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ORIGINAL ARTICLE

Integral and measure-turnpike properties for infinite-dimensional optimal control systems

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Abstract We first derive a general integral-turnpike property around a set for infinite-dimensional non-autonomous optimal control problems with any possible terminal state constraints, under some appropriate assumptions. Roughly speaking, the integral-turnpike property means that the time average of the distance from any optimal solution to the turnpike set converges to zero, as the time horizon tends to infinity. Then, we establish the measure-turnpike property for strictly dissipative optimal control systems, with state and control constraints. The measure-turnpike property, which is slightly stronger than the integral-turnpike property, means that any optimal solution remains essentially, during the time frame, close to an optimal solution of an associated static optimal control problem, except during a subset of time frame that is of small relative Lebesgue measure as the time horizon is large. Motivated by a specific example of optimal control problem for the heat equation with control constraints, we next prove that strict strong duality, which is a classical notion in optimization, implies strict dissipativity, and measure-turnpike. Finally, we conclude the paper with several comments and open problems.

Keywords Measure-turnpike · Strict dissipativity · Strong duality · State and control constraints

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1 Introduction

We start this paper with an intuitive idea in general terms. Consider the optimal control problem

$$\boxed{\begin{aligned}&\inf \frac{1}{T}\int_0^T f^0(y(t),u(t))\,\mathrm{d}t,\\ &\text{subject to } \ \dot y(t)=f(y(t),u(t)),\quad t\in[0,T],\end{aligned}}$$

under some terminal state conditions, with $T>0$ large. Setting $s=t/T$ and $\varepsilon=1/T$, we rewrite the above optimal control problem as

$$\boxed{\begin{aligned}&\inf \int_0^1 f^0(y(s),u(s))\,\mathrm{d}s,\\ &\text{subject to } \ \varepsilon\dot y(s)=f(y(s),u(s)),\quad s\in[0,1].\end{aligned}}$$

Then, we expect that, as $\varepsilon\to 0$, there is some convergence to the static problem

$$\boxed{\inf f^0(y,u),\quad \text{subject to } \ f(y,u)=0.}$$

This intuition has been turned into rigorous results in the literature, under some appropriate assumptions. These results say roughly that, if T is large, then any optimal solution $(y(\cdot),u(\cdot))$ on $[0,T]$ spends most of its time close to an optimal solution (y_s,u_s) of the static problem. This is the (neighborhood) turnpike phenomenon. We call the pair (y_s,u_s) a turnpike point.

This turnpike phenomenon was first observed and investigated by economists for discrete-time optimal control problems (see, e.g., [11,20]). In the last three decades, many turnpike results have been established in a large number of works (see, e.g., [1,2,6,7,9,12,16,19,25,33–35] and references therein), either for discrete-time or continuous-time problems involving control systems in finite-dimensional state spaces, and very few of them in the infinite-dimensional setting.

A more quantitative turnpike property, which is called the exponential turnpike property, has been established in [22,23,30] for both the linear and nonlinear continuous-time optimal controlled systems. It means that the optimal solution for the dynamic controlled problem remains exponentially close to an optimal solution for the corresponding static controlled problem within a sufficiently large time interval contained in the long-time horizon under consideration. We stress that in those works not only the optimal state and control, but also the corresponding adjoint vector, resulting from the application of the Pontryagin maximum principle, were shown to remain exponentially close to an extremal triple[1] for a corresponding static optimal control problem, except at the extremities of the time horizon. The main ingredient in the papers [22,23,30] is an exponential dichotomy transformation and the hyperbolicity feature of the Hamiltonian system derived from the Pontryagin maximum principle under some controllability and observability assumptions.

However, not all turnpike phenomena are around a single point. For instance, the turnpike theorem for calculus of variations problems in [25] is proved for the case when there are several turnpikes. More precisely, they show that there exists a competition between the several turnpikes for optimal trajectories with different initial states, and provide in particular a criterion for the choice of turnpikes that are in competition. On the other hand, for some classes of optimal control problems for periodic systems, the turnpike phenomenon may occur around a periodic trajectory, which is itself characterized as being the optimal solution of an appropriate periodic optimal control problem (cf., e.g., [26,29,34–36]).

In this paper, we first establish a more general turnpike result, valid for very general classes of optimal control problems settled in an infinite-dimensional state space, and where the turnpike phenomenon is around a set. This generalizes the standard case where the turnpike is a singleton, and the less standard case where the turnpike is a periodic trajectory. Between the case of one singleton and the periodic trajectory, however, there are, to our knowledge, very few examples for intermediate situations in the literature. Next, we recall the well-known notions of (strict) strong duality and of (strict) dissipativity and we define the measure-turnpike property. We establish that:

strict strong duality $\Rightarrow$ strict dissipativity $\Rightarrow$ measure-turnpike property

and we illustrate these notions and implications on various examples. We also show how dissipativity can be used to identify the long-time limit of optimal values.

The organization of the paper is as follows. In Sect. 2, we build up an abstract framework to derive a general turnpike phenomenon around a set. In Sect. 3, we enlighten the relationship between the above-mentioned abstract framework and the strict dissipativity property. Under the strict dissipativity assumption for optimal control problems, we establish the so-called measure-turnpike property. In Sect. 4, we provide some material to clarify the relationship between measure-turnpike, strict dissipativity and strong duality. Finally, Sect. 5 concludes the paper.

2 An abstract setting

In this section, we are going to derive a general turnpike phenomenon around a set. The framework is the following.

Let X (resp., U) be a reflexive Banach space endowed with the norm $\|\cdot\|_X$ (resp., $\|\cdot\|_U$). Let $f : \mathbb{R} \times X \times U \rightarrow X$ be a continuous mapping that is uniformly Lipschitz continuous in (y, u) for all $t \in \mathbb{R}$. Let $f^0 : \mathbb{R} \times X \times U \rightarrow \mathbb{R}$ be a continuous function

[1] In the statement of Pontryagin maximum principle for optimal control problems, the optimal state, optimal control and associated adjoint vector are called the extremal triple.

that is bounded from below. Let E and F be two subsets of X and U, respectively. Given any $t_0 \in \mathbb{R}$ and $t_1 \in \mathbb{R}$ with $t_0 < t_1$, we consider the non-autonomous optimal control problem

$$(P_{[t_0,t_1]}) \quad \begin{cases} J_{[t_0,t_1]} = \inf \frac{1}{t_1-t_0} \int_{t_0}^{t_1} f^0(t, y(t), u(t))\,\mathrm{d}t, \\ \text{subject to} \quad \dot{y}(t) = A(t)y + f(t, y(t), u(t)), \quad t \in [t_0, t_1], \\ R(t_0, y(t_0), t_1, y(t_1)) = 0, \quad (y(t), u(t)) \in E \times F, \quad t \in [t_0, t_1]. \end{cases} \tag{2.1}$$

Here, $(A(t), D(A(t)))$ is a family of unbounded operators on X such that the existence of the corresponding two-parameter evolution system $\Phi(t, s)$ is ensured (cf., e.g., [21, Chapter 5, Definition 5.3]), the controls are Lebesgue measurable functions $u(\cdot) : [t_0, t_1] \to F$, and Y is a Banach space, the mapping $R : \mathbb{R} \times X \times \mathbb{R} \times X \to Y$ stands for any possible terminal state conditions. Throughout the paper, the solutions $(y(\cdot), u(\cdot)) \in C([t_0, t_1]; X) \times L^2(t_0, t_1; U)$ are considered in the mild sense, meaning that

$$y(\tau) = \Phi(\tau, t_0)y(t_0) + \int_{t_0}^{\tau} \Phi(\tau, t) f(t, y(t), u(t))\,\mathrm{d}t, \qquad \forall \tau \in [t_0, t_1].$$

Remark 1 Typical examples of terminal conditions are the following:

- When both initial and final conditions are set free in $(P_{[t_0,t_1]})$, take $R = 0$.
- When the initial point is fixed and the final point is set free, take $R(s_0, z_0, s_1, z_1) = z_0 - y_0$.
- When both initial and final conditions are fixed (i.e., $y(t_0) = y_0$ and $y(t_1) = y_1$), take $R(s_0, z_0, s_1, z_1) = (z_0 - y_0, z_1 - y_1)$.
- When the final point is expected to coincide with the initial point (i.e., $y(t_0) = y(t_1)$ without any other constraint), for instance in a periodic optimal control problem, in which one assumes that there exists $T > 0$ such that $f(t+T, y, u) = f(t, y, u)$ and $f^0(t+T, y, u) = f^0(t, y, u), \forall (t, y, u) \in \mathbb{R} \times X \times U$, take $R(s_0, z_0, s_1, z_1) = (s_1 - s_0 - T, z_0 - z_1)$.

Hereafter, we call $(y(t), u(t))$, $t \in [t_0, t_1]$, an admissible pair if it verifies the state equation and the constraint $(y(t), u(t)) \in E \times F$ for almost every $t \in [t_0, t_1]$. We remark that the definition of admissible pair does not require that the terminal state condition $R(t_0, y(t_0), t_1, y(t_1)) = 0$ is satisfied. We denote by

$$C_{[t_0,t_1]}(y(\cdot), u(\cdot)) = \int_{t_0}^{t_1} f^0(t, y(t), u(t))\,\mathrm{d}t$$

the cost of an admissible pair $(y(\cdot), u(\cdot))$ on $[t_0, t_1]$. In other words, $J_{[t_0,t_1]}$ is the infimum with time average cost (*Cesàro mean*) over all admissible pairs satisfying the constraint on terminal points:

$$J_{[t_0,t_1]} = \inf \left\{ \frac{1}{t_1 - t_0} C_{[t_0,t_1]}(y(\cdot), u(\cdot)) \;\middle|\; (y(\cdot), u(\cdot)) \text{ admissible}, \; R(t_0, y(t_0), t_1, y(t_1)) = 0 \right\}.$$

Throughout the paper, we assume that the problem $(P_{[t_0,t_1]})$ has optimal solutions, and that an admissible pair $(y(\cdot), u(\cdot))$, with initial state $y(t_0)$, is said to be optimal for the problem $(P_{[t_0,t_1]})$ if $R(t_0, y(t_0), t_1, y(t_1)) = 0$ and $\frac{1}{t_1-t_0}C_{[t_0,t_1]}(y(\cdot), u(\cdot)) = J_{[t_0,t_1]}$. Existence of optimal solutions for optimal control problems is well-known under appropriate convexity assumptions on f^0, f and R with E and F convex and closed (see, for instance, [18, Chapter 3]).

We then consider the optimal control problem

$$(\bar{P}_{[t_0,t_1]}) \qquad \begin{cases} \bar{J}_{[t_0,t_1]} = \inf \frac{1}{t_1-t_0} C_{[t_0,t_1]}(y(\cdot), u(\cdot)), \\ \text{subject to} \quad \dot{y}(t) = A(t)y + f(t, y(t), u(t)), \quad t \in [t_0, t_1], \\ \quad (y(t), u(t)) \in E \times F, \quad t \in [t_0, t_1]. \end{cases} \tag{2.2}$$

Compared with the problem $(P_{[t_0,t_1]})$ as defined in (2.1), in the above problem there is no terminal state constraint, i.e., $R(\cdot) = 0$. In fact, $\bar{J}_{[t_0,t_1]}$ is the infimum with time average cost over all possible admissible pairs:

$$\bar{J}_{[t_0,t_1]} = \inf \left\{ \frac{1}{t_1 - t_0} C_{[t_0,t_1]}(y(\cdot), u(\cdot)) \;\middle|\; (y(\cdot), u(\cdot)) \text{ admissible} \right\}.$$

We say the problem $(\bar{P}_{[t_0,t_1]})$ has a limit value if

$$\lim_{t_1 \to +\infty} \bar{J}_{[t_0,t_1]}$$

exists as a real number. We refer to [15,24] for the sufficient conditions ensuring the existence of the limit value. More precisely, asymptotic properties of optimal values, as t_1 tends to infinity, have been studied in [24] under suitable non-expansivity assumptions, and in [15, Corollary 4 (iii)] by using occupational measures. In the sequel, we assume the limit value exists and is written as

$$\bar{J}_{[t_0,+\infty)} = \lim_{t_1 \to +\infty} \bar{J}_{[t_0,t_1]}.$$

Besides, given any $z \in X$ we define the value function

$$V_{[t_0,t_1]}(z) = \inf \left\{ \frac{1}{t_1 - t_0} C_{[t_0,t_1]}(y(\cdot), u(\cdot)) \;\middle|\; (y(\cdot), u(\cdot)) \text{ admissible},\ y(t_0) = z \right\}.$$

In fact, $V_{[t_0,t_1]}(z)$ is the optimal value of the optimal control problem with fixed initial data $y(t_0) = z$ (but free final point). Note that, if there exists no admissible trajectory starting at z (because E would not contain z), then we set $V_{[t_0,t_1]}(z) = +\infty$. For each $z \in X$, we say a limit value exists if

$$\lim_{t_1 \to +\infty} V_{[t_0,t_1]}(z)$$

exists as a real number. We now assume that, for each $z \in X$, the limit value exists and is written as

$$V_{[t_0,+\infty)}(z) = \lim_{t_1 \to +\infty} V_{[t_0,t_1]}(z).$$

Clearly, we have

$$\forall t_0 < t_1, \qquad J_{[t_0,t_1]} \geqslant \bar{J}_{[t_0,t_1]},$$

and thus

$$\liminf_{t_1 \to +\infty} J_{[t_0,t_1]} \geqslant \bar{J}_{[t_0,+\infty)}. \tag{2.3}$$

Meanwhile,

$$\forall t_0 < t_1, \quad \forall z \in X, \qquad V_{[t_0,t_1]}(z) \geqslant \bar{J}_{[t_0,t_1]},$$

and thus

$$\forall z \in X, \qquad V_{[t_0,+\infty)}(z) \geqslant \bar{J}_{[t_0,+\infty)}.$$

Remark 2 If the optimal control problem is autonomous (i.e., $A(\cdot) = A$, f and f^0 are independent of time variable), then it holds that $\bar{J}_{[t_0,+\infty)}$, as well as $V_{[t_0,+\infty)}(z)$, $\forall z \in X$, do not depend on $t_0 \in \mathbb{R}$.

Remark 3 Actually we have

$$\bar{J}_{[t_0,t_1]} = \inf_{z \in X} \; V_{[t_0,t_1]}(z).$$

This is obvious because we can split the infimum and write

$$\bar{J}_{[t_0,t_1]} = \inf_{z \in X} \; \inf_{\substack{(y(\cdot),u(\cdot)) \text{ admissible} \\ y(t_0)=z}} \; \frac{1}{t_1 - t_0} C_{[t_0,t_1]}(y(\cdot), u(\cdot)) = \inf_{z \in X} \; V_{[t_0,t_1]}(z).$$

In order to state the general turnpike result, we make the following assumptions:

(H_1). (Turnpike set) There exists a closed set $\mathcal{T} \times \mathcal{U} \subset X \times U$ (called turnpike set) such that

$$\forall t_0 \in \mathbb{R}, \quad \forall z \in \mathcal{T}, \qquad V_{[t_0,+\infty)}(z) = \bar{J}_{[t_0,+\infty)}.$$

(H_2). (Viability) The set $\mathcal{T}$ is *viable*, meaning that, for every $z \in \mathcal{T}$ and for every $t_0 \in \mathbb{R}$, there exists an admissible pair $(y(\cdot), u(\cdot))$ such that $y(t_0) = z$ and $y(t) \in \mathcal{T}$ for every $t \geqslant t_0$. Moreover, every admissible trajectory remaining in $\mathcal{T}$ is optimal in the following sense: for every $z \in \mathcal{T}$, for every $t_0 \in \mathbb{R}$, for every admissible pair $(y(\cdot), u(\cdot))$ such that $y(t_0) = z$ and $y(t) \in \mathcal{T}$ for every $t \geqslant t_0$, we have

$$V_{[t_0,+\infty)}(z) = \lim_{t \to +\infty} \frac{1}{t - t_0} C_{[t_0,t]}(y(\cdot), u(\cdot)).$$

(H_3). (Controllability) There exist $\bar{\delta}_0 > 0$ and $\bar{\delta}_1 > 0$ such that, for every $t_0 \in \mathbb{R}$ and every $t_1 \in \mathbb{R}$ with $t_1 > t_0 + \bar{\delta}_0 + \bar{\delta}_1$, and every optimal trajectory $y(\cdot)$ for the problem $(P_{[t_0,t_1]})$ as defined in (2.1),

- there exist $\delta_0 \in (0, \bar{\delta}_0]$ and an admissible pair $(y_0(\cdot), u_0(\cdot))$ on $[t_0, t_0 + \delta_0]$ such that $y_0(t_0) = y(t_0)$ and $y_0(t_0 + \delta_0) \in \mathcal{T}$,
- for every $z \in \mathcal{T}$, there exist $\delta_1 \in (0, \bar{\delta}_1]$ and an admissible pair $(y_1(\cdot), u_1(\cdot))$ on $[t_1 - \delta_1, t_1]$ such that $y_1(t_1 - \delta_1) = z$ and $y_1(t_1) = y(t_1)$.

(H_4). (Coercivity) There exist a monotone increasing continuous function $\beta : [0, +\infty) \to [0, +\infty)$ with $\beta(0) = 0$ and a distance $\mathrm{dist}(\cdot, \mathcal{T} \times \mathcal{U})$ to $\mathcal{T} \times \mathcal{U}$ such that for every t_0 and every $\hat{z} \in X$,

$$V_{[t_0,t_1]}(\hat{z}) \geqslant \inf_{z \in X} V_{[t_0,t_1]}(z) + \frac{1}{t_1 - t_0} \int_{t_0}^{t_1} \beta\big(\mathrm{dist}((\hat{y}(t), \hat{u}(t)), \mathcal{T} \times \mathcal{U})\big)\, \mathrm{d}t + \mathrm{o}(1),$$

holds for any optimal solution $(\hat{y}(\cdot), \hat{u}(\cdot))$ starting at $\hat{y}(t_0) = \hat{z}$ for the problem $(P_{[t_0,t_1]})$ as defined in (2.1), where the last term in the above inequality is an infinitesimal quantity as $t_1 \to +\infty$.

Hereafter, we speak of **Assumption (H)** in order to designate assumptions (H_1), (H_2), (H_3) and (H_4).

Remark 4 (i) Under (H_1), we actually have $\bar{J}_{[t_0,+\infty)} = \inf_{z \in X} V_{[t_0,+\infty)}(z)$, $\forall t_0 \in \mathbb{R}$.

(ii) (H_2) means that, starting at $z \in \mathcal{T}$, it is better to remain in $\mathcal{T}$ than to leave this set.

(iii) (H_3) is a specific controllability assumption. For instance, in the case that the initial point $y(t_0) = y_0$ and the final point $y(t_1) = y_1$ in the problem $(P_{[t_0,t_1]})$ are fixed, then (H_3) means that $\mathcal{T}$ is reachable from y_0 within time $\bar{\delta}_0$, and that y_1 is reachable from any point of $\mathcal{T}$ within time $\bar{\delta}_1$. When $\mathcal{T}$ is a single point, we refer the reader to [12] for a similar assumption.

(iv). (H_4) is a coercivity assumption involving the value function and the turnpike set $\mathcal{T} \times \mathcal{U}$. It may not be easy to verify this condition. However, under the strict dissipativity property (which will be introduced in the next section), (H_4) is satisfied. We refer the reader to Sect. 3 for more discussions about the relationship with strict dissipativity.

We first give a simple example which satisfies the **Assumption (H)**.

Example 1 Let $\Omega \subset \mathbb{R}^n$, $n \geqslant 1$, be a bounded domain with a smooth boundary $\partial\Omega$, and let $\mathcal{D} \subset \Omega$ be a non-empty open subset. We denote by $\chi_{\mathcal{D}}$ the characteristic function of $\mathcal{D}$. Let $M > 0$ and $y_0 \in L^2(\Omega)$ be arbitrarily given. For $t_0 < t_1$, consider the following optimal control problem for the heat equation:

$$\inf \frac{1}{t_1 - t_0} \int_{t_0}^{t_1} \Big(\|y(\cdot,t)\|^2_{L^2(\Omega)} + \|u(\cdot,t)\|^2_{L^2(\mathcal{D})} \Big) \, \mathrm{d}t$$

subject to

$$\begin{cases} y_t - \Delta y = \chi_{\mathcal{D}} u, & \text{in } \Omega \times (t_0, t_1), \\ y = 0, & \text{on } \partial\Omega \times (t_0, t_1), \\ y(\cdot, t_0) = y_0, \quad y(\cdot, t_1) = 0, & \text{in } \Omega, \\ \|u(\cdot,t)\|_{L^2(\mathcal{D})} \leqslant M, & \text{for a.e. } t \in (t_0, t_1). \end{cases}$$

Here, we take $X = L^2(\Omega)$, $U = L^2(\mathcal{D})$ and $F = \{u \in U \mid \|u\|_{L^2(\mathcal{D})} \leqslant M\}$. By the standard energy estimate, we can take $E = \{y \in X \mid \|y\|_{L^2(\Omega)} \leqslant \|y_0\|_{L^2(\Omega)} + M/\lambda_1\}$, where $\lambda_1 > 0$ is the first eigenvalue of the Laplace operator with zero Dirichlet boundary condition on $\partial\Omega$.

It is clear that $\bar{J}_{[t_0,+\infty)} = 0$. Let us define the turnpike set $\mathcal{T} \times \mathcal{U} = \{0\}$. By the L^∞-null controllability[2] and exponential decay of the energy of heat equations, the above control system with bounded controls is null controllable from each given point y_0 within a large time interval (see, e.g., [31]). Therefore, the assumptions (H_1), (H_2) and (H_3) are satisfied. Let $(y(\cdot), u(\cdot))$ be any optimal solution starting at $y(\cdot, t_0) = y_0$. By definition of the value function, we see that

$$V_{[t_0,t_1]}(y_0) \geqslant \frac{1}{t_1 - t_0} \int_{t_0}^{t_1} \Big(\|y(\cdot,t)\|^2_{L^2(\Omega)} + \|u(\cdot,t)\|^2_{L^2(\mathcal{D})} \Big) \, \mathrm{d}t.$$

Hence (H_4) is satisfied with $\beta(r) = r^2$, $r \geqslant 0$.

The main result of this paper is the following. It says that a general turnpike behavior occurs around the turnpike set $\mathcal{T} \times \mathcal{U}$, in terms of the time average of the distance from optimal solutions to $\mathcal{T} \times \mathcal{U}$.

Theorem 1 *Assume that f^0 is bounded on $\mathbb{R} \times E \times F$.*

(i) *Under (H_1), (H_2) and (H_3), for every $t_0 \in \mathbb{R}$ we have*

$$\lim_{t_1 \to +\infty} J_{[t_0,t_1]} = \bar{J}_{[t_0,+\infty)}. \tag{2.4}$$

(ii) *Furthermore, under the additional assumption (H_4) we have*

$$\lim_{t_1 \to +\infty} \frac{1}{t_1 - t_0} \int_{t_0}^{t_1} \beta\big(\mathrm{dist}((y(t), u(t)), \mathcal{T} \times \mathcal{U})\big) \, dt = 0, \tag{2.5}$$

[2] The L^∞-null controllability of the heat equation means that there exists a positive constant $C = C(\Omega, \mathcal{D})$ such that for any $z_0 \in L^2(\Omega)$, there is a control $f \in L^\infty(0, 1; L^2(\mathcal{D}))$ with $\|f\|_{L^\infty(0,1;L^2(\mathcal{D}))} \leqslant C\|z_0\|_{L^2(\Omega)}$ so that

$$\begin{cases} y_t - \Delta y = \chi_{\mathcal{D}} f, & \text{in } \Omega \times (0, 1), \\ y = 0, & \text{on } \partial\Omega \times (0, 1), \\ y(\cdot, 0) = z_0, \quad y(\cdot, 1) = 0, & \text{in } \Omega. \end{cases}$$

for any t_0 and any optimal solution $(y(\cdot), u(\cdot))$ of the problem $(P_{[t_0,t_1]})$ as defined in (2.1).

Remark 5 The boundedness assumption on f^0 in Theorem 1 can be removed in case the optimal control problem is autonomous, i.e., when A, f and f^0 do not depend on t, provided that the controllability assumption (H_3) be slightly reinforced, by assuming "controllability with finite cost": one can steer $y(t_0)$ to $\mathcal{T}$ within time δ_0 and steer any point of $\mathcal{T}$ to $y(t_1)$ within time δ_1 with a cost that is uniformly bounded with respect to every optimal trajectory $y(\cdot)$ and $z \in \mathcal{T}$. For non-autonomous control problems, see also Remark 7.

The property (2.5) is a weak turnpike property, which can be called the *β-integral-turnpike property*, and which is even weaker than the measure-turnpike property introduced further in Sect. 3.2. Indeed, from (2.5) we infer that for any $\delta > 0$, there exists $T_0 > t_0$ such that

$$\frac{1}{t_1 - t_0} \int_{t_0}^{t_1} \beta\big(\mathrm{dist}((y(t), u(t)), \mathcal{T} \times \mathcal{U})\big)\,\mathrm{d}t \leqslant \delta$$

for any $t_1 \geqslant T_0$. If, for any $\varepsilon > 0$, we set

$$Q^{\varepsilon}_{[t_0,t_1]} = \big\{t \in [t_0, t_1] \ \mid\ \mathrm{dist}\big((y(t), u(t)), \mathcal{T} \times \mathcal{U}\big) > \varepsilon\big\}, \qquad \forall t_1 \geqslant T_0.$$

Throughout the paper, we denote by $|Q|$ the Lebesgue measure of a subset $Q \subset \mathbb{R}$. Then, by Markov's inequality, one can easily derive that

$$\frac{\left|Q^{\varepsilon}_{[t_0,t_1]}\right|}{t_1 - t_0} \leqslant \frac{\delta}{\beta(\varepsilon)}, \qquad \forall t_1 \geqslant T_0.$$

This is weaker than the property (3.7) in Sect. 3.2.

Proof of Theorem 1 (i) Let $t_1 > t_0 + \bar{\delta}_0 + \bar{\delta}_1$, with $\bar{\delta}_0$ and $\bar{\delta}_1$ as in (H_3). Let $(y(\cdot), u(\cdot))$ be an optimal pair for the problem $(P_{[t_0,t_1]})$. By (H_2) and (H_3), there exist $\delta_0 \in (0, \bar{\delta}_0]$, $\delta_1 \in (0, \bar{\delta}_1]$ and an admissible pair $(\widetilde{y}(\cdot), \widetilde{u}(\cdot))$ such that

- $\widetilde{y}(\cdot)$ steers the control system from $y(t_0)$ to $\mathcal{T}$ within the time interval $[t_0, t_0 + \delta_0]$,
- $\widetilde{y}(\cdot)$ remains in $\mathcal{T}$ within the time interval $[t_0 + \delta_0, t_1 - \delta_1]$,
- $\widetilde{y}(\cdot)$ steers the control system from $\widetilde{y}(t_1 - \delta_1) \in \mathcal{T}$ to $y(t_1)$ within the time interval $[t_1 - \delta_1, t_1]$.

These trajectories are drawn in Fig. 1.

Its cost of time average within the time interval $[t_0, t_1]$ is

$$\begin{aligned}\frac{1}{t_1 - t_0} C_{[t_0,t_1]}(\widetilde{y}(\cdot), \widetilde{u}(\cdot)) &= \frac{1}{t_1 - t_0} C_{[t_0,t_0+\delta_0]}(\widetilde{y}(\cdot), \widetilde{u}(\cdot)) \\ &\quad + \frac{1}{t_1 - t_0} C_{[t_1-\delta_1,t_1]}(\widetilde{y}(\cdot), \widetilde{u}(\cdot)) + \frac{1}{t_1 - t_0} C_{[t_0+\delta_0,t_1-\delta_1]}(\widetilde{y}(\cdot), \widetilde{u}(\cdot)). \qquad (2.6)\end{aligned}$$

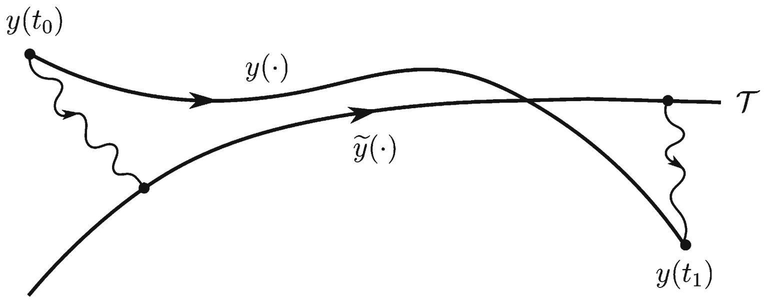

Fig. 1 Optimal trajectory $y(\cdot)$, and admissible trajectory $\widetilde{y}(\cdot)$ remaining along $\mathcal{T}$ as long as possible

Since f^0 is bounded on $\mathbb{R} \times E \times F$, the first two terms on the right-hand side of (2.6) converge to zero as $t_1 \to +\infty$. Since $\widetilde{y}(t_0 + \delta_0) \in \mathcal{T}$, by (H_2) we have

$$V_{[t_0+\delta_0,+\infty)}(\widetilde{y}(t_0+\delta_0)) = \lim_{t_1\to+\infty} \frac{1}{t_1 - \delta_1 - (t_0+\delta_0)} C_{[t_0+\delta_0, t_1-\delta_1]}(\widetilde{y}(\cdot), \widetilde{u}(\cdot)). \tag{2.7}$$

As $\widetilde{y}(t_0 + \delta_0) \in \mathcal{T}$, by (H_1) we infer

$$V_{[t_0+\delta_0,+\infty)}(\widetilde{y}(t_0+\delta_0)) = \bar{J}_{[t_0+\delta_0,+\infty)}. \tag{2.8}$$

We now claim that

$$\bar{J}_{[t_0+\delta_0,+\infty)} \leqslant \bar{J}_{[t_0,+\infty)}. \tag{2.9}$$

We postpone the proof of this claim and first see how it could be used in showing the convergence (2.4). Therefore, we derive from (2.8) and (2.9) that

$$V_{[t_0+\delta_0,+\infty)}(\widetilde{y}(t_0+\delta_0)) \leqslant \bar{J}_{[t_0,+\infty)}.$$

This, together with (2.6) and (2.7), indicate that

$$\lim_{t_1\to+\infty} \frac{1}{t_1 - t_0} C_{[t_0,t_1]}(\widetilde{y}(\cdot), \widetilde{u}(\cdot)) \leqslant \bar{J}_{[t_0,+\infty)}. \tag{2.10}$$

On the other hand, by the construction above, $(\widetilde{y}(\cdot), \widetilde{u}(\cdot))$ is an admissible pair satisfying the terminal state constraint $R(t_0, \widetilde{y}(t_0), t_1, \widetilde{y}(t_1)) = 0$, we have

$$J_{[t_0,t_1]} \leqslant \frac{1}{t_1 - t_0} C_{[t_0,t_1]}(\widetilde{y}(\cdot), \widetilde{u}(\cdot)).$$

This, combined with (2.10), infers that

$$\limsup_{t_1\to+\infty} J_{[t_0,t_1]} \leqslant \bar{J}_{[t_0,+\infty)}.$$

Which, along with (2.3), leads to (2.4).

Next, we present the proof of the claim (2.9). Let $(\bar{y}(\cdot), \bar{u}(\cdot))$ be an optimal pair for the problem $(\bar{P}_{[t_0,t_1]})$. Then

$$\begin{aligned}
\bar{J}_{[t_0,t_1]} - \bar{J}_{[t_0+\delta_0,t_1]} &= \frac{1}{t_1-t_0}\int_{t_0}^{t_0+\delta_0} f^0(t,\bar{y}(t),\bar{u}(t))\,\mathrm{d}t \\
&\quad + \frac{1}{t_1-t_0}\int_{t_0+\delta_0}^{t_1} f^0(t,\bar{y}(t),\bar{u}(t))\,\mathrm{d}t - \bar{J}_{[t_0+\delta_0,t_1]} \\
&= \frac{1}{t_1-t_0}\int_{t_0}^{t_0+\delta_0} f^0(t,\bar{y}(t),\bar{u}(t))\,\mathrm{d}t + \Big(\frac{t_1-t_0-\delta_0}{t_1-t_0} - 1\Big) \\
&\quad \times \frac{1}{t_1-t_0-\delta_0}\int_{t_0+\delta_0}^{t_1} f^0(t,\bar{y}(t),\bar{u}(t))\mathrm{d}t \\
&\quad + \frac{1}{t_1-t_0-\delta_0}\int_{t_0+\delta_0}^{t_1} f^0(t,\bar{y}(t),\bar{u}(t))\mathrm{d}t - \bar{J}_{[t_0+\delta_0,t_1]}.
\end{aligned}$$

Since $(\bar{y}(\cdot), \bar{u}(\cdot))$ is also admissible for the problem $(\bar{P}_{[t_0+\delta_0,t_1]})$,

$$\frac{1}{t_1-t_0-\delta_0}\int_{t_0+\delta_0}^{t_1} f^0(t,\bar{y}(t),\bar{u}(t))\mathrm{d}t \geqslant \bar{J}_{[t_0+\delta_0,t_1]},$$

we see that

$$\begin{aligned}
\bar{J}_{[t_0,t_1]} - \bar{J}_{[t_0+\delta_0,t_1]} &\geqslant \frac{1}{t_1-t_0}\int_{t_0}^{t_0+\delta_0} f^0(t,\bar{y}(t),\bar{u}(t))\,\mathrm{d}t \\
&\quad + \Big(\frac{t_1-t_0-\delta_0}{t_1-t_0} - 1\Big) \times \frac{1}{t_1-t_0-\delta_0}\int_{t_0+\delta_0}^{t_1} f^0(t,\bar{y}(t),\bar{u}(t))\mathrm{d}t.
\end{aligned}$$

By the boundedness of f^0 on $\mathbb{R} \times E \times F$ (i.e., there exists $M > 0$ such that $|f^0(\cdot)| \leqslant M$), we obtain

$$\bar{J}_{[t_0,t_1]} - \bar{J}_{[t_0+\delta_0,t_1]} \geqslant -\frac{2M\delta_0}{t_1-t_0},$$

which implies (2.9) as $t_1 \to +\infty$.

(ii). By the definition of the value function $V_{[t_0,t_1]}(\cdot)$, we obtain

$$V_{[t_0,t_1]}(y(0)) \leqslant \frac{1}{t_1-t_0} C_{[t_0,t_1]}(y(\cdot), u(\cdot)). \tag{2.11}$$

By (H_4) and Remark 3, we have

$$V_{[t_0,t_1]}(y(0)) \geqslant \bar{J}_{[t_0,t_1]} + \frac{1}{t_1-t_0}\int_{t_0}^{t_1} \beta\big(\mathrm{dist}((y(t),u(t)), \mathcal{T}\times\mathcal{U})\big)\,\mathrm{d}t + \mathrm{o}(1), \tag{2.12}$$

as $t_1 \to +\infty$. By (2.4) we infer

$$\lim_{t_1\to+\infty} \frac{1}{t_1-t_0} C_{[t_0,t_1]}(y(\cdot),u(\cdot)) = \lim_{t_1\to+\infty} J_{[t_0,t_1]} = \bar{J}_{[t_0,+\infty)}.$$

This, together with (2.11) and (2.12), indicates

$$\limsup_{t_1\to+\infty} \frac{1}{t_1-t_0} \int_{t_0}^{t_1} \beta\big(\mathrm{dist}((y(t),u(t)),\mathcal{T}\times\mathcal{U})\big)\,\mathrm{d}t = 0,$$

which completes the proof. □

Remark 6 In the proof of Theorem 1, the role of controllability assumption (H_3) is to ensure that there is an admissible trajectory $\widetilde{y}(\cdot)$ satisfying the terminal state condition $R(t_0,\widetilde{y}(t_0),t_1,\widetilde{y}(t_1)) = 0$ and with a comparable cost (i.e., (2.10)).

Note that (H_3) can be weakened to some cases where controllability may fail: take any control system that is asymptotically controllable to the set $\mathcal{T}$. This is the case for the heat equation which is asymptotically controllable for any given point (cf., e.g., [18, Chapter 7]). Then, if one waits for a certain time, one will arrive at some neighborhood of $\mathcal{T}$. Similarly, to run the proof as in Theorem 1, one needs an assumption which is stronger than (H_2). More precisely, one needs viability, not only along $\mathcal{T}$, but also in a neighborhood of $\mathcal{T}$. Under these assumptions, we believe that one can design a turnpike result for this control system with free final point.

In any case, note that, when the final point is free, having a turnpike property is more or less equivalent to having an asymptotic stabilization to $\mathcal{T}$ (see also an analogous discussion in [12, Remark 2]). If additionally one wants to fix the final point, then one would need the existence of a trajectory steering any point of the neighborhood of $\mathcal{T}$ to the final point.

Remark 7 As seen in the proof of Theorem 1, the assumption of boundedness of f^0 is used two times: the first one, in order to bound the first two terms of (2.6); the second one, in order to prove (2.9). For autonomous optimal control problems, on the one part we have $\bar{J}_{[t_0+\delta_0,+\infty)} = \bar{J}_{[t_0,+\infty)}$ (see Remark 2) and then (2.9) is true, and on the other part the first two terms at the right-hand side of (2.6) converge to zero as $t_1 \to +\infty$ under the "controllability with finite cost" assumption mentioned in Remark 5. In contrast, for non-autonomous optimal control problems the situation may be more complicated, in particular due to the dependence on time of f^0. The assumption of boundedness of f^0 is quite strong and could of course be weakened in a number of ways so as to ensure that the above proof still works. We prefer keeping this rather strong assumption in order to put light in the main line of the argument, not going into too technical details. Variants are easy to derive according to the context.

3 Relationship with (strict) dissipativity

In this section, we make precise the relationship between the strict dissipativity property (which we recall in Sect. 3.1) and the so-called measure-turnpike property (which we define in Sect. 3.2).

3.1 What is (strict) dissipativity

To fix ideas, in this section we only consider the autonomous case. Let X, U, E and F be the same as in Sect. 2. Let $A(\cdot) \equiv A$ generate a C_0 semigroup $\{e^{tA} : t \geqslant 0\}$ on X, and let f and f^0 be time-independent. To simplify the notation, for every $T > 0$, we here consider the optimal control problem

$$(\bar{P}_{[0,T]}) \qquad \begin{cases} \inf J^T(y(\cdot), u(\cdot)) = \dfrac{1}{T}\displaystyle\int_0^T f^0(y(t), u(t))\,\mathrm{d}t, \\ \text{subject to} \quad \dot{y}(t) = Ay(t) + f(y(t), u(t)), \quad t \in [0, T], \\ y(t) \in E, \quad u(t) \in F, \qquad t \in [0, T]. \end{cases} \tag{3.1}$$

Indeed, the above problem $(\bar{P}_{[0,T]})$ coincides with $(\bar{P}_{[t_0,t_1]})$ as defined in (2.2) (in Sect. 2) for $t_0 = 0$ and $t_1 = T$. Note that the terminal states $y(0)$ and $y(T)$ are set free in the problem $(\bar{P}_{[0,T]})$. Recall that the solutions $(y(\cdot), u(\cdot)) \in C([0, T]; X) \times L^2(0, T; U)$ are considered in the mild sense, meaning that

$$y(\tau) = e^{\tau A} y(0) + \int_0^\tau e^{(\tau - t)A} f(y(t), u(t))\,\mathrm{d}t, \qquad \forall \tau \in [0, T],$$

or equivalently,

$$\langle \varphi, y(\tau)\rangle_{X^*,X} - \langle \varphi, y(0)\rangle_{X^*,X} = \int_0^\tau \Big(\langle A^*\varphi, y(t)\rangle_{X^*,X} + \langle \varphi, f(y(t), u(t))\rangle_{X^*,X}\Big)\mathrm{d}t,$$

for each $\tau \in [0, T]$ and $\varphi \in D(A^*)$, where $A^* : D(A^*) \subset X^* \to X^*$ is the adjoint operator of A, and $\langle \cdot, \cdot \rangle_{X^*,X}$ is the dual paring between X and its dual space X^*.

Likewise, we say $(y(\cdot), u(\cdot))$ an admissible pair to the problem $(\bar{P}_{[0,T]})$ if it satisfies the state equation and the above state-control constraint. Assume that, for any $T > 0$, $(\bar{P}_{[0,T]})$ has at least one optimal solution denoted by $(y^T(\cdot), u^T(\cdot))$, and we set

$$\bar{J}^T = J^T(y^T(\cdot), u^T(\cdot)).$$

Note that $\bar{J}^T$ does not depend on the optimal solution under consideration.

In the finite-dimensional case where $X = \mathbb{R}^n$ and $U = \mathbb{R}^m$, without loss of generality, we may take $A = 0$, and then the control system is $\dot{y}(t) = f(y(t), u(t))$. We refer the reader to [12,30] for the asymptotic behavior of optimal solutions of such optimal control problems with constraints on the terminal states.

Consider the static optimal control problem

$$\boxed{(P_s) \qquad \begin{cases} \inf J_s(y,u) = f^0(y,u), \\ \text{subject to} \quad Ay + f(y,u) = 0, \\ y \in E, \quad u \in F, \end{cases}} \tag{3.2}$$

where the first equation means that

$$\langle A^*\varphi, y\rangle_{X^*,X} + \langle \varphi, f(y,u)\rangle_{X^*,X} = 0, \qquad \forall \varphi \in D(A^*).$$

As above, we assume that there exists at least one optimal solution (y_s, u_s) of (P_s). Such existence results are as well standard, for instance in the case where A is an elliptic differential operator (see [18, Chapter 3, Theorem 6.4]). We set

$$\bar{J}_s = J_s(y_s, u_s).$$

Note that $\bar{J}_s$ does not depend on the optimal solution that is considered. Of course, uniqueness of the minimizer cannot be ensured in general because the problem is not assumed to be convex. Note that (y_s, u_s) is admissible for the problem $(\bar{P}_{[0,T]})$ for any $T > 0$, meaning that it satisfies the constraints and is a solution of the control system.

We next define the notion of dissipativity for the infinite-dimensional controlled system, which is originally due to [32] for finite-dimensional dynamics (see also related definitions in [12]). Recall that the continuous function $\alpha : [0, +\infty) \to [0, +\infty)$ with $\alpha(0) = 0$ is said to be a $\mathcal{K}$-class function if it is monotone increasing.

Definition 1 We say that $\{(\bar{P}_{[0,T]}) \mid T > 0\}$ as defined in (3.1) is *dissipative* at an optimal stationary point (y_s, u_s) with respect to the *supply rate function*

$$\omega(y,u) = f^0(y,u) - f^0(y_s,u_s), \qquad \forall (y,u) \in E \times F, \tag{3.3}$$

if there exists a *storage function* $S : E \to \mathbb{R}$, locally bounded and bounded from below, such that, for any $T > 0$, the dissipation inequality

$$S(y(0)) + \int_0^\tau \omega(y(t), u(t))\,\mathrm{d}t \geqslant S(y(\tau)), \quad \forall \tau \in [0, T], \tag{3.4}$$

holds true, for any admissible pair $(y(\cdot), u(\cdot))$.

We say that $\{(\bar{P}_{[0,T]}) \mid T > 0\}$ is *strictly dissipative* at (y_s, u_s) with respect to the supply rate function ω if there exists a $\mathcal{K}$-class function $\alpha(\cdot)$ such that, for any $T > 0$, the strict dissipation inequality

$$\begin{aligned} S(y(0)) + \int_0^\tau \omega(y(t), u(t))\,\mathrm{d}t &\geqslant S(y(\tau)) \\ &+ \int_0^\tau \alpha\big(\|(y(t) - y_s, u(t) - u_s)\|_{X\times U}\big)\,\mathrm{d}t, \quad \forall \tau \in [0, T], \end{aligned} \tag{3.5}$$

holds true, for any admissible pair $(y(\cdot), u(\cdot))$. The function $d(\cdot) = \alpha(\|(y(\cdot) - y_s, u(\cdot) - u_s)\|_{X \times U})$ in (3.5) is called the dissipation rate.

Although there are many possible different notions of dissipativity introduced in the literature (such as the positivity or the local boundedness of the storage function in their definitions, cf., e.g., [5, Chapter 4]), they are proved to be equivalent in principle between with each other. Note that a storage function is defined up to an additive constant. We here define the storage function $S : E \to \mathbb{R}$ to take real values instead of positive real values. Since S is assumed to be bounded from below, one could as well consider $S : E \to [0, +\infty)$. We mention that no regularity is a priori required to define S. Actually, storage functions do possess some regularity properties, such as C^0 or C^1 regularity, under suitable assumptions. For example, the controllable and observable systems with positive transfer functions are dissipative with quadratic storage functions (see [5, Sect. 4.4.5] for instance).

When a system is dissipative with a given supply rate function, the question of finding a storage function has been extensively studied. This question is closely similar to the problem of finding a suitable Lyapunov function in the Lyapunov second method ensuring the stability of a system. For linear systems with a quadratic supply rate function, the existence of a storage function boils down to solve a Riccati inequality. In general, storage functions are closely related to viscosity solutions of a partial differential inequality, called a *Hamilton-Jacobi* inequality. We refer the reader to [5, Chapter 4] for more details on this subject.

An equivalent characterization of the dissipativity in [32] can be described by the so-called *available storage*, which is defined as

$$S_a(z) \triangleq \sup_{t \geqslant 0,\, (y(\cdot),u(\cdot))} \Big\{ - \int_0^t \omega(y(\tau), u(\tau))\, \mathrm{d}\tau \Big\}, \tag{3.6}$$

where the sup is taken over all admissible pairs $(y(\cdot), u(\cdot))$ (meaning that satisfy the dynamic controlled system and state-control constraints) with initial value $y(0) = z$. In fact, for every $z \in E$, $S_a(z)$ can be seen as the maximum amount of "energy" which can be extracted from the system with initial state $y(0) = z$. It has been shown by Willems [32] that the problem $(\bar{P}_{[0,T]})$ is dissipative at (y_s, u_s) with respect to the supply rate function $\omega(\cdot, \cdot)$ if and only if $S_a(z)$ is finite for every $z \in E$.

We provide a specific example of a (strictly) dissipative control system.

Example 2 Let $\Omega \subset \mathbb{R}^n$ $(n \geqslant 1)$ be a smooth and bounded domain, and let $\mathcal{D} \subset \Omega$ be a non-empty open subset. Denote by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ the inner product and norm in $L^2(\Omega)$, respectively. For each $T > 0$, consider the optimal control problem

$$\inf \int_0^T \Big(\langle y(t), \chi_{\mathcal{D}} u(t) \rangle + \|u(t)\|^2 \Big)\, \mathrm{d}t,$$

subject to

$$\begin{cases} y_t - \Delta y = \chi_{\mathcal{D}} u, & \text{in } \ \Omega \times (0, T), \\ y = 0, & \text{on } \ \partial\Omega \times (0, T), \\ \|y(t)\| \leqslant 1, \quad \|u(t)\| \leqslant 1, & \forall t \in [0, T]. \end{cases}$$

Notice that the corresponding static problem has a unique solution (0, 0). We show that this problem is strictly dissipative at (0, 0) with respect to the supply rate

$$\omega(y,u) = \langle y, \chi_{\mathcal{D}} u\rangle + \|u\|^2, \quad \forall (y,u) \in L^2(\Omega) \times L^2(\Omega).$$

In fact, integrating the heat equation by parts leads to

$$\int_0^\tau \langle y(t), \chi_{\mathcal{D}} u(t)\rangle \,\mathrm{d}t = \frac{\|y(\tau)\|^2 - \|y(0)\|^2}{2} + \int_0^\tau \|\nabla y(t)\|^2 \mathrm{d}t,$$

for any $\tau \in [0, T]$. This, together with the above definition of $\omega(\cdot,\cdot)$ and the Poincaré inequality, indicates that the strict dissipation inequality

$$S(y(\tau)) + c\int_0^\tau \left(\|y(t)\|^2 + \|u(t)\|^2\right) \mathrm{d}t \leqslant S(y(0)) + \int_0^\tau \omega(y(t), u(t))\,\mathrm{d}t, \quad \forall \tau \in [0, T],$$

holds with $\alpha(\gamma) = c\gamma^2$ for some constant $c > 0$, and a storage function $S(\cdot)$ given by

$$S(y) = \frac{1}{2}\|y\|^2, \quad \forall y \in L^2(\Omega).$$

Thus, this problem has the strict dissipativity property at (0, 0).

3.2 Strict dissipativity implies measure-turnpike

Next, we introduce a rigorous definition of measure-turnpike for optimal control problems.

Definition 2 We say that $\{(\bar{P}_{[0,T]}) \mid T > 0\}$ as defined in (3.1) enjoys the *measure-turnpike property* at (y_s, u_s) if, for every $\varepsilon > 0$, there exists $\Lambda(\varepsilon) > 0$ such that

$$|\mathcal{Q}_{\varepsilon,T}| \leqslant \Lambda(\varepsilon), \qquad \forall T > 0, \tag{3.7}$$

where

$$Q_{\varepsilon,T} = \left\{ t \in [0,T] \mid \left\| (y^T(t) - y_s, u^T(t) - u_s) \right\|_{X\times U} > \varepsilon \right\}. \tag{3.8}$$

We refer the reader to [7,12,34] (and references therein) for similar definitions. In this definition, the set $Q_{\varepsilon,T}$ measures the set of times at which the optimal trajectory and control stay outside an ε-neighborhood of (y_s, u_s) for the strong topology. We stress that the measure-turnpike property defined above concerns both state and control. In the existing literature (see, e.g., [7]), the turnpike phenomenon is often studied only for the state, meaning that, for each $\varepsilon > 0$, the (Lebesgue) measure of the set $\{t \in [0,T] \mid \|y^T(t) - y_s\|_X > \varepsilon\}$ is uniformly bounded for any $T > 0$.

In the following result, we establish the measure-turnpike property for optimal solutions of $(\bar{P}_{[0,T]})$ (as well for $(\bar{P}_{[t_0,t_1]})$) under the strict dissipativity assumption, as the parameter T goes to infinity. This implies that *any* optimal solution $(y^T(\cdot), u^T(\cdot))$ of $(\bar{P}_{[0,T]})$ remains *essentially* close to *some* optimal solution (y_s, u_s) of (P_s) as defined in (3.2). Our results can be seen in the stream of the recent works [12,22,23,30].

Theorem 2 *Let E be a bounded subset of X.*

(i) *If $\{(\bar{P}_{[0,T]}) \mid T > 0\}$ as defined in (3.1) is dissipative at (y_s, u_s) with respect to the supply rate function $\omega(\cdot,\cdot)$ given by (3.3), then*

$$\bar{J}^T = \bar{J}_s + O(1/T) \quad as \quad T \to +\infty. \tag{3.9}$$

(ii) *If $\{(\bar{P}_{[0,T]}) \mid T > 0\}$ is strictly dissipative at (y_s, u_s) with respect to the supply rate function $\omega(\cdot,\cdot)$ given by (3.3), then it satisfies the measure-turnpike property at (y_s, u_s).*

Remark 8 Note that $(\bar{P}_{[0,T]})$ is defined without any constraints on the terminal states. However, under appropriate controllability assumptions (similar to (H_3) in Sect. 2), one can also treat the case of terminal state constraint $R(\cdot) = 0$ (see also [12] and [13]).

Remark 9 From Theorem 2, we see that strict dissipativity is sufficient for the measure-turnpike property for the optimal control problem. This fact was observed in the previous works [9,12,13]. For the converse statements, i.e., results which show that the turnpike property implies strict dissipativity, we refer the reader to [14] and [12]. In [14], the authors first defined a turnpike-like behavior concerning all trajectories whose associated cost is close to the optimal one. This behavior is stronger than the measure-turnpike property, which only concerns the optimal trajectories. Then, the implication "turnpike-like behavior $\Rightarrow$ strict dissipativity" was proved in [14]. Besides, the implication "exact turnpike property $\Rightarrow$ strict dissipativity along optimal trajectories" was shown in [12], where the exact turnpike property means that the optimal solutions have to remain exactly at an optimal steady-state for most part of the long-time horizon.

Proof of Theorem 2 We first prove the second point of the theorem. Let $T > 0$ and let $(y^T(\cdot), u^T(\cdot))$ be any optimal solution of the problem $(\bar{P}_{[0,T]})$. By the strict dissipation inequality (3.5) applied to $(y^T(\cdot), u^T(\cdot))$, we have

$$\frac{1}{T}\int_0^T \alpha\big(\|(y^T(t)-y_s, u^T(t)-u_s)\|_{X\times U}\big)\,\mathrm{d}t \leqslant \bar{J}^T - \bar{J}_s + \frac{S(y^T(0)) - S(y^T(T))}{T}. \tag{3.10}$$

Note that $\alpha\big(\|(y^T(t)-y_s, u^T(t)-u_s)\|_{X\times U}\big) \geqslant \alpha(\varepsilon)$ whenever $t \in Q_{\varepsilon,T}$, where $Q_{\varepsilon,T}$ is defined by (3.8). Since $E \subset X$ is a bounded subset and $S(\cdot)$ is locally bounded, there exists $M > 0$ such that $|S(y)| \leqslant M$ for every $y \in E$. Therefore, it follows from (3.10) that

$$\frac{|Q_{\varepsilon,T}|}{T} \leqslant \frac{1}{\alpha(\varepsilon)}\left(\bar{J}^T - \bar{J}_s + \frac{2M}{T}\right). \tag{3.11}$$

On the other hand, noting that (y_s, u_s) is admissible for $(\bar{P}_{[0,T]})$ for any $T > 0$, we have

$$\bar{J}^T \leqslant \frac{1}{T}\int_0^T f^0(y_s, u_s)\,\mathrm{d}t = f^0(y_s, u_s) = \bar{J}_s. \tag{3.12}$$

This, combined with (3.11), leads to $|Q_{\varepsilon,T}| \leqslant \frac{2M}{\alpha(\varepsilon)}$ for every $T > 0$. The second point of the theorem follows.

Let us now prove the first point. On the one hand, it follows from (3.12) that

$$\limsup_{T\to\infty} \bar{J}^T \leqslant \bar{J}_s.$$

By the dissipation inequality (3.4) applied to any optimal solution $(y^T(\cdot), u^T(\cdot))$ of $(\bar{P}_{[0,T]})$, we get

$$S(y^T(0)) + \int_0^T f^0(y^T(t), u^T(t))\,\mathrm{d}t \geqslant T f^0(y_s, u_s) + S(y^T(T)),$$

which leads to

$$\bar{J}_s \leqslant \bar{J}^T + \frac{S(y^T(0)) - S(y^T(T))}{T}.$$

Since E is a bounded subset in X and since the storage function $S(\cdot)$ is locally bounded and bounded below, we infer that

$$\bar{J}_s \leqslant \liminf_{T\to\infty} \bar{J}^T.$$

Then (3.9) follows. □

Remark 10 The above proof borrows ideas from [13, Theorem 5.3] and [12]. We used in a crucial way the fact that any solution of the steady-state problem (P_s) is admissible for the problem $(\bar{P}_{[0,T]})$ under consideration. This is due to the fact that the terminal states are set free in $(\bar{P}_{[0,T]})$. Note that we only use the boundedness of $y^T(0)$ in the proof.

3.3 Dissipativity and assumption (H)

Under the (strict) dissipativity property, we can verify the abstract **Assumption (H)** for the autonomous case in Sect. 2.

Proposition 1 *Assume that, for any t_0 and t_1, the problem $(\bar{P}_{[t_0,t_1]})$ as defined in (2.2) is dissipative at (y_s, u_s) with the supply rate $\omega(y, u) = f^0(y, u) - f^0(y_s, u_s)$, and the associated storage function $S(\cdot)$ is bounded on E. Then*

(i) $\bar{J}_{[t_0,+\infty)} = \bar{J}_s$, $\forall t_0 \in \mathbb{R}$.
(ii) *There exists a turnpike set $\mathcal{T} \times \mathcal{U} = \{(y_s, u_s)\}$ such that (H_1) and (H_2) are satisfied.*

(iii) *Moreover, if* (H_3) *is satisfied and* $(\bar{P}_{[t_0,t_1]})$ *is strictly dissipative at* (y_s, u_s) *with dissipation rate* $d(\cdot) = \alpha(\|(y(\cdot) - y_s, u(\cdot) - u_s)\|_{X\times U})$*, then* (H_4) *is satisfied with* $\beta(\cdot) = \alpha(\cdot)$ *and* $\operatorname{dist}((y,u), \mathcal{T} \times \mathcal{U}) = \|(y - y_s, u - u_s)\|_{X\times U}$.

Proof (i) With a slight modification, the proof is the same as that of the first point of Theorem 2.

(ii) Since (y_s, u_s) is an equilibrium point, the constant pair (y_s, u_s) is admissible on any time interval. By the definition,

$$\bar{J}_{[t_0,t_1]} \leqslant V_{[t_0,t_1]}(y_s) \leqslant \frac{1}{t_1 - t_0}\int_{t_0}^{t_1} f^0(y_s, u_s)\,\mathrm{d}t = \bar{J}_s.$$

This, along with (i), indicates that

$$V_{[t_0,+\infty)}(y_s) = \lim_{t_1\to+\infty} V_{[t_0,t_1]}(y_s) = \bar{J}_s.$$

Hence, the assumptions (H_1) and (H_2) hold.

(iii) Let $(\tilde{y}(\cdot), \tilde{u}(\cdot))$ be an optimal solution to the problem $(P_{[t_0,t_1]})$ as defined in (2.1). Then, by the strict dissipativity property we have

$$S(\tilde{y}(t_1)) + \int_{t_0}^{t_1} \alpha(\|(\tilde{y}(t) - y_s, \tilde{u}(t) - u_s)\|_{X\times U})\,\mathrm{d}t \leqslant S(\tilde{y}(t_0)) + \int_{t_0}^{t_1} \big(f^0(\tilde{y}(t), \tilde{u}(t)) - f^0(y_s, u_s)\big)\,\mathrm{d}t.$$

Which is equivalent to

$$J_{[t_0,t_1]} \geqslant \bar{J}_s + \frac{1}{t_1 - t_0}\int_{t_0}^{t_1} \alpha(\|(\tilde{y}(t) - y_s, \tilde{u}(t) - u_s)\|_{X\times U})\,\mathrm{d}t + \frac{S(\tilde{y}(t_1)) - S(\tilde{y}(t_0))}{t_1 - t_0}. \tag{3.13}$$

Because

$$\begin{aligned} J_{[t_0,t_1]} &= \bar{J}_{[t_0,t_1]} + (J_{[t_0,t_1]} - \bar{J}_{[t_0,t_1]}) \\ &= \inf_{z\in X} V_{[t_0,t_1]}(z) + \big(J_{[t_0,t_1]} - \inf_{z\in X} V_{[t_0,t_1]}(z)\big) \\ &\leqslant V_{[t_0,t_1]}(\tilde{y}(t_0)) + \big(J_{[t_0,t_1]} - \inf_{z\in X} V_{[t_0,t_1]}(z)\big). \end{aligned}$$

The last inequality, along with (3.13), indicates that

$$V_{[t_0,t_1]}(\tilde{y}(t_0)) \geqslant \inf_{z\in X} V_{[t_0,t_1]}(z) + \frac{1}{t_1 - t_0}\int_{t_0}^{t_1} \alpha(\|(\tilde{y}(t) - y_s, \tilde{u}(t) - u_s)\|_{X\times U})\,\mathrm{d}t + \frac{S(\tilde{y}(t_1)) - S(\tilde{y}(t_0))}{t_1 - t_0} + \bar{J}_s - J_{[t_0,t_1]}. \tag{3.14}$$

As the storage function $S(\cdot)$ is bounded on E, by (i) and (2.4) we infer

$$\lim_{t_1\to+\infty} J_{[t_0,t_1]} = \bar{J}_s,$$

Hence, the sum of last three terms in (3.14) is an infinitesimal quantity as $t_1 \to +\infty$, and thus (H_4) holds.

Remark 11 Proposition 1 explains the role of dissipativity in the general turnpike phenomenon. It reflects that dissipativity allows one to identify the limit value $\bar{J}_{[t_0,+\infty)}$, that dissipativity implies (H_1) and (H_2), and that strict dissipativity, plus (H_3), implies (H_4). Recall that (H_3) is a controllability assumption.

3.4 Some comments on the periodic turnpike phenomenon

Inspired from [32] and [36], we introduce the concept of (strict) dissipativity with respect to a periodic trajectory. Let $A(\cdot)$, $f(\cdot)$ and $f^0(\cdot)$ be periodic in time with a period $\Pi > 0$.

Definition 3 We say the problem $(\bar{P}_{[t_0,t_1]})$ as defined in (2.2) is dissipative with respect to a Π-periodic trajectory $(\hat{y}(\cdot), \hat{u}(\cdot))$ with respect to the supply rate function

$$\omega(t, y, u) = f^0(t, y, u) - f^0(t, \hat{y}(t), \hat{u}(t)), \qquad \forall (t, y, u) \in \mathbb{R} \times E \times F,$$

if there exists a locally bounded and bounded from below storage function $S : \mathbb{R}\times E \to \mathbb{R}$, Π-periodic in time, such that

$$S(\tau_0, y(\tau_0)) + \int_{\tau_0}^{\tau_1} \omega(t, y(t), u(t))\,\mathrm{d}t \geqslant S(\tau_1, y(\tau_1)) \quad \text{for all} \;\; t_0 \leqslant \tau_0 < \tau_1 \leqslant t_1,$$

for any admissible pair $(y(\cdot), u(\cdot))$. If, in addition, there exists a $\mathcal{K}$-class function $\alpha(\cdot)$ such that

$$S(\tau_0, y(\tau_0)) + \int_{\tau_0}^{\tau_1} \omega(t, y(t), u(t))\,\mathrm{d}t \geqslant S(\tau_1, y(\tau_1)) + \int_{\tau_0}^{\tau_1} \alpha\big(\|(y(t) - \hat{y}(t), u(t) - \hat{u}(t))\|_{X\times U}\big)\,\mathrm{d}t,$$

we say that $(\bar{P}_{[t_0,t_1]})$ is strictly dissipative with respect to a Π-periodic trajectory $(\hat{y}(\cdot), \hat{u}(\cdot))$.

The definition of dissipativity for a general time-varying dynamic system with time-varying supply rate is introduced in [32, Sect. 6]. In fact, Definition 3 is a particular case of [32] when the dynamic system and supply rate are periodic in time. As mentioned in Sect. 3.1, finding a storage function is also the key ingredient for a time-varying dynamic system that satisfies the dissipativity property, and similarly to (3.6), a possible storage function of dissipativity for the general time-varying dynamic system can be characterized by the *available storage*, which is defined as (see [32, Sect. 6])

$$S_a(t_0, z) \triangleq \sup_{t\geqslant t_0,\,(y(\cdot),u(\cdot))} \Big\{ - \int_{t_0}^{t} \omega(\tau, y(\tau), u(\tau))\,\mathrm{d}\tau \Big\},$$

where the sup is taken over all admissible pairs $(y(\cdot), u(\cdot))$ with initial value $y(t_0) = z$.

In analogy to Example 2, we provide a particular example of a strictly dissipative control system with respect to a periodic trajectory.

Example 3 Let Ω and $\mathcal{D}$ be as in Example 2. Given any $\hat{u}(\cdot) \in L^2(0, \Pi; L^2(\Omega))$, we denote by $\hat{y}(\cdot) \in C([0, \Pi]; L^2(\mathcal{D}))$ the corresponding unique Π-periodic solution to the Dirichlet heat equation (see, e.g., [28, Lemma 1])

$$\begin{cases} \hat{y}_t - \Delta \hat{y} = \chi_{\mathcal{D}} \hat{u}, & \text{in } \Omega \times (0, \Pi), \\ \hat{y} = 0, & \text{on } \partial\Omega \times (0, \Pi), \\ \hat{y}(\cdot, 0) = \hat{y}(\cdot, \Pi), & \text{in } \Omega. \end{cases}$$

Extend $(\hat{y}(\cdot), \hat{u}(\cdot))$ to a Π-periodic pair in time satisfying $(\hat{y}(\cdot, t), \hat{u}(\cdot, t)) = (\hat{y}(\cdot, t + \Pi), \hat{u}(\cdot, t + \Pi))$ in Ω for any $t > 0$. Then, for each $T > 0$ and $y_0 \in L^2(\Omega)$, consider the optimal control problem

$$\inf_{u \in L^2(0,T;L^2(\Omega))} \int_0^T \left(\left\langle y(t) - \hat{y}(t), \chi_{\mathcal{D}}(u(t) - \hat{u}(t)) \right\rangle + \left\| u(t) - \hat{u}(t) \right\|^2 \right) \mathrm{d}t,$$

subject to

$$\begin{cases} y_t - \Delta y = \chi_{\mathcal{D}} u, & \text{in } \Omega \times (0, T), \\ y = 0, & \text{on } \partial\Omega \times (0, T), \\ y(\cdot, 0) = y_0, & \text{in } \Omega. \end{cases} \tag{3.15}$$

We next claim that this problem is strictly dissipative with respect to the Π-periodic trajectory $(\hat{y}(\cdot), \hat{u}(\cdot))$, with the supply rate

$$\omega(t, y, u) = \langle y - \hat{y}(t), \chi_{\mathcal{D}}(u - \hat{u}(t)) \rangle + \|u - \hat{u}(t)\|^2, \quad \forall (t, y, u) \in \mathbb{R}^+ \times L^2(\Omega) \times L^2(\Omega). \tag{3.16}$$

Indeed, for any $0 \leqslant s < \tau \leqslant T$, integrating by parts readily leads to

$$\int_s^\tau \langle y(t) - \hat{y}(t), \chi_{\mathcal{D}}(u(t) - \hat{u}(t)) \rangle \, \mathrm{d}t = \frac{\|y(\tau) - \hat{y}(\tau)\|^2 - \|y(s) - \hat{y}(s)\|^2}{2} + \int_s^\tau \|\nabla(y(t) - \hat{y}(t))\|^2 \mathrm{d}t$$

for any pair $(y(\cdot), u(\cdot))$ satisfying (3.15). The last inequality, along with (3.16) and the Poincaré inequality, implies that the strict dissipation inequality

$$S(\tau, y(\tau)) + c \int_s^\tau \left(\|y(t) - \hat{y}(t)\|^2 + \|u(t) - \hat{u}(t)\|^2 \right) \mathrm{d}t \leqslant S(s, y(s)) + \int_s^\tau \omega(t, y(t), u(t)) \, \mathrm{d}t$$

holds with $\alpha(\gamma)=c\gamma^2$ for some constant $c>0$, and a Π-periodic storage function

$$S(t,y)=\frac{1}{2}\|y-\hat{y}(t)\|^2,\quad \forall (t,y)\in\mathbb{R}^+\times L^2(\Omega).$$

Thus, this problem has the strict dissipativity property with respect to a periodic trajectory.

The notion of (strict) dissipativity with respect to a periodic trajectory in Definition 3 allows one to identify the optimal control problem $(\bar{P}_{[t_0,t_1]})$ as a periodic one. Consider the periodic optimal control problem

$$(\bar{P}_{\text{per}})\quad\begin{cases}\bar{J}_{\text{per}}=\inf\frac{1}{\Pi}C_{[0,\Pi]}(y(\cdot),u(\cdot)),\\ \text{subject to}\quad \dot{y}(t)=A(t)y+f(t,y(t),u(t)),\quad (y(t),u(t))\in E\times F,\quad t\in[0,\Pi],\\ y(0)=y(\Pi).\end{cases}$$

We assume that $(\bar{P}_{\text{per}})$ has at least one periodic optimal solution $(\bar{y}(\cdot),\bar{u}(\cdot))\in C([0,\Pi];X)\times L^\infty(0,\Pi;U)$ (see, e.g., [3] for the existence of periodic optimal solutions), and we set $\bar{J}_{\text{per}}=\frac{1}{\Pi}\int_0^\Pi f^0(t,\bar{y}(t),\bar{u}(t))\,\mathrm{d}t$ (optimal value of $(\bar{P}_{\text{per}})$). Let us extend $(\bar{y}(\cdot),\bar{u}(\cdot))$ to $\mathbb{R}$ by periodicity. Likewise, we have the following result.

Proposition 2 *Assume that, for any t_0 and t_1, the problem $(\bar{P}_{[t_0,t_1]})$ as defined in (2.2) is dissipative with respect to a Π-periodic optimal trajectory $(\bar{y}(\cdot),\bar{u}(\cdot))$, with the supply rate $\omega(t,y,u)=f^0(t,y,u)-f^0(t,\bar{y}(t),\bar{u}(t))$, and the associated storage function $S(\cdot)$ is bounded on E for all times. Then*

- (i) $\bar{J}_{[t_0,+\infty)}=\bar{J}_{per}$, $\forall t_0\in\mathbb{R}$.
- (ii) *There exists a turnpike set $\mathcal{T}\times\mathcal{U}=\{(\bar{y}(s),\bar{u}(s))\mid s\in[0,\Pi]\}$ such that (H_1) and (H_2) are satisfied.*
- (iii) *Moreover, if (H_3) is satisfied and $(\bar{P}_{[t_0,t_1]})$ is strictly dissipative with respect to the Π- periodic trajectory $(\bar{y}(\cdot),\bar{u}(\cdot))$, with dissipation rate $\alpha(\cdot)$, then (H_4) is satisfied with $\beta(\cdot)=\alpha(\cdot)$ and* $\operatorname{dist}((y,u),\mathcal{T}\times\mathcal{U})=\inf_{s\in[0,\Pi]}\|(y-\bar{y}(s),u-\bar{u}(s)\|_X$.

Proof We only show the proof of (i), as the rest is similar to the arguments in the proof of Proposition 1.

Since $(\bar{y}(\cdot),\bar{u}(\cdot))$ is an admissible trajectory in $[t_0,t_0+k\Pi]$ for any $k\in\mathbb{N}$, we have $\bar{J}_{[t_0,+\infty)}\leqslant\bar{J}_{\text{per}}$. Let us prove the converse inequality. By the periodic dissipativity in Definition 3, we have

$$S(t_0,y(t_0))+\int_{t_0}^{t_0+k\Pi}f^0(t,y(t),u(t))\,\mathrm{d}t\geqslant k\int_{t_0}^{t_0+\Pi}f^0(t,\bar{y}(t),\bar{u}(t))\,\mathrm{d}t\\ +S(t_0+k\Pi,y(t_0+k\Pi))$$

for any admissible trajectory $(y(\cdot),u(\cdot))$. Since $\bar{J}_{\text{per}}=\frac{1}{\Pi}\int_0^\Pi f^0(t,\bar{y}(t),\bar{u}(t))\,\mathrm{d}t$, it follows that

$$\bar{J}_{\text{per}}\leqslant\frac{1}{k\Pi}\int_{t_0}^{t_0+k\Pi}f^0(t,y(t),u(t))\,\mathrm{d}t+\frac{S(t_0,y(t_0))-S(t_0+k\Pi,y(t_0+k\Pi))}{k\Pi}.$$

Letting k tend to infinity, and taking the infimum over all possible admissible trajectories, we get that $\bar{J}_{\text{per}} \leqslant \bar{J}_{[t_0,+\infty)}$. □

4 Relationship with (strict) strong duality

After having detailed a motivating example in Sect. 4.1, we recall in Sect. 4.2 the notion of (strict) strong duality, and we establish in Sect. 4.3 that strict strong duality implies strict dissipativity (and thus measure-turnpike according to Sect. 3.2).

4.1 A motivating example

To illustrate the effect of Lagrangian function associated with the static problem when one derives the measure-turnpike property for the evolution control system, we consider the simplest model of heat equation with control constraints.

Let $\Omega \subset \mathbb{R}^n$, $n \geqslant 1$, be a bounded domain with a smooth boundary $\partial\Omega$, and let $\mathcal{D} \subset \Omega$ be a non-empty open subset. Throughout this subsection, we denote by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ the inner product and norm in $L^2(\Omega)$, respectively; by $\chi_{\mathcal{D}}$ the characteristic function of $\mathcal{D}$. For any $T > 0$, consider the optimal control problem for the heat equation with pointwise control constraints:

$$\bar{J}^T = \inf_{u(\cdot)\in L^2(0,T;\,\mathcal{U}_{ad})} \frac{1}{2T}\int_0^T \left(\|y(t)-y_{\text{d}}\|^2 + \|u(t)\|^2\right)\mathrm{d}t,$$

subject to

$$\begin{cases} y_t - \Delta y = \chi_{\mathcal{D}} u, & \text{in } \ \Omega\times(0,T),\\ y = 0, & \text{on } \ \partial\Omega\times(0,T),\\ y(\cdot,0) = y_0, & \text{in } \ \Omega,\end{cases}$$

where $y_{\text{d}} \in L^2(\Omega)$, $y_0 \in L^2(\Omega)$ and

$$\mathcal{U}_{ad} = \left\{u \in L^2(\Omega) \ \mid \ u_a(x) \leqslant u(x) \leqslant u_b(x) \ \text{ for a.e. } \ x\in\Omega\right\},$$

with u_a and u_b being in $L^2(\Omega)$. Assume that $(y^T(\cdot), u^T(\cdot))$ (the optimal pair obviously depends on the time horizon) is the unique optimal solution. We want to study the long-time behavior of optimal solutions, i.e., the optimal pair stays in a neighborhood of a static optimal solution at most of the long-time horizon.

As before, we consider the static optimal control problem stated below

$$\bar{J}_s = \inf_{u\in\mathcal{U}_{ad}} \frac{1}{2}\left(\|y-y_{\text{d}}\|^2 + \|u\|^2\right),$$

subject to

$$\begin{cases} -\Delta y = \chi_{\mathcal{D}} u, & \text{in } \ \Omega,\\ y = 0, & \text{on } \ \partial\Omega.\end{cases}$$

Assume that (y_s, u_s) is the unique optimal solution.

For this purpose, given every $\varepsilon > 0$, we define the set

$$Q_{\varepsilon,T} = \Big\{t \in [0, T] \ \mid \ \|y^T(t) - y_s\|^2 + \|u^T(t) - u_s\|^2 > \varepsilon\Big\},$$

which measures the time at which the optimal pair is outside of the ε-neighborhood of (y_s, u_s).

Proposition 3 *The following convergence hold*

$$\frac{1}{T}\int_0^T y^T(t)\,dt \to y_s \ \ and \ \ \frac{1}{T}\int_0^T u^T(t)\,dt \to u_s \ \ in \ \ L^2(\Omega), \ \ as \ \ T \to \infty.$$

Moreover, for each $\varepsilon > 0$, it holds true that

$$|Q_{\varepsilon,T}| \leqslant O\Big(\frac{1}{\varepsilon}\Big), \ \ for\ all \ \ T \geqslant 1,$$

i.e., the measure-turnpike property holds.

Proof The key point of the proof is to show that

$$\int_0^T \Big(\|y^T(t) - y_s\|^2 + \|u^T(t) - u_s\|^2\Big)\,\mathrm{d}t \leqslant C \quad \text{for all} \ \ T > 0, \tag{4.1}$$

where C is a constant independent of T. Once the inequality (4.1) is proved, the desired results hold automatically. To prove (4.1), the remaining part of the proof is proceeded into several steps as follows.

Step 1. We first introduce a Lagrangian function for the above stationary problem. According to the Karush-Kuhn-Tucker (KKT for short) optimality conditions (see, e.g., [27, Theorem 2.29]), there are functions p_s, μ_a and μ_b in $L^2(\Omega)$ such that

$$(KKT) \quad \begin{cases} -\Delta y_s = \chi_{\mathcal{D}} u_s, \quad -\Delta p_s = y_d - y_s, \ \text{ in } \ \Omega, \\ y_s = 0, \quad p_s = 0, \ \text{ on } \ \partial\Omega, \\ u_s - \chi_{\mathcal{D}} p_s - \mu_a + \mu_b = 0, \\ \mu_a \geqslant 0, \ \ \mu_b \geqslant 0, \ \ \mu_a(u_a - u_s) = \mu_b(u_s - u_b) = 0. \end{cases}$$

Now, we define the associated Lagrangian function $L : H_0^1(\Omega) \times L^2(\Omega) \to \mathbb{R}$ by setting

$$\begin{aligned} L(y, u) = \frac{1}{2}\Big(\|y - y_{\mathrm{d}}\|^2 + \|u\|^2\Big) + \langle \nabla y, \nabla p_s\rangle - \langle \chi_{\mathcal{D}} u, p_s\rangle \\ +\langle \mu_a, u_a - u\rangle + \langle \mu_b, u - u_b\rangle, \ \ \forall (y, u) \in H_0^1(\Omega) \times L^2(\Omega). \end{aligned} \tag{4.2}$$

From the above-mentioned KKT optimality conditions, we can see that

$$L(y_s, u_s) = \frac{1}{2}\Big(\|y_s - y_{\mathrm{d}}\|^2 + \|u_s\|^2\Big) = \bar{J}_s,$$
$$L'_{(y,u)}(y_s, u_s)\big((y - y_s, u - u_s)\big) = 0,$$
$$L''_{(y,u)}(y_s, u_s)\big((y - y_s, u - u_s), (y - y_s, u - u_s)\big) = \|y - y_s\|^2 + \|u - u_s\|^2.$$

Since L is a quadratic form, the Taylor expansion is

$$\begin{aligned} L(y, u) = L(y_s, u_s) &+ L'_{(y,u)}(y_s, u_s)\big((y - y_s, u - u_s)\big) \\ &+ \frac{1}{2} L''_{(y,u)}(y_s, u_s)\big((y - y_s, u - u_s), (y - y_s, u - u_s)\big), \\ &\forall (y, u) \in H_0^1(\Omega) \times L^2(\Omega), \end{aligned}$$

which means that

$$L(y, u) = \bar{J}_s + \frac{1}{2}\Big(\|y - y_s\|^2 + \|u - u_s\|^2\Big), \quad \forall (y, u) \in H_0^1(\Omega) \times L^2(\Omega). \tag{4.3}$$

Step 2. Noting that $\mu_a \geqslant 0$ and $\mu_b \geqslant 0$, we obtain from (4.2) and (4.3) that for each $(y, u) \in H_0^1(\Omega) \times \mathcal{U}_{ad}$,

$$\begin{aligned} \bar{J}_s + \frac{1}{2}\Big(\|y - y_s\|^2 + \|u - u_s\|^2\Big) \leqslant \frac{1}{2}\Big(\|y - y_{\mathrm{d}}\|^2 + \|u\|^2\Big) + \langle \nabla y, \nabla p_s\rangle \\ - \langle \chi_{\mathcal{D}} u, p_s\rangle. \end{aligned} \tag{4.4}$$

Since $(y^T(t), u^T(t)) \in H_0^1(\Omega) \times \mathcal{U}_{ad}$ for a.e. $t \in (0, T)$, we get from (4.4) that

$$\begin{aligned} \bar{J}_s + \frac{1}{2}\Big(\|y^T(t) - y_s\|^2 + \|u^T(t) - u_s\|^2\Big) \leqslant \frac{1}{2}\Big(\|y^T(t) - y_{\mathrm{d}}\|^2 + \|u^T(t)\|^2\Big) \\ + \langle \nabla y^T(t), \nabla p_s\rangle - \langle \chi_{\mathcal{D}} u^T(t), p_s\rangle, \quad \text{for a.e. } t \in (0, T). \end{aligned}$$

Integrating the above inequality over $(0, T)$ and then multiplying the resulting by $1/T$, we have

$$\begin{aligned} &\bar{J}_s + \frac{1}{2T}\int_0^T \Big(\|y^T(t) - y_s\|^2 + \|u^T(t) - u_s\|^2\Big)\,\mathrm{d}t \\ &\quad\leqslant \bar{J}^T + \frac{1}{T}\int_0^T \Big(\langle \nabla y^T(t), \nabla p_s\rangle - \langle \chi_{\mathcal{D}} u^T(t), p_s\rangle\Big)\,\mathrm{d}t. \end{aligned} \tag{4.5}$$

Observe that

$$-\big\langle y^T(T) - y_0, p_s\big\rangle = \int_0^T \Big(\big\langle \nabla y^T(t), \nabla p_s\big\rangle - \langle \chi_{\mathcal{D}} u^T(t), p_s\rangle\Big)\,\mathrm{d}t.$$

This, along with (4.5), implies that

$$\bar{J}_s + \frac{1}{2T}\int_0^T \left(\|y^T(t) - y_s\|^2 + \|u^T(t) - u_s\|^2\right) \mathrm{d}t \leqslant \bar{J}^T + \frac{\langle y_0 - y^T(T), p_s\rangle}{T}. \tag{4.6}$$

By the standard energy estimate for non-homogeneous heat equations, there is a constant $C > 0$ (independent of $T > 0$) such that

$$\|y^T(T)\| \leqslant C\Big(\|y_0\| + \max\big\{\|u_a\|, \|u_b\|\big\}\Big) \quad \text{for all } \ T > 0. \tag{4.7}$$

Hence, by the Cauchy–Schwarz inequality we have

$$\frac{\langle y_0 - y^T(T), p_s\rangle}{T} \leqslant \frac{C\|p_s\|}{T}\Big(\|y_0\| + \max\big\{\|u_a\|, \|u_b\|\big\}\Big) \leqslant O\Big(\frac{1}{T}\Big).$$

This, together with (4.6), indicates that

$$\bar{J}_s + \frac{1}{2T}\int_0^T \left(\|y^T(t) - y_s\|^2 + \|u^T(t) - u_s\|^2\right) \mathrm{d}t \leqslant \bar{J}^T + O\Big(\frac{1}{T}\Big). \tag{4.8}$$

Step 3. We claim that

$$\bar{J}^T \leqslant \bar{J}_s + O\Big(\frac{1}{T}\Big), \quad \text{when } \ T \geqslant 1. \tag{4.9}$$

Indeed, since u_s is always an admissible control, it holds that

$$\bar{J}^T \leqslant \frac{1}{2T}\int_0^T \left(\|y(t; u_s) - y_{\mathrm{d}}\|^2 + \|u_s\|^2\right) \mathrm{d}t, \tag{4.10}$$

where $y(\cdot; u_s)$ is the solution to

$$\begin{cases} y_t - \Delta y = \chi_{\mathcal{D}} u_s, & \text{in } \ \Omega \times (0, T), \\ y = 0, & \text{on } \ \partial\Omega \times (0, T), \\ y(\cdot, 0) = y_0, & \text{in } \ \Omega. \end{cases}$$

It can be readily checked that

$$\frac{1}{2T}\int_0^T \left(\|y(t; u_s) - y_{\mathrm{d}}\|^2 + \|u_s\|^2\right) \mathrm{d}t \leqslant \bar{J}_s + O\Big(\frac{1}{T}\Big), \quad \text{when } \ T \geqslant 1. \tag{4.11}$$

Which in turn, together with (4.10), implies that (4.9).

Step 4. End of the proof for the inequality (4.1). We obtain immediately from (4.8) and (4.9) that

$$\bar{J}^T = \bar{J}_s + O\Big(\frac{1}{T}\Big),$$

as well as

$$\frac{1}{2T}\int_0^T \Big(\|y^T(t) - y_s\|^2 + \|u^T(t) - u_s\|^2\Big)\,\mathrm{d}t \leqslant O\Big(\frac{1}{T}\Big),$$

which is equivalent to the inequality (4.1). □

Remark 12 Notice that the inequality (4.1) is stronger than the weak turnpike property (2.5) for $\mathcal{T} \times \mathcal{U} = \{(y_s, u_s)\}$. The above proof yields the convergence result for the long-time horizon control problems toward to the steady-state one in the measure-theoretical sense. It is an improved version of the case of *time-independent* controls [23, Sect. 4].

Remark 13 We remark that, in the steps 2 and 3 of the proof of Proposition 3, we have used the exponential stabilization of the heat equation to derive the upper bounds (4.7) and (4.11). See also Remark 6.

4.2 What is (strict) strong duality

In the above proof of Proposition 3, we have seen an important role played by the Lagrangian (4.3), which is closely related to the notion of strict strong duality introduced below. We recall that the notion of strong duality, well known in optimization (see, e.g., [4]).

Definition 4 We say that the static problem (P_s) as defined in (3.2) (in Sect. 3.1) has the *strong duality property* if there exists $\varphi_s \in D(A^*)$ (Lagrangian multiplier) such that (y_s, u_s) minimizes the *Lagrangian function* $L(\cdot, \cdot, \varphi_s) : E \times F \to \mathbb{R}$ defined by

$$L(y, u, \varphi_s) = f^0(y, u) + \langle A^*\varphi_s, y\rangle_{X^*,X} + \langle \varphi_s, f(y, u)\rangle_{X^*,X}.$$

We say (P_s) has the *strict strong duality property* if there exists a $\mathcal{K}$-class function $\alpha(\cdot)$ such that

$$L(y, u, \varphi_s) \geqslant L(y_s, u_s, \varphi_s) + \alpha\Big(\|(y - y_s, u - u_s)\|_{X\times U}\Big)$$

for all $(y, u) \in E \times F$.

Remark 14 Note that $L(y_s, u_s, \varphi_s) = \bar{J}_s$. If (y_s, u_s) is the unique minimizer of the Lagrangian function $L(\cdot, \cdot, \varphi_s)$, and if $E \times F$ is compact in $X \times U$, then (P_s) enjoys the strict strong duality property. However, it is generally a very strong assumption that $L(\cdot, \cdot, \varphi_s)$ has a unique minimizer. Note that uniqueness of minimizers for elliptic optimal control problems is still a long outstanding and difficult problem (cf., e.g., [27]).

In finite dimension, strong duality is introduced and investigated in optimization problems for which the primal and dual problems are equivalent. The notion of strong duality is closely related to the saddle point property of the Lagrangian function associated with the primal optimization problem (see, e.g., [4,27]). Note that the Karush-Kuhn-Tucker condition is sufficient to ensure the strong duality property for a convex problem (see [4, Chapter 5]). Similar assumptions are also considered for other purposes in the literature (see, for example, [6, Assumption 1], [7, Assumption 4.2 (ii)] and [8, Assumption 2].)

In infinite dimension, however, the usual strong duality theory cannot be applied because the underlying constraint set may have an empty interior. The corresponding strong duality theory, as well as the existence of Lagrange multipliers associated with optimization problems or to variational inequalities, have been developed only quite recently in [10]. The strict strong duality property is closely related to the second-order sufficient optimality condition, which guarantees the local optimality of (y_s, u_s) for the problem (P_s) (see, e.g., [27]).

We provide hereafter an example satisfying the strict strong duality property.

Example 4 Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with a smooth boundary $\partial\Omega$. Given any $y_{\mathrm{d}} \in L^2(\Omega)$, we consider the static optimal control problem

$$\inf \frac{1}{2}\big(\|y - y_{\mathrm{d}}\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)}\big),$$

over all $(y, u) \in H^1_0(\Omega) \times L^2(\Omega)$ satisfying

$$\begin{cases} -\triangle y + y^3 = u & \text{in } \Omega, \\ y = 0 & \text{on } \partial\Omega. \end{cases}$$

Let (y_s, u_s) be an optimal solution of this problem. According to first-order necessary optimality conditions (see, e.g., [17, Chapter 1] or [27, Chapter 6, Sect. 6.1.3]), there exists an adjoint state $\varphi_s \in H^2(\Omega) \cap H^1_0(\Omega)$ satisfying

$$\begin{cases} -\triangle \varphi_s + 3y_s^2\varphi_s = y_s - y_{\mathrm{d}} & \text{in } \Omega, \\ \varphi_s = 0 & \text{on } \partial\Omega, \end{cases}$$

such that $u_s = \varphi_s$. Moreover, since φ_s is a Lagrangian multiplier associated with (y_s, u_s) for the Lagrangian function $L(\cdot, \cdot, \varphi_s) : H^1_0(\Omega) \times L^2(\Omega) \to \mathbb{R}$ defined by

$$\begin{aligned} L(y, u, \varphi_s) = {} & \frac{1}{2}\big(\|y - y_{\mathrm{d}}\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)}\big) + \langle -\triangle\varphi_s, y\rangle_{L^2(\Omega), L^2(\Omega)} \\ & + \langle \varphi_s, y^3 - u\rangle_{L^2(\Omega), L^2(\Omega)}, \end{aligned}$$

we have

$$L(y_s, u_s, \varphi_s) \leqslant L(y, u, \varphi_s), \quad \forall (y, u) \in H^1_0(\Omega) \times L^2(\Omega).$$

It means that this problem has the strong duality property.

Next, we claim that the strict strong duality property is satisfied under the condition that $\|y_{\mathrm{d}}\|_{L^2(\Omega)}$ is small enough. Notice that

$$\frac{1}{2}\left(\|y_s - y_{\mathrm{d}}\|^2_{L^2(\Omega)} + \|u_s\|^2_{L^2(\Omega)}\right) \leqslant \frac{1}{2}\|y_{\mathrm{d}}\|^2_{L^2(\Omega)}.$$

Now, assuming that the norm of the target y_d is small enough guarantees the smallness of (y_s, u_s), which consequently belongs to a ball B_r in $H^1_0(\Omega) \times L^2(\Omega)$, centered at the origin and with a small radius $r > 0$. Moreover, by the elliptic regularity, we deduce that the norms of y_s and φ_s are small in $H^2(\Omega) \cap L^\infty(\Omega)$ (see [23, Sect. 3]). For the Lagrangian function $L(\cdot, \cdot, \varphi_s)$ defined above, its first-order Fréchet derivative is

$$L'(y_s, u_s, \varphi_s)\,((y - y_s, u - u_s)) = 0, \tag{4.12}$$

and its second-order Fréchet derivative is

$$\begin{aligned} &L''(y_s, u_s, \varphi_s)\,((y - y_s, u - u_s), (y - y_s, u - u_s)) \\ &\quad = \|y - y_s\|^2_{L^2(\Omega)} + \|u - u_s\|^2_{L^2(\Omega)} + 6\int_\Omega y_s \varphi_s (y - y_s)^2\,\mathrm{d}x, \end{aligned} \tag{4.13}$$

whenever $(y, u) \in B_r$ (see, for instance, [27, Chapter 6, p. 337-338]). Note that

$$\begin{aligned} L(y, u, \varphi_s) = L(y_s, u_s, \varphi_s) &+ L'(y_s, u_s, \varphi_s)\,((y - y_s, u - u_s)) \\ &+ L''(y_s, u_s, \varphi_s)\,((y - y_s, u - u_s), (y - y_s, u - u_s)) \\ &+ o(\|y - y_s\|^2_{L^2(\Omega)} + \|u - u_s\|^2_{L^2(\Omega)}), \end{aligned}$$

for all $(y, u) \in B_r$. This, together with (4.12), (4.13) and the smallness of (y_s, φ_s) in $L^\infty(\Omega)$, implies that

$$L(y, u, \varphi_s) \geqslant L(y_s, u_s, \varphi_s) + \frac{1}{2}(\|y - y_s\|^2_{L^2(\Omega)} + \|u - u_s\|^2_{L^2(\Omega)}), \quad \forall (y, u) \in B_r,$$

which proves the above claim.

Remark 15 Similarly to second-order gap conditions for local optimality [27], the positive semi-definiteness of Hessian matrix of the Hamiltonian is a necessary condition for local optimality, while its positive definiteness is a sufficient condition for the local optimality. The latter is also known as the strengthened Legendre-Clebsch condition.

4.3 Strict strong duality implies strict dissipativity

In this subsection, by means of strict strong duality, we extend Proposition 3 to general optimal control problems. More precisely, we establish sufficient conditions, in terms of (strict) strong duality for (P_s), under which (strict) dissipativity holds true with a specific storage function for $(\bar{P}_{[0,T]})$ in Sect. 3.1. As seen in Theorem 2, strict dissipativity implies measure-turnpike.

Theorem 3 *Let E be a bounded subset of X. Then, strong duality (resp., strict strong duality) for (P_s) as defined in (3.2) implies dissipativity (resp., strict dissipativity) for $(\bar{P}_{[0,T]})$ as defined in (3.1) , with the storage function given by $S(y) = -\langle \varphi_s, y\rangle_{X^*,X}$ for every $y \in E$. Consequently, $(\bar{P}_{[0,T]})$ has the measure-turnpike property under the strict strong duality property.*

Proof It suffices to prove that strong duality for (P_s) implies dissipativity for $(\bar{P}_{[0,T]})$ (the proof with the "strict" additional property is similar with only minor modifications).

By the definition of strong duality, there exists a Lagrangian multiplier $\varphi_s \in D(A^*)$ such that $L(y_s, u_s, \varphi_s) \leqslant L(y, u, \varphi_s)$ for all $(y, u) \in E \times F$, which means that

$$f^0(y_s, u_s) \leqslant f^0(y, u) + \langle A^*\varphi_s, y\rangle_{X^*,X} + \langle \varphi_s, f(y, u)\rangle_{X^*,X} \qquad \forall (y, u) \in E \times F.$$

Let $T > 0$. Assume that $(y(\cdot), u(\cdot))$ is an admissible pair for the problem $(\bar{P}_{[0,T]})$. Then,

$$f^0(y_s, u_s) \leqslant f^0(y(t), u(t)) + \langle A^*\varphi_s, y(t)\rangle_{X^*,X} + \langle \varphi_s, f(y(t), u(t))\rangle_{X^*,X}, \quad \text{for a.e. } t \in [0, T].$$

Integrating the above inequality over $(0, \tau)$, with $0 < \tau \leqslant T$, leads to

$$\tau f^0(y_s, u_s) \leqslant \int_0^\tau f^0(y(t), u(t))\,\mathrm{d}t + \int_0^\tau \langle A^*\varphi_s, y(t)\rangle_{X^*,X}\,\mathrm{d}t + \int_0^\tau \langle \varphi_s, f(y(t), u(t))\rangle_{X^*,X}\,\mathrm{d}t. \tag{4.14}$$

Notice that $(y(\cdot), u(\cdot))$ satisfies the state equation in the problem $(\bar{P}_{[0,T]})$, we have

$$\int_0^\tau \langle A^*\varphi_s, y(t)\rangle_{X^*,X}\,\mathrm{d}t + \int_0^\tau \langle \varphi_s, f(y(t), u(t))\rangle_{X^*,X}\,\mathrm{d}t = \langle \varphi_s, y(\tau)\rangle_{X^*,X} - \langle \varphi_s, y(0)\rangle_{X^*,X}.$$

This, together with (4.14), leads to

$$\int_0^\tau \Big(f^0(y(t), u(t)) - f^0(y_s, u_s)\Big)\,\mathrm{d}t + \langle \varphi_s, y(\tau)\rangle_{X^*,X} \geqslant \langle \varphi_s, y(0)\rangle_{X^*,X}.$$

Set $S(y) = -\langle \varphi_s, y\rangle_{X^*,X}$ for every $y \in E$. Since E is a bounded subset of X, we see that $S(\cdot)$ is locally bounded and bounded from below. Therefore, we infer that $\{(\bar{P}_{[0,T]}) \mid T > 0\}$ has the dissipativity property. □

Remark 16 Strong duality and dissipativity are equivalent in some situations:

- On one hand, we proved above that strong duality (resp. strict strong duality) implies dissipativity (resp., strict dissipativity). We refer also the reader to [12, Lemma 3] for a closely related result.
- On the other hand, it is easy to see that, if the storage function $S(\cdot)$ is continuously Fréchet differentiable, then strong duality (resp., strict strong duality) is the infinitesimal version of the dissipative inequality (3.4) (resp., of (3.5)). For this point, we also mention that [13, Assumption 5.2] is a discrete version of strict dissipativity, and that [12, Inequality (14)] is the infinitesimal version of strict dissipativity for the continuous system when the storage function is differentiable.

5 Conclusions and further comments

In this paper, we first have proved that a general turnpike phenomenon around a set holds for optimal control problems with terminal state constraints in an abstract framework. Next, we have obtained the following auxiliary result:

strict strong duality $\Rightarrow$ strict dissipativity $\Rightarrow$ measure-turnpike property.

We have also used dissipativity to identify the long-time limit of optimal values.

Now, several comments and perspectives are in order.

Measure-turnpike versus exponential turnpike In the paper [29], we establish the exponential turnpike property for general classes of optimal control problems in infinite dimension that are similar to the problem $(\bar{P}_{[0,T]})$ investigated in the present paper, but with the following differences:

(i) $E = X$ and $F = U$;
(ii) $y(0) = y_0 \in X$.

The item (i) means that, in [29], we consider optimal control problems without any state or control constraint. Under the additional assumption made in (ii), we are then able to apply the *Pontryagin maximum principle* in Banach spaces (see [18]), thus obtaining an extremal system that is *smooth*, which means in particular that the extremal control is a *smooth* function of the state and of the adjoint state. This smooth regularity is crucial in the analysis done in [29] (see also [30]), consisting of linearizing the extremal system around an equilibrium point, which is itself the optimal solution of an associated static optimal control problem, and then of analyzing from a spectral point of view the hyperbolicity properties of the resulting linear system. Adequately interpreted, this implies the local exponential turnpike property, saying that

$$\left\| y^T(t) - y_s \right\|_X + \left\| u^T(t) - u_s \right\|_U + \left\| \lambda^T(t) - \lambda_s \right\|_X \leqslant c \left(e^{-\mu t} + e^{-\mu(T-t)} \right),$$

for every $t \in [0, T]$, for some constants $\mu, c > 0$ not depending on T, where λ^T is the adjoint state coming from the Pontryagin maximum principle. There are many examples of control systems for which the measure-turnpike holds but not exponential turnpike. The exponential turnpike property is much stronger than the measure-turnpike property, not only because it gives an exponential estimate on the control and the state, instead of the softer estimate in terms of Lebesgue measure, but also because it gives the closeness property for the adjoint state. This leads us to the next comment.

Turnpike on the adjoint state As mentioned above, the exponential turnpike property established in [29] holds as well for the adjoint state coming from the application of the Pontryagin maximum principle. This property is particularly important when one wants to implement a numerical method in order to compute the optimal trajectories, may it be a direct method (classical optimization under constraints) or an indirect method (resolution of a shooting problem by a Newton algorithm). Indeed, the exponential closeness property of the adjoint state to the optimal static adjoint allows one to successfully initialize such a numerical method, as explained in [30] where in particular an appropriate modification and adaptation of the usual shooting method has been described and implemented.

The flaw of the linearization approach developed in [29] is that it does not a priori allow to take easily into account some possible control constraints (without speaking of state constraints).

The softer approach developed in the present paper leads to the weaker property of measure-turnpike, but permits to take into account some state and control constraints.

However, under the assumption (ii) above, one can as well apply the Pontryagin maximum principle, and thus obtain an adjoint state λ^T. Due to state and control constraints, of course, one cannot expect that the extremal control u^T be a smooth function of y^T and λ^T, but anyway our approach by dissipativity is soft enough to yield the measure-turnpike property for the optimal state y^T and for the optimal control u^T. Now, it is an open question to know whether the measure-turnpike property holds or not for the adjoint state λ^T. As mentioned above, having such a result is particularly important in view of numerical issues.

Local versus global properties It is interesting to stress on the fact that Theorem 2 (saying that strict dissipativity implies measure-turnpike) is of *global* nature, whereas Theorem 3 (saying that strict strong duality implies strict dissipativity) is rather of *local* nature. This is because, as soon as Lagrangian multipliers enter into play, except under strong convexity assumptions this underlies that one is performing reasonings that are local, such as applying first-order conditions for optimality. Therefore, although Theorem 3 provides a sufficient condition ensuring strict dissipativity and thus allowing one to apply the result of Theorem 2, in practice showing strict strong duality can in general only be done locally. In contrast, dissipativity is a much more general property, which is global in the sense that it reflects a global qualitative behavior of the dynamics, as in the Lyapunov theory. We insist on this global picture because this is also a big difference with the results of [29,30] on exponential turnpike, that are purely local and require smallness conditions. Here, in the framework of Theorem 2, no smallness condition is required. The price to pay however is that one has to know a storage function, ensuring strict dissipativity. In practical situations, this is often the case and storage functions often represent an energy that has a physical meaning.

Semilinear heat equation We end the paper with a still open problem, related to the above-mentioned smallness condition. Continuing with Example 4, given any $y_{\mathrm{d}} \in L^2(\Omega)$ we consider the evolution optimal control problem

$$\inf \frac{1}{2T} \int_0^T \left(\|y(t) - y_{\mathrm{d}}\|^2_{L^2(\Omega)} + \|u(t)\|^2_{L^2(\Omega)} \right) \mathrm{d}t$$

over all possible solutions of

$$\begin{cases} y_t - \triangle y + y^3 = u & \text{in } \Omega \times (0, T), \\ y = 0 & \text{on } \partial\Omega \times (0, T), \end{cases} \tag{5.1}$$

such that $(y(t), u(t)) \in H_0^1(\Omega) \times L^2(\Omega)$ for almost every $t \in (0, T)$. It follows from Example 4 and Theorem 3 that the problem is dissipative at an optimal stationary point (y_s, u_s) with the storage function $S(y) = -\langle \varphi_s, y \rangle_{L^2(\Omega), L^2(\Omega)}$. Under the additional smallness condition on $\|y_{\mathrm{d}}\|_{L^2(\Omega)}$, the strict strong duality holds and thus the measure-turnpike property follows. As said above, this assumption reflects the fact that Theorem 3 is rather of local nature. However, due to the fact that the nonlinear term in (5.1) has the "right sign", we do not know how to take advantage of this monotonicity of the control system (5.1) to infer the measure-turnpike property. It is interesting to compare this result with [23, Theorem 3.1], where the authors used a subtle analysis of optimality systems to establish an exponential turnpike property, under the same smallness condition. The question of whether the turnpike property actually holds or not for optimal solutions *but* without the smallness condition on the target, is still an interesting open problem.

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