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Morse-Sard type results in sub-Riemannian geometry

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Abstract. Let (M, Δ, g) be a sub-Riemannian manifold and $x_0 \in M$. Assuming that Chow's condition holds and that M endowed with the sub-Riemannian distance is complete, we prove that there exists a dense subset N_1 of M such that for every point x of N_1, there is a unique minimizing path steering x_0 to x, this trajectory admitting a normal extremal lift. If the distribution Δ is everywhere of corank one, we prove the existence of a subset N_2 of M of full Lebesgue measure such that for every point x of N_2, there exists a minimizing path steering x_0 to x which admits a normal extremal lift, is nonsingular, and the point x is not conjugate to x_0. In particular, the image of the sub-Riemannian exponential mapping is dense in M, and in the case of corank one is of full Lebesgue measure in M.

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1. Introduction and main results

The following general definition of a sub-Riemannian distance (also called Carnot-Carathéodory distance) is due to [3]. Let M be a connected smooth n-dimensional manifold, m an integer such that $1 \leq m \leq n$, and $f_1, \dots, f_m$ be smooth vector fields on the manifold M. For all $x \in M$ and $v \in T_xM$, set

$$g(x, v) := \inf \left\{ \sum_{i=1}^{m} u_i^2 \mid u_1, \dots, u_m \in \mathbb{R}, \sum_{i=1}^{m} u_i f_i(x) = v \right\}.$$

Then $g(x, \cdot)$ is a positive definite quadratic form on the subspace of T_xM spanned by $f_1(x), \dots, f_m(x)$. Outside this subspace we set $g(x, v) = +\infty$. The form g is called *sub-Riemannian metric* associated to the m-tuple of vector fields $(f_1, \dots, f_m)$. Let $\mathcal{AC}([0, 1], M)$ denote the set of absolutely continuous paths in M defined on $[0, 1]$, we define the *length* of $\gamma \in \mathcal{AC}([0, 1], M)$ as

$$l(\gamma) := \int_0^1 \sqrt{g(\gamma(t), \dot{\gamma}(t))}\, dt.$$

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We say that *Chow's condition* holds if the Lie algebra spanned by the vector fields $f_1, \dots, f_m$, is equal to the tangent space $T_x M$ at every point x of M. It is well-known that under this condition any two points of M can be joined by an absolutely continuous path with finite length.
The *sub-Riemannian distance* associated to the m-tuple of vector fields $(f_1, \dots, f_m)$, between two points x_0, x_1 in M, is defined as

$$d_{SR}(x_0, x_1) := \inf \{l(\gamma) \mid \gamma \in \mathcal{AC}([0, 1], M),\ \gamma(0) = x_0, \gamma(1) = x_1\}.$$

The *sub-Riemannian sphere* $S_{SR}(x_0, r)$ (resp. the *sub-Riemannian ball* $B_{SR}(x_0, r)$) centered at x_0 with radius r as the set of points $x \in M$ such that $d_{SR}(x_0, x) = r$ (resp. $d_{SR}(x_0, x) < r$). A path $\gamma \in \mathcal{AC}([0, 1], M)$ is said to be *minimizing* if it realizes the sub-Riemannian distance between its extremities.

Remark 1. If Chow's condition holds, then:

- the topology defined by the sub-Riemannian distance d_{SR} coincides with the original topology of M,
- sufficiently near points can be joined by a minimizing path,
- if the manifold M is moreover a complete metric space for the sub-Riemannian distance d_{SR}, then any two points can be joined by a minimizing path.

Consider on the other part the differential system on the tangent bundle TM of M

$$\dot{x}(t) = \sum_{i=1}^{m} u_i(t) f_i(x(t)) \quad \text{a.e. on } [0, 1], \tag{1}$$

where the function $u(\cdot) = (u_1(\cdot), \dots, u_m(\cdot))$, called *control function*, belongs to $L^2([0, 1], \mathbb{R}^m)$. Let $x_0 \in \mathbb{R}^n$, and let $\mathcal{U}$ denote the (open) subset of $L^2([0, 1], \mathbb{R}^m)$ such that the solution of (1) starting at x_0 and associated to a control $u(\cdot) \in \mathcal{U}$ is well-defined on [0, 1]. The mapping

$$\begin{aligned} E_{x_0} : \quad \mathcal{U} &\longrightarrow \mathbb{R}^n \\ u(\cdot) &\longmapsto x(1), \end{aligned}$$

which to a control $u(\cdot)$ associates the extremity $x(1)$ of the corresponding solution $x(\cdot)$ of (1) starting at x_0, is called *end-point mapping* at the point x_0; it is a smooth mapping. The trajectory $x(\cdot)$ is said to be *singular* if the associated control $u(\cdot)$ is a singular point of the end-point mapping (*i.e.* if the Fréchet derivative of E_{x_0} at $u(\cdot)$ is not onto); it is *minimizing* if it realizes the sub-Riemannian distance between its extremities.

Remark 2. A sub-Riemannian manifold is often defined as a triple (M, Δ, g), where M is a n-dimensional manifold, Δ is a distribution of rank $m \leq n$, and g is a Riemannian metric on Δ. If the vector fields $(f_1, \dots, f_m)$ are everywhere linearly independent, then controlled paths solutions of (1) coincide with absolutely continuous paths tangent to the distribution Δ, where

$$\Delta(x) = \text{Span}\,\{f_1(x), \ldots, f_m(x)\},$$

for all $x \in M$. These paths are said Δ-*horizontal*.

On the other part, for $x_0 \in M$, let $\Omega(x_0, \Delta)$ be the set of Δ-horizontal paths starting from x_0 whose derivative is square integrable for the metric g (and hence for any Riemannian metric on Δ). Endowed with the H^1-topology, $\Omega(x_0, \Delta)$ inherits of a Hilbert manifold structure, see [4]. For $(x_0, x_1) \in M \times M$, let $\Omega(x_0, x_1, \Delta)$ be the subset of paths $x(\cdot) \in \Omega(x_0, \Delta)$ such that $x(1) = x_1$. The set $\Omega(x_0, x_1, \Delta)$ is a submanifold of $\Omega(x_0, \Delta)$ in a neighborhood of any nonsingular path, but might fail to be a (global) manifold due to the possible existence of singular paths.

Let x_0 and x_1 in M. The sub-Riemannian problem of determining a minimizing trajectory steering x_0 to x_1 can be easily seen (up to reparametrization, and using the Cauchy-Schwarz inequality) to be equivalent to the *optimal control problem* of finding a control $u(\cdot) \in \mathcal{U}$ such that the solution of the control system (1) steers x_0 to x_1 in time 1, and minimizes the *cost function*

$$C(u(\cdot)) := \int_0^1 \sum_{i=1}^m u_i(t)^2\, dt. \tag{2}$$

If a control $u(\cdot)$ associated to a trajectory $x(\cdot)$ such that $x(0) = x_0$ is optimal, then there exists a nontrivial *Lagrange multiplier* $(\psi, \psi^0) \in T^*_{x(1)}M \times \mathbb{R}$ such that

$$\psi.dE_{x_0}(u(\cdot)) = -\psi^0 dC(u(\cdot)), \tag{3}$$

where $dE_{x_0}(u(\cdot))$ (resp. $dC(u(\cdot))$) denotes the Fréchet derivative of E_{x_0} (resp. C) at the point $u(\cdot)$. The well-known Pontryagin maximum principle (see [8]) parametrizes this condition and asserts that the trajectory $x(\cdot)$ is the projection of an *extremal*, that is a quadruple $(x(\cdot), p(\cdot), p^0, u(\cdot))$, solution of the constrained Hamiltonian system

$$\dot{x}(t) = \frac{\partial H}{\partial p}(x(t), p(t), p^0, u(t)),\ \ \dot{p}(t) = -\frac{\partial H}{\partial x}(x(t), p(t), p^0, u(t)),$$
$$\frac{\partial H}{\partial u}(x(t), p(t), p^0, u(t)) = 0,$$

almost everywhere on [0, 1], where

$$H(x, p, p^0, u) := \langle p, \sum_{i=1}^m u_i f_i(x)\rangle + p^0 \sum_{i=1}^m u_i^2$$

is the *Hamiltonian* of the optimal control problem, $p(\cdot)$ (called *adjoint vector*) is an absolutely continuous mapping on [0, 1] such that $p(t) \in T^*_{x(t)}M$, and p^0 is a real nonpositive constant. Moreover there holds

$$(p(1), p^0) = (\psi, \psi^0), \tag{4}$$

up to a multiplying scalar. If $p^0 < 0$ then the extremal is said to be *normal*, and in this case it is normalized to $p^0 = -1/2$. If $p^0 = 0$ then the extremal is said to be *abnormal*.

Remark 3. Any singular trajectory is the projection of an abnormal extremal, and conversely.

Furthermore, a singular trajectory is said to be *strict* (or *strictly singular*) if it does not admit a normal extremal lift; equivalently in that case we say that its abnormal extremal lift is *strictly abnormal*.

The *sub-Riemannian wave-front* $W_{SR}(x_0, r)$ centered at x_0 and with radius r is defined as the set of end-points $x(1)$, where $(x(\cdot), p(\cdot), p^0, u(\cdot))$ is an extremal such that $x(0) = x_0$ and $C(u(\cdot)) = r^2$. Under Chow's condition, it is clear from Remark 1 that $S_{SR}(x_0, r)$ is a subset of $W_{SR}(x_0, r)$.

Using the previous normalization, controls associated to normal extremals can be computed as

$$u_i(t) = \langle p(t), f_i(x(t))\rangle, \ i = 1, \dots, m.$$

Hence normal extremals are solutions of the Hamiltonian system

$$\dot{x}(t) = \frac{\partial H_1}{\partial p}(x(t), p(t)), \ \dot{p}(t) = -\frac{\partial H_1}{\partial x}(x(t), p(t)), \tag{5}$$

where

$$H_1(x, p) = \frac{1}{2}\sum_{i=1}^{m}\langle p, f_i(x)\rangle^2.$$

Notice that $H_1(x(t), p(t))$ is constant along each normal extremal and that the length of the path $x(\cdot)$ equals $(2\, H_1(x(0), p(0)))^{1/2}$. Actually, given a point x_0 of M, the differential system (5) has a well-defined smooth solution on [0, 1] such that $x(0) = x_0$ and $p(0) = p_0$, for $p_0 \in U$, where U is a connected open subset of $T^*_{x_0}M$. In what follows, the point x_0 is fixed.

Definition 1. *The smooth mapping*

$$\begin{aligned} exp_{x_0} : \ U &\longrightarrow M \\ p_0 &\longmapsto x(1) \end{aligned}$$

where $(x(\cdot), p(\cdot))$ *is the solution of the system (5) such that* $x(0) = x_0$ *and* $p(0) = p_0$*, is called exponential mapping at the point* x_0.

The exponential mapping parametrizes normal extremals. Notice that every minimizing trajectory steering x_0 to a point of $M \setminus \exp_{x_0}(U)$ is necessarily strictly singular.

Remark 4. Using notations of Definition 1, it is easy to see by reparametrization that $x(t) = \exp_{x_0}(tp_0)$, for all $t \in [0, 1]$.

Remark 5. For all $p_0 \in U$ such that $H_1(x_0, p_0) = \frac{r^2}{2}$, one has $\exp_{x_0}(p_0) \in W_{SR}(x_0, r)$. The space of normal extremals with length r is parametrized by the manifold $U_r = U \cap H_1^{-1}(\frac{r^2}{2})$, which is diffeomorphic to $\mathcal{S}^{m-1} \times \mathbb{R}^{n-m}$ if the distribution Δ has rank m at x_0.

A point $x \in \exp_{x_0}(U)$ is said *conjugate* to x_0 if it is a critical value of the mapping $\exp_{x_0}$, *i.e.* if there exists $p_0 \in U$ such that $x = \exp_{x_0}(p_0)$ and the differential $d\exp_{x_0}(p_0)$ is not onto. The *conjugate locus*, denoted by $\mathcal{C}(x_0)$, is defined as the set of all points conjugate to x_0.

Remark 6. By Sard Theorem applied to the mapping $\exp_{x_0}$, it is clear that the conjugate locus $\mathcal{C}(x_0)$ has Lebesgue measure zero in M.

Remark 7. Let $x \in \exp_{x_0}(U)$, and $p_0 \in U$ such that $x = \exp_{x_0}(p_0)$. We denote by $(x(\cdot, p_0), p(\cdot, p_0), -\frac{1}{2}, u(\cdot, p_0))$ the associated normal extremal. Then we have

$$\exp_{x_0}(p_0) = E_{x_0}(u(\cdot, p_0)).$$

Therefore if x is not conjugate to x_0 then the control $u(\cdot, p_0)$ is nonsingular. In particular, the set of endpoints of nonstrictly singular trajectories starting from x_0 has Lebesgue measure zero in M.

Remark 8. With notations of the previous remark, if x is not conjugate to x_0 then the path $x(\cdot) := x(\cdot, p_0)$ associated to the control $u(\cdot) := u(\cdot, p_0)$ admits a unique normal extremal lift. Indeed if it had two distinct normal extremals lifts $(x(\cdot), p_1(\cdot), -\frac{1}{2}, u(\cdot))$ and $(x(\cdot), p_2(\cdot), -\frac{1}{2}, u(\cdot))$, then the extremal $(x(\cdot), p_1(\cdot) - p_2(\cdot), 0, u(\cdot))$ would be an abnormal extremal lift of the path $x(\cdot)$, which is a contradiction since $u(\cdot)$ is nonsingular.

In the present paper we prove the two following theorems.

Theorem 1. *Suppose Chow's condition holds, and that the manifold M is complete for the sub-Riemannian distance d_{SR}. There exists a dense subset N_1 of M such that, for every point $x \in N_1$, there is a unique minimizing path joining x_0 to x; moreover this trajectory admits a normal extremal lift. In particular the image $\exp_{x_0}(U)$ of the exponential mapping is dense in M.*

For all $x \in M$, let $\Delta(x) := \text{Span}\,\{f_1(x), \cdots, f_m(x)\}$, and let μ denote the Lebesgue measure on M. Regarding the previous result, one can wonder whether almost every point of M belongs to $\exp_{x_0}(U)$. The following result gives a positive answer in the case of a corank-one distribution.

Theorem 2. *Suppose Chow's condition holds, and that the manifold M is complete for the sub-Riemannian distance d_{SR}. If the distribution Δ is everywhere of corank one, then there exists a subset N_2 of M of full Lebesgue measure such that, for every point $x \in N_2$, there exists a minimizing path joining x_0 to x and having a normal extremal lift. Moreover this trajectory is nonsingular, and x is not conjugate to x_0. In particular, the set $exp_{x_0}(U)$ is of full measure in M, i.e. $\mu(M \setminus exp_{x_0}(U)) = 0$.*

The next two sections are devoted to the proof of the latter results. In a last section we discuss some consequences and open problems.

2. Proof of Theorem 1

2.1. The proximal sub-differential

Let M be a smooth manifold of dimension n and Ω be an open subset of M. Let $f : \Omega \to \mathbb{R}$ be a continuous function on Ω; we call *proximal sub-differential* of the function f at the point $x \in \Omega$ the subset of T_x^*M defined by

$$\partial_P f(x) := \left\{ d\phi(x) \mid \phi \in C^\infty(M) \text{ and } f - \phi \text{ attains a local minimum at } x \right\}.$$

Note that since every local C^∞ function can be extended to a C^∞ function on M, the proximal sub-differential of f at x depends only on the local behavior of the function f near x. In addition, remark that $\partial_P f(x)$ is a convex subset of T_x^*M which may be empty; for instance the proximal sub-differential of the real function $t \mapsto -|t|$ at $t = 0$ is empty.

Remark 9. Notice that when $M = \mathbb{R}^n$, a vector ζ belongs to the proximal sub-differential of f at a point x if and only if there exists σ and $\delta > 0$ such that

$$f(y) - f(x) + \sigma \|y - x\|^2 \geq \langle \zeta, y - x \rangle, \quad \forall y \in x + \delta B.$$

This is the usual definition of proximal sub-differentials in Hilbert spaces; we refer the reader to [6] for further details on that subject.

In fact, an immediate application of the smooth variational principle of Borwein-Preiss (see [5]) implies the following result.

Theorem 3. *The proximal sub-differential of a continuous function $f : \Omega \to \mathbb{R}$ is nonempty on a dense subset of Ω.*

The proximal sub-differential of f defines a multivalued mapping from Ω into the cotangent bundle T^*M. It is said to be locally bounded on Ω if for each $x \in \Omega$ there exists a neighborhood $\mathcal{V}$ of x such that $\partial_P f(\mathcal{V})$ is relatively compact in T^*M. The following result is standard.

Proposition 1. *The function f is Lipschitz continuous on Ω if and only if the proximal sub-differentials of f are locally bounded on Ω.*

Remark 10. Notice that the Fréchet (or viscosity) sub-differential of f at x, defined by

$$D^- f(x) := \left\{ d\phi(x) \mid \phi \in C^1(M) \text{ and } f - \phi \text{ attains a local minimum at } x \right\},$$

is larger than the proximal sub-differential, but in fact both notions coincide locally; we refer the reader to [6, Prop. 4.5 p. 138, Prop. 4.12 p. 142] for a precise statement.

To conclude this preliminary section, we remark that there exists a complete calculus of proximal sub-differentials, one that extends all the theorems of the usual smooth calculus, see [6].

2.2. Application to the proof of Theorem 1

In what follows we denote $e(\cdot) := d_{SR}(x_0, \cdot)^2$.

Proposition 2. *Let $x \in M$ such that $\partial_P e(x) \neq \emptyset$. Then there exists a unique minimizing path $x(\cdot)$ joining x_0 to x. Moreover for every $\zeta \in \partial_P e(x)$, the path $x(\cdot)$ admits a normal extremal lift $(x(\cdot), p(\cdot), -\frac{1}{2}, u(\cdot))$ such that $p(1) = \frac{1}{2}\zeta$.*

Proof. We adopt the following notation: for every control $u(\cdot) \in \mathcal{U}$, we denote by $x_u(\cdot)$ the trajectory solution of (1) associated to the control $u(\cdot)$ and such that $x_u(0) = x_0$. Let $x \in M$ and $\zeta \in \partial_P e(x)$. We first prove that every minimizing path steering x_0 to x admits a normal extremal lift such that $p(1) = \frac{1}{2}\zeta$. Let $u(\cdot) \in \mathcal{U}$ be an optimal control such that the associated trajectory $x_u(\cdot)$ joins x_0 to x; there holds

$$e(x) = \int_0^1 \sum_{i=1}^m u_i(t)^2 dt.$$

On the other hand, since $\zeta \in \partial_P e(x)$, there exists a function ϕ of class C^∞ with $d\phi(x) = \zeta$ and such that $e - \phi$ attains a local minimum at x. Thus there exists a neighborhood $\mathcal{V}$ of $u(\cdot)$, contained in $\mathcal{U}$, such that

$$e(x) \leq e(x_v(1)) - \phi(x_v(1)) + \phi(x),$$

for every control $v(\cdot) \in \mathcal{V}$. Moreover it can be easily seen by definition of the distance function, that

$$e(x_v(1)) \leq \int_0^1 \sum_{i=1}^m v_i(t)^2 dt.$$

Therefore we obtain

$$e(x) \leq \int_0^1 \sum_{i=1}^m v_i(t)^2 dt - \phi(x_v(1)) + \phi(x),$$

for every control $v(\cdot) \in \mathcal{V}$. In particular, this means that $u(\cdot)$ is a solution of the minimization problem

$$\min_{v \in \mathcal{V}} \left(\int_0^1 \sum_{i=1}^m v_i(t)^2 dt - \phi(x_v(1)) + \phi(x) \right).$$

Hence $u(\cdot)$ is a critical point of the function

$$v(\cdot) \in \mathcal{V} \mapsto C(v(\cdot)) - \phi(E_{x_0}(v(\cdot))) + \phi(x),$$

and thus

$$dC(u(\cdot)) - \zeta . dE_{x_0}(u(\cdot)) = 0.$$

This leads to the existence of a normal extremal lift $(x_u(\cdot), p_u(\cdot), -\frac{1}{2}, u(\cdot))$ such that $(x_u(1), p_u(1)) = (x, \frac{1}{2}\zeta)$. In particular, uniqueness of a minimizing path joining x_0 to x follows. □

Th. 1 is a straightforward consequence of Prop. 2 together with Th. 3.

3. Proof of Theorem 2

3.1. The limiting sub-differential

Let M be a smooth manifold of dimension n and Ω be an open subset of M. Let $f : \Omega \to \mathbb{R}$ be a continuous function on Ω; we call *limiting sub-differential* of the function f at the point $x \in \Omega$ the subset of T_x^*M defined by

$$\partial_L f(x) := \{\lim \zeta_n \mid \zeta_n \in \partial_P f(x_n), x_n \to x\}.$$

As the proximal sub-differential, the limiting sub-differential of f at x depends only on the local behavior of f near x. Moreover by construction, $\partial_L f(x)$ is a closed subset of T_x^*M which contains $\partial_P f(x)$, which is not necessarily convex and which may be empty. In some situations, the limiting sub-differential of f at x can be proven to be nonempty; the result is as follows.

Proposition 3. *Let $x \in \Omega$. If there exists a Lipschitz continuous ϕ defined in a neighborhood of x such that $f - \phi$ attains a local minimum at x, then $\partial_L f(x)$ is nonempty.*

Proof. Without loss of generality, we can assume to be in $\mathbb{R}^n$. By assumption, the function $f-\phi$ attains a local minimum at x; this implies that $0 \in \partial_L(f-\phi)(x)$. By the sum rule on limiting sub-differentials (see [6, Prop. 10.1 p. 62]), the function $-\phi$ being Lipschitz continuous, there holds

$$\partial_L(f-\phi)(x) \subset \partial_L f(x) + \partial_L(-\phi)(x),$$

and hence $\partial_L f(x)$ is necessarily nonempty. □

This proposition will be the key result to prove Th. 2. Notice that there exist some continuous functions $f : \mathbb{R}^n \to \mathbb{R}, n \geq 2$, such that their limiting sub-differential is empty on a subset of positive Lebesgue measure. However if $n = 1$, it can be proven that the limiting sub-differential of any continuous function $f : \mathbb{R} \to \mathbb{R}$ is nonempty almost everywhere. Our proof of Th. 2 for corank-one distributions is in some way related to this latter result, but is not a consequence of it.

3.2. Application to the proof of Theorem 2

In what follows, we denote $e(\cdot) := d_{SR}(x_0, \cdot)^2$.

Proposition 4. *Let $x \in M$ such that $\partial_L e(x) \neq \emptyset$ and let $\zeta \in \partial_L e(x)$. Then there exists a minimizing trajectory joining x_0 to x which admits a normal extremal lift $(x(\cdot), p(\cdot), -\frac{1}{2}, u(\cdot))$ such that $p(1) = \frac{1}{2}\zeta$.*

Proof. By definition of the limiting sub-differential, there exists a sequence $(x_n)_{n\in\mathbf{N}}$ of points in M converging to x and a sequence $(\zeta_n)_{n\in\mathbf{N}} \in \partial_P e(x_n)$ such that $\lim \zeta_n = \zeta$. For each integer n, we denote by $u_n(\cdot)$ a minimizing control joining x_0 to x_n, and by $x_{u_n}(\cdot)$ its associated trajectory. From Prop. 2, for each integer n, we know that $x_{u_n}(\cdot)$ admits a normal extremal lift $(x_{u_n}(\cdot), p_{u_n}(\cdot), -\frac{1}{2}, u_n(\cdot))$ such that $p_{u_n}(1) = \frac{1}{2}\zeta_n$. Since the sub-Riemannian distance is continuous, the sequence of controls $(u_n(\cdot))_{n\in\mathbf{N}}$ is clearly bounded in $L^2([0,1], \mathbb{R}^m)$, and then up to a subsequence, it converges towards an element $u(\cdot)$ for the weak L^2-topology. As a consequence, since the end-point mapping E_{x_0} is continuous for the weak L^2-topology (see [9] for a proof), we deduce, passing to the limit, that $E_{x_0}(u(\cdot)) = x$. Furthermore, up to a subsequence the sequence $(x_{u_n}(\cdot))_{n\in\mathbf{N}}$ converges uniformly towards a minimizing path $x_u(.)$. This implies that the sequence $(p_{u_n}(\cdot))_{n\in\mathbf{N}}$ converges uniformly towards some $p_u(\cdot)$, where $p_u(\cdot)$ is an adjoint vector associated to the trajectory $x_u(\cdot)$, and $p_u(1) = \frac{1}{2}\lim_{n\to\infty}\zeta_n$. Finally the quadruple $(x_u(\cdot), p_u(\cdot), -\frac{1}{2}, u(\cdot))$ is a normal extremal lift of $x_u(\cdot)$. □

Analogously to Th. 3, we have the following result.

Proposition 5. *If the distribution is everywhere of corank one, then $\partial_L e(x) \neq \emptyset$ for almost every $x \in M$.*

Proof. In what follows, our point of view being local, we can assume to work in $\mathbb{R}^n$. Denote by P the set of points x of M such that

$$\liminf_{y\to x} \frac{e(y)-e(x)}{\|y-x\|} = -\infty.$$

We have $M = P \cup P^c$, where P^c denotes the complement of the set P in M. Note that if $x \in P^c$ then there exists $\alpha \in \mathbb{R}$ such that $\liminf_{y\to x} \frac{e(y)-e(x)}{\|y-x\|} = \alpha$, which means that there exists a neighborhood $\mathcal{V}$ of x such that

$$e(y) \geq e(x) + (\alpha - 1)\|y-x\|, \quad \forall y \in \mathcal{V}.$$

We infer that the function e has a Lipschitz continuous support function at x and hence from Prop. 3 that $\partial_L e(x)$ is nonempty. The rest of the proof is devoted to show that the set $\partial_L e(x)$ is nonempty for almost every point $x \in P$. We argue by contradiction: denote by A the subset of P where the limiting sub-differential of f is empty, and suppose that $\mu(A) > 0$.

For all $x \in M$, let $\nu(x)$ denote a vector of $T_x M$ transverse to the distribution $\Delta(x)$. We may assume the vector field $\nu(\cdot)$ to be smooth on M. Let us consider integral curves of the differential system

$$\dot{y}(t) = \nu(y(t)). \tag{6}$$

From Fubini's theorem, there exists an interval $I \subset \mathbb{R}$ and an integral curve $(y(t))_{t\in I}$ of (6) such that the set

$$T := \{t \in I \mid y(t) \in A\},$$

satisfies $\lambda(T) > 0$, where λ denotes the Lebesgue measure on $\mathbb{R}$. We are going to prove that some $\bar{t} \in T$ is the limit of local minima of the function $e(\cdot)$ restricted to the curve $y(t)$. To this aim we need different lemmas.

Lemma 1. *For all $x \in M$, there exist a neighborhood $\mathcal{V}_x$ of x in M, a neighborhood U_x of 0 in T_x^*M, and a submanifold D_x of codimension 1 in M, such that*

$$\mathcal{V}_x \cap D_x \subset exp_x(U_x).$$

Proof. Clearly the mapping $\exp_x$ is smooth on its domain of definition, and its differential at 0, denoted $d\exp_x(0)$, can be computed as

$$d\exp_x(0) \,.\, \delta p_0 = \delta x(1),$$

where $(\delta x(\cdot), \delta p(\cdot))$ is the solution of the linearized system of system (5) at the equilibrium point $(x, 0)$, such that $\delta x(0) = 0$ and $\delta p(0) = \delta p_0$. This linearized system writes

$$\delta\dot{x}(t) = \sum_{i=1}^{n-1} \langle \delta p(t), f_i(x)\rangle f_i(x), \quad \delta\dot{p}(t) = 0,$$

and thus $\delta p(t)$ is constant, equal to δp_0, whence

$$\delta x(1) = \sum_{i=1}^{n-1} \langle \delta p_0, f_i(x) \rangle f_i(x). \tag{7}$$

Therefore the mapping $\exp_x$ has rank $n-1$ at the point 0, and the conclusion follows. □

For each $x \in M$, let $(p_i^*(x))_{i=1,\dots,n}$ denote the dual basis in T_x^*M of the basis $(f_1(x), \dots, f_{n-1}(x), \nu(x))$ in T_xM. We define the mapping $\Phi : I \times O \to M$, where O is a neighborhood of 0 in $\mathbb{R}^{n-1}$, by the formula

$$\Phi(t, \alpha_1, \dots, \alpha_{n-1}) := \exp_{y(t)} \left(\sum_{i=1}^{n-1} \alpha_i p_i^*(y(t)) \right).$$

Using (7), it is quite easy to see that, for all $t_0 \in I$, the mapping Φ is a local diffeomorphism at $(t_0, 0)$. Thus the following lemma is straightforward.

Lemma 2. *Let $t_0 \in T$. There exist a neighborhood $\mathcal{V}$ of $y(t_0)$ in M and a smooth function $\rho : \mathcal{V} \longrightarrow I$ such that for every $z \in \mathcal{V}$, one has $z \in D_{y(\rho(z))}$, and such that for every $t \in I$ with $y(t) \in \mathcal{V}$, there holds $\rho(y(t)) = t$. Moreover, there exists a real number $\delta > 0$ such that*

$$|e(z) - e(y(\rho(z)))| \le \delta \|z - y(\rho(z))\|. \tag{8}$$

for all $z \in \mathcal{V}$.

Define the continuous function $g : I \to \mathbb{R}$ by $g(t) := e(y(t))$.

Lemma 3. *There exists $\bar{t} \in T$ and a sequence $(t_n)_{n \in \mathbf{N}}$ of I converging towards $\bar{t}$, such that the function g attains a local minimum at t_n, for every integer n.*

Proof. We argue by contradiction. If the conclusion of the lemma does not hold, this means that for every $t \in T$, there exists a neighborhood V_t of t in I on which g is monotonous. In particular g has bounded variations on V_t, and hence g is differentiable almost everywhere in V_t. On the other hand, since the set T has positive Lebesgue measure, there exists $t \in T$ such that $\lambda(V \cap T) > 0$ for any neighborhood V of t in I. Hence, this proves that for such a $t \in T$, the function g is differentiable almost everywhere in V_t which has positive measure. Fix some $s \in V_t$ where g is differentiable. As a consequence, there exists some Lipschitz continuous function $\psi : I \to \mathbb{R}$ such that

$$\psi(s) = g(s), \text{ and } \psi(s') \le g(s'), \quad \forall s' \in I.$$

On the other hand, by Lemma 2, there exists a neighborhood $\mathcal{V}$ of $y(s)$ in M and a smooth function $\rho : \mathcal{V} \to I$ which satisfy the property given in the lemma. By (8), we deduce that for any $x \in \mathcal{V}$, there holds

$$\begin{aligned} e(x) &\geq e(y(\rho(x))) - \delta\|x - y(\rho(x))\| \\ &\geq \psi(\rho(x)) - \delta\|x - y(\rho(x))\|. \end{aligned}$$

Therefore if we define locally $\phi(x) := -\delta\|x - y(\rho(x))\|$, the function $e - \psi \circ \rho - \phi$ attains a local minimum at $y(s)$. Since $\psi \circ \rho$ and ϕ are Lipschitz continuous, the sum rule on limiting sub-differentials (see [6, Prop. 10.1 p. 62]) implies that

$$0 \in \partial_L(e - \psi \circ \rho - \phi)(y(s)) \subset \partial_L e(y(s)) + \partial_L(-\psi \circ \rho - \phi)(y(s)).$$

Hence there exists $\zeta \in \partial_L e(y(s))$ and $\zeta' \in \partial_L(-\psi \circ \rho - \phi)(y(s))$ such that $0 = \zeta + \zeta'$, which proves that $\partial_L e(y(s))$ is nonempty and which contradicts the fact that $y(s) \in A$. □

Lemma 4. *There exists some constant $K > 0$ such that for every integer n, the limiting sub-differential $\partial_L e(y(t_n))$ contains an element with norm less than K.*

Proof. By construction of the sequence $(t_n)_{n\in\mathbf{N}}$, for every integer n the function g attains a minimum at t_n. This means that there exists an interval (a_n, b_n) containing t_n such that

$$\forall t \in (a_n, b_n) \quad g(t) \geq g(t_n).$$

On the other hand, by Lemma 2, there exists a neighborhood $\mathcal{V}$ of $y(\bar{t})$ such that for n large enough, any x close enough to $y(t_n)$ belongs to $D_{y(\rho(x))}$ where $\rho(x) \in (a_n, b_n)$. By (8), we deduce that for x close enough to $y(t_n)$, there holds

$$\begin{aligned} e(x) &\geq e(y(\rho(x))) - \delta\|x - y(\rho(x))\| \\ &\geq e(y(t_n)) - \delta\|x - y(\rho(x))\|. \end{aligned}$$

Therefore as before, if we define locally $\phi(x) := -\delta\|x - y(\rho(x))\|$, the function $e - \phi$ attains a local minimum at $y(t_n)$. Since ϕ is Lipschitz continuous, the sum rule on limiting sub-differentials implies that

$$0 \in \partial_L(e - \phi)(y(t_n)) \subset \partial_L e(y(t_n)) + \partial_L(-\phi)(y(t_n)).$$

Hence there exists $\zeta \in \partial_L e(y(t_n))$ and $\zeta' \in \partial_L(-\phi)(y(t_n))$ such that $0 = \zeta + \zeta'$. Finally $\|\zeta\| = \|\zeta'\|$ where $\|\zeta'\|$ is less than the Lipschitz constant of the function ϕ. This concludes the proof of the lemma. □

Returning to the proof of Prop. 5, we infer easily that $\partial_L e(y(\bar{t}))$ is nonempty. This yields a contradiction with the fact that $y(\bar{t}) \in A$, and ends the proof of the proposition. □

Propositions 4 and 5 imply the existence of a subset N of full Lebesgue measure in M such that, for every $x \in N$, there exists a minimizing trajectory steering x_0 to x and having a normal extremal lift. Let $N_2 := N \setminus \mathcal{C}(x_0)$. It is the set of points $x \in M$ which are not conjugate to x_0, and such that there exists a minimizing path $x(\cdot)$ joining x_0 to x and having a normal extremal lift. Remark 7 implies that the trajectory $x(\cdot)$ is moreover nonsingular. From Remark 6 it is clear that N_2 is of full Lebesgue measure in M. This ends the proof of Th. 2.

4. Consequences and open questions

In what follows, we assume that Chow's condition holds, and that the manifold M is complete for the sub-Riemannian distance. Let $x_0 \in M$ be fixed.

4.1. A formula for the sub-Riemannian distance

From Th. 1, there exists a dense subset N_1 of M such that every point of N_1 can be joined from x_0 by a unique minimizing trajectory, which moreover admits a normal extremal lift. This yields the following result.

Corollary 1. *For all point $x \in N_1$ one has*

$$d_{SR}(x_0, x) = \inf \left\{ (2\, H_1(x_0, p))^{1/2} \ \mid p \in U \ s.t.\ exp_{x_0}(p) = x \right\}.$$

Remark 11. Actually Th. 1 implies that for every $x \in N_1$ there exists a unique $p \in U$ such that the above infimum is attained.

As a consequence, we deduce that the function $g : M \to \mathbb{R} \cup \{\infty\}$ defined by

$$g(x) := \inf \left\{ (2\, H_1(x_0, p))^{1/2} \ \mid p \in U \text{ s.t. } \exp_{x_0}(p) = x \right\},$$

for all $x \in M$, coincides with the mapping $d_{SR}(x_0, \cdot)$ on a dense subset of the manifold M. In particular, since g is continuous on M, there holds

$$d_{SR}(x_0, x) = \inf \{\lim g(x_n) \mid x_n \to x\}$$

for all $x \in M$.

Remark 12. If the sub-Riemannian distance to x_0 is Lipschitz continuous outside x_0, then from Prop. 1 the limiting sub-differentials of $d_{SR}(x_0, \cdot)$ are always nonempty; hence the set of points x of M such that every minimizing trajectory joining x_0 to x is strictly singular, is empty. The converse is false; a counterexample is given by the so-called Martinet flat case, see [2]. To get a converse statement, the assumption has to be strengthened as follows: if there does not exist any nontrivial singular minimizing trajectory, then $d_{SR}(x_0, \cdot)$ is Lipschitz continuous outside x_0, see [1].

4.2. On the sub-Riemannian wave-front and sphere

The following result is a direct consequence of Th. 1.

Corollary 2. *The sub-Riemannian wave-front $W_{SR}(x_0, r)$ is connected, for all $r > 0$.*

Proof. Using notations of Remark 5, and from Th. 1, we have the inclusions

$$\exp_{x_0}(U_r) \subset W_{SR}(x_0, r) \subset \overline{\exp_{x_0}(U_r)},$$

where U_r is diffeomorphic to $\mathcal{S}^{m-1} \times \mathbb{R}^{n-m}$, and thus is connected. The conclusion follows readily. □

Proposition 6. *If the distribution Δ is everywhere of corank one, then the sub-Riemannian wave-front $W_{SR}(x_0, r)$, and thus the sub-Riemannian sphere $S_{SR}(x_0, r)$, has Lebesgue measure zero, for all $r > 0$.*

Proof. It suffices to notice that the image by a locally lipschitzian mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$ of a set of zero measure has zero measure, and to apply Th. 2. □

4.3. Sard type conjectures

Let $\mathcal{A}$ (resp. $\mathcal{A}_s$) denote the set of points x of M such that every minimizing trajectory joining x_0 to x is singular (resp. strictly singular). Obviously $\mathcal{A}_s \subset \mathcal{A}$. Th. 1 and 2 yield the following result.

Corollary 3. *The subset $\mathcal{A}_s$ has an empty interior in M. In the case of a corank-one distribution the subset $\mathcal{A}$ has Lebesgue measure zero in M.*

Let now $\mathcal{S}$ (resp. $\mathcal{S}_{min}$, resp. $\mathcal{S}_{min}^{strict}$) denote the set of points x of M such that there exists a singular trajectory (resp. a singular minimizing trajectory, resp. a strictly singular minimizing trajectory) steering x_0 to x. Notice that $\mathcal{S}$ is the set of critical values of the end-point-mapping E_{x_0}.

Corollary 4. *The set $\mathcal{S}_{min}^{strict}$ has an empty interior in M.*

Let N_3 be the set of points $x \in M$ such that there exists a unique minimizing path $x(\cdot)$ joining x_0 to x, which moreover admits a normal extremal lift, and such that x is not conjugate to x_0. Notice that from Remark 7, the path $x(\cdot)$ is non-singular. The set N_3 can be proven to be open in M; we formulate the following conjecture.

Conjecture 1. The subset N_3 is of full Lebesgue measure in M. In particular, the set $\mathcal{S}_{min}$ has Lebesgue measure zero in M.

We end the paper with the following open question.

Conjecture 2. The end-point mapping satisfies Sard's property, *i.e*. the set $\mathcal{S}$ has Lebesgue measure zero in M.

This conjecture has been formulated and discussed, among others, in [7]. Up to now, it is still open, even in the case of a corank-one distribution.

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