Uniform controllability of semidiscrete parabolic systems

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Heterogeneity in Control Systems, Nancy
Controlled PDE

- \( X, U : \) Hilbert spaces,
- \( A : D(A) \rightarrow X \) infinitesimal operator of a \( C_0 \) semi-group \( S(t) \),
- \( B \) control operator (in general, unbounded) on \( U \).

Control system

\[
\dot{y}(t) = Ay(t) + Bu(t), \quad y(0) = y_0, \quad (S)
\]

where \( y(t) \in X, u(t) \in U \).
Semi-discretization in space (e.g. finite differences, finite elements, ...)

- $X_h, U_h$: finite dimensional spaces
- $A_h : X_h \rightarrow X_h$ linear operator
- $B_h : U_h \rightarrow X_h$ linear operator

Family of finite dimensional control systems

\[
\dot{y}_h(t) = A_h y_h(t) + B_h u_h(t), \quad y_h(0) = y_{h,0}, \quad (S_h)
\]

where $y_h(t) \in X_h$, $u_h(t) \in U_h$. 

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Uniform controllability of semidiscrete parabolic systems
**Infinite dimensional control system** \((S)\):

\[
\dot{y}(t) = Ay(t) + Bu(t), \quad y(0) = y_0
\]

**Family of finite dimensional control systems** \((S_h)\):

\[
\dot{y}_h(t) = A_h y_h(t) + B_h u_h(t), \quad y_h(0) = y_{h,0}
\]

**Assumption**: \((S)\) is controllable (in time \(T\)).

Let \(y_1 \in X\). Then, \(\exists u\) s.t. the solution \(y(\cdot)\) of \((S)\) satisfies \(y(T) = y_1\).

**Question**

Is it possible to design controls \(u_h\), for \(0 < h < h_0\), converging to \(u\) as \(h \to 0\), such that the solutions \(y_h(\cdot)\) of \((S_h)\) converge to \(y(\cdot)\)? Efficient algorithm?

**Remark**. LQR case: Banks-Kunisch, Banks-Ito-Wang, Gibson, Kappel-Salamon, Lasiecka-Triggiani, Liu-Zheng, Ramdani-Takahashi-Tucsnak, ...

Context here: HUM
### Infinite dimensional control system $(S)$

$$\dot{y}(t) = Ay(t) + Bu(t), \quad y(0) = y_0$$

### Family of finite dimensional control systems $(S_h)$

$$\dot{y}_h(t) = A_h y_h(t) + B_h u_h(t), \quad y_h(0) = y_{h,0}$$

### Question

Is it possible to design controls $u_h$, for $0 < h < h_0$, converging to $u$ as $h \to 0$, such that the solutions $y_h(\cdot)$ of $(S_h)$ converge to $y(\cdot)$? Efficient algorithm?

### Some references

- Glowinski-Lions, 1996.
- 1D Schrödinger equation: Zuazua, 2005.
- 1D heat equation, discrete Carleman: Boyer - Hubert - Le Rousseau, 2009
- ...

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Uniform controllability of semidiscrete parabolic systems
Controllability of finite dimensional linear control systems

$T > 0$ fixed. Finite dimensional linear control system:

$$\dot{y}(t) = Ay(t) + Bu(t),$$

where $y(t) \in \mathbb{R}^n$, $A \in \mathcal{M}_n(\mathbb{R})$, $B \in \mathcal{M}_{n,m}(\mathbb{R})$, and $u(\cdot) \in L^2(0, T; \mathbb{R}^m)$.

**Definition**

Let $y_0 \in \mathbb{R}^n$. The system (1) is **controllable from $y_0$ in time $T$** if $\forall y_1 \in \mathbb{R}^n, \exists u(\cdot) \in L^2(0, T; \mathbb{R}^m)$ such that the corresponding solution $y(\cdot)$, with $y(0) = x_0$, satisfies $y(T) = x_1$.

**Theorem**

The system (1) is controllable in time $T$ if and only if the matrix

$$G = \int_0^T e^{(T-t)A}BB^*e^{(T-t)A^*} \, dt$$

(called **Gramian**) is invertible.

In finite dimension, this is equivalent to the existence of $\alpha > 0$ such that

$$\int_0^T \|B^*e^{(T-t)A^*}\psi\|^2 \, dt \geq \alpha \|\psi\|^2,$$

for every $\psi \in \mathbb{R}^n$ (observability inequality).
Controllability of infinite dimensional linear control systems

- $X, U :$ Hilbert
- $S(t) : C_0$ semi-group on $X$, of generator $(A, D(A))$
- $X_{-1}$ completion of $X$ for $\|x\|_{-1} = \|(\beta I - A)^{-1}x\|$, $(\beta \in \rho(A)$ fixed
- $X_{-1} \sim (D(A^*))'$, and $X \subset X_{-1}$ (continuous dense injection)
- $S(t)$ extends on $X_{-1}$, of generator an extension of $A : X \to X_{-1}$

**Definition (admissible control operator, Weiss 1989, Tucsnak Weiss 2009)**

$B \in L(U, X_{-1})$ is **admissible** for $S(t)$ if every solution of

$$\dot{y}(t) = Ay(t) + Bu(t),$$

with $y(0) = y_0 \in X$ and $u(\cdot) \in L^2(0, +\infty; U)$, satisfies $y(t) \in X$, $\forall t \geq 0$. In this case, $y(t) = S(t)y_0 + \int_0^t S(t - s)Bu(s)ds$. 
Controllability of infinite dimensional linear control systems

Definition (admissible control operator, Weiss 1989, Tucsnak Weiss 2009)

\( B \in L(U, X_{-1}) \) is *admissible* for \( S(t) \) if every solution of

\[
\dot{y}(t) = Ay(t) + Bu(t),
\]

with \( y(0) = y_0 \in X \) and \( u(\cdot) \in L^2(0, +\infty; U) \), satisfies \( y(t) \in X, \forall t \geq 0 \). In this case,

\[
y(t) = S(t)y_0 + \int_0^t S(t-s)Bu(s)ds.
\]

Definition (exact controllability)

For \( y_0 \in X \) and \( T > 0 \), the system (3) is *exactly controllable* from \( y_0 \) in time \( T \) if

\[ \forall y_1 \in X, \exists u(\cdot) \in L^2(0, T; U) \text{ such that the corresponding solution of (3), with } \]

\[ y(0) = y_0, \text{ satisfies } y(T) = y_1. \]

Theorem (observability inequality)

The system (3) is exactly controllable in time \( T \)

\[ \iff \exists \alpha > 0 \text{ s.t. } \int_0^T \| B^*S^*(t)\psi \|_U^2 dt \geq \alpha \| \psi \|_X^2, \quad \forall \psi \in D(A^*). \]
Controllability of infinite dimensional linear control systems

**Definition (admissible control operator, Weiss 1989, Tucsnak Weiss 2009)**

\[ B \in L(U, X_{-1}) \text{ is admissible for } S(t) \text{ if every solution of} \]
\[ \dot{y}(t) = Ay(t) + Bu(t), \quad (3) \]
with \( y(0) = y_0 \in X \) and \( u(\cdot) \in L^2(0, +\infty; U) \), satisfies \( y(t) \in X, \forall t \geq 0 \). In this case, \( y(t) = S(t)y_0 + \int_0^t S(t - s)Bu(s)ds \).

**Definition (exact null controllability)**

For \( T > 0 \), the system (3) is **exactly null controllable** in time \( T \) if \( \forall y_0 \in X, \exists u(\cdot) \in L^2(0, T; U) \) such that the corresponding solution of (3), with \( y(0) = y_0 \), satisfies \( y(T) = 0 \).

**Theorem (observability inequality)**

The system (3) is exactly null controllable in time \( T \)
\[
\iff \exists \alpha > 0 \text{ s.t. } \int_0^T \| B^* S^*(t)\psi \|_U^2 dt \geq \alpha \| S(T)^*\psi \|_X^2, \quad \forall \psi \in D(A^*). \]
For $\psi \in D(A^*)$, define

$$J(\psi) = \frac{1}{2} \int_0^T \| B^* S(t)^* \psi \|^2_U dt + \langle S(T)^* \psi, y_0 \rangle_x.$$  \hspace{1cm} (4)

$J$ is strictly convex and, using the observability inequality, is coercive. Let $\psi$ a minimizer, let

$$u(t) = B^* S(T - t)^* \psi,$$

for every $t \in [0, T]$, and let $y(\cdot)$ be the corresponding solution of (3), such that $y(0) = y_0$. Then, $y(T) = 0$.

(i.e., observability implies controllability : HUM method, J.-L. Lions, 1989)
Example of semi-discretization: 1-D heat equation with Dirichlet control (E. Zuazua)

**Continuous system**

\[
\begin{aligned}
    y_t &= y_{xx}, \\
    y(t, 0) &= 0, \quad y(t, 1) = u(t), \\
    y(0, x) &= y_0(x),
\end{aligned}
\]

This system is exactly null controllable, in any time \( T > 0 \), with \( X = L^2(0, 1) \) and \( U = L^2(0, T) \).

Actually, the following observability inequality holds:

\[
\forall T > 0 \quad \exists \alpha_T > 0 \quad \text{s.t.} \quad \int_0^T \psi_x(t, L)^2 dt \geq \alpha_T \int_0^L \psi(T, x)^2 dx,
\]

for every solution of

\[
\begin{aligned}
    \psi_t &= \psi_{xx}, \\
    \psi(t, 0) &= \psi(t, 1) = 0, \quad 0 < x < 1, \quad 0 < t < T,
\end{aligned}
\]
Example of semi-discretization: 1-D heat equation with Dirichlet control (E. Zuazua)

Continuous system

\[
\begin{aligned}
\begin{cases}
y_t &= y_{xx}, \\
y(t, 0) &= 0, \ y(t, 1) = u(t), \\
y(0, x) &= y_0(x),
\end{cases}
\quad 0 < x < 1, \ 0 < t < T,
\end{aligned}
\]

This system is exactly null controllable, in any time \( T > 0 \), with \( X = L^2(0, 1) \) and \( U = L^2(0, T) \).

Semidiscrete model (finite differences)

\[
\begin{aligned}
y_j'(t) &= \frac{y_{j+1}(t) - 2y_j(t) + y_{j-1}(t)}{h^2}, \quad 0 < t < T, \ j = 1, \ldots, N, \\
y_0(t) &= 0, \ y_{N+1}(t) = u(t), \quad 0 < t < T, \\
y_j(0) &= y_{0,j}, \quad j = 1, \ldots, N.
\end{aligned}
\]
Continuous system

\[
\begin{align*}
    y_t &= y_{xx}, \\
    y(t,0) &= 0, \quad y(t,1) = u(t), \\
    y(0,x) &= y_0(x).
\end{align*}
\]

Semidiscrete model

\[
\begin{align*}
    y'_j(t) &= \frac{y_{j+1}(t) - 2y_j(t) + y_{j-1}(t)}{h^2}, \\
    y_0(t) &= 0, \quad y_{N+1}(t) = u(t), \\
    y_j(0) &= y_{0,j}.
\end{align*}
\]

Theorem (Lopez Zuazua, 1997)

\[
\forall T > 0 \quad \exists c_T > 0 \text{ s.t.} \quad \int_0^T \left( \frac{\psi_N(t)}{h} \right)^2 dt \geq c_T h \sum_{j=1}^N \psi_j(T)^2,
\]

for every \( h > 0 \) and every solution of

\[
\begin{align*}
    \psi_j'(t) &= \frac{\psi_{j+1}(t) - 2\psi_j(t) + \psi_{j-1}(t)}{h^2}, \quad 0 < t < T, \quad j = 1, \ldots, N, \\
    \psi_0(t) &= 0, \quad \psi_{N+1}(t) = 0, \quad 0 < t < T.
\end{align*}
\]

This implies the uniform exact null controllability.
### Continuous system

\[
\begin{align*}
    &y_{tt} = y_{xx}, \\
    &y(t, 0) = 0, \quad y(t, 1) = u(t), \\
    &y(0, x) = y_0(x), \quad y_t(0, x) = y_1(x).
\end{align*}
\]

### Semidiscrete model

\[
\begin{align*}
    y''_j(t) &= \frac{y_{j+1}(t) - 2y_j(t) + y_{j-1}(t)}{h^2}, \\
    y_0(t) &= 0, \quad y_{N+1}(t) = u(t), \\
    y_j(0) &= y_{0,j}, \quad y'_j(0) = y_{1,j}.
\end{align*}
\]

### Theorem (Infante Zuazua, 1999)

\[
\forall T > 0 \quad \exists c_{T,h} > 0 \quad \text{s.t.} \quad \int_0^T \left(\frac{\psi_N(t)}{h}\right)^2 dt \geq c_{T,h} h \sum_{j=1}^N \left(\psi_j'(T)^2 + \left(\frac{\psi_{j+1}(T) - \psi_j(T)}{h}\right)^2\right),
\]

but

\[
c_{T,h} \to 0
\]

as \( h \to 0. \)

### Consequence

Banach-Steinhaus \( \Rightarrow \) \( \exists \) initial data for which discrete HUM diverges.

Under the following assumptions:

- the semi-group \( S(t) \) is **analytic** (i.e., parabolic case),
- the unboundedness degree of \( B \) is \(< 1/2\),
  \( (i.e., \exists \gamma \in [0, 1/2) \) s.t. \( B \in L(U, D((-\hat{A}^*)\gamma')) \))
- standard approximation assumptions,

there holds:

\( S \) is exactly null controllable \( \Leftrightarrow \) the family \((S_h)\) is uniformly controllable.

(in the sense that a uniform observability inequality holds)

In this case, the controls \( \text{HUM}_h \) converge to the HUM control.

Typical example:

heat equation (in \( \mathbb{R}^n \)) with boundary Neumann control.

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Uniform controllability of semidiscrete parabolic systems

Under the following assumptions:

- the semi-group $S(t)$ is *analytic* (i.e., parabolic case),
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  *(i.e., $\exists \gamma \in [0, 1/2]$ s.t. $B \in L(U, D((-\hat{A}^*)^\gamma)''))$*
- standard approximation assumptions,

there holds:

$S$ is exactly null controllable $\Leftrightarrow$ the family $(S_h)$ is uniformly controllable.

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**Typical example:**

heat equation (in $\mathbb{R}^n$) with boundary Neumann control.

Under the following assumptions:

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- the unboundedness degree of $B$ is $< 1/2$, 
  (i.e., $\exists \gamma \in [0, 1/2]$ s.t. $B \in L(U, D((-\hat{A}^*)\gamma)')$)
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Typical example:

heat equation (in $\mathbb{R}^n$) with boundary Neumann control.
Typical example: heat equation with Neumann boundary control

\( d \geq 1 \) integer, \( c \in \mathbb{R} \), \( \Omega \subset \mathbb{R}^d \) open connected bounded, \( \Gamma = \partial \Omega = \Gamma_1 \cup \Gamma_2 \) with \( \Gamma_1 \cap \Gamma_2 = \emptyset \).

The controlled PDE

\[
\begin{aligned}
\frac{\partial y}{\partial t} &= \triangle y + cy \quad \text{in } (0, T) \times \Omega, \\
y(0, \cdot) &= y_0(\cdot) \quad \text{in } \Omega, \\
\frac{\partial y}{\partial n} &= u \quad \text{on } [0, T] \times \Gamma_1, \\
\frac{\partial y}{\partial n} &= 0 \quad \text{on } [0, T] \times \Gamma_2, 
\end{aligned}
\]

where \( y_0 \in L^2(\Omega) \) and \( u \in L^2(0, T; L^2(\Gamma_1)) \).

This system is exactly null controllable in \( X = L^2(\Omega) \) with \( U = L^2(\Gamma_1) \).

Here, \( Ay = \triangle y + cy \), \( \text{on } D(A) = \left\{ y \in H^2(\Omega) \mid \frac{\partial y}{\partial n} = 0 \text{ on } \Gamma \right\} \),

and \( B = -AN \in L(U, D(A^*)') \), where \( N \) is the Neumann mapping: \( Nu = y \iff \begin{cases} 
Ay = 0 \text{ dans } \Omega, \\
\frac{\partial y}{\partial n} = u \text{ on } \Gamma_1, \\
\frac{\partial y}{\partial n} = 0 \text{ on } \Gamma_2.
\end{cases} \)

\( B \) is admissible (trace regularity result). The unboundedness degree of \( B \) is \( \gamma = 1/4 + \varepsilon \), for every \( \varepsilon > 0 \).
Typical example: heat equation with Neumann boundary control

\( d \geq 1 \) integer, \( c \in \mathbb{R} \), \( \Omega \subset \mathbb{R}^d \) open connected bounded, \( \Gamma = \partial \Omega = \Gamma_1 \cup \Gamma_2 \) with \( \Gamma_1 \cap \Gamma_2 = \emptyset \).

The controlled PDE

\[
\begin{align*}
\frac{\partial y}{\partial t} &= \Delta y + cy \quad \text{in} \ (0, T) \times \Omega, \\
y(0, \cdot) &= y_0(\cdot) \quad \text{in} \ \Omega, \\
\frac{\partial y}{\partial n} &= u \quad \text{on} \ [0, T] \times \Gamma_1, \\
\frac{\partial y}{\partial n} &= 0 \quad \text{on} \ [0, T] \times \Gamma_2,
\end{align*}
\]

where \( y_0 \in L^2(\Omega) \) and \( u \in L^2(0, T; L^2(\Gamma_1)) \).

Semidiscrete model (finite elements)

\[
M_h \dot{Y}(t) = A_h Y(t) + B_h V(t), \quad Y(0) = Y_0.
\]

→ slow convergence, but number of gradient iterations bounded
\( \Omega = (0, 1), \Gamma_1 = \{1\}, \Gamma_2 = \{0\}, c = 1, \text{ and } T = 1. \)

Regular subdivision of \( \Omega. \) Time step : 0.001.

Data :

<table>
<thead>
<tr>
<th>name</th>
<th>( S_h )</th>
<th>( h )</th>
<th>( y_0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1D_10</td>
<td>11</td>
<td>( 10^{-1} )</td>
<td>( y_0(x) = x )</td>
</tr>
<tr>
<td>1D_100</td>
<td>101</td>
<td>( 10^{-2} )</td>
<td>( y_0(x) = x )</td>
</tr>
<tr>
<td>1D_1000</td>
<td>1001</td>
<td>( 10^{-3} )</td>
<td>( y_0(x) = x )</td>
</tr>
<tr>
<td>sin1D_10</td>
<td>11</td>
<td>( 10^{-1} )</td>
<td>( y_0(x) = \sin^2(\pi x) )</td>
</tr>
<tr>
<td>sin1D_100</td>
<td>101</td>
<td>( 10^{-2} )</td>
<td>( y_0(x) = \sin^2(\pi x) )</td>
</tr>
<tr>
<td>sin1D_1000</td>
<td>1001</td>
<td>( 10^{-3} )</td>
<td>( y_0(x) = \sin^2(\pi x) )</td>
</tr>
</tbody>
</table>

Numerical results :

<table>
<thead>
<tr>
<th>name</th>
<th>( | \psi_h |_X )</th>
<th>( h^3 )</th>
<th>( | h^3 \psi_h + y_h(T) |_X )</th>
<th>( | y_h(T) |_{X_h} )</th>
<th>( | y_h^d(T) |_{X_h} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1D_10</td>
<td>0.413804</td>
<td>0.372</td>
<td>0.00305</td>
<td>0.15670</td>
<td>1.35778</td>
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<tr>
<td>1D_100</td>
<td>0.531854</td>
<td>0.126</td>
<td>0.00369</td>
<td>0.07000</td>
<td>1.35778</td>
</tr>
<tr>
<td>1D_1000</td>
<td>0.705366</td>
<td>0.044</td>
<td>0.00399</td>
<td>0.03397</td>
<td>1.35778</td>
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<tr>
<td>sin1D_10</td>
<td>0.413831</td>
<td>0.372</td>
<td>0.00308</td>
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<tr>
<td>sin1D_100</td>
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<td>1.35778</td>
</tr>
<tr>
<td>sin1D_1000</td>
<td>0.705650</td>
<td>0.044</td>
<td>0.00399</td>
<td>0.03398</td>
<td>1.35778</td>
</tr>
</tbody>
</table>

Slow convergence (here, \( \beta/2 = 0.225 \)).
\( \Omega \): unit disk of \( \mathbb{R}^2 \), \( c = 1 \), and \( T = 1 \). Time step : 0.001.

Data:

<table>
<thead>
<tr>
<th>name</th>
<th>( S_h )</th>
<th>( y_0 )</th>
<th>( \Gamma_1 )</th>
<th>( \Gamma_2 )</th>
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</thead>
<tbody>
<tr>
<td>disk_1</td>
<td>55</td>
<td>( y_0(x, y) = x + y )</td>
<td>( \Gamma )</td>
<td>( \emptyset )</td>
</tr>
<tr>
<td></td>
<td>104</td>
<td>( y_0(x, y) = x + y )</td>
<td>( \Gamma )</td>
<td>( \emptyset )</td>
</tr>
<tr>
<td>disk_2</td>
<td>55</td>
<td>( y_0(x, y) = x + y )</td>
<td>( { (x, y) \in \Gamma \mid x \geq 0 \text{ and } y \geq 0 } )</td>
<td>( \Gamma \setminus \Gamma_1 )</td>
</tr>
<tr>
<td></td>
<td>104</td>
<td>( y_0(x, y) = x + y )</td>
<td>( { (x, y) \in \Gamma \mid x \geq 0 \text{ and } y \geq 0 } )</td>
<td>( \Gamma \setminus \Gamma_1 )</td>
</tr>
</tbody>
</table>

Meshes:

Numerical results:

<table>
<thead>
<tr>
<th>name</th>
<th>( S_h )</th>
<th>( |\psi_h|_X )</th>
<th>( h^3 )</th>
<th>( h^3 \psi_h + y_h(T) )</th>
<th>( y_h(T) )</th>
<th>( y_h^\beta(T) )</th>
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</thead>
<tbody>
<tr>
<td>disk_1</td>
<td>55</td>
<td>0.15872</td>
<td>0.61424</td>
<td>0.01747</td>
<td>0.08030</td>
<td>0.10351</td>
</tr>
<tr>
<td></td>
<td>104</td>
<td>0.23091</td>
<td>0.45112</td>
<td>0.01953</td>
<td>0.08494</td>
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</tr>
<tr>
<td>disk_2</td>
<td>55</td>
<td>0.16497</td>
<td>0.61424</td>
<td>0.01162</td>
<td>0.09623</td>
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<tr>
<td></td>
<td>104</td>
<td>0.17513</td>
<td>0.45112</td>
<td>0.01062</td>
<td>0.10259</td>
<td>0.10861</td>
</tr>
</tbody>
</table>
What happens for the wave equation?

1D wave equation with Dirichlet control

\[
\begin{aligned}
\frac{\partial^2 y}{\partial t^2} - \frac{\partial^2 y}{\partial x^2} &= 0, \text{ in } ]0, T[ \times ]0, 1[,
\end{aligned}
\]
\[
\begin{aligned}
y(t, 0) &= 0, & y(t, 1) &= u(t), \text{ on } ]0, T[.
\end{aligned}
\]

This system is exactly controllable whenever \( T \geq 2 \). No analyticity.

**Discretization**: finite difference method.

1D wave equation : numerical comparison with heat equation

<table>
<thead>
<tr>
<th>( n_p )</th>
<th>gradient iter., heat</th>
<th>gradient iter., wave</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>5</td>
<td>15</td>
</tr>
<tr>
<td>10</td>
<td>5</td>
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<tr>
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<tr>
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<td>1515</td>
</tr>
<tr>
<td>50</td>
<td>4</td>
<td>–</td>
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</tbody>
</table>

→ number of iterations not uniformly bounded, even with "regular" initial and final data.
Open problems and ongoing research

- condition on the unboundedness degree $\gamma$ of $B$
- convergence rate of controls
  - Boyer - Hubert - Le Rousseau 2009: uniform discrete parabolic Carleman estimates, 1D heat equation
- remove analyticity assumption
  - Castro - Micu 2006: 1D wave equation, mixed finite elements;
- full discretization (w.r.t. space AND time)
- semilinear, nonlinear controlled PDE's

From January 2010:
Laurent Desvillettes, Emmanuel Trélat
Control of PDE's and Applications

October 1st -- December 31th 2010

Main topics

- Controllability properties of partial differential equations
- Optimal control of PDE's and applications
- Numerical approximations of controlled PDE's
- Fluid-structures interactions
- Control of infinite dimensional quantum mechanical systems
- Optimal design and boundary control of parabolic and hyperbolic PDE's
- Stability and stabilization of partial differential equations

Planned courses

- Controllability of linear and nonlinear PDE's (J.-M. Coron)
- Control of the wave equation (G. Lebeau)
- Control and discretization PDE's (E. Zuazua)
- Modelization and control of quantum systems (P. Rouchon)

Planned workshops

- Control of parabolic equations and systems, applications to fluids
- Hyperbolic systems and control in networks, applications
- Control of dispersive equations
- Quantum control

Seminars

A seminar on control and related problems, with two conferences every two weeks.

Organizers: Olivier Glass (Paris 6), Jérôme Le Rousseau (Orléans), Lionel Rosier (Nancy), Emmanuel Trélat (Orléans).


We acknowledge the financial support of the Agence Nationale de la Recherche, of the Fondation des Sciences Mathématiques de Paris, of the Institute Henri Poincaré.
\((X_h)_{0<h<h_0}\) and \((U_h)_{0<h<h_0}\) : families of finite dimensional vector spaces.

\((H_3)\) \(\forall h \in (0, h_0), \) there exist linear mappings \(P_h : D((-\hat{A}^*)^{1/2})' \to X_h\) and \(\tilde{P}_h : X_h \to D((-\hat{A}^*)^{1/2})\) (resp., \(Q_h : U \to U_h\) et \(\tilde{Q}_h : U_h \to U\)), satisfying:

\((H_{3.1})\) \(\forall h \in (0, h_0), \) \(P_h \tilde{P}_h = id_{X_h},\) and \(Q_h \tilde{Q}_h = id_{U_h}.\)

\((H_{3.2})\) \(\exists s, C_4 > 0 \text{ s.t. } \forall h \in (0, h_0),\)

\[
\| (I - \tilde{P}_h P_h) \psi \|_X \leq C_4 h^s \| A^* \psi \|_X,
\]

\[
\| (-\hat{A}^*)^{\gamma} (I - \tilde{P}_h P_h) \psi \|_X \leq C_4 h^{\gamma s (1-\gamma)} \| A^* \psi \|_X, \quad \forall \psi \in D(A^*),
\]

\[
\| (I - \tilde{Q}_h Q_h) u \|_U \to 0 \text{ as } h \to 0, \quad \forall u \in U,
\]

\[
\| (I - \tilde{Q}_h Q_h) B^* \psi \|_U \leq C_4 h^{\gamma s (1-\gamma)} \| A^* \psi \|_X, \quad \forall \psi \in D(A^*).
\]

\((H_{3.3})\) \(\forall h \in (0, h_0), \) \(P_h = \tilde{P}_h^*,\) and \(Q_h = \tilde{Q}_h^*.\)

\((H_{3.4})\) \(\exists C_6 > 0 \text{ s.t. } \| B^* \tilde{P}_h \psi_h \|_U \leq C_6 h^{-\gamma s} \| \psi_h \|_X_h, \quad \forall h \in (0, h_0), \forall \psi_h \in X_h.\)

\(\forall h \in (0, h_0),\) let \(A_h^* : X_h \to X_h\) and \(B_h^* : X_h \to U_h\) defined by

\[A_h^* = P_h A^* \tilde{P}_h,\] and \(B_h^* = Q_h B^* \tilde{P}_h.\)
(H$_4$) The following properties hold:

(H$_4.1$) The family $e^{tA^*_h}$ is uniformly analytic, i.e., $\exists C_7 > 0$ s.t.

$$\|e^{tA^*_h}\|_{L(X_h)} \leq C_7 e^{\omega t}, \text{ and } \|A_h e^{tA^*_h}\|_{L(X_h)} \leq C_7 \frac{e^{\omega t}}{t}, \ \forall t > 0.$$ (5)

(H$_4.2$) $\exists C_9 > 0$ s.t. $\forall f \in X, \forall h \in (0, h_0)$, the respective solutions of $\hat{A}^* \psi = f$ et $\hat{A}_h^* \psi_h = P_h f$ satisfy

$$\|P_h \psi - \psi_h\|_{X_h} \leq C_9 h^s \|f\|_X.$$ (6)

In other words, $\|P_h \hat{A}^*-1 - \hat{A}_h^*-1 P_h\|_{L(X,X_h)} \leq C_9 h^s$. 

**Back**

\[ \dot{y} = Ay + Bu \] is exactly null controllable in time \( T > 0 \)
\[ \iff \] the family \( \dot{y}_h = A_hy_h + B_hu_h \) is uniformly controllable in the following sense:

\[ \exists \beta, h_1, c, c' > 0 \mid \forall h \in (0, h_1), \forall \psi_h \in X_h, \]
\[ c \| e^{TA^*_h}\psi_h \|_{X_h}^2 \leq \int_0^T \| B_h e^{tA^*_h}\psi_h \|_{U_h}^2 dt + h^\beta \| \psi_h \|_{X_h}^2 \leq c' \| \psi_h \|_{X_h}^2. \]
In these conditions: \( \forall y_0 \in X, \ \forall h \in (0, h_1), \ \exists ! \psi_h \in X_h \)

minimizing

\[
J_h(\psi_h) = \frac{1}{2} \int_0^T \| B_h^* e^{tA_h^*} \psi_h \|^2 U_h \, dt + \frac{1}{2} h^\beta \| \psi_h \|^2_{X_h} + \langle e^{T A_h^*} \psi_h, P_h y_0 \rangle_{X_h},
\]

and the sequence \((\tilde{Q}_h u_h)_{0 < h < h_1}\), where \(u_h(t) = B_h^* e^{(T-t)A_h^*} \psi_h\), converges weakly (up to subsequence) to \(u \in L^2(0, T; U)\), the corresponding solution of which satisfies

\[
\dot{y} = Ay + Bu, \ y(0) = y_0, \ y(T) = 0.
\]
∀ \( h \in (0, h_1) \), let \( y_h(\cdot) \) be the solution of
\[
\dot{y}_h = A_h y_h + B_h u_h, \quad y_h(0) = P_h y_0.
\] (5)
Then,
- \( y_h(T) = -h^\beta \psi_h \);
- \( \forall t \in (0, T], (\tilde{P}_h y_h(t))_{0 < h < h_1} \rightarrow y(t) \) (up to subsequence) in \( X \).
- \( \exists M > 0 \mid \forall h \in (0, h_1), \)
  \[
  \int_0^T \| u_h(t) \|_{U_h}^2 dt \leq M^2 \| y_0 \|_X^2, \quad h^\beta \| \psi_h \|_{X_h}^2 \leq M^2 \| y_0 \|_X^2,
  \]
  et \( \| y_h(T) \|_{X_h} \leq M h^{\beta/2} \| y_0 \|_X \).
E. Trélat
Uniform controllability of semidiscrete parabolic systems