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OPTIMAL DESIGN OF SENSORS FOR A DAMPED WAVE EQUATION

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Abstract. In this paper we model and solve the problem of shaping and placing in an optimal way sensors for a wave equation with constant damping in a bounded open connected subset Ω of $\mathbb{R}^n$. Sensors are modeled by subdomains of Ω of a given measure $L|\Omega|$, with $0 < L < 1$. We prove that, if L is close enough to 1, then the optimal design problem has a unique solution, which is characterized by a finite number of low frequency modes. In particular the maximizing sequence built from spectral approximations is stationary.

1. **Introduction.** Throughout this paper, we consider an integer $n \geq 1$ and a bounded open connected subset Ω of $\mathbb{R}^n$. For any measurable subset $\omega \subset \Omega$, the notation χ_ω stands for the characteristic function of ω.

For every $u \in L^2(\Omega, \mathbb{C})$, we have $\|u\|_{L^2(\Omega,\mathbb{C})} = \left(\int_\Omega |u(x)|^2 \, dx\right)^{1/2}$. The Hilbert space $H^1(\Omega, \mathbb{C})$ is the space of functions of $L^2(\Omega, \mathbb{C})$ having a distributional derivative in $L^2(\Omega, \mathbb{C})$, endowed with the norm $\|u\|_{H^1(\Omega,\mathbb{C})} = \left(\|u\|^2_{L^2(\Omega,\mathbb{C})} + \|\nabla u\|^2_{L^2(\Omega,\mathbb{C})}\right)^{1/2}$. The Hilbert space $H^1_0(\Omega, \mathbb{C})$ is the closure in $H^1(\Omega, \mathbb{C})$ of the set of functions of class C^∞ on Ω and of compact support in Ω. It is endowed with the norm $\|u\|_{H^1_0(\Omega,\mathbb{C})} = \|\nabla u\|_{L^2(\Omega,\mathbb{C})}$.

Let $T > 0$ and $a > 0$ be arbitrary positive real numbers. We consider the wave equation with constant damping

$$\partial_{tt} y(t, x) - \triangle y(t, x) + a \, \partial_t y(t, x) = 0, \tag{1}$$

in $(0, T) \times \Omega$, with Dirichlet boundary conditions.

1.1. **Spectral expansions.** Let $(\phi_j)_{j\in\mathbb{N}^*}$ be a Hilbert basis of $L^2(\Omega, \mathbb{C})$ consisting of (real-valued) normalized eigenfunctions of the negative of the Dirichlet-Laplacian on Ω, associated with the eigenvalues $0 < \mu_1 \leq \cdots \leq \mu_j \leq \cdots$, with $\mu_j \to +\infty$ as $j \to +\infty$.

Let $(y^0, y^1) \in H^1_0(\Omega, \mathbb{C}) \times L^2(\Omega, \mathbb{C})$ be some arbitrary initial data. The unique solution $y \in C^0(0, T; H^1_0(\Omega, \mathbb{C})) \cap C^1(0, T; L^2(\Omega, \mathbb{C}))$ of (1) such that $y(0, \cdot) = y^0(\cdot)$ and $\partial_t y(0, \cdot) = y^1(\cdot)$ can be expanded as

$$y(t, x) = \sum_{j=1}^{+\infty} (a_j e^{\lambda_j^+ t} + b_j e^{\lambda_j^- t}) \phi_j(x), \tag{2}$$

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where

$$\lambda_j^- = \frac{-a - \sqrt{a^2 - 4\mu_j}}{2} \quad \text{and} \quad \lambda_j^+ = \frac{-a + \sqrt{a^2 - 4\mu_j}}{2},$$

and the sequences $(a_j)_{j\in\mathbb{N}^*}$ and $(b_j)_{j\in\mathbb{N}^*}$ are such that $(\mu_j a_j)_{j\in\mathbb{N}^*}$ and $(\mu_j b_j)_{j\in\mathbb{N}^*}$ belong to $\ell^2(\mathbb{C})$. Introducing the Fourier coefficients $y_j^0 = \int_\Omega y^0(x)\phi_j(x)\,dx$ and $y_j^1 = \int_\Omega y^1(x)\phi_j(x)\,dx$ of the initial data y^0 and y^1, we have

$$a_j = \frac{y_j^1 - \lambda_j^- y_j^0}{\lambda_j^+ - \lambda_j^-} \qquad \text{and} \qquad b_j = \frac{\lambda_j^+ y_j^0 - y_j^1}{\lambda_j^+ - \lambda_j^-}, \tag{3}$$

for every $j \in \mathbb{N}^*$. We have the following obvious lemma.

Lemma 1.1. *There exists $N_{LF} \in \mathbb{N}$ such that $a^2 - 4\mu_j \geq 0$ for every $j \leq N_{LF}$, with the agreement that $N_{LF} = 0$ if this inequality never occurs.*

If $N_{LF} > 0$, then λ_j^- and λ_j^+ are real for every $j \leq N_{LF}$, and λ_j^- and λ_j^+ are complex non real for every $j > N_{LF}$. If $N_{LF} = 0$, then λ_j^- and λ_j^+ are complex non real for every $j \in \mathbb{N}^*$.

1.2. **Observability inequalities, and optimal design problem.** In practice, the problem of optimizing the placement and shape of sensors arises when one wants to place some sensors in some cavity (domain Ω), in order to make some measurements of the signals propagating in Ω over a certain horizon of time, and submitted to some damping. It is desirable to place and shape these sensors in such a way to ensure that the reconstruction will be as good as possible. This is the question that we address in this paper.

We first recall some basic facts on the notion of observation of the equation (1). Let $T > 0$ and ω be a measurable subset of Ω. We say that (1) is *observable* on ω in time T if there exists $C > 0$ such that

$$C\|(y^0, y^1)\|^2_{H_0^1 \times L^2} \leq \int_0^T \int_\omega |\partial_t y(t,x)|^2\,dx\,dt, \tag{4}$$

for all $(y^0, y^1) \in H_0^1(\Omega, \mathbb{C}) \times L^2(\Omega, \mathbb{C})$, where y denotes the unique solution of (1) such that $y(0,\cdot) = y^0$ and $\partial_t y(0,\cdot) = y^1$. This is the *observability inequality*, of great importance in view of showing the well-posedness of some inverse problems. Note that, as is well known (see [19, Theorem 7.4.1 and Remark 7.3.6]), this inequality holds within the class of $\mathcal{C}^\infty$ domains Ω, if the pair (ω, T) satisfies the *Geometric Control Condition* in Ω (see [3, 6]), according to which every geodesic ray propagating in Ω and reflecting on its boundary according to the laws of geometrical optics intersects the observation set ω within time T. The largest possible constant for which (4) holds, called the *observability constant*, is defined by

$$C_T(\chi_\omega) = \inf_{(y^0,y^1)\in H_0^1(\Omega,\mathbb{C})\times L^2(\Omega,\mathbb{C})\setminus\{(0,0)\}} \frac{\int_0^T \int_\omega |\partial_t y(t,x)|^2\,dx\,dt}{\|(y^0,y^1)\|^2_{H_0^1\times L^2}}.$$

It can be equal to 0. It depends both on the time T (the horizon time of observation) and on the subset ω on which the measurements are done. Noting that

$$\|(y^0, y^1)\|^2_{H_0^1 \times L^2} = \sum_{j=1}^{+\infty} \left(\mu_j |a_j + b_j|^2 + |\lambda_j^+ a_j + \lambda_j^- b_j|^2\right),$$

where a_j and b_j are defined by (3), it follows that

$$C_T(\chi_\omega) \quad = \quad \inf \left\{ \int_0^T \int_\omega \left| \sum_{j=1}^{+\infty} \left(a_j \lambda_j^+ e^{\lambda_j^+ t} + b_j \lambda_j^- e^{\lambda_j^- t} \right) \phi_j(x) \right|^2 dx\,dt \right.$$

$$(a_j),(b_j) \in \ell^2(\mathbb{C}), \quad \sum_{j=1}^{+\infty}\left(\mu_j|a_j+b_j|^2+|\lambda_j^+a_j+\lambda_j^-b_j|^2\right) = 1\Bigg\}.$$

In addition to the fact that this quantity is difficult to compute, and a fortiori to characterize, in practice this constant corresponds to a worst case of observation, and does not necessarily reflect in a relevant way the desired situation, since one generally carries out a very large number of experiments. In view of that, in order to model the issue of optimizing the shape and location of sensors for (1), we use the same approach as in [16, 17], based on randomized initial data. According to [5, 7, 8]) (using early ideas of [12]), we randomize the coefficients a_j, b_j in (2) of the initial conditions, by multiplying each of them by some adequate random law. This random selection of all possible initial data for the wave equation (1) consists of replacing $C_T(\chi_\omega)$ with the randomized version

$$C_{T,\mathrm{rand}}(\chi_\omega) = \inf\Bigg\{\mathbb{E}\int_0^T\int_\omega\left|\sum_{j=1}^{+\infty}\left(\beta_{1,j}^\nu a_j\lambda_j^+e^{\lambda_j^+t}+\beta_{2,j}^\nu b_j\lambda_j^-e^{\lambda_j^-t}\right)\phi_j(x)\right|^2 dx\,dt,$$
$$(a_j),(b_j)\in\ell^2(\mathbb{C}), \quad \sum_{j=1}^{+\infty}\left(\mu_j|a_j+b_j|^2+|\lambda_j^+a_j+\lambda_j^-b_j|^2\right)=1\Bigg\}.$$

where $(\beta_{1,j}^\nu)_{j\in\mathbb{N}^*}$ and $(\beta_{2,j}^\nu)_{j\in\mathbb{N}^*}$ are two sequences of independent Bernoulli random variables on a probability space $(\mathcal{X},\mathcal{A},\mathbb{P})$, satisfying $\mathbb{P}(\beta_{1,j}^\nu=\pm1)=\mathbb{P}(\beta_{2,j}^\nu=\pm1)=\frac{1}{2}$ and $\mathbb{E}(\beta_{1,j}^\nu\beta_{2,k}^\nu)=0$, for all nonzero integers j and k and every $\nu\in\mathcal{X}$. Here, the notation $\mathbb{E}$ stands for the expectation over the space $\mathcal{X}$ with respect to the probability measure $\mathbb{P}$. In other words, instead of considering the deterministic observability inequality (4) for the wave equation (1), we consider the *randomized observability inequality*

$$C_{T,\mathrm{rand}}(\chi_\omega)\|(y^0,y^1)\|^2_{H_0^1\times L^2}\le\mathbb{E}\left(\int_0^T\int_\omega|y_\nu(t,x)|^2\,dx\,dt\right),$$

for all $(y^0,y^1)\in H_0^1(\Omega,\mathbb{C})\times L^2(\Omega,\mathbb{C})$, where

$$y_\nu(t,x)=\sum_{j=1}^{+\infty}\left(\beta_{1,j}^\nu a_je^{\lambda_j^+t}+\beta_{2,j}^\nu b_je^{\lambda_j^-t}\right)\phi_j(x).$$

In some sense, the randomization procedure corresponds to a random selection of the initial data y^0 and y^1 over the set of all possible initial data. This new constant $C_{T,\mathrm{rand}}(\chi_\omega)$ is called *randomized observability constant.*

Optimal design problem. According to the previous discussion, we model the best sensor shape and location problem for the damped wave equation (1) as the problem of maximizing the functional $\chi_\omega\mapsto C_{T,\mathrm{rand}}(\chi_\omega)$ over the set

$$\mathcal{U}_L=\{\chi_\omega\in L^\infty(\Omega;\{0,1\})\mid\omega\subset\Omega\text{ is measurable and }|\omega|=L|\Omega|\}.$$

This set models the fact that the quantity of sensors to be employed is limited and, hence, that we cannot measure the solution over Ω in its whole. The optimal design problem is then

$$\boxed{\sup_{\chi_\omega\in\mathcal{U}_L} C_{T,\mathrm{rand}}(\chi_\omega).} \tag{5}$$

1.3. **Existing literature.** The literature on optimal observation or sensor location problems is abundant in engineering applications (see, e.g., [2, 11, 18, 20, 21] and references therein), where the aim is often to optimize the number, the place and the type of sensors in order to improve the estimation of the state of the system. Fields of applications are very numerous and concern for example active structural acoustics, piezoelectric actuators, vibration control in mechanical structures, damage detection and chemical reactions. In most of these applications the method consists in approximating appropriately the problem by selecting a finite number of possible optimal candidates and of recasting the problem as a finite-dimensional combinatorial optimization problem.

Among the possible approaches, the closest one to ours consists of considering truncations of Fourier expansion representations. Adopting such a Fourier point of view, the authors of [9, 10] studied optimal stabilization issues of the one-dimensional wave equation and highlighted a kind of numerical instability result when truncating Fourier series in the optimization procedure, the so-called *spillover phenomenon*. In [4] the authors investigate the problem modeled in [18] of finding the best possible distributions of two materials with different elastic Young modulus in a rod in order to minimize the vibration energy in the structure. The authors of [1] also propose a convexification formulation of eigenfrequency optimization problems applied to optimal design. We also quote the article [14] where the problem of finding the optimal location of the support of the control for the one-dimensional wave equation is addressed. In [13] it is proved that, for fixed initial data as well, the problem of optimal shape and location of sensors is always well-posed for heat, wave or Schrödinger equations, and it is showed that the complexity of the optimal set depends on the regularity of the initial data; in particular even for smooth initial data the optimal set can be fractal.

In the recent works [15, 16], an optimal design problem similar to (5) (that is, independent on the initial data, but with a model built thanks to randomization considerations) is studied for the wave equation without damping, and in [17] this is done for general classes of parabolic equations such as the heat equation or the Stokes equation.

2. Main results.

In this section, we state our main results and make some comments. The proofs are done in the next sections.

Our first result provides a more explicit expression of $C_{T,\mathrm{rand}}(\chi_\omega)$.

Proposition 1. *We have*

$$C_{T,rand}(\chi_\omega) = \inf_{j\in\mathbb{N}^*} \frac{1}{\lambda_1(M_j)} \int_\omega \phi_j(x)^2\,dx,$$

where $\lambda_1(M_j)$ is the largest eigenvalue of the real symmetric 2×2 matrix

$$M_j = \begin{pmatrix} \frac{\mu_j+|\lambda_j^+|^2}{|\lambda_j^+|^2\int_0^T e^{2\mathrm{Re}\lambda_j^+ t}\,dt} & \frac{\left|\mu_j+\lambda_j^+\overline{\lambda_j^-}\right|}{\mu_j\sqrt{\int_0^T e^{2\mathrm{Re}\lambda_j^+ t}\,dt\int_0^T e^{2\mathrm{Re}\lambda_j^- t}\,dt}} \\ \frac{\left|\mu_j+\lambda_j^+\overline{\lambda_j^-}\right|}{\mu_j\sqrt{\int_0^T e^{2\mathrm{Re}\lambda_j^+ t}\,dt\int_0^T e^{2\mathrm{Re}\lambda_j^- t}\,dt}} & \frac{\mu_j+|\lambda_j^-|^2}{|\lambda_j^-|^2\int_0^T e^{2\mathrm{Re}\lambda_j^- t}\,dt} \end{pmatrix}.$$

We are going to state an existence result for the optimal design problem (5), provided that L is close enough to 1. It is first required to introduce a truncated version of the problem. For every $N\in\mathbb{N}^*$, we define

$$C^N_{T,\mathrm{rand}}(\chi_\omega) = \min_{1\le j\le N} \frac{1}{\lambda_1(M_j)} \int_\omega \phi_j(x)^2\,dx,$$

for every measurable subset ω of Ω. We consider the shape optimization problem

$$\sup_{\chi_\omega\in\mathcal{U}_L} C^N_{T,\mathrm{rand}}(\chi_\omega), \tag{6}$$

which is a spectral approximation of the problem (5),

Theorem 2.1. *For every $N\in\mathbb{N}^*$, the problem (6) has a unique solution $\chi_{\omega^N}\in\mathcal{U}_L$. Moreover, ω^N is semi-analytic*[1] *and thus it has a finite number of connected components.*

[1] A subset ω of a real analytic finite dimensional manifold M is said to be semi-analytic if it can be written in terms of equalities and inequalities of analytic functions. We recall that such semi-analytic subsets enjoy

We will not provide a proof of that result since it is exactly similar to the one of [16, Theorem 8] or of [17, Proposition 2].

The high frequencies play an essential role in the definition of the criterion $C_{T,\mathrm{rand}}(\chi_\omega)$. For this reason, we need to determine the asymptotic of the quantity $\lambda_1(M_j)$ defined in the statement of Proposition 1 as j tends to $+\infty$.

Lemma 2.2. *We have* $\dfrac{1}{\lambda_1(M_j)} = \dfrac{\int_0^T e^{-at}\,dt}{4}\left(1-\dfrac{a}{4\sqrt{\mu_j}}\right) + \mathrm{o}\left(\dfrac{1}{\mu_j}\right)$ *as* $j\to+\infty$.

Proof of Lemma 2.2. Let N_{LF} be the integer defined in the statement of Lemma 1.1, and $j > N_{LF}$. Then $\mathrm{Re}(\lambda_j^+) = \mathrm{Re}(\lambda_j^-) = -\frac{a}{2}$, $|\lambda_j^+| = |\lambda_j^-| = \sqrt{\mu_j}$, and therefore

$$M_j = \frac{1}{\int_0^T e^{-at}\,dt}\begin{pmatrix} 2 & \delta_j \\ \delta_j & 2 \end{pmatrix} \qquad \text{with} \qquad \delta_j = \frac{a}{2\sqrt{\mu_j}}.$$

We infer that $\lambda_1(M_j) = \frac{1}{\int_0^T e^{-at}\,dt}(2+\delta_j)$ and the conclusion follows from an easy computation. □

This lemma leads to define j_0 as the positive integer minimizing the sequence $(1/\lambda_1(M_j))_{j\in\mathbb{N}^*}$. We have

$$\frac{1}{\lambda_1(M_{j_0})} = \underset{j\in\mathbb{N}^*}{\mathrm{argmin}}\left(\frac{1}{\lambda_1(M_j)}\right) < \frac{\int_0^T e^{-at}\,dt}{4}.$$

Theorem 2.3. *Assume that*

(H1) *The sequence of probability measures* $\mu_j = \phi_j^2\,dx$ *converges vaguely to the uniform measure* $\frac{1}{|\Omega|}\,dx$.

(H2) *There exists* $A>0$ *such that* $\|\phi_j\|_{L^\infty(\Omega)} \leq A$, *for every* $j\in\mathbb{N}^*$.

Let $L \in \left(\frac{4}{\lambda_1(M_{j_0})\int_0^T e^{-at}\,dt}, 1\right)$. *Then there exists* $N_0\in\mathbb{N}^*$ *such that*

$$\max_{\chi_\omega\in\mathcal{U}_L} C_{T,rand}(\chi_\omega) = \max_{\chi_\omega\in\mathcal{U}_L} C_{T,rand}^N(\chi_\omega) < \frac{L}{4}\int_0^T e^{-at}\,dt,$$

for every $N\geq N_0$. *In particular, as a consequence, the problem* (5) *has a unique solution* $\chi_{\omega^{N_0}} \in \mathcal{U}_L$. *Moreover the set* ω^{N_0} *is semi-analytic and thus it has a finite number of connected components.*

Remark 1. The (strong) assumptions (H1) and (H2) have been widely commented in [16, Sections 3.2 and 3.3]. They are true in dimension one. Indeed, the eigenfunctions of the Dirichlet-Laplacian operator on $\Omega = (0,\pi)$ are given by $\phi_j(x) = \sqrt{\frac{2}{\pi}}\sin(jx)$, for every $j\in\mathbb{N}^*$, and $(\phi_j^2)_{j\in\mathbb{N}^*}$ converges weakly to $1/\pi$ for the weak star topology of $L^\infty(0,\pi)$. In larger dimension, the assumption (H2) fails in general, and for (H1) the situation is widely open and is related to quantum ergodicity properties of Ω.

We end this paper with a short series of concluding remarks.

First of all, it can be noted that we have addressed, in the present article, a wave equation with a constant damping. The fact that the damping is constant has been instrumental in Section 1.1 to derive spectral expansions within a Hilbert basis of eigenfunctions of the Dirichlet-Laplacian (that is, the operator without damping).

What may happen with a general nonconstant damping is open. Of course, it would be possible to make spectral expansions within a frame of eigenfunctions of the (nonselfadjoint) operator underlying the damped wave equation, but then there are (at least) two problems:

local finitetess properties, such that: local finite perimeter, local finite number of connected components, etc.

the first is that, in dimension more than one, this frame is even not a Riesz basis; the second one is that it is not completely clear whether the randomization procedure may produce data that are of full measure in the set of initial data (thus, leading to a relevant model).

A last remark is that, at the beginning, we have fixed a given basis of eigenfunctions of the Dirichlet-Laplacian. Then all results coming afterwards depend on this specific choice. As soon as the eigenvalues are not simple, there are other possible choices of such a basis, and then, although the existence results we have derived are of course still true, the optimal shape depends on the choice of the basis. This has been commented with more details in [17].

2.1. **Proof of Proposition 1.** Following the proofs of [16, Theorem 4] and [17, Proposition 1], we first claim that

$$C_{T,\mathrm{rand}}(\chi_\omega) = \inf_{(a_j),(b_j)} \frac{N(a,b)}{D(a,b)},$$

where

$$N(a,b) = \sum_{j=1}^{+\infty}\left(|a_j|^2|\lambda_j^+|^2\int_0^T e^{2\mathrm{Re}\lambda_j^+ t}\,dt + |b_j|^2|\lambda_j^-|^2\int_0^T e^{2\mathrm{Re}\lambda_j^- t}\,dt\right)\int_\omega \phi_j(x)^2\,dx,$$

$$D(a,b) = \sum_{j=1}^{+\infty}(\mu_j|a_j+b_j|^2 + \left|\lambda_j^+ a_j + \lambda_j^- b_j\right|^2).$$

Setting $a_j = \rho_{j,1}e^{i\theta_{j,1}}$ and $b_j = \rho_{j,2}e^{i\theta_{j,2}}$ for every $j \in \mathbb{N}^*$, we get

$$C_{T,\mathrm{rand}}(\chi_\omega) = \inf_{j\in\mathbb{N}^*}\ \inf_{\rho_{j,1},\rho_{j,2}\in\mathbb{R}_+}\ \inf_{\theta_{j,1},\theta_{j,2}\in\mathbb{R}/(2\pi\mathbb{N})} \frac{\tilde N(\rho_1,\rho_2)}{\tilde D(\rho_1,\rho_2,\theta_1,\theta_2)},$$

where $\rho_k = (\rho_{j,k})_{j\in\mathbb{N}^*}$ and $\theta_k = (\theta_{j,k})_{j\in\mathbb{N}^*}$ for $k = 1,2$, and

$$\tilde N(\rho_1,\rho_2) = \sum_{j=1}^{+\infty}\left(\rho_{j,1}^2|\lambda_j^+|^2\int_0^T e^{2\mathrm{Re}\lambda_j^+ t}\,dt + \rho_{j,2}^2|\lambda_j^-|^2\int_0^T e^{2\mathrm{Re}\lambda_j^- t}\,dt\right)\int_\omega \phi_j(x)^2\,dx,$$

$$\tilde D(\rho_1,\rho_2,\theta_1,\theta_2) = \sum_{j=1}^{+\infty}\Bigg(\rho_{j,1}^2(\mu_j + |\lambda_j^+|^2) + \rho_{j,2}^2(\mu_j+|\lambda_j^-|^2) + 2\rho_{j,1}\rho_{j,2}\mathrm{Re}\left(\left(\mu_j + \lambda_j^+\overline{\lambda_j^-}\right)e^{i(\theta_{j,1}-\theta_{j,2})}\right)\Bigg).$$

It follows that $\displaystyle\inf_{\theta_{j,1},\theta_{j,2}\in\mathbb{R}/(2\pi\mathbb{N})}\frac{\tilde N(\rho_1,\rho_2)}{\tilde D(\rho_1,\rho_2,\theta_1,\theta_2)} = \frac{\tilde N(\rho_1,\rho_2)}{\tilde D(\rho_1,\rho_2)}$, where

$$\tilde D(\rho_1,\rho_2) = \sum_{j=1}^{+\infty}\rho_{j,1}^2(\mu_j+|\lambda_j^+|^2) + \rho_{j,2}^2(\mu_j+|\lambda_j^-|^2) + 2\rho_{j,1}\rho_{j,2}\left|\mu_j + \lambda_j^+\overline{\lambda_j^-}\right|.$$

At this step, we need the following lemma.

Lemma 2.4. *Let x, y, u, v, w, c_1 and c_2 be positive constants. We have*

$$\inf_{(X,Y)\in\mathbb{R}_+^2}\frac{c_1 + xX^2 + yY^2}{c_2 + uX^2 + vY^2 + 2wXY} = \min\left\{\frac{c_1}{c_2}, \frac{1}{\lambda_1(M)}\right\},$$

where $\lambda_1(M)$ is the largest eigenvalue of $M = \begin{pmatrix} \frac{u}{x} & \frac{w}{\sqrt{xy}} \\ \frac{w}{\sqrt{xy}} & \frac{v}{y}\end{pmatrix}$.

Proof of Lemma 2.4. Using a polar change of coordinates leads to

$$\inf_{(X,Y)\in\mathbb{R}_+^2} \frac{c_1 + xX^2 + yY^2}{c_2 + uX^2 + vY^2 + 2wXY} = \inf_{\theta\in\mathbb{R}/(2\pi\mathbb{Z})} \inf_{\rho\in\mathbb{R}_+} \frac{\rho^2(x\cos^2\theta + y\sin^2\theta) + c_1}{c_2 + \rho^2(u\cos^2\theta + v\sin^2\theta + 2w\cos\theta\sin\theta)}.$$

A straightforward study of the variations of the function of the nonnegative variable ρ in the right-hand side shows that

$$\inf_{(X,Y)\in\mathbb{R}_+^2} \frac{c_1 + xX^2 + yY^2}{c_2 + uX^2 + vY^2 + 2wXY} = \inf_{\theta\in\mathbb{R}/(2\pi\mathbb{N})} \min\left\{\frac{c_1}{c_2}, \frac{x\cos^2\theta + y\sin^2\theta}{u\cos^2\theta + v\sin^2\theta + 2w\cos\theta\sin\theta}\right\},$$

and the conclusion follows easily. ☐

Using Lemma 2.4, we infer that

$$C_{T,\text{rand}}(\chi_\omega) = \min\left(\frac{\int_\omega \phi_1(x)^2\,dx}{\lambda_1(M_1)}, \inf_{\substack{j\in\mathbb{N}^*\\ j\neq 1}} \inf_{\rho_{j,1},\rho_{j,2}\in\mathbb{R}_+} \frac{\tilde{N}_1(\rho_1,\rho_2)}{\tilde{D}_1(\rho_1,\rho_2)}\right),$$

where M_1 is the matrix defined in the statement of the proposition, and the ratio $\frac{\tilde{N}_1(\rho_1,\rho_2)}{\tilde{D}_1(\rho_1,\rho_2)}$ has the same expression as the ratio $\frac{\tilde{N}(\rho_1,\rho_2)}{\tilde{D}(\rho_1,\rho_2)}$ except that the sum does not run anymore over all positive integers j, but over all positive integers j different from 1. Applying again Lemma 2.4 yields that

$$C_{T,\text{rand}}(\chi_\omega) = \min\left(\frac{\int_\omega \phi_1(x)^2\,dx}{\lambda_1(M_1)}, \frac{\int_\omega \phi_2(x)^2\,dx}{\lambda_1(M_2)}, \inf_{\substack{j\in\mathbb{N}^*\\ j\notin\{1,2\}}} \inf_{\rho_{j,1},\rho_{j,2}\in\mathbb{R}_+} \frac{\tilde{N}_2(\rho_1,\rho_2)}{\tilde{D}_2(\rho_1,\rho_2)}\right),$$

where M_2 is the matrix defined in the statement of the proposition, and the ratio $\frac{\tilde{N}_2(\rho_1,\rho_2)}{\tilde{D}_2(\rho_1,\rho_2)}$ has the same expression as the ratio $\frac{\tilde{N}(\rho_1,\rho_2)}{\tilde{D}(\rho_1,\rho_2)}$ except that the sum does not run anymore over all positive integers j, but over all positive integers j different from 1 and 2. The conclusion follows from a straightforward induction argument.

2.2. **Proof of Theorem 2.3.** We start by defining a convexified version of the optimal design problem (5). Since the set $\mathcal{U}_L$ does not have good compactness properties ensuring the existence of a solution of (5), we consider the convex closure of $\mathcal{U}_L$ for the weak star topology of L^∞, given by

$$\overline{\mathcal{U}}_L = \left\{ b \in L^\infty(\Omega,[0,1]) \;\middle|\; \int_\Omega b(x)\,dx = L|\Omega| \right\}.$$

Replacing $\chi_\omega \in \mathcal{U}_L$ with $a \in \overline{\mathcal{U}}_L$, we define the convexified formulation of the problem (5) by

$$\sup_{b\in\overline{\mathcal{U}}_L} C_{T,\text{rand}}(b), \tag{7}$$

where

$$C_{T,\text{rand}}(b) = \inf_{j\in\mathbb{N}^*} \frac{1}{\lambda_1(M_j)} \int_\Omega b(x)\phi_j(x)^2\,dx.$$

Note that

$$\sup_{\chi_\omega\in\mathcal{U}_L} C_{T,\text{rand}}(\chi_\omega) \leq \sup_{b\in\overline{\mathcal{U}}_L} C_{T,\text{rand}}(b).$$

Since $b \mapsto C_{T,\text{rand}}(b)$ is defined as the infimum of linear functionals that are continuous for the weak star topology of L^∞, it is upper semi-continuous for this topology. The following lemma is then obvious.

Lemma 2.5. *The problem* (7) *has at least one solution denoted* b^*.

The functional $C^N_{T,\text{rand}}$ is naturally extended to $\overline{\mathcal{U}}_L$ by

$$C^N_{T,\text{rand}}(b) = \min_{1\le j\le N} \frac{1}{\lambda_1(M_j)} \int_\Omega b(x)\phi_j(x)^2\,dx,$$

for every $b \in \overline{\mathcal{U}}_L$.

Let us first prove that there exists $N_0 \in \mathbb{N}^*$ such that $C_{T,\text{rand}}(b^*) = C^{N_0}_{T,\text{rand}}(b^*)$. Let $\varepsilon \in \left(0, L - \frac{4}{\lambda_1(M_{j_0})\int_0^T e^{-at}\,dt}\right)$. It follows from (H1), (H2) and from Lemma 2.2 that there exists $N_0 \in \mathbb{N}^*$ such that

$$\frac{1}{\lambda_1(M_j)} \int_\Omega b^*(x)\phi_j(x)^2\,dx \ge \frac{\int_0^T e^{-at}\,dt}{4}(L-\varepsilon), \tag{8}$$

for every $j > N_0$. Therefore, using the fact that $L - \varepsilon > \frac{4}{\lambda_1(M_{j_0})\int_0^T e^{-at}\,dt}$, we get

$$\begin{aligned} C_{T,\text{rand}}(b^*) &= \inf_{j\in\mathbb{N}^*} \frac{1}{\lambda_1(M_j)} \int_\Omega b^*(x)\phi_j(x)^2\,dx \\ &= \min\left(\inf_{1\le j\le N_0} \frac{1}{\lambda_1(M_j)} \int_\Omega b^*\phi_j^2, \inf_{j>N_0} \frac{1}{\lambda_1(M_j)} \int_\Omega b^*\phi_j^2 \right) \\ &\ge \min\left(C^{N_0}_{T,\text{rand}}(b^*), \frac{\int_0^T e^{-at}\,dt}{4}(L-\varepsilon)\right) = C^{N_0}_{T,\text{rand}}(b^*), \end{aligned}$$

since $C^{N_0}_{T,\text{rand}}(b^*) \le \frac{1}{\lambda_1(M_{j_0})}$. It follows that $C_{T,\text{rand}}(b^*) = C^{N_0}_{T,\text{rand}}(b^*)$.

Let us now prove that $C_{T,\text{rand}}(b^*) = C^{N_0}_{T,\text{rand}}(b^{N_0})$, where b^{N_0} is the unique maximizer of $C^{N_0}_{T,\text{rand}}$ (see Theorem 2.1). By definition of a maximizer, we have $C_{T,\text{rand}}(b^*) = C^{N_0}_{T,\text{rand}}(b^*) \le C^{N_0}_{T,\text{rand}}(b^{N_0})$. Reasoning by contradiction, assume that $C^{N_0}_{T,\text{rand}}(b^*) < C^{N_0}_{T,\text{rand}}(b^{N_0})$. Let us then design an admissible perturbation $b_t \in \overline{\mathcal{U}}_L$ of b^* such that $C_{T,\text{rand}}(b_t) > C_{T,\text{rand}}(b^*)$, which will then raise a contradiction with the optimality of b^*. For every $t \in [0,1]$, we set $b_t = b^* + t(b^{N_0} - b^*)$. Since $C^{N_0}_{T,\text{rand}}$ is concave, we get $C^{N_0}_{T,\text{rand}}(b_t) \ge (1-t)C^{N_0}_{T,\text{rand}}(b^*) + tC^{N_0}_{T,\text{rand}}(b^{N_0}) > C^{N_0}_{T,\text{rand}}(b^*)$, for every $t \in (0,1]$, which means that

$$\inf_{1\le j\le N_0} \frac{1}{\lambda_1(M_j)} \int_\Omega b_t(x)\phi_j(x)^2\,dx > \inf_{1\le j\le N_0} \frac{1}{\lambda_1(M_j)} \int_\Omega b^*(x)\phi_j(x)^2\,dx \ge C_{T,\text{rand}}(b^*), \tag{9}$$

for every $t \in (0,1]$. Besides, since $b^{N_0}(x) - b^*(x) \in (-2,2)$ for almost every $x \in \Omega$, it follows from (8) that

$$\begin{aligned} &\frac{1}{\lambda_1(M_j)} \int_\Omega b_t(x)\phi_j^2(x)\,dx \\ &\qquad = \frac{1}{\lambda_1(M_j)} \left(\int_\Omega b^*(x)\phi_j(x)^2\,dx + t\int_\Omega (b^{N_0}(x) - b^*(x))\phi_j(x)^2\,dx \right) \ge L - \varepsilon - 2t, \end{aligned}$$

for every $j \ge N_0$. Let us choose t such that $0 < t < \frac{1}{2}\left(L - \varepsilon - \frac{4}{\lambda_1(M_{j_0})\int_0^T e^{-at}\,dt}\right)$, so that the previous inequality yields

$$\frac{1}{\lambda_1(M_j)} \int_\Omega b_t(x)\phi_j(x)^2\,dx > \frac{1}{\lambda_1(M_{j_0})} \ge \frac{1}{\lambda_1(M_{j_0})} \int_\Omega b^*(x)\phi_{j_0}(x)^2\,dx \ge C_{T,\text{rand}}(b^*),$$

for every $j \geq N_0$. Combining the estimate (9) on the low modes with the above estimate on the high modes, we conclude that

$$C_{T,\text{rand}}(b_t) = \inf_{j\in\mathbb{N}^*} \frac{1}{\lambda_1(M_j)} \int_\Omega b_t(x)\phi_j(x)^2\,dx > C_{T,\text{rand}}(b^*),$$

which contradicts the optimality of b^*.

Therefore $C^{N_0}_{T,\text{rand}}(b^*) = C_{T,\text{rand}}(b^*) = C^{N_0}_{T,\text{rand}}(b^{N_0})$, and the result follows.

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