of solution. The procedure of construction of solution can be decomposed into two paragraphs: the first one deal with the case \( c > 0 \). The second one is devoted to the case when \( c < 0 \). Since the construction of solution is similar for the two parts, we shall present in the first situation the approach of construction of solution and in the last one i.e., when \( c < 0 \), we will just give the necessary a priori estimates.

\( \blacklozenge \) Case when \( c > 0 \).

In this paragraph we assume that the constant \( c \) appearing in System (17) is a positive constant.

Local existence of solution \((\rho, w)\)

- Firstly, fixed \( w \in C([-\infty, T], X_{n}) \) where \( X_{n} = \text{span}\{\phi_{i}\}_{i=1}^{n} \) and \( \{\phi_{i}\}_{i=1}^{\infty} \) is an orthonormal basis of \( V \) such that \( \phi_{i} \in (C^{\infty}(\Omega))^{3} \) with \( \text{div} \phi_{i} = 0 \) for all \( i \in \mathbb{N} \), one can show the existence of \( \rho \) solution of the continuity equation using the standard theory on the parabolic equations. Precisely, for \( \nu \in (0, 1) \), and initial condition \( \rho_{0}^{\delta} \in C^{2+\nu}(\Omega) \) is such that \( 0 < \tau \leq \rho_{0}^{\delta} \leq R \) and satisfies the periodic boundary conditions. Then problem (44) possesses a unique classical solution \( \rho \) from the class

\[
V[0, T] = \left\{ \rho \in C([-\infty, T]; C^{2+\nu}(\Omega) \cap C^{1}([0, T] \times \Omega)), \quad \partial_{t} \rho \in C^{\nu/2}([0, T], C(\Omega)) \right\}
\]

and satisfying a classical maximum principle

\[
0 < \tau \leq \rho(t, x) \leq R.
\]

Moreover, the mapping \( w \to \rho(w) \) maps bounded sets in \( C([-\infty, T]; X_{n}) \) into bounded sets in \( V[0, T] \) and is continuous with values in \( C([-\infty, T]; C^{2+\nu}(\Omega)) \), \( 0 < \nu' < \nu < 1 \).

- Having obtained \( \rho \), we construct an approximate local solution \((w, v)\) using a fixed point argument at the level of the Galerkin approximate system. Precisely, we rewrite equations (45)-(46) as a fixed point problem:

\[
\begin{align*}
(w(t), v(t)) & = \left( \mathcal{M}_{\rho(t)} \left[ P_{X_{n}}(\rho w)^{0} + \int_{0}^{t} \mathcal{K}(w)(s) \, ds \right], \mathcal{N}_{\rho(t)} \left[ P_{Y_{n}}(\rho v)^{0} + \int_{0}^{t} \mathcal{L}(v)(s) \, ds \right] \right) \\
& = \mathcal{F}[w, v](t),
\end{align*}
\]

where \( \rho = \rho(w) \) is a solution to the continuity equation as explained above,

\[
\begin{align*}
\mathcal{M}_{\rho(t)} : X_{n} \times Y_{n} & \to X_{n}, \quad \int_{\Omega} \rho \mathcal{M}_{\rho(t)}[\phi] \cdot \psi = \langle \phi, \psi \rangle, \quad \phi, \psi \in X_{n}, \\
\mathcal{N}_{\rho(t)} : Y_{n} \times X_{n} & \to Y_{n}, \quad \int_{\Omega} \rho \mathcal{N}_{\rho(t)}[\xi] \cdot \zeta = \langle \xi, \zeta \rangle, \quad \xi, \zeta \in Y_{n},
\end{align*}
\]
where \( P_{X_n}, P_{Y_n} \) denote the projections of \( L^2(\Omega) \) on \( X_n, Y_n \), respectively, and \( K(w), L(v) \) are the operators defined as

\[
K : X_n \times Y_n \to X_n,
\]

\[
\langle K(w), \phi \rangle = \int_0^T \int_\Omega ((\rho|w|_\delta - 2\kappa \nabla \rho) \otimes w) : \nabla \phi \, dx \, dt
\]

\[
-2(1 - \kappa) \int_0^T \int_\Omega \rho \, D(w) : \nabla \phi \, dx \, dt
\]

\[
-\varepsilon \int_0^T \int_\Omega (\Delta w \Delta \phi + (1 + |\nabla w|^2) \nabla w : \nabla \phi) \, dx \, dt - 2\kappa \int_0^T \int_\Omega \rho \, A(w) : \nabla \phi \, dx \, dt
\]

\[
+ 2\kappa(1 - \kappa) \int_0^T \int_\Omega \rho \nabla v : \nabla \phi \, dx \, dt - \frac{c}{2(2m + 1)} \int_0^T \int_\Omega \rho^{2m+1} \nabla v : \nabla \phi \, dx \, dt
\]

\[
L : Y_n \times X_n \to Y_n,
\]

\[
\langle L(v), \xi \rangle = \int_0^T \int_\Omega ((\rho|w|_\delta - 2\kappa \nabla \rho) \otimes v) : \nabla \xi \, dx \, dt - 2\kappa \int_0^T \int_\Omega \rho \nabla v : \nabla \xi \, dx \, dt
\]

\[
+ 2 \int_0^T \int_\Omega \rho \nabla^2 w : \nabla \xi \, dx \, dt.
\]

Observe that since \( \rho(t, x) \) is strictly positive, we have

\[
\|M_{\rho(t)}\|_{L(X_n, X_n)}, \|N_{\rho(t)}\|_{L(Y_n, Y_n)} \leq \frac{1}{\tau}.
\]

Moreover

\[
\|M_{\rho^1(t)} - M_{\rho^2(t)}\|_{L(X_n, X_n)} + \|N_{\rho^1(t)} - N_{\rho^2(t)}\|_{L(Y_n, Y_n)} \leq c(n, r^1, r^2)\|\rho^1 - \rho^2\|_{L^1(\Omega)},
\]

and by the equivalence of norms on the finite dimensional space we prove that

\[
\|K(w)\|_{X_n} + \|L(v)\|_{Y_n} \leq c(n, r, R, \|\nabla \rho\|_{L^2(\Omega)}, \|w\|_{X_n}, \|v\|_{Y_n}).
\]

Next, we consider a ball \( B \) in the space \( C([0, \tau]; X_n) \times C([0, \tau]; Y_n) \):

\[
B_{M, \tau} = \{ (w, v) \in C([0, \tau]; X_n) \times C([0, \tau]; Y_n) : \|w\|_{C([0, \tau]; X_n)} + \|v\|_{C([0, \tau]; Y_n)} \leq M \}
\]

Using estimates (51), (52), (47) and (48), one can check that \( \mathcal{F} \) is a continuous mapping of the ball \( B_{M, \tau} \) into itself and for sufficiently small \( \tau = T(n) \) it is a contraction. Therefore, it possesses a unique fixed point which is a solution to (45) and (46) for \( T = T(n) \).

**Global existence of solution.**

In order to extend the local in-time solution obtained above to the global in time one, we need to find uniform (in time) estimates, so that the above procedure can be iterated. First, let us note that \( w, v \) obtained in the previous paragraph have better regularity with respect to time. It follows by taking the time derivative of (49) and using the estimates (47), (48), that

\[
(w, v) \in C^1([0, \tau]; X_n) \times C^1([0, \tau]; Y_n).
\]
This is an important feature since now we can take time derivatives of (45) and (46) and use the test functions \( \phi = w \) and \( \xi = v \), respectively. We then obtain

\[
\frac{d}{dt} \int \Omega \rho \frac{|w|^2}{2} \, dx + 2(1 - \kappa) \int \Omega \rho |D(w)|^2 \, dx \\
+ \varepsilon \int \Omega |\Delta w|^2 + (1 + |\nabla w|^2)|\nabla w|^2 \, dx + 2\kappa \int \Omega \rho |A(w)|^2 \\
- 2\kappa(1 - \kappa) \int \Omega \rho \nabla v : \nabla w \, dx + \frac{c}{2(2m + 1)} \int_0^T \int \Omega \rho^{2m+1} \nabla v : \nabla w \, dx \, dt = 0, 
\]

and

\[
\frac{d}{dt} \int \Omega \rho \frac{|v|^2}{2} \, dx + 2\kappa \int_0^T \int \Omega \rho |\nabla v|^2 \, dx - 2 \int_0^T \int \Omega \rho \nabla^t w : \nabla v \, dx = 0. 
\]

Gathering the above Equations together to obtain

\[
\frac{d}{dt} \int \Omega \rho \left( \frac{|w|^2}{2} + \frac{|v|^2}{2} \right) \, dx + 2(1 - \kappa) \int \Omega \rho |D(w)|^2 \, dx \\
+ \varepsilon \int \Omega |\Delta w|^2 + (1 + |\nabla w|^2)|\nabla w|^2 \, dx \\
+ 2\kappa \int \Omega \rho |A(w)|^2 + 2\kappa \int_0^T \int \Omega \rho |\nabla v|^2 \, dx = -G, 
\]

with \( G \) is equal to

\[
G := 2\kappa(1 - \kappa) \int \Omega \rho \nabla v : \nabla w \, dx - \frac{c}{2(2m + 1)} \int_0^T \int \Omega \rho^{2m+1} \nabla v : \nabla w \, dx \, dt \\
+ 2 \int_0^T \int \Omega \rho \nabla^t w : \nabla v \, dx. 
\]

Using now Hölder inequality and choosing \( \varepsilon \) large enough, we obtain uniform estimate for \( w \) and \( v \) necessary to repeat the procedure described in the previous paragraph. Thus, we obtain a global in time unique solution \( (\rho, w, v) \) satisfying equations ((44), (45), (46)).

**Passage to the limit with respect to \( n, \delta \) and \( \varepsilon \).**

Below we present uniform estimates that will allow us to pass to the limit with respect to the above parameters.

First, observe that multiplying the continuity equation (44) by \( \rho \) and integrating by parts with respect to \( x \), we obtain

\[
||\rho||_{L^\infty(0,T;L^2(\Omega))} + ||\nabla \rho||_{L^2(0,T;L^2(\Omega))} \leq c. 
\]

Moreover, the standard maximum principle gives boundness of \( \rho \) from above and below, i.e.,

\[
0 < r \leq \rho(t, x) \leq R. 
\]

Taking in mind the following equality

\[
\int \Omega |D(u)|^2 = \int \Omega \rho |D(u) - \frac{1}{d} \text{div} \, u|^2 + \int_\Omega \frac{2}{d} |\text{div} \, u|^2, 
\]

\[
(57) 
\]
and the fact that $w$ is a divergence free we deduce form Estimate (55) that for $0 < \kappa < 1$ we have

$$
\|w\|_{L^\infty(0,T;H)} + \varepsilon^{1/2}\|w\|_{L^2(0,T;W^{2,2}(\Omega))} + \varepsilon^{1/4}\|\nabla w\|_{L^4(0,T;L^4(\Omega))} + \|v\|_{L^\infty(0,T;H)} + \|v\|_{L^2(0,T;W^{1,2}(\Omega))} \leq c. \tag{58}
$$

Obviously, this estimate is uniform with respect to $n$ and $\delta$ but not on $\varepsilon$ because we are chosen $\varepsilon$ is large enough in Equation (55) to control $G$. Later when we identify $v$ by $2\nabla \log \rho$, one can show using Proposition (1) that it’s also uniform with respect to $\varepsilon$. This is the first point of difference compared to [5]) (see Estimate (49) and Estimate (52) in [5]).

- **Passage to the limit with respect to $n$, $\delta$.** The limit passage in this paragraph can be performed exactly as in [5]. For a reason of completeness, we want just to make a sketch of proof. In fact, the biggest problem in this part is to pass to the limit in

$$
\nabla \rho_{n,\delta} \otimes w_{n,\delta}
$$

which requires the strong convergence of the gradient of the density or the strong convergence of $w_{n,\delta}$ and also in the convective term

$$
\rho_{n,\delta} w_{n,\delta} \otimes w_{n,\delta}
$$

which requires strong convergence of $\sqrt{\rho_{n,\delta}}w_{n,\delta}$. Following the same lines given in [5], we can obtain after seeking a time-derivative estimate for $\nabla \rho_{n,\delta}$ and employing Aubin-Simon Lemma the following convergence

$$
(\nabla \rho_{n,\delta}) \text{ converges strongly in } L^2(0,T;L^2(\Omega)) \tag{59}
$$

and therefore due of (32), we deduce

$$
(\rho_{n,\delta}) \text{ converges strongly in } L^p(0,T;L^p(\Omega)) \quad \forall p < \infty. \tag{60}
$$

In the same manner, we use Equation (45) to establish an estimate on the time-derivative of $\rho w$. We infer using Aubin-Lions lemma with (see [5])

$$
(\rho_{n,\delta}, w_{n,\delta}) \text{ converges strongly in } L^p(0,T;L^p(\Omega)) \quad \forall 1 < p < \infty, \tag{61}
$$

and therefore due of (60)-(58), we deduce

$$
(\nabla w_{n,\delta}) \text{ converges strongly in } L^p(0,T;L^p(\Omega)) \quad \forall 1 \leq p < 4, \tag{62}
$$

Clearly, the above convergence allows us to pass to the limit in (45)-(46).

The identification $w = u + 2\kappa \nabla \log \rho$ can be established using the continuity equation (44). Since this equation is exactly the same equation considered in [5], then we skip the proof of this part here.

- **Passage to the limit with respect to $\varepsilon$.** After identification $w$ to $u + 2\kappa \nabla \log \rho$, the important feature that can be obtained is that our Estimate (58) is also independent of $\varepsilon$. Indeed, testing Equation (12) with $w \in L^2(0,T;V)$, we get thanks to divergence free condition the following equation

$$
\int_\Omega \rho \nabla (\sqrt{\rho^{2m-1}} \Delta (\int_0^\rho \sqrt{s^{2m-1}})) \cdot w \, dx = - \frac{1}{(2m + 1)} \int_\Omega \rho^{2m+1} \nabla \nabla \log \rho : \nabla w \, dx. \tag{63}
$$
Employing now the fact that \( w = u + 2\kappa \nabla \log \rho \), we can write

\[
\frac{c}{(2m + 1)} \int_\Omega \rho^{2m+1} \nabla \nabla \log \rho : \nabla w \, dx \\
= -c \int_\Omega \rho \nabla \left( \sqrt{\rho^{2m-1}} \Delta \left( \int_0^\rho \sqrt{s^{2m-1}} \right) \right) \cdot u \, dx \\
- 2\kappa c \int_\Omega \rho \nabla \left( \sqrt{\rho^{2m-1}} \Delta \left( \int_0^\rho \sqrt{s^{2m-1}} \right) \right) \cdot \nabla \log \rho \, dx.
\]

(64)

Now, by virtue of (36)-(38), we infer with

\[
\frac{c}{(2m + 1)} \int_\Omega \rho^{2m+1} \nabla \nabla \log \rho : \nabla w \, dx \\
\geq c \frac{m^2}{2} \frac{d}{dt} \int_\Omega \rho^{2m+1} |\nabla \log \rho|^2 \, dx + \frac{8 \kappa c c_0}{(2m + 1)^2} \int_\Omega \left( \Delta \rho^{\frac{2m+1}{2}} \right)^2 \, dx
\]

(65)

Now, we come back to Equations (53)-(54). Multiplying (54) by \((1 - \kappa)\kappa\) and adding it to (53), we get

\[
\frac{d}{dt} \int_\Omega \rho \left( \frac{|w|^2}{2} + (1 - \kappa) |\nabla \log \rho|^2 \right) \, dx + 2(1 - \kappa) \int_\Omega \rho |D(w - 2\kappa \nabla \log \rho)|^2 \, dx \\
+ \varepsilon \int_\Omega |\Delta w|^2 + (1 + |\nabla w|^2) |\nabla w|^2 \, dx + 2\kappa \int_\Omega \rho |A(w)|^2 \\
+ c \frac{m^2}{2} \frac{d}{dt} \int_\Omega \rho^{2m+1} |\nabla \log \rho|^2 \, dx + \frac{8 \kappa c c_0}{(2m + 1)^2} \int_\Omega \left( \Delta \rho^{\frac{2m+1}{2}} \right)^2 \, dx \leq 0.
\]

(66)

Consequently, we deduce the global existence of solution without assuming \( \varepsilon \) to be large enough unlike the above paragraph. Furthermore, the estimate (58) is now also uniform with \( \varepsilon \).

We can move now to perform the limit \( \varepsilon \to 0 \). The biggest problem is again to pass to the limit in

\[ \rho \varepsilon \mathbf{w}_\varepsilon \otimes \mathbf{w}_\varepsilon, \]

which ask for strong convergence of \( \sqrt{\rho \varepsilon \mathbf{w}_\varepsilon} \). To establish this convergence, we can establish an estimate on fractional derivative of \( \mathbf{w}_\varepsilon \) in Nikolski space. Precisely, using Equation (45), we can show (see Lemma 8 in [4])

\[
\int_0^{T-\delta} \| \mathbf{w}_\varepsilon(t + \delta) - \mathbf{w}_\varepsilon(t) \|_{(L^2(\Omega))^2} \leq C \delta^{1/2},
\]

with \( C \) independent of \( \delta \). This estimate coupled with \( \mathbf{w} \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)) \) gives us

\[ (\mathbf{w}_\varepsilon) \text{ converges strongly in } L^2(0,T;L^2(\Omega)). \]

(67)

The reader can consult also the book of P.-L. Lions [17] for similar approach. Before we finish the proof, it remains just to examine in which sense are the initial conditions admitted. Since \( \nabla \rho \) satisfies

\[ \partial_t \nabla \rho = - \text{div}(\rho \mathbf{w}) - 2\kappa \Delta \rho, \]

we can infer using (58) that

\[ \partial_t \nabla \rho \in L^2(0,T;W^{-1,3/2}(\Omega)). \]
On the other hand since $\nabla \rho \in L^\infty(0,T;L^2(\Omega))$, hence we can use now the Arzela-Ascoli\textgreater{} theorem to verify that $^1$
\[ \nabla \rho_\varepsilon \to \nabla \rho \text{ in } C([0,T];L^2_{\text{weak}}(\Omega)). \] Besides, we have $\rho_\varepsilon w_\varepsilon \to \rho w$ in $C([0,T];L^2_{\text{weak}}(\Omega)).$ Finally, using a version of Aubin-Lions lemma we obtain that $\rho$ is strongly continuous, i.e. $\rho_\varepsilon \to \rho \text{ in } C([0,T];L^2(\Omega)).$

\section*{Case when $c < 0$.}

In this paragraph, we need just to establish the a priori estimate. The rest of the proof of existence of solutions would requires minor modifications solely. Recalling that our system under study reads as follow
\[
\begin{align*}
\partial_t \rho + \text{div}(\rho u) &= 0, \\
\partial_t (\rho w) + \text{div}(\rho u \otimes w) + \nabla \pi_1 - 2(1-\kappa) \text{div}(\rho D(u)) \\
-2\kappa \text{div}(\rho A(u)) &= \frac{c}{(2m+1)} \text{div}(\rho^{2m+1} \nabla \nabla \log \rho), \\
\text{div } w &= 0.
\end{align*}
\]

Following the computation in Proposition (1), we prove that the energy estimate associated to the above system may be written as
\[
\frac{d}{dt} \int_\Omega \rho \left( \frac{|w|^2}{2} + (1-\kappa) \frac{|2\nabla \log \rho|^2}{2} \right) dx + 2(1-\kappa) \int_\Omega |\rho D(u)|^2 dx \\
+ 2\kappa \int_\Omega \rho |A(w)|^2 dx + \frac{c}{(2m+1)} \int_\Omega \rho^{2m+1} \nabla \nabla \log \rho : \nabla w dx = 0.
\]

Notably, the only difference in this paragraph lies in the fact that we cannot prove a sign of the last term like the case when $c > 0$ (see (65)). To this purpose, we proceed to take this term to the right hand side and estimate it. After that we need to control the resulting terms by the terms of the left hand side. Indeed, we estimate
\[
\left| \frac{c}{(2m+1)} \int_\Omega \rho^{2m+1} \nabla \nabla \log \rho : \nabla w dx \right|
\leq \frac{|c|}{(2m+1)} R^{2m+1} \| \nabla^2 \log \rho \|_{L^2(\Omega)} \| \nabla w \|_{L^2(\Omega)}
\leq \frac{|c|}{(2m+1)} R^{2m+1} \left( \frac{1}{2\beta} \| \nabla^2 \log \rho \|_{L^2(\Omega)}^2 + \frac{\beta}{2} \| \nabla w \|_{L^2(\Omega)}^2 \right)
\]

Observing now that because of periodic boundary condition and the free divergence condition on $w$, we have
\[
\int_\Omega |A(w)|^2 dx = \int_\Omega |\nabla w|^2 dx.
\]

In the meantime, we can check also
\[
\int_\Omega |\rho D(u)|^2 dx = \int_\Omega |\rho D(u) - \frac{1}{d} \text{div } u|\|^2 dx + \int_\Omega \frac{|\text{div } u|^2 dx}{d},
\]

\[ f \in C([0,T];L^2_{\text{weak}}(\Omega)) \text{ iff } \lim_{t \to t_0} |g(f(t) - g(t_0))| = 0 \forall g \in L^2(\Omega), \forall t_0 \in [0,T] \]
Thus employing (74), (75), Estimate (77) remains

\[
\frac{d}{dt} \int_{\Omega} \left( \frac{w^2}{2} + (1 - \kappa) \kappa \frac{2 |\nabla \log \rho|^2}{2} \right) dx + 2 (1 - \kappa) \int_{\Omega} \rho |D(u) - \frac{1}{d} \text{div} u|^2 dx
\]

\[
+ 2 \kappa \int_{\Omega} \rho |A(w)|^2 dx - \frac{|c|}{2} \frac{\beta}{(2m + 1)} \int_{\Omega} |\nabla w|^2 dx
\]

\[
+ \frac{8(1 - \kappa)\kappa^2}{d} \int_{\Omega} \rho |\Delta \log \rho|^2 dx - \frac{|c|}{2\beta (2m + 1)} R^{2m+1} \int_{\Omega} |\Delta \log \rho|^2 dx \leq 0. \tag{77}
\]

To control the two terms with bad sign, it is sufficient to choose

\[
\frac{|c|}{(2m + 1)} \frac{\beta}{2} R^{2m+1} \leq 2\kappa r \quad \text{and} \quad \frac{|c|}{2\beta (2m + 1)} R^{2m+1} \leq \frac{8(1 - \kappa)\kappa^2}{d} r. \tag{78}
\]

The best constant of $\beta$ is to make an equivalence between these two preceding conditions, that gives

\[
\beta = \frac{1}{2\sqrt{\kappa(1 - \kappa)}},
\]

and hence (78) recasts on the following condition

\[
R^{2m+1} \leq \frac{8\kappa \sqrt{\kappa(1 - \kappa)} (2m + 1)}{|c|} r. \tag{79}
\]

**Remark 5.** We conclude this paper with few observation. As we observed that the inequality proved in Lemma 4.1 is the key point on the analysis of global existence. The framework developed here can be adapted easily to be available when $m = 0$ and $\varphi = \rho^m$, $0 \leq m \leq 1/2$. Our goal in a forthcoming work is to generalize the result of this paper for more general form of capillarity tensor and when an initial vacuum may exist. Finally, an interesting work left here is whether the results in this paper can be extended to the case of bounded domain.

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