Description of accessibility sets near an abnormal trajectory and consequences

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Abstract. We describe precisely, under generic conditions, the contact of the accessibility set at time T with an abnormal direction, first for a single-input affine control system with constraint on the control, and then as an application for a sub-Riemannian system of rank 2. As a consequence we obtain in sub-Riemannian geometry a new splitting-up of the sphere near an abnormal minimizer γ into two sectors, bordered by the first Pontryagin's cone along γ, called the L^∞-sector and the L^2-sector. Moreover we find again necessary and sufficient conditions of optimality of an abnormal trajectory for such systems, for any optimization problem.

1 Introduction

Consider a smooth control system on $\mathbb{R}^n$:

$$\dot{x}(t) = f(x(t), u(t)), \; x(0) = x_0 \tag{1}$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^n$ is smooth, $x_0 \in \mathbb{R}^n$, and the set of admissible controls $\mathcal{U}$ is made of measurable bounded functions $u \; : [0, T(u)] \longrightarrow \Omega \subset \mathbb{R}^m$.

Definition 1. Let $T > 0$. The *end-point mapping* at time T of system (1) is the mapping

$$E_T \; : \begin{array}{l} \mathcal{U} \longrightarrow \mathbb{R}^n \\ u \longmapsto x_u(T) \end{array}$$

where x_u is the trajectory associated to u.

The application E_T is smooth in the L^∞ topology if $\mathcal{U} \subset L^\infty([0,T])$.

Definition 2. A control u (or the corresponding trajectory x_u) is said to be *abnormal* on $[0,T]$ if it is a singular point of the mapping E_T anf if moreover the Hamiltonian of the system $H =< p, f(x,u) >$ is equal to 0 along the trajectory x_u.

Remark 1. If a control u is abnormal on $[0,T]$ then it is abnormal on $[0,t]$ for any $t \in [0,T]$.

Definition 3. Let u be an abnormal control on $[0,T]$, and x_u its associated trajectory. The subspace Im $dE_t(u)$ is called the *first Pontryagin's cone* at $x_u(t)$.

Definition 4. Consider the control system (1), and let $T>0$. The *accessibility set at time* T, denoted by $Acc(T)$, is the set of points that can be reached from x_0 in time T by solutions of system (1), i.e. this is the image of the end-point mapping E_T.

Let γ be a reference abnormal trajectory on $[0,T]$, solution of (1), associated to a control u. Our aim is to describe $Acc(T)$ near $\gamma(T)$.

2 Asymptotics of the accessibility sets

In this Section we describe precisely the boundary of accessibility sets for a single-input affine system with constraint on the input near a reference abnormal trajectory.

Consider a smooth *single-input affine control system* in $\mathbb{R}^n$, $n \geqslant 3$:

$$\dot{x}(t) = X(x(t)) + u(t)Y(x(t)),\ \ x(0)=0 \tag{2}$$

with the *constraint* on the control

$$|u(t)| \leqslant \eta \tag{3}$$

Let $Acc^\eta(T)$ denote the *accessibility set* at time T for this affine system with constraint η on the control. Let γ be a reference trajectory defined on $[0,T]$. In the sequel we make the following assumptions along γ :

(H_0) γ is injective, associated to $u=0$ on $[0,T]$.
(H_1) $\forall t \in [0,T]$ $\ K(t) = \text{Vect}\ \{ad^k X.Y(\gamma(t))\ /\ k \in \mathbb{N}\}$ (first Pontryagin's cone along γ) has codimension 1, and is spanned by the first $n-1$ vectors, i.e. :

$$K(t) = \text{Vect}\ \{ad^k X.Y(\gamma(t))\ /\ k = 0 \dots n-2\}$$

(H_2) $\forall t \in [0,T]$ $\ ad^2 Y.X(\gamma(t)) \notin K(t)$.
(H_3) $\forall t \in [0,T]$ $\ X(\gamma(t)) \notin \text{Vect}\ \{ad^k X.Y(\gamma(t))\ /\ k = 0 \dots n-3\}$.
(H_4) $\forall t \in [0,T]$ $\ X(\gamma(t)) \in K(t)$.

In these conditions γ is *abnormal* and its first Pontryagin's cone $K(t)$ is an hyperplane in $\mathbb{R}^n$. Actually assumptions $(H_1 - H_3)$ are generic along γ, see [4].

The following Theorem is founded on a very precise spectral analysis of the intrinsic second-order derivative of the end-point mapping along the abnormal direction γ (initialised in [3]), which leads actually to a contact theory of accessibility sets (see [7]).

Theorem 1. *Consider the affine system (2) with the constraint (3), and suppose that assumptions* ($H_0 - H_4$) *are fulfilled along the reference abnormal trajectory* γ *on* $[0, T]$. *Then there exist coordinates* $(x_1, \dots, x_n)$ *locally along* γ *such that in these coordinates :*

1. $\gamma(t) = (t, 0, \dots, 0)$, *and the first Pontryagin's cone along* γ *is :* $K(t) = \text{Vect}\,\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_{n-1}}\}_{|\gamma}$.
2. *If* T *is small enough then for any point* $(x_1, \dots, x_n)$ *of* $Acc^\eta(T) \backslash \{\gamma(T)\}$ *close to* $\gamma(T)$ *we have :* $x_n > 0$ *(see Fig. 1).*

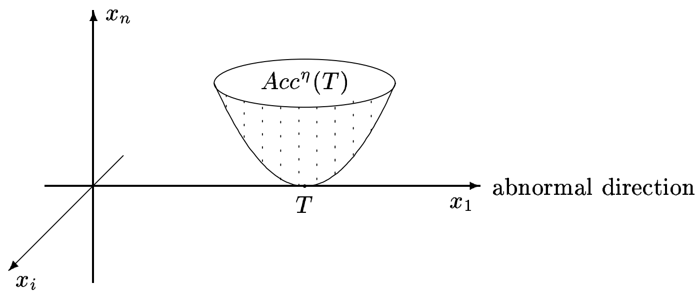

Fig. 1. Shape of $Acc^\eta(T)$, T small

3. *There exist two times* t_{cc}, t_c *such that* $0 < t_{cc} < t_c$, *called conjugate times or bifurcation times along* γ, *and such that the following holds.*
   If $T < t_c$, *then in the plane* (x_1, x_n), *near the point* $(T, 0)$, *the boundary of* $Acc^\eta(T)$ *does not depend on* η, *is a curve of class* C^2 *tangent to the abnormal direction, and its first term is :*

$$x_n = A_T(x_1 - T)^2 + o((x_1 - T)^2)$$

   The function $T \mapsto A_T$ *is continuous and strictly decreasing on* $[0, t_c[$. *It is positive on* $[0, t_{cc}[$ *and negative on* $]t_{cc}, t_c[$.
4. *If* $T > t_c$ *then* $Acc^\eta(T)$ *is open near* $\gamma(T)$.

The evolution in function of T of the intersection of $Acc^\eta(T)$ with the plane (x_1, x_n) is represented on Fig. 2. The contact with the abnormal direction is of order 2 ; the coefficient A_T describes the *concavity* of the curve. Beyond t_c the accessibility set is open.

Remark 2. The coefficient A_T can be explicitely computed. It is an invariant of the system, see [7]. Moreover the bifurcation times t_{cc} and t_c can be computed using an algorithm, see [3].

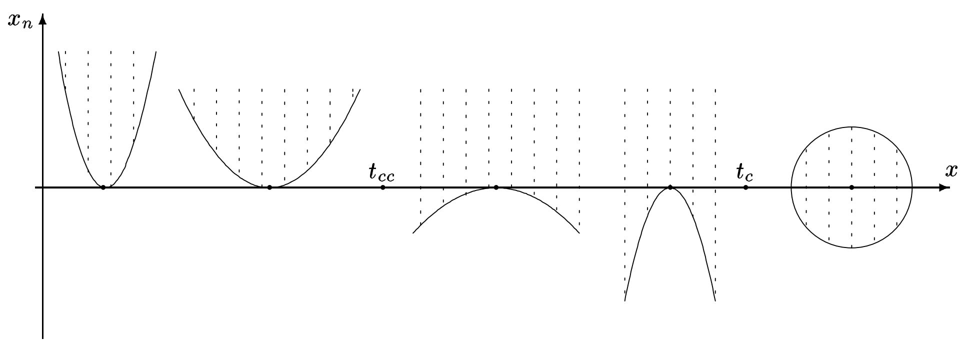

Fig. 2. Evolution of $Acc^{\eta}(T)$ in function of T

3 Applications

3.1 Application to the optimality status of an abnormal trajectory

In this Section we apply our previous theory on accessibility sets to studying optimality of abnormal trajectories ; this leads us to find again and to improve slightly some well-known results. Consider the single-input affine system (2) with constraint (3), and suppose assumptions $(H_0 - H_4)$ are fulfilled along a reference abnormal trajectory γ. We first investigate the time-optimal problem, and then the problem of minimizing any cost.

Time optimality The trajectory γ is said C^0*-time-minimal* on $[0, T]$ if there exists a C^0-neighborhood of γ such that T is the minimal time to steer $\gamma(0)$ to $\gamma(T)$ among the solutions of the system (2) with the constraint (3) that are entirely contained in this neighborhood.

We have the following result (compare with [1–3]) :

Theorem 2. *Under assumptions of Theorem 1, the trajectory γ is C^0-time-minimal if and only if $T < t_{cc}$.*

Optimization of any cost Let us now consider the problem of minimizing some cost $C(T, u)$, also denoted by $C_T(u)$, where C is a smooth function satisfying the following additional assumption along the reference abnormal trajectory γ :

(H_5) $\forall T$ rank $(dE_T(0), dC_T(0)) = n$

We distinguish between two optimization problems.

1. Final time not fixed The trajectory γ is said to be C^0-*cost-minimal* on $[0,T]$ if there exists a C^0-neighborhood of γ such that for any trajectory q contained in this neighborhood, with $q(0)=\gamma(0)$ and $q(t)=\gamma(T)$, we have : $C(t,v)\geqslant C(T,0)$, where v is the control associated to q.

We have the following (compare with [2]) :

Theorem 3. *Under assumptions* $(H_0 - H_5)$*, the trajectory* γ *is* C^0*-cost-minimal if and only if it is* C^0*-time-minimal, i.e. if and only if* $T < t_{cc}$.

2. Final time fixed The trajectory γ is said to be C^0-*cost-minimal* on $[0,T]$ if there exists a C^0-neighborhood of γ such that for any trajectory q contained in this neighborhood, with $q(0)=\gamma(0)$ and $q(T)=\gamma(T)$, we have : $C_T(v)\geqslant C_T(0)$, where v is the control associated to q.

Theorem 4. *The trajectory* γ *is* C^0*-cost-minimal if and only if* $T < t_c$.

Hence in this case the time-optimal problem is not equivalent to the problem of minimizing some cost. The trajectory γ ceases to be C^0-time-optimal *before* it ceases to be C^0-cost-optimal (since $t_{cc} < t_c$).

3.2 Application to the sub-Riemannian case

Consider a smooth sub-Riemannian structure (M,Δ,g) where M is a Riemannian n-dimensional manifold, $n\geqslant 3$, Δ is a rank 2 distribution on M, and g is a metric on Δ. Let $x_0\in M$; our point of view is local and we can assume that $M=\mathbb{R}^n$ and $x_0=0$. Suppose there exists a smooth injective abnormal trajectory γ passing through 0. Up to changing coordinates and reparametrizing we can assume that :

- $\gamma(t)=(t,0,\dots,0)$,
- $\Delta=\text{Span}\,\{X,Y\}$ where X,Y are g-orthonormal,
- γ is the integral curve of X passing through 0.

Under these assumptions, the sub-Riemannian problem is equivalent to the *time-optimal problem* for the system :

$$\dot{x}=vX(x)+uY(x),\ x(0)=0 \tag{4}$$

where the controls v,u satisfy the *constraint* :

$$v^2+u^2\leqslant 1 \tag{5}$$

The reference abnormal trajectory γ corresponds to the control : $v=1,u=0$.

The following result is a consequence of Theorem 1 and of a general statement proved in [6].

Theorem 5. *Consider the sub-Riemannian problem for the system* $\dot{x} = vX(x) + uY(x)$. *Let* γ *be an abnormal reference trajectory. Suppose assumptions* $(H_0 - H_4)$ *hold along* γ. *Then there exist coordinates* $(x_1, \dots, x_n)$ *locally along* γ *in which, if* T *is small enough :*

- $\gamma(t) = (t, 0, \dots, 0)$,
- *the first Pontryagin's cone along* γ *is* $K_\gamma(t) = (x_n = 0)$,
- *The sub-Riemannian sphere* $S(0,T)$ *splits into two sectors near* $\gamma(T)$ *:*
  1. *the* L^∞*-sector :* $(x_n > 0) \cap S(0,T)$, *made of end-points of minimizing trajectories associated to controls which are close to the abnormal reference control in* L^∞*-topology. Hence minimizing trajectories steering* 0 *to these points are close to* γ *in* C^1*-topology. Moreover in the plane* (x_1, x_n), *its graph is :*

     $$x_1 \geqslant T, \; x_n \sim A_T.(x_1 - T)^2$$

     where $T \mapsto A_T$ *is continuous, positive and decreasing.*
  2. *the* L^2*-sector :* $(x_n < 0) \cap S(0,T)$, *made of end-points associated to minimizing controls which are close to the abnormal reference control in* L^2*-topology, but not in* L^∞*-topology. Hence trajectories steering* 0 *to these points are close to* γ *in* C^0*-topology, but not in* C^1*-topology. This sector is tangent to the abnormal direction.*

  These two sectors are separated by the first Pontryagin's cone $x_n = 0$ *along* γ *(see Fig. 3).*

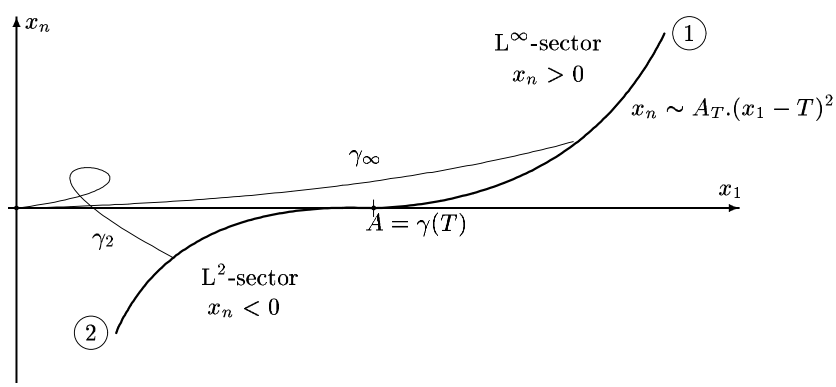

Fig. 3.

Remark 3. The abnormal trajectory γ is optimal for the sub-Riemannian problem if and only if $T < t_{cc}$.

Typical example : the Martinet case.

Consider the two following vector fields in $\mathbb{R}^3$:

$$X = \frac{\partial}{\partial x} + \frac{y^2}{2}\frac{\partial}{\partial z}\ ,\ Y = \frac{\partial}{\partial y}$$

and endow the distribution spanned by these vector fields with an analytic metric g of the type :

$$g = adx^2 + cdy^2$$

where $a = (1+\alpha y)^2$ and $c = (1+\beta x+\gamma y)^2$. The abnormal reference control for the sub-Riemannian system $\dot{x} = vX(x)+uY(x)$ with constraint $v^2+u^2 \leqslant 1$ is $v = 1, u = 0$, and corresponds to the trajectory $\gamma : x(t) = t, y(t) = z(t) = 0$. We have, see [5] :

Lemma 1. *Assumptions ($H_0 - H_4$) are fulfilled along γ if and only if $\alpha \neq 0$. In this case branches* 1 *and* 2 *(see Fig. 3 with $x_1 = x, x_n = z$) have the following contacts with the abnormal direction :*

- *branch* 1 : $x \geqslant T,\ z = \frac{1}{2T\alpha^2}(x-T)^2 + o((x-T)^2)$
- *branch* 2 : $x \leqslant T,\ z \sim \frac{1}{6}(1 + O(T))(x-T)^3$

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