PURE and APPLIED
ANALYSIS
Vol. 2, No. 1, 2020
dx.doi.org/10.2140/paa.2020.2.93

# CHARACTERIZATION BY OBSERVABILITY INEQUALITIES OF CONTROLLABILITY AND STABILIZATION PROPERTIES

EMMANUEL TRÉLAT, GENGSHENG WANG AND YASHAN XU

Given a linear control system in a Hilbert space with a bounded control operator, we establish a characterization of exponential stabilizability in terms of an observability inequality. Such dual characterizations are well known for exact (null) controllability. Our approach exploits classical Fenchel duality arguments and, in turn, leads to characterizations in terms of observability inequalities of approximate null controllability and of $\alpha$-null controllability. We comment on the relationships among those various concepts, at the light of the observability inequalities that characterize them.

## 1. Context and main result on stabilizability

***Framework.*** Let $X$ and $U$ be Hilbert spaces. We consider the linear control system

$$\dot{y}(t) = Ay(t) + Bu(t), \quad t \geqslant 0, \tag{1}$$

where $A : D(A) \to X$ is a linear operator generating a $C_0$ semigroup $(S(t))_{t\geq 0}$ on $X$ and $B \in L(U, X)$ is a control operator.

Given an initial state $y_0 \in X$ and a control $u \in L^2_{\text{loc}}(0, +\infty; U)$, the unique solution $y(t) = y(t; y_0, u)$ ($t \geqslant 0$) to (1), associated with $u$ and the initial condition $y(0) = y_0$, satisfies

$$y(T; y_0, u) = S(T)y_0 + L_T u \quad \text{for all } T > 0, \tag{2}$$

with $L_T \in L(L^2(0, T; U), X)$ defined by $L_T u = \int_0^T S(T-t)Bu(t)\,dt$. Note that $y \in C^0([0, +\infty), X) \cap H^1_{\text{loc}}(0, +\infty; X_{-1})$, with $X_{-1} = D(A^*)'$, the dual of $D(A^*)$ with respect to the pivot space $X$; see, e.g., [Engel and Nagel 2000; Tucsnak and Weiss 2009]. We recall that the dual mapping $L_T^* \in L(X, L^2(0, T; U))$ is given by $(L_T^*\psi)(t) = B^*S(T-t)^*\psi$ for every $\psi \in X$.

Throughout the paper we identify $X$ (resp., $U$) with its dual $X'$ (resp., $U'$). We denote by $\|\cdot\|_X$ and $\langle\cdot,\cdot\rangle_X$ (resp. $\|\cdot\|_U$ and $\langle\cdot,\cdot\rangle_U$) the Hilbert norm and scalar product in $X$ (resp., in $U$).

It is well known that, for the control system (1), exact (null) controllability is equivalent by duality to an observability inequality. In the existing results (e.g., heat, wave, Schrödinger equations), such inequalities are instrumental to establish controllability properties; see the textbooks [Curtain and Zwart 1995; Lasiecka and Triggiani 2000a; Lions 1988; Staffans 2005; Tucsnak and Weiss 2009; Zabczyk 1995]. Additionally, exponential stabilizability, meaning that there exists a feedback operator $K$ such that $A + BK$ generates an exponentially stable semigroup, is characterized in the existing literature in

*MSC2010:* 93B05, 93B07, 93C20.
*Keywords:* observability inequality, stabilizability, controllability.

terms of infinite-horizon linear quadratic optimal control and algebraic Riccati theory (see the previous references). But, to our knowledge, a dual characterization of exponential stabilizability in terms of an observability inequality is not known.

***Stabilizability.*** The control system (1) is exponentially stabilizable if there exists a feedback operator $K \in L(X,U)$ such that the operator $A+BK$, of domain $D(A+BK)=D(A)$, generates an exponentially stable $C_0$ semigroup $(S_K(t))_{t\geqslant 0}$, i.e., there exists $M \geqslant 1$ and $\omega < 0$ such that

$$\|S_K(t)\|_{L(X)} \leqslant M e^{\omega t} \quad \text{for all } t \geqslant 0. \tag{3}$$

The infimum $\omega_K$ of all possible real numbers $\omega$ such that (3) is satisfied for some $M \geqslant 1$ is the growth bound of the semigroup $(S_K(t))_{t\geqslant 0}$ and is given, see [Engel and Nagel 2000; Pazy 1983], by

$$\omega_K = \inf_{t>0} \frac{1}{t} \ln \|S_K(t)\|_{L(X)} = \lim_{t\to+\infty} \frac{1}{t} \ln \|S_K(t)\|_{L(X)}.$$

Exponential stabilizability means that there exists $K \in L(X,U)$ such that $\omega_K < 0$.

When the control system (1) is exponentially stabilizable, the best stabilization decay rate is defined by

$$\omega^* = \inf\{\omega_K \mid K \in L(X,U) \text{ such that } (S_K(t))_{t\geqslant 0} \text{ is exponentially stable}\}. \tag{4}$$

When $\omega^* = -\infty$, the control system (1) is said to be *completely stabilizable*: this means that stabilization can be achieved at any decay rate. We also speak of *rapid stabilization*.

***Main result.*** Hereafter, we set

$$\|B^*S(T-\cdot)^*\psi\|_{L^2(0,T;U)} = \left(\int_0^T \|B^*S(T-t)^*\psi\|_U^2\,dt\right)^{1/2}.$$

Given $\alpha \geqslant 0$, $T>0$ and $y_0 \in X$, we define

$$\mu^T_{y_0,\alpha} = \inf\{C \geqslant 0 \mid \langle \psi, S(T)y_0\rangle_X - \alpha\|y_0\|_X\|\psi\|_X \leqslant C\|B^*S(T-\cdot)^*\psi\|_{L^2(0,T;U)} \text{ for all } \psi \in X\}, \tag{5}$$

with the convention that $\inf\varnothing = +\infty$. Actually, when the set in (5) is not empty, the infimum is reached (see Section 4). By definition, we have

$$\begin{aligned}
&\mu^T_{y_0,\alpha} \in [0,+\infty], \\
&\mu^T_{y_0,\alpha_2} \leqslant \mu^T_{y_0,\alpha_1} && \text{if } \alpha_1 \leqslant \alpha_2, \\
&\mu^T_{y_0,\alpha} = 0 && \text{if } \alpha \geqslant \|S(T)\|_{L(X)}, \\
&\mu^T_{\lambda y_0,\alpha} = \lambda \mu^T_{y_0,\alpha} && \text{for every } \lambda > 0
\end{aligned}$$

(and thus $\mu^T_{y_0,\alpha} = \mu^T_{y_0/\|y_0\|_X,\alpha}\|y_0\|_X$). This homogeneity property leads us to define

$$\mu^T_\alpha = \sup_{\|y_0\|_X=1} \mu^T_{y_0,\alpha}. \tag{6}$$

We claim that

$$\mu^T_\alpha = \inf\{C \geqslant 0 \mid \|S(T)^*\psi\|_X - \alpha\|\psi\|_X \leqslant C\|B^*S(T-\cdot)^*\psi\|_{L^2(0,T;U)} \text{ for all } \psi \in X\} \tag{7}$$

(see Lemma 29 in Section 4.2) and we have as well

$$\begin{aligned} &\mu_\alpha^T \in [0,+\infty], \\ &\mu_{\alpha_2}^T \leqslant \mu_{\alpha_1}^T \quad \text{if } \alpha_1 \leqslant \alpha_2, \\ &\mu_\alpha^T = 0 \qquad \text{if } \alpha \geqslant \|S(T)\|_{L(X)}. \end{aligned}$$

**Theorem 1.** *The following items are equivalent*:

(i) *The control system* (1) *is exponentially stabilizable.*

(ii) *For every* $\alpha \in (0,1)$, *there exists* $T > 0$ *such that* $\mu_\alpha^T < +\infty$.

(iii) *There exist* $\alpha \in (0,1)$ *and* $T > 0$ *such that* $\mu_\alpha^T < +\infty$.

(iv) *For every* $\alpha \in (0,1)$, *there exists* $T > 0$ *such that the control system* (1) *is cost-uniformly* $\alpha$*-null controllable*[1] *in time* $T$; *i.e., there exists* $C = C(\alpha,T) \geqslant 0$ *such that, for every* $y_0 \in X$, *there exists* $u \in L^2(0,T;U)$ *such that*

$$\|y(T;y_0,u)\|_X \leqslant \alpha\|y_0\|_X \quad \textit{and} \quad \|u\|_{L^2(0,T;U)} \leqslant C\|y_0\|_X. \tag{8}$$

(v) *There exist* $\alpha \in (0,1)$ *and* $T > 0$ *such that the control system* (1) *is cost-uniformly* $\alpha$*-null controllable in time* $T$.

(vi) *For every* $\alpha \in (0,1)$, *there exist* $T > 0$ *and* $C \geqslant 0$ *such that*

$$\|S(T)^*\psi\|_X \leqslant C\|B^*S(T-\cdot)^*\psi\|_{L^2(0,T;U)} + \alpha\|\psi\|_X \quad \textit{for all } \psi \in X. \tag{9}$$

(vii) *There exist* $\alpha \in (0,1)$, $T > 0$ *and* $C \geqslant 0$ *such that inequality* (9) *is satisfied.*

*When one of these items is satisfied, the smallest possible constant* $C$ *in* (8) *and in the observability inequality* (9) *is* $C = \mu_\alpha^T$; *moreover, for every* $\alpha \in (0,1)$, *the real number* $T > 0$ *in* (ii), (iii) *and* (iv) *above can be taken to be the same.*

*Furthermore, the best stabilization decay rate defined by* (4) *is*

$$\omega^* = \inf\left\{\frac{\ln\alpha}{T} \;\middle|\; (\alpha,T) \in \mathcal{A}\right\} = \lim_{\alpha\to 0+} \inf\left\{\frac{\ln\alpha}{T} \;\middle|\; T \in \mathcal{T}(\alpha)\right\}, \tag{10}$$

*where*

$$\begin{aligned} \mathcal{A} &= \{(\alpha,T) \in (0,1)\times(0,+\infty) \mid \mu_\alpha^T < +\infty\}, \\ \mathcal{T}(\alpha) &= \{T > 0 \mid \mu_\alpha^T < +\infty\} \quad \textit{for all } \alpha \in (0,1). \end{aligned} \tag{11}$$

Theorem 1 will be proved in Section 4.

**Remark 2.** If the semigroup $(S(t))_{t\geqslant 0}$ is exponentially stable then the observability inequality (9) is obviously satisfied with $B = 0$. Indeed, the semigroup $(S(t))_{t\geqslant 0}$ is exponentially stable if and only if there exists $T > 0$ such that $\|S(T)\|_{L(X)} < 1$. This is in accordance with the fact that, in this case, no control is required to stabilize the control system.

[1] This definition and some characterizations will be given with more details in Section 2.1.

When the semigroup is not exponentially stable, the observability inequality (9) can be seen as a weakened version of the observability inequality corresponding to exact null controllability (see (14) in Remark 8), by adding the term $\alpha\|\psi\|_X$ at the right-hand side for some $\alpha \in (0, 1)$: this appears as a kind of compromise between the lack of exponential stability of $(S(t))_{t\geqslant 0}$ and the feedback action needed to exponentially stabilize the control system (1).

**Remark 3.** The equality (10) is comparable with the results on the stability rate given in [Engel and Nagel 2000, Chapter 4, Proposition 2.2].

By the second item of Theorem 1, if the control system (1) is exponentially stabilizable, then

$$\{\alpha \mid (\alpha, T) \in \mathcal{A} \text{ for some } T > 0\} = (0, 1).$$

A number of comments, in relation to other controllability concepts, are provided in Sections 2 and 3. It can already be noted that, in the weak observability inequality (9) which characterizes the stabilizability property, the coefficient $\alpha$ satisfies $0 < \alpha < 1$. The limit case $\alpha = 1$ is critical. Also, it is interesting to underline that, in some sense, the constant $\mu_\alpha^T$ quantifies the stabilizability property.

The proof of Theorem 1 follows a series of easy arguments essentially exploiting Fenchel duality. In turn, these arguments allow us to obtain characterizations, in terms of observability inequalities, of the concepts of $\alpha$-null controllability and of approximate null controllability, that we gather in the next section.

## 2. Several results on null and approximate controllability

**2.1.** ***α*-null controllability.** Let $T > 0$ and $\alpha \geqslant 0$ be arbitrary.

**Definition 4.** Given some $y_0 \in X$, the control system (1) is *$\alpha$-null controllable from $y_0$ in time $T$* if there exists $u \in L^2(0, T; U)$ such that $\|y(T; y_0, u)\|_X \leqslant \alpha\|y_0\|_X$, i.e., $y(T; y_0, u) \in \alpha\|y_0\|_X\mathcal{B}$, where $\mathcal{B}$ is the closed unit ball in $X$.

The control system (1) is *$\alpha$-null controllable in time $T$* if, for every $y_0 \in X$, the system is $\alpha$-null controllable from $y_0$ in time $T$.

The control system (1) is *cost-uniformly $\alpha$-null controllable in time $T$* if there exists $C = C(\alpha, T) \geqslant 0$ such that, for every $y_0 \in X$, there exists $u \in L^2(0, T; U)$ such that $\|y(T; y_0, u)\| \leqslant \alpha\|y_0\|_X$ and $\|u\|_{L^2(0,T;U)} \leqslant C\|y_0\|_X$.

Note that, for $\alpha = 0$, the notion of 0-null controllability coincides with the usual notion of exact null controllability. We have the following results.

**Proposition 5.** *Let $T > 0$, $\alpha \geqslant 0$ and $y_0 \in X$ be arbitrary. The following items are equivalent*:

(i) *The control system (1) is $\alpha$-null controllable from $y_0$ in time $T > 0$.*

(ii) *We have $\mu_{y_0,\alpha}^T < +\infty$.*

(iii) *There exists $C \geqslant 0$ such that*

$$\langle\psi, S(T)y_0\rangle_X \leqslant C\|B^*S(T-\cdot)^*\psi\|_{L^2(0,T;U)} + \alpha\|y_0\|_X\|\psi\|_X \quad \textit{for all } \psi \in X. \tag{12}$$

*When one of these items is satisfied, the smallest possible constant $C$ in the observability inequality (12) is $C = \mu^T_{y_0,\alpha}$. Moreover, $\mu^T_{y_0,\alpha}$ is the minimal $L^2$ norm of the control required to steer the control system (1) from $y_0$ to the target set $\alpha\|y_0\|_X\mathcal{B}$.*

**Proposition 6.** *Let $T > 0$ and $\alpha \geqslant 0$ be arbitrary. The following items are equivalent*:

(i) *The control system (1) is cost-uniformly $\alpha$-null controllable in time $T > 0$.*

(ii) *We have $\mu^T_\alpha < +\infty$.*

(iii) *There exists $C \geqslant 0$ such that*

$$\|S(T)^*\psi\|_X \leqslant C\|B^*S(T-\cdot)^*\psi\|_{L^2(0,T;U)} + \alpha\|\psi\|_X \quad \textit{for all } \psi \in X. \tag{13}$$

*When one of these items is satisfied, the smallest possible constant $C$ in the observability inequality (13) is $C = \mu^T_\alpha$.*

Propositions 5 and 6 will be proved in Section 4.

**Remark 7.** The control system (1) is exponentially stabilizable if and only if there exists (or, for every) $\alpha \in (0, 1)$, there exists $T > 0$ such that the control system (1) is cost-uniformly $\alpha$-null controllable in time $T$ (i.e., $\mu^T_\alpha < +\infty$).

**Remark 8.** As said above, for $\alpha = 0$ we recover the usual notion of exact null controllability. Recall that the control system (1) is exactly null controllable in time $T > 0$ if, for every $y_0 \in X$, there exists $u \in L^2(0, T; U)$ such that $y(T; y_0, u) = 0$. This is equivalent to $\mathrm{Ran}(S(T)) \subset \mathrm{Ran}(L_T)$ (see (2)), and also, by duality, to the observability inequality

$$\|S(T)^*\psi\|_X \leqslant \mu^T_0\|B^*S(T-\cdot)^*\psi\|_{L^2(0,T;U)} \quad \text{for all } \psi \in X, \tag{14}$$

which is (13) with $\alpha = 0$.

**Remark 9.** If the control system (1) is exactly null controllable in time $T$ then $\mu^T_\alpha < +\infty$ for every $\alpha > 0$, and $\mu^T_\alpha$ has a limit as $\alpha \to 0^+$, denoted by $\mu^T_0$ (these facts are easily seen by considering the optimal controls $\bar{u}_{y_0,\alpha}$). In particular, we have the observability inequality

$$\langle\psi, S(T)y_0\rangle_X \leqslant \mu^T_0\|B^*S(T-\cdot)^*\psi\|_{L^2(0,T;U)}$$

for every $\psi \in X$ and every $y_0 \in X$ of norm 1. Taking $y_0 = S(T)^*\psi/\|S(T)^*\psi\|_X$, we recover the observability inequality (14). Actually

$$\text{exact null controllable in time } T \quad \Longleftrightarrow \quad \mu^T_0 = \lim_{\alpha\to 0^+}\mu^T_\alpha < +\infty.$$

**Remark 10.** We claim that, for $\alpha = 0$,

$$\text{cost-uniformly 0-null controllable in time } T \quad \Longleftrightarrow \quad \text{0-null controllable in time } T,$$

but for $\alpha > 0$ we only have

$$\text{cost-uniformly } \alpha\text{-null controllable in time } T \quad \underset{\not\Leftarrow}{\Rightarrow} \quad \alpha\text{-null controllable in time } T.$$

Indeed, the first claim is a classical fact of the HUM theory (see [Lions 1988]; see also [Boyer 2013, Proposition 1.19] where the uniform boundedness principle is used to get the result).

In the second claim, the converse is wrong because it may happen that $\mu^T_{y_0,\alpha} < +\infty$ for every $y_0 \in X$, while $\mu^T_\alpha = \sup_{\|y_0\|_X=1} \mu^T_{y_0,\alpha} = +\infty$: see an example in Section 3.2.1. See also Remark 19.

**Remark 11.** Let $\alpha \geqslant 0$ and $T > 0$ be fixed. Summing up, we have seen that

- $\alpha$-null controllable from $y_0$ in time $T \iff \mu^T_{y_0,\alpha} < +\infty$,
- $\alpha$-null controllable in time $T \iff$ for all $y_0 \in X$, $\mu^T_{y_0,\alpha} < +\infty$,
- cost-uniformly $\alpha$-null controllable in time $T \iff \mu^T_\alpha < +\infty$,

and that none of these properties are equivalent when $\alpha > 0$. We have also seen that

the system is exponentially stabilizable

$$\begin{aligned}
&\iff \forall \alpha \in (0,1),\ \exists T > 0 \text{ such that the system is cost-uniform } \alpha\text{-null controllable in time } T \\
&\iff \forall \alpha \in (0,1),\ \exists T > 0 \text{ such that } \mu^T_\alpha < +\infty \\
&\iff \exists \alpha \in (0,1),\ \exists T > 0 \text{ such that the system is cost-uniform } \alpha\text{-null controllable in time } T \\
&\iff \exists \alpha \in (0,1),\ \exists T > 0 \text{ such that } \mu^T_\alpha < +\infty.
\end{aligned}$$

**Remark 12.** The constant $\mu^T_{y_0,\alpha}$ quantifies the $\alpha$-null controllability property: actually, as established in the proof in Section 4.1, when $\mu^T_{y_0,\alpha} < +\infty$ we have

$$\mu^T_{y_0,\alpha} = \|\bar{u}_{y_0,\alpha}\|_{L^2(0,T;U)},$$

where $\bar{u}_{y_0,\alpha}$ is the (unique) control of minimal $L^2$ norm steering in time $T$ the control system (1) from $y_0$ to the ball $\alpha\|y_0\|_X\mathcal{B}$.

### 2.2. *Approximate controllability.* Let $T > 0$ be arbitrary.

**Definition 13.** Given some $y_0 \in X$, the control system (1) is *approximately null controllable from $y_0$ in time $T$* if, for every $\alpha > 0$, the system is $\alpha$-null controllable from $y_0$ in time $T$. Equivalently, for every $\varepsilon > 0$ there exists $u \in L^2(0,T;U)$ such that $\|y(T;y_0,u)\|_X \leqslant \varepsilon$.

The control system (1) is *approximately null controllable in time $T$* if, for every $y_0 \in X$, it is approximately null controllable from $y_0$ in time $T$.

**Proposition 14.** *Let $y_0 \in X$ be arbitrary. The following items are equivalent*:

(i) *The control system* (1) *is approximately null controllable from $y_0$ in time $T > 0$.*

(ii) *For every $\alpha > 0$, we have $\mu^T_{y_0,\alpha} < +\infty$.*

(iii) *For every $\alpha > 0$, there exists $C \geqslant 0$ such that*

$$\langle \psi, S(T)y_0\rangle_X \leqslant C\|B^*S(T-\cdot)^*\psi\|_{L^2(0,T;U)} + \alpha\|y_0\|_X\|\psi\|_X \quad \text{for all } \psi \in X. \tag{15}$$

(iv) $B^*S(T-t)^*\psi = 0$ *for all $t \in [0,T]$ implies* $\langle \psi, S(T)y_0\rangle = 0$.

*When one of these items is satisfied, the smallest possible constant $C$ in the observability inequality* (15) *is $C = \mu^T_{y_0,\alpha}$.*

Equivalently, one can replace "every $\alpha > 0$" with "every $\alpha \in (0, \|S(T)\|_{L(X)})$" in Proposition 14. This proposition will be proved in Section 4.

**Remark 15.** This proposition is not new and can be found, in a slightly different form, in [Boyer 2013].

Proposition 14(iv) (unique continuation property) was suggested by the referee, for which we are grateful. The proof of the equivalence of (iii) and (iv) follows [Boyer 2013, Proposition 1.17]: (iii) obviously implies (iv) by taking $\alpha \to 0$. That (iv) implies (iii) is proved by contradiction: If the inequality (15) does not hold, then there exists $\alpha > 0$ such that, for every $n \in \mathbb{N}^*$, there exists $\psi_n \in X$ such that $\langle \psi_n, S(T)y_0\rangle_X = 1 > n\|B^*S(T-\cdot)^*\psi_n\|_{L^2(0,T;U)} + \alpha\|y_0\|_X\|\psi_n\|_X$. Hence $\psi_n$ is bounded and hence, up to some subsequence, it converges weakly to some $\psi \in X$, which must satisfy $\langle \psi, S(T)y_0\rangle_X = 1$ (and thus $\psi \neq 0$) and also $B^*S(T-\cdot)^*\psi = 0$ on $[0, T]$, whence $\psi = 0$, which raises a contradiction.

It is interesting to note that the unique continuation property (Proposition 14(iv)) is a qualitative way of expressing approximate controllability, while the observability inequality (15) is a quantitative way of expressing it: the constant $\mu^T_{y_0,\alpha}$ quantifies approximate controllability — it gives an account for the "quality" of the approximate controllability property.

**Remark 16.** The control system (1) is approximately null controllable in time $T > 0$ if and only if $\mathrm{Ran}(S(T))$ (or its closure) is contained in the closure of $\mathrm{Ran}(L_T)$, or, by duality, given any $\psi \in X$, if $B^*S(T-t)^*\psi = 0$ for every $t \in [0, T]$ then $S(T)^*\psi = 0$; see [Zabczyk 1995, Theorem 2.1, page 207].

**Remark 17.** Until now, we have spoken only of approximate null controllability. Let us comment about the more usual concept of approximate controllability.

The control system (1) is *approximately controllable in time* $T > 0$ if, for every $\varepsilon > 0$, for all $y_0, y_1 \in X$, there exists $u \in L^2(0, T; U)$ such that $\|y(T; y_0, u) - y_1\|_X \leqslant \varepsilon$.

Equivalently, $\mathrm{Ran}(L_T)$ is dense in $X$, or, by duality, $L_T^*$ is injective, which means that, given any $\psi \in X$, if $B^*S(T-t)^*\psi = 0$ for every $t \in [0, T]$ then $\psi = 0$ (unique continuation property; compare with the one given in Remark 16 and with the one given in Proposition 14).

It is interesting to note the following result:

> *Let $T > 0$ be arbitrary. Assume that $S(T)^*$ is injective, i.e., that* $\mathrm{Ran}(S(T))$ *is dense in $X$. Then approximate controllability in time $T$ is equivalent to approximate null controllability in time $T$.*

The assumption that $S(T)^*$ is injective is satisfied when $(S(t))_{t\geqslant 0}$ is either a group (obviously) or an analytic $C_0$ semigroup.

To prove the latter fact, by taking the adjoint, let us prove that if $(S(t))_{t\geqslant 0}$ is an analytic semigroup then for every $T \geqslant 0$ the operator $S(T)$ is injective: Let $T \geqslant 0$ and $x \in X$ be such that $S(T)x = 0$. Then $S(t)x = S(t-T)S(T)x = 0$ for every $t \geqslant T$, and by analyticity we infer that $S(t)x = 0$ for every $t \geqslant 0$, whence $x = 0$.

In contrast, when the semigroup is neither a group nor analytic, $S(T)$ may fail to be injective: for instance, the left-shift semigroup on the positive half-line is such that, for every $y_0$, there exists $T = T(y_0)$ such that $S(T)y_0 = 0$.

**Remark 18.** By the same approach, we obtain as well the following characterization of approximate controllability by an observability inequality:

*Define* $\mu^T_{y_0,y_1,\alpha} \in [0,+\infty]$ *similarly to* $\mu^T_{y_0,\alpha}$, *replacing the term* $S(T)y_0$ *with* $S(T)y_0 - y_1$. *The control system* (1) *is approximately controllable in time* $T>0$ *if and only if, for all* $y_0, y_1 \in X$ *and for every* $\alpha>0$, *there exists* $C \geqslant 0$ *such that*

$$\langle \psi, S(T)y_0 - y_1\rangle_X \leqslant C\|B^*S(T-\cdot)^*\psi\|_{L^2(0,T;U)} + \alpha\|y_0\|_X\|\psi\|_X \quad \text{for all } \psi \in X.$$

*When* $\mu^T_{y_0,y_1,\alpha} < +\infty$, *it is the smallest constant* $C$ *in the observability inequality above.*

This statement underlines in an unusual way the difference between approximate controllability and approximate null controllability (as said in Remark 17, both notions coincide if $S(T)^*$ is injective, or equivalently if $\mathrm{Ran}(S(T))$ is dense in $X$).

Similar statements can be given as well for $\alpha$-controllability to some target point $y_1$. We do not give details.

**Remark 19.** There is no relationship in general between approximate controllability and exponential stabilizability:

- There exist control systems that are exponentially stabilizable but that are not approximately null controllable in any time $T$.

For instance, take $B=0$ and take $A$ generating an exponentially stable semigroup, i.e., $\|S(t)\|_{L(X)} \leqslant Me^{-\beta t}$ for some $M \geqslant 1$ and $\beta > 0$. Then a minimal time $T_\alpha = (1/\beta)\ln(M/\alpha)$ is at least required to realize $\alpha$-null controllability, which cannot be bounded uniformly with respect to every $\alpha>0$ arbitrarily small.

- There exist control systems that are approximately controllable in some time $T$ but that are not exponentially stabilizable (see [Curtain and Zwart 1995, Example 5.2.2, page 228; Pritchard and Zabczyk 1981, Example 3.16; Triggiani 1975; Zabczyk 1995, Theorem 3.3(ii), page 227], which all give the same example; see also the example provided in Section 3.2.1, already commented on in Remark 10).

We mention however [Badra and Takahashi 2014, Theorem 1.6], which states that, when $(S(t))_{t\geqslant 0}$ is analytic, under some spectral assumptions, approximate controllability is equivalent to exponential stabilizability and to a Fattorini–Hautus criterion.[2]

**Remark 20.** Given $T>0$, recall that $\mu^T_\alpha = \mu^T_{y_0,\alpha} = 0$ when $\alpha \geqslant \|S(T)\|_{L(X)}$. When $\alpha \to 0^+$, $\mu^T_\alpha$ and $\mu^T_{y_0,\alpha}$ may tend to $+\infty$. Assuming that $\mu^T_\alpha$ is uniformly bounded with respect $\alpha \in (0, \|S(T)\|_{L(X)}]$, approximate null controllability in time $T$ is equivalent to exact null controllability in time $T$.

## 3. Further comments

**3.1.** ***Null controllability implies stabilizability.*** Inspecting the observability inequalities (14) and (9), we recover the well known fact that if the control system (1) is exactly null controllable in some time $T$ then it is exponentially stabilizable; see, e.g., [Zabczyk 1995, Theorem 3.3, page 227]. The converse is wrong in general: as said in Remark 9, the control system (1) is exactly null controllable in time $T$ if and only if

[2]We thank Guillaume Olive for having indicated this reference to us.

$\mu_0^T = \lim_{\alpha\to 0^+} \mu_\alpha^T < +\infty$; it may happen that when the system is exponentially stabilizable, as $\alpha \to 0^+$, the infimum of times $T$ such that $\mu_\alpha^T < +\infty$ tends to $+\infty$.

***Complete stabilizability.*** Complete stabilizability means that, given any $\omega \in \mathbb{R}$, one can find a feedback $K \in L(X, U)$ such that $\omega_K < \omega$, or equivalently, that the best stabilization rate given by (10) is $-\infty$. According to the expression (10), we have complete stabilizability if either, for a given $\alpha \in (0, 1)$, we have $\mu_\alpha^T < +\infty$ for every $T > 0$ arbitrarily small, or, for a given $T > 0$, we have $\mu_\alpha^T < +\infty$ for every $\alpha > 0$ arbitrarily small (which is equivalent, by Remark 9, to exact null controllability in time $T$). Several remarks on complete stabilizability are in order.

**Proposition 21.** *If the control system (1) is exactly null controllable in some time $T > 0$ then it is completely stabilizable.*

This result is proved in Appendix A.1. It also follows by Remark 9: since $\mu_\alpha^T$ remains uniformly bounded as $\alpha \to 0$ for some $T > 0$ fixed, we see from (10) that $\omega^* = -\infty$.

**Proposition 22.** *When $(S(t))_{t\geq 0}$ is a group, the following properties are equivalent*:

(i) *Exact controllability in some time $T$.*

(ii) *Exact null controllability in some time $T$.*

(iii) *Complete stabilizability.*

This result is already known; see [Slemrod 1974; Urquiza 2005; Zabczyk 1995, Theorem 3.4, page 229]. We provide in Appendix A.2 a proof of it that uses Theorem 1.

The strategy developed in [Komornik 1997] (applying also, to some extent, to unbounded admissible control operators) consists of taking $K_\lambda = -B^* C_\lambda^{-1}$, where $C_\lambda$ is defined by

$$C_\lambda = \int_0^{T+1/(2\lambda)} f_\lambda(t) S(-t) B B^* S(-t)^* \, dt$$

(variant of the Gramian operator), with $\lambda > 0$ arbitrary,

$$f_\lambda(t) = \begin{cases} e^{-2\lambda t} & \text{if } t \in [0, T], \\ 2\lambda e^{-2\lambda T}(T + 1/(2\lambda) - t) & \text{if } t \in [T, T + 1/(2\lambda)]. \end{cases}$$

The function $V(y) = \langle y, C_\lambda^{-1} y\rangle$ is a Lyapunov function (as noticed in [Coron 2007]), and the feedback $K_\lambda$ yields exponential stability with rate $-\lambda$. We also refer to [Coron and Lü 2014; Coron and Trélat 2004; Phung et al. 2017; Russell 1978] for issues on rapid stabilization.

Let us assume that $A$ is skew-adjoint, which is equivalent, by the Stone theorem, to the fact that $A$ generates a unitary group $(S(t))_{t\in\mathbb{R}}$; see [Engel and Nagel 2000]. Then $\|S(T)^*\psi\|_X = \|\psi\|_X$ for every $\psi \in X$ and therefore the observability inequality (9) characterizing exponential stabilizability is equivalent to the observability inequality characterizing exact controllability and can be achieved, for a given $T > 0$, with arbitrarily small values of $\alpha > 0$, which implies that $\omega^* = -\infty$. Therefore we recover a result of [Liu 1997]:

**Proposition 23.** *When $A$ is skew-adjoint, the following properties are equivalent:*

(i) *exact controllability in time $T$,*

(ii) *exact null controllability in time $T$,*

(iii) *exponential stabilizability,*

(iv) *complete stabilizability.*

### 3.2. *Examples.*

**3.2.1.** *First example.* We take $X = U = \ell^2(\mathbb{N}, \mathbb{R})$, $A$ the infinite-dimensional identity matrix, and $B$ the diagonal infinite-dimensional matrix $B = \operatorname{diag}\big(1, \frac{1}{2}, \frac{1}{3}, \ldots\big)$. We claim that, with this choice:

- For every $T > 0$ and every $\alpha \in (0, 1)$, the control system (1) is $\alpha$-null controllable in time $T$; i.e.,

$$\mu^T_{y_0,\alpha} < +\infty \quad \text{for all } y_0 \in X, \text{ for all } \alpha \in (0,1), \text{ for all } T > 0.$$

Hence, the control system (1) is approximately null controllable in any time $T > 0$.

- The control system (1) is not exponentially stabilizable; i.e.,

$$\mu^T_\alpha = +\infty \quad \text{for all } \alpha \in (0,1), \text{ for all } T > 0.$$

By Remark 7, the control system (1) is not cost-uniformly $\alpha$-null controllable in time $T$.

Let us first prove the $\alpha$-null controllability property. Let $(e_n)_{n\in\mathbb{N}^*}$ be the canonical base of $X$. For every $n \in \mathbb{N}^*$, we denote by $P_n$ the orthogonal projection of $X$ onto $\operatorname{Span}(e_1, \ldots, e_n)$. Let $y_0 \in X$ be arbitrary. Taking $n \in \mathbb{N}^*$ large enough such that $\|y_0 - P_n y_0\|_X \leqslant \alpha e^{-T}\|y_0\|_X$, and taking the control $u = 0$, we have

$$\|y(T; (\mathrm{id} - P_n)y_0, 0)\|_X \leqslant \alpha \|y_0\|_X. \tag{16}$$

By the Duhamel formula, for every $u \in L^2(0, T; U)$ we have

$$y(T; y_0, u) = y(T; P_n y_0, u) + y(T; (\mathrm{id} - P_n)y_0, 0). \tag{17}$$

Note that, taking $u$ such that $u(t) \in \operatorname{Ran}(P_n)$ for almost every $t \in [0, T]$, we have $y(t; P_n y_0, u) \in \operatorname{Ran}(P_n)$ for every $t \in [0, T]$. Since the control system in $\mathbb{R}^n$ given by $\dot{x}(t) = A_n x(t) + B_n u(t)$, with $A_n$ the identity matrix and $B_n = \operatorname{diag}\big(1, \frac{1}{2}, \ldots, \frac{1}{n}\big)$, is controllable (it satisfies the Kalman condition), there exists $u \in L^2(0, T; U)$ such that $y(T; P_n y_0, u) = 0$. Using (17) and (16), it follows that $\|y(T; y_0, u)\|_X \leqslant \alpha \|y_0\|_X$; i.e., we have proved that the system is $\alpha$-null controllable in time $T$.

Let us now prove that the system is not exponentially stabilizable. By contradiction, let us assume that there exist $K \in L(X, U)$, $M > 0$ and $\rho > 0$ such that $\|S_K(t)\| \leqslant M e^{-\rho t}$ for every $t \geqslant 0$ (we use the notation of (3)). Denoting by $a_n(t)$ the $n$-th component of $y_K(t; y_0) = S_K(t) y_0$, we have

$$\dot{a}_n(t) = a_n(t) + \langle e_n, BK y_K(t; y_0)\rangle = a_n(t) + \langle Be_n, K y_K(t; y_0)\rangle.$$

Hence,

$$
\begin{aligned}
|a_n(t)| &= \left| e^t a_n(0) + \int_0^t e^{t-s} \langle Be_n, Ky_K(s; y_0)\rangle \, ds \right| \\
&\geqslant e^t |a_n(0)| - \int_0^t e^{t-s} \|Be_n\| \|K\| \|y_K(s; y_0)\| \, ds \\
&\geqslant e^t |a_n(0)| - \int_0^t e^{t-s} \frac{1}{n} \|K\| M e^{-\rho s} \|y_0\| \, ds \geqslant e^t \left( |a_n(0)| - \frac{M\|K\|}{(1+\rho)n} \|y_0\| \right).
\end{aligned} \tag{18}
$$

Take $m \in \mathbb{N}$ satisfying

$$
\frac{M\|K\|}{(1+\rho)m} \leqslant \frac{1}{2}
$$

and take $y_0 = e_m$. Then $a_m(0) = \|y_0\| = 1$. It follows from (18) with $n = m$ that

$$
\|a_m(t)\| \geq e^t \left( 1 - \frac{M\|K\|}{(1+\rho)m} \right) \geq \frac{1}{2} e^t.
$$

Since $\lim_{m \to +\infty} |a_m(t)| = +\infty$, we obtain that $\|y_K(t; y_0)\| \to +\infty$, which is a contradiction.

**3.2.2.** *Second example.* Consider the coupled control system with Dirichlet boundary conditions

$$
\begin{aligned}
&\partial_{tt} z = \Delta z + w + u, \\
&\partial_t w = \Delta w + w + \frac{2}{\pi} \sin x \int_0^\pi (\partial_t z(t,s) + u(t,s)) \sin s \, ds, \\
&z(t,0) = z(t,\pi) = w(t,0) = w(0,\pi) = 0,
\end{aligned} \tag{19}
$$

where $z = z(t,x)$, $w = w(t,x)$, and $u = u(t,x)$ for $t \geqslant 0$ and $x \in (0,\pi)$. We take initial data $z(0,\cdot) = z_0 \in H_0^1(0,\pi)$, $\partial_t z(0,\cdot) = z_1 \in L^2(0,\pi)$, and $w(0,\cdot) = w_0 \in L^2(0,\pi)$.

To write (19) in the form (1), we define $X = H_0^1(0,\pi) \times L^2(0,\pi) \times L^2(0,\pi)$, $U = L^2(0,\pi)$, and

$$
y = \begin{pmatrix} z \\ \partial_t z \\ w \end{pmatrix}, \quad A = \begin{pmatrix} 0 & \mathrm{id} & 0 \\ \Delta & 0 & \mathrm{id} \\ 0 & P & \Delta + \mathrm{id} \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \\ P \end{pmatrix},
$$

where $D(A) = (H^2(0,\pi) \cap H_0^1(0,\pi)) \times H_0^1(0,\pi) \times (H^2(0,\pi) \cap H_0^1(0,\pi))$ and $P$ is the selfadjoint bounded operator defined on $L^2(0,\pi)$ by

$$
(Pv)(x) = \frac{2}{\pi} \sin x \int_0^\pi v(s) \sin s \, ds \quad \text{for all } v \in L^2(0,\pi).
$$

Note that $B \in L(X,U)$ is a bounded control operator. We claim that:

- With $u = 0$, the control system (19) is not asymptotically stable.
- The control system (19) satisfies the observability inequality (9) of Theorem 1, and hence it is stabilizable.
- The control system (19) is not exactly null controllable in any time $T > 0$. In particular, it is not completely stabilizable.

The fact that, with $u = 0$, the control system (19) is not asymptotically stable, can be seen from the fact that $(z, \partial_t z, w) = (\sin x, 0, \sin x)$ is a steady-state solution to (19) with $u = 0$.

Let us prove that the control system (19) is not exactly null controllable in any time $T > 0$. By contradiction, if there were some $T > 0$ such that the system (19) is exactly null controllable, then by taking the initial data $(z_0, z_1, w_0) = (0, 0, \sin 2x)$, we could find a control $u \in L^2(0, T; L^2(0, \pi))$ steering the system (19) from $(z_0, z_1, w_0)$ to zero in time $T$. However, setting $a(t) = \int_0^\pi w(t, x) \sin(2x)\, dx$, from the second equation of (19), we obtain

$$\dot{a}(t) = \langle w(t, x), (\triangle + \mathrm{id}) \sin(2x)\rangle_{L^2(0,\pi)} = -3\langle w(t, x), \sin(2x)\rangle_{L^2(0,\pi)} = -3a(t).$$

Since $a(0) = \frac{\pi}{2} \neq 0$, we must have $a(T) \neq 0$, which raises a contradiction.

Let us finally prove that the control system (19) is stabilizable. By Theorem 1, it suffices to establish:

*There exist* $\alpha \in (0, 1)$, $T > 0$, *and* $C > 0$ *such that*

$$\|S(T)^*\Psi\|_X^2 \leqslant C\|B^*S(T - \cdot)^*\Psi\|^2_{L^2(0,\pi;U)} + \alpha^2\|\Psi\|_X^2 \quad \textit{for all } \Psi \in X. \tag{20}$$

To prove (20), we first observe that $D(A^*) = D(A)$ and

$$A^* = \begin{pmatrix} 0 & -\mathrm{id} & 0 \\ -\triangle & 0 & P \\ 0 & \mathrm{id} & \triangle + \mathrm{id} \end{pmatrix}, \quad B^* \begin{pmatrix} \phi \\ \psi \\ \xi \end{pmatrix} = \psi + P\xi.$$

We define the adjoint system

$$\begin{aligned} \partial_t \phi &= \psi, \\ \partial_t \psi &= \triangle\phi - P\xi, \\ \partial_t \xi &= -\psi - \triangle\xi - \xi, \end{aligned} \tag{21}$$

with $(\phi(T), \psi(T), \xi(T)) = (\phi_T, \psi_T, \xi_T) \in D(A^*)$. Inequality (20) is then equivalent to

$$\left\| \begin{pmatrix} \phi(0) \\ \psi(0) \\ \xi(0) \end{pmatrix} \right\|_X^2 \leqslant C \int_0^\pi \|\psi(t) + P\xi(t)\|^2\, dt + \alpha^2 \left\| \begin{pmatrix} \phi(T) \\ \psi(T) \\ \xi(T) \end{pmatrix} \right\|_X^2 \quad \text{for all } (\phi_T, \psi_T, \xi_T) \in D(A^*). \tag{22}$$

Let us establish (22) for $T = \pi$, $\alpha = \sqrt{2}e^{-3T} < 1$, and for some $C > 0$ which will be given later. To this end, we arbitrarily fix $(\phi_T, \psi_T, \xi_T) \in D(A^*)$. We write

$$\begin{pmatrix} \phi_T \\ \psi_T \\ \xi_T \end{pmatrix} = \begin{pmatrix} P\phi_T \\ P\psi_T \\ P\xi_T \end{pmatrix} + \begin{pmatrix} (\mathrm{id}-P)\phi_T \\ (\mathrm{id}-P)\psi_T \\ (\mathrm{id}-P)\xi_T \end{pmatrix} = \begin{pmatrix} \phi_{T,1} \\ \psi_{T,1} \\ \xi_{T,1} \end{pmatrix} + \begin{pmatrix} \phi_{T,2} \\ \psi_{T,2} \\ \xi_{T,2} \end{pmatrix}.$$

Denote respectively by

$$\begin{pmatrix} \phi(\cdot) \\ \psi(\cdot) \\ \xi(\cdot) \end{pmatrix}, \quad \begin{pmatrix} \phi_1(\cdot) \\ \psi_1(\cdot) \\ \xi_1(\cdot) \end{pmatrix}, \quad \begin{pmatrix} \phi_2(\cdot) \\ \psi_2(\cdot) \\ \xi_2(\cdot) \end{pmatrix}$$

the solutions to (21) with final data (at time $T$)

$$\begin{pmatrix}\phi_T\\ \psi_T\\ \xi_T\end{pmatrix}, \quad \begin{pmatrix}\phi_{T,1}\\ \psi_{T,1}\\ \xi_{T,1}\end{pmatrix}, \quad \begin{pmatrix}\phi_{T,2}\\ \psi_{T,2}\\ \xi_{T,2}\end{pmatrix}.$$

Then we have

$$\begin{pmatrix}\phi\\ \psi\\ \xi\end{pmatrix} = \begin{pmatrix}\phi_1\\ \psi_1\\ \xi_1\end{pmatrix} + \begin{pmatrix}\phi_2\\ \psi_2\\ \xi_2\end{pmatrix}.$$

Since $(\phi_1(t,\cdot), \psi_1(t,\cdot), \xi_1(t,\cdot)) \in \mathrm{Span}(\sin x)$ and since the pair

$$\left(\begin{pmatrix}0 & 1 & 0\\ -1 & 0 & -1\\ 0 & -1 & 0\end{pmatrix}, (0,1,1)\right)$$

satisfies the (observability) Kalman condition, there exists $C_1 > 0$ independent of the final data such that

$$\left\| \begin{pmatrix}\phi_1(0)\\ \psi_1(0)\\ \xi_1(0)\end{pmatrix} \right\|_X^2 \leqslant C_1 \int_0^\pi \|\psi_1(t,\cdot) + \xi_1(t,\cdot)\|^2\, dt = C_1 \int_0^\pi \|\psi_1(t,\cdot) + P\xi_1(t,\cdot)\|^2\, dt. \tag{23}$$

Now, since $(\phi_2(t,\cdot), \psi_2(t,\cdot), \xi_2(t,\cdot)) \in (\mathrm{Span}(\sin x))^\perp$ and $P\xi_2(t,\cdot) = 0$, it follows from (21) that

$$\begin{aligned} \partial_t\phi_2 &= \psi_2, & \partial_t\psi_2 &= \triangle\phi_2,\\ \phi_2(T) &= \phi_{T,2}, & \psi_2(T) &= \psi_{T,2}. \end{aligned} \tag{24}$$

From (24) and by observability of the wave equation (which is true because $T > \pi$), there exists $C_2 > 0$ independent of the final data such that

$$\left\| \begin{pmatrix}\phi_2(0)\\ \psi_2(0)\end{pmatrix} \right\|_{H_0^1\times L^2}^2 \leqslant C_2 \int_0^\pi \|\psi_2(t,\cdot)\|^2\, dt = C_2 \int_0^\pi \|\psi_2(t,\cdot) + P\xi_2(t,\cdot)\|^2\, dt. \tag{25}$$

Additionally, it follows from the third equation of (21) that

$$\xi_2(0) = e^{T(\Delta+\mathrm{id})}\xi_2(T) + \int_0^T e^{t(\Delta+\mathrm{id})}\psi_2(t,\cdot)\, dt,$$

which implies

$$\begin{aligned} \|\xi_2(0)\|^2 &\leqslant \left(e^{-3T}\|\xi_2(T)\| + \int_0^T \|\psi_2(t,\cdot)\|\, dt\right)^2\\ &\leqslant 2e^{-6T}\|\xi_2(T)\|^2 + 2\left(\int_0^T \|\psi_2(t,\cdot)\|\, dt\right)^2\\ &\leqslant 2e^{-6T}\|\xi_2(T)\|^2 + 2T\int_0^\pi \|\psi_2(t,\cdot) + P\xi_2(t,\cdot)\|^2\, dt. \end{aligned} \tag{26}$$

From (25) and (26), we infer that

$$\left\| \begin{pmatrix} \phi_2(0) \\ \psi_2(0) \\ \xi_2(0) \end{pmatrix} \right\|_X^2 \leqslant (C_2 + 2T) \int_0^\pi \|\psi_2(t,\cdot) + P\xi_2(t,\cdot)\|^2 \, dt + 2e^{-6T} \|\xi_2(T)\|^2$$
$$\leqslant (C_2 + 2T) \int_0^\pi \|\psi_2(t,\cdot) + P\xi_2(t,\cdot)\|^2 \, dt + 2e^{-6T} \left\| \begin{pmatrix} \phi_2(T) \\ \psi_2(T) \\ \xi_2(T) \end{pmatrix} \right\|_X^2. \tag{27}$$

The desired inequality (22) follows from (23) and (27), with $T = \pi$, $\alpha = \sqrt{2}e^{-3T} < 1$ and $C = \max(C_1, C_2 + 2T)$.

**3.3.** ***Extension to unbounded admissible control operators.*** Throughout the paper we have assumed that the control operator $B$ is bounded, i.e., is linear continuous with values in $X$, that is, $B \in L(U, X)$. This assumption covers the case of internal controls, but not, in general, of boundary controls. For the latter case, we speak of *unbounded control operators*, which are operators $B$ that are not continuous from $U$ to $X$ but are continuous from $U$ to some larger space $X_{-\alpha}$ which can be defined by extrapolation (scale of Hilbert spaces; see [Engel and Nagel 2000; Staffans 2005; Tucsnak and Weiss 2009]). Given such a control operator $B \in L(U, X_{-\alpha})$ with $\alpha > 0$, the range of the operator $L_T$ may then fail to be contained in $X$. We say that $B$ is *admissible* when $\mathrm{Ran}(L_T) \subset X$ for some (and thus for all) $T > 0$; see [Tucsnak and Weiss 2009]. Note that $X_{-1}$ is isomorphic to $D(A^*)$ (with $X$ as a pivot space) and that if $B$ is admissible then $B \in L(U, X_{-1/2})$.

For an admissible control operator, since $\mathrm{Ran}(L_T) \subset X$, all arguments of Section 4.1 (Fenchel duality) remain valid.[3] One should anyway avoid using $\mathcal{G}_T^{1/2}$ (where $\mathcal{G}_T := \int_0^T S(T-t)BB^*S(T-t)^* \, dt$ is the usual Gramian operator when $B$ is bounded) in this more general context and replace $\|\mathcal{G}_T^{1/2}\psi\|_X$ with $\|B^*S(T-\cdot)^*\psi\|_{L^2(0,T;U)}$ everywhere throughout the proof. Therefore:

**Proposition 24.** *Propositions 5, 6 and 14 remain true in the more general context where the control operator $B$ is admissible (and may be unbounded).*

When the control operator $B$ is not admissible, the question of knowing whether Propositions 5, 6 and 14 may be extended in some way is open.

Now, concerning the exponential stabilizability result, the only critical fact is at the end of the proof of Lemma 31 where we invoke the Riccati theory: indeed this theory is well established in the general case only for admissible control operators and analytic semigroups (see Remark 32 at the end of the proof in Section 4.2). Therefore:

**Theorem 25.** *Theorem 1 is true, without any change, in the more general context where the control operator $B$ is admissible (and may be unbounded) and the semigroup $(S(t))_{t\geqslant 0}$ is analytic.*

For instance, this situation covers the case of heat equations in a $C^2$ bounded open subset $\Omega \subset \mathbb{R}^n$, with $X = L^2(\Omega)$, with Neumann control at the boundary of $\Omega$ (for which, at best, $B \in L(U, X_{-1/4-\varepsilon})$

[3] When the control operator is not admissible, we have $\mathrm{Ran}(L_T) \subset X_{-\alpha}$ for some $\alpha > 0$ and then the closed unit ball $\mathcal{B}$ should be the one in $X_{-\alpha}$, while the considered controllability concepts are in the space $X$.

for every $\varepsilon > 0$), but not with Dirichlet control at the boundary (for which, at best, $B \in L(U, X_{-3/4-\varepsilon})$ for every $\varepsilon > 0$). We refer to [Lasiecka and Triggiani 2000a] for details.

Actually, as indicated to us by Marius Tucsnak, we do not need to use the Riccati operator but only the fact that the finite-cost property (also called optimizability) implies stabilizability, which is true (as well as the converse implication) for a bounded control operator.

Let us recall that the control system (1) is *optimizable* (or, enjoys the *finite-cost property*) if, for every $y_0 \in X$, there exists $u \in L^2(0, +\infty; U)$ such that $y(\cdot\,; y_0, u) \in L^2(0, +\infty; X)$.

The argument of the proof of Lemma 31 can easily be extended (see Remark 32) to the case of an admissible control operator $B$ (which may be unbounded), and we obtain the following result:

**Theorem 26.** *For an admissible control operator $B$ (which may be unbounded), the following items are equivalent:*

(i) *The control system (1) is optimizable.*

(ii) *For every $\alpha \in (0, 1)$ there exists $T > 0$ such that $\mu_\alpha^T < +\infty$.*

(iii) *There exist $\alpha \in (0, 1)$ and $T > 0$ such that $\mu_\alpha^T < +\infty$.*

(iv) *For every $\alpha \in (0, 1)$, there exist $T > 0$ and $C \geqslant 0$ such that, for every $y_0 \in X$, there exists $u \in L^2(0, T; U)$ such that*

$$\|y(T; y_0, u)\|_X \leqslant \alpha \|y_0\|_X \quad \text{and} \quad \|u\|_{L^2(0,T;U)} \leqslant C \|y_0\|_X. \tag{28}$$

(v) *There exist $\alpha \in (0, 1)$, $T > 0$ and $C \geqslant 0$ such that, for every $y_0 \in X$, there exists $u \in L^2(0, T; U)$ such that inequality (28) is satisfied.*

(vi) *For every $\alpha \in (0, 1)$ (equivalently, there exists $\alpha \in (0, 1)$), there exist $T > 0$ and $C \geqslant 0$ such that*

$$\|S(T)^* \psi\|_X \leqslant C \|B^* S(T - \cdot)^* \psi\|_{L^2(0,T;U)} + \alpha \|\psi\|_X \quad \text{for all } \psi \in X. \tag{29}$$

(vii) *There exist $\alpha \in (0, 1)$, $T > 0$ and $C > 0$ such that inequality (29) is satisfied.*

*When one of these items is satisfied, the smallest possible constant $C$ in (8) and in the observability inequality (9) is $C = \mu_\alpha^T$; moreover, for every $\alpha \in (0, 1)$, the real number $T > 0$ in* (ii), (iii) *and* (iv) *above can be taken to be the same.*

By inspecting, in light of Remark 32, the proof of Lemma 31 and in particular (34), we note that, for an admissible control operator $B$, any of the items of Theorem 26 is equivalent to the following fact:

> *For every $\alpha \in (0, 1)$, there exist $T > 0$ and $M = M(\alpha, T) > 0$ such that, for every $y_0 \in X$, there exists $u \in L^2(0, +\infty; U)$ such that $\|y(t; y_0, u)\|_X \leqslant M \|y_0\|_X e^{((\ln \alpha)/T)t}$ for every $t \geqslant 0$.*

This is only an *open-loop* stabilizability property (or *asymptotic null controllability*), in the sense that the control $u$, depending on $y_0$, is not determined by a feedback. The usual concept of stabilizability underlies the existence of a feedback.

Hence, we should now discuss how the concepts of optimizability and of stabilizability are related to each other. As said above, optimizability is equivalent to exponential stabilizability (defined in (3)) when the control operator $B$ is bounded. For unbounded admissible control operators, the concept of exponential stabilizability is more difficult to define.

The definition of exponential stabilizability given in [Weiss and Rebarber 2000] is the following. Let $B \in L(U, X_{-1})$. The control system (1) is said to be exponentially stabilizable if $B$ is admissible and if there exists $K \in L(D(A), U)$ such that $\mathrm{id} - K_\Lambda(\lambda\,\mathrm{id} - A)^{-1}B$ is boundedly invertible for $\lambda > 0$ large enough (and the inverse is uniformly bounded) and such that the operator $A + BK_\Lambda$, of domain $D(A + BK_\Lambda) = \{y \in D(K_\Lambda) \mid (A + BK_\Lambda)y \in X\}$, generates on $X$ an exponentially stable $C_0$ semigroup $(S_{K_\Lambda}(t))_{t \geqslant 0}$. Here, $K_\Lambda \in L(D(K_\Lambda), U)$ is the $\Lambda$-extension (or Yosida extension) of $K$, defined by $K_\Lambda y = \lim_{\lambda \to +\infty} K\lambda(\lambda\,\mathrm{id} - A)^{-1}y$ for every $y \in D(K_\Lambda)$. The domain $D(K_\Lambda)$ of $K_\Lambda$, which is the set of all $y \in X$ such that the limit (in $X$) exists, can be equipped with a norm making it a Banach space, and we have $D(A) \subset D(K_\Lambda) \subset X$ with continuous embeddings, and, for every $x \in X$, we have $S(t)x \in D(K_\Lambda)$ and $S_{K_\Lambda}(t)x \in D(K_\Lambda)$ for almost every $t \geqslant 0$. Moreover, we have $S_{K_\Lambda}(t) = S(t) + \int_0^t S(t-s)BK_\Lambda S_{K_\Lambda}(s)\,ds$ for every $t \geqslant 0$, and the control operator $B$ is admissible for the semigroup $(S_{K_\Lambda}(t))_{t \geqslant 0}$.

Exponential stabilizability in the above sense implies optimizability. Indeed, given any $y_0 \in X$, take $u(t) = K_\Lambda S_{K_\Lambda}(t)y_0$, which is in $L^2(0, +\infty; U)$, and note that $y(t; y_0, u) = S_{K_\Lambda}(t)y_0 \in L^2(0, +\infty; X)$.

The converse statement is more involved: it is true but only in a weaker sense. Assume that the control system (1) is optimizable. It is proved in [Weiss and Rebarber 2000, Propositions 3.2, 3.3 and 3.4] (see also [Flandoli et al. 1988; Staffans 2005; Zwart 1996]) that, for every $y_0 \in X$, there exists a unique $\hat{u} \in L^2(0, +\infty; U)$ minimizing the functional

$$J(u, y_0) = \int_0^{+\infty} (\|u(t)\|_U^2 + \|y(t; y_0, u)\|_X^2)\,dt$$

over all possible $u \in L^2(0, +\infty; U)$. Moreover, we have $\hat{u}(t) = \widehat{K}\widehat{S}(t)y_0$ for every $y_0 \in D(\widehat{A})$ and almost every $t \geqslant 0$, where:

- $(\widehat{S}(t))_{t \geqslant 0}$ is an exponentially stable $C_0$ semigroup on $X$, of infinitesimal generator $\widehat{A} : D(\widehat{A}) \to X$, such that $\widehat{S}(t)y_0 = S(t)y_0 + \int_0^t S(t-s)B\hat{u}(s)\,ds$ for every $y_0 \in X$ and every $t \geqslant 0$.

- The feedback operator $\widehat{K} \in L(D(\widehat{A}), U)$ is defined by $\widehat{K} = -B^*P$, where $P \in L(X)$ is a positive symmetric definite operator mapping $D(\widehat{A})$ to $D(A^*)$, such that $J(\hat{u}, y_0) = \langle Py_0, y_0 \rangle_X$ for every $y_0 \in X$. Note that $P$ satisfies a Riccati equation on $D(\widehat{A})$ and possibly also another one on $D(A)$; see [Weiss and Zwart 1998].

- We have $\widehat{A} = A + B\widehat{K}$ on $D(\widehat{A})$, where $A$ in this formula stands for the extension $A : X \to X_{-1}$ of the original infinitesimal operator $A : D(A) \to X$. Moreover, considering the $\Lambda$-extension $\widehat{K}_\Lambda = \lim_{\lambda \to +\infty} \widehat{K}\lambda(\lambda\,\mathrm{id} - A)^{-1}$ of $\widehat{K}$, we have $\hat{u}(t) = \widehat{K}_\Lambda \widehat{S}(t)y_0$ for every $y_0 \in X$ and almost every $t \geqslant 0$.

In contrast to the above definition of exponential stabilizability, however, it is not known whether the control operator $B$ is admissible or not for the semigroup $(\widehat{S}(t))_{t \geqslant 0}$.

We may then define the best stabilization decay rate much as in (4), but, with the above results, we do not know if (10) remains true. We leave this problem as an open question.

**3.4.** ***Extension to Banach spaces.*** Throughout the paper we have assumed that the state space $X$ and the control space $U$ are Hilbert spaces. All our results can be extended without difficulty to the case where $X$ and $U$ are reflexive Banach spaces. One has to be careful to replace, in all observability inequalities, the scalar product $\langle \cdot , \cdot \rangle_X$ with the duality bracket $\langle \cdot , \cdot \rangle_{X',X}$.

**3.5.** ***Open problems.*** We end the section with several open issues.

***Systems with an observation.*** Throughout the paper we have focused on the control system (1), without observation. We could add to the system an observation $z(t) = Cy(t)$, where $C \in L(X, Y)$ is an observation operator, with $Y$ another Hilbert space. Corresponding notions of controllability and of stabilizability are classically defined as well. Extending our main results to that context is an interesting issue.

***Discretization problems.*** The observability inequality (9) may certainly be exploited to recover results on uniform semidiscretizations (or full discretizations). The problem is the following. Consider a spatial semidiscrete model of (1), written as

$$\dot{y}_N = A_N y_N + B_N u_N, \tag{30}$$

with $y_N(t) \in \mathbb{R}^N$ (see [Boyer 2013; Labbé and Trélat 2006; Lasiecka and Triggiani 2000a; Lasiecka and Triggiani 2000b] for a general framework on discretization issues). Assuming that the control system (1) is exponentially stabilizable (equivalently, that the observability inequality (9) is satisfied), how can one ensure that the family of control systems (30) is *uniformly* exponentially stabilizable? Uniform means here that, for each $N$, there exists a feedback matrix $K_N$ such that $\| \exp(t(A_N + B_N K_N))\| \leqslant M e^{\omega t}$ for some $M \geqslant 1$ and some $\omega < 0$ that are *uniform* with respect to $N$.

This problem has been much studied in the literature with various approaches. In [Banks and Ito 1997; Lasiecka and Triggiani 2000a; Liu and Zheng 1999], the convergence of the Riccati matrix $P_N$ (corresponding to (30)) to the Riccati operator $P$ (corresponding to (1)) is proved in the general parabolic case, even for unbounded control operators, that is, when $A : D(A) \to X$ generates an analytic semigroup, $B \in L(U, D(A^*)')$, and $(A, B)$ is exponentially stabilizable. Uniform exponential stability is also proved under uniform Huang–Prüss conditions in [Liu and Zheng 1999], allowing them to obtain convergence of Riccati operators for second-order systems $\ddot{y} + Ay = Bu$, with $A : D(A) \to X$ positive selfadjoint with compact inverse and $B$ bounded control operator. General conservative equations are treated in [Ervedoza and Zuazua 2009] where it is proved that adding a viscosity term in the numerical scheme helps to recover a uniform exponential decay, provided a uniform observability inequality holds true for the corresponding conservative equation (see also [Alabau-Boussouira and Ammari 2011; Alabau-Boussouira et al. 2017; Trélat 2018] for equations with nonlinear damping). Of course, given a specific equation, the difficulty is to establish the uniform observability inequality. This is a difficult issue, investigated in some particular cases (see [Zuazua 2005] for a survey).

Anyway, the observability inequality (9) is written in the semidiscrete case as

$$\|S_N(T)^* \psi_N\| \leqslant C \left( \int_0^T \|B_N^* S(T-t)^* \psi_N\|^2 \, dt \right)^{1/2} + \alpha \|\psi_N\|,$$

with $\alpha \in (0,1)$ and $C > 0$ uniform with respect to $N$. Such a uniform inequality is likely to be true in many cases. For instance it is nicely shown in [Zuazua 2004, Section 5] that approximate boundary controllability for one-dimensional waves is satisfied, in a uniform way, for finite-difference semidiscretization schemes. This is shown by using a functional similar to the one (31) used here (see also [Boyer 2013; Labbé and Trélat 2006] for $\alpha$-null controllability in the parabolic case where $\alpha$ is a decreasing function of the discretization parameter, and see [Boyer and Le Rousseau 2014; Boyer et al. 2019] for more recent results in the semilinear case).

But such considerations go beyond the scope of the present paper. We think that the observability inequalities derived in our main results may be used, at least to recover some known results on uniform convergence, and maybe to establish new ones.

***Hautus test.*** There exist many results with variants of the Hautus test. For instance, when $(S(t))_{t\geqslant 0}$ is a normal $C_0$ group, the Hautus test property, there exists $C > 0$ such that

$$\|(\lambda\,\mathrm{id} - A^*)\psi\|_X^2 + |\operatorname{Re}(\lambda)|^2\,\|B^*\psi\|_X^2 \geqslant C|\operatorname{Re}(\lambda)|^2\|\psi\|_X^2 \quad \text{for all } \lambda \in \mathbb{C}_-, \text{ for all } \psi \in D(A^*),$$

where $\mathbb{C}_-$ is the open left complex half-plane, is sufficient to ensure exponential stabilizability; see [Jacob and Zwart 2009]. It is interesting to investigate the question of how Hautus tests are related to the observability inequalities derived in our paper (see also [Weiss and Rebarber 2000, Proposition 3.5] for an extension of the Hautus test for stabilizability).

***Polynomial stabilizability.*** We have provided in Theorem 1 a characterization of exponential stabilizability. The $C_0$ semigroup $(S(t))_{t\geqslant 0}$ is said to be polynomially stable when there exist constants $\gamma, \delta > 0$ such that $\|S(t)(A - \beta\,\mathrm{id})^{-\gamma}\|_{L(X)} \leqslant M t^{-\delta}$ for every $t \geqslant 1$, for some $M > 0$ and some $\beta \in \rho(A)$ (see [Alabau-Boussouira 2002; 2003; Jacob and Schnaubelt 2007], where polynomial stability is compared with observability; see also [Borichev and Tomilov 2010]). Finding a dual characterization of polynomial stabilizability in terms of an observability inequality is an open issue, which may be related to the previous question on Hautus tests.

***Shape optimization.*** Let $T > 0$ and $\alpha \geqslant 0$ be arbitrary. The observability inequality (13) characterizes cost-uniform $\alpha$-null controllability in time $T$, and we have seen that, when $\alpha \in (0,1)$, it also characterizes exponential stabilizability. The best constant in the inequality (13) is $\mu_\alpha^T$, which, as already said, quantifies the $\alpha$-null controllability or the stabilizability property. One can then address the problem of optimizing this constant, by choosing an adequate control operator $B$ in a certain class. Let us give an example of such a problem.

The Dirichlet wave equation in a $C^2$ bounded open subset $\Omega \subset \mathbb{R}^n$ with internal control

$$\partial_{tt} y = \triangle y + \chi_O u, \quad y_{|\partial\Omega} = 0,$$

where $O$ is a measurable subset of $\Omega$, is well known to be exponentially stabilizable as soon as $O$ is an open subset satisfying the geometric control condition; see [Bardos et al. 1992; Le Rousseau et al. 2017]. Here, the control operator is $B = \chi_O$, and the constant $\mu_\alpha^T$ depends on $O$. Following [Privat et al. 2015;

2016], fixing some $L \in (0,1)$ and defining $\mathcal{U}_L$ as the set of all measurable subsets $O$ of $\Omega$ of Lebesgue measure $|O| = L|\Omega|$, one can consider the problem

$$\inf_{O \in \mathcal{U}_L} \mu_\alpha^T(O),$$

i.e., the problem of searching the best possible subdomain that minimizes $\mu_\alpha^T(O)$ over all possible measurable subdomains $O$ of $\Omega$ of Lebesgue measure $L|\Omega|$. We mention [Privat and Trélat 2015] for a similar problem consisting of maximizing the exponential decay rate.

Such shape optimization issues may also be raised in the above-mentioned context of polynomial stabilizability. We are not aware of any existing results in this direction.

## 4. Proofs

We now provide proofs of Propositions 5, 6 and 14 and of Theorem 1, following Fenchel duality arguments. In turn, we establish intermediate results which may be of interest themselves for other purposes.

We recall that the Gramian operator $\mathcal{G}_T \in L(X)$ is the symmetric positive semidefinite operator defined by

$$\mathcal{G}_T = \int_0^T S(T-t)BB^*S(T-t)^*\,dt.$$

We note that $\langle \mathcal{G}_T\psi, \psi\rangle_X = \|\mathcal{G}_T^{1/2}\psi\|_X^2 = \int_0^T \|B^*S(T-t)^*\psi\|_U^2\,dt$.

### 4.1. *Fenchel duality arguments.*

We start our analysis by considering the $\alpha$-null controllability problem or the approximate null controllability problem for the control system (1). What is written hereafter in this first subsection essentially follows the classical analysis by Fenchel duality done in [Lions 1992] (see also [Glowinski et al. 2008]), but is written in a more general framework.

Let $\alpha \geqslant 0$, let $T > 0$ and let $y_0 \in X$ be arbitrary. Let $\mathcal{B}$ be the closed unit ball in $X$. We consider the $\alpha$-null controllability problem from $y_0$ in time $T$, i.e., the problem of steering the control system (1) from $y_0$ to $\alpha\|y_0\|_X\mathcal{B}$ in time $T$, meaning that

$$\|y(T; y_0, u)\|_X \leqslant \alpha\|y_0\|_X.$$

When the control system (1) is $\alpha$-null controllable from $y_0$ in time $T$, we search the control of minimal $L^2$ norm (which is unique by strict convexity). We set

$$S_{y_0,\alpha}^T = \inf\left\{\tfrac{1}{2}\|u\|_{L^2(0,T;U)}^2 \;\middle|\; y(T; y_0, u) \in \alpha\|y_0\|_X\mathcal{B}\right\},$$

with the convention that $S_{y_0,\alpha}^T = +\infty$ whenever $\alpha\|y_0\|_X\mathcal{B}$ is not reachable from $y_0$ in time $T$.

The following lemma is obvious.

**Lemma 27.** *Let $y_0 \in X$ and $T > 0$ be arbitrary*:

- *Given some $\alpha \geqslant 0$, the control system (1) is $\alpha$-null controllable from $y_0$ in time $T$ if and only if $S_{y_0,\alpha}^T < +\infty$.*

- *The control system* (1) *is approximately null controllable from* $y_0$ *in time* $T$ *if and only if* $S^T_{y_0,\alpha} < +\infty$ *for every* $\alpha > 0$.

**4.1.1.** *Application of Fenchel duality.* Following [Lions 1992], we define the convex and lower semicontinuous functions $F : L^2(0,T;U) \to [0,+\infty)$ and $G : X \to [0,+\infty]$ by

$$F(u) = \tfrac{1}{2}\|u\|^2_{L^2(0,T;U)}$$

and

$$G(\phi) = \begin{cases} 0 & \text{if } \phi \in -S(T)y_0 + \alpha\|y_0\|_X \mathcal{B}, \\ +\infty & \text{otherwise,} \end{cases}$$

and we note that

$$S^T_{y_0,\alpha} = \inf_{u\in L^2(0,T;U)} (F(u) + G(L_T u)).$$

The Fenchel conjugates $F^* : L^2(0,T;U) \to [0,+\infty)$ and $G^* : X \to [0,+\infty]$ are given by

$$F^*(v) = \sup_{u\in L^2(0,T;U)} (\langle v,u\rangle_X - F(u)) = \tfrac{1}{2}\|v\|^2_{L^2(0,T;U)} = F(v)$$

and

$$G^*(\psi) = \sup_{\phi\in X} (\langle \phi,\psi\rangle_X - G(\phi)) = \sup_{\phi\in -S(T)y_0+\alpha\|y_0\|_X\mathcal{B}} \langle \phi,\psi\rangle_X = -\langle \psi, S(T)y_0\rangle_X + \alpha\|y_0\|_X\|\psi\|_X,$$

where the latter equality is obtained by applying the Cauchy–Schwarz inequality.

Noting that $L_T\,\mathrm{dom}(F) = \mathrm{Ran}(L_T)$ intersects the set of points at which $G$ is continuous when (1) is either $\alpha$-null controllable in time $T$ (with $\alpha > 0$ fixed) or approximately null controllable in time $T$, we infer from the Fenchel–Rockafellar duality theorem, see [Fenchel 1949; Ekeland and Témam 1999; Rockafellar 1967], that

$$S^T_{y_0,\alpha} = -\inf_{\phi\in X} (F^*(L_T^*\phi) + G^*(-\phi)) = -\inf_{\psi\in X} (F^*(L_T^*\psi) + G^*(\psi)) = -\inf_{\psi\in X} J^T_{y_0,\alpha}(\psi),$$

where we have set

$$J^T_{y_0,\alpha}(\psi) = F^*(L_T^*\psi) + G^*(\psi) = \tfrac{1}{2}\langle \mathcal{G}_T\psi,\psi\rangle_X - \langle \psi, S(T)y_0\rangle_X + \alpha\|y_0\|_X\|\psi\|_X \tag{31}$$

for every $\psi \in X$. Note that $J^T_{y_0,\alpha}$ is differentiable except at $\psi = 0$.

This result is still valid for $\alpha = 0$ but is not a consequence of the Fenchel–Rockafellar duality theorem (because we have used in a critical way that $\alpha > 0$): for $\alpha = 0$ this is the usual procedure in the Hilbert uniqueness method; see [Glowinski et al. 2008; Lions 1988].

**4.1.2.** *Computation of the minimizer.* As said above, when $S^T_{y_0,\alpha} < +\infty$, there is a unique minimizer $\bar{u}_{y_0,\alpha} \in L^2(0,T;U)$ and there is also a unique minimizer $\bar{\psi}_{y_0,\alpha} \in X$ of $J^T_{y_0,\alpha}$ (this follows from (32) below), and

$$S^T_{y_0,\alpha} = \tfrac{1}{2}\|\bar{u}_{y_0,\alpha}\|^2_{L^2(0,T;U)} = -J^T_{y_0,\alpha}(\bar{\psi}_{y_0,\alpha}).$$

We have either $\bar\psi_{y_0,\alpha}=0$ and then $\bar u_{y_0,\alpha}=0$ and $S^T_{y_0,\alpha}=0$, or $\bar\psi_{y_0,\alpha}\neq 0$ and then $\nabla J^T_{y_0,\alpha}(\bar\psi_{y_0,\alpha})=0$, which gives

$$\mathcal{G}_T\bar\psi_{y_0,\alpha}-S(T)y_0+\alpha\|y_0\|_X\frac{\bar\psi_{y_0,\alpha}}{\|\bar\psi_{y_0,\alpha}\|_X}=0. \tag{32}$$

Given any $\psi\in X$, we set $\psi=r\sigma$ with $r=\|\psi\|_X$ and $\sigma\in X$ of norm 1 (polar coordinates). For the minimizer $\bar\psi_{y_0,\alpha}=\bar r_{y_0,\alpha}\bar\sigma_{y_0,\alpha}$, we infer from (32) that

$$\bar r_{y_0,\alpha}=\frac{\langle S(T)y_0,\bar\sigma_{y_0,\alpha}\rangle_X-\alpha\|y_0\|_X}{\langle\mathcal{G}_T\bar\sigma_{y_0,\alpha},\bar\sigma_{y_0,\alpha}\rangle_X},\qquad \bar\sigma_{y_0,\alpha}=(\bar r_{y_0,\alpha}\mathcal{G}_T+\alpha\|y_0\|_X)^{-1}S(T)y_0.$$

Here, we used the facts that $\langle\mathcal{G}_T\bar\sigma_{y_0,\alpha},\bar\sigma_{y_0,\alpha}\rangle_X\neq 0$ (which follows from (32)) and that $\bar u_{y_0,\alpha}\neq 0$. Note that, necessarily, $\langle S(T)y_0,\bar\sigma_{y_0,\alpha}\rangle_X-\alpha\|y_0\|_X\geqslant 0$.

Note also that, in the Fenchel duality argument, the optimal control $\bar u_{y_0,\alpha}$ is given as a function of $\bar\psi_{y_0,\alpha}$ by

$$\bar u_{y_0,\alpha}(t)=(L_T^*\bar\psi_{y_0,\alpha})(t)=B^*S(T-t)^*\bar\psi_{y_0,\alpha}.$$

Until that step, there is nothing new with respect to the existing literature. Up to our knowledge, the novelty is in the next step, with a simple remark leading to an observability inequality.

**4.1.3.** *An alternative optimization problem.* Following the above arguments, we first note that

$$J^T_{y_0,\alpha}(\psi)=J^T_{y_0,\alpha}(r\sigma)=\tfrac12 r^2\langle\mathcal{G}_T\sigma,\sigma\rangle_X-r(\langle\sigma,S(T)y_0\rangle_X-\alpha\|y_0\|_X)$$

and that $J^T_{y_0,\alpha}(0)=0$, and given $a\geqslant 0$ and $b\in\mathbb{R}$, we have

$$\inf_{r>0}(ar^2-br)=\begin{cases}0 & \text{if } b\leqslant 0,\\ -\dfrac{b^2}{4a}\ (-\infty \text{ if } a=0), & \text{if } b>0,\ \text{reached at } r=\dfrac{b}{2a}.\end{cases}$$

Hence for any fixed $\sigma\in X$,

$$\inf_{r>0}J^T_{y_0,\alpha}(r\sigma)=\begin{cases}0 & \text{if } \langle\sigma,S(T)y_0\rangle_X-\alpha\|y_0\|_X\leqslant 0,\\ -\dfrac12\dfrac{(\langle\sigma,S(T)y_0\rangle_X-\alpha\|y_0\|_X)^2}{\langle\mathcal{G}_T\sigma,\sigma\rangle_X} & \text{if } \mathcal{G}_T\sigma\neq 0 \text{ and } \langle\sigma,S(T)y_0\rangle_X-\alpha\|y_0\|_X>0,\\ -\infty & \text{if } \mathcal{G}_T\sigma=0 \text{ and } \langle\sigma,S(T)y_0\rangle_X-\alpha\|y_0\|_X>0.\end{cases}$$

Additionally, by the definition (5) of $\mu^T_{y_0,\alpha}$, we have

$$\mu^T_{y_0,\alpha}=\sup_{\psi\in X}\begin{cases}0 & \text{if } \langle\psi,S(T)y_0\rangle_X-\alpha\|y_0\|_X\|\psi\|_X\leqslant 0,\\ \dfrac{\langle\psi,S(T)y_0\rangle_X-\alpha\|y_0\|_X\|\psi\|_X}{\|\mathcal{G}_T^{1/2}\psi\|_X} & \text{if } \mathcal{G}_T\psi\neq 0 \text{ and } \langle\psi,S(T)y_0\rangle_X-\alpha\|y_0\|_X\|\psi\|_X>0,\\ +\infty & \text{if } \mathcal{G}_T\psi=0 \text{ and } \langle\psi,S(T)y_0\rangle_X-\alpha\|y_0\|_X\|\psi\|_X>0.\end{cases}$$

It follows that

$$S^T_{y_0,\alpha}=-\inf_{\psi\in X}J^T_{y_0,\alpha}(\psi)=-\inf_{\|\sigma\|_X=1}\inf_{r>0}J^T_{y_0,\alpha}(r\sigma)=\tfrac12(\mu^T_{y_0,\alpha})^2.$$

Note that, when $0 < \mu^T_{y_0,\alpha} < +\infty$, we have

$$\mu^T_{y_0,\alpha} = \|\bar{u}_{y_0,\alpha}\|_{L^2(0,T;U)} = \frac{\langle \bar{\psi}_{y_0,\alpha}, S(T)y_0\rangle_X - \alpha\|y_0\|_X\|\bar{\psi}_{y_0,\alpha}\|_X}{\|\mathcal{G}_T^{1/2}\bar{\psi}_{y_0,\alpha}\|_X}.$$

**Remark 28.** According to the above discussion, if $\mu^T_{y_0,\alpha} < +\infty$, then $\mu^T_{y_0,\alpha}$ is the smallest constant $C \in [0,+\infty)$ such that there exists $u \in L^2(0,T;U)$ such that $\|y(T;y_0,u)\|_X \leqslant \alpha\|y_0\|_X$ and $\|u\|_{L^2(0,T;U)} \leqslant C$. When $C = \mu^T_{y_0,\alpha}$, one has $u = \bar{u}_{y_0,\alpha}$.

**4.1.4.** *Proofs of Propositions 5 and 14.* Propositions 5 and 14 follow, using Lemma 27 and the fact that $S^T_{y_0,\alpha} = \frac{1}{2}(\mu^T_{y_0,\alpha})^2$.

**4.2.** ***Fenchel dualization of the exponential stabilization property.***

**4.2.1.** *Proof of (7).*

**Lemma 29.** *Given $\alpha > 0$ and $T > 0$, $\mu^T_\alpha$ is the smallest possible $C \in [0,+\infty]$ such that the following weak observability inequality is satisfied*:

$$\|S(T)^*\psi\|_X - \alpha\|\psi\|_X \leqslant C\|\mathcal{G}_T^{1/2}\psi\|_X \quad \textit{for all } \psi \in X,$$

*and this is independent of the sign of $\|S(T)^*\psi\|_X - \alpha\|\psi\|_X$. Moreover, when $\mu^T_\alpha < +\infty$, it is the smallest constant $C \in [0,+\infty)$ such that the above weak observability inequality holds.*

*Proof.* Using (5), we have

$$\mu^T_{y_0,\alpha} = \sup_{\psi\in X} F^T_\alpha(y_0,\psi), \tag{33}$$

where

$$F^T_\alpha(y_0,\psi) = \begin{cases} \max\Big(\dfrac{\langle\psi, S(T)y_0\rangle_X - \alpha\|y_0\|_X\|\psi\|_X}{\|\mathcal{G}_T^{1/2}\psi\|_X}, 0\Big) & \text{if } (y_0,\psi) \in D_1, \\ +\infty & \text{if } (y_0,\psi) \in D_2, \\ 0 & \text{if } (y_0,\psi) \in D_3, \end{cases}$$

with

$$D_1 = \{(y_0,\psi) \in X\times X \mid \mathcal{G}_T\psi \neq 0\},$$
$$D_2 = \{(y_0,\psi) \in X\times X \mid \mathcal{G}_T\psi = 0 \text{ and } \langle\psi, S(T)y_0\rangle_X - \alpha\|y_0\|_X\|\psi\|_X > 0\},$$
$$D_3 = \{(y_0,\psi) \in X\times X \mid \mathcal{G}_T\psi = 0 \text{ and } \langle\psi, S(T)y_0\rangle_X - \alpha\|y_0\|_X\|\psi\|_X \leqslant 0\}.$$

Since

$$\sup_{\|y_0\|_X=1}\sup_{\psi\in X} F^T_\alpha(y_0,\psi) = \sup_{\psi\in X}\sup_{\|y_0\|_X=1} F^T_\alpha(y_0,\psi)$$

and

$$\sup_{\|y_0\|_X=1} F^T_\alpha(y_0,\psi) = \begin{cases} \max\Big(\dfrac{\|S(T)^*\psi\|_X - \alpha\|\psi\|_X}{\|\mathcal{G}_T^{1/2}\psi\|_X}, 0\Big) & \text{if } \mathcal{G}_T\psi \neq 0, \\ +\infty & \text{if } \mathcal{G}_T\psi = 0,\ \exists\, y_0 \text{ such that } (y_0,\psi) \in D_2, \\ 0 & \text{if } \mathcal{G}_T\psi = 0,\ \nexists\, y_0 \text{ such that } (y_0,\psi) \in D_2, \end{cases}$$

we derive the desired result from (6) and (33). □

**4.2.2.** *Another interpretation of* $\mu_\alpha^T$. We now give another interpretation of $\mu_\alpha^T$, useful for addressing exponential stabilizability.

**Lemma 30.** *Let* $\alpha > 0$ *and* $T > 0$ *be such that* $\mu_\alpha^T < +\infty$. *Then* $\mu_\alpha^T$ *is the smallest constant* $C \geqslant 0$ *such that, for every* $y_0 \in X$, *there exists* $u \in L^2(0,T;U)$ *such that* $\|y(T;y_0,u)\|_X \leqslant \alpha\|y_0\|_X$ *and* $\|u\|_{L^2(0,T;U)} \leqslant C\|y_0\|_X$.

*Proof.* The result follows from the fact that $S^T_{y_0,\alpha} = \frac{1}{2}(\mu^T_{y_0,\alpha})^2 = \frac{1}{2}\|\bar u_{y_0,\alpha}\|^2_{L^2(0,T;U)}$, that $\mu^T_{y_0,\alpha} = \mu^T_{y_0/\|y_0\|_X,\alpha}\|y_0\|_X \leqslant \mu^T_\alpha\|y_0\|_X$ and from Remark 28. □

Lemma 30 is closely related to exponential stabilizability when $\alpha < 1$. We indeed have the following result (an easy consequence of well-known results; however, we provide a proof).

**Lemma 31.** *The control system* (1) *is exponentially stabilizable if and only if, for every* $\alpha \in (0,1)$ (*equivalently, there exists* $\alpha \in (0,1)$), *there exist* $T > 0$ *and* $C > 0$ *such that, for every* $y_0 \in X$, *there exists* $u \in L^2(0,T;U)$ *such that* $\|y(T;y_0,u)\|_X \leqslant \alpha\|y_0\|_X$ *and* $\|u\|_{L^2(0,T;U)} \leqslant C\|y_0\|_X$.

*When this is satisfied, the best stabilization decay rate* $\omega^*$ *is the infimum of* $(\ln\alpha)/T$ *over all possible couples* $(T,\alpha)$ *for which the above inequalities are satisfied for some constant* $C > 0$.

*Proof.* Assume that the control system (1) is exponentially stabilizable. Then there exists $K \in L(X,U)$ such that $\|S_K(t)\|_{L(X)} \leqslant Me^{\omega_K t}$ for every $t \geqslant 0$, with $M \geqslant 1$ and $\omega_K < 0$. Let $y_0 \in X$. We set $u(t) = KS_K(t)y_0$ and $y(t) = S_K(t)y_0$. We have

$$\|y(T;y_0,u)\|_X = \|S_K(T)y_0\|_X \leqslant Me^{\omega_K T}\|y_0\|_X = \alpha\|y_0\|_X$$

for $T = (1/\omega_K)\ln(\alpha/M)$ and we compute

$$\|u\|_{L^2(0,T;U)} \leqslant \frac{\|K\|_{L(X,U)}M}{\sqrt{-2\omega_K}}\|y_0\|_X,$$

whence the result.

Conversely, we proceed by iteration. For the initial condition $y_0$, there exists a control $u_0 \in L^2(0,T;U)$ such that $\|y(T;y_0,u_0)\|_X \leqslant \alpha\|y_0\|_X$ and $\|u_0\|_{L^2(0,T;U)} \leqslant C\|y_0\|_X$. We set $y_1 = y(T;y_0,u_0)$ and we repeat the argument for this new initial condition $y_1$, and then we iterate, obtaining that $\|y_{j+1}\|_X = \|y((j+1)T;y_j,u_j)\|_X \leqslant \alpha\|y_j\|_X$ and $\|u_j\|_{L^2(0,T;U)} \leqslant C\|y_j\|_X$. The control $u$ defined as the concatenation of the controls $u_j \in L^2(jT,(j+1)T;U)$ generates the trajectory $y(\cdot\,;y_0,u)$ satisfying, at time $kT$, $\|y(kT;y_0,u)\|_X \leqslant \alpha^k\|y_0\|_X$, and

$$\|u\|^2_{L^2(0,+\infty;U)} \leqslant \sum_{j=0}^{+\infty} C^2\|y(jT;y_0,u)\|^2_X \leqslant C^2\sum_{j=0}^{+\infty}\alpha^{2j}\|y_0\|^2_X = \frac{C^2}{1-\alpha^2}\|y_0\|^2_X.$$

Let us prove that $\|y(t;y_0,u)\|_X$ decreases exponentially. The argument is standard. Taking $t \geqslant 0$ such that $kT \leqslant t < (k+1)T$ for some $k \in \mathbb{N}$, we have

$$y(t;y_0,u) = S(t-kT)y(kT) + \int_{kT}^t S(t-s)Bu_k(s)\,ds.$$

The semigroup $(S(t))_{t\geqslant 0}$ satisfies $\|S(t)\|_{L(X)} \leqslant M$ for every $t \in [0, T]$ for some $M \geqslant 1$. Therefore

$$\|y(t; y_0, u)\|_X \leqslant M(1 + \|B\|_{L(U,X)}\sqrt{T})C\|y_k\|_X \leqslant \frac{M}{\alpha}(1 + \|B\|_{L(U,X)}\sqrt{T})C\alpha^{k+1}\|y_0\|_X$$

and since $\alpha^{k+1} = e^{(k+1)T((\ln\alpha)/T)} \leqslant e^{((\ln\alpha)/T)t}$ we infer that

$$\|y(t; y_0, u)\|_X \leqslant \frac{M}{\alpha}(1 + \|B\|_{L(U,X)}\sqrt{T})C\|y_0\|_X e^{((\ln\alpha)/T)t} \quad \text{for all } t > 0. \tag{34}$$

We have therefore found a control $u \in L^2(0, +\infty; U)$ such that

$$\int_0^{+\infty} (\|u(t)\|_U^2 + \|y(t; y_0, u)\|_X^2)\, dt < +\infty. \tag{35}$$

Hence, by the classical Riccati theory, see [Zabczyk 1995, Theorem 4.3, page 240], the control system (1) is exponentially stabilizable.

The first equality in (10) concerning the best stabilization rate is obvious by inspecting the above argument, and in particular (34). □

**Remark 32.** To prove Theorems 25 and 26 in Section 3.3, we note that all arguments in the above proof still work (before the final step where Riccati theory is invoked) for an unbounded admissible control operator $B$, because the operators $L_t = \int_0^t S(t-s)Bu(s)\, ds$ are bounded in $X$ and we can write

$$\begin{aligned}\left\|\int_{kT}^t S(t-s)Bu_k(s)\, ds\right\|_X &= \left\|\int_0^{t-kT} S(t-kT-s)Bu_k(s+kT)\, ds\right\|_X \\ &= \|L_{t-kT}u_k(kT+\cdot)\|_X \leqslant C\|u_k\|_{L^2(kT,(k+1)T;U)}\end{aligned}$$

and the rest is unchanged: we find $u$ satisfying (35), i.e., the finite-cost condition (optimizability: see Section 3.3).

In other words, we obtain that, for an admissible control operator $B$, the control system (1) is optimizable if and only if, for every $\alpha \in (0, 1)$ (equivalently, there exists $\alpha \in (0, 1)$), there exist $T > 0$ and $C > 0$ such that, for every $y_0 \in X$, there exists $u \in L^2(0, T; U)$ such that $\|y(T; y_0, u)\|_X \leqslant \alpha\|y_0\|_X$ and $\|u\|_{L^2(0,T;U)} \leqslant C\|y_0\|_X$.

**4.2.3.** *Proofs of Proposition 6 and Theorem 1.* Proposition 6 follows from Lemmas 29 and 30. We note that in Lemma 29 (resp., in Lemma 30), for every $\alpha \in (0, 1)$, the time $T$ in items (ii) and (iv) (resp., items (ii) and (iii)) of Theorem 1 can be taken to be the same. Hence, Theorem 1 follows from Proposition 6 and Lemma 31, except the second equality in (10), which we prove next.

**4.3.** ***Proof of (10).*** To establish the second equality of (10), one way would consist of modifying the statement of Lemma 31 and to observe that, in this lemma, one can moreover choose $T \geqslant 1$. Anyway, we provide hereafter another argument of proof, which is, we believe, of independent interest.

The proof goes in two steps.

*Step 1: We prove that if* $(\alpha, T) \in \mathcal{A}$, *then* $(\alpha^n, nT) \in \mathcal{A}$ *for any* $n \in \mathbb{N}^*$.

Let $(\alpha, T) \in \mathcal{A}$. By (11), the fact that $\mu_\alpha^T < +\infty$ is equivalent to exponential stabilizability (see the two first items of Theorem 1). Using the equivalence of items (ii) and (iv) of the theorem and the fact that for every $\alpha \in (0,1)$ the time $T$ in those items can be taken to be the same, there exists $C \geqslant 0$ such that

$$\|S(T)^*\psi\|_X \leqslant C\|B^*S(T-\cdot)^*\psi\|_{L^2(0,T;U)} + \alpha\|\psi\|_X \quad \text{for all } \psi \in X. \tag{36}$$

For every $n \in \mathbb{N}^*$, we set $\widehat{T}_n = nT$, $\hat{\alpha}_n = \alpha^n$ and $\widehat{C}_n^2 = \sum_{j=0}^{n-1} C^2\alpha^{2j}$. We claim that

$$\|S(\widehat{T}_n)^*\xi\|_X \leqslant \widehat{C}_n\|B^*S(\widehat{T}_n-\cdot)^*\xi\|_{L^2(0,\widehat{T}_n;U)} + \hat{\alpha}_n\|\xi\|_X \quad \text{for all } \xi \in X. \tag{37}$$

Indeed, for an arbitrarily fixed $\xi \in X$, we use (36) with $\psi$ equal, respectively, to $\xi$, $S(T)^*\xi$, $\dots$, $(S(T)^*)^{n-1}\xi$, and we find that

$$\begin{aligned}
\|S(\widehat{T}_n)^*\xi\|_X &= \|S(T)^*(S(T)^*)^{n-1}\xi\|_X \\
&\leqslant C\|B^*S(T-\cdot)^*(S(T)^*)^{n-1}\xi\|_{L^2(0,T;U)} + \alpha\|(S(T)^*)^{n-1}\xi\|_X \\
&\leqslant C\|B^*S(T-\cdot)^*(S(T)^*)^{n-1}\xi\|_{L^2(0,T;U)} \\
&\qquad + C\alpha\|B^*S(T-\cdot)^*(S(T)^*)^{n-2}\xi\|_{L^2(0,T;U)} + \alpha^2\|(S(T)^*)^{n-2}\xi\|_X \\
&\ \ \vdots \\
&\leqslant \sum_{j=0}^{n-1} C\alpha^j\|B^*S(T-\cdot)^*(S(T)^*)^{n-j-1}\xi\|_{L^2(0,T;U)} + \alpha^n\|\xi\|_X.
\end{aligned}$$

By the Cauchy–Schwarz inequality, we have

$$\begin{aligned}
&\sum_{j=0}^{n-1} C\alpha^j\|B^*S(T-\cdot)^*(S(T)^*)^{n-j-1}\xi\|_{L^2(0,T;U)} \\
&\qquad \leqslant \Big(\sum_{j=0}^{n-1} C^2\alpha^{2j}\sum_{j=0}^{n-1}\|B^*S(T-\cdot)^*(S(T)^*)^{n-j-1}\xi\|^2_{L^2(0,T;U)}\Big)^{1/2} \\
&\qquad = \Big(\sum_{j=0}^{n-1} C^2\alpha^{2j}\Big)^{1/2}\|B^*S(nT-\cdot)^*\xi\|_{L^2(0,nT;U)} = \widehat{C}_n\|B^*S(\widehat{T}_n-\cdot)^*\xi\|_{L^2(0,\widehat{T}_n;U)}.
\end{aligned}$$

The previous inequality leads to (37). Hence $(\alpha^n, nT) \in \mathcal{A}$ for every $n \in \mathbb{N}^*$.

*Step 2: We prove the second equality in (10).*

Take an arbitrary $(\alpha_0, T_0) \in \mathcal{A}$. By Step 1, we have $(\alpha_0^n, nT_0) \in \mathcal{A}$. Additionally, for every $\alpha \in (0, \alpha_0)$, there exists $n = n(\alpha) \in \mathbb{N}^*$ such that $\alpha_0^n \leqslant \alpha < \alpha_0^{n-1}$. Hence $\mu_\alpha^{nT_0} \leqslant \mu_{\alpha_0^n}^{nT_0} < +\infty$ and thus $(\alpha, nT_0) \in \mathcal{A}$. Therefore, for every $\alpha \in (0, \alpha_0)$, we have

$$\inf\left\{\frac{\ln\alpha}{T} \;\middle|\; T \in \mathcal{T}(\alpha)\right\} \leqslant \frac{\ln\alpha}{nT_0} \leqslant \frac{\ln\alpha_0^{n-1}}{nT_0} = \frac{n-1}{n}\frac{\ln\alpha_0}{T_0}$$

and hence

$$\limsup_{\alpha\to 0+}\inf\left\{\frac{\ln\alpha}{T} \;\middle|\; T \in \mathcal{T}(\alpha)\right\} \leqslant \frac{\ln\alpha_0}{T_0}. \tag{38}$$

Since $(\alpha_0, T_0)$ was taken arbitrarily in $\mathcal{A}$, we infer from (38) that

$$\limsup_{\alpha\to 0+} \inf\left\{\frac{\ln\alpha}{T}\;\middle|\;(\alpha, T)\in\mathcal{A}\right\} \leqslant \inf\left\{\frac{\ln\alpha}{T}\;\middle|\;(\alpha, T)\in\mathcal{A}\right\}. \tag{39}$$

On the other hand, one can easily check that

$$\inf\left\{\frac{\ln\alpha}{T}\;\middle|\;(\alpha, T)\in\mathcal{A}\right\} \leqslant \liminf_{\alpha\to 0+} \inf\left\{\frac{\ln\alpha}{T}\;\middle|\;T\in\mathcal{T}(\alpha)\right\}. \tag{40}$$

Finally, from (39) and (40), it follows that

$$\inf\left\{\frac{\ln\alpha}{T}\;\middle|\;(\alpha, T)\in\mathcal{A}\right\} = \lim_{\alpha\to 0+} \inf\left\{\frac{\ln\alpha}{T}\;\middle|\;T\in\mathcal{T}(\alpha)\right\},$$

which leads to the second equality of (10).

## Appendix

**A.1.** ***Exact null controllability implies complete stabilizability (proof of Proposition 21).*** Let us prove Proposition 21, i.e., if the control system (1) is exactly null controllable in some time $T > 0$ then it is completely stabilizable.

*Proof of Proposition 21.* We first note that $(A, B)$ is exactly controllable in time $T$ if and only if $(A + \omega\,\mathrm{id}, B)$ is exactly controllable in time $T$ for any $\omega \in \mathbb{R}$. This follows straightforwardly by using the equivalence in terms of the observability inequality and the fact that $S_{A+\omega\,\mathrm{id}}(t) = e^{\omega t} S_A(t)$ (with obvious notation).

Now, let $\omega > 0$ be arbitrary. Since $(A + \omega\,\mathrm{id}, B)$ is exactly controllable in time $T$, there exists $K_\omega \in L(X, U)$ such that $A+\omega\,\mathrm{id}+BK_\omega$ generates the exponentially stable semigroup $S_{A+\omega\,\mathrm{id}+BK_\omega}(t) = e^{\omega t} S_{A+BK_\omega}(t)$, and thus $\|S_{A+BK_\omega}(t)\|_{L(X)} \leqslant Me^{-\omega t}$ for some $M \geqslant 1$. The result follows. □

Surprisingly, we have not found this result explicitly stated in the existing literature (except, in a rather indirect way, in [Curtain and Zwart 1995, Exercise 6.18, page 312], as kindly indicated to us by Guillaume Olive). What can usually be found is that exact null controllability in some time $T$ implies exponential stabilizability (see, e.g., [Zabczyk 1995]) and that, when $(S(t))_{t\geqslant 0}$ is a group, exact null controllability in some time $T$ implies complete stabilizability (see [Slemrod 1974]) and the converse is true (see [Zabczyk 1995]).

**A.2.** ***The case of a group (proof of Proposition 22, using Theorem 1).*** Let us prove Proposition 22, i.e., when $(S(t))_{t\geqslant 0}$ is a group, we have equivalence of exact controllability in some time $T$, exact null controllability in some time $T$, complete stabilizability.

*Proof of Proposition 22.*. Since $(S(t))_{t\geqslant 0}$ is a group, we have (i) ⇔ (ii). By Proposition 21, we have (ii) ⇒ (iii).

Let us now prove that (iii) ⇒ (ii), by using Theorem 1. Since $(S(t))_{t\geqslant 0}$ is a group, there exists $M \geqslant 1$ and $\omega \geqslant 0$ such that $\|S(t)\|_{L(X)} \leqslant Me^{\omega|t|}$ for every $t \in \mathbb{R}$; hence

$$\|\psi\|_X \leqslant \|S(-T)\|_{L(X)}\|S(T)^*\psi\|_X \leqslant Me^{\omega T}\|S(T)^*\psi\|_X \quad \text{for all } \psi\in X, \text{ for all } T > 0. \tag{41}$$

We set $\alpha_n = 1/n$, for every $n \in \mathbb{N}^*$. Since the control system (1) is completely stabilizable, by using the equivalence of items (i) and (ii) in Theorem 1, for every $n \in \mathbb{N}^*$ there exists $T_n > 0$ such that $\mu_{\alpha_n}^{T_n} < +\infty$ and such that, setting $\beta_n = (\ln n)/T_n$, we have $\beta_n \to +\infty$ as $n \to +\infty$. Hence

$$\lim_{n\to+\infty} \frac{Me^{\omega T_n}}{n} = M \lim_{n\to+\infty} \frac{e^{\omega(\ln n)/\beta_n}}{n} = M \lim_{n\to+\infty} \frac{(e^{\ln n})^{\omega/\beta_n}}{n} = M \lim_{n\to+\infty} n^{\omega/\beta_n - 1} = 0$$

and therefore there exists $N \in \mathbb{N}^*$ such that

$$\frac{Me^{\omega T_N}}{\alpha_N} \leqslant \frac{1}{2}. \tag{42}$$

Now, for every $n \in \mathbb{N}^*$, since $\mu_{\alpha_n}^{T_n} < +\infty$, using the equivalence of items (ii) and (iv) of Theorem 1 and the fact that for every $\alpha \in (0,1)$, the time $T$ in those items can be taken to be the same, there exists $C_n > 0$ such that

$$\|S(T_n)^*\psi\|_X \leqslant C_n \|B^*S(T_n - \cdot)^*\psi\|_{L^2(0,T_n;U)} + \frac{1}{n}\|\psi\|_X \quad \text{for all } \psi \in X. \tag{43}$$

We infer from (41) and (43) that

$$\|S(T_n)^*\psi\|_X \leqslant C_n \|B^*S(T_n - \cdot)^*\psi\|_{L^2(0,T_n;U)} + \frac{Me^{\omega T_n}}{n}\|S(T_n)^*\psi\|_X \quad \text{for all } \psi \in X,$$

and then, taking $n = N$ and using (42), we find that

$$\|S(T_N)^*\psi\|_X \leqslant 2C_N \|B^*S(T_N - \cdot)^*\psi\|_{L^2(0,T_N;U)} \quad \text{for all } \psi \in X,$$

which implies that the control system (1) is exactly null controllable in time $T_N$. □

## Acknowledgement

This work was partially funded by the Natural Science Foundation of China 11871166, 11631004.

## References

[Alabau-Boussouira 2002] F. Alabau-Boussouira, "Indirect boundary stabilization of weakly coupled hyperbolic systems", *SIAM J. Control Optim.* **41**:2 (2002), 511–541. MR Zbl

[Alabau-Boussouira 2003] F. Alabau-Boussouira, "A two-level energy method for indirect boundary observability and controllability of weakly coupled hyperbolic systems", *SIAM J. Control Optim.* **42**:3 (2003), 871–906. MR Zbl

[Alabau-Boussouira and Ammari 2011] F. Alabau-Boussouira and K. Ammari, "Sharp energy estimates for nonlinearly locally damped PDEs via observability for the associated undamped system", *J. Funct. Anal.* **260**:8 (2011), 2424–2450. MR Zbl

[Alabau-Boussouira et al. 2017] F. Alabau-Boussouira, Y. Privat, and E. Trélat, "Nonlinear damped partial differential equations and their uniform discretizations", *J. Funct. Anal.* **273**:1 (2017), 352–403. MR Zbl

[Badra and Takahashi 2014] M. Badra and T. Takahashi, "On the Fattorini criterion for approximate controllability and stabilizability of parabolic systems", *ESAIM Control Optim. Calc. Var.* **20**:3 (2014), 924–956. MR Zbl

[Banks and Ito 1997] H. T. Banks and K. Ito, "Approximation in LQR problems for infinite-dimensional systems with unbounded input operators", *J. Math. Systems Estim. Control* **7**:1 (1997), art. id. 62459. MR Zbl

[Bardos et al. 1992] C. Bardos, G. Lebeau, and J. Rauch, "Sharp sufficient conditions for the observation, control, and stabilization of waves from the boundary", *SIAM J. Control Optim.* **30**:5 (1992), 1024–1065. MR Zbl

[Borichev and Tomilov 2010] A. Borichev and Y. Tomilov, "Optimal polynomial decay of functions and operator semigroups", *Math. Ann.* **347**:2 (2010), 455–478. MR Zbl

[Boyer 2013] F. Boyer, "On the penalised HUM approach and its applications to the numerical approximation of null-controls for parabolic problems", pp. 15–58 in *CANUM 2012, 41e Congrès National d'Analyse Numérique*, edited by L. Chupin and A. Münch, ESAIM Proc. **41**, EDP Sci., Les Ulis, 2013. MR Zbl

[Boyer and Le Rousseau 2014] F. Boyer and J. Le Rousseau, "Carleman estimates for semi-discrete parabolic operators and application to the controllability of semi-linear semi-discrete parabolic equations", *Ann. Inst. H. Poincaré Anal. Non Linéaire* **31**:5 (2014), 1035–1078. MR Zbl

[Boyer et al. 2019] F. Boyer, V. Hernández-Santamaría, and L. de Teresa, "Insensitizing controls for a semilinear parabolic equation: a numerical approach", *Math. Control Relat. Fields* **9**:1 (2019), 117–158. MR

[Coron 2007] J.-M. Coron, *Control and nonlinearity*, Mathematical Surveys and Monographs **136**, American Mathematical Society, Providence, RI, 2007. MR Zbl

[Coron and Lü 2014] J.-M. Coron and Q. Lü, "Local rapid stabilization for a Korteweg-de Vries equation with a Neumann boundary control on the right", *J. Math. Pures Appl.* (9) **102**:6 (2014), 1080–1120. MR Zbl

[Coron and Trélat 2004] J.-M. Coron and E. Trélat, "Global steady-state controllability of one-dimensional semilinear heat equations", *SIAM J. Control Optim.* **43**:2 (2004), 549–569. MR Zbl

[Curtain and Zwart 1995] R. F. Curtain and H. Zwart, *An introduction to infinite-dimensional linear systems theory*, Texts in Applied Mathematics **21**, Springer, 1995. MR Zbl

[Ekeland and Témam 1999] I. Ekeland and R. Témam, *Convex analysis and variational problems*, Classics in Applied Mathematics **28**, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1999. MR Zbl

[Engel and Nagel 2000] K.-J. Engel and R. Nagel, *One-parameter semigroups for linear evolution equations*, Graduate Texts in Mathematics **194**, Springer, 2000. MR Zbl

[Ervedoza and Zuazua 2009] S. Ervedoza and E. Zuazua, "Uniformly exponentially stable approximations for a class of damped systems", *J. Math. Pures Appl.* (9) **91**:1 (2009), 20–48. MR Zbl

[Fenchel 1949] W. Fenchel, "On conjugate convex functions", *Canadian J. Math.* **1** (1949), 73–77. MR Zbl

[Flandoli et al. 1988] F. Flandoli, I. Lasiecka, and R. Triggiani, "Algebraic Riccati equations with nonsmoothing observation arising in hyperbolic and Euler–Bernoulli boundary control problems", *Ann. Mat. Pura Appl.* (4) **153** (1988), 307–382. MR Zbl

[Glowinski et al. 2008] R. Glowinski, J.-L. Lions, and J. He, *Exact and approximate controllability for distributed parameter systems: a numerical approach*, Encyclopedia of Mathematics and its Applications **117**, Cambridge University Press, 2008. MR Zbl

[Jacob and Schnaubelt 2007] B. Jacob and R. Schnaubelt, "Observability of polynomially stable systems", *Systems Control Lett.* **56**:4 (2007), 277–284. MR Zbl

[Jacob and Zwart 2009] B. Jacob and H. Zwart, "On the Hautus test for exponentially stable $C_0$-groups", *SIAM J. Control Optim.* **48**:3 (2009), 1275–1288. MR Zbl

[Komornik 1997] V. Komornik, "Rapid boundary stabilization of linear distributed systems", *SIAM J. Control Optim.* **35**:5 (1997), 1591–1613. MR Zbl

[Labbé and Trélat 2006] S. Labbé and E. Trélat, "Uniform controllability of semidiscrete approximations of parabolic control systems", *Systems Control Lett.* **55**:7 (2006), 597–609. MR Zbl

[Lasiecka and Triggiani 2000a] I. Lasiecka and R. Triggiani, *Control theory for partial differential equations: continuous and approximation theories, I: Abstract parabolic systems*, Encyclopedia of Mathematics and its Applications **74**, Cambridge University Press, 2000. MR Zbl

[Lasiecka and Triggiani 2000b] I. Lasiecka and R. Triggiani, *Control theory for partial differential equations: continuous and approximation theories, II: Abstract hyperbolic-like systems over a finite time horizon*, Encyclopedia of Mathematics and its Applications **75**, Cambridge University Press, 2000. MR Zbl

[Le Rousseau et al. 2017] J. Le Rousseau, G. Lebeau, P. Terpolilli, and E. Trélat, "Geometric control condition for the wave equation with a time-dependent observation domain", *Anal. PDE* **10**:4 (2017), 983–1015. MR Zbl

[Lions 1988] J.-L. Lions, *Contrôlabilité exacte, perturbations et stabilisation de systèmes distribués, I: Contrôlabilité exacte*, Recherches en Mathématiques Appliquées **8**, Masson, 1988. MR Zbl

[Lions 1992] J.-L. Lions, "Remarks on approximate controllability", *J. Anal. Math.* **59**:1 (1992), 103–116. MR Zbl

[Liu 1997] K. Liu, "Locally distributed control and damping for the conservative systems", *SIAM J. Control Optim.* **35**:5 (1997), 1574–1590. MR Zbl

[Liu and Zheng 1999] Z. Liu and S. Zheng, *Semigroups associated with dissipative systems*, Chapman & Hall/CRC Research Notes in Mathematics **398**, Chapman & Hall/CRC, Boca Raton, FL, 1999. MR Zbl

[Pazy 1983] A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, Applied Mathematical Sciences **44**, Springer, 1983. MR Zbl

[Phung et al. 2017] K. D. Phung, G. Wang, and Y. Xu, "Impulse output rapid stabilization for heat equations", *J. Differential Equations* **263**:8 (2017), 5012–5041. MR Zbl

[Pritchard and Zabczyk 1981] A. J. Pritchard and J. Zabczyk, "Stability and stabilizability of infinite-dimensional systems", *SIAM Rev.* **23**:1 (1981), 25–52. MR Zbl

[Privat and Trélat 2015] Y. Privat and E. Trélat, "Optimal design of sensors for a damped wave equation", *Discrete Contin. Dyn. Syst.* **2015**:Dynamical systems, differential equations and applications. 10th AIMS Conference. Suppl. (2015), 936–944. MR Zbl

[Privat et al. 2015] Y. Privat, E. Trélat, and E. Zuazua, "Optimal shape and location of sensors for parabolic equations with random initial data", *Arch. Ration. Mech. Anal.* **216**:3 (2015), 921–981. MR Zbl

[Privat et al. 2016] Y. Privat, E. Trélat, and E. Zuazua, "Optimal observability of the multi-dimensional wave and Schrödinger equations in quantum ergodic domains", *J. Eur. Math. Soc.* (*JEMS*) **18**:5 (2016), 1043–1111. MR Zbl

[Rockafellar 1967] R. T. Rockafellar, "Duality and stability in extremum problems involving convex functions", *Pacific J. Math.* **21** (1967), 167–187. MR Zbl

[Russell 1978] D. L. Russell, "Controllability and stabilizability theory for linear partial differential equations: recent progress and open questions", *SIAM Rev.* **20**:4 (1978), 639–739. MR Zbl

[Slemrod 1974] M. Slemrod, "A note on complete controllability and stabilizability for linear control systems in Hilbert space", *SIAM J. Control* **12** (1974), 500–508. MR Zbl

[Staffans 2005] O. Staffans, *Well-posed linear systems*, Encyclopedia of Mathematics and its Applications **103**, Cambridge University Press, 2005. MR Zbl

[Trélat 2018] E. Trélat, "Stabilization of semilinear PDEs, and uniform decay under discretization", pp. 31–76 in *Evolution equations: long time behavior and control*, edited by K. Ammari and S. Gerbi, London Math. Soc. Lecture Note Ser. **439**, Cambridge Univ. Press, 2018. MR Zbl

[Triggiani 1975] R. Triggiani, "On the stabilizability problem in Banach space", *J. Math. Anal. Appl.* **52**:3 (1975), 383–403. MR Zbl

[Tucsnak and Weiss 2009] M. Tucsnak and G. Weiss, *Observation and control for operator semigroups*, Birkhäuser, Basel, 2009. MR Zbl

[Urquiza 2005] J. M. Urquiza, "Rapid exponential feedback stabilization with unbounded control operators", *SIAM J. Control Optim.* **43**:6 (2005), 2233–2244. MR Zbl

[Weiss and Rebarber 2000] G. Weiss and R. Rebarber, "Optimizability and estimatability for infinite-dimensional linear systems", *SIAM J. Control Optim.* **39**:4 (2000), 1204–1232. MR Zbl

[Weiss and Zwart 1998] G. Weiss and H. Zwart, "An example in linear quadratic optimal control", *Systems Control Lett.* **33**:5 (1998), 339–349. MR Zbl

[Zabczyk 1995] J. Zabczyk, *Mathematical control theory: an introduction*, Birkhäuser, Boston, MA, 1995.

[Zuazua 2004] E. Zuazua, "Optimal and approximate control of finite-difference approximation schemes for the 1D wave equation", *Rend. Mat. Appl.* (7) **24**:2 (2004), 201–237. MR Zbl

[Zuazua 2005] E. Zuazua, "Propagation, observation, and control of waves approximated by finite difference methods", *SIAM Rev.* **47**:2 (2005), 197–243. MR Zbl

[Zwart 1996] H. J. Zwart, "Linear quadratic optimal control for abstract linear systems", pp. 175–182 in *Modelling and optimization of distributed parameter systems* (Warsaw, 1995), edited by K. Malanowski et al., Chapman & Hall, New York, 1996. MR Zbl

Received 30 Apr 2019. Revised 19 Jun 2019. Accepted 22 Jul 2019.

EMMANUEL TRÉLAT: emmanuel.trelat@sorbonne-universite.fr
*Sorbonne Université, CNRS, Université de Paris, Inria, Laboratoire Jacques-Louis Lions, Paris, France*

GENGSHENG WANG: wanggs@yeah.net
*Center for Applied Mathematics, Tianjin University, Tianjin, China*

YASHAN XU: yashanxu@fudan.edu.cn
*School of Mathematical Sciences, Fudan University, KLMNS, Shanghai, China*