Continuation from a flat to a round Earth model in the coplanar orbit transfer problem

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The coplanar orbit transfer problem

- Spherical Earth
- Central gravitational field \( g(r) = \frac{\mu}{r^2} \)

System in cylindrical coordinates

\[
\begin{align*}
\dot{r}(t) &= v(t) \sin \gamma(t) \\
\dot{\varphi}(t) &= \frac{v(t)}{r(t)} \cos \gamma(t) \\
\dot{v}(t) &= -g(r(t)) \sin \gamma(t) + \frac{T_{\text{max}}}{m(t)} u_1(t) \\
\dot{\gamma}(t) &= \left( \frac{v(t)}{r(t)} - \frac{g(r(t))}{v(t)} \right) \cos \gamma(t) + \frac{T_{\text{max}}}{m(t)v(t)} u_2(t) \\
\dot{m}(t) &= -\beta T_{\text{max}} \|u(t)\| 
\end{align*}
\]

- Thrust: \( T(t) = u(t) T_{\text{max}} \) \( (T_{\text{max}} \text{ large: strong thrust}) \)
- Control: \( u(t) = (u_1(t), u_2(t)) \) satisfying \( \|u(t)\| = \sqrt{u_1(t)^2 + u_2(t)^2} \leq 1 \)
The coplanar orbit transfer problem

<table>
<thead>
<tr>
<th>Initial conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r(0) = r_0$, $\varphi(0) = \varphi_0$, $v(0) = v_0$, $\gamma(0) = \gamma_0$, $m(0) = m_0$,</td>
</tr>
</tbody>
</table>

<table>
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<tr>
<th>Final conditions</th>
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<tbody>
<tr>
<td>a point of a specified orbit: $r(t_f) = r_f$, $v(t_f) = v_f$, $\gamma(t_f) = \gamma_f$,</td>
</tr>
<tr>
<td>or</td>
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<tr>
<td>an elliptic orbit of energy $K_f &lt; 0$ and eccentricity $e_f$:</td>
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<tr>
<td>$\xi_{K_f} = \frac{v(t_f)^2}{2} - \frac{\mu}{r(t_f)} - K_f = 0$,</td>
</tr>
<tr>
<td>$\xi_{e_f} = \sin^2 \gamma + \left(1 - \frac{r(t_f)v(t_f)^2}{\mu}\right)^2 \cos^2 \gamma - e_f^2 = 0$.</td>
</tr>
</tbody>
</table>

(orientation of the final orbit not prescribed: $\varphi(t_f)$ free; in other words: argument of the final perigee free)

<table>
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<tr>
<th>Optimization criterion</th>
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<tbody>
<tr>
<td>$\max m(t_f)$ (note that $t_f$ has to be fixed)</td>
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</table>
Application of the Pontryagin Maximum Principle

**Hamiltonian**

\[
H(q, p, p^0, u) = p_r v \sin \gamma + p_\varphi \frac{v}{r} \cos \gamma + p_v \left( -g(r) \sin \gamma + \frac{T_{\text{max}}}{m} u_1 \right) \\
+ p_\gamma \left( \left( \frac{v}{r} - \frac{g(r)}{v} \right) \cos \gamma + \frac{T_{\text{max}}}{mv} u_2 \right) - p_m \beta T_{\text{max}} \|u\|,
\]

**Extremal equations**

\[
\dot{q}(t) = \frac{\partial H}{\partial p}(q(t), p(t), p^0, u(t)), \quad \dot{p}(t) = -\frac{\partial H}{\partial q}(q(t), p(t), p^0, u(t)),
\]

**Maximization condition**

\[
H(q(t), p(t), p^0, u(t)) = \max_{\|w\| \leq 1} H(q(t), p(t), p^0, w)
\]
Application of the Pontryagin Maximum Principle

**Hamiltonian**

\[
H(q, p, p^0, u) = p_r v \sin \gamma + p_\phi \frac{v}{r} \cos \gamma + p_\nu \left( -g(r) \sin \gamma + \frac{T_{\text{max}}}{m} u_1 \right)
+ p_\gamma \left( \left( \frac{v}{r} - \frac{g(r)}{v} \right) \cos \gamma + \frac{T_{\text{max}}}{mv} u_2 \right) - p_m \beta T_{\text{max}} \|u\|,
\]

**Maximization condition leads to**

- \( u(t) = (u_1(t), u_2(t)) = (0, 0) \) whenever \( \Phi(t) < 0 \)
- \( u_1(t) = \frac{p_\nu(t)}{\sqrt{p_\nu(t)^2 + \frac{p_\gamma(t)^2}{v(t)^2}}} \), \( u_2(t) = \frac{p_\gamma(t)}{v(t) \sqrt{p_\nu(t)^2 + \frac{p_\gamma(t)^2}{v(t)^2}}} \) whenever \( \Phi(t) > 0 \)

where

\[
\Phi(t) = \frac{1}{m(t)} \sqrt{p_\nu(t)^2 + \frac{p_\gamma(t)^2}{v(t)^2}} - \beta p_m(t) \quad \text{(switching function)}
\]
Application of the Pontryagin Maximum Principle

**Hamiltonian**

\[
H(q, p, p^0, u) = p_r v \sin \gamma + p_\varphi \frac{v}{r} \cos \gamma + p_v \left( -g(r) \sin \gamma + \frac{T_{\text{max}}}{m} u_1 \right) \\
+ p_\gamma \left( \left( \frac{v}{r} - \frac{g(r)}{v} \right) \cos \gamma + \frac{T_{\text{max}}}{mv} u_2 \right) - p_m \beta T_{\text{max}} \|u\|,
\]

**Transversality conditions**

- case of a fixed point of a specified orbit: \( p_\varphi(t_f) = 0, \ p_m(t_f) = -p^0 \)
- case of an orbit of given energy and eccentricity:

\[
\partial_r \xi_K_f (p_\gamma \partial_v \xi_e - p_v \partial_\gamma \xi_e) + \partial_v \xi_K_f (p_r \partial_\gamma \xi_e - p_\gamma \partial_r \xi_e) = 0
\]

**Remark**

- \( p^0 \neq 0 \) (no abnormal) \( \Rightarrow p^0 = -1 \)
- no singular arc (Bonnard - Caillau - Faubourg - Gergaud - Haberkorn - Noailles - Trélat)
Shooting method

Find a zero of

\[ S(t_f, p_0) = \begin{pmatrix} r(t_f, p_0) - r_f \\ v(t_f, p_0) - v_f \\ \gamma(t_f, p_0) - \gamma_f \\ p_\varphi(t_f, p_0) \\ p_m(t_f, p_0) - 1 \end{pmatrix} \] or

\[ \begin{pmatrix} \xi_{K_f}(p_0) \\ \xi_{e_f}(p_0) \\ \ast \ast \ast \\ p_\varphi(t_f, p_0) \\ p_m(t_f, p_0) - 1 \end{pmatrix}, \]

Main problem: how to make the shooting method converge?

- initialization of the shooting method
- discontinuities of the optimal control
Main problem: how to make the shooting method converge?

- initialization of the shooting method
- discontinuities of the optimal control

Several methods:

- use first a direct method to provide a good initial guess, e.g. AMPL combined with IPOPT:

but usual flaws of direct methods (computationally demanding, lack of numerical precision).
Main problem: how to make the shooting method converge?

- initialization of the shooting method
- discontinuities of the optimal control

Several methods:

- use the impulse transfer solution to provide a good initial guess:

  but valid only for nearly circular initial and final orbits. See also:
Main problem: how to make the shooting method converge?

- initialization of the shooting method
- discontinuities of the optimal control

Several methods:

- multiple shooting method parameterized by the number of thrust arcs:

Shooting method

Main problem: how to make the shooting method converge?
- initialization of the shooting method
- discontinuities of the optimal control

Several methods:
- differential or simplicial continuation method linking the minimization of the $L^2$-norm of the control to the minimization of the fuel consumption:

but not adapted for high-thrust transfer.
Observation:
Solving the optimal control problem for a flat Earth model with constant gravity is simple and algorithmically very efficient.

In view of that:

Continuation from this simple model to the initial round Earth model.
**Simplified flat Earth model**

**System**

\[
\begin{align*}
\dot{x}(t) &= v_x(t) \\
\dot{h}(t) &= v_h(t) \\
\dot{v}_x(t) &= \frac{T_{\text{max}}}{m(t)} u_x(t) \\
\dot{v}_h(t) &= \frac{T_{\text{max}}}{m(t)} u_h(t) - g_0 \\
\dot{m}(t) &= -\beta T_{\text{max}} \sqrt{u_x(t)^2 + u_h(t)^2}
\end{align*}
\]

**Control**

Control \((u_x(\cdot), u_h(\cdot))\) such that \(u_x(\cdot)^2 + u_h(\cdot)^2 \leq 1\)

- **initial conditions:** \(x(0) = x_0, \ h(0) = h_0, \ v_x(0) = v_{x0}, \ v_h(0) = v_{h0}, \ m(0) = m_0\)
- **final conditions:** \(h(t_f) = h_f, \ v_x(t_f) = v_{xf}, \ v_h(t_f) = 0\)
Modified flat Earth model

Idea: mapping circular orbits to horizontal trajectories

\[
\begin{align*}
    x &= r \varphi \\
    h &= r - r_T \\
    v_x &= v \cos \gamma \\
    v_h &= v \sin \gamma \\
\end{align*}
\]

\[
\begin{align*}
    r &= r_T + h \\
    \varphi &= \frac{x}{r_T + h} \\
    v &= \sqrt{v_x^2 + v_h^2} \\
    \gamma &= \arctan \frac{v_h}{v_x}
\end{align*}
\]

\[
\begin{pmatrix}
    u_x \\
    u_h
\end{pmatrix} =
\begin{pmatrix}
    \cos \gamma & -\sin \gamma \\
    \sin \gamma & \cos \gamma
\end{pmatrix}
\begin{pmatrix}
    u_1 \\
    u_2
\end{pmatrix}
\]
Modified flat Earth model

Plugging this change of coordinates into the initial round Earth model:

\[
\begin{align*}
\dot{r}(t) &= v(t) \sin \gamma(t) \\
\dot{\phi}(t) &= \frac{v(t)}{r(t)} \cos \gamma(t) \\
\dot{v}(t) &= -g(r(t)) \sin \gamma(t) + \frac{T_{\text{max}}}{m(t)} u_1(t) \\
\dot{\gamma}(t) &= \left( \frac{v(t)}{r(t)} - \frac{g(r(t))}{v(t)} \right) \cos \gamma(t) + \frac{T_{\text{max}}}{m(t)v(t)} u_2(t) \\
\dot{m}(t) &= -\beta T_{\text{max}} \|u(t)\| 
\end{align*}
\]

leads to...
Modified flat Earth model

\[ \dot{x}(t) = v_x(t) + v_h(t) \frac{x(t)}{r_T + h(t)} \]
\[ \dot{h}(t) = v_h(t) \]
\[ \dot{v}_x(t) = \frac{T_{\text{max}}}{m(t)} u_x(t) - \frac{v_x(t)v_h(t)}{r_T + h(t)} \]
\[ \dot{v}_h(t) = \frac{T_{\text{max}}}{m(t)} u_h(t) - g(r_T + h(t)) + \frac{v_x(t)^2}{r_T + h(t)} \]
\[ \dot{m}(t) = -\beta T_{\text{max}} \| u(t) \| \]

Differences with the simplified flat Earth model (with constant gravity):
- the term in green: variable (usual) gravity.
- the terms in red: "correcting terms" allowing the existence of horizontal (periodic up to translation in \( x \)) trajectories with no thrust.
Continuation procedure

Simplified flat Earth model (with constant gravity) $\xrightarrow{\text{continuation procedure}}$ modified flat Earth model:

\[
\begin{align*}
\dot{x}(t) &= v_x(t) + \lambda_2 v_h(t) \frac{x(t)}{r_T + h(t)} \\
\dot{h}(t) &= v_h(t) \\
\dot{v}_x(t) &= \frac{T_{\text{max}}}{m(t)} u_x(t) - \lambda_2 \frac{v_x(t)v_h(t)}{r_T + h(t)} \\
\dot{v}_h(t) &= \frac{T_{\text{max}}}{m(t)} u_h(t) - \frac{\mu}{(r_T + \lambda_1 h(t))^2} + \lambda_2 \frac{v_x(t)^2}{r_T + h(t)} \\
\dot{m}(t) &= -\beta T_{\text{max}} \sqrt{u_x(t)^2 + u_h(t)^2}
\end{align*}
\]

- $\lambda_1 = \lambda_2 = 0$: simplified flat Earth model with constant gravity
- $\lambda_1 = 1, \lambda_2 = 0$: simplified flat Earth model with usual gravity
- $\lambda_1 = \lambda_2 = 1$: modified flat Earth model (equivalent to usual round Earth)
Continuation procedure

Simplified flat Earth model (with constant gravity) \(\xrightarrow{\text{continuation procedure}}\) modified flat Earth model:

\[
\begin{align*}
\dot{x}(t) &= v_x(t) + \lambda_2 v_h(t) \frac{x(t)}{r_T + h(t)} \\
\dot{h}(t) &= v_h(t) \\
\dot{v}_x(t) &= \frac{T_{\text{max}}}{m(t)} u_x(t) - \lambda_2 \frac{v_x(t)v_h(t)}{r_T + h(t)} \\
\dot{v}_h(t) &= \frac{T_{\text{max}}}{m(t)} u_h(t) - \frac{\mu}{(r_T + \lambda_1 h(t))^2} + \lambda_2 \frac{v_x(t)^2}{r_T + h(t)} \\
\dot{m}(t) &= -\beta T_{\text{max}} \sqrt{u_x(t)^2 + u_h(t)^2}
\end{align*}
\]

\(\Rightarrow\) Two-parameters family of optimal control problems: \((\text{OCP})_{\lambda_1, \lambda_2}\)
Continuation procedure

(OCP)_{0,0} 
flat Earth model, constant gravity

(\text{linear) continuation on } \lambda_1 \in [0, 1] \quad \text{final time } t_f \text{ free}

(OCP)_{1,0} 
flat Earth model, usual gravity

\downarrow 
(\text{linear) continuation on } \lambda_2 \in [0, 1] \quad \text{final time } t_f \text{ fixed}

(OCP)_{1,1} 
modified flat Earth model (equivalent to round Earth model)

Application of the PMP to (OCP)_{\lambda_1, \lambda_2} \Rightarrow \text{series of shooting problems.}
Remark: Once the continuation process has converged, we obtain the initial adjoint vector for \((OCP)_{1,1}\) in the modified coordinates.

To recover the adjoint vector in the usual cylindrical coordinates, we use the general fact:

**Lemma**

Change of coordinates \(x_1 = \phi(x)\) and \(u_1 = \psi(u)\)

\[ \Rightarrow \text{dynamics } f_1(x_1, u_1) = d\phi(x) \cdot f(\phi^{-1}(x_1), \psi^{-1}(u_1)) \]

and for the adjoint vectors:

\[ p_1(\cdot) = t d\phi(x(\cdot))^{-1} p(\cdot). \]

Here, this yields:

\[ p_r = \frac{x}{r_T + h} p_x + p_h \]
\[ p_\varphi = (r_T + h) p_x \]
\[ p_v = \cos \gamma p_{v_x} + \sin \gamma p_{v_h} \]
\[ p_\gamma = v(-\sin \gamma p_{v_x} + \cos \gamma p_{v_h}). \]
### Analysis of the flat Earth model

#### System

\[
\begin{align*}
\dot{x} &= v_x \\
\dot{h} &= v_h \\
\dot{v}_x &= \frac{T_{\text{max}}}{m} u_x \\
\dot{v}_h &= \frac{T_{\text{max}}}{m} (u_h - g_0) \\
\dot{m} &= -\beta T_{\text{max}} \sqrt{u_x^2 + u_h^2}
\end{align*}
\]

#### Initial conditions

- \(x(0) = x_0\)
- \(h(0) = h_0\)
- \(v_x(0) = v_{x0}\)
- \(v_h(0) = v_{h0}\)
- \(m(0) = m_0\)

#### Final conditions

- \(x(t_f)\) free
- \(h(t_f) = h_f\)
- \(v_x(t_f) = v_{xf}\)
- \(v_h(t_f) = 0\)
- \(m(t_f)\) free

#### Theorem

If \(h_f > h_0 + \frac{v_{x0}^2}{2g_0}\), then the optimal trajectory is a succession of at most two arcs, and the thrust \(\|u(\cdot)\| T_{\text{max}}\) is

- either constant on \([0, t_f]\) and equal to \(T_{\text{max}}\),
- or of the type \(T_{\text{max}} \rightarrow 0\),
- or of the type \(0 \rightarrow T_{\text{max}}\).
Main ideas of the proof:

- Application of the PMP
- The switching function \( \Phi = \frac{1}{m} \sqrt{p_{v_x}^2 + p_{v_h}^2} - \beta p_m \) satisfies:

\[
\dot{\Phi} = \frac{-p_h p_{v_h}}{m \sqrt{p_{v_x}^2 + p_{v_h}^2}}
\]

\[
\ddot{\Phi} = \frac{\beta \| u \|}{m} \dot{\Phi} - \frac{m}{\sqrt{p_{v_x}^2 + p_{v_h}^2}} \dot{\Phi}^2 + \frac{p_h^2}{m \sqrt{p_{v_x}^2 + p_{v_h}^2}}
\]

\( \Rightarrow \) \( \Phi \) has at most one minimum

\( \Rightarrow \) strategies \( T_{\text{max}}, T_{\text{max}} - 0, 0 - T_{\text{max}}, \) or \( T_{\text{max}} - 0 - T_{\text{max}} \)

- The strategy \( T_{\text{max}} - 0 - T_{\text{max}} \) cannot occur
A priori, we have:

5 unknowns
\( p_h, p_{v_x}, p_{v_h}(0), p_m(0), \) and \( t_f \)

5 equations
\[
\begin{align*}
  h(t_f) &= h_f, \\
  v_x(t_f) &= v_{xf}, \\
  v_h(t_f) &= 0, \\
  p_m(t_f) &= 1, \\
  H(t_f) &= 0
\end{align*}
\]

but using several tricks and some system analysis, the shooting method can be simplified to:

3 unknowns
\( p_{v_x}, p_{v_h}(0), \) and the first switching time \( t_1 \)

3 equations
\[
\begin{align*}
  h(t_f) &= h_f, \\
  v_x(t_f) &= v_{xf}, \\
  v_h(t_1) + g_0 t_1 &= g_0 p_{v_h}(0)
\end{align*}
\]

⇒ very easy and efficient (instantaneous) algorithm

and the initialization of the shooting method is automatic (CV for any initial adjoint vector)

⇒ automatic tool for initializing the continuation procedure
Introduction Flattening the Earth Continuation procedure Flat Earth Numerical simulations

Numerical simulations

\( T_{\text{max}} = 180 \text{ kN} \)

\( \text{Isp} = 450 \text{ s} \)

Initial conditions

\( \varphi_0 = 0 \) (SSO)

\( h_0 = 200 \text{ km} \)

\( v_0 = 5.5 \text{ km/s} \)

\( \gamma_0 = 2 \text{ deg} \)

\( m_0 = 40000 \text{ kg} \)

Final conditions

\( h_f = 800 \text{ km} \)

\( v_f = 7.5 \text{ km/s} \)

\( \gamma_f = 0 \text{ deg} \)  

(nearly circular final orbit)

Evolution of the shooting function unknowns \((p_h, p_{v_x}, p_{v_h}, p_m)\) (abscissa) with respect to homotopic parameter \(\lambda_2\) (ordinate)

\( \rightarrow \) continuous but not \(C^1\) path: \(\lambda_2 \approx 0.01, \lambda_2 \approx 0.8, \) and \(\lambda_2 \approx 0.82\):

- \(0 \leq \lambda_2 \lesssim 0.01: T_{\text{max}} - 0\)
- \(0.01 \lesssim \lambda_2 \lesssim 0.8: T_{\text{max}} - 0 - T_{\text{max}}\)
- \(0.8 \lesssim \lambda_2 \lesssim 0.82: T_{\text{max}} - 0\)
- \(0.82 \lesssim \lambda_2 \leq 1: T_{\text{max}} - 0 - T_{\text{max}}\)
Numerical simulations

Trajectory and control strategy of $(\text{OCP})_{1,0}$ (dashed) and $(\text{OCP})_{1,1}$ (plain). $t_f \approx 1483\ s$

Remark

In the case of a final orbit (no injecting point): additional continuation on transversality conditions.
Numerical simulations

Comparison with a direct method:
- Heun (RK2) discretization with $N$ points
- combination of AMPL with IPOPT
- needs however a careful initial guess

**Continuation method**

3 seconds:
- $(\text{OCP})_{0,0}$: instantaneous
- from $(\text{OCP})_{0,0}$ to $(\text{OCP})_{1,0}$: 0.5 second
- from $(\text{OCP})_{1,0}$ to $(\text{OCP})_{1,1}$: 2.5 seconds

→ Accuracy: $10^{-12}$

**Direct method**

- $N = 100$: 5 seconds
- $N = 1000$: 165 seconds

→ Accuracy: $10^{-6}$
Conclusion

- Algorithmic procedure to solve the problem of minimization of fuel consumption for the coplanar orbit transfer problem by shooting method approach
- Does not require any careful initial guess

Open questions

- Is this procedure systematically efficient, for any possible coplanar orbit transfer?
- Extension to 3D