Optimal control theory and some applications to aerospace problems

Emmanuel Trélat

Abstract. In this proceedings article we first shortly report on some classical techniques of nonlinear optimal control such as the Pontryagin Maximum Principle and the conjugate point theory, and on their numerical implementation. We illustrate these issues with problems coming from aerospace applications such as the orbit transfer problem which is taken as a motivating example. Such problems are encountered in a longstanding collaboration with the european space industry *EADS Astrium*. On this kind of nonacademic problem it is shown that the knowledge resulting from the maximum principle is insufficient for solving adequately the problem, in particular due to the difficulty of initializing the shooting method, which is an approach for solving the boundary value problem resulting from the application of the maximum principle. On the orbit transfer problem we show how the shooting method can be successfully combined with a numerical continuation method in order to improve significantly its performances. We comment on assumptions ensuring the feasibility of continuation or homotopy methods , which consist of deforming continuously a problem towards a simpler one, and then of solving a series of parametrized problems to end up with the solution of the initial problem. Finally, in view of designing low cost interplanetary space missions, we show how optimal control can be also combined with dynamical system theory, which allows to put in evidence nice properties of the celestial dynamics around Lagrange points that are of great interest for mission design.

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1. Introduction

Let n and m be nonzero integers. Consider on $\mathbb{R}^n$ the control system

$$\dot{x}(t) = f(t, x(t), u(t)), \tag{1}$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^n$ is smooth, and where the controls are bounded and measurable functions, defined on intervals $[0, T(u)]$ of $\mathbb{R}^+$, and taking their values in a subset U of $\mathbb{R}^m$. Let M_0 and M_1 be two subsets of $\mathbb{R}^n$. Denote by $\mathcal{U}$ the set of *admissible controls*, so that the corresponding trajectories steer the system from an initial point of M_0 to a final point in M_1. For such a control u, the *cost* of the corresponding trajectory $x_u(\cdot)$ is defined by

$$C(t_f, u) = \int_0^{t_f} f^0(t, x_u(t), u(t))dt + g(t_f, x_u(t_f)), \tag{2}$$

where $f^0 : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}$ and $g : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ are smooth. We investigate the *optimal control problem* of determining a trajectory $x_u(\cdot)$ solution of (1), associated with a control u on $[0, t_f]$, such that $x_u(0) \in M_0$, $x_u(t_f) \in M_1$, and minimizing the cost C. The final time t_f can be fixed or not.

When the optimal control problem has a solution, we say that the corresponding control (or the corresponding trajectory) is minimizing or optimal.

As a motivating example, we consider the orbit transfer problem with low thrust, modeled with the controlled Kepler equations

$$\ddot{q}(t) = -q(t)\frac{\mu}{r(t)^3} + \frac{T(t)}{m(t)}, \quad \dot{m}(t) = -\beta\|T(t)\|, \tag{3}$$

where $q(t) \in \mathbb{R}^3$ is the position of the engine at time t, $r(t) = \|q(t)\|$, $T(t)$ is the thrust at time t, and $m(t)$ is the mass, with $\beta = 1/I_{\text{sp}}g_0$. Here g_0 is the usual gravitational constant and I_{sp} is the specific impulsion of the engine. The thrust is submitted to the constraint $\|T(t)\| \leqslant T_{\max}$, where the typical value of the maximal thrust $T_{\max}$ is around 0.1 Newton, for low-thrust engines. The orbit transfer problem consists of steering the engine from a given initial orbit (e.g. an initial excentric inclinated orbit) to a final one (e.g., the geostationary orbit), either in minimal time or by minimizing the fuel consumption. This is an optimal control problem of the form settled above, where the state is then $x = (q, \dot{q}, m) \in \mathbb{R}^7$, the control is the thrust, the set U of constraints on the control is the closed ball of $\mathbb{R}^3$ centered at the origin and with radius $T_{\max}$. If one considers the minimal time problem, then one can choose $f^0 = 1$ and $g = 0$, and if one considers the minimal fuel consumption problem, then one can choose $f^0(t, x, u) = \beta\|u\|$ and $g = 0$. Note that controllability properties, ensuring the feasibility of the problem, have been studied in [10, 13], based on a careful analysis of the Lie algebra generated by the vector fields of the system (3).

The purpose of this proceedings article is to shortly report on some of the main issues of optimal control theory, with a special attention to applications to aerospace problems. We will show that the most classical techniques of optimal control, namely, the Pontryagin Maximum Principle, the conjugate point theory, and the associated numerical methods, are in general insufficient to solve efficiently a given optimal control problem. They can however be significantly improved by combining them with other tools like numerical continuation (homotopy) methods, geometric optimal control, or results of dynamical system theory. These items will be illustrated with the minimal time or minimal consumption orbit transfer problem with strong or low thrust (mentioned above), and space mission design using the dynamics around Lagrange points. We mention the recent survey [50] for a more detailed exposition of optimal control applied to aerospace.

2. Classical optimal control theory and numerical approaches

Throughout this section, we assume that the optimal control problem (1)-(2) has an optimal solution. Note that there exists a large literature on existence results for optimal controls. Such results usually require some convexity assumptions on the dynamics (see e.g. [18]). Here, we are not concerned with such issues. The aim of this section is to provide the most classical first- and second-order necessary and/or sufficient conditions allowing one to characterize and compute optimal trajectories.

2.1. First-order necessary optimality conditions. The set of admissible controls on $[0, t_f]$ is denoted $\mathcal{U}_{t_f,\mathbb{R}^m}$, and the set of admissible controls on $[0, t_f]$ taking their values in U is denoted $\mathcal{U}_{t_f,U}$.

Definition 2.1. The *end-point mapping* $E: \mathbb{R}^n \times \mathbb{R}^+ \times \mathcal{U} \to \mathbb{R}^n$ of the system is defined by $E(x_0, T, u) = x(x_0, T, u)$, where $t \mapsto x(x_0, t, u)$ is the trajectory solution of (1), corresponding to the control u, such that $x(x_0, 0, u) = x_0$.

The set $\mathcal{U}_{t_f,\mathbb{R}^m}$, endowed with the standard topology of $L^\infty([0, t_f], \mathbb{R}^m)$, is open, and the end-point mapping is smooth on $\mathcal{U}_{t_f,\mathbb{R}^m}$. In terms of the end-point mapping, the optimal control problem (1)-(2) can be recast as the infinite dimensional minimization problem

$$\min\{C(t_f, u) \mid x_0 \in M_0,\ E(x_0, t_f, u) \in M_1, u \in L^\infty(0, t_f; U)\}. \tag{4}$$

Although it is in infinite dimension, this is a classical optimization problem with constraints. It is well-known that in such a problem it is usually important that the set of constraints have (at least locally) the structure of a manifold. This is one of the motivations of the following definition.

Definition 2.2. Assume that $M_0 = \{x_0\}$. A control u defined on $[0, t_f]$ is said singular if and only if the differential $\frac{\partial E}{\partial u}(x_0, t_f, u)$ is not of full rank.

Singular controls are one of the main notions in optimal control theory. Note that, in the above constrained minimization problem, the set of constraints is a local manifold around a given control u provided u is nonsingular.

Assume temporarily that we are in the simplified situation where $M_0 = \{x_0\}$, $M_1 = \{x_1\}$, T is fixed, $g = 0$ and $U = \mathbb{R}^m$. The optimal control problem the consists of steering the system (1) from the initial point x_0 to the final point x_1 in time T and minimizing the cost (2) among controls $u \in L^\infty([0, T], \mathbb{R}^m)$. Assuming that the extremities are fixed is not a big simplification and it is not difficult to extend the following statements to the case of general subsets. Here, the main (important) simplification is the fact that the controls are unconstrained. In that case, the optimization problem (4) reduces to

$$\min_{E(x_0,T,u)=x_1} C(T, u). \tag{5}$$

If u is optimal then there exists a Lagrange multiplier $(\psi, \psi^0) \in \mathbb{R}^n \times \mathbb{R} \setminus \{0\}$ such that

$$\psi.dE_{x_0,T}(u) = -\psi^0 dC_T(u), \tag{6}$$

written equivalently as $\frac{\partial L_T}{\partial u}(u,\psi,\psi^0) = 0$ where $L_T(u,\psi,\psi^0) = \psi E_{x_0,T}(u) + \psi^0 C_T(u)$ is the so-called Lagrangian of the optimization problem (5). It can be noted that with such an approach it would be more difficult to take into account control (or even state) constraints. One of the main contributions of Pontryagin and his collaborators is the consideration of needle-like variations, allowing one to deal efficiently with control constraints. The Pontryagin Maximum Principle, which is the milestone of optimal control theory, is a far-reaching issue of the first-order necessary condition (6). In some sense it is a parametrization of (6) punctually along the trajectory. The following statement is the most usual Pontryagin Maximum Principle, valuable for general nonlinear optimal control problems (1)-(2), with control constraints but without state constraint. Usual proofs rely on a fixed point argument and on the use of Pontryagin cones (see e.g. [42, 35]).

Theorem 2.3 (Pontryagin Maximum Principle). *If the trajectory $x(\cdot)$, associated to the optimal control u on $[0,t_f]$, is optimal, then it is the projection of an extremal $(x(\cdot),p(\cdot),p^0,u(\cdot))$ (called extremal lift), where $p^0 \leqslant 0$ and $p(\cdot):[0,t_f]\to \mathbb{R}^n$ is an absolutely continuous mapping called adjoint vector, with $(p(\cdot),p^0)\neq(0,0)$, such that*

$$\dot{x}(t) = \frac{\partial H}{\partial p}(t,x(t),p(t),p^0,u(t)),\ \dot{p}(t) = -\frac{\partial H}{\partial x}(t,x(t),p(t),p^0,u(t)),$$

almost everywhere on $[0,t_f]$, where $H(t,x,p,p^0,u) = \langle p, f(t,x,u)\rangle + p^0 f^0(t,x,u)$ is called the Hamiltonian of the optimal control problem, and there holds

$$H(t,x(t),p(t),p^0,u(t)) = \max_{v\in U} H(t,x(t),p(t),p^0,v) \tag{7}$$

almost everywhere on $[0,t_f]$. Moreover, if the final time t_f to reach the target M_1 is not fixed, then one has the condition at the final time t_f

$$\max_{v\in U} H(t_f,x(t_f),p(t_f),p^0,v) = -p^0\frac{\partial g}{\partial t}(t_f,x(t_f)). \tag{8}$$

Additionally, if M_0 and M_1 (or just one of them) are submanifolds of $\mathbb{R}^n$ locally around $x(0)\in M_0$ and $x(t_f)\in M_1$, then the adjoint vector can be built in order to satisfy the transversality conditions at both extremities (or just one of them)

$$p(0)\perp T_{x(0)}M_0, \quad p(t_f) - p^0\frac{\partial g}{\partial x}(t_f,x(t_f)) \perp T_{x(t_f)}M_1, \tag{9}$$

where T_xM_i denotes the tangent space to M_i at the point x.

The relation between the Lagrange multipliers and $(p(\cdot),p^0)$ is that the adjoint vector can be constructed so that $(\psi,\psi^0) = (p(t_f),p^0)$ up to some multiplicative scalar. In particular, the Lagrange multiplier ψ is unique (up to a multiplicative scalar) if and only if the trajectory $x(\cdot)$ admits a unique extremal lift (up to a multiplicative scalar).

If $p^0<0$ then the extremal is said *normal*, and in this case, since the Lagrange multiplier is defined up to a multiplicative scalar, it is usual to normalize it so that $p^0=-1$. If $p^0=0$ then the extremal is said *abnormal*.

It can be also noted (using (6)) that, in the absence of control constraints, abnormal extremals project exactly onto singular trajectories. The scalar p^0 is a Lagrange multiplier associated with the instantaneous cost. Abnormal extremals, corresponding to $p^0 = 0$, are not detected with the usual Calculus of Variations approach, because this approach postulates at the very beginning that, in a neighborhood of some given reference trajectory, there are other trajectories having the same terminal points, whose respective costs can be compared (and this leads to Euler-Lagrange equations). But this postulate fails whenever the reference trajectory is isolated: it may indeed happen that there is only one trajectory joining the terminal points under consideration. A typical situation where this phenomenon occurs is when there is a unique trajectory joining the desired extremities: then obviously it will be optimal, for any possible optimization criterion. In this case the trajectory is singular and the corresponding extremal is abnormal. In many situations, where some qualification conditions hold, abnormal extremals do not exist in the problem under consideration, but in general it is impossible to say whether, given some initial and final conditions, these qualification conditions hold or not.

Finally, we mention that in the normal case the Lagrange multiplier ψ (or the adjoint vector $p(t_f)$ at the final time) coincides up to some multiplicative scalar with the gradient of the value function (solution of a Hamilton-Jacobi equation); see e.g. [22] for precise results.

Remark 2.4. The Pontryagin Maximum Principle withstands many possible generalizations: intrinsic version on manifolds (see [2]), wider classes of functionals and boundary conditions, delayed systems, hybrid or nonsmooth systems, etc. An important extension is the case of state constraints (see [21, 32]): in that case the adjoint vector becomes a bounded variation measure and may have some jumps when the trajectory meets the boundary of the allowed state domain.

In practice in order to compute optimal trajectories with the Pontryagin Maximum Principle the first step is to make explicit the maximization condition. Under the usual strict Legendre assumption, that is, the Hessian $\frac{\partial^2 H}{\partial u^2}(t, x, p, p^0, u)$ is negative definite, a standard implicit function argument allows one to express (locally) the optimal control u as a function of x and p. To simplify, below we assume that we are in the normal case ($p^0 = -1$). Plugging the resulting expression of the control in the Hamiltonian equations, and defining $H_r(t, x, p) = H(t, x, p, -1, u(x, p))$, it follows that every normal extremal is solution of

$$\dot{x}(t) = \frac{\partial H_r}{\partial p}(t, x(t), p(t)), \quad \dot{p}(t) = -\frac{\partial H_r}{\partial x}(t, x(t), p(t)). \tag{10}$$

The exponential mapping is then defined by $\exp_{x_0}(t, p_0) = x(t, x_0, p_0)$, where the solution of (10) starting from (x_0, p_0) at $t = 0$ is denoted as $(x(t, x_0, p_0), p(t, x_0, p_0))$. It parametrizes the (normal) extremal flow, and is the natural extension to optimal control theory of the Riemannian exponential mapping. In Riemannian geometry the extremal equations correspond to the cotangent formulation of the geodesics equations. Note as well that the equations (10) are the cotangent version of the usual Euler-Lagrange equations of the calculus of variations.

The abnormal extremal flow can be parametrized as well provided that there holds such a kind of Legendre assumption in the abnormal case.

When the Hessian of the Hamiltonian is degenerate the situation is more intricate. This is the case for instance when one considers the minimal time problem for single-input control affine systems $\dot{x}(t) = f_0(x(t)) + u(t)f_1(x(t))$ without constraint on controls. In that case, the maximization condition leads to $\frac{\partial H}{\partial u} = 0$, that is, there must hold $\langle p(t), f_1(x(t))\rangle = 0$ along the corresponding extremal. To compute the control, the method consists of differentiating two times this relation with respect to t, which leads at first to $\langle p(t), [f_0, f_1](x(t))\rangle = 0$ and then at $\langle p(t), [f_0, [f_0, f_1]](x(t))\rangle + u(t)\langle p(t), [f_1, [f_0, f_1]](x(t))\rangle = 0$, where $[\cdot,\cdot]$ denotes the Lie bracket of vector fields. This permits as well to express the optimal control $u(t)$ as a function of $x(t)$ and $p(t)$, provided that the quantity $\langle p(t), [f_1, [f_0, f_1]](x(t))\rangle$ does not vanish along the extremal (strong generalized Legendre-Clebsch condition, see [12]). It can also be shown that this kind of computation is valid in a generic situation (see [19, 20]).

Note that, when facing with an optimal control problem, in general we have to deal with two extremal flows, distinguished by the binary variable $p^0 \in \{0, -1\}$. Due to additional constraints it is however expected that the abnormal flow fills less space than the normal one, in the sense that almost every point of the accessible set should be reached by a normal extremal. Such a statement is however difficult to derive. We refer to [43, **?**, 48] for precise results for control-affine systems.

2.2. Second-order optimality conditions. We stress that the Pontryagin Maximum Principle is first-order necessary condition for optimality. Conversely, in order to ensure that a given extremal is indeed optimal (at least locally), sufficient second-order conditions are required. For the sake of simplicity we assume that we are in the simplified situation where $M_0 = \{x_0\}$, $M_1 = \{x_1\}$, $g = 0$ and $U = \mathbb{R}^m$.

In this simplified situation where there is no constraint on the control, conditions of order two are standard. Defining the usual intrinsic second order derivative Q_T of the Lagrangian as the Hessian $\frac{\partial^2 L_T}{\partial^2 u}(u, \psi, \psi^0)$ restricted to the subspace $\ker \frac{\partial L_T}{\partial u}$, it is well-known that a second-order necessary condition for optimality is that Q_T be nonpositive (recall the agreement $\psi^0 \leqslant 0$), and a second-order sufficient condition for *local* optimality is that Q_T be negative definite. As previously with the Pontryagin Maximum Principle, these second-order conditions can be parametrized along the extremals, as derived in the theory of conjugate points, whose main issues are the following. Under the strict Legendre assumption, the quadratic form Q_T is negative definite whenever $T > 0$ is small enough. This leads naturally to define the *first conjugate time* t_c along a given extremal as the infimum of times $t > 0$ such that Q_t has a nontrivial kernel. Under the strict Legendre assumption, there holds $t_c > 0$, and this first conjugate time characterizes the (local) optimality status of the trajectory, in the sense that the trajectory $x(\cdot)$ is locally optimal (in L^∞ topology) on $[0, t]$ if and only if $t < t_c$ (see [2, 12] for more details on that theory). The following result is important in view of practical computations of conjugate times.

Theorem 2.5. *The time t_c is a conjugate time along $x(\cdot)$ if and only if the mapping $\exp_{x_0}(t_c, \cdot)$ is not an immersion at p_0 (that is, its differential is not injective).*

Essentially it states that computing a first conjugate time reduces to compute the vanishing of some determinant along the extremal. Indeed, the fact that the exponential mapping is not an immersion can be translated in terms of so-called *vertical Jacobi fields*. Note however that the domain of definition of the exponential mapping requires a particular attention in order to define properly these Jacobi fields according to the context: normal or abnormal extremal, final time fixed or not. A more complete exposition can be found in the survey article [11], which provides also some algorithms to compute first conjugate times in various contexts.

Remark 2.6. The conjugate point theory sketched above can be extended to more general situations, such as general initial and final sets (notion of focal point), discontinuous controls (see [37, 3]) by using the notion of extremal field (see [1]). We refer the reader to [46] where a brief survey with a unifying point of view of different approaches has been written in the introduction. Up to now a complete theory of conjugate points, that would cover all possible cases (with trajectories involving singular arcs and/or boundary arcs) still does not exist.

2.3. Numerical methods in optimal control. It is usual to distinguish between direct and indirect numerical approaches in optimal control.

Direct methods consist of discretizing the state and the control and after this discretization the problem is reduced to a nonlinear finite dimensional optimization problem with constraints of the form

$$\min\{F(Z) \mid g(Z) = 0,\ h(Z) \leqslant 0\}. \tag{11}$$

The dimension is of course larger as the discretization is finer. There exist many ways to carry out such discretizations (collocation, spectral or pseudospectral methods, etc). In any case, one has to choose finite dimensional representations of the control and of the state, and then express in a discrete way the differential equation representing the system. Once all static or dynamic constraints have been transcribed into a problem with a finite number of variables, one is ought to solve the resulting optimization problem with constraints, using some adapted optimization method (gradient methods, penalization, dual methods, etc).

Note that this kind of method is easy to implement in the sense that it does not require a precise a priori knowledge of the optimal control problem. Moreover, it is easy to take into account some state constraints or any other kinds of constraints in the optimal control problem. In this sense, this approach is not so much sensitive to the model. In practice, note that it is possible to combine automatic differentiation softwares (such as the modeling language `AMPL`, see [25]) with expert optimization routines (such as the open-source package `IPOPT`, see [51]), which allows one to implement in a simple way with few lines of code even difficult optimal control problems. We refer the reader to [6] for an excellent survey on direct methods with a special interest to applications in aerospace.

Another approach, that we only mention quickly, is to solve numerically the Hamilton-Jacobi equation satisfied (in the sense of viscosity) by the value function of the optimal control problem. We refer to [45] for level set methods but mention that this approach can only be applied to problems of very low dimension.

Indirect methods consist of solving numerically the boundary value problem derived from the application of the Pontryagin Maximum Principle, and lead to the shooting methods. More precisely, since every optimal trajectory is the projection of an extremal, the problem is reduced to finding an extremal, solution of the extremal equations and satisfying some boundary conditions:

$$\dot z(t) = F(t, z(t)), \quad R(z(0), z(t_f)) = 0.$$

Denoting by $z(t, z_0)$ the solution of the Cauchy problem $\dot z(t) = F(t, z(t))$, $z(0) = z_0$, and setting $G(z_0) = R(z_0, z(t_f, z_0))$, this boundary value problem is then equivalent to solving $G(z_0) = 0$, that is, one should determine a zero of the so-called *shooting function* G. This can be achieved in practice by using a Newton like method. This approach is called a *shooting method*. It has many possible refinements, among which the multiple shooting method, consisting of subdividing the time interval in N intervals, and of considering as unknowns the values at each node (with some gluing conditions). The aim is to improve the stability of the method (see e.g.[47]). From the practical implementation point of view, there exist many variants of Newton methods, among which the Broyden method or the Powell hybrid method are quite competitive. Moreover, as for direct methods the shooting methods can be combined with automatic differentiation, used in order to generate the Hamiltonian equations of extremals[1].

It should be noted that, when implementing a shooting method, the structure of the trajectory has to be known a priori, particularly in the case where the trajectory involves singular arcs (see e.g. [8]). This raises the question of being able to determine at least locally the structure of optimal trajectories, as investigated in geometric optimal control theory (see further).

Note that the shooting method is well-posed if and only if the Jacobian of the shooting function is invertible. In the simplest situation this is equivalent to requiring the fact that the exponential mapping is a local immersion or, in other words, that the point under consideration is not a conjugate point.

Remark 2.7. As sketched above, direct methods consist of discretizing first, and then of dualizing (by applying to the discretized problem (11) some necessary condition for optimality), whereas indirect methods consist of dualizing first (by applying the Pontryagin Maximum Principle which is a necessary condition for optimality), and then of discretizing (by applying a shooting method, that is a Newton method composed with a numerical integration method). It must be noted that this diagram may fail to be commutative in general (see [31] for very simple counterexamples), due to a possible lack of coerciveness in the discrete scheme. In

[1] We refer to the code `COTCOT` (Conditions of Order Two and COnjugate Times), available for free on the web, documented in [11], implementing such issues as well as efficient algorithms to compute conjugate points.

other words, although usual assumptions of consistency and stability (Lax scheme) allow one to show convergence of the indirect approach, they may be insufficient to ensure the convergence of the direct approach. Up to now very few conditions do exist on the numerical schemes that can ensure the commutativity of this diagram (see [31, 7, 44]) and this problem is still widely open in general.

Remark 2.8. Direct and indirect approaches have their own advantages and drawbacks and are complementary. Roughly speaking, one can say that direct methods are less precise, more computationally demanding than indirect methods, but less sensitive and more robust in particular with respect to the model (for instance it is easy to add state constraints in a direct approach). Indirect methods inherit of the main advantage of the Newton methods, namely, its extremely quick convergence and good accuracy, but also suffer from its main drawback, namely, its sensitivity to the initial guess. Moreover, the structure of the trajectory has to be known a priori in an indirect approach. We refer to [6, 40] for details on these methods and some solutions to bypass the difficulties. The reader can also find many other references in the recent survey [50].

In this article we focus on applications of optimal control to aerospace, and in such problems indirect methods are often priviledged because, although they are difficult to initialize and thus to make converge, they offer an extremely good numerical accuracy and a very quick execution time. In the sequel we will show how this initialization difficulty can be overcome by combining optimal control tools with numerical continuation and geometric optimal control for orbit transfer problems (Section 3), and with dynamical systems theory for interplanetary mission design (Section 4).

3. Solving the orbit transfer problem by continuation

Coming back to the orbit transfer problem (3) mentioned in the introduction as a motivating example, one immediately realizes that the main difficulty of this problem is the fact that the maximal authorized modulus of thrust is very low. It is actually not surprising to observe numerically that the lower is the maximal thrust, the smallest is the domain of convergence of the Newton method in the shooting problem. In these conditions it is natural to consider first a larger value of $T_{\max}$ (e.g., 60 N), so that in that case the domain of convergence of the shooting method is much larger and thus the shooting method is easy to initialize successfully, and then to make decrease the value of $T_{\max}$ step by step in order to reach down the value $T_{\max} = 0.1$ N. At each step, the shooting method is then initialized with the solution obtained from the previous step.

This strategy was implemented in [15] in order to realize the minimal time 3D transfer of a satellite from a low and eccentric inclinated initial orbit towards the geostationary orbit.

From the mathematical point of view, this method consists of implementing a so-called *continuation*, or *homotopy*, on the parameter $T_{\max}$. To ensure its feasibility, one should at least ensure that, along the continuation path, the optimal solution depends continuously on the continuation parameter. Let us provide several mathematical details on the continuation method.

The objective of continuation or homotopy methods is to solve a problem step by step from a simpler one by parameter deformation. There exists a well-developed theory and many algorithms and numerical methods implementing these ideas, and the field of applications encompasses Brouwer fixed point problems, polynomial and nonlinear systems of equations, boundary value problems in many diverse forms, etc. We refer the reader to [4] for a complete report on these theories and methods.

Here we use the continuation or homotopy approach in order to solve the shooting problem resulting from the application of the Pontryagin Maximum Principle to an optimal control problem. More precisely, the method consists of deforming the problem into a simpler one that we are able to solve (without any careful initialization of the shooting method), and then of solving a series of shooting problems parametrized by some parameter $\lambda \in [0,1]$, step by step, to come back to the original problem. The continuation method consists of tracking the set of zeros of the shooting function, as the parameter λ evolves. Numerical continuation can fail whenever the path of zeros which is tracked has bifurcation points or more generally singularities, or whenever this path fails to exist globally and does not reach the final desired value of the parameter (say, $\lambda = 1$). Let us provide shortly the basic arguments ensuring the local feasibility of the continuation method. From the theoretical point of view, regularity properties require at least that the optimal solution be continuous, or differentiable, with respect to the parameter λ. This kind of property is usually derived using an implicit function argument, which is encountered in the literature as *sensitivity analysis*. Let us explain what is the general reasoning of sensitivity analysis, in the simplified situation where $M_0 = \{x_0\}$, $M_1 = \{x_1\}$ and $U = \mathbb{R}^m$. We are faced with a family of optimal control problems, parametrized by λ, that can be as in (5) written in the form of

$$\min_{E_{x_0,T,\lambda}(u_\lambda)=x_1} C_{T,\lambda}(u), \tag{12}$$

that is, in the form of an optimization problem depending on the parameter λ. According to the Lagrange multipliers rule, if u_λ is optimal then there exists $(\psi_\lambda, \psi_\lambda^0) \in \mathbb{R}^n \times \mathbb{R} \setminus \{0\}$ such that $\psi_\lambda dE_{x_0,T,\lambda}(u_\lambda) + \psi_\lambda^0 dC_{T,\lambda}(u) = 0$. Assume that there are no minimizing abnormal extremals in the problem. Under this assumption, since the Lagrange multiplier $(\psi_\lambda, \psi_\lambda^0)$ is defined up to a multiplicative scalar we can definitely assume that $\psi_\lambda^0 = -1$. Then, we are seeking $(u_\lambda, \psi_\lambda)$ such that $F(\lambda, u_\lambda, \psi_\lambda) = 0$, where the function F is defined by

$$F(\lambda, u, \psi) = \begin{pmatrix} \psi dE_{x_0,T,\lambda}(u) - dC_{T,\lambda}(u) \\ E_{x_0,T,\lambda}(u) - x_1 \end{pmatrix} = \begin{pmatrix} \frac{\partial L_{T,\lambda}}{\partial u}(u,\psi) \\ E_{x_0,T,\lambda}(u) - x_1 \end{pmatrix},$$

where $L_{T,\lambda}(u,\psi) = \psi E_{x_0,T,\lambda}(u) - C_{T,\lambda}(u)$. Let $(\bar\lambda, u_{\bar\lambda}, \psi_{\bar\lambda})$ be a zero of F. Assume that F is of class C^1. If the Jacobian of F with respect to (u,ψ), taken at the point $(\bar\lambda, u_{\bar\lambda}, \psi_{\bar\lambda})$, is invertible, then according to a usual implicit function argument one can solve the equation $F(\lambda, u_\lambda, \psi_\lambda) = 0$, and the solution $(u_\lambda, \psi_\lambda)$ depends in a C^1 way on the parameter λ. The Jacobian of F with respect to (u, ψ) is

$$\begin{pmatrix} Q_{T,\lambda} & dE_{x_0,T,\lambda}(u)^* \\ dE_{x_0,T,\lambda}(u) & 0 \end{pmatrix}, \tag{13}$$

where $Q_{T,\lambda}$ is is defined as the restriction of the Hessian $\frac{\partial^2 L_{T,\lambda}}{\partial^2 u}(u, \psi, \psi^0)$ to the subspace ker $\frac{\partial L_{T,\lambda}}{\partial u}$, and $dE_{x_0,T,\lambda}(u)^*$ is the transpose of $dE_{x_0,T,\lambda}(u)$. The matrix (13) (which is a matrix of operators) is called *sensitivity matrix* in sensitivity analysis. It is easy to prove that it is invertible if and only if the linear mapping $dE_{x_0,T,\lambda}(u)$ is surjective and the quadratic form Q_T is nondegenerate. The surjectivity of $dE_{x_0,T,\lambda}(u)$ exactly means that the control u is not singular (see Definition 2.2). The nondegeneracy of $Q_{T,\lambda}$ is exactly related with the concept of conjugate point (see Section 2.2). Note that, as long as we do not encounter any conjugate time along the continuation path, the extremals that are computed are locally optimal. It follows that, to ensure the surjectivity of $dE_{x_0,T,\lambda}(u)$ along the continuation process, it suffices to assume the absence of singular minimizing trajectory (in the simplified problem considered here, with unconstrained controls, singular trajectories are exactly the projections of abnormal extremals).

Therefore, we conclude that, as long as we do not encounter any minimizing singular control nor conjugate point along the continuation procedure, then the continuation method is *locally* feasible, and the extremal solution $(u_\lambda, \psi_\lambda)$ which is locally computed as above is of class C^1 with respect to the parameter λ.

Remark 3.1. The absence of conjugate point can be tested numerically, as explained previously. The assumption of the absence of minimizing singular trajectories is of a much more geometric nature. Such results exist for some classes of control-affine systems under some strong Lie bracket assumptions (see [2, 12]). Moreover, it is proved in [19, 20] that for generic (in the Whitney sense) control-affine systems with more than two controls, there is no minimizing singular trajectory; hence for such kinds of systems the assumption of the absence of minimizing singular trajectory is automatically satisfied.

To ensure the global feasibility of the continuation procedure, we ought to ensure that the path of zeros is globally defined on $[0, 1]$. It could indeed happen that the path either reaches some singularity or wanders off to infinity before reaching $\lambda = 1$. To eliminate the first possibility, we can make the assumption of the absence of minimizing singular trajectory and of conjugate point over all the domain under consideration (not only along the continuation path). As remarked above, the absence of singular minimizing trajectory over the whole space is generic for large classes of systems, hence this is a reasonable assumption; however the global absence of conjugate point is a strong assumption and there exist other issues (see Remark 3.3 below). To eliminate the second possibility, we ought to provide sufficient conditions ensuring that the tracked paths remain bounded. In other words, we have to ensure that the initial adjoint vectors that are computed along the continuation procedure remain bounded, uniformly with respect to the homotopic parameter λ. This fact is exactly ensured by assuming the absence of minimizing abnormal extremals over the domain (see [50] for details, see also[48]), hence by one of the (generic) assumptions done above.

Proposition 3.2. *In the simplified case where* $M_0 = \{x_0\}$*,* $M_1 = \{x_1\}$ *and* $U = \mathbb{R}^m$*, if there is no minimizing singular trajectory nor conjugate points over all the domain, then the continuation procedure is globally feasible on* $[0, 1]$.

Remark 3.3. There exist some other possibilities to eliminate the first possibility above. Singularities due to conjugate points may be either detected and then handled with specific methods (see [4]), or can be removed generically by Sard arguments, by considering a global perturbation of the homotopy function (probability-one homotopy method, see the survey [52]). We also refer the reader to [50] for a more detailed discussion of continuation and homotopy methods.

The continuation method is a powerful tool significantly improving the efficiency of shooting approaches and making them realistic in many applications. This combination has been applied successfully to a number of applications in aerospace problems (see e.g. [9, 15, 26]), and to the application below.

Automatic solving of the optimal flight of the last stage of Ariane launchers. We mention here our recent work with EADS Astrium (les Mureaux, France), consisting of solving the minimal consumption transfer for the last stage of Ariane V and next Ariane VI launchers, the objective being to obtain a robust software able to provide automatically (that is, without any careful initialization) and instantaneously (actually, within one second) the optimal solution of that problem, for any possible initial and final conditions prescribed by the user. This software is operational on a very large range of possible values covering in particular the domain of applications usually treated at EADS Astrium for civilian launchers. Successfully integrated to the global optimization tools of EADS Astrium, this real-time algorithm brought a significant improvement for Ariane V trajectory planning and also allows one to consider new strategies for the forthcoming Ariane VI launchers. From the mathematical point of view, the approach is based on a combination of the Pontryagin Maximum Principle with numerical continuation methods and with a refined geometric analysis of the extremal flow and the use of recent items of geometric optimal control (see below).

Although we are not allowed to describe the precise method we employed, we mention some byproduct works realized in collaboration with EADS. In [17] we provide an alternative approach to the strong thrust minimal consumption orbit transfer planification problem, consisting of considering at first the problem for a flat model of the Earth with constant gravity (which is extremely easy to solve), and then of introducing step by step, by continuation, the variable gravity and the curvature of the Earth, in order to end up with the true model. In [30] we show how one can take into account a shadow cone (eclipse) constraint in the orbit transfer problem, by defining an hybridization of the problem, considering that the controlled vector fields are zero when crossing the shadow cone. A regularization procedure consisting of smoothing the system, combined with a continuation, is also implemented and strong convergence properties of the smoothing procedure are derived.

We end this section by providing a very short insight of what is geometric optimal control.

Geometric optimal control. Geometric optimal control can be described as the combination of the knowledge inferred from the Pontryagin Maximum Principle with geometric considerations such as the use of Lie brackets, of subanalytic sets, of differential geometry on manifolds, of symplectic geometry and Hamiltonian systems, of singularity theory, with the ultimate objective of deriving *optimal synthesis* results, permitting to describe in a precise way the structure of optimal trajectories. In other words, the objective is to derive results saying that, according to the class of control systems we are considering, the optimal trajectories have a precise structure and are of such and such kind. The geometric tools mentioned above are used to provide a complement to the Pontryagin Maximum Principle whenever its application alone happens to be insufficient to adequately solve an optimal control problem, due to a lack of information. We refer the reader to the textbooks [2, 12, 33] and to [50] for a more detailed exposition on geometric optimal control and other applications to aerospace (in particular, the atmospheric re-entry of a space shuttle), where it is shown how results of geometric optimal control theory can help to make converge a shooting method, or at least can simplify its implementation by describing precisely the structure of the optimal trajectory.

4. Dynamical systems theory and mission design

We now switch to another problem, also realized in collaboration with EADS Astrium, on mission design planification with the help of Lagrange points. The objective is here to show how dynamical systems theory can supply new directions for control issues.

4.1. Dynamics around Lagrange points. Consider the so-called circular restricted three-body problem, in which a body with negligible mass evolves in the gravitational field of two masses m_1 and m_2 called primaries and assumed to have circular coplanar orbits with the same period around their center of mass. The gravitational forces exerted by any other planet or body are neglected. In the solar system this problem provides a good approximation for studying a large class of problems. In a rotating frame the equations are of the form

$$\ddot{x} - 2\dot{y} = \frac{\partial \Phi}{\partial x}, \quad \ddot{y} + 2\dot{x} = \frac{\partial \Phi}{\partial y}, \quad \ddot{z} = \frac{\partial \Phi}{\partial z}$$

with $\Phi(x, y, z) = \frac{x^2+y^2}{2} + (1-\mu)((x+\mu)^2 + y^2 + z^2)^{-1/2} + \mu((x-1+\mu)^2 + y^2 + z^2)^{-1/2} + \frac{\mu(1-\mu)}{2}$. These equations have the first integral (called Jacobi first integral) $J = 2\Phi - (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$, hence the solutions evolve on a five-dimensional energy manifold, the topology of which determines the so-called Hill's region of possible motions (see e.g. [34]).

It is well-known that the above dynamics admit five equilibrium points called Lagrange points: three collinear ones L_1, L_2 and L_3 (already known by Euler), and the equilateral ones L_4 and L_5 (see Figure 1). The linearized system around

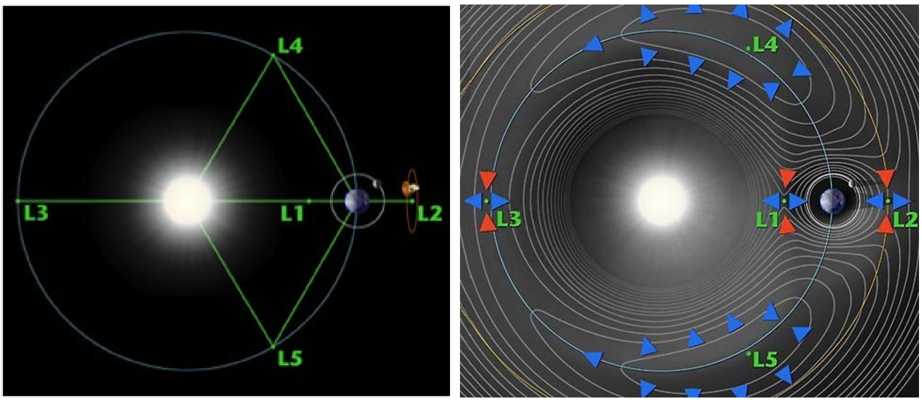

Figure 1. Lagrange points in the system Sun–Earth, and phaseportrait around them

these equilibrium points admits eigenvalues with zero real part, hence the study of their stability is not obvious. It follows from a generalization of a theorem of Lyapunov (due to Moser [39]) that, for a value of the Jacobi integral a bit less than the one of the Lagrange points, the solutions have the same qualitative behavior as the solutions of the linearized system around the Lagrange points. It was then established in [36] that the three collinear Lagrange points are always unstable, whereas L_4 and L_5 are stable under some conditions (that are satisfied in the solar system for instance for the Earth-Moon system, or for the system formed by the Sun and any other planet).

These five equilibrium points are naturally priviledged sites for space observation. At a first step one may think of the stability feature of L_4 and L_5 as a good news: indeed if one places an observation engine near one these points, then the engine will remain in a neighborhood of the Lagrange point. However, because of this stability, it happens that the neighborhood of these stable points is full of small particles that have been trapped into the potential well due to astronomic hazards[2]. These small particles are of course extremely dangerous for a space engine, and finally these sites must be avoided. In view of that, the instability of L_1, L_2 and L_3 is finally a good news be cause the vicinity of these points is clean in some sense. The counterpart is that, since they are unstable, any engine located in the neighborhood of such a point must be stabilized by means of a control. However, stabilizing a control system near an unstable equilibrium point requires only

[2]As a striking example, we mention the troyan asteroids, located near the points L_4 and L_5 in the Sun-Jupiter system.

little energy and hence this appears as a good strategy for astronomic observation. The Lagrange points L_1 and L_2 of the Sun-Earth system are indeed used for such issues since years: the satellite SOHO, whose mission is to observe the surface of the Sun, is located near[3] the point L_1; the James Webb Space Telescope (JWST) will be launched within next years and placed near the point L_2, which is an ideal site to observe the cosmos. There are many other objects near Lagrange points.

The dynamics around these Lagrange points have particularly interesting features for space mission design. Using Lyapunov-Poincaré's Theorem, it is shown that there exists a two-parameter family of periodic trajectories around every Lagrange point (see [36], see also [13]), among which the well-known halo orbits are periodic orbits that are diffeomorphic to circles (see [14]) whose interest for mission design was put in evidence by Farquhar (see [23, 24]). There exist many other families of periodic orbits (called Lissajous orbits) and quasi-periodic orbits around Lagrange points (see [28, 29]). The invariant (stable and unstable) manifolds of these periodic orbits, consisting of all trajectories converging to the orbit (as the time tends to $\pm\infty$), are four-dimensional tubes, topologically equivalent to $S^3 \times \mathbb{R}$, in the five-dimensional energy manifold (see [27]). Hence they play the role of separatrices. Roughly speaking, these tubes can be seen as gravity currents, similar to ocean currents except that their existence is due to gravity effects. Therefore they can be used for mission design and space exploration, since a trajectory starting inside such a tube (called transit orbit) stays inside this tube. Many recent studies have been done on this subject. It can be noted however that the invariant manifolds of halo orbits (which can be really seen as tubes) are chaotic in large time: they do not keep their aspect of tube and behave in a chaotic way, far from the halo orbit (see [34]). In contrast, the invariant manifolds of eight-shaped Lissajous orbits[4] (which are eight-shaped tubes) are numerically shown in [5] to keep their regular structure globally in time (see Figure 2 on the left). Moreover, in the Earth-Moon system, it is shown that they permit to fly over almost all the surface of the Moon, passing very close to the surface (between 1500 and 5000 kilometers, see Figure 2 on the right). These features are of particular interest in view of designing low-cost space missions to the Moon. Indeed in the future space exploration the Moon could serve as an intermediate point (with a lunar space station) for farther space missions.

4.2. Applications to mission design and challenges. The idea of using the specific properties of the dynamics around Lagrange points in order to explore lunar regions is far from new but has recently received a renewal of interest. In [34, 38], the authors combine the use of low-thrust propulsion with the use of the nice properties of invariant manifolds of periodic orbits around Lagrange points in order to design low-cost trajectories for space exploration. Their techniques consist of stating an optimal control problem that is numerically solved using either

[3]Actually not exactly: the satellite SOHO is satbilized along a halo orbit of quite large amplitude around the point L_1.

[4]Eight-shaped Lissajous orbits are the Lissajous orbits of the second kind. They are diffeomorphic to a curve having the shape of an eight. They are chiefly investigated in [5].

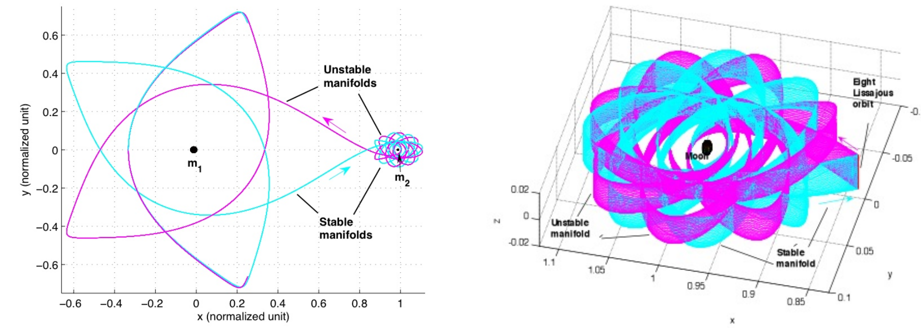

Figure 2. Invariant manifolds of an Eight Lissajous orbit, in the Earth–Moon system (left), and a zoom around the Moon (right)

a direct or an indirect transcription, carefully initialized with the trajectories of the previously studied system (with no thrust). In such a way they are able to realize a reasonable compromise between fuel consumption and time of transfer, and design trajectories requiring moderate propellant mass and reaching the target within reasonable time.

In these studies the previously studied circular restricted three-body problem approximation is used to provide an appropriate first guess for carefully initializing an optimal control method (for instance, a shooting method) applied to a more precise model. In view of that, and having in mind the previous methodology based on continuation, it is natural to develop an optimal planification method based on the combination of the dynamics of the three-body problem with a continuation on the value of the maximal authorized thrust. This idea opens new directions for future investigations and is a promising method for designing efficiently fuel low-consumption space missions. Although the properties of the dynamics around Lagrange points have been widely used for developing planification strategies, up to now, and to the best of our knowledge they have not been combined with continuation procedures that would permit to introduce, for instance, the gravitational effects of other bodies, or values of the maximal thrust that are low or mild, or other more complex models. This is a challenge for potential future studies.

Note that, in [41], the author implements a numerical continuation procedure to compute minimal-energy trajectories with low thrust steering the engine from the Earth to the Lagrange point L_1 in the Earth-Moon system, by making a continuation on the gravitational constant of the Moon. The continuation procedure is initialized with the usual Kepler transfer, in which the Moon coincides with the point L_1, and ends up with a trajectory reaching the point L_1 with a realistic gravitational effect of the Moon.

In view of designing future mission design it should be done a precise cartography of all invariant manifolds generated by all possible periodic orbits (not only halo or eight-shaped orbits) around Lagrange points. The existence of such invariant manifolds indeed makes possible the design of low-cost interplanetary missions. The design of trajectories taking advantage of these gravity currents, of gravitational effects of celestial bodies of the solar system, of ”swing-by” strategies, is a difficult problem related to techniques of continuous and discrete optimization (multidisciplinary optimization). Is is an open challenge to design a tool combining refined techniques of nonlinear optimal control, continuation procedures, mixed optimization, and global optimization procedures.

Another challenge, which is imperative to be solved within next years, is the problem of debris cleaning. Indeed, recently it has been observed a drastic growth of space debris in the space around the Earth, in particular near the SSO orbit and polar orbits with altitude between 600 and 1200 km (indeed these orbits are intensively used for Earth observation): there are around 22000 debris of more than 10 cm (which are cataloged), around 500000 debris between 1 and 10 cm (which are not cataloged), and, probably, millions of smaller debris that cannot be detected. These debris are due to former satellites that were abandoned, and now cause high collision risks for future space flights. It has become an urgent challenge to clean the space at least from its biggest debris in order to stabilize the debris population, otherwise it will soon become impossible to launch additional satellites. At present, all space agencies in the world are aware of that problem and are currently working to provide efficient solutions for designing space debris collecting missions. One of them, currently led with EADS Astrium (see [16]), consists of deorbiting five heavy debris per year, selected in a list of debris (in the LEO region) so that the required fuel consumption for the mission is minimized. The problem to be solved turns into a global optimization problem consisting of several continuous transfer problems and of a combinatorial path problem (selection of the debris and of the collecting order). It is not obvious to solve since it must combine continuous optimal control methods with combinatorial optimization, and other considerations that are specific to the problem. The results of [16] (which are valuable for high-thrust engines) provide first solutions in this direction, and open new problems for further investigation. For instance it is an open problem to design efficient space cleaning missions for low-thrust engines, taking benefit of the gravitational effects due to Lagrange points and to invariant manifolds associated with their periodic orbits. Such studies can probably be carried out with appropriate continuation procedures, carefully initialized with trajectories computed from the natural dynamics of the three-body problem. It is one of the top priorities in the next years is to clean the space from the biggest fragments. Although we have at our disposal a precise catalog of fragments, wreckage, scraps, it is a challenging problem to design optimally a space vehicle able to collect in minimal time a certain number of fragments, themselves being chosen in advance in the catalog in an optimal way. This problem combines techniques of continuous optimal control in order to determine a minimal time trajectory between two successive fragments, and techniques of discrete optimization for the best possible choice of the fragments to be collected.

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Emmanuel Trélat, Université Pierre et Marie Curie (Univ. Paris 6) and Institut Universitaire de France, CNRS UMR 7598, Laboratoire Jacques-Louis Lions, 75005, Paris, France.

E-mail: emmanuel.trelat@upmc.fr