Optimality of singular trajectories and asymptotics of accessibility sets under generic assumptions

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Abstract

We investigate minimization problems along a singular trajectory of a single-input affine control system with constraint on the control, and then as an application of a sub-Riemannian system of rank 2. Under generic assumptions we get necessary and sufficient conditions for optimality of such a singular trajectory. Moreover we describe precisely the contact of the accessibility sets at time T with the singular direction. As a consequence we obtain in sub-Riemannian geometry a new splitting-up of the sphere near an abnormal minimizer γ into two sectors, bordered by the first Pontryagin's cone along γ, called the L^∞-sector and the L^2-sector.

1 Introduction

1.1 Statement of the problems

Consider a control system on $\mathbb{R}^n$:

$$\dot{x}_u(t) = f(x(t), u(t)),\ \ x_u(0) = x_0 \tag{1}$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^n$ is smooth, $x_0 \in \mathbb{R}^n$, and the set of admissible controls $\mathcal{U}$ is made of measurable bounded functions $u : [0, T(u)] \longrightarrow \Omega \subset \mathbb{R}^m$. Let $f^0 : \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}$ be a smooth function, $T > 0$, and set $C_T(u) = \int_0^T f^0(x_u(t), u(t))dt$: it is called the *cost* of the trajectory x_u associated to the control u on $[0, T]$.

Definition 1.1. Let $T > 0$. The *end-point mapping* at time T of system (1) is the mapping

$$E_T : \begin{array}{ccc} \mathcal{U} & \longrightarrow & \mathbb{R}^n \\ u & \longmapsto & x_u(T) \end{array}$$

where x_u is the trajectory associated to u.

Definition 1.2. A control u on $[0,T]$ (or the corresponding trajectory x_u) is said to be *singular* if it is a singularity of the end-point mapping E_T, that is if there exists a non trivial vector ψ in $\mathbb{R}^n$ such that $\psi.dE_T(u) = 0$.

Definition 1.3. Let u be a singular control on $[0,T]$. The subspace Im $dE_T(u)$ is called the *first Pontryagin's cone along u* (or along x_u). The control u is said to be of *corank 1* if Im $dE_T(u)$ has codimension 1 in $\mathbb{R}^n$, that is if ψ is unique up to a scalar.

1.1.1 Optimization problems

Let γ be a solution of system (1) such that $\gamma(0) = x_0$, $\gamma(T) = x_1$. The problem is the following :

> Among all solutions of system (1) steering x_0 to x_1, is γ minimizing the cost ?

Here we consider a *single-input affine system with constraint on the control* :

$$\begin{aligned} &\dot{x}(t) = f_0(x(t)) + u(t) f_1(x(t)), \quad |u(t)| \leqslant \eta \\ &x(0) = x_0 \end{aligned} \tag{2}$$

Let γ be a reference singular trajectory of this system, associated to a corank 1 control u, and let $\psi \in \mathbb{R}^n \setminus \{0\}$ such that $\psi.dE_T(u) = 0$. We suppose that $u = 0$.

Definition 1.4.
- If f_0 and $[f_1,[f_1,f_0]]$ are on the same side with respect to the first Pontryagin's cone along γ, then γ is said to be *elliptic*.
- If they are on opposite sides, γ is called *hyperbolic*.
- If $f_0 \in$ Im $dE_T(u)$ along γ, then γ is said to be *exceptional*.

Actually due to the well-known *Legendre-Clebsh condition*, elliptic trajectories are never time-minimizing.

The basic object we have to study is the so-called *intrinsic second-order derivative* :

Definition 1.5. The *intrinsic second-order derivative* along γ is the real quadratic form :

$$E''_u(v) = \psi.d^2E_T(u).(v,v)$$

where $v \in$ Ker $dE_T(u)$.

Roughly speaking, if the latter quadratic form is positive (or negative) definite[1] then γ is locally isolated and thus locally optimal. Conversely if γ is optimal then E''_u is positive (or negative), see [5]. Actually this reasoning works for hyperbolic trajectories (see [16]). For exceptional trajectories the situation is a bit more complicated, and we have to study the intrinsic second-order derivative on a larger domain (see the *time×input/state mapping* in [5], *reduced operator* in [7]).

[1] A real quadratic form $q(x)$ is said *positive definite* if $x \neq 0 \implies q(x) > 0$, and *indefinite* if there exist $x, y \neq 0$ such that $q(x)q(y) < 0$.

1.1.2 Accessibility sets

Definition 1.6. Consider the general control system (1), and let $T > 0$. The *accessibility set at time* T, denoted by $Acc(T)$ is the set of points that can be reached from x_0 in time T by solutions of system (1), i.e. this is the image of the end-point mapping E_T.

Let γ be a reference trajectory on $[0, T]$, solution of (1), associated to a singular control u. Using the formalism of [7] we are able to describe precisely the boundary of $Acc(T)$ along γ for the single-input affine system (2).

2 Optimality of singular trajectories

Consider the single-input affine control system with constraint on the control :

$$\dot{x} = X + uY(x), \quad |u| \leqslant \eta \tag{3}$$

Let γ be a reference singular trajectory, defined on $[0, T]$ and such that $\gamma(0) = x_0$. If γ is injective we may assume that it is associated to the control $u = 0$. In the sequel we make the following assumptions along γ :

(H_0) γ is injective, associated to $u = 0$ on $[0, T]$.

(H_1) $\forall t \in [0, T]$ $\quad K(t) = \text{Vect } \{ad^k X.Y(\gamma(t)) \ / \ k \in \mathbb{N}\}$ (first Pontryagin's cone along γ) has codimension 1, and is spanned by the first $n-1$ vectors :

$$K(t) = \text{Vect } \{ad^k X.Y(\gamma(t)) \ / \ k = 0 \dots n-2\}$$

(H_2) $\forall t \in [0, T]$ $\quad ad^2 Y.X(\gamma(t)) \notin K(t)$.

(H_3) If $n = 2$ then : $\forall t \in [0, T]$ $\quad X(\gamma(t))$ and $Y(\gamma(t))$ are independant. If $n = 3$ then : $\forall t \in [0, T]$ $\quad X(\gamma(t)) \notin \text{Vect } \{ad^k X.Y(\gamma(t)) \ / \ k = 0 \dots n-3\}$.

In these conditions γ is *of corank 1*, and moreover we get normal forms which allow to express easily the differential operator representating the intrinsic second-order derivative and make easier the computation of conjugate times (see [7]).

We first investigate the time-optimal problem, and then the problem of minimizing some cost.

2.1 Time optimality

Definition 2.1. • The trajectory γ is said C^0*-time-minimal* on $[0, T]$ if there exists a C^0-neighborhood of γ such that T is the minimal time to steer $\gamma(0)$ to $\gamma(T)$ among the solutions of the system (3) that are entirely contained in this neighborhood.

- Recall that γ is associated to the control $u=0$. Let $\delta>0$. The trajectory γ is said *L^∞-time-minimal* on $[0,T]$ if there exists a neighborhood of 0 in $L^\infty([0,T+\delta])$ such that T is the minimal time to steer $\gamma(0)$ to $\gamma(T)$ among trajectories associated to controls of this neighborhood.

Obviously if γ is C^0-time-minimal then it is L^∞-time-minimal.

We have to distinguish between hyperbolic and exceptional cases. The following results generalize those of [7] which concern an affine system (3) *without any constraint on the control.*

2.1.1 Hyperbolic case

Lemma 2.1. *[7] Suppose that γ is hyperbolic and $n \geqslant 2$. Then the system (3) is in a C^0-neighborhood of γ feedback-equivalent to :*

$$f_0 = \frac{\partial}{\partial x_1} + \sum_{i=2}^{n-1} x_{i+1}\frac{\partial}{\partial x_i} + \sum_{i,j=2}^{n} a_{ij}(x_1)x_i x_j \frac{\partial}{\partial x_1} + R$$
$$f_1 = \frac{\partial}{\partial x_n}$$

where $a_{n,n}(t)<0$ on $[0,T]$ and R can be neglected in our work.

In these conditions, the controllable part of the system is $(x_2,\dots,x_n)$, the singular reference trajectory is $\gamma(t)=(t,0,\dots,0)$, and the intrinsic second-order derivative $d^2E_1^T(0)_{/\mathrm{Ker}\ dE_T(0)}$ along γ is identified to :

$$\int_0^T \sum_{i,j=2}^{n} a_{ij}(t)\xi_i(t)\xi_j(t)\ dt\ , \text{ where :}$$
$$\dot\xi_2 = \xi_3,\dots,\dot\xi_{n-1}=\xi_n, \dot\xi_n = v$$

Set $y=\xi_2$. Then it can be written as $Q_{T/G}$, where :

$$Q_T(y) = \int_0^T q_T(y)dt \quad \text{and} \quad q_T(y) = \sum_{i,j=0}^{n-2} b_{ij} y^{(i)} y^{(j)}$$

with $b_{i-2,j-2} = \frac{a_{ij}+a_{ji}}{2}$, and where G is the following space corresponding to the kernel of the first derivative :

$$G = \{y\ /\ y^{(2(n-2))} \in L^2([0,T]),\ y^{(i)}(0) = y^{(i)}(T) = 0,\ i=0\dots n-2\}$$

Integrating by parts we get :

Lemma 2.2 ([7]). *The quadratic form Q_T is represented on G by the operator D_T so that :*

$$Q_T(y) = (D_T y, y)_{L^2}$$

where $(\ ,\ )_{L^2}$ *is the usual scalar product in* $L^2([0,T])$*, and the operator* D_T *is :*

$$D_T \;=\; \frac{1}{2}\sum_{i=0}^{n-2}(-1)^i \frac{d^i}{dt^i}\frac{\partial q}{\partial y^{(i)}} \;=\; \sum_{i,j=0}^{n-2}(-1)^j \frac{d^j}{dt^j} b_{ij} \frac{d^i}{dt^i}$$

Our aim is to study the sign of Q_T, thus we are lead to make a spectral analysis of D_T. Unfortunately the spectrum of D_T on G is empty. Hence we have to enlarge this space so that the spectrum is not trivial and that the representation lemma 2.2 is still valid. That's why we set :

$$F = \{y \;/\; y^{(n-2)} \in L^2([0,T]),\ y^{(i)}(0) = y^{(i)}(T) = 0,\ i = 0 \dots n-3\}$$

Definition 2.2. We call T a conjugate time of Q along γ if there exists $y \in F$ such that $y^{(2(n-2))} \in L^2([0,T])$ and $D_T y = 0$.

Lemma 2.3. *For any* $f \in L^2([0,T])$*, if* T *is not a conjugate time, there exists* $y \in F$ *unique such that* $y^{(2(n-2))} \in L^2([0,T])$ *and* $D_T y = f$*. Let* L *denote the operator* $f \mapsto y$ *considered as an operator from* $L^2([0,T])$ *into* $L^2([0,T])$ *; it is selfadjoint and compact.*

Let t_c be the first conjugate time of the operator D. It is known (see for instance [7], [16], [4]) that $t_c > 0$ or $t_c = +\infty$. We have the following result :

Theorem 2.4. *The trajectory* γ *is* C^0*-time-minimal if and only if* $T < t_c$*. Moreover* γ *is not* L^∞*-time-minimal if* $T > t_c$*.*

The shape of accessibility sets at time T is represented on fig. 1.

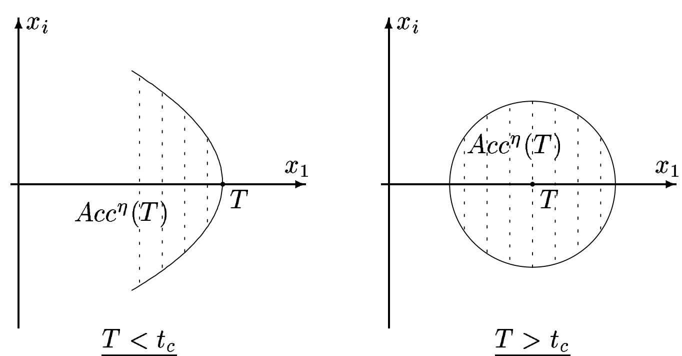

Figure 1: Hyperbolic case

Remark 2.1. In dimension 2, the operator D is equal to $b_0 Id$, and thus $t_c = +\infty$ (provided assumptions $(H_0 - H_3)$ are fulfilled on $\mathbb{R}^+$), i.e. γ is C^0-time-minimal on $\mathbb{R}^+$.

2.1.2 Exceptional case

Lemma 2.5. *[7] Consider the affine system $\dot{q} = X + wY$, $q(0) = 0$ under the assumptions $(H_0 - H_3)$, and suppose γ is exceptional. Then in a C^0-neighborhood of γ the system (X, Y) is feedback-equivalent to :*

$$\begin{aligned} f_0 &= \frac{\partial}{\partial x_1} + \sum_{i=1}^{n-2} x_{i+1} \frac{\partial}{\partial x_i} + \sum_{i,j=2}^{n} a_{ij}(x_1) x_i x_j \frac{\partial}{\partial x_n} + R \\ f_1 &= \frac{\partial}{\partial x_{n-1}} \end{aligned} \tag{4}$$

where $a_{n-1,n-1}(t) > 0$ on $[0, T]$ and R can be neglected in our work.

Set $x_1 = t + \xi$. The controllable part of the system is $(\xi, x_2, \ldots, x_{n-1})$, the reference singular trajectory is $\gamma(t) = (t, 0, \ldots, 0)$, and the intrinsic second-order derivative $d^2 E_n^T(0)_{\text{Ker } dE_n^T(0)}$ along γ is identified to :

$$\int_0^T \sum_{i,j=2}^{n-1} a_{ij}(t) \xi_i(t) \xi_j(t) \; dt \text{ , where :}$$

$$\dot{\xi}_1 = \xi_2, \ldots, \dot{\xi}_{n-2} = \xi_{n-1}, \dot{\xi}_{n-1} = v, \quad \text{and} \quad \xi_i(0) = \xi_i(T) = 0, i = 1 \ldots, n-1$$

Contrarily to the hyperbolic case where only one differential operator is in a natural way associated to the intrinsic second-order derivative, here in the exceptional case we get *two* natural operators in a natural way :

1. If $\xi = x_1 - t$, it can be written as $Q_{/G}$, where :

   $$Q(\xi) = \int_0^T q(\xi) dt \quad \text{and} \quad q(\xi) = \sum_{i,j=1}^{n-2} b_{ij} \xi^{(i)} \xi^{(j)}$$

   with $b_{i-1,j-1} = \frac{a_{ij} + a_{ji}}{2}$, and where G is the following space corresponding to the kernel of the first derivative :

   $$G = \{\xi \; / \; \xi^{(2(n-2))} \in L^2([0, T]), \; \xi^{(i)}(0) = \xi^{(i)}(T) = 0, \; i = 0 \ldots n-2\}$$

   Let D be the operator representing Q. We have :

   $$Q(\xi) = (\xi, D\xi)_{L^2}$$

   where

   $$D \;=\; \frac{1}{2} \sum_{i=1}^{n-2} (-1)^i \frac{d^i}{dt^i} \frac{\partial q}{\partial y^{(i)}} \;=\; \sum_{i,j=1}^{n-2} (-1)^j \frac{d^j}{dt^j} b_{ij} \frac{d^i}{dt^i} \tag{5}$$

2. It can be expressed in function of x_2 as Q_{1/G_1}, where :

$$Q_1(x_2) = \int_0^T q_1(x_2)dt \quad \text{and} \quad q_1(x_2) = \sum_{i,j=0}^{n-3} b_{i+1,j+1} x_2^{(i)} x_2^{(j)}$$

and where G_1 is the space corresponding to the kernel of the first derivative :

$$G_1 = \{x_2 \ / \ x_2^{(2(n-3))} \in L^2([0,T]),\ x_2^{(i)}(0) = x_2^{(i)}(T) = 0,\ i = 0 \ldots n-3, \\ \text{and} \int_0^T x_2 \ dt = 0\}$$

Let D_1 be the operator representing Q_1. We have :

$$Q_1(x_2) = (x_2, D_1 x_2)_{L^2}$$

where

$$D_1 \ = \ \frac{1}{2}\sum_{i=0}^{n-3}(-1)^i \frac{d^i}{dt^i}\frac{\partial q}{\partial y^{(i)}} \ = \ \sum_{i,j=0}^{n-3}(-1)^j \frac{d^j}{dt^j} b_{i+1,j+1}\frac{d^i}{dt^i} \tag{6}$$

Note that $Q(\xi) = Q_1(\dot{\xi})$ and $D = -\frac{d}{dt}D_1\frac{d}{dt}$. As previously the spectral study of these operators has to be made on larger spaces :

- $F = \{\xi \ / \ \xi^{(n-2)} \in L^2([0,T]),\ \xi^{(i)}(0) = \xi^{(i)}(T) = 0,\ i = 0 \ldots n-3\}$ for the operator D.
- $F_1 = \{x_2 \ / \ x_2^{(n-3)} \in L^2([0,T]),\ x_2^{(i)}(0) = x_2^{(i)}(T) = 0,\ i = 0 \ldots n-4\}$ for D_1 if $n \geqslant 4$ (if $n = 3$, no condition is imposed).

The following lemma is an improvement of [7], where only a non strict inequality is proved :

Lemma 2.6. *Let t_c (resp. t_{cc}) denote the first conjugate time of Q on F (resp. Q_1 on F_1). Then : $0 < t_{cc} < t_c$.*

We have the following result :

Theorem 2.7. *The trajectory γ is C^0-time-minimal if and only if $T < t_{cc}$. Moreover γ is not L^∞-time-minimal if $T > t_{cc}$.*

The shape of accessibility sets in function of T is represented on fig. 2.

Remark 2.2. If $n = 3$, we have $t_{cc} = +\infty$ provided assumptions $(H_0 - H_3)$ are fulfilled on $\mathbb{R}^+$. Hence in this case γ is C^0-time-minimal on $\mathbb{R}^+$.

Remark 2.3. In both hyperbolic and exceptional cases, the notion of conjugate time and the optimality of γ *do not depend on the constraint* on the control. It comes from the fact that singular reference control belongs to the *interior* of the domain of constraints.

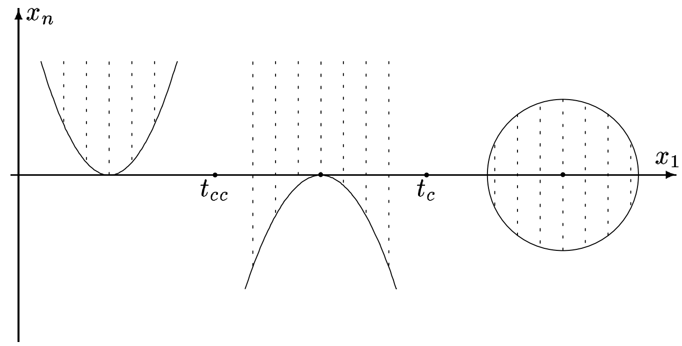

Figure 2: Exceptional case

2.2 Optimality for some cost

Let us now investigate the problem of minimizing some cost $C(T,u)$, also denoted by $C_T(u)$, where C is a smooth function satisfying the following additional assumption along the reference singular trajectory γ :

(H_4) $\forall T \quad \text{rank}\ (dE_T(0), dC_T(0)) = n$

i.e. the singularity of the end-point mapping of the extended system has codimension 1, and in particular the cost is independant from the end-point mapping along γ. We investigate several optimization problems :

1. final time not fixed : the aim is to steer the system from x_0 to x_1 in some time T (not preassigned) and minimizing the cost C.

2. final time fixed : let $T > 0$; the aim is to steer the system from x_0 to x_1 in time T and minimizing the cost C_T.

2.2.1 Final time not fixed

Definition 2.3.

- The trajectory γ is said to be C^0-*cost-minimal* on $[0,T]$ if there exists a C^0-neighborhood of γ such that for any trajectory q contained in this neighborhood, with $q(0) = \gamma(0)$ and $q(t) = \gamma(T)$, we have : $C(t,v) \geqslant C(T,0)$, where v is the control associated to q.

- Let $\delta > 0$. The trajectory γ is said to be L^∞-*cost-minimal* on $[0,T]$ if there exists a neighborhood of 0 in $L^\infty([0,T+\delta])$ such that, for any trajectory q associated to a control v of this neighborhood, with $q(0) = \gamma(0)$ and $q(t) = \gamma(T)$, we have : $C(t,v) \geqslant C(T,0)$.

Obviously the C^0-cost-minimality implies the L^∞-cost-minimality.

We have the following result (compare with [5]) :

Theorem 2.8. *1. If γ is hyperbolic, γ is never L^∞-cost-minimal.*

2. If γ is exceptional, then γ is C^0-cost-minimal if and only if it is C^0-time-minimal. Actually, γ is C^0-cost-minimal if $T < t_{cc}$, and is not L^∞-cost-minimal if $T > t_{cc}$.

Hence in the exceptional case, both problems of cost-minimization and time-minimization are equivalent.

2.2.2 Final time fixed

Definition 2.4.
- The trajectory γ is said to be C^0-*cost-minimal* on $[0,T]$ if there exists a C^0-neighborhood of γ such that for any trajectory q contained in this neighborhood, with $q(0) = \gamma(0)$ and $q(T) = \gamma(T)$, we have : $C_T(v) \geqslant C_T(0)$, where v is the control associated to q.
- The trajectory γ is said to be L^∞-*cost-minimal* on $[0,T]$ if there exists a neighborhood of 0 in $L^\infty([0,T])$ such that, for any trajectory q associated to a control v of this neighborhood, with $q(0) = \gamma(0)$ and $q(T) = \gamma(T)$, we have : $C_T(v) \geqslant C_T(0)$.

We have the following :

Theorem 2.9. *1. If γ is hyperbolic, then γ is C^0-cost-minimal if and only if it is C^0-time-minimal. Actually, γ is C^0-cost-minimal if $T < t_c$, and is not L^∞-cost-minimal if $T > t_c$, where t_c denotes the first conjugate time of γ (see theorem 2.4).*

2. If γ is exceptional, then γ is C^0-cost-minimal if and only if $T < t_c$. Moreover, γ is not L^∞-cost-minimal if $T > t_c$ (whereas γ is C^0-time-minimal if and only if $T < t_{cc}$), where t_{cc} and t_c denote the two types of first conjugate times of γ (see lemma 2.6).

Hence in the hyperbolic case, the times at which γ ceases to be minimizing are the same in both time-optimal and cost-optimal problems. At the contrary in the exceptional case they are different : γ ceases to be C^0-time-optimal *before* it ceases to be C^0-cost-optimal (since $t_{cc} < t_c$, see lemma 2.6).

2.3 Application to the sub-Riemannian case

Consider a smooth sub-Riemannian structure (M, Δ, g) where M is a Riemannian n-dimensional manifold, $n \geqslant 3$, Δ is a rank 2 distribution on M, and g is a metric on Δ. Let $q_0 \in M$; our point of view is local and we can assume that $M = \mathbb{R}^n$ and $q_0 = 0$. Suppose there exists a smooth injective abnormal (or singular) trajectory γ passing through 0. Up to changing coordinates and reparametrizing we can assume that :

- $\gamma(t) = (t, 0, \dots, 0)$,

- $\Delta = \text{Span}\ \{X, Y\}$ where X, Y are g-orthonormal,
- γ is the integral curve of X passing through 0.

Under these assumptions, the sub-Riemannian problem is equivalent to the *time-optimal problem* for the system with constraint :

$$\begin{aligned} &\dot{q} = vX + uY,\ q(0) = 0 \\ &v^2 + u^2 \leqslant 1 \end{aligned} \tag{7}$$

and the trajectory γ corresponds to the control : $v = 1, u = 0$.

Definition 2.5. We call *affine system associated to the sub-Riemannian system (7)* the following system :

$$\dot{x} = X(x) + wY(x) \tag{8}$$

where the control w satisfies a constraint of the form : $|w| \leqslant \eta$.

In order to investigate the optimality of the trajectory γ for the sub-Riemannian system (7), we compare this system with its associated affine system (8). The fact that the optimality of γ for the affine system does not depend on the constraint is crucial. We obtain the following result :

Theorem 2.10. *Suppose that assumptions ($H_0 - H_3$) are fulfilled along γ for the system (X, Y). Then γ is C^0-optimal for the sub-Riemannian system (7) if and only if it is C^0-time-minimal for its associated affine system (8). Moreover γ is exceptional for this affine system ; actually γ is C^0-optimal if $T < t_{cc}$ and is not L^∞-optimal if $T > t_{cc}$.*

In particular conjugate times are the same along γ for both systems. Therefore the whole formalism that was introduced for affine systems (the differential operator D_1) is still valid in sub-Riemannian geometry. Hence the conjugate time of the sub-Riemannian problem can be computed using an algorithm. This result makes a link between works of [7] and [3], [4].

Example 2.1. The Martinet case (see section 3.2.2) is in dimension 3, hence $t_{cc} = +\infty$ (see remark 2.2). The abnormal trajectory is optimal on $\mathbb{R}^+$.

Remark 2.4. As proved in [2] the C^0-optimality is in sub-Riemannian geometry equivalent to the optimality in the sense of L^2 on controls.

Remark 2.5. If T is small enough (depending on the choice of the Riemannian structure, and lower than t_{cc}), then as first noted by [3] γ is moreover *globally* optimal among all sub-Riemannian trajectories steering 0 to $\gamma(T)$.

Remark 2.6. It should be noted that the loss of optimality is in L^∞. Hence controls L^2-close to the reference abnormal control have no influence on the optimality of the abnormal trajectory (see splitting-up in sectors, section 3.2.2).

3 Asymptotics of the accessibility sets

In this section we describe very precisely the boundary of accessibility sets for a single-input affine system near a reference singular trajectory. These boundaries are the level sets of the value function associated to the time-optimal problem. Then we apply our results to the sub-Riemannian case of rank 2, where we describe precisely the *contact* of the sphere with the abnormal direction. As a consequence we obtain a *splitting-up of the sphere into two sectors near the abnormal minimizer.*

3.1 Single-input affine control systems

3.1.1 Hyperbolic case

In this case the shape of the accessibility set *depends on the constraint*, as shown in the following example :

$$\begin{array}{rcl} \dot{x} &=& 1+y^2 \\ \dot{y} &=& u \end{array} \qquad \text{where } |u| \leqslant \eta$$

The accessibility set $Acc^\eta(T)$ is represented on fig. 3.

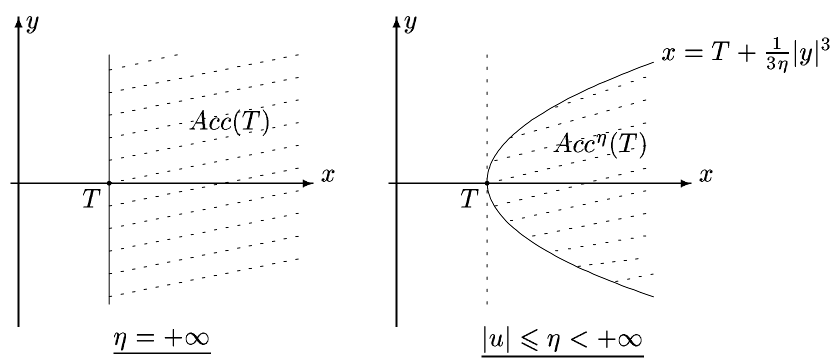

Figure 3: Hyperbolic case

3.1.2 Exceptional case

Contrarily to the previous case, in the exceptional case the boundary of $Acc^\eta(T)$ *does not depend on the constraint* near $\gamma(T)$. Precisely we have the following :

Theorem 3.1. *Consider the affine system with constraint (3) and suppose that assumptions ($H_0 - H_3$) are fulfilled along the reference singular trajectory γ on*

$[0,T]$. *We suppose that γ is exceptional. Let t_{cc} and t_c denote the first conjugate times associated to γ, see section 2.1.2. Then :*

1. *There exist coordinates $(x_1,\dots,x_n)$ locally along γ such that in these coordinates : $\gamma(t)=(t,0,\dots,0)$, and the first Pontryagin's cone along γ is : $K(t)=$ Vect $\{\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_{n-1}}\}_{|\gamma}$.*

2. *If T is small enough then for any point $(x_1,\dots,x_n)$ of $Acc^\eta(T)\backslash\{(T,0,\dots,0)\}$ close to $\gamma(T)$ we have : $x_n>0$ (see fig. 4).*

3. *If $T<t_c$, then in the plane (x_1,x_n), near the point $(T,0)$, the boundary of $Acc^\eta(T)$ does not depend on η, is a curve of class C^2 tangent to the singular direction, and its first term is :*

$$x_n = A_T(x_1-T)^2 + o((x_1-T)^2)$$

   The function $T\mapsto A_T$ is continuous and strictly decreasing on $[0,t_c[$. It is positive on $[0,t_{cc}]$ and negative on $[t_{cc},t_c[$.
   Moreover, if η depends on x_1-T then the result is still valid providing : $x_1-T=o(\eta)$ as $x_1\to T$.

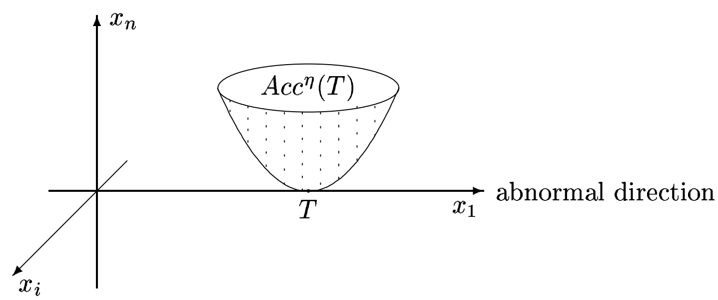

Figure 4: Shape of $Acc^\eta(T)$, T small

The evolution of $Acc^\eta(T)$ in function of T in the plane (x_1,x_n) is represented on fig. 5. The contact with the singular direction is of order 2 ; the coefficient A_T describes the concavity of the curve. Beyond t_c the accessibility set is open.

Remark 3.1. The coefficient A_T can be computed in the following way (see [7], [18]). Let D denote the operator (5) introduced in section 2.1.2 and Q the quadratic form associated to D, representing the intrinsic second-order derivative along γ. There exists a function J of class $C^{2(n-2)}$ on $[0,T]$ such that $DJ=0$ and satisfying the limit conditions :

$$\forall k\in\{0,\dots,n-3\}\quad J^{(k)}(0)=0,\ J^{(k)}(T)=\delta_0^k$$

Then :

$$A_T = Q(J) \tag{9}$$

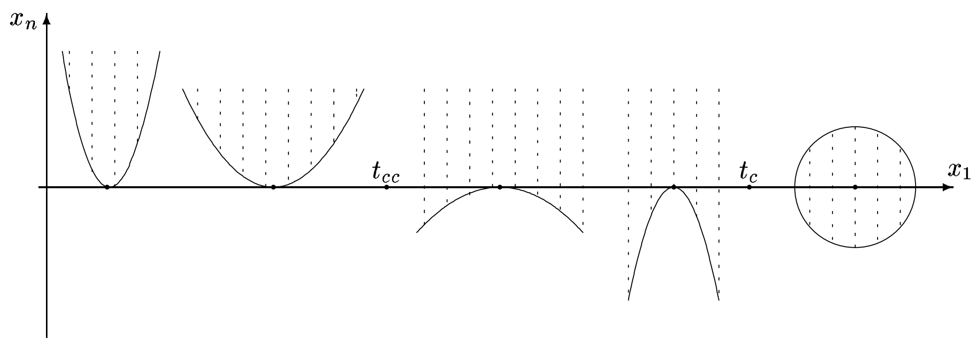

Figure 5:

3.2 Application to the sub-Riemannian case

3.2.1 Asymptotics of the sub-Riemannian sphere along an abnormal direction

Let us consider the framework introduced in section 2.3, and let us now define a notion of *constrained accessibility set* :

Definition 3.1. Let $0 < \alpha < 1$. We denote by $Acc^{\alpha}_{SR}(T)$ the accessibility set at time T for the sub-Riemannian system (7) with the additional constraint on the control :

$$v^2 + u^2 \leqslant 1 \ , \ \ 1 - \alpha \leqslant v \leqslant 1 \ , \ \ |u| \leqslant \alpha$$

(see fig. 6)

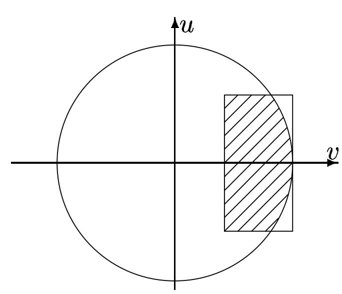

Figure 6:

Note that controls steering 0 to points of $Acc^{\alpha}_{SR}(T)$ are in a L^{∞}*-neighborhood* of the abnormal reference control $v = 1, u = 0$.

Consider the *associated affine system* :

$$\dot{x} = X(x) + wY(w) \tag{10}$$

and denote by $Acc_A^{\eta}(T)$ the accessibility set at time T for this system with the constraint : $|w| \leqslant \eta$. The reference singular trajectory γ corresponds to $w = 0$, and is *exceptional* for this affine system.

Theorem 3.2. *Suppose assumptions* $(H_0 - H_3)$ *are fulfilled along the reference abnormal trajectory* γ *for the system* (X, Y). *Let* t_{cc} *and* t_c *denote the first conjugate times of* γ *for the associated affine system. Let* $\alpha \in]0, 1[$. *Then :*

1. *There exist coordinates* $(x_1, \dots, x_n)$ *locally along* γ *such that in these coordinates :* $\gamma(t) = (t, 0, \dots, 0)$, *and the first Pontryagin's cone along* γ *is :* $K(t) = \text{Vect}\,\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_{n-1}}\}_{|\gamma}$.

2. *If* T *is small enough then for any point* $(x_1, \dots, x_n)$ *of* $Acc_{SR}^{\alpha}(T)$ *close to* $\gamma(T)$ *we have* $x_n \geqslant 0$ *(see fig. 4).*

3. *If* $T < t_{cc}$, *then in the plane* (x_1, x_n), *close to the point* $(T, 0)$, *the boundary of* $Acc_{SR}^{\alpha}(T)$ *does not depend on* α, *is a curve of class* C^2 *outside* $(T, 0)$, *tangent to the abnormal direction, whose first term is :*

   - *if* $x_1 \leqslant T$ *then* $x_n = 0$.
   - *if* $x_1 \geqslant T$ *then* $x_n = A_T(x_1 - T)^2 + o((x_1 - T)^2)$.

   The function $T \mapsto A_T$ *is the same as in theorem 3.1.*

Figure 7 represents the evolution of $Acc_{SR}^{\alpha}(T)$ in function of T in the plane (x_1, x_n). It is open in a neighborhood of $\gamma(T)$ if $T > t_{cc}$, contrarily to the affine case where it becomes open only beyond t_c.

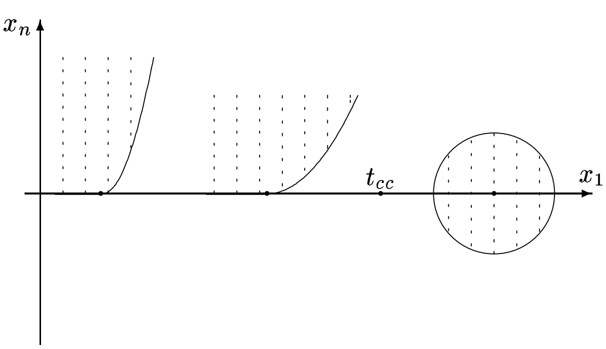

Figure 7:

Remark 3.2. To compare the system (7) with its associated affine system (10) we need the following reparametrizing :

$$\frac{ds}{dt} = v$$

which only holds if v does not vanish. This condition is satisfied when the control (v, u) is in a L^∞-neighborhood of the abnormal reference control $(1, 0)$, for in this case v is close to 1 in the L^∞ sense. Hence using this method it is only possible to describe a constrained accessibility set, i.e. in a L^∞-neighborhood of the reference abnormal control.

3.2.2 Splitting-up of the sphere near an abnormal direction

Let $T > 0$ small enough so that properties 2 and 3 of theorem 3.2 are satisfied. In particular the reference abnormal trajectory γ is minimizing. Then $A = \gamma(T)$ belongs to the sub-Riemannian sphere $S(0, T)$ with radius T. If controls steering 0 to points of the boundary of $Acc^{\alpha}_{SR}(T)$ in $x_n > 0$ (that are *L^∞-optimal*) are actually *globally optimal*, then this boundary is included in the sphere $S(0, T)$. In this case the sphere splits into *two sectors* near $\gamma(T)$, bordered by the first Pontryagin's cone $x_n = 0$:

- sector $x_n > 0$ corresponding to the previous description,
- sector $x_n < 0$.

According to the previous results, final points at time T associated to controls which are L^∞-close to the reference abnormal control are in the first sector : $x_n > 0$. Obviously due to controllability of the system the sector $x_n < 0$ is accessible. In fact a basic calculus shows :

Lemma 3.3. *For any neighborhood V of the point A in $\mathbb{R}^n$ we have :*

$$S(0, T) \cap V \cap (x_n < 0) \neq \emptyset$$

These points in $(x_n < 0)$ are reached by controls which are close to the reference control *in the L^2 sense but not in the L^∞ sense.* More precisely :

Lemma 3.4. *Let $M_n = E_T(u_n) \in S(0, T)$ whose last coordinate x_n is strictly negative. Let u denote the abnormal reference control. We suppose that M_n converges to $A = E(u)$ in $\mathbb{R}^n$. Then u_n converges to u in $L^2([0, T])$ but not in $L^\infty([0, T])$.*

Hence near the abnormal direction the sphere is made of two sectors : a *L^∞-sector* $(x_n > 0)$ described by theorem 3.2, and a *L^2-sector* $(x_n < 0)$. The contact of the first sector is known, but not the second one a priori. Anyway according to the *tangency theorem* (see [17]), under some nice stratification assumptions, this sector *ramifies tangently* to the Pontryagin cone $x_n = 0$, see fig. 8.

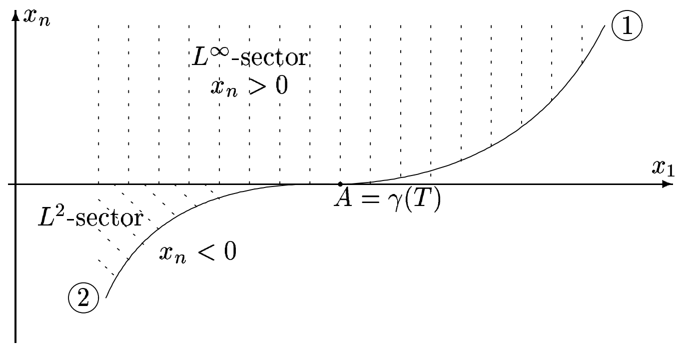

Figure 8:

Typical example : the Martinet case. Consider the two following vector fields in $\mathbb{R}^3$:

$$X = \frac{\partial}{\partial x} + \frac{y^2}{2}\frac{\partial}{\partial z} \ , \ \ Y = \frac{\partial}{\partial y}$$

and endow the distribution spanned by these vector fields with an analytic metric g of the type :

$$g = adx^2 + cdy^2$$

where $a = (1+\alpha y)^2$ and $c = (1+\beta x+\gamma y)^2$. The abnormal reference control for the sub-Riemannian system $\dot{x} = vX(x) + uY(x)$ with constraint $v^2+u^2 \leqslant 1$ is $v=1, u=0$, and corresponds to the trajectory $\gamma : x(t) = t, y(t) = z(t) = 0$. We have, see [9] :

Lemma 3.5. *Assumptions ($H_0 - H_3$) are fulfilled along γ if and only if $\alpha \neq 0$. In this case branches 1 et 2 (see fig. 8 with $x_1 = x, x_n = z$) have the following contacts with the abnormal direction :*

- *branch 1 : $x \geqslant T,\ z = \frac{1}{2T\alpha^2}(x-T)^2 + o((x-T)^2)$*
- *branch 2 : $x \leqslant T,\ z \sim \frac{1}{6}(1+O(T))(x-T)^3$*

Remark 3.3. The coefficient A_T of the first branch can be computed directly or using formula (9) (see remark 3.1).

As we are in dimension 3, results of theorem 3.2 are in fact available on $\mathbb{R}^+$, see remark 2.2. The L^∞-sector is $z>0$ and corresponds to controls that are globally minimizing.

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