Sparse control and stabilization to consensus of collective behavior models

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Graz, June 2015
Cucker-Smale consensus model

Prototypical model for the interaction of \( N \) agents:

\[
\begin{align*}
\dot{x}_i(t) &= v_i(t) \\
\dot{v}_i(t) &= \frac{1}{N} \sum_{j=1}^{N} \frac{v_j(t) - v_i(t)}{(1 + \|x_j(t) - x_i(t)\|^2)^\beta} \\
\end{align*}
\]

\( \beta > 0, \ x_i \in \mathbb{R}^d, \ v_i \in \mathbb{R}^d \)

\( x_i \): main state of the agent \( i \)

\( v_i \): consensus parameter of the agent \( i \)

- initially introduced to describe the formation and evolution of languages
- then for describing the flocking of a swarm of birds
Cucker-Smale consensus model

Prototypical model for the interaction of $N$ agents:

**General (finite-dimensional) Cucker-Smale model**

\[
\begin{align*}
\dot{x}_i(t) &= v_i(t), \\
\dot{v}_i(t) &= \frac{1}{N} \sum_{j=1}^{N} a(\|x_j(t) - x_i(t)\|)(v_j(t) - v_i(t)),
\end{align*}
\]

$i = 1, \ldots, N$

$x_i \in \mathbb{R}^d, v_i \in \mathbb{R}^d$

$a \in C^1([0, +\infty))$ nonincreasing positive function (interaction potential, rate of communication)

\[
a(s) = \frac{1}{(1+s^2)^\beta}
\]

in the classical Cucker-Smale model

There are other classes of models, cf social dynamics...

Bellomo, Bertozi, Boudin, Carrillo, Couzin, Cristiani, Cucker, Degond, Desvillettes, Dong, Fornasier, Frasca, Ha, Haskovec, Kim, Lee, Leonard, Levy, Mordecki, Motsch, Parrish, Perthame, Piccoli,Salvarani, Smale, Tadmor, Toscani, Tosin, Vecil, Vicsek,........
Cucker-Smale consensus model

Prototypical model for the interaction of $N$ agents:

**General (finite-dimensional) Cucker-Smale model**

\[
\begin{align*}
\dot{x}_i(t) &= v_i(t), \\
\dot{v}_i(t) &= \frac{1}{N} \sum_{j=1}^{N} a(\|x_j(t) - x_i(t)\|)(v_j(t) - v_i(t)), \\
\end{align*}
\]

$i = 1, \ldots, N$

$x_i \in \mathbb{R}^d$, $v_i \in \mathbb{R}^d$

$a \in C^1([0, +\infty))$ nonincreasing positive function (interaction potential, rate of communication)

\[a(s) = \frac{1}{(1+s^2)^\beta}\] in the classical Cucker-Smale model

Objectives:

- Model **self-organization and consensus emergence** in a group of agents
- **Organization via intervention** (social forces), enforce or facilitate pattern formation or convergence to consensus
Cucker-Smale consensus model

\[
\begin{align*}
\dot{x}_i(t) &= v_i(t) \\
\dot{v}_i(t) &= \frac{1}{N} \sum_{j=1}^{N} a(\|x_j(t) - x_i(t)\|)(v_j(t) - v_i(t)) & i = 1, \ldots, N
\end{align*}
\]

Mean consensus vector: \( \bar{v} = \frac{1}{N} \sum_{i=1}^{N} v_i(t) \rightarrow \text{constant} \)
Cucker-Smale consensus model

\[
\dot{x}_i(t) = v_i(t) \\
\dot{v}_i(t) = \frac{1}{N} \sum_{j=1}^{N} a(\|x_j(t) - x_i(t)\|)(v_j(t) - v_i(t)) \
i = 1, \ldots, N
\]

With matrix notations:

\[
\dot{v}_i = \frac{1}{N} \left( \sum_{j=1}^{N} a_{ij}(v_j - v_i) \right) = \frac{1}{N} \left( (Av)_i - \left( \sum_{j=1}^{N} a_{ij} \right)v_i \right)
\]

hence

\[
\dot{x} = v, \quad \dot{v} = -Lv
\]

with

\[
L = \frac{1}{N} (D - A) \quad \text{Laplacian,} \quad D = \text{diag} \left( \sum_{j=1}^{N} a_{ij} \right)
\]

N.B.: \( L \geq 0 \) and \( L(v, \ldots, v) = 0 \)
Cucker-Smale consensus model

\[ \dot{x}_i(t) = v_i(t) \]

\[ \dot{v}_i(t) = \frac{1}{N} \sum_{j=1}^{N} a(\|x_j(t) - x_i(t)\|)(v_j(t) - v_i(t)) \quad i = 1, \ldots, N \]

\((\mathbb{R}^d)_N = V_f \oplus V_\perp\) with \(V_f = \{(v_1, \ldots, v_N) \in (\mathbb{R}^d)_N \mid v_1 = \cdots = v_N \in \mathbb{R}^d\}\)

\(V_\perp = \{(v_1, \ldots, v_N) \in (\mathbb{R}^d)_N \mid \sum_{i=1}^{N} v_i = 0\}\)

\[ X(t) = \frac{1}{2N^2} \sum_{i,j=1}^{N} \|x_i(t) - x_j(t)\|^2 \]

\[ V(t) = \frac{1}{2N^2} \sum_{i,j=1}^{N} \|v_i(t) - v_j(t)\|^2 = \frac{1}{N} \sum_{i=1}^{N} \|v_i(t) - \bar{v}\|^2 = \frac{1}{N} \sum_{i=1}^{N} \|v(t)_{\perp_i}\|^2 \]

\[ \dot{V}(t) = -\frac{1}{N} \sum_{i,j=1}^{N} a(\|x_j(t) - x_i(t)\|)\|v_i(t) - v_j(t)\|^2 \leq 0 \]
Cucker-Smale consensus model

\[ \dot{x}_i(t) = v_i(t) \]

\[ \dot{v}_i(t) = \frac{1}{N} \sum_{j=1}^{N} a(\|x_j(t) - x_i(t)\|)(v_j(t) - v_i(t)) \quad i = 1, \ldots, N \]

Definitions

- **Consensus point** = steady configuration = \((x, v) \in (\mathbb{R}^d)^N \times V_f\).
- **Consensus manifold** = \((\mathbb{R}^d)^N \times V_f\).
- \((x(t), v(t))\) tends to **consensus** (or **flocking**) if \(v_i(t) \xrightarrow{t \to +\infty} \bar{v} \quad \forall i = 1, \ldots, N\), or, equivalently, if \(V(t) \xrightarrow{t \to +\infty} 0\).
Cucker-Smale consensus model

\[ \dot{x}_i(t) = v_i(t) \]
\[ \dot{v}_i(t) = \frac{1}{N} \sum_{j=1}^{N} a(\|x_j(t) - x_i(t)\|)(v_j(t) - v_i(t)) \quad i = 1, \ldots, N \]

Note that

\[ \dot{V}(t) = -\frac{1}{N} \sum_{i,j=1}^{N} a(\|x_j(t) - x_i(t)\|)\|v_i(t) - v_j(t)\|^2 \]
\[ \leq -2a(\sqrt{2NX(t)}) V(t) \]

and hence \textit{if } \text{X}(t) \text{ remains bounded} \text{ then } \dot{V} \leq -\alpha V, \text{ whence flocking.} \]
Cucker-Smale consensus model

\[ \dot{x}_i(t) = v_i(t) \]
\[ \dot{v}_i(t) = \frac{1}{N} \sum_{j=1}^{N} a(\|x_j(t) - x_i(t)\|)(v_j(t) - v_i(t)) \]
\[ i = 1, \ldots, N \]

Proposition (Ha-Ha-Kim, 2010)

Let \((x_0, v_0) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N\) be such that

\[ \int_{\sqrt{X(0)}}^{+\infty} a(\sqrt{2Nr}) \, dr \geq \sqrt{V(0)}. \]

Then the solution with initial data \((x_0, v_0)\) tends to consensus.

→ self-organization of the group in this so-called consensus region (region of natural asymptotic stability to consensus). Sharp estimate.

Example: \(a(x) = \frac{1}{(1+x^2)^\beta}\) with \(0 < \beta \leq 1/2\) (initial model by Cucker-Smale, 2007)

→ globally stable
Kinetic Cucker-Smale model

By passing to the limit (hydrodynamic, or mean-field limit):

\[
\partial_t \mu + \langle \mathbf{v}, \nabla_x \mu \rangle + \text{div}_\mathbf{v} (\xi[\mu] \mu) = 0
\]

- \(\mu(t)\): probability measure on \(\mathbb{R}^d \times \mathbb{R}^d\)
  (if \(\mu(t, x, v) = f(t, x, v) \, dx \, dv\), then \(f\) is the density of the crowd)
- \(\xi[\mu]\): interaction kernel

\[
\xi[\mu](x, v) = \int_{\mathbb{R}^d \times \mathbb{R}^d} a(\|x - y\|)(w - v) \, d\mu(y, w)
\]

Link with finite dimension:

\[
\mu(t) = \frac{1}{N} \sum_{i=1}^{N} \delta(x_i(t), v_i(t))
\]
Kinetic Cucker-Smale model

\[ \partial_t \mu + \langle \nu, \nabla \mu \rangle + \text{div}_v (\xi[\mu] \mu) = 0 \]

Space and velocity barycenters:

\[
\bar{x}(t) = \int_{\mathbb{R}^d \times \mathbb{R}^d} x \, d\mu(t)(x, \nu), \quad \bar{v}(t) = \int_{\mathbb{R}^d \times \mathbb{R}^d} v \, d\mu(t)(x, \nu)
\]

Then:

\[ \dot{\bar{x}}(t) = \bar{v} \quad \bar{v} = \text{Cst} \]

Variance:

\[ V(t) = \int_{\mathbb{R}^d \times \mathbb{R}^d} |v - \bar{v}|^2 \, d\mu(t) \]
Kinetic Cucker-Smale model

\[ \partial_t \mu + \langle v, \nabla_x \mu \rangle + \text{div}_v (\xi[\mu] \mu) = 0 \]

A solution \( \mu \in C^0(\mathbb{R}, \mathcal{P}_c(\mathbb{R}^d \times \mathbb{R}^d)) \) converges to flocking if:

1. \( \exists X_M > 0 \mid \text{supp}(\mu(t)) \subseteq B(\vec{x}(t), X_M) \times \mathbb{R}^d \ \forall t > 0 \)
2. \( V(t) = \int_{\mathbb{R}^d \times \mathbb{R}^d} |v - \bar{v}|^2 \, d\mu(t) \xrightarrow{t \to +\infty} 0 \)

\( \rightarrow \) similar results: consensus region...
When consensus is not achieved by self-organization: is it possible to control the group to consensus by means of an external action?

→ organization via intervention

**Controlled Cucker-Smale model**

\[
\begin{align*}
\dot{x}_i(t) &= v_i(t) \\
\dot{v}_i(t) &= \frac{1}{N} \sum_{j=1}^{N} a(\|x_j(t) - x_i(t)\|)(v_j(t) - v_i(t)) + u_i(t) \\
\end{align*}
\]

\[i = 1, \ldots, N\]

with

\[\sum_{i=1}^{N} \|u_i(t)\| \leq M\]

for \(M > 0\) fixed.

(Caponigro Fornasier Piccoli Trélat, M3AS 2015)
Control of the Cucker-Smale model

\[ \dot{x}_i(t) = v_i(t) \]

\[ \dot{v}_i(t) = \frac{1}{N} \sum_{j=1}^{N} a(\|x_j(t) - x_i(t)\|)(v_j(t) - v_i(t)) + u_i(t) \]

with \( \sum_{i=1}^{N} \|u_i(t)\| \leq M \)

Objectives:

- Design a \textit{feedback control} steering ”optimally” the system to consensus, with:
  (i) a minimal amount of components active at each time
  (ii) a minimal amount of switchings in time
- Control the system to any prescribed consensus
Control of the Cucker-Smale model

\[ \dot{x}_i(t) = v_i(t) \]

\[ \dot{v}_i(t) = \frac{1}{N} \sum_{j=1}^{N} a(\|x_j(t) - x_i(t)\|)(v_j(t) - v_i(t)) + u_i(t) \quad i = 1, \ldots, N \]

with \( \sum_{i=1}^{N} \|u_i(t)\| \leq M \)

Objectives:

- Design a feedback control steering "optimally" the system to consensus, with:
  (i) a minimal amount of components active at each time
  (ii) a minimal amount of switchings in time
- Control the system to any prescribed consensus

→ concept of sparse control
Control of the Cucker-Smale model

\[ \dot{x}_i(t) = v_i(t) \]

\[ \dot{v}_i(t) = \frac{1}{N} \sum_{j=1}^{N} a(\|x_j(t) - x_i(t)\|)(v_j(t) - v_i(t)) + u_i(t) \]

\[ i = 1, \ldots, N \]

with \( \sum_{i=1}^{N} \|u_i(t)\| \leq M \)

Objectives:

- Design a feedback control steering ”optimally” the system to consensus, with:
  
  (i) a minimal amount of components active at each time → componentwise sparse control
  
  (ii) a minimal amount of switchings in time → time sparse control
  
- Control the system to any prescribed consensus

→ concept of sparse (feedback) control
Control of the kinetic Cucker-Smale model

Controlled kinetic Cucker-Smale model

\[ \partial_t \mu + \langle v, \nabla_x \mu \rangle + \text{div}_v ((\xi[\mu] + \chi \omega u) \mu) = 0 \]

with \( u \in L^\infty(\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d) \) and \( \omega(t) \subset \mathbb{R}^d \) measurable, s.t.

\[ \|u(t, \cdot, \cdot)\|_{L^\infty(\mathbb{R}^d \times \mathbb{R}^d)} \leq 1 \quad \text{(bounded external action)} \]

and

\[ \mu(t)(\omega(t)) = \int_{\omega(t)} d\mu(t)(x, v) \leq c \]

(\text{one is allowed to act only on a given proportion } c \text{ of the crowd})

Variant:

\[ |\omega(t)| = \int_{\omega(t)} dx \, dv \leq c \]

(limit on the space of configurations)
\[ \begin{align*}
\dot{x}_i(t) &= v_i(t) \\
\dot{v}_i(t) &= \frac{1}{N} \sum_{j=1}^{N} a(\|x_j(t) - x_i(t)\|)(v_j(t) - v_i(t)) + u_i(t) \\
\text{with } \sum_{i=1}^{N} \|u_i(t)\| &\leq M
\end{align*} \]

with \(\alpha > 0\) small enough, stabilizes the system to consensus (in infinite time)

Indeed:
\[ \dot{V} \leq -\frac{2}{N} \sum_i \langle v_{\perp i}, u_i \rangle = -2\alpha V \]

BUT this control is far from sparse!
Sparse stabilization of the Cucker-Smale model

\[ \dot{x}_i(t) = v_i(t) \]

\[ \dot{v}_i(t) = \frac{1}{N} \sum_{j=1}^{N} a(\|x_j(t) - x_i(t)\|)(v_j(t) - v_i(t)) + u_i(t) \]

with \( \sum_{i=1}^{N} \|u_i(t)\| \leq M \)

Idea:

\[ \min_{\sum_{i=1}^{N} \|u_i\| \leq M} \left( \frac{1}{2N^2} \sum_{i,j=1}^{N} \langle v_i - v_j, u_i - u_j \rangle \right) = \min_{\sum_{i=1}^{N} \|u_i\| \leq M} \left( \frac{1}{N} \sum_{i=1}^{N} \langle v_{\perp i}, u_{\perp i} \rangle \right) \]

\[ \Rightarrow u = -\alpha v_{\perp} \]
Sparse stabilization of the Cucker-Smale model

\[
\dot{x}_i(t) = v_i(t)
\]
\[
\dot{v}_i(t) = \frac{1}{N} \sum_{j=1}^{N} a(\|x_j(t) - x_i(t)\|)(v_j(t) - v_i(t)) + u_i(t) \\
i = 1, \ldots, N
\]

with \( \sum_{i=1}^{N} \|u_i(t)\| \leq M \)

Idea:

\[
\min_{\sum_{i=1}^{N} \|u_i\| \leq M} \left( \frac{1}{2N^2} \sum_{i,j=1}^{N} \langle v_i - v_j, u_i - u_j \rangle + \gamma(X) \frac{1}{N} \sum_{i=1}^{N} \|u_i\| \right),
\]

where

\[
\gamma(X) = \int_{\sqrt{X}}^{+\infty} a(\sqrt{2Nr}) \, dr
\]

- \( \ell^1 \) norm \( \Rightarrow \) enforce sparsity
- \( \gamma(X) \) threshold \( \Rightarrow \) the control switches off when entering the consensus region
Sparse stabilization of the Cucker-Smale model

\[ \dot{x}_i(t) = v_i(t) \]

\[ \dot{v}_i(t) = \frac{1}{N} \sum_{j=1}^{N} a(\|x_j(t) - x_i(t)\|)(v_j(t) - v_i(t)) + u_i(t) \]

\[ i = 1, \ldots, N \]

with \( \sum_{i=1}^{N} \|u_i(t)\| \leq M \)

This leads to the componentwise sparse feedback control \( u \):

- if \( \max_{1 \leq i \leq N} \|v_{\perp i}(t)\| \leq \gamma(X(t))^2 \), then \( u(t) = 0 \)

- if \( \|v_{\perp j}(t)\| = \max_{1 \leq i \leq N} \|v_{\perp i}(t)\| > \gamma(X(t))^2 \) (with \( j \) be the smallest index) then

\[ u_j(t) = -M \frac{v_{\perp j}(t)}{\|v_{\perp j}(t)\|}, \quad \text{and} \quad u_i(t) = 0 \quad \text{for every } i \neq j \]

**Theorem**

This feedback control stabilizes the system to consensus
Sparse stabilization of the Cucker-Smale model

\[
\begin{align*}
\dot{x}_i(t) &= v_i(t) \\
\dot{v}_i(t) &= \frac{1}{N} \sum_{j=1}^{N} a(\|x_j(t) - x_i(t)\|)(v_j(t) - v_i(t)) + u_i(t) \\
&\quad \text{for } i = 1, \ldots, N
\end{align*}
\]

with \( \sum_{i=1}^{N} \|u_i(t)\| \leq M \)

**Theorem**

This feedback control stabilizes the system to consensus

Indeed:

\[
\dot{V} \leq \frac{2}{N} \sum_{i} \langle v_{\perp i}, u_i \rangle = -2 \frac{M}{N} \|v_{\perp j}\|
\]

with \( \|v_{\perp j}\| = \max_{1 \leq i \leq N} \|v_{\perp i}\| \geq \sqrt{V} \) \implies \dot{V} \leq -2 \frac{M}{N} \sqrt{V}

hence in finite time we enter the consensus region, and then \( u = 0 \) (forever)
\[ \dot{x}_i(t) = v_i(t) \]
\[ \dot{v}_i(t) = \frac{1}{N} \sum_{j=1}^{N} a(\|x_j(t) - x_i(t)\|)(v_j(t) - v_i(t)) + u_i(t) \quad i = 1, \ldots, N \]

with \( \sum_{i=1}^{N} \|u_i(t)\| \leq M \)

**Remarks**

- This feedback control is componentwise sparse.
- This feedback control is however not necessarily time sparse: it may chatter

\[
\rightarrow \text{To avoid possible chattering in time: sampling in time (sample-and-hold)}
\]

\[\Rightarrow \text{we obtain a time sparse and componentwise sparse feedback control}\]
Proposition

For every time $t$, $u(t)$ minimizes $\frac{d}{dt} V(t)$ over all possible feedbacks controls.

In other words, the feedback control $u(t)$ is the best choice in terms of the rate of convergence to consensus

→ ”Sparse is better”:

A policy maker, who is not allowed to have prediction on future developments, should always consider more favorable to intervene with stronger actions on the fewest possible instantaneous optimal leaders than trying to control more agents with minor strength.
3 agents in flocking position:

\[ x_1 = (2, 0), \quad v_1 = (5, 0) \]
\[ x_2 = (1, -2), \quad v_2 = (5, 0) \]
\[ x_3 = (0, 1), \quad v_3 = (5, 0) \]

plus one agent to be controlled:

\[ x_4 = (5, -5), \quad v_4 = (0, 5) \]

--- uncontrolled trajectories
--- controlled trajectories
**** active control
3 agents in flocking position:

\[ x_1 = (2, 0), \quad v_1 = (5, 0) \]
\[ x_2 = (1, -2), \quad v_2 = (5, 0) \]
\[ x_3 = (0, 1), \quad v_3 = (5, 0) \]

plus 2 agents to be controlled:

\[ x_4 = (5, -5), \quad v_4 = (0, 5) \]
\[ x_5 = (5, -1), \quad v_5 = (-5, 0) \]
20 agents
random initial positions (on a circle)
random initial speeds (sufficiently large so that the uncontrolled system does not flock)

- uncontrolled trajectories
- controlled trajectories
- active control

Other videos: 10 agents, 50 agents
Kinetic Cucker-Smale model

**Theorem**

\[ \forall \mu^0 \in \mathcal{P}_c^{ac}(\mathbb{R}^d \times \mathbb{R}^d) \quad \exists \chi_\omega u \text{ satisfying } \|u(t, \cdot, \cdot)\|_{L^\infty(\mathbb{R}^d \times \mathbb{R}^d)} \leq 1 \text{ and } \]

\[ \mu(t)(\omega(t)) = \int_{\omega(t)} d\mu(t)(x, v) \leq c \quad \text{or} \quad |\omega(t)| = \int_{\omega(t)} dx \, dv \leq c \]

and \[ \exists! \mu \in C^0(\mathbb{R}; \mathcal{P}_c^{ac}(\mathbb{R}^d \times \mathbb{R}^d)) \text{ solution, s.t. } \mu(0) = \mu^0, \text{ and converging to flocking} \]

**Remarks**

- \( \mu \) remains AC and of compact support (but becomes singular in infinite time)
- the control \( \chi_\omega u \) is designed in an explicit way, and
  - \( \omega(t) \) piecewise constant (p.c.) in \( t \)
  - \( u(t, x, v) \) p.c. in \( t \) for \( (x, v) \) fixed, \( C^0 \) and piecewise linear in \( (x, v) \) for \( t \) fixed
  - \( \forall \mu^0 \in \mathcal{P}_c^{ac}(\mathbb{R}^d \times \mathbb{R}^d) \quad \exists T(\mu^0) \geq 0 \quad | \quad \forall t > T(\mu^0) \quad u(t, x, v) = 0 \)
Kinetic Cucker-Smale model

Theorem

\[ \forall \mu^0 \in \mathcal{P}^{ac}_c(\mathbb{R}^d \times \mathbb{R}^d) \quad \exists \chi \omega u \text{ satisfying } \|u(t, \cdot, \cdot)\|_{L^\infty(\mathbb{R}^d \times \mathbb{R}^d)} \leq 1 \text{ and } \]

\[
\mu(t)(\omega(t)) = \int_{\omega(t)} d\mu(t)(x, v) \leq c \quad \text{or} \quad |\omega(t)| = \int_{\omega(t)} dx \, dv \leq c
\]

and \[ \exists! \mu \in C^0(\mathbb{R}; \mathcal{P}^{ac}_c(\mathbb{R}^d \times \mathbb{R}^d)) \text{ solution, s.t. } \mu(0) = \mu^0, \text{ and converging to flocking } \]

\[
\partial_t \mu + \text{div}(x, v) \left( V_{\omega, u}[\mu] \right) = 0 \quad \text{with} \quad V_{\omega, u}[\mu] = \left( \xi[\mu] + \chi \omega u \right)
\]

Controlled particle flow \( \Phi_{\omega, u}(t) \) generated by \( V_{\omega, u}[\mu(t)] \): characteristics

\[
\dot{x}(t) = v(t), \quad \dot{v}(t) = \xi[\mu(t)](x(t), v(t)) + \chi \omega(t) u(t, x(t), v(t))
\]

\[ \Rightarrow \mu(t) = \Phi_{\omega, u}(t) \# \mu^0 \]

Shepherd control design strategy: choose \( \omega(t) \) and \( u(t) \) such that \( V_{\omega, u}[\mu(t)] \) points inwards the invariant domain (confining the population) i.e., the size of \( \text{supp}_v(\mu(t)) \) decreases (exponentially) in time
In finite dimension only:

**Proposition**

Local controllability holds at almost every point of the consensus manifold. Moreover, controllability can be realized with time sparse and componentwise sparse controls.

**Corollary**

Any point of \((\mathbb{R}^d)^N \times (\mathbb{R}^d)^N\) can be steered to almost any point of the consensus manifold in finite time by means of a time sparse and componentwise sparse control.

(by stabilization and then iterated local controllability along a path of consensus points)
Another way of designing sparse controls: by optimal control

Optimal control problem with a fixed initial point and free final point, where the cost to be minimized is

$$\int_0^T \sum_{i=1}^N \left( v_i(t) - \frac{1}{N} \sum_{j=1}^N v_j(t) \right)^2 dt + \gamma \sum_{i=1}^N \int_0^T \| u_i(t) \| dt$$

where $\gamma > 0$ is fixed, under the constraint $\sum_{i=1}^N \| u_i(t) \| \leq M$.

The $\ell^1$-norm in the red term implies componentwise sparsity features of the optimal control.

(proof by applying the Pontryagin maximum principle + genericity arguments)
Generalizations

There are many generalizations of the Cucker-Smale model:

- **general potentials (friction, attraction/repulsion, ...):**
  Cucker, Dong, Ha, Ha, Kim, Leonard, Motsch, Slemrod, Tadmor

- **stochastic aspects (adding noise):** Carrillo, Cucker, Fornasier, Ha, Lee, Levy, Mordecki, Toscani

- **delay:** ongoing work with Cristina Pignotti

- **Models in infinite dimension (hydrodynamic, kinetic, mean field limit):**
  Carrillo, Degond, Fornasier, Ha, Hascoët, Motsch, Rosado, Tadmor, Toscani

Other open problems:
cluster control, control of opinion formation, black swan, cancerology